A Simple Method to Test the Zeros of Riemann Zeta Function

Zhengjun Cao

Department of Mathematics, Shanghai University, Shangda Road 99,
Shanghai, 200444, China. caozhj@shu.edu.cn.

Abstract

The zeta function $\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}$ is convergent only for $\mathrm{Re}(z)>1$. The Riemann-Siegel function is $Z(t)=e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$. If $Z(t_1)$ and $Z(t_2)$ have opposite signs, $Z(t)$ vanishes between t_1 and t_2, and $\zeta(z)$ has a zero on the critical line between $\frac{1}{2}+it_1$ and $\frac{1}{2}+it_2$. This method to test zeros is too hard to practice for newcomers. The eta function $\eta(z)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^z}$ is convergent for $\mathrm{Re}(z)>0$, and $\eta(z)=\left(1-\frac{2}{2^z}\right)\zeta(z)$ for the critical strip $0<\mathrm{Re}(z)<1$. So, $\eta(z)$ and the analytic continuation of $\zeta(z)$ have the same zeros in the critical strip, and the alternating series can be directly used to test the zeros.

Keywords: Riemann zeta function, Dirichlet eta function, partial sum, absolute convergence.

1 Introduction

The Riemann zeta function $\zeta(z)=\sum\limits_{n=1}^{\infty}\frac{1}{n^z}, z\in\mathbb{C}$, is absolutely convergent in the region $\mathrm{Re}(z)>1$. It is well known [1] that $\zeta(2)=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$. By the famous functional equation [2], $\zeta(z)=2^z\pi^{z-1}\sin\frac{z\pi}{2}\Gamma(1-z)\zeta(1-z)$, we have $\zeta(-1)=2^{-1}\pi^{-2}\sin\frac{-\pi}{2}\Gamma(2)\zeta(2)=\frac{1}{2\pi^2}\times(-1)\times 1\times\frac{\pi^2}{6}=-\frac{1}{12}$. But by the original series, we have $\zeta(-1)=1+2+3+\cdots\to\infty$.

Actually, in the above functional equation, $\zeta(z)$ *and* $\zeta(1-z)$ *cannot be concurrently convergent*, because at least one of $\mathrm{Re}(z)$ and $\mathrm{Re}(1-z)$ is strictly smaller than 1. So, $\zeta(z)$ and $\zeta(1-z)$ must be two different branches of the analytic continuation of the original series on the complex plane.

The famous Riemann zeros are not for the original series, instead for a branch of its analytic continuation. The general method to test these zeros needs to use the Riemann-Siegel function, which is defined by $Z(t) = e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$. If $Z(t_1)$ and $Z(t_2)$ have opposite signs, $Z(t)$ vanishes between t_1 and t_2, and so $\zeta(z)$ has a zero on the critical line between $\frac{1}{2}+it_1$ and $\frac{1}{2}+it_2$. Clearly, this method is too hard to practice for newcomers, and the mysterious zeros have not been broadly exhibited to the average person.

In this paper, we present a simple method to test these famous zeros. The method is based on that $\eta(z) = \left(1 - \frac{2}{2^z}\right)\zeta(z)$ in the critical strip $0 < \mathrm{Re}(z) < 1$, where the Dirichlet eta function $\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$ is convergent for $\mathrm{Re}(z) > 0$. This relationship shows that $\eta(z)$ and the analytic continuation of $\zeta(z)$ have the same zeros in the critical strip, and we can use the alternating series to test the zeros.

2 Zeta function and Eta function

The Riemann zeta function is further represented as [3]

$$
\begin{aligned}
\zeta(z) &= \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} e^{-z\ln n} \xlongequal[a,b\in\mathbb{R}]{z=a+ib} \sum_{n=1}^{\infty} e^{-(a+ib)\ln n} \\
&= \sum_{n=1}^{\infty} e^{-a\ln n} e^{-ib\ln n} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \cos(b\ln n) - i\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \sin(b\ln n)
\end{aligned} \tag{1}
$$

If $a > 1$, both $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \cos(b\ln n)$ and $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^a \sin(b\ln n)$ are absolutely convergent. Therefore, $\zeta(z)$ has no zeros for $a > 1$.

The Dirichlet eta function [4] is the alternating series

$$\eta(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}, \quad z \in \mathbb{C}. \tag{2}$$

$\eta(0)$ is defined to be $1/2$. $\eta(1) = \ln 2$, $\eta(2) = \frac{\pi^2}{12}$. Notice that, for $\mathrm{Re}(z) > 1$

$$
\begin{aligned}
\frac{2}{2^z}\zeta(z) &= \frac{2}{2^z}\left(1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots\right) = \frac{2}{2^z} + \frac{2}{4^z} + \frac{2}{6^z} + \frac{2}{8^z} + \cdots, \\
\left(1 - \frac{2}{2^z}\right)\zeta(z) &= \left(1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots\right) - \left(\frac{2}{2^z} + \frac{2}{4^z} + \frac{2}{6^z} + \frac{2}{8^z} + \cdots\right) \\
&\xlongequal{\text{rearranged}} 1 + \left(\frac{1}{2^z} - \frac{2}{2^z}\right) + \frac{1}{3^z} + \left(\frac{1}{4^z} - \frac{2}{4^z}\right) + \cdots \\
&= 1 - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z} = \eta(z).
\end{aligned}
$$

Extending this relationship $\eta(z) = (1-2^{1-z})\zeta(z)$ to the complex plane, we can obtain the functional equation $\zeta(z) = 2^z\pi^{z-1}\sin\frac{z\pi}{2}\Gamma(1-z)\zeta(1-z)$. If $z = -2, -4, -6, \cdots$,

$\sin\frac{z\pi}{2}=0$. These values are called simple zeros of $\zeta(z)$. Since $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$, we know $\Gamma(1-z)$ has no zeros. So, $\zeta(z)=0$ iff $\zeta(1-z)=0$, which also implies that $\zeta(\bar{z})=\zeta(1-\bar{z})=0$. The famous Riemann hypothesis [5] claims that all the complex zeros of $\zeta(z)$ lie on the critical line $\mathrm{Re}(z)=1/2$.

In history, the zeros of $\zeta(z)$ were very hard to calculate [6]. Nowadays, several million zeros have been obtained [7]. We refer to the table of zeros https://www-users.cse.umn.edu/~odlyzko/zeta_tables/index.html. It is worth noting that the symbol $\zeta(z)$ didn't refer to the original series, instead its analytic continuation.

3 The general method to test zeros

The general method to test zeros is based on the famous functional equation. Define the functions

$$\chi(z)=2^{z-1}\pi^z\sec\frac{z\pi}{2}\,/\,\Gamma(z),\quad \vartheta=\vartheta(t)=-\frac{|\chi(\frac{1}{2}+it)|}{2}\arg\chi(\frac{1}{2}+it),$$

where $\Gamma(z)=\int_0^\infty x^{z-1}e^{-x}\mathrm{d}x$, and the Riemann-Siegel function

$$Z(t)=e^{i\vartheta(t)}\zeta(\frac{1}{2}+it) \tag{3}$$

which is real for real values of t. The Riemann-Siegel theta function appearing above is also defined by

$$\vartheta(t)=\arg[\Gamma(\frac{1}{4}+\frac{1}{2}it)]-\frac{t}{2}\ln\pi. \tag{4}$$

If $Z(t_1)$ and $Z(t_2)$ have opposite signs, $Z(t)$ vanishes between t_1 and t_2, and so $\zeta(z)$ has a zero on the critical line between $\frac{1}{2}+it_1$ and $\frac{1}{2}+it_2$.

To calculate the first nontrivial zero, one needs to determine the sign of $Z(0)=e^{i\vartheta(\frac{1}{2})}\zeta(\frac{1}{2})$. If $z=1/2$, $\eta(1/2)=1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\cdots$, which converges to a positive number. Since $\eta(1/2)=(1-2^{1/2})\zeta(1/2)$ and $1-\sqrt{2}<0$, it claims that $\zeta(1/2)<0$ (page 388, Ref.[2]). Define

$$\xi(z)=\frac{1}{2}z(z-1)\pi^{-\frac{z}{2}}\Gamma(\frac{z}{2})\zeta(z). \tag{5}$$

Hence, $\xi(1/2)=-\frac{1}{8}\pi^{-\frac{1}{4}}\Gamma(\frac{1}{4})\zeta(1/2)$. Since $\zeta(\frac{1}{2})<0$ and $\Gamma(\frac{1}{4})>0$, then $\xi(\frac{1}{2})>0$, which implies $Z(0)<0$. By numerical analysis, it shows that $Z(6\pi)>0$. Therefore, there is one zero at least on the critical line between $t=0$ and $t=6\pi$. We currently know that the first zero approximates to $1/2+14.1347251\,i$.

4 A simple method to test zeros

For the first three zeros

$$r_1=1/2+14.134725\,i,\quad r_2=1/2+21.0220396\,i,\quad r_3=1/2+25.01085758\,i,$$

we have the following numerical calculations (see Table 1).

Table 1: Numerical calculations for the first three zeros

Partial-sum	$\eta(1/2+14.134725\,i)=c-di$	$\lvert c\rvert+\lvert d\rvert$
100000	-0.00127695-0.000932693 I	0.00220965
200000	0.00108099 +0.000285084 I	0.00136607
300000	0.000629314 +0.000661002 I	0.00129032
400000	-0.000785402+0.0000901101 I	0.000875512
500000	0.000701391 -0.0000898594 I	0.000791251
600000	-0.00058472-0.000273739 I	0.00085846
700000	0.00010144 +0.000588673 I	0.000690113
800000	0.00049395 -0.000261985 I	0.000755936
900000	-0.000289405-0.000440756 I	0.000730162
Partial-sum	$\eta(1/2+21.0220396\,i)=c-di$	$\lvert c\rvert+\lvert d\rvert$
100000	0.00156992 -0.000194757 I	0.00176467
200000	-0.000590181-0.000950496 I	0.00154068
300000	-0.000307477+0.00085796 I	0.00116544
400000	-0.000432487+0.000660018 I	0.0010925
500000	-0.000582426-0.000401353 I	0.000983779
600000	0.000643547 -0.0000592712 I	0.000702818
700000	-0.000586334+0.000110852 I	0.000697186
800000	0.000553754 +0.0000797987 I	0.000633552
900000	-0.000362282-0.000383504 I	0.000745786
Partial-sum	$\eta(1/2+25.010857\,i)=c-di$	$\lvert c\rvert+\lvert d\rvert$
100000	-0.000747071-0.00139243 I	0.0021395
200000	0.000953789 -0.000583543 I	0.00153733
300000	-0.000273221+0.000871604 I	0.00114482
400000	0.00045176 +0.000649957 I	0.00110172
500000	-0.0000663671+0.000704695 I	0.000771062
600000	-0.000625278-0.000157189 I	0.000782467
700000	0.000534317 -0.000268088 I	0.000802405
800000	-0.000439384+0.000345612 I	0.000784996
900000	0.000470723 -0.00023747 I	0.000708193

With the finite precision, we have the faith in that the three values are really zeros of eta function. Of course, they are not for the original zeta series. Practically, the three series $\zeta(r_1), \zeta(r_2), \zeta(r_3)$ are divergent, not convergent.

Theorem 1. *Let $z=\frac{1}{2}+bi, b>0$. Denote the partial sum $\sum_{n=1}^{k}\frac{(-1)^{n-1}}{n^z}$ by $c-di$, for some positive integer k. Then the modulus $|c-di|$ is continuous with respect to b.*

Proof. It is easy to see that

$$c-di=\sum_{n=1}^{k}\frac{(-1)^{n-1}}{n^z}=\sum_{n=1}^{K}(-1)^{n-1}e^{-z\ln n}\overset{z=\frac{1}{2}+ib}{=\!=\!=}\sum_{n=1}^{k}(-1)^{n-1}e^{-(\frac{1}{2}+ib)\ln n}$$

$$= \sum_{n=1}^{k} (-1)^{n-1} \sqrt{\frac{1}{n}} \cos(b \ln n) - i \sum_{n=1}^{k} (-1)^{n-1} \sqrt{\frac{1}{n}} \sin(b \ln n),$$

$$|c - di| = \sqrt{\left(\sum_{n=1}^{k} (-1)^{n-1} \sqrt{\frac{1}{n}} \cos(b \ln n) \right)^2 + \left(\sum_{n=1}^{k} (-1)^{n-1} \sqrt{\frac{1}{n}} \sin(b \ln n) \right)^2}.$$

Since all $\cos(b \ln n), \sin(b \ln n), n = 1, \cdots, k$, are continuous with respect to b, the above modulus is also continuous with respect to b. □

Based on this theorem, we now present a new method (see Algorithm 1) to search for a zero in a short interval. Let $s_k := \sum_{n=1}^{k} (-1)^{n-1} e^{-(\frac{1}{2}+ib)\ln n}$. We compute the mean of partial sums $s_{k_1}, s_{k_2}, \cdots, s_{k_\ell}$, so as to partly offset the roundoff errors.

Algorithm 1: Testing zeros of Dirichlet eta series in the critical strip

---

Input: $(b_1, b_2), b_2 > b_1 > 0$, which contains at least one zero of eta series, and a set of positive integers $K = \{k_1, k_2, \cdots, k_\ell\}$, $k_1 < k_2 < \cdots < k_\ell$.

Output: $(c, d) \subset (b_1, b_2)$, which contains at least one zero of eta series.

$steplen \leftarrow 1/4$ (or 1/32, 1/256, etc), $stepnum \leftarrow (b_2 - b_1)/steplen$

$l \leftarrow 0$, $r \leftarrow 0$, $T \leftarrow \{\ \}$ // T is the empty set

for $j = 0, j \leq stepnum$ **do**

  $b \leftarrow b_1 + steplen * j$, $S \leftarrow \{\ \}$ // S is the empty set

  for $n = 1, n \leq k_\ell$ **do**

    $l \leftarrow l + (-1)^{n-1} \sqrt{\frac{1}{n}} \cos(b \ln n)$

    $r \leftarrow r + (-1)^{n-1} \sqrt{\frac{1}{n}} \sin(b \ln n)$

    if $n \in K$ **then**

      $s \leftarrow l - ri$ // $i^2 = -1$

      $S \leftarrow S \cup \{|s|\}$ // $|s|$ is the modulus of s

  $t \leftarrow$ the mean value of S

  $T \leftarrow T \cup \{(b, t)\}$

Find $(\hat{b}, \hat{t}) \in T$, with a local minimum $\hat{t}$

$c \leftarrow \hat{b} - steplen, d \leftarrow \hat{b} + steplen$

Theorem 2. *The computational cost for Algorithm 1 is $O(50k_\ell(3.32p+\log_2(k_\ell))^2)$, where p is the accuracy, i.e., the effective number of these digits which appear to the right of the decimal point.*

Proof. The longest binary length of operands in the procedure is $\log_2(k_\ell)$ (for integer part) plus $\log_2(10^p)$ (for fractional part). The total iteration number is $stepnum \times k_\ell$. Usually, $stepnum = 50$ which suffices to determine the local minimums in a short interval. Note that $\log_2(10) \approx 3.32$. So, the computational cost for a multiplication is $O((3.32p + \log_2(k_\ell))^2)$, and the total cost is $O(50k_\ell(3.32p + \log_2(k_\ell))^2)$. □

The following Mathematica code can be directly used to test the zeros, in which we take $k_\ell = 80000$.

```
Eta[b_,k_,mylist_]:= Module[{a,n,l,r,s,t,U,V,precision},
   l = r = 0; U = V = {}; a = 1/2; precision = 10;
   For[n = 1, n <=k, n++,
      l = N[l + (-1)^(n - 1)/(n^a)*Cos[b*Log[n]], precision];
      r = N[r + (-1)^(n - 1)/(n^a)*Sin[b*Log[n]], precision];
      If[MemberQ[mylist, n], s = l - r*I; t = Abs[s];
        U = AppendTo[U, {n, s, t}]]];
   V = U];
Eta2[b1_, b2_, steplen_, k_, mylist_] :=
  Module[{A, B, stepnum, b, j, W, v, precision},
  A = B = W = {}; precision = 10; stepnum = (b2 - b1)/steplen;
  For[j = 0, j <= stepnum, j++, b = b1 + steplen*j;
       A = Eta[b, k, mylist]; v = N[Mean[A[[All, 3]]], precision];
       B = AppendTo[B, {b, v}]];
  W = B]
```

```
 k = 80000; mylist = Table[j*10^4, {j, 3, 8}];
 b1 = 60.0; b2 = 70.0; steplen = 1/4;
 A = Eta2[b1, b2, steplen, k, mylist]; Print[A];
 ListLinePlot[A, Mesh -> Full]
```

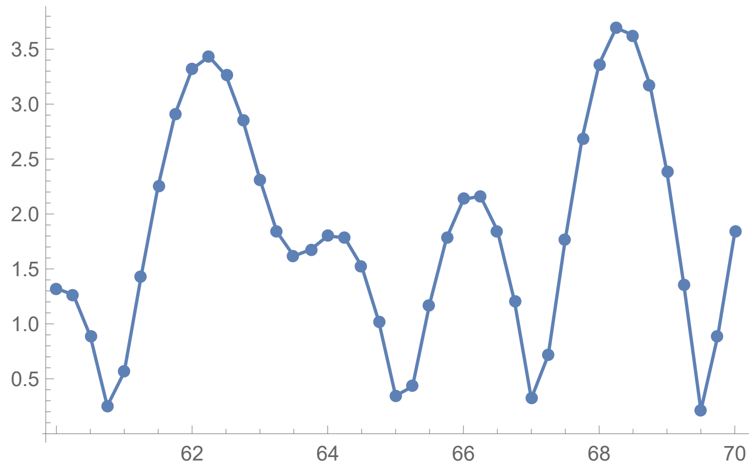

Fig. 1 The local minimums for the interval (60, 70)

```
{{60.,1.32234},{60.25,1.26547},{60.5,0.889173},{60.75,0.252221},
{61.,0.560395},{61.25,1.4362},{61.5,2.25372},{61.75,2.90991},
{62.,3.31871},{62.25,3.4328},{62.5,3.26017},{62.75,2.8505},
{63.,2.31537},{63.25,1.83795},{63.5,1.61724},{63.75,1.67664},
```

```
{64.,1.80111},{64.25,1.78278},{64.5,1.52398},{64.75,1.02702},
{65.,0.345057},{65.25,0.431093},{65.5,1.17698},{65.75,1.77798},
{66.,2.13208},{66.25,2.16306},{66.5,1.85003},{66.75,1.21334},
{67.,0.32057},{67.25,0.715488},{67.5,1.76543},{67.75,2.69039},
{68.,3.36349},{68.25,3.69287},{68.5,3.62832},{68.75,3.17258},
{69.,2.38021},{69.25,1.3504},{69.5,0.212189},{69.75,0.89441},
{70.,1.83443}}
```

By the Fig.1, we see there are four local minimums of modulus, corresponding to the tuples $(60.75, 0.252221), (65, 0.345057), (67, 0.32057), (69.5, 0.212189)$. So, the four possible intervals are $(60.5, 61), (64.75, 65.25), (66.75, 67.25), (69.25, 69.75)$. In fact, the target zeros are $60.8317785, 65.112544, 67.07981, 69.546401$.

For the first interval $(60.5, 61)$, we have the following results.

```
b1 = 60.5; b2 = 61.0; steplen=1/32;
A = Eta2[b1, b2, steplen, k, mylist]; Print[A];
ListLinePlot[A, Mesh -> Full]
```

```
{{60.5,0.889173},{60.5313,0.822285},{60.5625,0.751447},{60.5938,0.676809},
{60.625,0.598526},{60.6563,0.516757},{60.6875,0.431669},{60.7188,0.34343},
{60.75,0.252221},{60.7813,0.158227},{60.8125,0.0616484},{60.8438,0.0373803},
{60.875,0.138516},{60.9063,0.241628},{60.9375,0.346471},{60.9688,0.452808},
{61.,0.560395}}
```

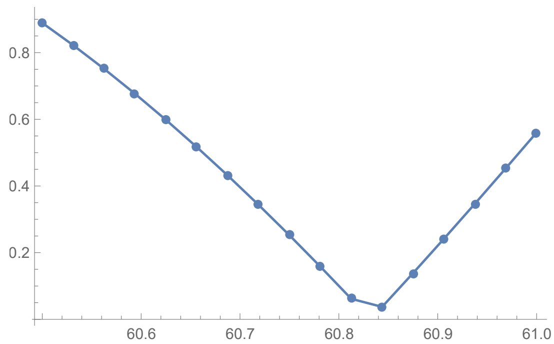

Fig. 2 The local minimum for the interval (60.5, 61)

By the Fig.2, it is easy to see that the local minimum of modulus is 0.0373803, corresponding to the tuple $(60.8438, 0.0373803)$. So, the shorter interval is $(60.8125, 60.875)$, which still contains the target zero 60.8317785.

```
b1 = 60.8125; b2 = 60.875; steplen=1/256;
A = Eta2[b1, b2, steplen, k, mylist]; Print[A];
ListLinePlot[A, Mesh -> Full]
```

```
{{60.8125,0.0616484},{60.8164,0.049407},{60.8203,0.0371318},
{60.8242,0.0248276},{60.8281,0.0125183},{60.832,0.00206596},
```

```
{60.8359,0.0124851},{60.8398,0.0248998},{60.8438,0.0373803},
{60.8477,0.0499026},{60.8516,0.0624616},{60.8555,0.0750554},
{60.8594,0.0876828},{60.8633,0.100343},{60.8672,0.113036},
{60.8711,0.12576},{60.875,0.138516}}
```

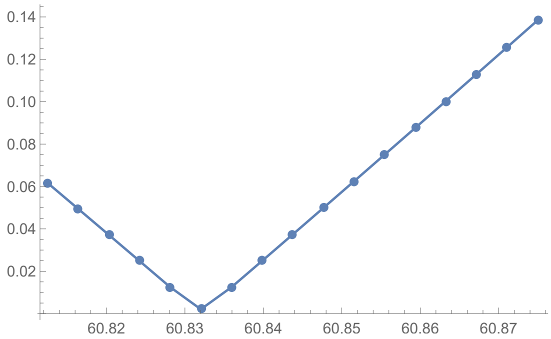

Fig. 3 The local minimum for the interval (60.8125, 60.875)

With the shorter step length 1/256, we find the local minimum of modulus is 0.00206596, corresponding to the tuple $(60.832, 0.00206596)$. So, the shorter interval is $(60.8281, 60.8359)$, which still contains the target zero 60.8317785.

By the similar procedure, we obtain the strictly decreasing modulus chain

$$\text{modulus}: \quad 0.252221 > 0.0373803 > 0.00206596 > \cdots$$

corresponding to the nested intervals

$$\text{intervals}: \quad (60.5, 61) \supset (60.8125, 60.875) \supset (60.8281, 60.8359) \supset \cdots$$

Finally, we can obtain a more accurate approximation of the target zero.

5 Conclusion

We show that the Dirichlet eta function and the analytic continuation of Riemann zeta function have the same zeros in the critical strip. Based on this relationship and that the partial sum of eta series is continuous, we present a simple method to test zeros. The programming code is also presented, which is very easy to execute. To the best of our knowledge, it is the first time to invent such a simple method to test the famous zeros.

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