



LeOPaRd: Towards Practical Post-Quantum Oblivious PRFs via Interactive Lattice Problems

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Abstract. In this work, we introduce a more efficient post-quantum oblivious PRF (OPRF) design, called LeOPaRd. Our proposal is round-optimal and supports verifiability and partial obliviousness, all of which are important for practical applications. The main technical novelty of our work is a new method for computing samples of MLWE (module learning with errors) in a two-party setting. To do this, we introduce a new family of interactive lattice problems, called *interactive MLWE and rounding with re-use* (iMLWER-RU). We rigorously study the hardness of iMLWER-RU and reduce it (under some natural “idealized” setting) to a more standard MLWE-like problem where the adversary is additionally given access to a randomized MLWE PRF oracle. We believe iMLWER-RU can be of independent interest for other interactive protocols.

LeOPaRd exploits this new iMLWER-RU assumption to realize a lattice-based OPRF design without relying on heavy machinery such as noise flooding and fully homomorphic encryption used in earlier works. LeOPaRd can feature around 136 KB total communication, compared to 300+ KB in earlier works. We also identify gaps in some existing constructions and models, and propose appropriate fixes.

Keywords: Oblivious PRF · Post-Quantum · Lattice · Zero-Knowledge

1 Introduction

An oblivious pseudorandom function (OPRF) extends a standard PRF in the following manner. In a standard PRF, a party in possession of a secret key k , can evaluate the PRF F , on any given input x to obtain the output $y = F(k, x)$. The pseudorandomness property of a PRF states that it is infeasible to distinguish the outputs of this function from those of a random function. Now, in OPRF, the goal is to compute the PRF output y in a two-party server-client model, where the server holds the secret key k and the client has the input x . After the client and the server run the two-party OPRF protocol, we want the client to learn the PRF output y without learning anything about the secret key k , and the server does not learn any information on the input x or the output y . We say that an OPRF is *verifiable* OPRF (VOPRF) if the client is guaranteed that the output received is indeed evaluated under the committed key. As discussed in [CHL22, TCR⁺22], many applications of OPRFs such as for password-authenticated key exchange, checking compromised credentials and spam detection require a part of the client’s input to be public to allow for domain separation for the PRF computation. In this case, we divide the client’s input into a private part, x , and a public part, t , called the tag, and we call the resulting primitive a *partial* OPRF (POPRF).

(VP)OPRFs have become an important tool for privacy-preserving protocols and have numerous applications, including but not limited to private lightweight authentication mechanisms [DGS⁺18], private set intersection for checking compromised credentials [LPA⁺19, TPY⁺19], secure data de-duplication [KBR13], password-protected secret sharing [JKK14, JKKX16] and password-authenticated key exchange [JKX18].

Despite the widespread use of (VP)OPRFs, the existing constructions are either efficient but based on classical assumptions [FIPR05, JL09, JKK14, TCR⁺22], or based on (plausibly) post-quantum assumptions [ADDS21, ADDG24, BDFH24, AG24] but practically inefficient. Given the recent standardization efforts by the IETF³ for the DH-based OPRFs of [JKK14, TCR⁺22] and by NIST for other post-quantum primi-

³ <https://datatracker.ietf.org/doc/rfc9497/>

| Scheme | Assumption | Rounds | Communication | Model |
|--|--------------|--------|---------------|--------------|
| [ADDS21] | RLWE + SIS | 2 | >128GB | strong, QROM |
| [ADDG24] | heuristic | 2 | 15MB + 633KB | strong, ROM |
| [BDFH24] | Legendre PRF | 9 | 2.8MB + 110KB | strong, ROM |
| [AG24], $Q = 2^{16}$ | RLWE | 2 | 108KB + 189KB | weak, ROM |
| [AG24], $Q = 2^{32}$ | RLWE | 2 | 114KB + 198KB | weak, ROM |
| LeOPaRd, $Q_x = 2^{16}$ | iMLWER-RU + | 2 | 10KB + 126KB | strong, ROM |
| LeOPaRd, $Q_x = 2^{32}$ (Section 4) | MLWE + MSIS | 2 | 20KB + 159KB | strong, ROM |

Table 1: Comparison of post-quantum verifiable (P)OPRF constructions. When bandwidth is reported as a sum, this is for a one-time offline cost and online costs per query, respectively. We do not consider amortization over batched queries in this table. The server-to-client communication in LeOPaRd, in particular, reduces to just 2-3 KB per query for a batch of 64 queries (see Table 3). Q denotes a *global* upperbound on the number of OPRF queries; while Q_x denotes the maximum number of OPRF queries *per fixed* PRF tag/input pair (t, x) .

tives⁴, it is imperative to find efficient post-quantum OPRF constructions, which can pave the way for future post-quantum OPRF standards. We summarize the state of the art in post-quantum verifiable (P)OPRFs in Table 1 (extending a table from [AG24]). All schemes here provide security against malicious clients and servers (as needed in many applications). We also distinguish between the security models used in the schemes as “strong” if a stronger model as in [TCR+22] or UC framework is used⁵, and as “weak” for otherwise weaker models.

1.1 Our Contributions

Novel post-quantum VPOPRF construction. Our first contribution in this work is a construction of a lattice-based VPOPRF, called LeOPaRd, that is more efficient than the current state-of-the-art. The comparison of our construction with the existing post-quantum VPOPRFs is given in Table 1, and a more thorough performance analysis is provided in Section 6. The efficiency gains of our construction come not only from utilizing the state-of-the-art lattice-based NIZK proof systems, such as LaBRADOR⁶ [BS23], but also by using a novel interactive assumption that we introduce, and cleverly combining this with some additional lattice-based techniques that we explain in Section 1.3. We prove our construction secure in the random oracle model in Section 4.2. Specifically, we show that our construction achieves the strong security definitions from [TCR+22]: pseudorandomness, which provides security in the presence of malicious clients (POPRF security), and request privacy against malicious servers (POPRIV2 security). Moreover, in Appendix C.2 we show that our construction also achieves uniqueness, which in the verifiable setting ensures to the clients that the server honestly and consistently performs the blind evaluations.

Interactive MLWE assumption. We introduce a novel interactive assumption, called *interactive Module Learning with Errors and Rounding with Re-Use* (iMLWER-RU), which we use to prove the security of our construction (against malicious clients). We believe introduction of such an interactive MLWE-like assumption is a natural next step given the interactive nature of protocols we deal with. This assumption not only allows us to prove the security of our OPRF construction, but also allows us to circumvent the shortcomings of the prior lattice-based OPRF constructions [ADDS21, ADDG24], such as removing the need for noise

⁴ <https://csrc.nist.gov/projects/post-quantum-cryptography>, <https://csrc.nist.gov/projects/pqc-dig-sig>

⁵ We note that the UC model of OPRF is even stronger than the one given in [TCR+22].

⁶ At the time of writing, there is no general-purpose implementation of LaBRADOR [BS23] available publicly. We are aware of the LaZer library [SS24] (that includes LaBRADOR) for such purpose, but it is not yet publicly available. We plan to use this library, when available in future, to implement our OPRF proposal in full.

flooding or fully homomorphic encryption. The interactive MLWE assumption can be seen as a family of assumptions, and in Section 3, we show that for some “idealized” version of iMLWER-RU, defined under appropriately chosen parameters, we can provide a reduction to (a mild variant of) the MLWE problem. Our ‘idealized’ variant of iMLWER-RU makes two natural idealized assumptions, which we believe still capture the hardness of iMLWER-RU. The first assumption is to model the deterministic rounding error in the underlying lattice-based PRF by a random discrete Gaussian error of the same standard deviation. The second assumption is that the two interactive oracles available to the attacker in iMLWER-RU are merged into one oracle that runs both of the original oracles. Our reduction provides an initial evidence regarding the conjectured hardness of the iMLWER-RU assumption, and we encourage more research into investigating its hardness. We note that the iMLWER-RU assumption might be of independent interest and find use in proving the security of other types of interactive protocols, such as blind signatures, as an alternative to one-more type assumptions.

Issues with existing constructions and models. As an additional contribution, we uncover flaws in the security proofs of two existing lattice-based OPRF constructions, [ADDS21] and [ADDG24]. We refer to Appendix C.1 for a more detailed exposition of these flaws and their potential fixes. We also observe that a reduction made in [TCR+22], that correctness and POPRIV2 security together imply uniqueness, does not necessary hold unless we have a one-to-one correspondence between the OPRF secret and public keys. In Appendix C.2, we patch this reduction by introducing a new property, called *key binding*, and show that correctness, key binding and POPRIV2 are sufficient for uniqueness. Along the way we also prove that LeOPaRd achieves key binding.

1.2 Related Work

The first post-quantum, and also lattice-based, VOPRF construction was given in [ADDS21], which although relied on a standard RLWE and SIS assumptions, was totally impractical as it required communication costs in the order of GBs. This was improved in [ADDG24] by combining fully homomorphic encryption with the weak PRF given in [BIP+18], but the reported bandwidth per query was in the order of MBs. Moreover, in order to achieve verifiability they needed to rely on a heuristic assumption about the hardness of evaluating deep circuits in an FHE scheme supporting only shallow circuits. Recently, Beullens et al. [BDFH24] provided a VOPRF construction that is based on the Legendre PRF and which makes use of oblivious transfer (OT) and ZK proofs that can be instantiated from lattice assumptions, resulting also in a plausibly post-quantum secure VOPRF. However, this construction also has communication costs in the order of MBs. Apart from these constructions, Basso [Bas24] provided the first isogeny-based VOPRF construction based on a new ad-hoc assumption called uniSIDH, which was later discovered to be insecure as stated in the author’s revised draft at the time of the writing [Bas24].

Comparison with concurrent work [AG24]. In a concurrent and independent work published on IACR’s ePrint just a few days before we uploaded this paper to ePrint, Albrecht and Gur [AG24] provided a new lattice-based (threshold) VOPRF construction. Their construction significantly improves, both assumption and performance wise, the state-of-the-art lattice-based VOPRF given in [ADDG24] as shown in Table 1. Similar to ours, their starting point is the OPRF construction given in [ADDS21], however, unlike us, they take a different technical route. They improve the construction given in [ADDS21] by first removing the reliance on the 1D-SIS assumption by borrowing a trick from [GdKQ+23] and making use of Rényi divergence to avoid noise flooding, which together remove a superpolynomial factor from their modulus q . However, switching to Rényi divergence forbids them from using an indistinguishability-based security model, and in turn forces them to use a *weaker* security model than the one we use, which is the well-established OPRF security model given by Tyagi et al. [TCR+22]. Specifically, the model given in [TCR+22] is a simulation-based indistinguishability model that provides a careful and granular tracking of the oracle calls and comes close to the UC security model for blind protocols given in [JKK14]. On the other hand, the security model used by Albrecht and Gur [AG24] is akin to the one-more unforgeability definition used in the context of blind signatures, and is comparatively weaker than the model of Tyagi et al. [TCR+22]. Moreover, the use of Rényi divergence forces them to use a *global* bound on number of OPRF queries, which

can potentially hinder the practicality. Our proposal, on the other hand, relies on a bound on the number of queries *per input/tag pair* (x, t) .

Apart from the aforementioned improvements over the prior work [ADDS21], Albrecht and Gur [AG24] also make use of the newer lattice-based proof systems, specifically LNP22 [LNP22] and LaBRADOR [BS23], analogous to us. However, in order for their security proof to go through they require the underlying proof system to be straight-line extractable. While this can be done for LNP22 using the Katsumata transform [Kat21] (albeit by incurring a significant performance penalty), it is an open problem how to obtain straight-line extractability for LaBRADOR. We note that we do not require straight-line extractability from the underlying NIZK proof systems for our security proofs. Lastly, the additional overhead that comes from the straight-line extractability, along with the fact that they aim only for 90-100 bit security level, are not properly reflected in their reported numbers, which are also shown in Table 1. Given that they [AG24] rely on (fully structured) Ring LWE problem, they have less flexibility in choosing their parameters, and would need to incur almost $2\times$ performance penalty if their parameters aimed at 128-bit or higher security level. To match the security level of [AG24], our reported results in Table 1 also aim at 90-95 bits of security, and we provide parameters and performance results at 128-bit security level in Table 3.

1.3 Technical Overview

VOPRF construction. The 2HashDH OPRF paradigm [JKK14] (and its extension to partially oblivious setting in [TCR+22]) has been shown to be highly effective in the discrete-log setting. Therefore, our goal is to efficiently realize this high-level paradigm in the lattice setting. An initial attempt was already done in [ADDS21], which serves as a starting point for our scheme. For simplicity, our discussion here focuses on the regular OPRF setting (without partial obliviousness).

The high-level idea of the 2HashDH paradigm is to first hash the PRF input x to some group element $b_x = \mathcal{G}_1(x)$. This term is then blinded via a randomness r and the resulting value $c_x = \mathcal{G}_1(x)^r$ is sent to the server. The server then blindly performs PRF evaluation using its key k as $u_x = c_x^k$. The blinded evaluation result u_x is sent to the client, who unblinds the value using their randomness r to obtain a PRF value $z = \mathcal{G}_1(x)^k$. The final PRF output is computed via a second hashing as $y = \mathcal{G}_2(x, z)$. The blind/unblind operations rely on the homomorphic properties of the computations performed in the protocol.

To realize this idea in the lattice setting over a ring R_q with modulus q , we want to encrypt the matrix $\mathbf{B}_x = \mathcal{G}(x)$ using homomorphic encryption. We can do this as follows.

1. The client computes $\mathbf{C}_x = \mathbf{R}\mathbf{A}_r + \mathbf{B}_x$ for randomness \mathbf{R} and public matrix \mathbf{A}_r .
2. Upon receiving \mathbf{C}_x , the server can compute $\mathbf{u}_x = \mathbf{C}_x\mathbf{k} + \mathbf{e}'_s$ using its key \mathbf{k} with some error vector \mathbf{e}'_s . Observe that $\mathbf{u}_x = \mathbf{R}\mathbf{A}_r\mathbf{k} + \mathbf{B}_x\mathbf{k} + \mathbf{e}'_s$.
3. Given \mathbf{u}_x and additionally $\mathbf{v}_k = \mathbf{A}_r\mathbf{k} + \mathbf{e}_s$, the client can compute $\mathbf{u}_x - \mathbf{R}\mathbf{v}_k = \mathbf{B}_x\mathbf{k} + \mathbf{e}_f$ for some final error term \mathbf{e}_f .

Provided the final error is small enough, the client can round it off to arrive at $\lfloor \mathbf{B}_x\mathbf{k} \rfloor_p$ where $\lfloor \cdot \rfloor_p$ denotes dividing each coefficient by p/q and rounding to the nearest integer. Both parties additionally use NIZK proofs to show the correctness of their computations to provide security and verifiability. What we have discussed so far is effectively the high-level blueprint in [ADDS21].

The main technical difficulty in realizing this idea *efficiently* is that we need to argue that \mathbf{u}_x does not leak information about the server's key \mathbf{k} . Observe that even though \mathbf{u}_x looks like an MLWE sample, the matrix \mathbf{C}_x is controlled by the (malicious) client, and therefore, is not guaranteed to be uniformly random. To get around this problem, [ADDS21] relies on noise flooding (a.k.a. smudging) with an exponentially large error \mathbf{e}'_s with coefficients of size $O(2^\lambda)$ to make sure that no secret information is leaked. However, then the final error term that we need to get rid of is also of size $O(2^\lambda)$, meaning that we need $q = O(2^{2\lambda})$ to arrive at the correct PRF output with overwhelming probability. Noting that the overall communication size is *quadratic* in $\log q$, such a large modulus leads to an inefficient construction⁷.

⁷ We note here that [ADDS21] also used earlier NIZK proof systems, which are very inefficient compared to more state-of-the-art proofs of today. However, even without considering NIZKs, [ADDS21] incurs communication in the order of MBs.

Beyond using state-of-the-art NIZK proofs in our scheme, the main technical novelty of our LeOPaRd proposal is that we introduce a new method to argue server security *without* relying on noise flooding. In particular, we want to force the client into committing to its input x and randomness \mathbf{R} *before* seeing the public matrix \mathbf{A}_r . To this end, the client first commits to (\mathbf{R}, x) , resulting in \mathbf{c}_r , and uses a hash function (modeled as a random oracle) to derive the matrix \mathbf{A}_r . One can immediately observe that the client’s control in \mathbf{C}_x is now significantly limited and intuitively revealing \mathbf{u}_x as above should be secure based on the randomness of \mathbf{A}_r and the fact that the client in any case obtains a noisy vector close to $\mathbf{B}_x \mathbf{k}$ in their final step.

Formal security analysis via an interactive MLWE assumption. We go beyond intuitive thinking and analyze the security of the construction rigorously. The client’s commitment \mathbf{c}_r to (\mathbf{R}, x) can be seen as the client submitting an oracle query with input (\mathbf{R}, x) to obtain back a random matrix \mathbf{A}_r . To realize this in the security reduction while adhering to client security, we use an *extractable* commitment scheme that enables extraction/decryption of the committed message using a trapdoor (that the reduction knows in the security analysis). We then use our new interactive MLWE-like assumption, iMLWER-RU, where the adversary is given access to a SampMLWE-RU oracle that, on input (\mathbf{R}, x) returns a random matrix \mathbf{A}_r , an MLWE sample \mathbf{c} under \mathbf{A}_r , and an MLWE-like sample \mathbf{u} under \mathbf{C}_x akin to the OPRF paradigm as above. Note that \mathbf{A}_r is re-used in both \mathbf{c} and \mathbf{u} , and hence the naming of the assumption. To enable simulation in the pseudorandomness analysis, the adversary is also given access to a SampPRF oracle that outputs a rounding sample $\lfloor \mathbf{B}_x \mathbf{k} \rfloor_p$ for a given x .

Security reduction for iMLWE-RU-id from MLWE-PRF. We define an “idealized” version of iMLWER-RU, denoted as iMLWE-RU-id, which is derived from iMLWER-RU by applying two idealized assumptions (see Section 3), and then give a hardness reduction for iMLWE-RU-id from a mild variant of MLWE that we call MLWE-PRF. The MLWE-PRF assumption is that the randomized function $x \mapsto \mathcal{G}(x) \cdot \mathbf{k} + \mathbf{e}$, where \mathbf{e} is a small error following a Gaussian distribution is pseudorandom even when given additional MLWE samples for \mathbf{k} with respect to uniformly random matrices \mathbf{A}_r . Therefore, the MLWE-PRF assumption is a mild (randomized) variant of the pseudorandomness of the MLWR-based PRF $x \mapsto \mathcal{G}(x) \cdot \mathbf{k}$, which has been proved based on the MLWE assumption when \mathcal{G} is implemented appropriately as in [BLMR13] and [BP14].

In the iMLWE-RU-id problem, each of the attacker’s oracle queries $(x, \{\mathbf{R}_j\}_j)$ is answered with MLWE samples $(\mathbf{B}_x, \mathbf{y}_B = \mathbf{B}_x \mathbf{k} + \mathbf{e}_B)$ and $(\mathbf{A}_{r,j}, \mathbf{c}_j = \mathbf{A}_{r,j} \mathbf{k} + \mathbf{e}_j)$ where $\mathbf{B}_x := \mathcal{G}(x)$, with respect to “fresh” matrices along with a “re-used” matrix MLWE samples of the form $(\mathbf{C}_{x,j} := \mathbf{R}_j \mathbf{A}_{r,j} + \mathbf{B}_x, \mathbf{u}_j = \mathbf{C}_{x,j} \mathbf{k} + \mathbf{e}'_j)$, reminiscent of the “Re-Used A LWE” [MS23] and Hint-MLWE problem [KLSS23]. Accordingly, our security reduction from iMLWE-RU-id to MLWE-PRF proceeds in two steps. In the first step, we reduce a variant of the Hint-MLWE problem called HintMLWE-PRF to iMLWE-RU-id. In the HintMLWE-PRF problem, besides the MLWE samples w.r.t. \mathbf{B}_x and $\mathbf{A}_{r,j}$ matrices, the adversary is also provided with appropriate linear combinations of the error vectors as hints of the form $\mathbf{h}_j := \mathbf{R}_j \mathbf{e}_j + \mathbf{e}_B - \mathbf{e}'_j$. Note that in HintMLWE-PRF each hint contains a linear combination of several error vectors (rather than hints containing only scalar multiples of the error vector as in [KLSS23]). Hence HintMLWE-PRF can be viewed as a special case of the Extended Hint-MLWE [WSE24] that generalizes Hint-MLWE [KLSS23] to the case of general hint matrices. In the second step, we present a reduction from MLWE-PRF to HintMLWE-PRF by optimizing the reduction from MLWE to Extended Hint-MLWE [WSE24] to the special form of hint matrices in HintMLWE-PRF. This reduces to (see Section 3.2) finding an optimized upper bound on the matrix norm B_r of our HintMLWE-PRF hint matrix.

Overall, we achieve a reduction from MLWE-PRF to iMLWE-RU-id, provided that the necessary parameter constraints are satisfied. Our reduction is essentially tight in the sense that our reduction’s parameter condition on the underlying error width parameter is *necessary* to prevent an existing *attack* on iMLWE-RU-id. In particular, our reduction requires the “masking ratio” lower bound $\sigma_1/\sigma \geq \sqrt{B_r} \approx \sqrt{Q_x^\infty}$ for large Q_x^∞ , where σ_1 and σ denote the width (std dev) of the masking errors \mathbf{e}'_j and MLWE errors \mathbf{e}_j , respectively, and Q_x^∞ denotes an upper bound on the oracle samples queried by the adversary *per input* x (corresponding to input/tag pair (x, t) in our OPRF protocol). On the other hand, this lower bound is necessary to protect against the following simple error ‘averaging’ attack on iMLWE-RU-id (and our OPRF protocol as well as those using the same blueprint [ADDS21, AG24]). In this attack, the adversary uses the Q_x^∞ samples on

same x queried to the oracle to compute $\mathbf{z}'_j := \mathbf{u}_j - \mathbf{R}_j \mathbf{c}_j = \mathbf{B}_x \mathbf{k} + \mathbf{e}'_j - \mathbf{R}_j \mathbf{e}_j$ for $j \in [Q_x^\infty]$. The attacker then computes the average (over the rationals) of these noisy secrets to get $\bar{\mathbf{z}} := \frac{1}{Q_x^\infty} \sum_{j \in [Q_x^\infty]} \mathbf{z}'_j = \mathbf{B}_x \mathbf{k} + \bar{\mathbf{e}}$ in which the error $\bar{\mathbf{e}} := \frac{1}{Q_x^\infty} \sum_{j \in [Q_x^\infty]} (\mathbf{e}'_j - \mathbf{R}_j \mathbf{e}_j)$ has a reduced standard deviation $\bar{\sigma} \approx \sigma_1 / \sqrt{Q_x^\infty}$. If $\bar{\sigma}$ is a constant factor smaller than $1/2$, all the $\bar{\mathbf{e}}$ error coordinates will likely be $< 1/2$ and rounding $\bar{\mathbf{z}}$ to the nearest integers will give the attacker the secret $\mathbf{B}_x \mathbf{k}$ (and hence \mathbf{k}) with high probability. Consequently, to prevent the averaging attack, a necessary condition is $\sigma_1 \geq \sqrt{Q_x^\infty}/c$ for a small constant $c > 1$, and more generally, a necessary condition to prevent the averaging attack from reducing the error standard deviation (and MLWE security) below the σ standard deviation used to set MLWE security in the samples \mathbf{c}_j , is that $\sigma_1/\sigma \geq \sqrt{Q_x^\infty}/c$. This lower bound matches (up to the small factor c) our iMLWE-RU-id security reduction parameter lower bound discussed above (for large Q_x^∞), showing the optimality of our reduction condition up to a small factor. Note that the attack and our security reduction bounds only depend on the maximum no. of samples queried per x value. This is in contrast to the suboptimal Rényi-divergence-based security analysis in [AG24], which seems to inherently lose a factor proportional to \sqrt{Q} , where Q is the total number of oracle queries over *all* x values.

2 Preliminaries

Notation. We denote the security parameter by $\lambda \in \mathbb{N}$, and $\kappa \in \mathbb{N}$ denotes the statistical correctness parameter. We will aim to obtain the correct PRF value except with probability at most $2^{-\kappa}$. For $n \in \mathbb{N}$, set $[n] = \{1, 2, \dots, n\}$. We denote by $x \stackrel{\$}{\leftarrow} X$ the uniform sampling of the variable x from the set X , and we denote the uniform distributions on a set X by $\mathcal{U}(X)$. We write $x \leftarrow \mathbf{A}(y)$ to denote that a probabilistic polynomial time (PPT) algorithm \mathbf{A} on input y , outputs x . If \mathbf{A} is a deterministic polynomial time (DPT) algorithm, we use the notation $x := \mathbf{A}(y)$. We use the same notation for the projection of tuples, e.g., we write $\sigma := (\sigma_1, \sigma_2)$ for a tuple σ composed of two elements σ_1 and σ_2 . We define *polynomial* functions as $\text{poly}(\lambda) = \bigcup_{d \in \mathbb{N}} O(\lambda^d)$ and *negligible* functions as $\text{negl}(\lambda) = \bigcap_{d \in \mathbb{N}} o(\lambda^{-d})$.

We denote by $R_q = \mathbb{Z}_q[X]/(X^d + 1)$. We write R_{bin} to denote the set of polynomials in R_q that has binary coefficients. For $a \in \mathbb{Z}^+$, we use \mathbb{S}_a to denote the set of polynomials in R_q with infinity norm at most a . The rounding operation $\lfloor a \rfloor_p : R_q \rightarrow R_p$ is defined as multiplying a by p/q and rounding the result to the nearest integer. When \mathbf{a} is a vector, then by $\lfloor \mathbf{a} \rfloor_p$ we consider coordinate-wise rounding. For a matrix \mathbf{R} , we denote by $\|\mathbf{R}\|$, $\|\mathbf{R}\|_1$ and $\|\mathbf{R}\|_{\infty, M}$ its matrix 2-norm (largest singular value), matrix 1-norm and matrix ∞ -norm, respectively and by $\sigma_{\min}(\mathbf{R})$ and $\sigma_{\max}(\mathbf{R})$ the smallest (resp. largest) singular values of \mathbf{R} . We denote by $\|\mathbf{R}\|_\infty$ the entry-wise maximum absolute value of \mathbf{R} .

As the NIZK relations to be proven get more complex for various protocols, we believe it is imperative to have a more explicit and concise notation. We introduce the notation “ $\mathfrak{D}(x); \mathfrak{K}(w)$ ” to describe a NIZK relation to mean “given a statement x and knowledge of witness w ”.

Next, we provide some of the preliminaries needed for the rest of the paper, and refer the reader to Appendix A for additional preliminaries.

2.1 (Partially) Oblivious PRF

We use the game-based definitions given in [TCR+22] and [ADDG24]. An OPRF is a protocol between a server \mathbf{S} that has a private key k and a client \mathbf{C} that wants to obtain an evaluation of PRF F_k on inputs of its choice. We say that an OPRF is a partial OPRF (POPRF) if part of the client’s input is given to the server, i.e., $y = F_k(t, x)$, where t is in the public part and x is the private part hidden from \mathbf{S} . When the client can verify the correctness and consistency of the PRF evaluations, then we consider verifiable OPRF (VOPRF).

Definition 1 (Partial Oblivious PRF [TCR+22]). *A partial oblivious PRF (POPRF) \mathbf{F} is a tuple of PPT algorithms*

(Setup, KeyGen, Request, BlindEval, Finalize, Eval).

The `Setup` algorithm and the `KeyGen` algorithm generate public parameters pp and a public/secret key pair (pk, sk) , respectively. Oblivious evaluation is carried out as an interactive protocol between \mathcal{C} and \mathcal{S} , presented as algorithms `F.Request`, `F.BlindEval`, `F.Finalize`, which work as follows (where the random oracles are denoted by RO):

1. First, \mathcal{C} runs the algorithm `F.Request` $_{\text{pp}}^{\text{RO}}(\text{pk}, t, x)$ taking as input a public key pk , a (public) tag t and a private input x . It outputs a local state st and a request message req , which is sent to \mathcal{S} .
2. \mathcal{S} runs `F.BlindEval` $_{\text{pp}}^{\text{RO}}(\text{sk}, t, \text{req})$ taking as input a secret key sk , a (public) tag t and the request message req . It produces a response message rep , which is sent to \mathcal{C} .
3. Finally, \mathcal{C} runs `F.Finalize` $_{\text{pp}}^{\text{RO}}(\text{rep}, \text{st})$ taking as input the response message rep and its previously constructed state st . It outputs either a PRF evaluation y or \perp if rep is rejected.

The unblinded evaluation algorithm `F.Eval` is deterministic and takes as input a secret key sk , an input pair (t, x) , and outputs a PRF evaluation y .

We also define sets $\mathcal{SK}, \mathcal{PK}, \mathcal{T}, \mathcal{X}, \mathcal{Z}$, representing the secret key, public key, (public) tag, private input, and output spaces, respectively. We define the input space as $\mathcal{T} \times \mathcal{X}$. We assume there exist efficient algorithms for sampling and membership queries on these sets.

We remark that fixing $t = \perp$, gives us the definition of (plain) OPRF. We use the correctness definition from [ADDG24], which allows for a small failure probability, and then define *pseudorandomness* against malicious clients as in [TCR+22].

Definition 2 (Correctness [ADDG24]). A partial oblivious PRF (POPRF) \mathcal{F} is correct, if the following holds for $\text{pp} \leftarrow \mathcal{F}.\text{Setup}(1^\lambda)$ and $(\text{pk}, \text{sk}) \leftarrow \mathcal{F}.\text{KeyGen}_{\text{pp}}^{\text{RO}}(1^\lambda)$

$$\Pr \left[y = \mathcal{F}.\text{Eval}_{\text{pp}}^{\text{RO}}(\text{sk}, t, x) \mid \begin{array}{l} (\text{st}, \text{req}) \leftarrow \mathcal{F}.\text{Request}_{\text{pp}}^{\text{RO}}(\text{pk}, t, x) \\ \text{rep} \leftarrow \mathcal{F}.\text{BlindEval}_{\text{pp}}^{\text{RO}}(\text{sk}, t, \text{req}) \\ y \leftarrow \mathcal{F}.\text{Finalize}_{\text{pp}}^{\text{RO}}(\text{rep}, \text{st}) \end{array} \right] = 1 - \text{negl}(\lambda).$$

Definition 3 (Pseudorandomness [TCR+22]). We say that a partial oblivious PRF (POPRF) \mathcal{F} is pseudorandom if for all PPT adversaries \mathcal{A} , there exists a PPT simulator \mathcal{S} and a negligible function $\text{negl}(\cdot)$, such that the following holds,

$$\text{Adv}_{\mathcal{F}, \mathcal{A}, \mathcal{S}, \text{RO}}^{\text{po-prf}}(\lambda) = \left| \Pr [\text{POPRF}_{\mathcal{F}, \mathcal{A}, \mathcal{S}, \text{RO}}^1(\lambda) = 1] - \Pr [\text{POPRF}_{\mathcal{F}, \mathcal{A}, \mathcal{S}, \text{RO}}^0(\lambda) = 1] \right| \leq \text{negl}(\lambda),$$

where the experiment `POPRF` is defined in Figure 1.

The oracle `Prim` captures access to the random oracle(s) used in the POPRF construction. For $b = 0$ (i.e., when the adversary interacts with a simulator and a truly random function), the simulator may only use a limited number of random function queries to simulate the random oracle accessed via `Prim`.

Lastly, we define the *request privacy*, which provides security against malicious servers. We present the stronger `POPRIV2` definition satisfied by our proposal, and refer the reader to [TCR+22] for the weaker `POPRIV1` definition.

Definition 4 (Request Privacy [TCR+22]). We say that a partial oblivious PRF (POPRF) \mathcal{F} has request privacy against malicious servers if for all PPT adversaries, there exists a negligible function $\text{negl}(\cdot)$, such that the following holds,

$$\text{Adv}_{\mathcal{F}, \mathcal{A}, \text{RO}}^{\text{po-priv-k}}(\lambda) = \left| \Pr [\text{POPRIV}[k]_{\mathcal{F}, \mathcal{A}, \text{RO}}^1(\lambda) = 1] - \Pr [\text{POPRIV}[k]_{\mathcal{F}, \mathcal{A}, \text{RO}}^0(\lambda) = 1] \right| \leq \text{negl}(\lambda),$$

where the experiment `POPRIV` $[k]$ is defined in Figure 1.

| | |
|---|--|
| <p>POPFRF_{F,A,S,RO}^b(λ)</p> <hr/> <pre> 1 : q_{t,s} := 0, q_t := 0 2 : st_F ← RO_F.Init(1^λ) // T × X → Z 3 : st_{RO} ← RO.Init(1^λ) 4 : pp₁ ← F.Setup(1^λ) 5 : (st_S, pk₀, pp₀) ← S.Init(pp₁) 6 : (sk, pk₁) ← F.KeyGen_{pp₁}^{RO}(1^λ) 7 : b' ← A^{Eval, BlindEval, Prim}(pp_b, pk_b) 8 : return b' </pre> <p>BlindEval(t, req)</p> <hr/> <pre> 1 : q_t := q_t + 1 2 : (rep₀, st_S) ← S.BlindEval^{LimitEval}(t, req, st_S) 3 : rep₁ ← F.BlindEval_{pp₁}^{RO}(sk, t, req) 4 : return rep_b </pre> | <p>Eval(t, x)</p> <hr/> <pre> 1 : z₀ ← RO_F.Eval((t, x), st_F) 2 : z₁ ← F.Eval_{pp₁}^{RO}(sk, t, x) 3 : return z_b </pre> <p>LimitEval(t, x)</p> <hr/> <pre> 1 : q_{t,s} := q_{t,s} + 1 2 : if q_{t,s} ≤ q_t then 3 : return Eval(t, x) 4 : return ⊥ </pre> <p>Prim(x)</p> <hr/> <pre> 1 : (h₀, st_S) ← S.Eval^{LimitEval}(x, st_S) 2 : h₁ ← RO.Eval(x, st_{RO}) 3 : return h_b </pre> |
| <p>POPRIV2_{F,A,RO}^b(λ)</p> <hr/> <pre> 1 : pp ← F.Setup(1^λ) 2 : i := 0 3 : b' ← A^{Request, Finalize, RO}(pp) 4 : return b' </pre> <p>Request(pk, t, x₀, x₁)</p> <hr/> <pre> 1 : i := i + 1 2 : (st_{i,0}, req₀) ← F.Request_{pp}^{RO}(pk, t, x₀) 3 : (st_{i,1}, req₁) ← F.Request_{pp}^{RO}(pk, t, x₁) 4 : return (req_b, req_{1-b}) </pre> | <p>Finalize(j, rep₀, rep₁)</p> <hr/> <pre> 1 : if j > i then 2 : return ⊥ 3 : y_b ← F.Finalize_{pp}^{RO}(rep₀, st_{j,b}) 4 : y_{1-b} ← F.Finalize_{pp}^{RO}(rep₁, st_{j,1-b}) 5 : if y₀ = ⊥ ∨ y₁ = ⊥ then 6 : return ⊥ 7 : return (y₀, y₁) </pre> |

Fig. 1: POPRF and POPRIV2 Experiments.

2.2 Lattice Preliminaries

We start with the definitions of standard problems in lattice-based cryptography.

Definition 5 (MSIS_{n,m,β}). Let $\mathbf{A} \xleftarrow{\$} R_q^{n \times m}$, where $n, m > 0$ and let $0 < \beta < q$. We say $\mathbf{z} \in R_q^m$ is a solution for MSIS_{n,m,B} problem if $\mathbf{A}\mathbf{z} = 0$ over R_q and $0 < \|\mathbf{z}\| < \beta$. If the following inequality holds for an adversary \mathcal{A}

$$\Pr \left[0 < \|\mathbf{z}\| < \beta \wedge \mathbf{A}\mathbf{z} = 0 \mid \mathbf{A} \xleftarrow{\$} R_q^{n \times m}, \mathbf{z} \leftarrow \mathcal{A}(\mathbf{A}) \right] \geq \epsilon,$$

then we say \mathcal{A} has advantage ϵ in solving MSIS_{n,m,B}.

Definition 6 (MLWE_{n,m,χ}). Let χ be an error distribution over R , $\mathbf{A} \xleftarrow{\$} R_q^{m \times n}$, where $m, n > 0$, let $\mathbf{s} \xleftarrow{\$} \chi^n$ be a secret vector and $\mathbf{e} \xleftarrow{\$} \chi^m$ be an error vector. The MLWE_{n,m,χ} problem asks an adversary \mathcal{A} to distinguish between $(\mathbf{A}, \mathbf{A}\mathbf{s} + \mathbf{e})$ and (\mathbf{A}, \mathbf{b}) , for $\mathbf{b} \xleftarrow{\$} R_q^m$. We say \mathcal{A} has advantage ϵ against the MLWE_{n,m,χ}

problem if

$$\Pr \left[b = 1 \mid \mathbf{A} \stackrel{\$}{\leftarrow} R_q^{m \times n}, \mathbf{s} \stackrel{\$}{\leftarrow} \chi^n, \mathbf{e} \stackrel{\$}{\leftarrow} \chi^m, b \leftarrow \mathcal{A}(\mathbf{A}, \mathbf{A}\mathbf{s} + \mathbf{e}) \right] \\ - \Pr \left[b = 1 \mid \mathbf{A} \stackrel{\$}{\leftarrow} R_q^{m \times n}, \mathbf{b} \stackrel{\$}{\leftarrow} R_q^m, b \leftarrow \mathcal{A}(\mathbf{A}, \mathbf{b}) \right] \geq \epsilon.$$

Definition 7 (knMLWE $_{n,m,h,\chi}$). Let χ be an error distribution over R , $\mathbf{A} \stackrel{\$}{\leftarrow} R_q^{n \times m}$, where $n > m > 0$, and let $\mathbf{S} \stackrel{\$}{\leftarrow} \chi^{h \times n}$. The knMLWE $_{n,m,h,\chi}$ problem asks an adversary \mathcal{A} to distinguish between $(\mathbf{A}, \mathbf{S}\mathbf{A})$ and (\mathbf{A}, \mathbf{U}) , for $\mathbf{U} \stackrel{\$}{\leftarrow} R_q^{h \times m}$. We say \mathcal{A} has advantage ϵ against the knMLWE $_{n,m,h,\chi}$ problem if

$$\Pr \left[b = 1 \mid \mathbf{A} \stackrel{\$}{\leftarrow} R_q^{n \times m}, \mathbf{S} \stackrel{\$}{\leftarrow} \chi^{h \times n}, b \leftarrow \mathcal{A}(\mathbf{A}, \mathbf{S}\mathbf{A}) \right] \\ - \Pr \left[b = 1 \mid \mathbf{A} \stackrel{\$}{\leftarrow} R_q^{n \times m}, \mathbf{U} \stackrel{\$}{\leftarrow} R_q^{h \times m}, b \leftarrow \mathcal{A}(\mathbf{A}, \mathbf{U}) \right] \geq \epsilon.$$

The duality between the above Knapsack form of MLWE and regular MLWE has been established already. That is, from a practical security perspective, knMLWE $_{n,m,h,\chi}$ is as hard as MLWE $_{n-m,m,\chi}$, which we will use to estimate the practical hardness of knMLWE $_{n,m,h,\chi}$. We refer to [EZS⁺19, Appendix C] for a summary discussion (and references therein for more details). Also, note that our knMLWE definition is in the ‘‘multi-secret’’ setting where multiple vectors are multiplied by the matrix \mathbf{A} . A standard hybrid argument of replacing each $\mathbf{s}_i^\top \mathbf{A}$ with a uniformly random \mathbf{u}_i^\top reduces its hardness to standard Knapsack form of MLWE.

The second parameter m in the definitions of MSIS, MLWE, and knMLWE does not play a critical role in our practical hardness estimations of these problems against state-of-the-art attacks (i.e., we assume that the adversary has the optimal choice of m). Therefore, we sometimes simply omit this parameter.

Definition 8 (Discrete Gaussian Distribution). For any $s > 0$ and dimension $n \in \mathbb{Z}^+$, the spherical n -dimensional Gaussian function with parameter s^2 is defined as $\rho_s(\mathbf{x}) := \exp(-\pi \|\mathbf{x}\|^2 / s^2)$ for $\mathbf{x} \in \mathbb{R}^n$. More generally, given a positive definite symmetric covariance parameter matrix $\Sigma \in \mathbb{R}^{n \times n}$ and center \mathbf{c} , the n -dimensional Gaussian function with covariance parameter Σ and center \mathbf{c} is defined as $\rho_{\Sigma, \mathbf{c}}(\mathbf{x}) := \exp(-\pi(\mathbf{x} - \mathbf{c})^\top \Sigma^{-1}(\mathbf{x} - \mathbf{c}))$ for $\mathbf{x} \in \mathbb{R}^n$. The discrete Gaussian distribution $\mathcal{D}_{\Lambda, \Sigma, \mathbf{c}}$ over an n -dimensional lattice $\Lambda \subseteq \mathbb{R}^n$ with covariance parameter Σ , centre \mathbf{c} and support Λ is defined as $D_{\Lambda, \Sigma, \mathbf{c}}(\mathbf{x}) := \frac{\rho_{\Sigma, \mathbf{c}}(\mathbf{x})}{\sum_{\mathbf{y} \in \Lambda} \rho_{\Sigma, \mathbf{c}}(\mathbf{y})}$ for $\mathbf{x} \in \Lambda$. In the spherical case, where $\Sigma = s^2 \mathbf{I}_n$, we write $D_{\Lambda, s, \mathbf{c}}$, and we omit \mathbf{c} if it is $\mathbf{0}$.

Lemma 1 ([DPS23], [BLP⁺13, Le. 2.3]). There exists a ppt algorithm that, given a basis $\mathbf{B} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ of a full rank n -dimensional lattice Λ , a positive definite symmetric matrix Σ and a center $\mathbf{c} \in \mathbb{R}^n$, returns a sample from $\mathcal{D}_{\Lambda, \Sigma, \mathbf{c}}$, assuming the condition $\sqrt{\ln(2n+4)/\pi} \cdot \max_i \|\Sigma^{-1/2} \mathbf{b}_i\| \leq 1$.

Lemma 2 ([KLSS23], [Pei10]). For $n \in \mathbb{Z}^+$, $\epsilon \in \mathbb{R}^+$, let Σ_1, Σ_2 be positive definite symmetric matrices such that $\Sigma_3^{-1} := \Sigma_1^{-1} + \Sigma_2^{-1}$ satisfies $\sqrt{\Sigma_3} \geq \eta_\epsilon(\mathbb{Z}^n)$ for $0 < \epsilon < 1/2$. Then for an arbitrary center $\mathbf{c} \in \mathbb{Z}^n$, the distribution

$$\{\mathbf{x}_1 + \mathbf{x}_2 : \mathbf{x}_1 \stackrel{\$}{\leftarrow} \mathcal{D}_{\mathbb{Z}^n, \Sigma_1}, \mathbf{x}_2 \stackrel{\$}{\leftarrow} \mathcal{D}_{\mathbb{Z}^n, \Sigma_2, \mathbf{c}}\}$$

is within statistical distance $\leq 2\epsilon$ of $\mathcal{D}_{\mathbb{Z}^n, \Sigma_1 + \Sigma_2, \mathbf{c}}$.

Definition 9 (Smoothing Parameter [MR04, Pei10]). For an n -dimensional lattice Λ and $\epsilon > 0$, the smoothing parameter $\eta_\epsilon(\Lambda)$ is the smallest s such that $\rho_{1/s}(\Lambda^* \setminus \mathbf{0}) \leq \epsilon$, where Λ^* denotes the dual lattice of Λ . More generally, for a positive definite symmetric matrix Σ , we say that $\sqrt{\Sigma} \geq \eta_\epsilon(\Lambda)$ if $\eta_\epsilon(\sqrt{\Sigma}^{-1} \cdot \Lambda) \leq 1$.

Lemma 3 ([MR04]). For $n \in \mathbb{Z}^+$, $\epsilon \in \mathbb{R}^+$, and n dimensional lattice Λ , we have

$$\eta_\epsilon(\Lambda) \leq \sqrt{\frac{\ln(2n(1+1/\epsilon))}{\pi}} \cdot \lambda_n(\Lambda), \quad (1)$$

where $\lambda_n(\Lambda)$ is the smallest radius of an n -dimensional ball that contains n linearly independent vectors in Λ .

⁸ Note that the parameter s is related to the standard deviation σ by $s = \sqrt{2\pi} \cdot \sigma$.

Lemma 4 ([KLSS23]). For $n \in \mathbb{Z}^+, \epsilon \in \mathbb{R}^+$, n dimensional lattice Λ , and a positive definite symmetric matrix Σ , we have $\sqrt{\Sigma} \geq \eta_\epsilon(\Lambda)$ if $\|\Sigma^{-1}\| \leq \eta_\epsilon(\Lambda)^{-2}$.

Lemma 5 ([WSE24], Generalization of [KLSS23, Le. 7]). Fix $s_1 \in \mathbb{R}^+, n, \ell \in \mathbb{Z}^+$, a positive definite symmetric matrix Σ_1 , and a matrix $\bar{\mathbf{R}} \in \mathbb{Z}^{\ell \times n}$. Furthermore, let $\Sigma_0 := \left(\Sigma_1^{-1} + \frac{1}{s_1^2} \bar{\mathbf{R}}^\top \bar{\mathbf{R}}\right)^{-1}$. Then, the following two probability distributions over $\mathbb{Z}^{n+\ell}$ are equal:

$$\begin{aligned} \mathcal{D}_1 &:= \left\{ (\mathbf{e}, \mathbf{h}) : \mathbf{e} \stackrel{\$}{\leftarrow} \mathcal{D}_{\mathbb{Z}^n, \Sigma_1}, \mathbf{e}' \stackrel{\$}{\leftarrow} \mathcal{D}_{\mathbb{Z}^n, s_1}, \mathbf{h} = \bar{\mathbf{R}}\mathbf{e} + \mathbf{e}' \right\} \\ \mathcal{D}_2 &:= \left\{ (\tilde{\mathbf{e}}, \mathbf{h}) : \mathbf{e} \stackrel{\$}{\leftarrow} \mathcal{D}_{\mathbb{Z}^n, \Sigma_1}, \mathbf{e}' \stackrel{\$}{\leftarrow} \mathcal{D}_{\mathbb{Z}^n, s_1}, \mathbf{h} = \bar{\mathbf{R}}\mathbf{e} + \mathbf{e}' \right\} \\ &\quad \mathbf{c} := \frac{1}{s_1^2} \Sigma_0 \bar{\mathbf{R}} \mathbf{h}, \tilde{\mathbf{e}} \stackrel{\$}{\leftarrow} \mathcal{D}_{\mathbb{Z}^n, \Sigma_0, \mathbf{c}}. \end{aligned}$$

Proof. Fix any $(\mathbf{v}, \mathbf{w}) \in \mathbb{Z}^n \times \mathbb{Z}^\ell$. For the first distribution we have:

$$\begin{aligned} \mathcal{D}_1(\mathbf{v}, \mathbf{w}) &= \mathcal{D}_{\mathbb{Z}^n, \Sigma_1}(\mathbf{v}) \cdot \mathcal{D}_{\mathbb{Z}^n, s_1}(\mathbf{w} - \bar{\mathbf{R}}\mathbf{v}) \\ &\propto \exp \left[-\pi \left(\mathbf{v}^\top \Sigma_1^{-1} \mathbf{v} - \frac{1}{s_1^2} (\mathbf{w} - \bar{\mathbf{R}}\mathbf{v})^\top (\mathbf{w} - \bar{\mathbf{R}}\mathbf{v}) \right) \right] \\ &= \exp \left[-\pi \left((\mathbf{v} - \mathbf{c})^\top \Sigma_0^{-1} (\mathbf{v} - \mathbf{c}) + \frac{1}{s_1^2} \mathbf{w}^\top \mathbf{w} - \mathbf{c}^\top \Sigma_0^{-1} \mathbf{c} \right) \right], \end{aligned}$$

where $\mathbf{c} := \frac{1}{s_1^2} \Sigma_0 \bar{\mathbf{R}}^\top \mathbf{w}$. Since $\frac{1}{s_1^2} \mathbf{w}^\top \mathbf{w} - \mathbf{c}^\top \Sigma_0^{-1} \mathbf{c}$ is a constant that does not depend on \mathbf{v} , it follows that the conditional distribution $\Pr_{(\mathbf{e}, \mathbf{h}) \stackrel{\$}{\leftarrow} \mathcal{D}_1}[\mathbf{e} = \mathbf{v} | \mathbf{h} = \mathbf{w}] = \mathcal{D}_{\mathbb{Z}^n, \Sigma_0, \mathbf{c}}(\mathbf{v})$, which is (by construction of \mathcal{D}_2) exactly equal to the conditional distribution $\Pr_{(\tilde{\mathbf{e}}, \mathbf{h}) \stackrel{\$}{\leftarrow} \mathcal{D}_2}[\tilde{\mathbf{e}} = \mathbf{v} | \mathbf{h} = \mathbf{w}]$ in \mathcal{D}_2 . Since the marginal distribution of \mathbf{h} is also exactly the same in \mathcal{D}_1 and \mathcal{D}_2 , we conclude that $\mathcal{D}_1 = \mathcal{D}_2$. \square

3 iMLWER-RU: New Interactive Lattice Problem

We introduce the following new interactive variant of the MLWE assumption, called Interactive MLWE (iMLWER-RU).

Definition 10 (Interactive MLWE - iMLWER-RU) $_{\mathcal{G}, Q_M, Q_P, Q_x^\infty, q, m, N, h, L, \beta_r, p, \bar{\chi}}$. Let $\bar{\chi} = (\chi, \chi_1, \chi_k)$ be a discrete distribution over R , \mathcal{G} be a hash family, and $Q, Q_x^\infty, q, m, N, h, L, \beta_r, p \in \mathbb{Z}^+$. We say that the interactive MLWE assumption $\text{iMLWER-RU}_{\text{param}}$ holds, for $\text{param} := (\mathcal{G}, Q_M, Q_P, Q_x^\infty, q, m, N, h, L, \beta_r, p, \bar{\chi})$, if for all PPT adversaries \mathcal{A} , there exists a negligible function $\text{negl}(\cdot)$, such that the following holds,

$$\text{Adv}_{\text{param}}^{\text{iMLWER-RU}}(\lambda) = \left| \Pr[\text{iMLWER-RU}^1(\lambda) = 1] - \Pr[\text{iMLWER-RU}^0(\lambda) = 1] \right| \leq \text{negl}(\lambda),$$

where the experiment iMLWER-RU^b is defined below, Q_M, Q_P denote the total number of queries to SampPRF , and SampMLWE-RU , respectively, made by \mathcal{A} during the experiment, and Q_x^∞ is the maximum allowed upper bound on the number of SampMLWE-RU queries per queried x .

| $\text{iMLWER-RU}_{\mathcal{G}, Q_M, Q_P, Q_x^\infty, q, m, N, h, L, \beta_r, p, \bar{\chi}}^b(\lambda)$ | $\text{SampPRF}^b(x)$ |
|---|--|
| $1 : \mathbf{k} \stackrel{\$}{\leftarrow} \chi_k^m$ $2 : b' \leftarrow \mathcal{A}^{\text{SampPRF}^b(\cdot), \text{SampMLWE-RU}^b(\cdot, \cdot, \cdot)}(1^\lambda)$ $3 : \text{return } b = b'$ | $1 : \text{if } b = 0 \text{ then}$ $2 : \quad \text{if } \mathcal{Z}[x] = \perp \text{ then}$ $3 : \quad \quad \mathbf{z}' \stackrel{\$}{\leftarrow} R_q^h, \mathcal{Z}[x] := \mathbf{z}'$ $4 : \quad \quad \mathbf{z} = \lfloor \mathcal{Z}[x] \rfloor_p$ |
| $\text{SampMLWE-RU}^b(\mathbf{R} \in R^{h \times N}, x \in \{0, 1\}^L)$ | $5 : \text{if } b = 1 \text{ then}$ $6 : \quad \mathbf{B}_x = \mathcal{G}(x) \in R_q^{h \times m}$ $7 : \quad \mathbf{z} = \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p \in R_p^h$ $8 : \text{return } \mathbf{z}$ |
| $1 : \text{if } \ \mathbf{R}\ _\infty > \beta_r \text{ then return } \perp$ $2 : \mathbf{A}_r \stackrel{\$}{\leftarrow} R_q^{N \times m}, \mathbf{e} \stackrel{\$}{\leftarrow} \chi^N$ $3 : \mathbf{B}_x = \mathcal{G}(x) \in R_q^{h \times m}$ $4 : \mathbf{C}_x = \mathbf{R} \mathbf{A}_r + \mathbf{B}_x \in R_q^{h \times m}, \mathbf{e}' \stackrel{\$}{\leftarrow} \chi_1^h$ $5 : \text{if } \mathcal{Z}[x] = \perp \text{ then SampPRF}^0(x)$ $6 : \mathbf{z}' := \mathcal{Z}[x]$ $7 : \tilde{\mathbf{e}}' \stackrel{\$}{\leftarrow} \chi_1^h, \tilde{\mathbf{e}} \stackrel{\$}{\leftarrow} \chi^N \quad // \text{ simulated errors}$ $8 : \mathbf{e}_{\text{sim}} = \tilde{\mathbf{e}}' - \mathbf{R} \tilde{\mathbf{e}} \in R^h$ $9 : \text{if } b = 0 \text{ then}$ $10 : \quad \mathbf{c} \stackrel{\$}{\leftarrow} R_q^N, \mathbf{u} = \mathbf{R} \mathbf{c} + \mathbf{z}' + \mathbf{e}_{\text{sim}} \in R_q^h$ $11 : \text{if } b = 1 \text{ then}$ $12 : \quad \mathbf{c} = \mathbf{A}_r \mathbf{k} + \mathbf{e} \in R_q^N, \mathbf{u} = \mathbf{C}_x \mathbf{k} + \mathbf{e}' \in R_q^h$ $13 : \text{return } (\mathbf{A}_r, \mathbf{c}, \mathbf{u})$ | |

Note that the iMLWER-RU security game uses a table \mathcal{Z} to store previous answers to SampPRF^0 queries, to ensure that the same random answer is returned if the oracle is queried again at the same point x . We are currently unable to give a security hardness reduction from existing variants of MLWE to our iMLWER-RU problem, and obtaining such a reduction is an interesting open problem. However, as initial evidence for the conjectured hardness of iMLWER-RU, we now define an “idealized” version of iMLWER-RU called iMLWE-RU-id, for which we prove, under appropriate parameter settings, a hardness security reduction from (a mild variant of) the MLWE problem we call MLWE-PRF.

The “idealized” problem iMLWE-RU-id is obtained from iMLWER-RU by making the following two idealized assumptions:

- **Idealized Assumption 1 (Rounding error to Random error):** The deterministic rounding error \mathbf{e}_B in $\mathbf{y}_B := \frac{q}{p} \mathbf{z} = \frac{q}{p} \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p := \mathbf{B}_x \mathbf{k} + \mathbf{e}_B \in R_q^h$ in the SampPRF^b oracle of iMLWER-RU is replaced in iMLWE-RU-id by an independent random error $\mathbf{e}_B \stackrel{\$}{\leftarrow} \chi_B^h$ from a distribution χ_B^h . In our security reduction from MLWE-PRF to iMLWE-RU-id, we instantiate χ_B^h by a discrete Gaussian $\mathcal{D}_{\mathbb{Z}^{hd}, s_B}$ with standard deviation $\sigma_B = s_B / \sqrt{2\pi} := \frac{q}{\sqrt{12p}}$ matching the standard deviation of a uniformly random rounding error in the interval $[-\frac{q}{2p}, \frac{q}{2p}]$. Our security reduction also assumes that χ and χ_1 are discrete Gaussian distributions.
- **Idealized Assumption 2 (Merged SampPRF^b and SampMLWE-RU^b Oracles):** The SampPRF^b and SampMLWE-RU^b oracles of iMLWER-RU are merged into a single SampPRF-MLWE-RU^b oracle that can be queried only once for each x with any number Q_x of matrices $\mathbf{R}_1, \dots, \mathbf{R}_{Q_x}$, and responds by running $\text{SampPRF}^b(x)$ once and calling $\text{SampMLWE-RU}^b(N, \mathbf{R}_j, x)$ for $j \in [Q_x]$, and returns the results returned by these oracle calls. Similar to iMLWER-RU^b, we allow the adversary a total of $Q := \sum_x Q_x$ underlying queries to SampMLWE-RU^b .

For clarity, we give the precise definition of iMLWE-RU-id in Definition 11.

| <u>iMLWE-RU-id$_{\mathcal{G}, Q, Q_x^\infty, q, m, N, h, L, \beta_r, \bar{\chi}}^b(\lambda)$</u> | <u>SampPRF$^b(x)$</u> |
|---|--|
| 1 : $\mathbf{k} \stackrel{\$}{\leftarrow} \chi_k^m$ | 1 : if $b = 0$ then |
| 2 : $b' \leftarrow \mathcal{A}^{\text{SampPRF-MLWE-RU}^b(\cdot, \cdot)}(1^\lambda)$ | 2 : if $\mathcal{Z}[x] = \perp$ then |
| 3 : return $b = b'$ | 3 : $\mathbf{z}' \stackrel{\$}{\leftarrow} R_q^h, \mathcal{Z}[x] := \mathbf{z}'$ |
| <u>SampPRF-MLWE-RU$^b(Q_x \in \mathbb{Z}^+, (\mathbf{R}_j \in R^{h \times N})_{j \in [Q_x]}, x \in \{0, 1\}^L)$</u> | 4 : if $b = 1$ then |
| 1 : if $\exists j \in [Q_x]$ s.t. $\ \mathbf{R}_j\ _\infty > \beta_r$ or $\mathcal{Z}[x] \neq \perp$ or $Q_x > Q_x^\infty$ | 5 : if $\mathcal{Z}[x] = \perp$ then |
| 2 : then return \perp | 6 : $\mathbf{B}_x = \mathcal{G}(x) \in R_q^{h \times m}$ |
| 3 : $\mathbf{B}_x = \mathcal{G}(x) \in R_q^{h \times m}$ | 7 : $\mathbf{e}_B \stackrel{\$}{\leftarrow} \chi_B^h$ |
| 4 : $\mathbf{y}_B := \text{SampPRF}^b(x)$ | 8 : $\mathbf{y}_B = \mathbf{B}_x \mathbf{k} + \mathbf{e}_B \in R_q^h$ |
| 5 : $\widetilde{\mathbf{e}}_B \stackrel{\$}{\leftarrow} \chi_B^h$ // simulated error for $b = 0$ | 9 : $\mathcal{Z}[x] := \mathbf{y}_B$ |
| 6 : for $j \in [Q_x]$ do | 10 : $\mathbf{y}_B = \mathcal{Z}[x]$ |
| 7 : $\mathbf{A}_{r,j} \stackrel{\$}{\leftarrow} R_q^{N \times m}, \mathbf{e}_j \stackrel{\$}{\leftarrow} \chi^N$ | 11 : return \mathbf{y}_B |
| 8 : $\mathbf{C}_{x,j} = \mathbf{R}_j \mathbf{A}_{r,j} + \mathbf{B}_x \in R_q^{h \times m}, \mathbf{e}'_j \stackrel{\$}{\leftarrow} \chi_1^h$ | |
| 9 : $\widetilde{\mathbf{e}}'_j \stackrel{\$}{\leftarrow} \chi_1^h, \widetilde{\mathbf{e}}_j \stackrel{\$}{\leftarrow} \chi^N$ // simulated errors for $b = 0$ | |
| 10 : $\mathbf{e}_{\text{sim},j} = \widetilde{\mathbf{e}}_j - \mathbf{R}_j \widetilde{\mathbf{e}}_j \in R^h$ | |
| 11 : if $b = 0$ then | |
| 12 : $\mathbf{c}_j \stackrel{\$}{\leftarrow} R_q^N, \mathbf{u}_j = \mathbf{R}_j \mathbf{c}_j + \mathbf{y}_B - \widetilde{\mathbf{e}}_B + \mathbf{e}_{\text{sim},j} \in R_q^h$ | |
| 13 : // $\mathbf{u}_j = \mathbf{R}_j \mathbf{c}_j + \mathbf{y}_B - (\mathbf{R}_j \widetilde{\mathbf{e}}_j + \widetilde{\mathbf{e}}_B) + \widetilde{\mathbf{e}}'_j$ | |
| 14 : if $b = 1$ then | |
| 15 : $\mathbf{c}_j = \mathbf{A}_{r,j} \mathbf{k} + \mathbf{e}_j \in R_q^N, \mathbf{u}_j = \mathbf{C}_{x,j} \mathbf{k} + \mathbf{e}'_j \in R_q^h$ | |
| 16 : // $\mathbf{u}_j = \mathbf{R}_j \mathbf{c}_j + \mathbf{y}_B - (\mathbf{R}_j \mathbf{e}_j + \mathbf{e}_B) + \mathbf{e}'_j$ | |
| 17 : endfor | |
| 18 : return $(\mathbf{y}_B, (\mathbf{A}_{r,j}, \mathbf{c}_j, \mathbf{u}_j)_{j \in [Q_x]})$ | |

Fig. 2: iMLWE-RU-id security game

Definition 11 (Idealized Interactive MLWE iMLWE-RU-id $_{\mathcal{G}, Q, Q_x^\infty, q, m, N, h, L, \beta_r, \bar{\chi}}$). Let $\bar{\chi} = (\chi, \chi_B, \chi_1, \chi_k)$ be discrete distributions over R , let \mathcal{G} be a hash family, and let $Q, Q_x^\infty, q, m, N, h, L, \beta_r \in \mathbb{Z}^+$. We say that the idealized interactive MLWE assumption iMLWE-RU-id $_{\text{param}}$ holds, for $\text{param} := (\mathcal{G}, Q, Q_x^\infty, q, m, N, h, L, \beta_r, \bar{\chi})$, if for all PPT adversaries \mathcal{A} , there exists a negligible function $\text{negl}(\cdot)$, such that the following holds,

$$\text{Adv}_{\text{param}}^{\text{iMLWE-RU-id}}(\lambda) = \left| \Pr [\text{iMLWE-RU-id}^1(\lambda) = 1] - \Pr [\text{iMLWE-RU-id}^0(\lambda) = 1] \right| \leq \text{negl}(\lambda),$$

where the experiment iMLWE-RU-id b is defined below, Q_x denotes the number of MLWE sample matrix pairs $(\mathbf{A}_{r,j}, \mathbf{C}_{x,j})_{j \in [Q_x]}$ requested from the oracle for a queried x , $Q := \sum_x Q_x$ denotes the total number of requested matrix pairs in the experiment, and Q_x^∞ is the maximum allowed upper bound on $\max_x Q_x$. Each x is only allowed to be queried at most once to SampPRF-MLWE-RU b .

The MLWE-PRF problem is the assumption that the randomized function $x \mapsto \mathcal{G}(x) \cdot \mathbf{k} + \mathbf{e}$ (for “small” Gaussian distributed error \mathbf{e}) is pseudorandom, even given some additional MLWE samples $(\mathbf{A}_r, \mathbf{A}_r \mathbf{k} + \mathbf{e}_{A,r})$ with \mathbf{A}_r having uniformly random entries in R_q and $\mathbf{e}_{A,r}$ are “small” Gaussian errors. Thus, MLWE-PRF is a variant of the standard pseudorandomness assumption for the PRF using the function \mathcal{G} , i.e. $x \mapsto [\mathcal{G}(x) \cdot \mathbf{k}]$, which has a security reduction from MLWR when \mathcal{G} is instantiated as in [BLMR13] or [BP14], respectively.

Definition 12 (MLWE-PRF $_{\mathcal{G}, Q_P, Q_M, q, m, h, L, \bar{\chi}}$). Let $\bar{\chi} = (\chi, \chi_B, \chi_k)$ be discrete distributions over R , let \mathcal{G} be a hash family, and let $Q_P, Q_M, q, m, h, L \in \mathbb{Z}^+$. We say that the MLWE-PRF assumption MLWE-PRF $_{\text{param}}$ holds, for $\text{param} := (\mathcal{G}, Q_P, Q_M, q, m, h, L, \bar{\chi})$, if for all PPT adversaries \mathcal{A} , there exists a negligible function $\text{negl}(\cdot)$, such that the following holds,

$$\text{Adv}_{\text{param}}^{\text{MLWE-PRF}}(\lambda) = |\Pr[\text{MLWE-PRF}^1(\lambda) = 1] - \Pr[\text{MLWE-PRF}^0(\lambda) = 1]| \leq \text{negl}(\lambda),$$

where the experiment MLWE-PRF b is defined below, Q_P denotes the number of queries to the MLWE PRF oracle SampMLWE-PRF b , while Q_M denotes the number of queries to the MLWE oracle SampMLWE b . Each x is only allowed to be queried at most once to SampMLWE-PRF b .

| MLWE-PRF $_{\mathcal{G}, Q_P, Q_M, q, m, h, L, \bar{\chi}}(\lambda)$ | SampMLWE-PRF $^b(x)$ |
|---|--|
| $1 : \mathbf{k} \xleftarrow{\$} \chi_k^m$ $2 : b' \leftarrow \mathcal{A}^{\text{SampMLWE-PRF}^b(\cdot), \text{SampMLWE}^b(\cdot)}(1^\lambda)$ $3 : \text{return } b = b'$ | $1 : \text{if } b = 0 \text{ then}$ $2 : \quad \text{if } \mathcal{Z}[x] = \perp \text{ then}$ $3 : \quad \quad \mathbf{z}' \xleftarrow{\$} R_q^h, \mathcal{Z}[x] := \mathbf{z}'$ $4 : \text{if } b = 1 \text{ then}$ $5 : \quad \text{if } \mathcal{Z}[x] = \perp \text{ then}$ $6 : \quad \quad \mathbf{B}_x = \mathcal{G}(x) \in R_q^{h \times m}$ $7 : \quad \quad \mathbf{e}_B \xleftarrow{\$} \chi_B^h$ $8 : \quad \quad \mathbf{y}_B = \mathbf{B}_x \mathbf{k} + \mathbf{e}_B \in R_q^h$ $9 : \quad \quad \mathcal{Z}[x] := \mathbf{y}_B$ $10 : \mathbf{y}_B = \mathcal{Z}[x]$ $11 : \text{return } \mathbf{y}_B$ |
| <hr/> $1 : \mathbf{a} \xleftarrow{\$} R_q^m, e \xleftarrow{\$} \chi$ $2 : \text{if } b = 0 \text{ then}$ $3 : \quad c \xleftarrow{\$} R_q$ $4 : \text{if } b = 1 \text{ then}$ $5 : \quad c = \mathbf{a}^\top \mathbf{k} + e \in R_q$ $6 : \text{return } (\mathbf{a}, c)$ | |

We present our reduction from MLWE-PRF to iMLWE-RU-id via an intermediate problem, which is a “hint” variant of the iMLWE-RU-id problem, where instead of the re-used MLWE samples we provide the adversary with MLWE hints of the form $\mathbf{h}_i := \mathbf{R}_i \mathbf{e}_i + \mathbf{e}_B + \mathbf{e}'_i$ for $i \in [Q_x]$. We call the latter variant of iMLWE-RU-id the HintMLWE-PRF problem. Our HintMLWE-PRF problem can be viewed as a special case of the “Extended Hint-MLWE” problem introduced in [WSE24] (with a reduction from MLWE), which in turn generalizes the Hint-MLWE problem introduced in [KLSS23], to allow for general hint matrices, rather than block diagonal hint matrices used in [KLSS23].

Definition 13 (HintMLWE-PRF $_{\mathcal{G}, Q, Q_x^\infty, q, m, N, h, L, \beta_r, \bar{\chi}}$). Let $\bar{\chi} = (\chi, \chi_B, \chi_1, \chi_k)$ be discrete distributions over R , let \mathcal{G} be a hash family, and let $Q, Q_x^\infty, q, m, N, h, L, \beta_r \in \mathbb{Z}^+$. We say that the HintMLWE-PRF $_{\text{param}}$ holds, for $\text{param} := (\mathcal{G}, Q, Q_x^\infty, q, m, N, h, L, \beta_r, \bar{\chi})$, if for all PPT adversaries \mathcal{A} , there exists a negligible function $\text{negl}(\cdot)$, such that the following holds,

$$|\Pr[\text{HintMLWE-PRF}^1(\lambda) = 1] - \Pr[\text{HintMLWE-PRF}^0(\lambda) = 1]| \leq \text{negl}(\lambda),$$

where the experiment HintMLWE-PRF b is defined below, Q_x denotes the number of MLWE sample matrix/hint pairs $(\mathbf{A}_{r,j}, \mathbf{h}_j)_{j \in [Q_x]}$ requested from the oracle for a queried x , $Q := \sum_x Q_x$ denotes the total number of requested hint vectors during the experiment, and Q_x^∞ is the maximum allowed upper bound on $\max_x Q_x$. Each x is only allowed to be queried at most once to SampHMLWE b .

3.1 Hardness Reduction from HintMLWE-PRF to iMLWE-RU-id

As an intermediate conceptual step in our reduction from MLWE-PRF to iMLWE-RU-id, we observe that we can reduce a “hint” variant HintMLWE-PRF of MLWE-PRF (inspired by the Hint-MLWE problem [KLSS23]) to iMLWE-RU-id. The idea is that in the HintMLWE-PRF problem, in addition to the MLWE-PRF samples

| HintMLWE-PRF $_{\mathcal{G}, Q, Q_x^\infty, q, m, N, h, L, \beta_r, \bar{\chi}}^b(\lambda)$ | SampPRF $^b(x)$ |
|--|---|
| 1 : $\mathbf{k} \xleftarrow{\$} \chi_k^m$ | 1 : if $b = 0$ then |
| 2 : $b' \leftarrow \mathcal{A}^{\text{SampHMLWEP}^b(\cdot, \cdot)}(1^\lambda)$ | 2 : if $\mathcal{Z}[x] = \perp$ then |
| 3 : return $b = b'$ | 3 : $\mathbf{z}' \xleftarrow{\$} R_q^h, \mathcal{Z}[x] := \mathbf{z}'$ |
| <hr/> | |
| SampHMLWEP $^b(Q_x \in \mathbb{Z}^+, (\mathbf{R}_j \in R^{h \times N})_{j \in [Q_x]}, x \in \{0, 1\}^L)$ | |
| 1 : if $\exists j \in [Q_x]$ s.t. $\ \mathbf{R}_j\ _\infty > \beta_r$ or $\mathcal{Z}[x] \neq \perp$ or $Q_x > Q_x^\infty$ | 4 : if $b = 1$ then |
| 2 : then return \perp | 5 : if $\mathcal{Z}[x] = \perp$ then |
| 3 : $\mathbf{B}_x = \mathcal{G}(x) \in R_q^{h \times m}$ | 6 : $\mathbf{B}_x = \mathcal{G}(x) \in R_q^{h \times m}$ |
| 4 : $\mathbf{y}_B := \text{SampPRF}^b(x)$ | 7 : $\mathbf{e}_B \xleftarrow{\$} \chi_B^h, \mathcal{E}[x] := \mathbf{e}_B$ |
| 5 : if $b = 0$ then $\tilde{\mathbf{e}}_B \xleftarrow{\$} \chi_B^h$ // simulated error for $b = 0$ | 8 : $\mathbf{y}_B = \mathbf{B}_x \mathbf{k} + \mathbf{e}_B \in R_q^h$ |
| 6 : if $b = 1$ then $\mathbf{e}_B = \mathcal{E}[x]$ // real error for $b = 1$ | 9 : $\mathcal{Z}[x] := \mathbf{y}_B$ |
| 7 : for $j \in [Q_x]$ do | 10 : $\mathbf{y}_B = \mathcal{Z}[x]$ |
| 8 : $\mathbf{A}_{r,j} \xleftarrow{\$} R_q^{N \times m}$ | 11 : return \mathbf{y}_B |
| 9 : $\mathbf{e}'_j \xleftarrow{\$} \chi_1^h, \mathbf{e}_j \xleftarrow{\$} \chi^N$ // real errors for $b = 1$ | |
| 10 : $\tilde{\mathbf{e}}'_j \xleftarrow{\$} \chi_1^h, \tilde{\mathbf{e}}_j \xleftarrow{\$} \chi^N$ // simulated errors for $b = 0$ | |
| 11 : if $b = 0$ then | |
| 12 : $\mathbf{c}_j \xleftarrow{\$} R_q^N, \mathbf{h}_j = (\mathbf{R}_j \tilde{\mathbf{e}}_j + \tilde{\mathbf{e}}_B) + \tilde{\mathbf{e}}'_j$ | |
| 13 : if $b = 1$ then | |
| 14 : $\mathbf{c}_j = \mathbf{A}_{r,j} \mathbf{k} + \mathbf{e}_j \in R_q^N, \mathbf{h}_j = (\mathbf{R}_j \mathbf{e}_j + \mathbf{e}_B) + \mathbf{e}'_j$ | |
| 15 : return $(\mathbf{y}_B, (\mathbf{A}_{r,j}, \mathbf{c}_j, \mathbf{h}_j)_{j \in [Q_x]})$ | |

Fig. 3: HintMLWE-PRF security game

$(\mathbf{B}_x, \mathbf{y}_B = \mathbf{B}_x \mathbf{k} + \mathbf{e}_B)$ and $(\mathbf{A}_r, \mathbf{c} = \mathbf{A}_r \mathbf{k} + \mathbf{e})$, the adversary is also provided with appropriate linear combinations of the error vectors as “hints” with respect to the \mathbf{R} matrix provided by the adversary, i.e. $\mathbf{h} := \mathbf{R}\mathbf{e} + \mathbf{e}_B - \mathbf{e}'$. The hints \mathbf{h} then allow the adversary to simulate the “re-used” iMLWE-RU-id samples of the form $(\mathbf{C}_x := \mathbf{R}\mathbf{A}_r + \mathbf{B}_x, \mathbf{u} = \mathbf{C}_x \mathbf{k} + \mathbf{e}')$ from the given MLWE-PRF samples, since $\mathbf{R}\mathbf{c} + \mathbf{y}_B - \mathbf{h} = \mathbf{C}_x \mathbf{k} + \mathbf{e}'$.

Lemma 6. *Let $\bar{\chi} := (\chi, \chi_B, \chi_1, \chi_k)$ be discrete distributions over R , let \mathcal{G} be a hash family, and let $Q, Q_x^\infty, q, m, N, h, L, \beta_r \in \mathbb{Z}^+$. Moreover, let $\bar{\chi}_h = (\chi, \chi_B, -\chi_1, \chi_k)$, where $-\chi_1$ denotes the distribution obtained by sampling \mathbf{e}' from χ_1 and outputting $-\mathbf{e}'$. The iMLWE-RU-id $_{\mathcal{G}, Q, Q_x^\infty, q, m, N, h, L, \beta_r, \bar{\chi}}$ assumption holds if the HintMLWE-PRF $_{\mathcal{G}, Q, Q_x^\infty, q, m, N, h, L, \beta_r, \bar{\chi}_h}$ assumption holds. More precisely, for any PPT adversary \mathcal{A} against the iMLWE-RU-id problem, there exists a PPT adversary \mathcal{B} against the HintMLWE-PRF problem, such that*

$$\text{Adv}_{\mathcal{A}}^{\text{iMLWE-RU-id}}(\lambda) \leq \text{Adv}_{\mathcal{B}}^{\text{HintMLWE-PRF}}(\lambda).$$

Proof. Given an adversary \mathcal{A} against iMLWE-RU-id, we construct an adversary \mathcal{B} against HintMLWE-PRF that runs as shown in Figure 4.

Next, to analyze the advantage of \mathcal{B} against HintMLWE-PRF, we consider first the case that $b = 1$. In this case, the response $(\mathbf{y}_B, (\mathbf{A}_{r,j}, \mathbf{c}_j, \mathbf{h}_j)_{j \in [Q_x]})$ returned by the SampHMLWEP b oracle has $\mathbf{y}_B = \mathbf{B}_x \mathbf{k} + \mathbf{e}_B$ with $\mathbf{B}_x = \mathcal{G}(x)$, $\mathbf{c}_j = \mathbf{A}_{r,j} \mathbf{k} + \mathbf{e}_j \in R_q^N$ and hints $\mathbf{h}_j = (\mathbf{R}_j \mathbf{e}_j + \mathbf{e}_B) + \mathbf{e}'_j$, with \mathbf{e}'_j samples from $-\chi_1$, so it follows that $\mathbf{u}_j = \mathbf{R}_j \mathbf{c}_j + \mathbf{y}_B - \mathbf{h}_j = \mathbf{C}_x \mathbf{k} - \mathbf{e}'_j$ with $\mathbf{C}_x := \mathbf{R}_j \mathbf{A}_{r,j} + \mathbf{B}_x$ and $-\mathbf{e}'_j$ sampled from the distribution χ_1 , since \mathbf{e}'_j is sampled from the distribution $-\chi_1$. Therefore, in the HintMLWE-PRF 1 game, the view of \mathcal{A} is simulated with the same distribution as in the real iMLWE-RU-id 1 game.

| |
|--|
| $\mathcal{B}_{\mathcal{G}, Q, Q_x^\infty, q, m, N, h, L, \beta_r, \bar{\chi}_h}^{\text{SampPRF-MLWE-RU}^b(\cdot, \cdot, \cdot)}(1^\lambda)$ |
| $1 : b' \leftarrow \mathcal{A}^{\text{SampPRF-MLWE-RU}^b(\cdot, \cdot, \cdot)}(1^\lambda)$ |
| $2 : \text{return } b = b'$ |
| $\text{SimSampPRF-MLWE-RU}^b(Q_x \in \mathbb{Z}^+, (\mathbf{R}_j \in R^{h \times N})_{j \in [Q_x]}, x \in \{0, 1\}^L)$ |
| $1 : (\mathbf{y}_B, (\mathbf{A}_{r,j}, \mathbf{c}_j, \mathbf{h}_j)_{j \in [Q_x]}) := \text{SampHMLWEP}^b(Q_x \in \mathbb{Z}^+, (\mathbf{R}_j \in R^{h \times N})_{j \in [Q_x]}, x \in \{0, 1\}^L)$ |
| $2 : \text{for } j \in [Q_x] \text{ do}$ |
| $3 : \quad \mathbf{u}_j = \mathbf{R}_j \mathbf{c}_j + \mathbf{y}_B - \mathbf{h}_j$ |
| $4 : \text{endfor}$ |
| $5 : \text{return } (\mathbf{y}_B, (\mathbf{A}_{r,j}, \mathbf{c}_j, \mathbf{u}_j)_{j \in [Q_x]})$ |

Fig. 4: HintMLWE-PRF algorithm \mathcal{B} in the security reduction.

In the other case that $b = 0$, the response $(\mathbf{y}_B, (\mathbf{A}_{r,j}, \mathbf{c}_j, \mathbf{h}_j)_{j \in [Q_x]})$ returned by the SampHMLWEP^b oracle has \mathbf{y}_B uniformly random in R_q^h and \mathbf{c}_j uniformly random and independent in R_q^N , but hints are still given by $\mathbf{h}_j = (\mathbf{R}_j \mathbf{e}_j + \mathbf{e}_B) + \mathbf{e}'_j$, with \mathbf{e}'_j samples from $-\chi_1$, so it follows that $\mathbf{u}_j = \mathbf{R}_j \mathbf{c}_j + \mathbf{y}_B - \mathbf{h}_j = -(\mathbf{R}_j \mathbf{e}_j + \mathbf{e}_B) - \mathbf{e}'_j$, where $-\mathbf{e}'_j$ is sampled from the distribution χ_1 , and $\mathbf{e}_j, \mathbf{e}_B$ sampled from χ, χ_B respectively, which is exactly the same as the distribution in the real iMLWE-RU-id^0 game. We conclude that the advantage of \mathcal{B} against HintMLWE-PRF is equal to the advantage of \mathcal{A} against iMLWE-RU-id , as claimed. \square

3.2 Hardness Reduction from MLWE-PRF to HintMLWE-PRF

Our reduction from MLWE-PRF to HintMLWE-PRF can be viewed as a special case of the reduction for the Extended Hint MLWE problem [WSE24] from the MLWE problem, which is a generalization of the reduction from MLWE to Hint-MLWE in [KLSS23]. For completeness, since the manuscript [WSE24] is not currently published, we provide below a complete proof of the reduction from MLWE-PRF to HintMLWE-PRF based on an optimization of the general result in [WSE24] to the special form of hint matrices in our HintMLWE-PRF problem. In particular, our “re-use” hints have the form $\mathbf{h}_i := \mathbf{R}_i \mathbf{e}_i + \mathbf{e}_B + \mathbf{e}'_i$, for $i \in [Q_x]$, where Q_x denotes the number of oracle queries on the same x values, which re-use the same PRF output, and hence the same \mathbf{e}_B PRF error term.

Theorem 1. *Let $\bar{\chi}_h := (\chi, \chi_B, \chi_1, \chi_k)$ be discrete distributions over R , where $\chi := \mathcal{D}_{\mathbb{Z}^d, s}$, $\chi_B := \mathcal{D}_{\mathbb{Z}^d, s_B}$ and $\chi_1 := \mathcal{D}_{\mathbb{Z}^d, s_1}$ for some $s, s_B, s_1, \eta, \delta, \epsilon \in \mathbb{R}^+$. Let \mathcal{G} be a hash family, and let $Q, Q_x^\infty, q, m, N, h, L, \beta_r \in \mathbb{Z}^+$. Moreover, let $\bar{\chi} = (\chi_0, \chi_0, \chi_k)$, where $\chi_0 := \mathcal{D}_{\mathbb{Z}^d, s_0}$. The HintMLWE-PRF $_{\mathcal{G}, Q, Q_x^\infty, q, m, N, h, L, \beta_r, \bar{\chi}_h}$ assumption holds if the MLWE-PRF $_{\mathcal{G}, Q, Q_x^\infty, q, m, h, L, \bar{\chi}}$ assumption holds and the following conditions are satisfied:*

$$s_1/s \geq \eta \cdot \sqrt{Q_x^\infty + (\beta_r d)^2 h N}, \quad (2)$$

$$s_0/s \leq \sqrt{\frac{1 - 1/(1 + \eta^2) - \epsilon_p}{(1 + \delta)}}, \quad (3)$$

$$s_0 \geq \sqrt{\frac{(1 + 1/\delta) \cdot \ln(2(Q_x^\infty N + h)d(1 + 1/\epsilon))}{\pi}}, \quad (4)$$

$$s_B \geq s, \quad \text{where} \quad (5)$$

$$\epsilon_p \leq \frac{1}{(1 + 1/\delta) \cdot (1 + \ln((1 + 1/\epsilon)/2)/\ln(4(Q_x^\infty N + h)d))}. \quad (6)$$

More precisely, for any PPT adversary \mathcal{A} against the HintMLWE-PRF problem, there exists a PPT adversary \mathcal{B} against the MLWE-PRF problem, such that

$$\text{Adv}_{\mathcal{A}}^{\text{HintMLWE-PRF}}(\lambda) \leq \text{Adv}_{\mathcal{B}}^{\text{MLWE-PRF}}(\lambda) + 2Q \cdot \epsilon,$$

where Q and Q_x^∞ denote the total number of queries and maximum number of hints requested per x query, respectively, that the adversary \mathcal{A} makes.

The proof of the above theorem follows the approach of the MLWE to Hint-MLWE reduction in [KLSS23] and in particular its generalization in [WSE24] to hints that contain linear combinations of error coordinates, as is the case in our hint matrices. Namely, the security reduction simulates the hints with their correct marginal distribution and then computes the conditional distribution of the concatenated error vector $\bar{\mathbf{e}}^\top := (\mathbf{e}^\top, \mathbf{e}_B^\top)$ given the concatenated hints vector \mathbf{h} . The latter conditional distribution turns out to be a non-zero centered skewed Gaussian with covariance matrix Σ_0 that depends on the width parameter s_1 of the hint “masking” errors \mathbf{e}' and the width parameter s of the error vector \mathbf{e} , as well as the norm of the concatenated hint matrix $\bar{\mathbf{R}}$. The reduction then transforms its input MLWE samples spherical Gaussian error distribution with width parameter $s_0 \approx s$ to the required skewed Gaussian with covariance matrix Σ_0 by adding an independent Gaussian error with a compensating covariance matrix $\Sigma_0 - s_0^2 \mathbf{I}$. However, sampling a Gaussian with the latter compensating covariance is only possible if the compensating covariance matrix is positive definite, and satisfying the latter requirement requires the “masking” ratio s_1/s to exceed the largest singular value of the concatenated hint matrix $\bar{\mathbf{R}}$. Accordingly, we provide in Lemma 7 below an optimized upper bound on the matrix norm of our form of hint matrix. It results in a lower bound on the ratio $s_1/s \approx \sqrt{Q_x^\infty}$ that is nearly optimal for large Q_x , as it approximately matches the minimum value of s_1/s value required to thwart the known “averaging” attack (see Section 1).

Lemma 7. Fix Q, N, h , and $\beta_r \in \mathbb{Z}^+$, matrices $\mathbf{R}_i \in \mathbb{Z}^{h \times N}$ for $i \in [Q]$ with entries in $[-\beta_r, \dots, \beta_r]$, and let

$$\bar{\mathbf{R}} := \begin{pmatrix} \mathbf{R}_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{R}_2 & \dots & \mathbf{0} & \mathbf{I} \\ \vdots & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{R}_Q & \mathbf{I} \end{pmatrix} \in \mathbb{Z}^{Qh \times (QN+h)}.$$

Then we have

$$\|\bar{\mathbf{R}}\| \leq \sqrt{Q + B_r^2}, \text{ where } B_r := \max_i \|\mathbf{R}_i\| \leq \beta_r \sqrt{Nh}. \quad (7)$$

Proof. Let $\mathbf{u}^\top := (\mathbf{u}_1^\top, \dots, \mathbf{u}_Q^\top, \mathbf{u}_B^\top) \in \mathbb{R}^{QN+h}$, with $\mathbf{u}_i \in \mathbb{R}^N$ for $i \in [Q]$ and $\mathbf{u}_B \in \mathbb{R}^h$. We have $\|\bar{\mathbf{R}}\| = \max_{\mathbf{u}: \|\mathbf{u}\|=1} \|\bar{\mathbf{R}}\mathbf{u}\|$. Take \mathbf{u} such that $\|\mathbf{u}\| = 1$. We have:

$$\begin{aligned} \|\bar{\mathbf{R}}\mathbf{u}\| &= \left\| \begin{pmatrix} \mathbf{R}_1 \mathbf{u}_1 + \mathbf{u}_B \\ \vdots \\ \mathbf{R}_Q \mathbf{u}_Q + \mathbf{u}_B \end{pmatrix} \right\| \\ &\leq \left\| \begin{pmatrix} \mathbf{R}_1 \mathbf{u}_1 \\ \vdots \\ \mathbf{R}_Q \mathbf{u}_Q \end{pmatrix} \right\| + \left\| \begin{pmatrix} \mathbf{u}_B \\ \vdots \\ \mathbf{u}_B \end{pmatrix} \right\| \\ &= \sqrt{\sum_{i \in [Q]} \|\mathbf{R}_i \mathbf{u}_i\|^2} + \sqrt{Q} \|\mathbf{u}_B\| \\ &\leq \sqrt{\sum_{i \in [Q]} \|\mathbf{R}_i\|^2 \|\mathbf{u}_i\|^2} + \sqrt{Q} \|\mathbf{u}_B\| \\ &\leq B_r \sqrt{\sum_{i \in [Q]} \|\mathbf{u}_i\|^2} + \sqrt{Q} \|\mathbf{u}_B\| \\ &= B_r \sqrt{1 - \|\mathbf{u}_B\|^2} + \sqrt{Q} \|\mathbf{u}_B\|. \end{aligned} \quad (8)$$

The first inequality above is using the triangle inequality, the second inequality is by the definition of $\|\mathbf{R}_i\|$'s, the third inequality follows from the definition of $B_r := \max_i \|\mathbf{R}_i\|$, and the last equality is due to $\|\mathbf{u}\| = 1$. It follows that $\|\bar{\mathbf{R}}\mathbf{u}\| \leq \max_{\ell \in [0,1]} f(\ell)$, where $f(\ell) := B_r \sqrt{1 - \ell^2} + \sqrt{Q}\ell$. It is easy to verify by differentiation that the function f attains its maximum on $[0, 1]$ at the point $\ell_{\max} := \sqrt{\frac{Q}{Q+B_r^2}}$ and the maximum is $f(\ell_{\max}) = \sqrt{Q + B_r^2}$. This proves the claimed upper bound on $\|\bar{\mathbf{R}}\|$. The claimed upper bound on B_r follows from the inequality $\|\mathbf{R}_i\| \leq \sqrt{\|\mathbf{R}_i\|_1 \cdot \|\mathbf{R}_i\|_{\infty, M}}$, where $\|\mathbf{R}_i\|_1$ (resp. $\|\mathbf{R}_i\|_{\infty, M}$) is equal to the maximum column 1-norm (resp. maximum row 1-norm) of \mathbf{R}_i and upper bounded by $\beta_r h$ (resp. $\beta_r N$). This completes the proof. \square

We are now ready to prove our security reduction from MLWE-PRF to HintMLWE-PRF.

Proof (of Theorem 1). Given an adversary \mathcal{A} against HintMLWE-PRF, we construct an adversary \mathcal{B} against MLWE-PRF that runs as shown in Figure 5.

Note that for the discrete Gaussian sampling in line 15 to be PPT sampleable, it is sufficient by Lemma 1 that Σ is positive definite, i.e. $\sigma_{\min}(\Sigma) > 0$, where $\Sigma := \Sigma_0 - s_0^2 \mathbf{I}$, and that⁹ $\delta_p \cdot M_{\Sigma} \leq 1$, where $\delta_p := \sqrt{\frac{\ln(2(Q_x N + h)d) + 4}{\pi}}$ and M_{Σ} denotes the max column norm of $\sqrt{\Sigma^{-1}}$, which is bounded as $M_{\Sigma} \leq \sqrt{\sigma_{\max}(\Sigma^{-1})} \leq \frac{1}{\sqrt{\sigma_{\min}(\Sigma)}} = \frac{1}{\sqrt{\sigma_{\min}(\Sigma_0) - s_0}}$. So the sufficient sampleability condition is

$$\sigma_{\min}(\Sigma_0) \geq (1 + \delta) \cdot s_0^2 + \delta_p^2, \quad (9)$$

for some $\delta \geq 0$. On the other hand, we have: $\sigma_{\min}(\Sigma_0) = \frac{1}{\|\Sigma_0^{-1}\|}$ and

$$\begin{aligned} \|\Sigma_0^{-1}\| &= \|\Sigma_1^{-1} + \frac{1}{s_1^2} \bar{\mathbf{R}}^{\top} \bar{\mathbf{R}}\| \\ &\leq \|\Sigma_1^{-1}\| + \frac{1}{s_1^2} \|\bar{\mathbf{R}}^{\top} \bar{\mathbf{R}}\| \\ &\leq \frac{1}{s^2} + \frac{1}{s_1^2} \|\bar{\mathbf{R}}\|^2 \\ &\leq \frac{1}{s^2} + \frac{1}{s_1^2} (Q_x^{\infty} + (\beta_r d)^2 h(m + \ell)) \\ &\leq \frac{1}{s^2} \cdot (1 + 1/\eta^2). \end{aligned} \quad (10)$$

In the first inequality above we use the triangle inequality. The second inequality uses the identity $\|\bar{\mathbf{R}}^{\top} \bar{\mathbf{R}}\| = \|\bar{\mathbf{R}}\|^2$ and the assumed lower bound $s_B \geq s$ in (4), which implies $\|\Sigma_1^{-1}\| = \max(\frac{1}{s^2}, \frac{1}{s_B^2}) \leq \frac{1}{s^2}$. The third inequality uses the upper bound on $\bar{\mathbf{R}}$ from Lemma 7 and that $Q_x \leq Q_x^{\infty}$, and the fourth inequality uses the assumed lower bound (2) on s_0/s . Therefore, using (10), we see that the sampleability condition (9) is implied by the assumed upper bound (3) on s_0/s , as required.

Next, to analyze the advantage of \mathcal{B} against MLWE-PRF, we consider first the case that $b = 1$. In this case, the response vector \mathbf{y}' consisting of the concatenation of the samples returned by the SampMLWE^1 and SampMLWE-PRF^1 oracles has the form

$$\mathbf{y}' = \begin{pmatrix} \bar{\mathbf{A}}_r \\ \mathbf{B}_x \end{pmatrix} \mathbf{k} + \mathbf{e}', \quad (11)$$

where $\bar{\mathbf{A}}_r^{\top} := [\mathbf{A}_{r,1}^{\top}, \dots, \mathbf{A}_{r,Q_x}^{\top}]$ and \mathbf{e}' is sampled from the distribution $\mathcal{D}_{\mathbb{Z}(Q_x N + h)d, s_0^2 \mathbf{I}}$ and hence

$$\mathbf{y} = \begin{pmatrix} \bar{\mathbf{A}}_r \\ \mathbf{B}_x \end{pmatrix} \mathbf{k} + \mathbf{e}, \quad (12)$$

⁹ We remark that there is a minor omission in the Hint-MLWE security reduction analysis in [KLSS23][Thm 1] which omitted this extra PPT sampleability condition and only required positive definiteness; it leads to the extra term ϵ_p in the bound.

| |
|---|
| $\mathcal{B}_{\mathcal{G}, Q, Q_x^\infty, q, m, N, h, L, \beta_r, \bar{\chi}}^{\text{SampMLWE-PRF}^b(\cdot), \text{SampMLWE}^b(\cdot)}(1^\lambda)$ |
| $1 : b' \leftarrow \mathcal{A}^{\text{SimSampHMLWEP}^b(\cdot, \cdot, \cdot)}(1^\lambda)$ |
| $2 : \text{return } b'$ |
| $\text{SimSampHMLWEP}^b(Q_x \in \mathbb{Z}^+, \{\mathbf{R}_j \in R^{h \times N}\}_{j \in [Q_x]}, x \in \{0, 1\}^L)$ |
| $1 : \text{if } \exists j \in [Q_x] \text{ s.t. } \ \mathbf{R}_j\ _\infty > \beta_r \text{ or } \mathcal{Z}[x] \neq \perp \text{ or } Q_x > Q_x^\infty \text{ then}$ |
| $2 : \text{return } \perp$ |
| $3 : \mathbf{y}'_B := \text{SampMLWE-PRF}^b(x), \mathcal{Z}[x] := \mathbf{y}'_B$ |
| $4 : \widetilde{\mathbf{e}}_B \stackrel{\$}{\leftarrow} \chi_B^h \quad // \text{ simulated error}$ |
| $5 : \text{for } j \in [Q_x] \text{ do}$ |
| $6 : \text{Call } \text{SampMLWE}^b(\cdot) \text{ } N \text{ times to get } (\mathbf{A}_{r,j}, \mathbf{c}'_j) \in R_q^{N \times m} \times R_q^N$ |
| $7 : \tilde{\mathbf{e}}'_j \stackrel{\$}{\leftarrow} \chi_1^h, \tilde{\mathbf{e}}_j \stackrel{\$}{\leftarrow} \chi^N \quad // \text{ simulated errors}$ |
| $8 : \mathbf{h}_j = (\mathbf{R}_j \tilde{\mathbf{e}}_j + \widetilde{\mathbf{e}}_B) + \tilde{\mathbf{e}}'_j$ |
| $9 : \text{endfor}$ |
| $10 : \mathbf{h}^\top := [\mathbf{h}_1^\top, \dots, \mathbf{h}_{Q_x}^\top] \in \mathbb{Z}^{Q_x h d}, (\mathbf{y}')^\top := [\mathbf{c}'_1^\top, \dots, \mathbf{c}'_{Q_x}^\top, \mathbf{y}'_B^\top] \in \mathbb{Z}^{(Q_x N + h)d}$ |
| $11 : \bar{\mathbf{R}} := \begin{pmatrix} \mathbf{R}_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{R}_2 & \dots & \mathbf{0} & \mathbf{I} \\ \vdots & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{R}_{Q_x} & \mathbf{I} \end{pmatrix} \in \mathbb{Z}^{Q_x h d \times (Q_x N + h)d}$ |
| $12 : \Sigma_1 := \begin{pmatrix} s^2 \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & s^2 \mathbf{I} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \mathbf{0} & \ddots & \mathbf{0} & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & s^2 \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & s_B^2 \mathbf{I} \end{pmatrix} \in \mathbb{Z}^{(Q_x N + h)d \times (Q_x N + h)d}$ |
| $13 : \Sigma_0 := \left(\Sigma_1^{-1} + \frac{1}{s_1^2} \bar{\mathbf{R}}^\top \bar{\mathbf{R}} \right)^{-1}$ |
| $14 : \mathbf{c} := \frac{1}{s_1^2} \Sigma_0 \bar{\mathbf{R}}^\top \mathbf{h}$ |
| $15 : \mathbf{t} \stackrel{\$}{\leftarrow} \mathcal{D}_{\mathbb{Z}^{(Q_x N + h)d}, \Sigma_0 - s_0^2 \mathbf{I}, \mathbf{c}}$ |
| $16 : \mathbf{y} := \mathbf{y}' + \mathbf{t} \in R_q^{Q_x N + h}$ |
| $17 : \text{Parse } \mathbf{y}^\top = (\mathbf{c}_1^\top, \dots, \mathbf{c}_{Q_x}^\top, \mathbf{y}_B)$ |
| $18 : \text{return } (\mathbf{y}_B, (\mathbf{A}_{r,j}, \mathbf{c}_j, \mathbf{h}_j)_{j \in [Q_x]})$ |

Fig. 5: MLWE-PRF algorithm \mathcal{B} in the security reduction.

where $\mathbf{e} := \mathbf{e}' + \mathbf{t}$ is sampled from the distribution $\mathcal{D}_{\text{conv}} := \mathcal{D}_{\mathbb{Z}^{(Q_x N+h)d}, s_0^2 \mathbf{I}} + \mathcal{D}_{\mathbb{Z}^{(Q_x N+h)d}, \mathbf{\Sigma}_0 - s_0^2 \mathbf{I}, \mathbf{c}}$. By the convolution Lemma 2, the distribution $\mathcal{D}_{\text{conv}}$ is within statistical distance $\leq 2\epsilon$ of $\mathcal{D}_{\mathbb{Z}^{(Q_x N+h)d}, \mathbf{\Sigma}_0, \mathbf{c}}$, assuming that the following smoothing condition holds:

$$\sqrt{\mathbf{\Sigma}_3} \geq \eta_\epsilon(\mathbb{Z}^{(Q_x N+h)d}), \text{ where } \mathbf{\Sigma}_3 := \frac{1}{s_0^2} \mathbf{I} + (\mathbf{\Sigma}_0 - s_0^2 \mathbf{I})^{-1}. \quad (13)$$

We show below that (13) is satisfied. Therefore, by Lemma 5, the output $(\mathbf{y}_B, (\mathbf{A}_{r,j}, \mathbf{c}_j, \mathbf{h}_j)_{j \in [Q_x]})$ returned by the SimSampHMLWEP¹ simulator for each queried x is within statistical distance $\leq 2\epsilon$ of the distribution of the output returned by the SampHMLWEP¹ oracle in the real HintMLWE-PRF game. Adding up over all (at most Q) x queries, we conclude that the distribution of the view of \mathcal{B} simulated by \mathcal{A} in the MLWE-PRF game with $b = 1$ is within statistical distance $\leq 2Q \cdot \epsilon$ of the distribution of \mathcal{B} 's view in the real HintMLWE-PRF game with $b = 1$.

We now consider the second case that $b = 0$. In this case, the response vector \mathbf{y}' is sampled from the uniform distribution on $\mathbb{Z}_q^{(Q_x N+h)d}$ and hence $\mathbf{y} = \mathbf{y}' + \mathbf{t}$ is also uniformly distributed on $\mathbb{Z}_q^{(Q_x N+h)d}$ independently of the hint vector \mathbf{h} , while the marginal distribution of the \mathbf{h} remains the same as in the $b = 1$ game. Therefore, the distribution of the view of \mathcal{B} simulated by \mathcal{A} in the MLWE-PRF game with $b = 0$ is exactly equal to the distribution of \mathcal{B} 's view in the real HintMLWE-PRF game with $b = 0$. From the above analysis, we conclude that $\text{Adv}_{\mathcal{B}}^{\text{MLWE-PRF}}(\lambda) \geq \text{Adv}_{\mathcal{B}}^{\text{MLWE-PRF}}(\lambda) - 2Q \cdot \epsilon$, which is the claimed advantage bound.

It remains to show that condition (13) is satisfied. By Lemma 4, the condition (13) holds if

$$\|\mathbf{\Sigma}_3^{-1}\| \leq \eta_\epsilon(\mathbb{Z}^{(Q_x N+h)d})^{-2}. \quad (14)$$

From (9), we have $\sigma_{\min}(\mathbf{\Sigma}_0 - s_0^2 \mathbf{I}) \geq \delta \cdot s_0^2$. Therefore, the triangle inequality gives

$$\|\mathbf{\Sigma}_3^{-1}\| \leq \frac{1}{s_0^2} + \|(\mathbf{\Sigma}_0 - s_0^2 \mathbf{I})^{-1}\| = \frac{1}{s_0^2} + \frac{1}{\sigma_{\min}(\mathbf{\Sigma}_0 - s_0^2 \mathbf{I})} \leq \frac{1 + 1/\delta}{s_0^2}. \quad (15)$$

Combining (15) and (14), we conclude that (13) is satisfied if $s_0 \geq \sqrt{1 + 1/\delta} \cdot \eta_\epsilon(\mathbb{Z}^{(Q_x N+h)d})$, and the latter condition is implied by the assumed lower bound (4) on s_0 , using Lemma 3, as required. \square

4 LeOPaRd : Our Lattice-Based VPOPRF Proposal

At a very high-level, our goal is to realise the 2HashDH OPRF idea [JKK14] in the lattice setting as first done by Albrecht et al. [ADDS21]. Our construction ensures that a client with a tag/input pair (t, x) can recover the PRF value $[\mathbf{B}_x \mathbf{k}]_p$ after interacting with the server, where $\mathbf{B}_x = \mathcal{G}(t, x)$, for some *message mapping* \mathcal{G} , and \mathbf{k} is the secret key of the server. We discuss potential instantiations of the function \mathcal{G} and its relation to the existing lattice-based PRF schemes in Section 5.2. As discussed before, our OPRF construction, LeOPaRd, is round-optimal, and supports partial obliviousness and verifiability.

We would like the OPRF evaluation to begin with the client linearly encrypting the matrix \mathbf{B}_x such that $\mathbf{C}_x := \mathbf{R} \mathbf{A}_r + \mathbf{B}_x$, followed by the server computing $\mathbf{u}_x := \mathbf{C}_x \mathbf{k} + \mathbf{e}'_s$ for some error vector \mathbf{e}'_s (so that \mathbf{u}_x contains the term $\mathbf{B}_x \mathbf{k}$). However, to minimize the client's control over \mathbf{C}_x so that \mathbf{u}_x does not leak server's secret key, we ask the client to commit to the pair (\mathbf{R}, x) , and generate the matrix \mathbf{A}_r via a random oracle using the resulting commitment \mathbf{c}_r . This way, the client is forced to pick their randomness \mathbf{R} and the input x *before* seeing the random matrix \mathbf{A}_r . This step is critical to ensuring that we can reduce pseudorandomness of LeOPaRd to the new iMLWER-RU problem (without requiring smudging). The client sends the pair $(\mathbf{c}_r, \mathbf{C}_x)$ (and their well-formedness NIZK proof) to the server.

Upon receiving the values from the client, the server recomputes \mathbf{A}_r given the commitment \mathbf{c}_r , and responds to the client with the pair $(\mathbf{u}_x := \mathbf{C}_x \mathbf{k} + \mathbf{e}'_s, \mathbf{v}_k := \mathbf{A}_r \mathbf{k} + \mathbf{e}_s)$, where \mathbf{e}_s and \mathbf{e}'_s denote some appropriately distributed errors. At this point, the client can use its secret randomness \mathbf{R} to recover the PRF output $[\mathbf{u}_x - \mathbf{R} \mathbf{v}_k]_p = [\mathbf{B}_x \mathbf{k} + \mathbf{e}_f]_p = [\mathbf{B}_x \mathbf{k}]_p$, where $\mathbf{e}_f := \mathbf{e}'_s - \mathbf{R} \mathbf{e}_s$ is some final error term with small

| | |
|---|---|
| F.Setup(1^λ) <hr/> 1 : $\forall i \in [2], \text{ck}_i \leftarrow \text{COM.Setup}(1^\lambda)$ 2 : $\forall i \in [2], \text{crs}_i \leftarrow \text{NIZK}_i.\text{Setup}(1^\lambda)$ 3 : $\text{pp} := (\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2)$ 4 : return pp | F.KeyGen(pp) <hr/> 1 : parse pp := (ck ₁ , ck ₂ , crs ₁ , crs ₂) 2 : $\mathbf{k} \xleftarrow{\$} \chi_k^m$ 3 : $\mathbf{c}_k \leftarrow \text{COM.Commit}(\text{ck}_2, \mathbf{k}; \rho_k)$ 4 : return (pk := c _k , sk := k) |
| F.Request(pp, pk, t, x) <hr/> 1 : parse pp := (ck ₁ , ck ₂ , crs ₁ , crs ₂) 2 : $\mathbf{R} \xleftarrow{\$} \chi_r^{h \times (\ell+m)}$ 3 : $\mathbf{c}_r \leftarrow \text{COM.Commit}(\text{ck}_1, (\mathbf{R}, x); \rho_r)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 6 : $\mathbf{C}_x := \mathbf{R}\mathbf{A}_r + \mathbf{B}_x \in R_q^{h \times m}$ 7 : $\text{stmt}_1 := (\mathbf{c}_r, \mathbf{C}_x, \text{ck}_1, \mathbf{A}_r, \mathcal{G}, t)$ 8 : $\text{wit}_1 := (\mathbf{R}, x, \rho_r)$ 9 : $\pi_c \leftarrow \text{NIZK}_1.\text{P}(\text{crs}_1, \text{stmt}_1, \text{wit}_1)$ 10 : $\text{st} := (t, x, \mathbf{R}, \mathbf{C}_x, \text{pk}, \mathbf{A}_r)$ 11 : $\text{req} := (\mathbf{c}_r, \mathbf{C}_x, \pi_c)$ 12 : return (st, req) | F.BlindEval(pp, pk, sk, t, req) <hr/> 1 : parse pp := (ck ₁ , ck ₂ , crs ₁ , crs ₂) 2 : parse pk := c _k , sk := k 3 : parse req := (c _r , C _x , π _c) 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : if NIZK ₁ .V(crs ₁ , π _c , stmt ₁) ≠ 1 then 6 : return ⊥ 7 : $\mathbf{e}_s \xleftarrow{\$} \chi^{\ell+m}, \mathbf{e}'_s \xleftarrow{\$} \chi_1^h$ 8 : $\mathbf{v}_k := \mathbf{A}_r \mathbf{k} + \mathbf{e}_s \in R_q^{\ell+m}$ 9 : $\mathbf{u}_x := \mathbf{C}_x \mathbf{k} + \mathbf{e}'_s \in R_q^h$ 10 : $\text{stmt}_2 := (\mathbf{u}_x, \mathbf{C}_x, \mathbf{c}_k, \mathbf{v}_k, \text{ck}_2, \mathbf{A}_r)$ 11 : $\text{wit}_2 := (\mathbf{k}, \mathbf{e}_s, \mathbf{e}'_s, \rho_k)$ 12 : $\pi_s \leftarrow \text{NIZK}_2.\text{P}(\text{crs}_2, \text{stmt}_2, \text{wit}_2)$ 13 : return rep := (u _x , v _k , π _s) |
| F.Finalize(pp, rep, st) <hr/> 1 : parse pp := (ck ₁ , ck ₂ , crs ₁ , crs ₂) 2 : parse rep := (u _x , v _k , π _s) 3 : parse st := (t, x, R, C _x , pk := c _k , A _r) 4 : if NIZK ₂ .V(crs ₂ , π _s , stmt ₂) ≠ 1 then 5 : return ⊥ 6 : $\mathbf{z} := \lfloor \mathbf{u}_x - \mathbf{R}\mathbf{v}_k \rfloor_p \in R_p^h \quad // = \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p$ 7 : $y := \text{RO}_z(t, x, \mathbf{z})$ 8 : return y | F.Eval(sk, t, x) <hr/> 1 : parse sk := k 2 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 3 : $\mathbf{z} := \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p \in R_p^h$ 4 : $y := \text{RO}_z(t, x, \mathbf{z})$ 5 : return y |
| RO_r(t, c_r) <hr/> 1 : if H[t, c _r] = ⊥ then 2 : $\mathcal{H}[t, \mathbf{c}_r] \xleftarrow{\$} R_q^{(\ell+m) \times m}$ 3 : return H[t, c _r] | RO_z(t, z, z) <hr/> 1 : if F[t, x, z] = ⊥ then 2 : $\mathcal{F}[t, x, \mathbf{z}] \xleftarrow{\$} \{0, 1\}^\lambda$ 3 : return F[t, x, z] |

Fig. 6: LeOPaRd : Our verifiable POPRF construction.

The server proof π_s proves the following,

$$\pi_s := \left\{ \begin{array}{l} \mathfrak{D}(\mathbf{u}_x, \mathbf{C}_x, \mathbf{c}_k, \mathbf{v}_k, \mathbf{ck}_2, \mathbf{A}_r, \beta_k, \beta_e); \mathfrak{N}(\mathbf{k}, \mathbf{e}_s, \mathbf{e}'_s, \rho_k) \mid \|\mathbf{k}\|_\infty \leq \beta_k \wedge \\ \|\mathbf{e}_s\|_\infty \leq \beta \wedge \|\mathbf{e}'_s\|_\infty \leq \beta_1 \wedge \mathbf{c}_k = \text{COM.Commit}(\mathbf{ck}_2, \mathbf{k}; \rho_k) \wedge \\ \mathbf{v}_k = \mathbf{A}_r \mathbf{k} + \mathbf{e}_s \text{ mod } q \wedge \mathbf{u}_x = \mathbf{C}_x \mathbf{k} + \mathbf{e}'_s \text{ mod } q \end{array} \right\}.$$

The client proof π_c proves the following,

$$\pi_c := \left\{ \begin{array}{l} \mathfrak{D}(\mathbf{c}_r, \mathbf{C}_x, \mathbf{ck}_1, \mathbf{A}_r, \mathcal{G}, t, \beta_r); \mathfrak{N}(\mathbf{R}, x, \rho_r) \mid \|\mathbf{R}\|_\infty \leq \beta_r \wedge \mathbf{B}_x := \mathcal{G}(t, x) \wedge \\ \mathbf{c}_r = \text{COM.Commit}(\mathbf{ck}_1, (\mathbf{R}, x); \rho_r) \wedge \mathbf{C}_x = \mathbf{R} \mathbf{A}_r + \mathbf{B}_x \text{ mod } q \end{array} \right\}.$$

Fig. 7: Relations for NIZK proofs performed in LeOPaRd.

| Notation | Description |
|------------------|---|
| λ | security parameter |
| κ | correctness parameter |
| q | system modulus |
| p | rounding modulus |
| d | ring dimension of $R = \mathbb{Z}[X]/(X^d + 1)$ |
| χ, χ_1 | server's error distributions |
| β, β_1 | ℓ_∞ -norm bounds on server's errors $\mathbf{e}_s, \mathbf{e}'_s$ |
| χ_k | server's key distribution |
| χ_r | client's randomness distribution |
| β_r | ℓ_∞ -norm bound on client's randomness \mathbf{R} |
| s_0 | error std. dev. in server's MLWE security |
| m | dimension of server key \mathbf{k} (over R_q) |
| h | # of rows of \mathbf{B}_x |
| ℓ | client MLWE dimension parameter |
| n_c, n_s | dimensions of trapdoor keys used by client, server |
| γ | base parameter for Gadget matrix |

Table 2: Summary of main notations/parameters used for LeOPaRd.

coefficients (relative to q). To additionally achieve verifiability and protect against malicious adversaries, we require the client and server to prove the well-formedness of their computations with NIZK proofs.

Our full POPRF construction is given in Figure 6, and the NIZK relations of client and server proofs (π_c, π_s) are given in Figure 7. In the protocol description, NIZK_1 and NIZK_2 denote the NIZK argument system of the client and server, respectively. We also summarize the main notations/parameters in Table 2. We instantiate the commitment scheme in LeOPaRd using Regev-style encryption [Reg05], which can also be seen as an extractable version of the BDLOP commitment [BDL⁺18]. Given such an encryption is quite standard by now, we defer the details to Appendix E.

Remark 1. We note that certain NIZK proof systems such as LNP22 [LNP22] already constructs a commitment to its witness inside the NIZK proof. In fact, as discussed in Appendix E, the trapdoor commitment scheme we use is the trapdoor version of the BDLOP commitment [BDL⁺18] used in [LNP22]. Therefore, by unboxing the NIZK proofs, it may be possible (in certain cases) to optimize our LeOPaRd design. However, we forego such optimizations in this work as they would make the overall protocol significantly more complex.

Remark 2. In our iMLWER-RU security reduction from Section 3, it is important to know the maximum number of OPRF evaluation queries the adversary makes with the *same* $\mathbf{B}_x = \mathcal{G}(t, x)$. In the partially oblivious setting, since the server already knows t , we can let the server keep track of queried tags; and either never allow the same tag be queried twice or restrict the number of queries under the same tag, e.g., to 2^{16} . This way, the adversary would be restricted to seeing a limited number of OPRF evaluation results for a particular \mathbf{B}_x . In this case, the impact of the term Q_x^∞ in Theorem 1 would effectively diminish.

| F.Request(pp, pk, t, x) | F.BlindEval(pp, pk, sk, t, req) |
|--|---|
| 1 : parse $\mathbf{t} := (t_1, \dots, t_K)$ | 1 : parse pp := (ck ₁ , ck ₂ , crs ₁ , crs ₂) |
| 2 : parse $\mathbf{x} := (x_1, \dots, x_K)$ | 2 : parse pk := $\mathbf{c}_k, \mathbf{sk} := \mathbf{k}$ |
| 3 : parse pp := (ck ₁ , ck ₂ , crs ₁ , crs ₂) | 3 : parse req := ($\hat{\mathbf{c}}_r, \hat{\mathbf{C}}_x, \pi_c$) |
| 4 : $\hat{\mathbf{R}} \xleftarrow{\$} \chi_r^{Kh \times (\ell+m)}$ | 4 : $\mathbf{A}_r := \text{RO}_r(\mathbf{t}, \hat{\mathbf{c}}_r) \in R_q^{(\ell+m) \times m}$ |
| 5 : $\hat{\mathbf{c}}_r \leftarrow \text{COM.Commit}(\text{ck}_1, (\hat{\mathbf{R}}, \mathbf{x}); \rho_r)$ | 5 : if NIZK ₁ .V(crs ₁ , π _c , stmt ₁) ≠ 1 then |
| 6 : $\mathbf{A}_r := \text{RO}_r(\mathbf{t}, \hat{\mathbf{c}}_r) \in R_q^{(\ell+m) \times m}$ | 6 : return ⊥ |
| 7 : $\forall i \in [K], \mathbf{B}_{x_i} := \mathcal{G}(x_i, t_i) \in R_q^{h \times m}$ | 7 : $\mathbf{e}_s \xleftarrow{\$} \chi^{\ell+m}, \hat{\mathbf{e}}'_s \xleftarrow{\$} \chi_1^{Kh}$ |
| 8 : $\hat{\mathbf{B}}_x := \begin{pmatrix} \mathbf{B}_{x_1} \\ \vdots \\ \mathbf{B}_{x_K} \end{pmatrix}$ | 8 : $\mathbf{v}_k := \mathbf{A}_r \mathbf{k} + \mathbf{e}_s \in R_q^{\ell+m}$ |
| 9 : $\hat{\mathbf{C}}_x := \hat{\mathbf{R}} \mathbf{A}_r + \hat{\mathbf{B}}_x \in R_q^{Kh \times m}$ | 9 : $\hat{\mathbf{u}}_x := \hat{\mathbf{C}}_x \mathbf{k} + \hat{\mathbf{e}}'_s \in R_q^{Kh}$ |
| 10 : $\text{stmt}_1 := (\hat{\mathbf{c}}_r, \hat{\mathbf{C}}_x, \text{ck}_1, \mathbf{A}_r, \mathcal{G}, \mathbf{t})$ | 10 : $\text{stmt}_2 := (\hat{\mathbf{u}}_x, \hat{\mathbf{C}}_x, \mathbf{c}_k, \mathbf{v}_k, \text{ck}_2, \mathbf{A}_r)$ |
| 11 : $\text{wit}_1 := (\hat{\mathbf{R}}, \mathbf{x}, \rho_r)$ | 11 : $\text{wit}_2 := (\mathbf{k}, \mathbf{e}_s, \hat{\mathbf{e}}'_s, \rho_k)$ |
| 12 : $\pi_c \leftarrow \text{NIZK}_1.\text{P}(\text{crs}_1, \text{stmt}_1, \text{wit}_1)$ | 12 : $\pi_s \leftarrow \text{NIZK}_2.\text{P}(\text{crs}_2, \text{stmt}_2, \text{wit}_2)$ |
| 13 : $\text{st} := (\mathbf{x}, \hat{\mathbf{R}}, \hat{\mathbf{C}}_x, \text{pk}, \mathbf{A}_r)$ | 13 : return rep := ($\hat{\mathbf{u}}_x, \mathbf{v}_k, \pi_s$) |
| 14 : $\text{req} := (\hat{\mathbf{c}}_r, \hat{\mathbf{C}}_x, \pi_c)$ | |
| 15 : return (st, req) | |

Fig. 8: Batched LeOPaRd evaluation protocol.

4.1 Batched Queries

Some applications such as Privacy Pass [DGS⁺18] can benefit from batching multiple queries in one go. It is not difficult to see that LeOPaRd can support this. Suppose the client wants to get evaluations on $(x_1, \dots, x_K) =: \mathbf{x}$ with tags $(t_1, \dots, t_K) =: \mathbf{t}$ for some $K \geq 1$. Then, we can run Request and BlindEval procedures as shown in Figure 8. Finalize would compute $\hat{\mathbf{z}} := \left[\hat{\mathbf{u}}_x - \hat{\mathbf{R}} \mathbf{v}_k \right]_p$ and then output $y_i := \text{RO}_z(x_i, t_i, \mathbf{z}_i)$ for $\hat{\mathbf{z}} = (\mathbf{z}_1, \dots, \mathbf{z}_K)$ and $i = 1, \dots, K$.

The batched LeOPaRd has two major advantages over running K independent evaluation queries:

1. Both the client and the server compute a single NIZK proof attesting to the validity of the whole batched query. When using a succinct argument system like LaBRADOR [BS23], the proof size will increase by a very small factor. For example, as discussed in [ADDG24, App. A.2], their LaBRADOR proof size increases from 45KB (for one single query) to just 79KB for a batch of 64 queries ($< 1.8 \times$ increase).
2. Observe that the term \mathbf{v}_k (on the server's side) in our protocol is *not* affected by batching at all. Therefore, its size remains constant for any $K \geq 1$. When using the message mapping from BP14 PRF (see Section 5.2), we have $h = 1$ and therefore, \mathbf{t}_x is a single ring element. The linear communication cost by the server in the batched setting then boils down K ring elements.

The significant practical advantage in the batched setting can be observed from the results presented in Table 3 and Table 4.

4.2 Correctness and Security Analyses

Correctness analysis. We upper bound the correctness error probability of our protocol in Lemma 8, in terms of the protocol parameters. Due to space limitations, we only sketch the proofs and refer the reader to Appendix B.1 for full proofs and details.

Lemma 8. Fix κ, h, d . Let $B_f(\kappa, d)$ denote an upper bound on a fixed coordinate of $\mathbf{e}_f = \mathbf{e}'_s - \mathbf{R}\mathbf{e}_s$ (as in (26)) that holds except with probability $p_e \leq 2^{-(\kappa+2+\log(hd))}$. Also, assume that the function family \mathcal{G} satisfies ϵ_u -uniformity (as per Definition 19) with

$$\epsilon_u \leq 2^{-(\kappa+2+\log(hd))}. \quad (16)$$

Then, for any fixed $x \in \{0, 1\}^L$, $2^{-\kappa}$ -correctness holds (as per Definition 2) if

$$q/p \geq 2^{\kappa+2} \cdot hd \cdot (2B_f(\kappa, d) + 1). \quad (17)$$

Remarks:

- Assume that $\chi = \mathcal{U}(\mathbb{S}_\beta)$, $\chi_r = \mathcal{U}(\mathbb{S}_{\beta_r})$ and $\chi_1 = \mathcal{U}(\mathbb{S}_{\beta_1})$. Using the central limit theorem Gaussian approximation for the distribution of the coordinates of the term $\mathbf{R}\mathbf{e}_s$ in (26), their standard deviation is given by $\sigma\sigma_r\sqrt{(m+\ell)d}$, where $\sigma := \sqrt{\frac{1}{12}((2\beta+1)^2-1)}$ and $\sigma_r := \sqrt{\frac{1}{12}((2\beta_r+1)^2-1)}$ are the standard deviations of the \mathbf{R} and \mathbf{e}_s coordinates, respectively. Using a Gaussian tail bound and the upper bound β_1 for the coordinates of \mathbf{e}'_s , we have the approximate upper bound

$$B_f(\kappa, d) \leq \beta_1 + \sigma\sigma_r\sqrt{(m+\ell)d \cdot 2\ln(2)(\kappa+2+\log(hd))}. \quad (18)$$

Note that $B_f(\kappa, d)$ is also upper bounded by the worst-case bound ($p_e = 0$):

$$B_f(\kappa, d) \leq (m+\ell)d\beta\beta_r + \beta_1 \quad (19)$$

- The existing correctness definition in Definition 2 assumes only 1 key generation run and 1 OPRF protocol evaluation. For a modified correctness definition with Q_{eval} total key generation and protocol evaluation pairs, the bound in Equation (17) will be multiplied by Q_{eval} .

In our evaluation protocol, the client obtains $\mathbf{u}_x - \mathbf{R}\mathbf{v}_k = \mathbf{B}_x\mathbf{k} + \mathbf{e}_f \in R_q^h$, where $\mathbf{e}_f = \mathbf{R}\mathbf{e}_s + \mathbf{e}'_s$ is an error term due to the server and client's randomness, and then rounds the result to the nearest multiple of q/p . The correctness proof bounds the probability of a PRF rounding error $\Pr[\lfloor \mathbf{B}_x\mathbf{k} + \mathbf{e}_f \rfloor_p \neq \lfloor \mathbf{B}_x\mathbf{k} \rfloor_p]$, which occurs only if $\mathbf{B}_x\mathbf{k}$ falls “close” to (within distance B_f), of an upper bound on the coordinates of error term \mathbf{e}_f . For this, we exploit the uniformity, up to negligible statistical distance ϵ_u , of coordinates of $\mathbf{B}_x\mathbf{k}$ (a property of \mathcal{G} which we call ϵ_u -uniformity) and choose q/p sufficiently large by a factor $\approx 2^\kappa$ compared to B_f to achieve $2^{-\kappa}$ correctness error. We upper bound the ϵ_u -uniformity of the BLMR13 and BP14 PRF instantiations of \mathcal{G} for this purpose.

Security analysis. We first prove pseudorandomness against malicious clients with Theorem 2 and then request privacy against malicious servers (POPRIV2) with Theorem 3. Due to space constraints, we give only the sketch of the main proof steps, and refer the reader to Appendix B.2 and Appendix B.3 for the full proofs, respectively. In Appendix C.2, we discuss an issue in a proof of [TCR⁺22], where they prove that correctness and POPRIV2 together implies uniqueness (a stronger form of verifiability). We show that this implication also requires a *key binding* property, and for completeness, we provide a full proof of the statement that correctness, POPRIV2 and key binding implies uniqueness. Our LeOPaRd proposal satisfies all these properties. Although our correctness error is not necessarily less than 2^{-128} , the verifiability property is not affected by the correctness error. We may only have a case where a different PRF output is computed in Finalize with probability $2^{-\kappa}$.

Theorem 2. The POPRF construction \mathbf{F} from Figure 6 satisfies pseudorandomness given in Definition 3, with random oracles RO_r and RO_z , if:

- The client argument system NIZK_1 is computationally sound and the server argument system NIZK_2 is computationally zero-knowledge (Definition 18),
- The commitment scheme COM is computationally hiding and extractable (Definitions 15 and 17), and

- The $\text{iMLWER-RU}_{\text{param}}$ assumption, for $\text{param} := (\mathcal{G}, Q_M, Q_P, Q_x^\infty, q, m, \ell + m, h, 2\lambda, \beta_r, p, \bar{\chi})$, holds (Definition 10).

More precisely, for any PPT adversary \mathcal{A} , there exist PPT adversaries $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$ and \mathcal{B}_5 against the computational zero-knowledge of NIZK_2 , computational soundness of NIZK_1 , hiding of COM , extractability of COM and $\text{iMLWER-RU}_{\text{param}}$ assumption, respectively, such that

$$\begin{aligned} \text{Adv}_{\mathcal{F}, \mathcal{A}, \mathcal{S}, \text{RO}_r, \text{RO}_z}^{\text{po-prf}}(\lambda) &\leq \text{Adv}_{\text{NIZK}_2, \mathcal{B}_1}^{\text{CZK}}(\lambda) + Q_B \cdot \text{Adv}_{\text{NIZK}_1, \mathcal{B}_2}^{\text{CS}}(\lambda) + \text{Adv}_{\text{COM}, \mathcal{B}_3}^{\text{CH}}(\lambda) \\ &\quad + Q_{\text{RO}_r} \cdot \text{Adv}_{\text{COM}, \mathcal{B}_4}^{\text{Ext}}(\lambda) + \text{Adv}_{\mathcal{B}_5}^{\text{iMLWE}_{\text{param}}}(\lambda) + Q_z \left(\frac{1}{p} + \frac{1}{q} \right)^{dh}, \end{aligned}$$

where Q_B and Q_{RO_r} denote the number of BlindEval and RO_r queries, respectively, that the adversary \mathcal{A} makes.

Our proof essentially revolves around the idea of removing traces of the server’s secret key sk from the computation of the POPRF output, such that the simulator \mathcal{S} of the pseudorandomness game does not need any secret key dependent information. In order to do so, we first simulate the server proof π_s , such that from that point onwards we do not need sk for the generation of the proof. Next, we commit to an all zero vector $\mathbf{0}_m$ during the computation of the public key pk , which allows us to remove sk from the computation of pk . Lastly, we rely on our iMLWER-RU assumption to remove all traces of sk from the computation of \mathbf{u}_x and \mathbf{v}_k , and yet maintain the correctness of the scheme. After applying these changes, we end up in a position where we do not require the secret key sk anymore, and hence, the simulator \mathcal{S} can be constructed easily. We note that to make these argument go through, we also need to extract the client’s randomness \mathbf{R} and carefully program the random oracles RO_r and RO_z throughout the proof.

Theorem 3. *The POPRF construction \mathcal{F} from Figure 6 satisfies the request privacy against malicious servers (POPRIV2), given in Definition 4, with random oracles RO_r and RO_z , if:*

- The client argument system NIZK_1 is computationally zero-knowledge and the server argument system NIZK_2 is computationally sound (Definition 18),
- The commitment scheme COM is computationally hiding and extractable (Definitions 15 and 17), and
- The $\text{knMLWE}_{\ell+m, m, h, \chi}$ assumption holds (Definition 7).

More precisely, for any PPT adversary \mathcal{A} , there exist PPT adversaries $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$ and \mathcal{B}_5 against the computational zero-knowledge of NIZK_1 , computational hiding of COM , computational soundness of NIZK_2 , extractability of COM and $\text{knMLWE}_{m, \ell+m, h, \chi}$ assumption, respectively, such that

$$\begin{aligned} \text{Adv}_{\mathcal{F}, \mathcal{A}, \text{RO}_r, \text{RO}_z}^{\text{po-priv-2}}(\lambda) &\leq \text{Adv}_{\text{NIZK}_1, \mathcal{B}_1}^{\text{CZK}}(\lambda) + 2Q_R \cdot \text{Adv}_{\text{COM}, \mathcal{B}_2}^{\text{CH}}(\lambda) + 2Q_F \cdot \text{Adv}_{\text{NIZK}_2, \mathcal{B}_3}^{\text{CS}}(\lambda) \\ &\quad + Q_R^{\text{pk}} \cdot \text{Adv}_{\text{COM}, \mathcal{B}_4}^{\text{Ext}}(\lambda) + 2Q_R \cdot \text{Adv}_{\mathcal{B}_5}^{\text{knMLWE}_{\ell+m, m, h, \chi}}(\lambda), \end{aligned}$$

where Q_R and Q_F denote the number of Request and Finalize oracle queries, respectively, and Q_R^{pk} denotes the number of queries to Request with different pk inputs that the adversary \mathcal{A} makes.

In order to prove request privacy against malicious servers, we aim to remove traces of the input x , such that at the end the transcript observed by the adversary is independent of the challenge bit b . To this end, we first simulate the client proof π_c , such that we do not require anymore the input x for computing the proof. Next, we change the client commitment \mathbf{c}_r to a commitment of all zero vector $\mathbf{0}_{h \cdot (\ell+m)+1}$ instead of the pair (\mathbf{R}, x) , which removes another occurrence of the input x . Finally, we rely on the Knapsack MLWE (knMLWE) assumption to replace computation of \mathbf{C}_x with a uniformly random matrix. We note that in order to maintain the correctness of the scheme, during this last change we rely on the extractability of the commitment scheme. More precisely, we extract the secret key sk from the public key pk , and use sk to correct the POPRF evaluation in Finalize . Importantly, we can use Regev-style [Reg05] *lightweight* commitments, as opposed to requiring *heavy* full trapdoors to extract from a commitment of the form $\mathbf{A}\mathbf{k} + \mathbf{e}$ used in [ADDS21]. At this point we removed all occurrences of the input x , and hence, the adversary’s view is independent of the input bit b .

5 Instantiating the NIZK Proofs and Message Mapping \mathcal{G}

In this section, we discuss possible ways to instantiate the message mapping \mathcal{G} and the underlying non-interactive zero-knowledge (NIZK) proofs performed by the server and the client.

5.1 Proof by the Server

The NIZK proof, π_s , (see Figure 7) conducted by the server is the most typical proof needed in lattice-based cryptography, and there are various proof systems that can be used to instantiate it. Some notable ones are the LANES [ALS20, ENS20, LNS20], LANES⁺ [ESLR23], LNP22 [LNP22] and LaBRADOR [BS23] proof systems. As discussed in Appendix E, we use a lattice-based commitment scheme, and thus, these proof systems can natively support its well-formedness proof. Given we require a relatively large modulus q , our experiments show that LaBRADOR offers the best communication performance among these options. Thanks to its succinctness features, it also scales very efficiently in the batched setting. We report on the performance results in Section 6.

5.2 Instantiating the Message Mapping \mathcal{G}

In this section, we primarily focus on the *secret-dependent* part of the message mapping \mathcal{G} and discuss our main option on how the message mapping $\mathcal{G} : x \mapsto \mathbf{B}_x \in R_q^{h \times m}$ can be instantiated. That is, we assume the public input part t to be empty. When the tag t is present, the mapping would simply be computed on $\hat{x} := x||t$ as $\mathcal{G}(\hat{x})$ and it is easy to verify the computation w.r.t. this input extended with *public* information (which we discuss more in Section 5.3). More message mapping options are discussed in Appendix D. There are mainly two considerations: (i) security of iMLWER-RU w.r.t. to \mathcal{G} chosen, and (ii) efficiency of our OPRF design, particularly the underlying NIZK proofs especially by the client (i.e., “ZK-friendliness”). The latter is the primary consideration as we believe iMLWER-RU remains secure for all instantiations discussed in this section.

We believe that a natural way to instantiate \mathcal{G} is to use the mappings in existing lattice-based PRFs so that the SampPRF oracle in iMLWER-RU becomes an oracle outputting PRF samples (in the case of $b = 1$). Therefore, we discuss options based on existing lattice-based PRFs.

Using the mapping from BP14 PRF [BP14] for \mathcal{G} . The option we see as the most suitable one is using the message mapping employed in BP14 PRF [BP14]. Here, the PRF computation involves a gadget matrix \mathbf{G} and its (non-linear) inverse computation $G^{-1} : R_q^{1 \times m} \rightarrow R_q^{m \times m}$. The G^{-1} mapping is the standard bit decomposition operation. In this case, we have two vectors $\mathbf{a}_0, \mathbf{a}_1 \stackrel{\$}{\leftarrow} R_q^m$ published as public parameters, and as a result, $h = 1$. Then, for $\mathbf{k} \leftarrow \chi_k^m$ for some distribution χ_k ¹⁰ on R_q , we compute

$$F_{\mathbf{k}}^{\text{bp}}(x) = \lfloor \mathbf{b}_x^\top \cdot \mathbf{k} \rfloor_p, \quad (20)$$

where the message mapping \mathcal{G}^{bp} is defined as

$$\mathcal{G}^{\text{bp}} : x \mapsto \mathbf{b}_x^\top = \mathbf{a}_{x_0}^\top \cdot G^{-1}(\mathbf{a}_{x_1} \cdots G^{-1}(\mathbf{a}_{x_{L-2}} \cdot G^{-1}(\mathbf{a}_{x_{L-1}}))). \quad (21)$$

For the computation to correctly work, we need $m = \lceil \log q \rceil$. For the client’s NIZK proof in this case, we will need to treat each $\mathbf{B}_i := G^{-1}(\mathbf{a}_{x_i}^\top \cdot \mathbf{B}_{i+1}) \in R_q^{m \times m}$ for $i = L - 1, \dots, 0$ (with $\mathbf{B}_L := \mathbf{I}_m$) as a variable. Therefore, there are $m^2 L$ variable polynomials, but with the correctness restriction that $m = \lceil \log q \rceil$.

We also consider a generalized version of this PRF proposal where the gadget matrix works with base $\gamma \geq 2$ (instead of $\gamma = 2$ as before), i.e., $\mathbf{g} = (1, \gamma, \gamma^2, \dots)$, and $\mathbf{G} = \mathbf{g} \otimes \mathbf{I}$. In this case, we would require that $m = \lceil \log_\gamma q \rceil$, and hence, a smaller m parameter may suffice.

¹⁰ Note that in this case we may have small secret key coefficients.

5.3 Proof by the Client

In this section, we discuss how the client’s NIZK proof can be instantiated. One can observe from the NIZK relation proven by the client (given in Figure 7) that the majority of the proof components are the most common relations proven in lattice-based cryptography (when the commitment scheme is instantiated using a suitable lattice-based extractable commitment scheme). There is perhaps one exception to this: (i) proving $\mathbf{B}_x = \mathcal{G}(t, x)$. Accordingly, we next discuss the proof of $\mathbf{B}_x = \mathcal{G}(t, x)$. Depending on the instantiation of the message mapping \mathcal{G} , the client’s proof may vary significantly. We look more closely at the first two instantiations of \mathcal{G} discussed in Section 5.2.

NIZK for BP14 PRF mapping. Here, recall that we have $h = 1$. We first discuss the case when no tag t is present, i.e., $\mathbf{B}_x = \mathcal{G}(x)$. The case of $\mathbf{B}_x = \mathcal{G}(t, x)$ is discussed at the end of this section. For a known \mathbf{A}_r and \mathbf{C}_x , the prover wants to prove knowledge of short matrix \mathbf{R} and a bit-string $x = (x_0, x_1, \dots, x_{L-1}) \in \{0, 1\}^L$ such that $\mathbf{C}_x = \mathbf{R}\mathbf{A}_r + \mathbf{B}_x \bmod q$ where \mathbf{B}_x is defined in (21).

Using a similar strategy as in [ADDS21] (see “**Proof system 1: Proofs of Masked Partial PRF Computation**”), define variables $\mathbf{B}_i \in R_q^{m \times m}$ for $i = L - 1, \dots, 0$ as $\mathbf{B}_{L-1} := G^{-1}(\mathbf{a}_{x_{L-1}}^\top)$ and $\mathbf{B}_i := G^{-1}(\mathbf{a}_{x_i}^\top \cdot \mathbf{B}_{i+1})$ for $i = L - 2, \dots, 0$. Using this, we have $\mathbf{B}_x = \mathbf{G} \cdot \mathbf{B}_0$. Then the statement being proven now can be transformed to prove the following system with $\mathbf{B}_L := \mathbf{I}$:

$$\begin{aligned} \mathbf{G} \cdot \mathbf{B}_i &= \mathbf{a}_{x_i} \cdot \mathbf{B}_{i+1}, \quad i = 0, \dots, L - 1 \\ \mathbf{C}_x &= \mathbf{R}\mathbf{A}_r + \mathbf{G} \cdot \mathbf{B}_0. \end{aligned}$$

This system of equations above is not linear due to the x_i and \mathbf{B}_{i+1} terms and the fact that G^{-1} is not a linear operator. We can manipulate the system of equations above to obtain

$$\begin{aligned} \mathbf{G} \cdot \mathbf{B}_{L-1} &= \mathbf{a}_0 \cdot (1 - x_{L-1}) + \mathbf{a}_1 \cdot x_{L-1}, \\ \mathbf{G} \cdot \mathbf{B}_{L-2} &= \mathbf{a}_0 \cdot (1 - x_{L-2}) \cdot \mathbf{B}_{L-1} + \mathbf{a}_1 \cdot x_{L-2} \cdot \mathbf{B}_{L-1}, \\ &\vdots \\ \mathbf{G} \cdot \mathbf{B}_0 &= \mathbf{a}_0 \cdot (1 - x_0) \cdot \mathbf{B}_1 + \mathbf{a}_1 \cdot x_0 \cdot \mathbf{B}_1, \\ \mathbf{C}_x &= \mathbf{R}\mathbf{A}_r + \mathbf{G} \cdot \mathbf{B}_0. \end{aligned}$$

The required proof now boils down to proving that a sequence of quadratic equations holds, which can be easily handled by LaBRADOR [BS23]. Observe that the witness dimension (over R_q) is in the order of m^2L . As discussed in Section 5.2, we consider a generalized version of the BP14 PRF with the gadget matrix with base $\gamma \geq 2$. Therefore, the witness dimension (over R_q) is in the order of $m^2L \approx (\log_\gamma q)^2L$. In our parameter settings (see Section 6), the largest m is around 64; hence $m^2L \approx 2^{18}$ (or less) for $L = 64$. In [NS24], the authors report a 53 KB LaBRADOR-based proof size for a polynomial evaluation proof of degree 2^{30} with all running times around a couple of minutes or less. Therefore, we believe the proof we require will be reasonably efficient.

Note that when a tag t is used in computing $\mathbf{B}_x = \mathcal{G}(t, x)$ (i.e., $\mathbf{B}_x = \mathcal{G}(\hat{x})$ for $\hat{x} := x||t$), the above system of equations will have a minor change in the first expression such that we will have $\mathbf{G} \cdot \mathbf{B}_{L-1} = \mathbf{a}_0 \cdot (1 - x_{L-1}) \cdot \mathbf{B}_t + \mathbf{a}_1 \cdot x_{L-1} \cdot \mathbf{B}_t$ for some *public* matrix \mathbf{B}_t dependent on the tag t . The NIZK proof can equivalently prove this similar system of equations.

6 Practical Performance Analysis

Our performance analysis focuses on estimating the sizes of various components of LeOPaRd. Once protocol components are small enough, a practically-acceptable computational performance is often also achieved given that lattice-based schemes involve quite simple operations like matrix-vector multiplications (see examples in [ESLL19, EZS⁺19, LNS20, NS24]). As discussed in the introduction, we aim to implement our scheme once the LaZeR library [SS24] becomes available.

One of the most important factors impacting the performance of LeOPaRd is the size of the modulus q . Since the dimension parameters for suitable MLWE security grow linearly with $\log q$, the overall communication is in fact *quadratic* in $\log q$. Particularly, the sizes of commitments/encryptions (i.e., $(\mathbf{c}_r, \mathbf{C}_x)$ on the client side and \mathbf{v}_k on the server side for LeOPaRd) quickly become large for very large q . This is not just specific to our scheme, but true in general for lattice-based commitment/encryption schemes (as also used e.g. in [ADDS21]). Therefore, our first goal is to minimize q as much as possible.

As shown in the correctness analysis (Lemma 8), the size of q itself is heavily dominated by the 2^κ term. Therefore, given this bound is *statistical* and for correctness (rather than a security property), we consider a range of κ values between 16 and 64. One may wonder what if a (malicious) server wants to stop the client from getting the correct OPRF result. There are two parts to consider here. First, as discussed in Remark 4, our correctness analysis can be easily extended to cover such a malicious server setting as the NIZK proof π_s proves shortness of the server’s error terms, and also we can simply increase κ . Second, OPRFs are often used in the server-client setting, and if clients do not receive the correct PRF output and therefore end up not receiving the intended service from the system, then the server would harm its own reputation and lose clients. Therefore, we believe a rational server would not intentionally try to stop the client from getting the correct OPRF result.

For the performance analysis, we focus on the instantiation of the message mapping \mathcal{G}^{bp} with the BP14 PRF since this choice leads to a better performance overall. As discussed in Section 5, we consider a general base γ for the gadget matrix \mathbf{G} , i.e., $\mathbf{g} = (1, \gamma, \gamma^2, \dots)$, and $\mathbf{G} = \mathbf{g} \otimes \mathbf{I}$. As a result, we only require $m \geq \lceil \log_\gamma q \rceil$ ¹¹. We can first consider the size of q (i.e., $\log q$), then determine m based on the other requirements such as MLWE security and then simply set $\gamma = \lceil 2^{\log q/m} \rceil$.

As done in the correctness analysis, we set $\chi = \mathcal{U}(\mathbb{S}_\beta)$, $\chi_r = \mathcal{U}(\mathbb{S}_{\beta_r})$ and $\chi_1 = \mathcal{U}(\mathbb{S}_{\beta_1})$ for parameters β, β_1 and β_r . Since the client’s randomness \mathbf{R} is one-time, we simply fix $\beta_r = 1$. Note that β_r does not impact the iMLWER-RU security reduction in Section 3. Since BP14 PRF allows for a small secret key, we sample the server’s secret key \mathbf{k} from the same χ distribution, i.e., $\chi_k = \chi = \mathcal{U}(\mathbb{S}_\beta)$.

To estimate the practical hardness of iMLWER-RU problem, we rely on the reduction in Section 3, particularly Theorem 1. Based on this theorem, we require the hardness of MLWE with the same secret dimension m as in iMLWER-RU and error Gaussian parameter s_0 . We also require the size of the rounding errors to be greater than the s parameter given in Theorem 1, which is easily satisfied since q/p is very large in our case.

Our parameter setting then proceeds as follows. Fix a value for $\kappa \in \{16, 32, 64\}$. We iterate over different values of $d \in \{64, 128, 256\}$ that optimize the communication efficiency for both the server and the client. Recall that for BP14 PRF, we have $h = 1$. Therefore, for a fixed d , we also set p as the smallest integer satisfying $(1/p+1/q)^{dh} \leq 2^{-128}$ (to satisfy a requirement from the pseudorandomness analysis in Theorem 2). Here, the term $1/q$ is much smaller (since q is always larger than 2^{32}) and therefore does not really play an important role. Then, based on Theorem 1, we first fix $\delta = \eta = 1$, $Q = 2^{64}$ and $\epsilon = 2^{-128}$, and compute the smallest (s_0, s, s_1) values. We then convert them to standard deviations $(\sigma_0, \sigma, \sigma_1)$ (by dividing by $\sqrt{2\pi}$). Here we set $Q_x^\infty = 2^\kappa$ for the fully oblivious setting, and $Q_x^\infty = 2^{16}$ for the partially oblivious setting. We explain the reasoning behind this towards the end of this section. We then find the corresponding smallest β (and β_1) such that uniform distribution on $\{-\beta, \dots, \beta\}$ (and $\{-\beta_1, \dots, \beta_1\}$) has standard deviation at least σ (and σ_1). In this procedure, we have $N = m + \ell$ and we estimate the values for m and ℓ initially, and correct the estimate iteratively until the estimate is checked to be accurate.

Now we consider the size of q based on the correctness analysis (Lemma 8) and set $\log q$ as small as possible. Then, we set the smallest value for m so that MLWE security with error standard deviation σ_0 at 128 bits is achieved (server security). As discussed above, this comes from the security reduction in Section 3. Similarly, we set the smallest value for ℓ so that MLWE $_{\ell, \chi_r}$ security at 128 bits is achieved (client security). We measure the practical hardness of MLWE using the lattice estimator [APS15] and aim for a “root Hermite factor” (RHF) of around 1.0045 as in earlier works [ESLR23, ESZ22b, LNP22, ESLL19]. RHF is a commonly used measure to estimate the quality of lattice reduction to solve a particular lattice problem. Many earlier works such as [ESLR23, ESZ22b, LNP22, ESLL19] considered the same RHF value for 128-bit security level. The same MLWE dimension parameters $w_c = \ell$ (by client) and $w_s = m$ (by server) can be used to establish

¹¹ Note that if $m > \lceil \log_\gamma q \rceil$, then we can simply pad the output vectors of G^{-1} decomposition mapping with zeros.

the hiding property of commitment scheme used by the client and the server as they have the same $\log q$ and their randomnesses are sampled from the same distributions as the earlier MLWE error terms. We also note that given many MLWE and iMLWER-RU samples may be leaked due to many OPRF evaluation queries, we also need to take into account combinatorial attacks (e.g., using Gröbner basis) against MLWE. We verified using the lattice estimator [APS15] that the combinatorial attack complexity is always above 2^{128} for our parameter settings.

Once the majority of the parameters are selected as above, what remains is to consider (i) the binding property of the commitment scheme, (ii) the indistinguishability of a commitment key with a trapdoor from a regular commitment key, and (iii) decryption/extraction correctness for the commitments. Given that we are dealing with (relatively) large moduli, we see that the third requirement is easily met. Again due to (relatively) large moduli and that the MSIS solution norm bounds are quite small (due to tight norm-bound NIZK proofs), the second requirement turns out to be more dominant than the first one in setting the values for (n_s, n_c) (the n parameters from Appendix E used by the server and client, respectively). As discussed in Appendix E, we require the same MLWE assumptions over R_q with errors with standard deviation σ_0 (for server’s commitment) and errors sampled from χ_r (for client’s commitment). As a result, we get that $n_s = m$ and $n_c = \ell$. Note that the dimensions of the randomnesses used inside the commitments by the server and the client do not have a significant impact in our parameter setting and efficiency estimates since the well-formedness of the commitments are proven by LaBRADOR, and therefore, this proof cost gets amortized along with everything else (see below for more discussion on the proof costs). For the MSIS hardness of our parameter settings, it turns out that module ranks (from both the client and the server’s side) as small as 1-2 is already sufficient. Therefore, we can comfortably use the standard low-order bit-dropping technique from [DKL⁺18]. For simplicity, we assume $D = 12$ bits can be dropped in the ‘binding’ parts of the commitments \mathbf{c}_r and \mathbf{c}_k (i.e., top n_c and n_s rows of \mathbf{c}_r and \mathbf{c}_k , respectively). A similar D parameter has been used in earlier works with smaller moduli such as [ESLR23, LNP22]. We believe that a larger D can be used, but the saving is not very significant, and therefore, we opt to not pursue further analysis here so as not to over-complicate the discussions. This concludes the setting of all parameters for LeOPaRd except the underlying NIZK proofs.

To estimate the sizes of LaBRADOR [BS23], one can observe that its proof size is barely impacted by the witness size (see, e.g., [BS23, Fig. 1]) thanks to its recursive nature. In [ADDG24], the authors report that their well-formedness LaBRADOR-based NIZK proof under a 75-bit modulus for FHE ciphertexts is about 45 KB for a single query and 79 KB for a batch of 64 queries. We believe the proofs needed for LeOPaRd are even simpler than those in [ADDG24], and therefore, we take a LaBRADOR-based proof cost to be 45 KB for a single query and 79 KB for a batch of 64 queries when estimating sizes for LeOPaRd. We note that the dimension of the polynomial ring used in the core LeOPaRd parts and that in the underlying NIZK proofs do not necessarily need to be the same as discussed in [ADDG24, LNPS21]. However, our results turn out to be optimal for $d = 64$, which is the dimension already used in LaBRADOR.

In Table 3, we present some example parameter settings and communication sizes aiming at 128-bit security level. The offline communication simply involves communication of the server’s public key, while online communication is those by the client and server. Here, we consider two cases: (i) fully oblivious setting where $Q_x^\infty = 2^\kappa$ (for Q_x^∞ in Theorem 1); and (ii) partially oblivious setting where $Q_x^\infty = 2^{16}$. In the former case, we are restricting the total number of OPRF queries to 2^κ , matching also the correctness error. In the latter case, we consider that the server has the ability to see a part of the PRF input, i.e. tag t , and therefore, can limit the number of OPRF queries under the same tag to be at most 2^{16} . If the tag corresponds to a user identifier as in [ECS⁺15], then this would mean 2^{16} queries per identifier, which we believe is quite reasonable (for honest users) in practice. We also note that our parameter setting aiming at 90-95 bits of security (to compare with [AG24] in Table 1) is provided in Table 4.

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| Obliv. mode | Query batch size | κ | $m = n_s$ | $\ell = n_c$ | σ_0 | β | β_1 | $\log q$ | Server commun. (per query) | Server PK size | Client commun. (per query) |
|-------------|------------------|----------|-----------|--------------|------------|---------|------------------|----------|----------------------------|----------------|----------------------------|
| Partial | 1 | 16 | 24 | 27 | 3.95 | 15 | 7782 | 42 | 62.06 | 13.5 | 76.27 |
| | | 32 | 33 | 37 | 3.95 | 15 | 8838 | 58 | 77.17 | 26.81 | 105.42 |
| | | 64 | 53 | 56 | 3.96 | 15 | 10691 | 91 | 123.20 | 70.39 | 195.45 |
| | 64 | 16 | 24 | 27 | 3.95 | 15 | 7782 | 42 | 1.82 | 13.50 | 26.27 |
| | | 32 | 33 | 37 | 3.95 | 15 | 8838 | 58 | 2.18 | 26.81 | 48.57 |
| | | 64 | 53 | 56 | 3.96 | 15 | 10691 | 91 | 3.16 | 70.39 | 117.66 |
| Full | 1 | 16 | 24 | 27 | 3.95 | 15 | 7782 | 42 | 62.06 | 13.50 | 76.27 |
| | | 32 | 37 | 41 | 4.09 | 16 | 1064838 | 65 | 85.12 | 34.11 | 120.88 |
| | | 64 | 66 | 69 | 4.36 | 20 | $\approx 2^{36}$ | 113 | 165.06 | 110.34 | 277.77 |
| | 64 | 16 | 24 | 27 | 3.95 | 15 | 7782 | 42 | 1.82 | 13.50 | 26.27 |
| | | 32 | 37 | 41 | 4.09 | 16 | 1064838 | 65 | 2.36 | 34.11 | 60.41 |
| | | 64 | 66 | 69 | 4.36 | 20 | $\approx 2^{36}$ | 113 | 3.98 | 110.34 | 180.41 |

Table 3: Summary of our parameters aiming around 128 bit security and communication cost results. All communication and PK sizes are in KB. For all settings, we have $d = 64$, $p = 8$, $h = 1$, $\gamma = 4$, and $\beta_r = 1$.

| Obliv. mode | Query batch size | κ | $m = n_s$ | $\ell = n_c$ | σ_0 | β | β_1 | $\log q$ | Server commun. (per query) | Server PK size | Client commun. (per query) |
|-------------|------------------|----------|-----------|--------------|------------|---------|------------------|----------|----------------------------|----------------|----------------------------|
| Partial | 1 | 16 | 18 | 20 | 3.94 | 15 | 6971 | 42 | 57.80 | 10.12 | 68.39 |
| | | 32 | 25 | 27 | 3.95 | 15 | 7841 | 58 | 69.02 | 20.31 | 90.05 |
| | | 64 | 39 | 41 | 3.95 | 15 | 9347 | 90 | 101.95 | 51.19 | 154.36 |
| | 64 | 16 | 18 | 20 | 3.94 | 15 | 6971 | 42 | 1.76 | 10.12 | 20.01 |
| | | 32 | 25 | 27 | 3.95 | 15 | 7841 | 58 | 2.06 | 20.31 | 36.73 |
| | | 64 | 39 | 41 | 3.95 | 15 | 9347 | 90 | 2.82 | 51.19 | 86.00 |
| Full | 1 | 16 | 18 | 20 | 3.94 | 15 | 6971 | 42 | 57.80 | 10.12 | 68.39 |
| | | 32 | 28 | 30 | 4.09 | 16 | 1062331 | 65 | 74.96 | 25.81 | 101.60 |
| | | 64 | 49 | 52 | 4.36 | 20 | $\approx 2^{36}$ | 113 | 135.05 | 81.92 | 219.34 |
| | 64 | 16 | 18 | 20 | 3.94 | 15 | 6971 | 42 | 1.76 | 10.12 | 20.01 |
| | | 32 | 28 | 30 | 4.09 | 16 | 1062331 | 65 | 2.20 | 25.81 | 45.61 |
| | | 64 | 49 | 52 | 4.36 | 20 | $\approx 2^{36}$ | 113 | 3.51 | 81.92 | 135.18 |

Table 4: Summary of our parameters aiming around 90-95 bit security and communication cost results. All communication and PK sizes are in KB. For all settings, we have $d = 64$, $p = 8$, $h = 1$, $\gamma \in \{5, 6\}$, and $\beta_r = 1$.

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A Additional Preliminaries

A.1 Extractable Commitment Scheme

In this work, we consider an extractable commitment scheme [Di 03], which is formally defined as follows.

Definition 14 (Extractable Commitment Scheme). A (non-interactive) extractable commitment scheme consists of a tuple of algorithms $\text{COM} = (\text{Setup}, \text{Commit}, \text{Verify})$ defined as follows:

$\text{Setup}(1^\lambda)$: is a PPT algorithm that on input a (unary encoded) security parameter λ , outputs a commitment key ck .

$\text{Commit}(\text{ck}, m)$: is a PPT algorithm that on input a commitment key ck and a message $m \in \{0, 1\}^\lambda$, outputs a commitment c and an opening information d .

$\text{Verify}(\text{ck}, c, d, m)$: is a DPT algorithm that on input a commitment key ck , commitment c , an opening information d and a message $m \in \{0, 1\}^\lambda$, outputs a bit $b \in \{0, 1\}$.

In the above Commit definition, we leave the randomness part implicit and assume that it is generated internally inside the Commit function. When proving well-formedness of commitments via a zero-knowledge proof, the randomness will also be part of the prover's witness. When we need to specify the randomness r in such cases, we will write $\text{Commit}(\text{ck}, m; r)$ to explicitly refer to the internally generated randomness of the commitment.

We require the standard notion of *correctness*, which says that for every $\lambda \in \mathbb{N}$, every $\text{ck} \leftarrow \text{Setup}(1^\lambda)$ and every message $m \in \{0, 1\}^\lambda$, it holds that

$$\Pr [\text{Verify}(\text{ck}, \text{Commit}(\text{ck}, m), m) = 1] = 1.$$

In terms of security, we require the commitment scheme to satisfy *computational hiding* and *perfect binding* properties, along with *extractability*.

Definition 15 (Computational Hiding). A commitment scheme COM is computationally hiding if for every PPT adversary \mathcal{A} , there exists a negligible function $\text{negl}(\lambda)$, such that

$$\Pr [\text{CH}_{\text{COM}, \mathcal{A}}(\lambda) = 1] \leq \frac{1}{2} + \text{negl}(\lambda),$$

where the experiment $\text{CH}_{\text{COM}, \mathcal{A}}$ is defined as follows

| $\text{CH}_{\text{COM}, \mathcal{A}}(\lambda)$ |
|---|
| 1 : $\text{ck} \leftarrow \text{Setup}(1^\lambda)$ |
| 2 : $(m_0, m_1) \leftarrow \mathcal{A}(\text{ck})$ |
| 3 : $b \xleftarrow{\$} \{0, 1\}$ |
| 4 : $(c, d) \leftarrow \text{Commit}(1^\lambda, m_b)$ |
| 5 : $b' \leftarrow \mathcal{A}(c)$ |
| 6 : return $b = b'$ |

Definition 16 (Computational Binding). A commitment scheme COM is computationally binding if for every PPT adversary \mathcal{A} , there exists a negligible function $\text{negl}(\lambda)$, such that

$$\Pr [\text{CB}_{\text{COM}, \mathcal{A}}(\lambda) = 1] \leq \text{negl}(\lambda),$$

where the experiment $\text{CB}_{\text{COM}, \mathcal{A}}$ is defined as follows

| |
|---|
| $\text{CB}_{\text{COM}, \mathcal{A}}(\lambda)$ |
| $1 : \text{ck} \leftarrow \text{Setup}(1^\lambda)$ |
| $2 : (c, d, d', m, m') \leftarrow \mathcal{A}(\text{ck})$ |
| $3 : \text{assert } m \neq m'$ |
| $4 : \text{return } \text{Verify}(\text{ck}, c, d, m) = 1 \wedge \text{Verify}(\text{ck}, c, d', m') = 1$ |

Definition 17 (Extractability). A commitment scheme COM is extractable, if there exists a pair of PPT algorithm $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2)$, called the extractor, such that for all $m \in \{0, 1\}^\lambda$ and for every PPT adversary \mathcal{A} , there exists a negligible function $\text{negl}(\lambda)$, such that

$$\left| \Pr [\text{ck} \leftarrow \text{Setup}(1^\lambda), (c, d) \leftarrow \mathcal{A}(\text{ck}, m) : \text{Verify}(\text{ck}, c, d, m) = 1] - \Pr [(c, d) \leftarrow \mathcal{E}_1(1^\lambda), (c, d) \leftarrow \mathcal{A}(\text{ck}), m \leftarrow \mathcal{E}_2(\text{ck}, \text{td}, c) : \text{Verify}(\text{ck}, c, d, m) = 1] \right| \leq \text{negl}(\lambda).$$

A.2 Non-Interactive Zero-Knowledge Arguments

Let R be a binary relation and L the language consisting of statements in R . We formally define it as follows a non-interactive zero-knowledge (NIZK) argument system [BFM88] as follows.

Definition 18 (Non-Interactive Zero-Knowledge Argument System). A non-interactive zero-knowledge (NIZK) argument system NIZK for a language $L \in \text{NP}$ (with witness relation R) is a tuple of algorithms $\text{NIZK} = (\text{PGen}, \text{P}, \text{V})$, such that:

$\text{PGen}(1^\lambda)$: is a PPT algorithm that on input a (unary encoded) security parameter λ , outputs a common reference string crs .

$\text{P}(\text{crs}, x, w)$: is a PPT algorithm that on input a common reference string crs , a statement x and a witness w , outputs a proof π .

$\text{V}(\text{crs}, x, \pi)$: is a DPT algorithm that on input a common reference string crs , a statement x and a proof π , outputs a bit b .

We require NIZK to meet the following properties:

Perfect Completeness. For every $(x, w) \in R$ we have that

$$\Pr [\text{crs} \leftarrow \text{PGen}(1^\lambda), \pi \leftarrow \text{P}(\text{crs}, x, w) : \text{V}(\text{crs}, x, \pi) = 1] = 1.$$

Computational Soundness. For every $x \notin L$, and every PPT adversary \mathcal{A} , we have that

$$\Pr [\text{crs} \leftarrow \text{PGen}(1^\lambda), \pi \leftarrow \mathcal{A}(\text{crs}, x) : \text{V}(\text{crs}, x, \pi) = 1] \leq \text{negl}(\lambda).$$

Computational Zero-Knowledge. There exists a PPT algorithm $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ such that for every PPT adversary \mathcal{A} ,

$$\left| \Pr [\text{crs} \leftarrow \text{PGen}(1^\lambda) : \mathcal{A}^{\text{P}(\text{crs}, \cdot, \cdot)}(\text{crs}) = 1] - \Pr [(\text{crs}_s, \tau_s) \leftarrow \mathcal{S}_1(1^\lambda) : \mathcal{A}^{\mathcal{O}(\text{crs}_s, \tau_s, \cdot, \cdot)}(\text{crs}) = 1] \right| \leq \text{negl}(\lambda),$$

where $\mathcal{O}(\text{crs}_s, \tau_s, \cdot, \cdot)$ is an oracle that outputs \perp on input (x, w) when $(x, w) \notin R$ and outputs $\pi \leftarrow \mathcal{S}_2(\text{crs}_s, \tau_s, x)$ when $(x, w) \in R$.

B Deferred Proofs

B.1 Correctness Analysis

Definition 19. We say that function family \mathcal{G} satisfies ϵ_u -uniformity if, for each fixed $x \in \{0, 1\}^L$, the distribution of any fixed \mathbb{Z}_q coordinate of $\mathbf{B}_x \mathbf{k} := \mathcal{G}(x) \mathbf{k} \in R_q^h$ over the choice of \mathcal{G} (in $\text{F.Setup}(1^\lambda)$) and $\mathbf{k} \xleftarrow{\$} \chi_k^m$ is within statistical distance

$$\epsilon_u \leq 2^{-(\kappa+2+\log(hd))} \quad (22)$$

from the uniform distribution on \mathbb{Z}_q .

Lemma 8. Fix κ, h, d . Let $B_f(\kappa, d)$ denote an upper bound on a fixed coordinate of $\mathbf{e}_f = \mathbf{e}'_s - \mathbf{R}\mathbf{e}_s$ (as in (26)) that holds except with probability $p_e \leq 2^{-(\kappa+2+\log(hd))}$. Also, assume that the function family \mathcal{G} satisfies ϵ_u -uniformity (as per Definition 19) with

$$\epsilon_u \leq 2^{-(\kappa+2+\log(hd))}. \quad (16)$$

Then, for any fixed $x \in \{0, 1\}^L$, $2^{-\kappa}$ -correctness holds (as per Definition 2) if

$$q/p \geq 2^{\kappa+2} \cdot hd \cdot (2B_f(\kappa, d) + 1). \quad (17)$$

Proof. Fix (t, x) , and let E and δ denote respectively the event that a correctness error occurs and the correctness error probability. We have

$$\delta := \Pr[E] := \Pr[\lfloor \mathbf{u}_x - \mathbf{R}\mathbf{v}_k \rfloor_p \neq \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p] \quad (23)$$

$$= \Pr[\lfloor \mathbf{B}_x \mathbf{k} + \mathbf{e}_f \rfloor_p \neq \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p] \quad (24)$$

$$(25)$$

where the final error \mathbf{e}_f satisfies

$$\mathbf{e}_f := \mathbf{u}_x - \mathbf{R}\mathbf{v}_k - \mathbf{B}_x \mathbf{k} = \mathbf{e}'_s - \mathbf{R}\mathbf{e}_s. \quad (26)$$

For $i \in [h]$ and $j \in [d]$, let $\mathbf{e}_{f,i} \in R$ denote the i th ring element of $\mathbf{e}_f \in R^h$ and let $e_{f,i,j} \in \mathbb{Z}$ denote the j 'th integer coefficient of $\mathbf{e}_{f,i}$ in the coefficient embedding of R . Similarly, viewing \mathbf{B}_x in its representation over \mathbb{Z} , we denote by $\mathbf{b}_{x,i,j}^\top \in \mathbb{Z}_q^{md}$ the j 'th row the column rot (negacyclic) matrix corresponding to \mathbf{B}_x . Let $B_p \subset [0, q-1]$ denote the set of rounding mod p interval boundary points, i.e. $B_p = \{\frac{q}{2p}, \frac{q}{2p} + \frac{q}{p}, \dots, \frac{q}{2p} + (p-1) \cdot \frac{q}{p}\}$. If event E occurs, then there must exist some $i \in [h]$ and $j \in [d]$ such $\mathbf{b}_{x,i,j}^\top \mathbf{k} \in \mathbb{Z}_q$ lands within distance $|e_{f,i,j}|$ of a rounding interval boundary point in B_p . Therefore, we have

$$\begin{aligned} \delta &\leq \Pr[\exists i \in [h], j \in [d] \text{ s.t. } \mathbf{b}_{x,i,j}^\top \mathbf{k} \in B_p \pm |e_{f,i,j}|] \\ &\leq \sum_{i \in [h], j \in [d]} \delta_{i,j}, \text{ where } \delta_{i,j} := \Pr[\mathbf{b}_{x,i,j}^\top \mathbf{k} \in B_p \pm |e_{f,i,j}|]. \end{aligned} \quad (27)$$

Now, for each fixed $i \in [h]$ and $j \in [d]$, we have

$$\begin{aligned} \delta_{i,j} &\leq \Pr_{u \xleftarrow{\$} \mathbb{Z}_q} [u \in B_p \pm |e_{f,i,j}|] + \epsilon_u \\ &\leq \Pr_{u \xleftarrow{\$} \mathbb{Z}_q} [u \in B_p \pm B_f(\kappa, d)] + p_e + \epsilon_u \\ &= \frac{|(B_p \pm B_f(\kappa, d)) \cap \mathbb{Z}_q|}{|\mathbb{Z}_q|} + p_e + \epsilon_u \\ &\leq \frac{p \cdot (2B_f(\kappa, d) + 1)}{q} + p_e + \epsilon_u \\ &\leq 2^{-\kappa} / (hd), \end{aligned}$$

where the first inequality above uses the assumed ϵ_u -uniformity of \mathcal{G} , the second inequality uses the bound $|e_{f,i,j}| \leq B_f(\kappa, d)$ that holds except with probability $\leq p_e$, the third inequality uses $|B_p| = p$ and the last inequality uses the assumed upper bounds $\leq 2^{-\kappa}/(3hd)$ on the three terms on the right-hand side of the third inequality. It follows that $\delta \leq \sum_{i \in [h], j \in [d]} \delta_{i,j} \leq 2^{-\kappa}$, as claimed. \square

Bounds for BLMR13 PRF Instantiation of \mathcal{G} . For BLMR13 (see [SSS23] for the module instantiation), we have $h = m$, $\chi_k := U(R_q)$, $\mathcal{G}(\mathbf{x}) := \prod_{i=0}^{L-1} \mathbf{A}_{x_i}$, where $\text{F.Setup}(1^\lambda)$ samples $\mathbf{A}_0, \mathbf{A}_1 \xleftarrow{\$} R_{\text{bin}}^{m \times m}$ and restarts if \mathbf{A}_0 or \mathbf{A}_1 are not invertible over R_q .

Lemma 9. *For BLMR13, the function family \mathcal{G} satisfies 0-uniformity. In particular, the distribution of $\mathbf{B}_x \mathbf{k} := \mathcal{G}(x) \mathbf{k} \in R_q^{h \times m}$ over the choice of \mathcal{G} (in $\text{F.Setup}(1^\lambda)$) and $\mathbf{k} \xleftarrow{\$} \chi_k^m$ is the uniform distribution on $R_q^{h \times m}$.*

Proof. Follows immediately from the uniformly random distribution χ_k and the invertibility of the matrix $\mathbf{B}_x = \mathcal{G}(x) = \prod_{i=0}^{L-1} \mathbf{A}_{x_i}$ over R_q , thanks to the invertibility of \mathbf{A}_0 and \mathbf{A}_1 over R_q . \square

Bounds for BP14 PRF Instantiation of \mathcal{G} . For BP14 with $\chi_k := U(R_q)$, $\mathcal{G}(\mathbf{x}) := (\mathbf{A}_{x_0} \mathbf{B}'_1)^\top \in R_q^{h \times m}$, where $\mathbf{B}_{L-1} := G^{-1}(\mathbf{A}_{x_{L-1}}) \in R_q^{m \times m}$, $\mathbf{B}_i := G^{-1}(\mathbf{A}_{x_i} \mathbf{B}_{i+1}) \in R_q^{m \times m}$ for $i = L-2, \dots, 1$, \mathbf{B}'_1 consists of the leftmost h columns of \mathbf{B}_1 , and $\text{F.Setup}(1^\lambda)$ samples $\mathbf{A}_0, \mathbf{A}_1 \xleftarrow{\$} R_q^{m \times m}$.

Lemma 10. *Assume that q is prime. If the BP14 function family \mathcal{G} does not satisfy ϵ_u -uniformity for some non-negligible ϵ_u , then there exists a poly-time 1-query adversary against the pseudorandomness of the BP14 PRF with non-negligible advantage $\geq (1 - 1/p) \cdot \epsilon_u$.*

Proof. By hypothesis, there exists $x \in \{0, 1\}^L$, $i \in [h]$ and $j \in [d]$ such that the statistical distance ϵ between the distribution of $\mathbf{b}_{x,i,j}^\top \mathbf{k} \in \mathbb{Z}_q$ and $U(\mathbb{Z}_q)$ is $\geq \epsilon_u$, where $\mathbf{b}_{x,i,j}^\top$ denotes the j 'th row of the column rot (negacyclic) matrix representation of \mathbf{B}_x over \mathbb{Z} of the i 'th row of $\mathbf{B}_x \in R_q^{h \times m}$, and similarly \mathbf{k} is viewed over \mathbb{Z} as the concatenation of ring element coefficient vectors. Let Z denote the event that $\mathbf{b}_{x,i,j}^\top = \mathbf{0}$. Then we have $\epsilon = \epsilon_{nz} \cdot (1 - \Pr[Z]) + \epsilon_z \cdot \Pr[Z]$, where ϵ_{nz} (resp. ϵ_z) denotes the statistical distance between the distribution of $\mathbf{b}_{x,i,j}^\top \mathbf{k} \in \mathbb{Z}_q$ and $U(\mathbb{Z}_q)$ conditioned on the event that Z does not occur (resp. Z occurs). Conditioned on Z not occurring, we have $\mathbf{b}_{x,i,j}^\top \neq \mathbf{0}$, and hence $\mathbf{b}_{x,i,j}^\top$ has a non-zero and hence (by primality of q) invertible component in \mathbb{Z}_q . Thanks to the independence and uniformity of \mathbf{k} 's components in \mathbb{Z}_q , it follows that in this case $\mathbf{b}_{x,i,j}^\top \mathbf{k}$ is uniformly distributed in \mathbb{Z}_q , so $\epsilon_{nz} = 0$. Therefore $\epsilon = \epsilon_z \cdot \Pr[Z] \leq \Pr[Z]$. It follows that $\Pr[Z] \geq \epsilon \geq \epsilon_u$ is non-negligible. Then there exists a pseudorandomness adversary \mathcal{A} against the BP14 PRF that works as follows: it queries x to its oracle to get $\mathbf{z} \in R_q^h$. If Z occurs then \mathcal{A} outputs 1 if $z_{i,j} = 0$ and output 0 else. If Z does not occur, \mathcal{A} outputs a random coin. In the case that \mathcal{A} 's oracle is the BP14 PRF, if the event Z occurs, the oracle will output 0 with probability 1 and hence \mathcal{A} will output 1 with probability $p_P = \Pr[Z] + 1/2(1 - \Pr[Z])$. In the other case that \mathcal{A} 's oracle is a uniformly random function with range \mathbb{Z}_p , the event $z_{i,j} = 0$ will occur with probability $1/p$ independently of event Z , and hence in this case \mathcal{A} will output 1 with probability $p_U = \Pr[Z] \cdot 1/p + 1/2(1 - \Pr[Z])$. It follows that the distinguishing advantage of \mathcal{A} against the BP14 PRF security is lower bounded as $|p_P - p_U| \geq (1 - 1/p) \cdot \Pr[Z] > (1 - 1/p) \cdot \epsilon_u$, as claimed. \square

Remark 3. We note that the ϵ_u -uniformity Lemma above assumes χ_k has uniformly random coordinates in \mathbb{Z}_q , but the BP14 PRF could also be instantiated with χ_k restricted to small coefficients. In the latter case, the ϵ_u -uniformity seems more difficult to prove but we heuristically expect it to still be satisfied.

Remark 4. We note that our correctness analysis extends to a *malicious* server setting by relying on the fact that the NIZK proof π_s by the server ensures that the error terms $(\mathbf{e}_s, \mathbf{e}'_s)$ have small coefficients (say bounded by β and β_1 , respectively). Particularly, using the worst-case bound in (19) for B_f gives the correctness analysis requirement against a malicious server.

B.2 Pseudorandomness Analysis

Theorem 2. *The POPRF construction F from Figure 6 satisfies pseudorandomness given in Definition 3, with random oracles RO_r and RO_z , if:*

- *The client argument system $NIZK_1$ is computationally sound and the server argument system $NIZK_2$ is computationally zero-knowledge (Definition 18),*
- *The commitment scheme COM is computationally hiding and extractable (Definitions 15 and 17), and*
- *The $iMLWER\text{-}RU_{\text{param}}$ assumption, for $\text{param} := (\mathcal{G}, Q_M, Q_P, Q_x^\infty, q, m, \ell + m, h, 2\lambda, \beta_r, p, \bar{\chi})$, holds (Definition 10).*

More precisely, for any PPT adversary \mathcal{A} , there exist PPT adversaries $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$ and \mathcal{B}_5 against the computational zero-knowledge of $NIZK_2$, computational soundness of $NIZK_1$, hiding of COM , extractability of COM and $iMLWER\text{-}RU_{\text{param}}$ assumption, respectively, such that

$$\begin{aligned} \text{Adv}_{F, \mathcal{A}, S, RO_r, RO_z}^{\text{po-prf}}(\lambda) &\leq \text{Adv}_{NIZK_2, \mathcal{B}_1}^{\text{CZK}}(\lambda) + Q_B \cdot \text{Adv}_{NIZK_1, \mathcal{B}_2}^{\text{CS}}(\lambda) + \text{Adv}_{COM, \mathcal{B}_3}^{\text{CH}}(\lambda) \\ &\quad + Q_{RO_r} \cdot \text{Adv}_{COM, \mathcal{B}_4}^{\text{Ext}}(\lambda) + \text{Adv}_{\mathcal{B}_5}^{\text{iMLWE}_{\text{param}}}(\lambda) + Q_z \left(\frac{1}{p} + \frac{1}{q} \right)^{dh}, \end{aligned}$$

where Q_B and Q_{RO_r} denote the number of BlindEval and RO_r queries, respectively, that the adversary \mathcal{A} makes.

Proof. We consider the POPRF pseudorandomness game given in Definition 3.

We note that the adversary \mathcal{A} has access to oracles $\text{Eval}, \text{BlindEval}$ and Prim , where Prim denotes the random oracles RO_r and RO_z , and the security proof follows in a series of hybrids, where we start with the pseudorandomness game $\text{POPRF}_{F, \mathcal{A}, S, RO_r, RO_z}^1$, i.e., $b = 1$, and we gradually apply changes until we end up with the game $\text{POPRF}_{F, \mathcal{A}, S, RO_r, RO_z}^0$, i.e., $b = 0$ and the oracle calls are answered by the simulator \mathcal{S} . Let $\text{Adv}_{\mathcal{A}, i}(\lambda)$ denote the advantage of \mathcal{A} in **Hybrid_i** and **Hybrid_i** \approx **Hybrid_{i+1}** denote $|\Pr[\mathbf{Hybrid}_i = 1] - \Pr[\mathbf{Hybrid}_{i+1} = 1]| \leq \text{negl}(\lambda)$. We consider the following hybrids.

Hybrid₀: This corresponds to the pseudorandomness game $\text{POPRF}_{F, \mathcal{A}, S, RO_r, RO_z}^1$ (Definition 3), and the interaction follows as in Figure 6. Hence, it follows that

$$\text{Adv}_{F, \mathcal{A}, S, RO_r, RO_z}^{\text{po-prf}}(\lambda) = \text{Adv}_{\mathcal{A}, 0}(\lambda).$$

Hybrid₁: In this hybrid, we replace crs_2 and proof π_s with a simulated CRS and proof $(\text{crs}_2^*, \pi_s^*)$, as shown in Figure 9.

In order to show that **Hybrid₀** \approx **Hybrid₁**, we construct a reduction \mathcal{B}_1 to the computational zero-knowledge property of $NIZK_2$. We note that all queries are answered as in **Hybrid₀** with the exception of BlindEval queries. The only difference here is that \mathcal{B}_1 receives a CRS crs_2^* from its zero-knowledge challenger, and sets $\text{crs}_2 := \text{crs}_2^*$, instead of generating crs_2 via the $NIZK_2.\text{Setup}$ algorithm. When \mathcal{A} makes a BlindEval query, \mathcal{B}_1 proceeds as in **Hybrid₀** to compute \mathbf{v}_k and \mathbf{u}_x , but it makes an oracle call to its zero-knowledge challenger with the input $((\mathbf{u}_x, \mathbf{C}_x, \mathbf{c}_k, \mathbf{v}_k, \text{ck}_2, \mathbf{A}_r), (\mathbf{k}, \mathbf{e}, \mathbf{e}_s, \mathbf{e}'_s))$ to obtain the proof π_s^* .

If the zero-knowledge challenger of $NIZK_2$ used the honest setup and prover algorithms to generate crs_2^* and π_s^* , then we are exactly in **Hybrid₀**, and if it used the simulator, then we are in **Hybrid₁**. Therefore, if \mathcal{A} can distinguish between the two hybrids with non-negligible advantage, then \mathcal{B}_1 can break the computational zero-knowledge property of $NIZK_2$. Hence, it follows that

$$|\text{Adv}_{\mathcal{A}, 0}(\lambda) - \text{Adv}_{\mathcal{A}, 1}(\lambda)| \leq \text{Adv}_{NIZK_2, \mathcal{B}_1}^{\text{CZK}}(\lambda),$$

and in particular **Hybrid₀** \approx **Hybrid₁**.

Hybrid₂: Let BAD_1 be the event that for a blind evaluation query $(t, \text{req} := (\mathbf{c}_r, \mathbf{C}_x, \pi_c))$, it holds that $NIZK_1.\text{V}(\text{crs}_1, \pi_c, (\mathbf{c}_r, \mathbf{C}_x, \text{ck}_1, \mathbf{A}_r, \mathcal{G}, t)) = 1$, but $(\mathbf{C}_x \neq \mathbf{R}\mathbf{A}_r + \mathcal{G}(x) \vee \mathbf{c}_r \neq \text{COM.Commit}(\text{ck}_1, (\mathbf{r}_0, \dots, \mathbf{r}_{h-1}, x); \rho_r))$. If BAD_1 happens, then the challenger aborts, as depicted in Figure 10.

| | |
|---|---|
| F.Setup(1^λ) <hr/> 1 : $\forall i \in [2], \text{ck}_i \leftarrow \text{COM.Setup}(1^\lambda)$ 2 : $\text{crs}_1 \leftarrow \text{NIZK}_1.\text{Setup}(1^\lambda)$ 3 : $\text{crs}_2^*, \tau_2 \leftarrow \text{NIZK}_2.\mathcal{S}_1(1^\lambda)$ 4 : $\text{pp} := (\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 5 : return pp | F.KeyGen(pp) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : $\mathbf{k} \xleftarrow{\$} \chi_k^m$ 3 : $\mathbf{c}_k \leftarrow \text{COM.Commit}(\text{ck}_2, \mathbf{k}; \rho_k)$ 4 : return $(\text{pk} := \mathbf{c}_k, \text{sk} := \mathbf{k})$ |
| F.Request(pp, pk, t, x) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : $\mathbf{R} \xleftarrow{\$} \chi_r^{h \times (\ell+m)}$ 3 : $\mathbf{c}_r \leftarrow \text{COM.Commit}(\text{ck}_1, (\mathbf{R}, x); \rho_r)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 6 : $\mathbf{C}_x := \mathbf{R}\mathbf{A}_r + \mathbf{B}_x \in R_q^{h \times m}$ 7 : $\text{stmt}_1 := (\mathbf{c}_r, \mathbf{C}_x, \text{ck}_1, \mathbf{A}_r, \mathcal{G}, t)$ 8 : $\text{wit}_1 := (\mathbf{R}, x, \rho_r)$ 9 : $\pi_c \leftarrow \text{NIZK}_1.\text{P}(\text{crs}_1, \text{stmt}_1, \text{wit}_1)$ 10 : $\text{st} := (t, x, \mathbf{R}, \mathbf{C}_x, \text{pk}, \mathbf{A}_r)$ 11 : $\text{req} := (\mathbf{c}_r, \mathbf{C}_x, \pi_c)$ 12 : return (st, req) | F.BlindEval(pp, pk, sk, t, req) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : parse pk := $\mathbf{c}_k, \text{sk} := \mathbf{k}$ 3 : parse req := $(\mathbf{c}_r, \mathbf{C}_x, \pi_c)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : if $\text{NIZK}_1.\text{V}(\text{crs}_1, \pi_c, \text{stmt}_1) \neq 1$ then 6 : return \perp 7 : $\mathbf{e}_s \xleftarrow{\$} \chi^{\ell+m}, \mathbf{e}'_s \xleftarrow{\$} \chi_1^h$ 8 : $\mathbf{v}_k := \mathbf{A}_r \mathbf{k} + \mathbf{e}_s \in R_q^{\ell+m}$ 9 : $\mathbf{u}_x := \mathbf{C}_x \mathbf{k} + \mathbf{e}'_s \in R_q^h$ 10 : $\text{stmt}_2 := (\mathbf{u}_x, \mathbf{C}_x, \mathbf{c}_k, \mathbf{v}_k, \text{ck}_2, \mathbf{A}_r)$ 11 : $\pi_s^* \leftarrow \text{NIZK}_2.\mathcal{S}_2(\text{crs}_2^*, \tau_2, \text{stmt}_2)$ 12 : return $\text{rep} := (\mathbf{u}_x, \mathbf{v}_k, \pi_s^*)$ |
| F.Finalize(pp, rep, st) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : parse rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s^*)$ 3 : parse st := $(t, x, \mathbf{R}, \mathbf{C}_x, \text{pk} := \mathbf{c}_k, \mathbf{A}_r)$ 4 : if $\text{NIZK}_2.\text{V}(\text{crs}_2^*, \pi_s^*, \text{stmt}_2) \neq 1$ then 5 : return \perp 6 : $\mathbf{z} := \lfloor \mathbf{u}_x - \mathbf{R}\mathbf{v}_k \rfloor_p \in R_p^h \quad // = \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p$ 7 : $y := \text{RO}_z(t, x, \mathbf{z})$ 8 : return y | F.Eval(sk, t, x) <hr/> 1 : parse sk := \mathbf{k} 2 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 3 : $\mathbf{z} := \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p \in R_p^h$ 4 : $y := \text{RO}_z(t, x, \mathbf{z})$ 5 : return y |
| RO_r(t, c_r) <hr/> 1 : if $\mathcal{H}[t, \mathbf{c}_r] = \perp$ then 2 : $\mathcal{H}[t, \mathbf{c}_r] \xleftarrow{\$} R_q^{(\ell+m) \times m}$ 3 : return $\mathcal{H}[t, \mathbf{c}_r]$ | RO_z(t, x, z) <hr/> 1 : if $\mathcal{F}[t, x, \mathbf{z}] = \perp$ then 2 : $\mathcal{F}[t, x, \mathbf{z}] \xleftarrow{\$} \{0, 1\}^\lambda$ 3 : return $\mathcal{F}[t, x, \mathbf{z}]$ |

Fig. 9: **Hybrid₁** of pseudorandomness proof.

| | |
|---|---|
| F.Setup(1^λ) <hr/> 1 : $\forall i \in [2], \text{ck}_i \leftarrow \text{COM.Setup}(1^\lambda)$ 2 : $\text{crs}_1 \leftarrow \text{NIZK}_1.\text{Setup}(1^\lambda)$ 3 : $\text{crs}_2^*, \tau_2 \leftarrow \text{NIZK}_2.\mathcal{S}_1(1^\lambda)$ 4 : $\text{pp} := (\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 5 : return pp | F.KeyGen(pp) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : $\mathbf{k} \xleftarrow{\$} \chi_k^m$ 3 : $\mathbf{c}_k \leftarrow \text{COM.Commit}(\text{ck}_2, \mathbf{k}; \rho_k)$ 4 : return $(\text{pk} := \mathbf{c}_k, \text{sk} := \mathbf{k})$ |
| F.Request(pp, pk, t, x) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : $\mathbf{R} \xleftarrow{\$} \chi_r^{h \times (\ell+m)}$ 3 : $\mathbf{c}_r \leftarrow \text{COM.Commit}(\text{ck}_1, (\mathbf{R}, x); \rho_r)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 6 : $\mathbf{C}_x := \mathbf{R}\mathbf{A}_r + \mathbf{B}_x \in R_q^{h \times m}$ 7 : $\text{stmt}_1 := (\mathbf{c}_r, \mathbf{C}_x, \text{ck}_1, \mathbf{A}_r, \mathcal{G}, t)$ 8 : $\text{wit}_1 := (\mathbf{R}, x, \rho_r)$ 9 : $\pi_c \leftarrow \text{NIZK}_1.\text{P}(\text{crs}_1, \text{stmt}_1, \text{wit}_1)$ 10 : $\text{st} := (t, x, \mathbf{R}, \mathbf{C}_x, \text{pk}, \mathbf{A}_r)$ 11 : $\text{req} := (\mathbf{c}_r, \mathbf{C}_x, \pi_c)$ 12 : return (st, req) | F.BlindEval(pp, pk, sk, t, req) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : parse pk := $\mathbf{c}_k, \text{sk} := \mathbf{k}$ 3 : parse req := $(\mathbf{c}_r, \mathbf{C}_x, \pi_c)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : if $\text{NIZK}_1.\text{V}(\text{crs}_1, \pi_c, \text{stmt}_1) = 1$ 6 : $\wedge (\mathbf{C}_x \neq \mathbf{R}\mathbf{A}_r + \mathcal{G}(x))$ 7 : $\vee \mathbf{c}_r \neq \text{COM.Commit}(\text{ck}_1, (\mathbf{R}, x); \rho_r)$ 8 : then abort 9 : $\mathbf{e}_s \xleftarrow{\$} \chi^{\ell+m}, \mathbf{e}'_s \xleftarrow{\$} \chi_1^h$ 10 : $\mathbf{v}_k := \mathbf{A}_r \mathbf{k} + \mathbf{e}_s \in R_q^{\ell+m}$ 11 : $\mathbf{u}_x := \mathbf{C}_x \mathbf{k} + \mathbf{e}'_s \in R_q^h$ 12 : $\text{stmt}_2 := (\mathbf{u}_x, \mathbf{C}_x, \mathbf{c}_k, \mathbf{v}_k, \text{ck}_2, \mathbf{A}_r)$ 13 : $\pi_s^* \leftarrow \text{NIZK}_2.\mathcal{S}_2(\text{crs}_2^*, \tau_2, \text{stmt}_2)$ 14 : return rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s^*)$ |
| F.Finalize(pp, rep, st) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : parse rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s^*)$ 3 : parse st := $(t, x, \mathbf{R}, \mathbf{C}_x, \text{pk} := \mathbf{c}_k, \mathbf{A}_r)$ 4 : if $\text{NIZK}_2.\text{V}(\text{crs}_2^*, \pi_s^*, \text{stmt}_2) \neq 1$ then 5 : return \perp 6 : $\mathbf{z} := \lfloor \mathbf{u}_x - \mathbf{R}\mathbf{v}_k \rfloor_p \in R_p^h \quad // = \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p$ 7 : $y := \text{RO}_z(t, x, \mathbf{z})$ 8 : return y | F.Eval(sk, t, x) <hr/> 1 : parse sk := \mathbf{k} 2 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 3 : $\mathbf{z} := \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p \in R_p^h$ 4 : $y := \text{RO}_z(t, x, \mathbf{z})$ 5 : return y |
| RO_r(t, c_r) <hr/> 1 : if $\mathcal{H}[t, \mathbf{c}_r] = \perp$ then 2 : $\mathcal{H}[t, \mathbf{c}_r] \xleftarrow{\$} R_q^{(\ell+m) \times m}$ 3 : return $\mathcal{H}[t, \mathbf{c}_r]$ | RO_z(t, x, z) <hr/> 1 : if $\mathcal{F}[t, x, \mathbf{z}] = \perp$ then 2 : $\mathcal{F}[t, x, \mathbf{z}] \xleftarrow{\$} \{0, 1\}^\lambda$ 3 : return $\mathcal{F}[t, x, \mathbf{z}]$ |

Fig. 10: **Hybrid₂** of pseudorandomness proof.

Clearly, we have that **Hybrid₁** and **Hybrid₂** are identical until the event BAD_1 happens. Hence, we show that we can bound the probability $\Pr[\text{BAD}_1]$ by constructing a reduction \mathcal{B}_2 to the computational soundness of NIZK_1 argument system. We note that all queries are answered as in **Hybrid₁**. Let Q_B denote the maximum number of BlindEval queries that \mathcal{A} performs. \mathcal{B}_2 randomly chooses a $q^* \in [Q_B]$ to serve as its guess for the query where the proof π_c will be forged by the adversary \mathcal{A} . For $\text{req} := (\mathbf{c}_r, \mathbf{C}_x, \pi_c)$ that corresponds to the q^* -th blind evaluation query, \mathcal{B}_2 simply outputs π_c and aborts. We note that \mathcal{B}_2 avoids recovering \mathbf{R} and x (which are computationally infeasible to compute), and instead just guesses which of the proofs would have caused the event BAD_1 to happen. Hence, we have that

$$|\text{Adv}_{\mathcal{A},1}(\lambda) - \text{Adv}_{\mathcal{A},2}(\lambda)| \leq Q_B \cdot \text{Adv}_{\text{NIZK}_1, \mathcal{B}_2}^{\text{CS}}(\lambda),$$

and in particular **Hybrid₁** \approx **Hybrid₂**.

Hybrid₃: In this hybrid, during key generation we compute the commitment $\mathbf{c}_k^* \leftarrow \text{COM.Commit}(\text{ck}_2, \mathbf{0}_m)$, where $\mathbf{0}_m$ is an all zero vector of length m , instead of computing the commitment $\mathbf{c}_k \leftarrow \text{COM.Commit}(\text{ck}_2, \mathbf{k})$ as in the previous hybrid. This change is shown in Figure 11.

In order to show that **Hybrid₂** \approx **Hybrid₃**, we construct a reduction \mathcal{B}_3 to the computational hiding property of COM . For computing the public key pk , \mathcal{B}_3 samples a random $\mathbf{k} \xleftarrow{\$} \chi_k^m$, sets $(m_0 := \mathbf{k}, m_1 := \mathbf{0}_m)$ and submits the pair (m_0, m_1) to the computational hiding challenger of COM , which responds with \mathbf{c}_k^* . At this point \mathcal{B}_3 sets $\text{pk} := \mathbf{c}_k^*$, and proceeds to answer the oracle queries as in the previous hybrid.

If the computational hiding challenger of COM provided a commitment to m_0 , then we are exactly in **Hybrid₂**, and if it provided a commitment to m_1 , then we are in **Hybrid₃**. Therefore, if \mathcal{A} can distinguish between the two hybrids with non-negligible advantage, then \mathcal{B}_3 can break the computational hiding property of COM . Hence, it follows that

$$|\text{Adv}_{\mathcal{A},2}(\lambda) - \text{Adv}_{\mathcal{A},3}(\lambda)| \leq \text{Adv}_{\text{COM}, \mathcal{B}_3}^{\text{CH}}(\lambda),$$

and in particular **Hybrid₂** \approx **Hybrid₃**.

Hybrid₄: In this hybrid, we replace the commitment key ck_1 with a trapdoor variant $(\text{ck}_1^*, \text{td}) \leftarrow \text{COM.E}_1(1^\lambda)$, and use the trapdoor td to extract the committed values queried to RO_r oracle, i.e., run $(\mathbf{R}, x) := \text{COM.E}_2(\text{ck}_1^*, \text{td}, \mathbf{c}_r)$, compute \mathbf{v}_k and \mathbf{u}_x inside RO_r and store these values in a table for use during the blind evaluation queries, as depicted in Figure 12.

In order to prove that **Hybrid₃** \approx **Hybrid₄** we construct an adversary \mathcal{B}_4 to the extractability property of COM . We note that all queries are answered as in **Hybrid₃** with the exception of BlindEval and RO_r queries. \mathcal{B}_4 samples a trapdoor commitment key $(\text{ck}_1^*, \text{td}) \leftarrow \text{COM.E}_1(1^\lambda)$, instead of generating ck_1 via the COM.Setup algorithm. Let Q_{RO_r} denote the number of RO_r queries that the adversary \mathcal{A} makes. For each such query (t, \mathbf{c}_r) , \mathcal{B}_4 uses the trapdoor td to extract $(\mathbf{R}, x) \leftarrow \text{COM.E}_2(\text{ck}_1^*, \text{td}, \mathbf{c}_r)$ and compute the pair $(\mathbf{u}_x, \mathbf{v}_k)$ accordingly, which gets stored in the table $\mathcal{T}[t, \mathbf{c}_r] := (\mathbf{u}_x, \mathbf{v}_k)$. Upon receiving a BlindEval query $(t, \text{req} := (\mathbf{c}_r, \mathbf{C}_x, \pi_c))$, \mathcal{B}_4 uses the already stored values in $\mathcal{T}[t, \mathbf{c}_r] := (\mathbf{u}_x, \mathbf{v}_k)$ in order to simulate the proof π_s^* and answer the query with the tuple $(\mathbf{u}_x, \mathbf{v}_k, \pi_s^*)$.

Due to the soundness of the client proof π_c we know that the adversarial inputs, and especially commitments, are well-formed. Hence, if \mathcal{A} can distinguish between the two hybrids with non-negligible advantage, then \mathcal{B}_4 can break the extractability property of COM . Moreover, since the adversary \mathcal{A} makes at most Q_{RO_r} queries to RO_r oracle, it follows that

$$|\text{Adv}_{\mathcal{A},3}(\lambda) - \text{Adv}_{\mathcal{A},4}(\lambda)| \leq Q_{\text{RO}_r} \cdot \text{Adv}_{\text{COM}, \mathcal{B}_4}^{\text{Ext}}(\lambda),$$

and in particular **Hybrid₃** \approx **Hybrid₄**.

Hybrid₅: In this hybrid, we change the way we respond to RO_r and Eval queries. Concretely, for answering Eval queries, instead of computing $\mathbf{z} := \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p$, we sample a uniform $\mathbf{z}' \xleftarrow{\$} R_q^h$ and return $\mathbf{z} := \lfloor \mathbf{z}' \rfloor_p$. For answering RO_r queries, instead of computing $\mathbf{v}_k := \mathbf{A}_r \mathbf{k} + \mathbf{e}_s$ and $\mathbf{u}_x := \mathbf{C}_x \mathbf{k} + \mathbf{e}'_s$, we sample a uniform

| | |
|---|---|
| F.Setup(1^λ) 1 : $\forall i \in [2], \text{ck}_i \leftarrow \text{COM.Setup}(1^\lambda)$ 2 : $\text{crs}_1 \leftarrow \text{NIZK}_1.\text{Setup}(1^\lambda)$ 3 : $\text{crs}_2^*, \tau_2 \leftarrow \text{NIZK}_2.\mathcal{S}_1(1^\lambda)$ 4 : $\text{pp} := (\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 5 : return pp | F.KeyGen(pp) 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : $\mathbf{k} \xleftarrow{\$} \chi_k^m$ 3 : $\mathbf{c}_k^* \leftarrow \text{COM.Commit}(\text{ck}_2, \mathbf{0}_m; \rho_k)$ 4 : return $(\text{pk} := \mathbf{c}_k^*, \text{sk} := \mathbf{k})$ |
| F.Request(pp, pk, t, x) 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : $\mathbf{R} \xleftarrow{\$} \chi_r^{h \times (\ell+m)}$ 3 : $\mathbf{c}_r \leftarrow \text{COM.Commit}(\text{ck}_1, (\mathbf{R}, x); \rho_r)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 6 : $\mathbf{C}_x := \mathbf{R}\mathbf{A}_r + \mathbf{B}_x \in R_q^{h \times m}$ 7 : $\text{stmt}_1 := (\mathbf{c}_r, \mathbf{C}_x, \text{ck}_1, \mathbf{A}_r, \mathcal{G}, t)$ 8 : $\text{wit}_1 := (\mathbf{R}, x, \rho_r)$ 9 : $\pi_c \leftarrow \text{NIZK}_1.\text{P}(\text{crs}_1, \text{stmt}_1, \text{wit}_1)$ 10 : $\text{st} := (t, x, \mathbf{R}, \mathbf{C}_x, \text{pk}, \mathbf{A}_r)$ 11 : $\text{req} := (\mathbf{c}_r, \mathbf{C}_x, \pi_c)$ 12 : return (st, req) | F.BlindEval(pp, pk, sk, t, req) 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : parse pk := $\mathbf{c}_k^*, \text{sk} := \mathbf{k}$ 3 : parse req := $(\mathbf{c}_r, \mathbf{C}_x, \pi_c)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : if $\text{NIZK}_1.\text{V}(\text{crs}_1, \pi_c, \text{stmt}_1) = 1$ 6 : $\wedge (\mathbf{C}_x \neq \mathbf{R}\mathbf{A}_r + \mathcal{G}(x))$ 7 : $\vee \mathbf{c}_r \neq \text{COM.Commit}(\text{ck}_1, (\mathbf{R}; \rho_r))$ 8 : then abort 9 : $\mathbf{e}_s \xleftarrow{\$} \chi^{\ell+m}, \mathbf{e}'_s \xleftarrow{\$} \chi_1^h$ 10 : $\mathbf{v}_k := \mathbf{A}_r \mathbf{k} + \mathbf{e}_s \in R_q^{\ell+m}$ 11 : $\mathbf{u}_x := \mathbf{C}_x \mathbf{k} + \mathbf{e}'_s \in R_q^h$ 12 : $\text{stmt}_2 := (\mathbf{u}_x, \mathbf{C}_x, \mathbf{c}_k^*, \mathbf{v}_k, \text{ck}_2, \mathbf{A}_r)$ 13 : $\pi_s^* \leftarrow \text{NIZK}_2.\mathcal{S}_2(\text{crs}_2^*, \tau_2, \text{stmt}_2)$ 14 : return $\text{rep} := (\mathbf{u}_x, \mathbf{v}_k, \pi_s^*)$ |
| F.Finalize(pp, rep, st) 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : parse rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s^*)$ 3 : parse st := $(t, x, \mathbf{R}, \mathbf{C}_x, \text{pk} := \mathbf{c}_k^*, \mathbf{A}_r)$ 4 : if $\text{NIZK}_2.\text{V}(\text{crs}_2^*, \pi_s^*, \text{stmt}_2) \neq 1$ then 5 : return \perp 6 : $\mathbf{z} := \lfloor \mathbf{u}_x - \mathbf{R}\mathbf{v}_k \rfloor_p \in R_p^h \quad // = \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p$ 7 : $y := \text{RO}_z(t, x, \mathbf{z})$ 8 : return y | F.Eval(sk, t, x) 1 : parse sk := \mathbf{k} 2 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 3 : $\mathbf{z} := \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p \in R_p^h$ 4 : $y := \text{RO}_z(t, x, \mathbf{z})$ 5 : return y |
| RO_r(t, c_r) 1 : if $\mathcal{H}[t, \mathbf{c}_r] = \perp$ then 2 : $\mathcal{H}[t, \mathbf{c}_r] \xleftarrow{\$} R_q^{(\ell+m) \times m}$ 3 : return $\mathcal{H}[t, \mathbf{c}_r]$ | RO_z(t, x, z) 1 : if $\mathcal{F}[t, x, \mathbf{z}] = \perp$ then 2 : $\mathcal{F}[t, x, \mathbf{z}] \xleftarrow{\$} \{0, 1\}^\lambda$ 3 : return $\mathcal{F}[t, x, \mathbf{z}]$ |

Fig. 11: **Hybrid₃** of pseudorandomness proof.

| F.Setup(1^λ) | F.KeyGen(pp) | F.Eval(sk, t, x) |
|--|---|---|
| 1 : $(ck_1^*, td) \leftarrow \text{COM.E}_1(1^\lambda)$ 2 : $ck_2 \leftarrow \text{COM.Setup}(1^\lambda)$ 3 : $crs_1 \leftarrow \text{NIZK}_1.\text{Setup}(1^\lambda)$ 4 : $crs_2^*, \tau_2 \leftarrow \text{NIZK}_2.\mathcal{S}_1(1^\lambda)$ 5 : $pp := (ck_1^*, ck_2, crs_1, crs_2^*)$ 6 : return pp | 1 : parse pp := $(ck_1^*, ck_2, crs_1, crs_2^*)$ 2 : $\mathbf{k} \xleftarrow{\$} \chi_k^m$ 3 : $\mathbf{c}_k^* \leftarrow \text{COM.Commit}(ck_2, \mathbf{0}_m; \rho_k)$ 4 : return (pk := \mathbf{c}_k^* , sk := \mathbf{k}) | 1 : parse sk := \mathbf{k} 2 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 3 : $\mathbf{z} := \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p \in R_p^h$ 4 : $y := \text{RO}_z(t, x, \mathbf{z})$ 5 : return y |
| F.Request(pp, pk, t, x) | F.BlindEval(pp, pk, sk, t, req) | |
| 1 : parse pp := $(ck_1^*, ck_2, crs_1, crs_2^*)$ 2 : $\mathbf{R} \xleftarrow{\$} \chi_r^{h \times (\ell+m)}$ 3 : $\mathbf{c}_r \leftarrow \text{COM.Commit}(ck_1^*, (\mathbf{R}, x); \rho_r)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 6 : $\mathbf{C}_x := \mathbf{R}\mathbf{A}_r + \mathbf{B}_x \in R_q^{h \times m}$ 7 : $\text{stmt}_1 := (\mathbf{c}_r, \mathbf{C}_x, ck_1, \mathbf{A}_r, \mathcal{G}, t)$ 8 : $\text{wit}_1 := (\mathbf{R}, x, \rho_r)$ 9 : $\pi_c \leftarrow \text{NIZK}_1.\text{P}(crs_1, \text{stmt}_1, \text{wit}_1)$ 10 : $\text{st} := (t, x, \mathbf{R}, \mathbf{C}_x, \text{pk}, \mathbf{A}_r)$ 11 : $\text{req} := (\mathbf{c}_r, \mathbf{C}_x, \pi_c)$ 12 : return (st, req) | 1 : parse pp := $(ck_1^*, ck_2, crs_1, crs_2^*)$ 2 : parse pk := \mathbf{c}_k^* , sk := \mathbf{k} 3 : parse req := $(\mathbf{c}_r, \mathbf{C}_x, \pi_c)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : if $\text{NIZK}_1.\text{V}(crs_1, \pi_c, \text{stmt}_1) = 1$ 6 : $\wedge (\mathbf{C}_x \neq \mathbf{R}\mathbf{A}_r + \mathcal{G}(x))$ 7 : $\vee \mathbf{c}_r \neq \text{COM.Commit}(ck_1, (\mathbf{R}, x); \rho_r)$ 8 : then abort 9 : $(\mathbf{u}_x, \mathbf{v}_k) := \mathcal{T}[t, \mathbf{c}_r]$ 10 : $\text{stmt}_2 := (\mathbf{u}_x, \mathbf{C}_x, \mathbf{c}_k^*, \mathbf{v}_k, ck_2, \mathbf{A}_r)$ 11 : $\pi_s^* \leftarrow \text{NIZK}_2.\mathcal{S}_2(crs_2^*, \tau_2, \text{stmt}_2)$ 12 : return rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s^*)$ | |
| F.Finalize(pp, rep, st) | $\text{RO}_r(t, \mathbf{c}_r)$ | |
| 1 : parse pp := $(ck_1^*, ck_2, crs_1, crs_2^*)$ 2 : parse rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s^*)$ 3 : parse st := $(t, x, \mathbf{R}, \mathbf{C}_x, \text{pk} := \mathbf{c}_k^*, \mathbf{A}_r)$ 4 : if $\text{NIZK}_2.\text{V}(crs_2^*, \pi_s^*, \text{stmt}_2) \neq 1$ then 5 : return \perp 6 : $\mathbf{z} := \lfloor \mathbf{u}_x - \mathbf{R}\mathbf{v}_k \rfloor_p \in R_p^h \quad // = \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p$ 7 : $y := \text{RO}_z(t, x, \mathbf{z})$ 8 : return y | 1 : if $\mathcal{H}[t, \mathbf{c}_r] = \perp$ then 2 : $\mathbf{A}_r := \mathcal{H}[t, \mathbf{c}_r] \xleftarrow{\$} R_q^{(\ell+m) \times m}$ 3 : $(\mathbf{R}, x) \leftarrow \text{COM.E}_2(ck_1^*, td, \mathbf{c}_r)$ 4 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 5 : $\mathbf{e}_s \xleftarrow{\$} \chi^{\ell+m}, \mathbf{e}'_s \xleftarrow{\$} \chi_1^h$ 6 : $\mathbf{v}_k := \mathbf{A}_r \mathbf{k} + \mathbf{e}_s \in R_q^{\ell+m}$ 7 : $\mathbf{C}_x := \mathbf{R}\mathbf{A}_r + \mathbf{B}_x \in R_q^{h \times m}$ 8 : $\mathbf{u}_x := \mathbf{C}_x \mathbf{k} + \mathbf{e}'_s \in R_q^h$ 9 : $\mathcal{T}[t, \mathbf{c}_r] := (\mathbf{u}_x, \mathbf{v}_k)$ 10 : return $\mathcal{H}[t, \mathbf{c}_r]$ | |
| $\text{RO}_z(t, x, \mathbf{z})$ | | |
| 1 : if $\mathcal{F}[t, x, \mathbf{z}] = \perp$ then 2 : $\mathcal{F}[t, x, \mathbf{z}] \xleftarrow{\$} \{0, 1\}^\lambda$ 3 : return $\mathcal{F}[t, x, \mathbf{z}]$ | | |

Fig. 12: **Hybrid**₄ of pseudorandomness proof.

$\mathbf{v}_k \stackrel{\$}{\leftarrow} R_q^{\ell+m}$ and set $\mathbf{u}_x := \mathbf{R}\mathbf{v}_k + \mathbf{z}' + \mathbf{e}^*$, where \mathbf{e}^* is computed as $\mathbf{e}^* = \tilde{\mathbf{e}}' - \mathbf{R}\tilde{\mathbf{e}} \in R^h$, such that $\tilde{\mathbf{e}}' \stackrel{\$}{\leftarrow} \chi_1^h$, $\tilde{\mathbf{e}} \stackrel{\$}{\leftarrow} \chi^{\ell+m}$. These changes are depicted in Figure 13.

In order to show that **Hybrid₄** \approx **Hybrid₅** we construct an adversary \mathcal{B}_5 against the iMLWER-RU_{param} assumption, for $\text{param} := (\mathcal{G}, Q_M, Q_P, Q_x^\infty, q, m, \ell + m, h, 2\lambda, \beta_r, p, \bar{\chi})$. For answering Eval queries (t, x) , \mathcal{B}_5 makes an oracle query $\mathbf{z} \leftarrow \text{SampPRF}(t||x)$ to iMLWER-RU challenger, and returns $y := \text{RO}_z(t, x, \mathbf{z})$. For answering RO_r queries (t, \mathbf{c}_r) , \mathcal{B}_5 uses the commitment trapdoor to extract $(\mathbf{R}, x) := \text{COM.E}_2(\text{ck}_1^*, \text{td}, \mathbf{c}_r)$ (as in the previous hybrid), and makes an oracle call $(\mathbf{A}_r, \mathbf{v}_k, \mathbf{u}_x) \leftarrow \text{SampMLWE-RU}(\mathbf{R}, t||x)$ to the iMLWER-RU challenger. Then, \mathcal{B}_5 sets $\mathcal{H}[t, \mathbf{c}_r] := \mathbf{A}_r$ and stores $\mathcal{T}[t, \mathbf{c}_r] := (\mathbf{u}_x, \mathbf{v}_k)$. The rest of the queries are answered as in the previous hybrid.

If the iMLWER-RU challenger used $b = 1$, then the view of \mathcal{A} simulated by \mathcal{B}_5 is exactly as in **Hybrid₄**, and if it used $b = 0$, then the view of \mathcal{A} is as in **Hybrid₅**. Therefore, if \mathcal{A} can distinguish between the two hybrids with non-negligible advantage, then \mathcal{B}_5 can break the iMLWER-RU_{param} assumption, where Q_M and Q_P in the iMLWER-RU assumption denote the number of Eval and RO_r queries, and Q_x^∞ denotes the maximum number of identical $(t||x)$ queries, respectively, that the adversary \mathcal{A} makes. Hence, it follows that

$$|\text{Adv}_{\mathcal{A},4}(\lambda) - \text{Adv}_{\mathcal{A},5}(\lambda)| \leq \text{Adv}_{\mathcal{B}_5}^{\text{iMLWE}_{\text{param}}}(\lambda),$$

and in particular **Hybrid₄** \approx **Hybrid₅**.

Hybrid₆: In this hybrid we replace the Eval algorithm with a random oracle, which is depicted in Figure 14. This corresponds to the pseudorandomness game $\text{POPRF}_{\mathcal{F},\mathcal{A},\mathcal{S},\text{RO}_r,\text{RO}_z}^0$ (Definition 3).

Let BAD_2 be the event that the adversary \mathcal{A} queries (t, x, \mathbf{z}) to RO_z oracle, for some $\mathbf{z} := \lfloor \mathcal{Z}[t, x] \rfloor_p$, before ever making a query to the RO_r oracle. Clearly, we have that **Hybrid₅** and **Hybrid₆** are identical until the event BAD_2 happens. We observe that the only distinguishing advantage the adversary \mathcal{A} has between **Hybrid₅** and **Hybrid₆** is if \mathcal{A} queries RO_z oracle with some input (t, x, \mathbf{z}) , where initially it holds that $\mathbf{z} \neq \lfloor \mathcal{Z}[t, x] \rfloor_p$, but after making a call to RO_r oracle, we have that it is set to $\mathcal{Z}[t, x] := \mathbf{z}'$, where $\mathbf{z} := \lfloor \mathbf{z}' \rfloor_p$. In this case, \mathcal{A} will obtain two different outputs for the same (t, x, \mathbf{z}) input to RO_z oracle (before and after making a call to RO_r oracle), and it can trivially distinguish the hybrids. The probability of this happening corresponds to the aforementioned BAD_2 event happening, which we now argue that it can only happen with a negligible probability.

Note that inside the RO_r oracle, we sample \mathbf{v}_k and \mathbf{z}' uniformly from $R_q^{\ell+m}$ and R_q^h , respectively. Then, we compute $\mathbf{u}_x := \mathbf{R}\mathbf{v}_k + \mathbf{z}' + \mathbf{e}^* \in R_q^h$, for some \mathbf{e}^* random in R^h , and we set $\mathcal{Z}[t, x] := \mathbf{z}'$. Therefore, the only value that is under the control of the adversary is \mathbf{R} , and since the rest of the values are chosen uniformly randomly, we have that \mathbf{u}_x is distributed uniformly over R_q^h .

Concretely, we have that the adversarially controlled term of $\mathbf{R}\mathbf{v}_k$ will actually cancel out during the computation of $\lfloor \mathbf{u}_x - \mathbf{R}\mathbf{v}_k \rfloor_p$ (in the Finalize algorithm). Since \mathbf{z}' is sampled independently of \mathbf{e}^* and after \mathbf{R} is fixed by the adversary, the distribution of each coefficient of $\mathbf{z}' + \mathbf{e}^*$ will be uniform mod q . Then, the chance that the adversary can hit the correct value for a coefficient after rounding is at most $1/p + 1/q$ (the $1/q$ terms appears when q is not a multiple of p). Since \mathcal{A} is a PPT algorithm, it can only make polynomially many queries to RO_z oracle. Letting Q_z denote the total number of queries that \mathcal{A} makes to RO_z oracle, we have that

$$\Pr[\text{BAD}_2] \leq Q_z \cdot \left(\frac{1}{p} + \frac{1}{q} \right)^{dh} =: \epsilon_{\text{BAD}}.$$

Hence, it follows that

$$|\text{Adv}_{\mathcal{A},5}(\lambda) - \text{Adv}_{\mathcal{A},6}(\lambda)| \leq \epsilon_{\text{BAD}},$$

and in particular **Hybrid₅** \approx **Hybrid₆**.

| F.Setup(1^λ) | F.KeyGen(pp) | F.Eval(sk, t, x) |
|---|--|--|
| 1 : $(\text{ck}_1^*, \text{td}) \leftarrow \text{COM.E}_1(1^\lambda)$ 2 : $\text{ck}_2 \leftarrow \text{COM.Setup}(1^\lambda)$ 3 : $\text{crs}_1 \leftarrow \text{NIZK}_1.\text{Setup}(1^\lambda)$ 4 : $\text{crs}_2^*, \tau_2 \leftarrow \text{NIZK}_2.\mathcal{S}_1(1^\lambda)$ 5 : $\text{pp} := (\text{ck}_1^*, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 6 : return pp | 1 : parse pp := $(\text{ck}_1^*, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : $\mathbf{c}_k^* \leftarrow \text{COM.Commit}(\text{ck}_2, \mathbf{0}_m; \rho_k)$ 3 : return (pk := \mathbf{c}_k^* , sk := \perp) | 1 : if $\mathcal{Z}[t, x] = \perp$ then 2 : $\mathbf{z}' \xleftarrow{\$} R_q^h$ 3 : $\mathcal{Z}[t, x] := \mathbf{z}'$ 4 : $\mathbf{z} := \lfloor \mathcal{Z}[t, x] \rfloor_p$ 5 : $y := \text{RO}_z(t, x, \mathbf{z})$ 6 : return y |
| F.Request(pp, pk, t, x) | F.BlindEval(pp, pk, sk, t, req) | |
| 1 : parse pp := $(\text{ck}_1^*, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : $\mathbf{R} \xleftarrow{\$} \chi_r^{h \times (\ell+m)}$ 3 : $\mathbf{c}_r \leftarrow \text{COM.Commit}(\text{ck}_1^*, (\mathbf{R}, x); \rho_r)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 6 : $\mathbf{C}_x := \mathbf{R}\mathbf{A}_r + \mathbf{B}_x \in R_q^{h \times m}$ 7 : $\text{stmt}_1 := (\mathbf{c}_r, \mathbf{C}_x, \text{ck}_1, \mathbf{A}_r, \mathcal{G}, t)$ 8 : $\text{wit}_1 := (\mathbf{R}, x, \rho_r)$ 9 : $\pi_c \leftarrow \text{NIZK}_1.\text{P}(\text{crs}_1, \text{stmt}_1, \text{wit}_1)$ 10 : $\text{st} := (t, x, \mathbf{R}, \mathbf{C}_x, \text{pk}, \mathbf{A}_r)$ 11 : $\text{req} := (\mathbf{c}_r, \mathbf{C}_x, \pi_c)$ 12 : return (st, req) | 1 : parse pp := $(\text{ck}_1^*, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : parse pk := \mathbf{c}_k^* , sk := \perp 3 : parse req := $(\mathbf{c}_r, \mathbf{C}_x, \pi_c)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : if $\text{NIZK}_1.\text{V}(\text{crs}_1, \pi_c, \text{stmt}_1) = 1$ 6 : $\wedge (\mathbf{C}_x \neq \mathbf{R}\mathbf{A}_r + \mathcal{G}(x))$ 7 : $\vee \mathbf{c}_r \neq \text{COM.Commit}(\text{ck}_1, (\mathbf{R}, x); \rho_r)$ 8 : then abort 9 : $(\mathbf{u}_x, \mathbf{v}_k) := \mathcal{T}[t, \mathbf{c}_r]$ 10 : $\text{stmt}_2 := (\mathbf{u}_x, \mathbf{C}_x, \mathbf{c}_k^*, \mathbf{v}_k, \text{ck}_2, \mathbf{A}_r)$ 11 : $\pi_s^* \leftarrow \text{NIZK}_2.\mathcal{S}_2(\text{crs}_2^*, \tau_2, \text{stmt}_2)$ 12 : return rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s^*)$ | |
| F.Finalize(pp, rep, st) | $\text{RO}_r(t, \mathbf{c}_r)$ | |
| 1 : parse pp := $(\text{ck}_1^*, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : parse rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s^*)$ 3 : parse st := $(t, x, \mathbf{R}, \mathbf{C}_x, \text{pk} := \mathbf{c}_k^*, \mathbf{A}_r)$ 4 : if $\text{NIZK}_2.\text{V}(\text{crs}_2^*, \pi_s^*, \text{stmt}_2) \neq 1$ then 5 : return \perp 6 : $\mathbf{z} := \lfloor \mathbf{u}_x - \mathbf{R}\mathbf{v}_k \rfloor_p \in R_p^h \quad // = \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p$ 7 : $y := \text{RO}_z(t, x, \mathbf{z})$ 8 : return y | 1 : if $\mathcal{H}[t, \mathbf{c}_r] = \perp$ then 2 : $\mathbf{A}_r := \mathcal{H}[t, \mathbf{c}_r] \xleftarrow{\$} R_q^{(\ell+m) \times m}$ 3 : $(\mathbf{R}, x) \leftarrow \text{COM.E}_2(\text{ck}_1^*, \text{td}, \mathbf{c}_r)$ 4 : $\mathbf{v}_k \xleftarrow{\$} R_q^{\ell+m}$ 5 : if $\mathcal{Z}[t, x] = \perp$ then 6 : $\mathbf{z}' \xleftarrow{\$} R_q^h$ 7 : $\mathcal{Z}[t, x] := \mathbf{z}'$ 8 : $\tilde{\mathbf{e}}' \xleftarrow{\$} \chi_1^h, \tilde{\mathbf{e}} \xleftarrow{\$} \chi^{\ell+m}$ 9 : $\mathbf{e}^* := \tilde{\mathbf{e}}' - \mathbf{R}\tilde{\mathbf{e}} \in R^h$ 10 : $\mathbf{u}_x := \mathbf{R}\mathbf{v}_k + \mathcal{Z}[t, x] + \mathbf{e}^* \in R_q^h$ 11 : $\mathcal{T}[t, \mathbf{c}_r] := (\mathbf{u}_x, \mathbf{v}_k)$ 12 : return $\mathcal{H}[t, \mathbf{c}_r]$ | |
| $\text{RO}_z(t, x, \mathbf{z})$ | | |
| 1 : if $\mathcal{F}[t, x, \mathbf{z}] = \perp$ then 2 : $\mathcal{F}[t, x, \mathbf{z}] \xleftarrow{\$} \{0, 1\}^\lambda$ 3 : return $\mathcal{F}[t, x, \mathbf{z}]$ | | |

Fig. 13: **Hybrid₅** of pseudorandomness proof.

| F.Setup(1^λ) | F.KeyGen(pp) | F.Eval(sk, t, x) |
|---|--|--|
| 1 : $(\text{ck}_1^*, \text{td}) \leftarrow \text{COM.E}_1(1^\lambda)$ 2 : $\text{ck}_2 \leftarrow \text{COM.Setup}(1^\lambda)$ 3 : $\text{crs}_1 \leftarrow \text{NIZK}_1.\text{Setup}(1^\lambda)$ 4 : $\text{crs}_2^*, \tau_2 \leftarrow \text{NIZK}_2.\mathcal{S}_1(1^\lambda)$ 5 : $\text{pp} := (\text{ck}_1^*, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 6 : return pp | 1 : parse pp := $(\text{ck}_1^*, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : $\mathbf{c}_k^* \leftarrow \text{COM.Commit}(\text{ck}_2, \mathbf{0}_m; \rho_k)$ 3 : return (pk := \mathbf{c}_k^* , sk := \perp) | 1 : if $\mathcal{Y}[t, x] = \perp$ then 2 : $\mathcal{Y}[t, x] \xleftarrow{\$} \{0, 1\}^\lambda$ 3 : return $\mathcal{Y}[t, x]$ |
| F.Request(pp, pk, t, x) | F.BlindEval(pp, pk, sk, t, req) | |
| 1 : parse pp := $(\text{ck}_1^*, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : $\mathbf{R} \xleftarrow{\$} \chi_r^{h \times (\ell+m)}$ 3 : $\mathbf{c}_r \leftarrow \text{COM.Commit}(\text{ck}_1^*, (\mathbf{R}, x); \rho_r)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 6 : $\mathbf{C}_x := \mathbf{R}\mathbf{A}_r + \mathbf{B}_x \in R_q^{h \times m}$ 7 : $\text{stmt}_1 := (\mathbf{c}_r, \mathbf{C}_x, \text{ck}_1, \mathbf{A}_r, \mathcal{G}, t)$ 8 : $\text{wit}_1 := (\mathbf{R}, x, \rho_r)$ 9 : $\pi_c \leftarrow \text{NIZK}_1.\text{P}(\text{crs}_1, \text{stmt}_1, \text{wit}_1)$ 10 : st := $(t, x, \mathbf{R}, \mathbf{C}_x, \text{pk}, \mathbf{A}_r)$ 11 : req := $(\mathbf{c}_r, \mathbf{C}_x, \pi_c)$ 12 : return (st, req) | 1 : parse pp := $(\text{ck}_1^*, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : parse pk := \mathbf{c}_k^* , sk := \perp 3 : parse req := $(\mathbf{c}_r, \mathbf{C}_x, \pi_c)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : if $\text{NIZK}_1.\text{V}(\text{crs}_1, \pi_c, \text{stmt}_1) = 1$ 6 : $\wedge (\mathbf{C}_x \neq \mathbf{R}\mathbf{A}_r + \mathcal{G}(x))$ 7 : $\vee \mathbf{c}_r \neq \text{COM.Commit}(\text{ck}_1, (\mathbf{R}, x); \rho_r)$ 8 : then abort 9 : $(\mathbf{u}_x, \mathbf{v}_k) := \mathcal{T}[t, \mathbf{c}_r]$ 10 : $\text{stmt}_2 := (\mathbf{u}_x, \mathbf{C}_x, \mathbf{c}_k^*, \mathbf{v}_k, \text{ck}_2, \mathbf{A}_r)$ 11 : $\pi_s^* \leftarrow \text{NIZK}_2.\mathcal{S}_2(\text{crs}_2^*, \tau_2, \text{stmt}_2)$ 12 : return rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s^*)$ | |
| F.Finalize(pp, rep, st) | RO _r (t, c _r) | |
| 1 : parse pp := $(\text{ck}_1^*, \text{ck}_2, \text{crs}_1, \text{crs}_2^*)$ 2 : parse rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s^*)$ 3 : parse st := $(t, x, \mathbf{R}, \mathbf{C}_x, \text{pk} := \mathbf{c}_k^*, \mathbf{A}_r)$ 4 : if $\text{NIZK}_2.\text{V}(\text{crs}_2^*, \pi_s^*, \text{stmt}_2) \neq 1$ then 5 : return \perp 6 : $\mathbf{z} := \lfloor \mathbf{u}_x - \mathbf{R}\mathbf{v}_k \rfloor_p \in R_p^h \quad // = \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p$ 7 : $y := \text{RO}_z(t, x, \mathbf{z})$ 8 : return y | 1 : if $\mathcal{H}[t, \mathbf{c}_r] = \perp$ then 2 : $\mathbf{A}_r := \mathcal{H}[t, \mathbf{c}_r] \xleftarrow{\$} R_q^{(\ell+m) \times m}$ 3 : $(\mathbf{R}, x) \leftarrow \text{COM.E}_2(\text{ck}_1^*, \text{td}, \mathbf{c}_r)$ 4 : $\mathbf{v}_k \xleftarrow{\$} R_q^{\ell+m}$ 5 : if $\mathcal{Z}[t, x] = \perp$ then 6 : $\mathbf{z}' \xleftarrow{\$} R_q^h$ 7 : $\mathcal{Z}[t, x] := \mathbf{z}'$ 8 : $\tilde{\mathbf{e}}' \xleftarrow{\$} \chi_1^h, \tilde{\mathbf{e}} \xleftarrow{\$} \chi^{\ell+m}$ 9 : $\mathbf{e}^* := \tilde{\mathbf{e}}' - \mathbf{R}\tilde{\mathbf{e}} \in R^h$ 10 : $\mathbf{u}_x := \mathbf{R}\mathbf{v}_k + \mathcal{Z}[t, x] + \mathbf{e}^* \in R_q^h$ 11 : $\mathcal{T}[t, \mathbf{c}_r] := (\mathbf{u}_x, \mathbf{v}_k)$ 12 : return $\mathcal{H}[t, \mathbf{c}_r]$ | |
| RO _z (t, x, z) | | |
| 1 : if $\mathcal{F}[t, x, \mathbf{z}] = \perp$ then 2 : if $\mathcal{Z}[t, x] \neq \perp$ and $\mathbf{z} = \lfloor \mathcal{Z}[t, x] \rfloor_p$ then 3 : if $\mathcal{Y}[t, x] = \perp$ then 4 : $\mathcal{Y}[t, x] \xleftarrow{\$} \{0, 1\}^\lambda$ 5 : $\mathcal{F}[t, x, \mathbf{z}] := \mathcal{Y}[t, x]$ 6 : else 7 : $\mathcal{F}[t, x, \mathbf{z}] \xleftarrow{\$} \{0, 1\}^\lambda$ 8 : return $\mathcal{F}[t, x, \mathbf{z}]$ | | |

Fig. 14: **Hybrid₆** of pseudorandomness proof.

Finally, we note that the simulator \mathcal{S} answers the oracle calls as in **Hybrid**₆, and putting everything together, we obtain

$$\begin{aligned} \text{Adv}_{\mathcal{F}, \mathcal{A}, \mathcal{S}, \text{RO}_r, \text{RO}_z}^{\text{po-prf}}(\lambda) &\leq \text{Adv}_{\text{NIZK}_2, \mathcal{B}_1}^{\text{CZK}}(\lambda) + Q_{\mathcal{B}} \cdot \text{Adv}_{\text{NIZK}_1, \mathcal{B}_2}^{\text{CS}}(\lambda) + \text{Adv}_{\text{COM}, \mathcal{B}_3}^{\text{CH}}(\lambda) \\ &\quad + Q_{\text{RO}_r} \cdot \text{Adv}_{\text{COM}, \mathcal{B}_4}^{\text{Ext}}(\lambda) + \text{Adv}_{\mathcal{B}_5}^{\text{iMLWE}_{\text{param}}}(\lambda) + \epsilon_{\text{BAD}}, \end{aligned}$$

as claimed. This completes the proof of Theorem 2. \square

B.3 Request Privacy Analysis

Theorem 3. *The POPRF construction \mathcal{F} from Figure 6 satisfies the request privacy against malicious servers (POPRIV2), given in Definition 4, with random oracles RO_r and RO_z , if:*

- The client argument system NIZK_1 is computationally zero-knowledge and the server argument system NIZK_2 is computationally sound (Definition 18),
- The commitment scheme COM is computationally hiding and extractable (Definitions 15 and 17), and
- The $\text{knMLWE}_{\ell+m, m, h, \chi}$ assumption holds (Definition 7).

More precisely, for any PPT adversary \mathcal{A} , there exist PPT adversaries $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$ and \mathcal{B}_5 against the computational zero-knowledge of NIZK_1 , computational hiding of COM , computational soundness of NIZK_2 , extractability of COM and $\text{knMLWE}_{m, \ell+m, h, \chi}$ assumption, respectively, such that

$$\begin{aligned} \text{Adv}_{\mathcal{F}, \mathcal{A}, \text{RO}_r, \text{RO}_z}^{\text{po-priv-2}}(\lambda) &\leq \text{Adv}_{\text{NIZK}_1, \mathcal{B}_1}^{\text{CZK}}(\lambda) + 2Q_{\mathcal{R}} \cdot \text{Adv}_{\text{COM}, \mathcal{B}_2}^{\text{CH}}(\lambda) + 2Q_{\mathcal{F}} \cdot \text{Adv}_{\text{NIZK}_2, \mathcal{B}_3}^{\text{CS}}(\lambda) \\ &\quad + Q_{\mathcal{R}}^{\text{pk}} \cdot \text{Adv}_{\text{COM}, \mathcal{B}_4}^{\text{Ext}}(\lambda) + 2Q_{\mathcal{R}} \cdot \text{Adv}_{\mathcal{B}_5}^{\text{knMLWE}_{\ell+m, m, h, \chi}}(\lambda), \end{aligned}$$

where $Q_{\mathcal{R}}$ and $Q_{\mathcal{F}}$ denote the number of Request and Finalize oracle queries, respectively, and $Q_{\mathcal{R}}^{\text{pk}}$ denotes the number of queries to Request with different pk inputs that the adversary \mathcal{A} makes.

Proof. We consider the POPRF request privacy against malicious servers game given in Definition 4. We note that the adversary \mathcal{A} has access to oracles Request, Finalize and the random oracles RO_r and RO_z . The security proof follows in a series of hybrids, where we start with the request privacy against malicious servers game $\text{POPRIV2}_{\mathcal{F}, \mathcal{A}, \text{RO}_r, \text{RO}_z}^b$, for a random challenge bit $b \in \{0, 1\}$, and we gradually apply changes until we end up with a game where the transcript observed by \mathcal{A} is independent of the challenge bit b . Let $\text{Adv}_{\mathcal{A}, i}(\lambda)$ denote the advantage of \mathcal{A} in **Hybrid** _{i} and **Hybrid** _{i} \approx **Hybrid** _{$i+1$} denote

$$|\Pr[\mathbf{Hybrid}_i = 1] - \Pr[\mathbf{Hybrid}_{i+1} = 1]| \leq \text{negl}(\lambda).$$

Hybrid₀: This corresponds to the request privacy against malicious servers game $\text{POPRIV2}_{\mathcal{F}, \mathcal{A}, \text{RO}_r, \text{RO}_z}^b$ (Definition 4), for a random challenge bit $b \in \{0, 1\}$, and the interaction follows as in Figure 6. Hence, it follows that

$$\text{Adv}_{\mathcal{F}, \mathcal{A}, \text{RO}_r, \text{RO}_z}^{\text{po-priv-2}}(\lambda) = \text{Adv}_{\mathcal{A}, 0}(\lambda).$$

Hybrid₁: In this hybrid, we replace crs_1 and proof π_c with a simulated CRS and proof $(\text{crs}_1^*, \pi_c^*)$, as shown in Figure 15.

In order to show that **Hybrid**₀ \approx **Hybrid**₁, we construct a reduction \mathcal{B}_1 to the computational zero-knowledge property of NIZK_1 . We note that all queries are answered as in **Hybrid**₀ with the exception of the Request queries. The only difference here is that \mathcal{B}_1 prior to passing the public parameters pp to \mathcal{A} , it receives a CRS crs_1^* from its zero-knowledge challenger, and sets $\text{crs}_1 := \text{crs}_1^*$, instead of generating crs_1 via the $\text{NIZK}_1.\text{Setup}$ algorithm. When \mathcal{A} makes a Request query (pk, t, x_0, x_1) , \mathcal{B}_1 proceeds as in **Hybrid**₀ to compute $\mathbf{c}_{r, i}$ and \mathbf{C}_{x_i} , for $i \in \{0, 1\}$, but it makes two oracle calls to its zero-knowledge challenger with the inputs $((\mathbf{c}_{r, i}, \mathbf{C}_{x_i}, \mathbf{A}_{r, 1, i}, \mathcal{G}, t), (\mathbf{R}_i, x_i))$, for $i \in \{0, 1\}$, to obtain the proofs $(\pi_{c, 0}^*, \pi_{c, 1}^*)$. Finally, \mathcal{B}_1 responds to the Request query with $(\text{req}_b := (\mathbf{c}_{r, b}, \mathbf{C}_{x_b}, \pi_{c, b}^*), \text{req}_{1-b} := (\mathbf{c}_{r, 1-b}, \mathbf{C}_{x_{1-b}}, \pi_{c, 1-b}^*))$.

| | |
|--|--|
| F.Setup(1^λ) <hr/> 1 : $\forall i \in [2], \text{ck}_i \leftarrow \text{COM.Setup}(1^\lambda)$ 2 : $(\text{crs}_1^*, \tau) \leftarrow \text{NIZK}_1.\mathcal{S}_1(1^\lambda)$ 3 : $\text{crs}_2 \leftarrow \text{NIZK}_2.\text{Setup}(1^\lambda)$ 4 : $\text{pp} := (\text{ck}_1, \text{ck}_2, \text{crs}_1^*, \text{crs}_2)$ 5 : return pp | F.KeyGen(pp) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1^*, \text{crs}_2)$ 2 : $\mathbf{k} \xleftarrow{\$} \chi_k^m$ 3 : $\mathbf{c}_k \leftarrow \text{COM.Commit}(\text{ck}_2, \mathbf{k}; \rho_k)$ 4 : return $(\text{pk} := \mathbf{c}_k, \text{sk} := \mathbf{k})$ |
| F.Request(pp, pk, t, x) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1^*, \text{crs}_2)$ 2 : $\mathbf{R} \xleftarrow{\$} \chi_r^{h \times (\ell+m)}$ 3 : $\mathbf{c}_r \leftarrow \text{COM.Commit}(\text{ck}_1, (\mathbf{R}, x); \rho_r)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 6 : $\mathbf{C}_x := \mathbf{R}\mathbf{A}_r + \mathbf{B}_x \in R_q^{h \times m}$ 7 : $\text{stmt}_1 := (\mathbf{c}_r, \mathbf{C}_x, \text{ck}_1, \mathbf{A}_r, \mathcal{G}, t)$ 8 : $\pi_c^* \leftarrow \text{NIZK}_1.\mathcal{S}_2(\text{crs}_1^*, \tau, \text{stmt}_1)$ 9 : $\text{st} := (t, x, \mathbf{R}, \mathbf{C}_x, \text{pk}, \mathbf{A}_r)$ 10 : $\text{req} := (\mathbf{c}_r, \mathbf{C}_x, \pi_c^*)$ 11 : return (st, req) | F.BlindEval(pp, pk, sk, t, req) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1^*, \text{crs}_2)$ 2 : parse pk := $\mathbf{c}_k, \text{sk} := \mathbf{k}$ 3 : parse req := $(\mathbf{c}_r, \mathbf{C}_x, \pi_c^*)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r) \in R_q^{(\ell+m) \times m}$ 5 : if $\text{NIZK}_1.\text{V}(\text{crs}_1^*, \pi_c^*, \text{stmt}_1) \neq 1$ then 6 : return \perp 7 : $\mathbf{e}_s \xleftarrow{\$} \chi^{\ell+m}, \mathbf{e}'_s \xleftarrow{\$} \chi_1^h$ 8 : $\mathbf{v}_k := \mathbf{A}_r \mathbf{k} + \mathbf{e}_s \in R_q^{\ell+m}$ 9 : $\mathbf{u}_x := \mathbf{C}_x \mathbf{k} + \mathbf{e}'_s \in R_q^h$ 10 : $\text{stmt}_2 := (\mathbf{u}_x, \mathbf{C}_x, \mathbf{c}_k, \mathbf{v}_k, \text{ck}_2, \mathbf{A}_r)$ 11 : $\text{wit}_2 := (\mathbf{k}, \mathbf{e}, \mathbf{e}_s, \mathbf{e}'_s, \rho_k)$ 12 : $\pi_s \leftarrow \text{NIZK}_2.\text{P}(\text{crs}_2, \text{stmt}_2, \text{wit}_2)$ 13 : return $\text{rep} := (\mathbf{u}_x, \mathbf{v}_k, \pi_s)$ |
| F.Finalize(pp, rep, st) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1^*, \text{crs}_2)$ 2 : parse rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s)$ 3 : parse st := $(t, x, \mathbf{R}, \mathbf{C}_x, \text{pk} := \mathbf{c}_k, \mathbf{A}_r)$ 4 : if $\text{NIZK}_2.\text{V}(\text{crs}_2, \pi_s, \text{stmt}_2) \neq 1$ then 5 : return \perp 6 : $\mathbf{z} := [\mathbf{u}_x - \mathbf{R}\mathbf{v}_k]_p \in R_p^h \quad // = [\mathbf{B}_x \mathbf{k}]_p$ 7 : $\mathbf{y} := \text{RO}_z(t, x, \mathbf{z})$ 8 : return y | F.Eval(sk, t, x) <hr/> 1 : parse sk := \mathbf{k} 2 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 3 : $\mathbf{z} := [\mathbf{B}_x \mathbf{k}]_p \in R_p^h$ 4 : $\mathbf{y} := \text{RO}_z(t, x, \mathbf{z})$ 5 : return y |
| RO_r(t, c_r) <hr/> 1 : if $\mathcal{H}[t, \mathbf{c}_r] = \perp$ then 2 : $\mathcal{H}[t, \mathbf{c}_r] \xleftarrow{\$} R_q^{(\ell+m) \times m}$ 3 : return $\mathcal{H}[t, \mathbf{c}_r]$ | RO_z(t, x, z) <hr/> 1 : if $\mathcal{F}[t, x, \mathbf{z}] = \perp$ then 2 : $\mathcal{F}[t, x, \mathbf{z}] \xleftarrow{\$} \{0, 1\}^\lambda$ 3 : return $\mathcal{F}[t, x, \mathbf{z}]$ |

Fig. 15: **Hybrid₁** of request privacy proof.

If the zero-knowledge challenger of NIZK_1 used the honest setup and prover algorithms to generate crs_1^* and $(\pi_{c,0}^*, \pi_{c,1}^*)$, then we are exactly in **Hybrid₀**, and if it used the simulator, then we are in **Hybrid₁**. Therefore, if \mathcal{A} can distinguish between the two hybrids with non-negligible advantage, then \mathcal{B}_1 can break the zero-knowledge property of NIZK_1 . Hence, it follows that

$$|\text{Adv}_{\mathcal{A},0}(\lambda) - \text{Adv}_{\mathcal{A},1}(\lambda)| \leq \text{Adv}_{\text{NIZK}_1, \mathcal{B}_1}^{\text{CZK}}(\lambda),$$

and in particular **Hybrid₀** \approx **Hybrid₁**.

Hybrid₂: In this hybrid, we commit to $\mathbf{c}_r^* \leftarrow \text{COM.Commit}(\text{ck}_1, \mathbf{0}_{h \cdot (\ell+m)+1})$, where $\mathbf{0}_{h \cdot (\ell+m)+1}$ is an all zero vector of length $h \cdot (\ell+m) + 1$, instead of computing the commitment $\mathbf{c}_r \leftarrow \text{COM.Commit}(\text{ck}_1, (\mathbf{R}, x))$ as in the previous hybrid. This change is shown in Figure 16.

In order to show that **Hybrid₁** \approx **Hybrid₂**, we construct a reduction \mathcal{B}_2 to the computational hiding property of COM. We note that the public parameters pp and the responses to all queries are computed as in **Hybrid₁**, with the exception of the Request queries. Let Q_R denote the number of Request queries that the adversary \mathcal{A} makes. Upon receiving a Request query (pk, t, x_0, x_1) , \mathcal{B}_2 samples $\mathbf{R}_i \xleftarrow{\$} \chi_r^{h \times (\ell+m)}$, for $i \in \{0, 1\}$, sets $(m_{i,0} := (\mathbf{R}_i, x_i), m_{i,1} := \mathbf{0}_{h \cdot (\ell+m)+1})$, and submits the pairs $(m_{i,0}, m_{i,1})$ to the computational hiding challenger of COM, which responds with \mathbf{c}_i^* . At this point \mathcal{B}_2 sets $\mathbf{c}_{r,i} := \mathbf{c}_i^*$, and proceeds to compute $\mathbf{C}_{x,i}$ as in **Hybrid₁** and to simulate $\pi_{c,i}$, for $i \in \{0, 1\}$. Finally, \mathcal{B}_2 responds to the Request query with $(\text{req}_b := (\mathbf{c}_{r,b}, \mathbf{C}_{x_b}, \pi_{c,b}), \text{req}_{1-b} := (\mathbf{c}_{r,1-b}, \mathbf{C}_{x_{1-b}}, \pi_{c,1-b}))$.

If the computational hiding challenger of COM responds with a commitment to $m_{i,0}$, then we are exactly in **Hybrid₁**, and if it responds with a commitment to $m_{i,1}$, then we are in **Hybrid₂**. Therefore, if \mathcal{A} can distinguish between the two hybrids with non-negligible advantage, then \mathcal{B}_2 can break the computational hiding of COM. Since we consider here a multi-challenge variant of the computational hiding property (concretely, the 2-challenge variant) and the adversary \mathcal{A} makes at most Q_R queries to the Request oracle, it follows by a standard argument that

$$|\text{Adv}_{\mathcal{A},1}(\lambda) - \text{Adv}_{\mathcal{A},2}(\lambda)| \leq 2Q_R \cdot \text{Adv}_{\text{COM}, \mathcal{B}_2}^{\text{CH}}(\lambda),$$

and in particular **Hybrid₁** \approx **Hybrid₂**.

Hybrid₃: Let BAD be the event that for a Finalize query $(j, \text{rep}_0 := (\mathbf{u}_{x,0}, \mathbf{v}_{k,0}, \pi_{s,0}), \text{rep}_1 := (\mathbf{u}_{x,1}, \mathbf{v}_{k,1}, \pi_{s,1}))$, it holds that either

- $\text{NIZK}_2.V(\text{crs}_2, \pi_{s,0}, (\mathbf{u}_{x,0}, \mathbf{C}_{x,j,b}, \mathbf{c}_k, \mathbf{v}_{k,0}, \text{ck}_2, \mathbf{A}_{r,j,b})) = 1$, but $(\mathbf{c}_k \neq \text{COM.Commit}(\text{ck}_2, \mathbf{k}; \rho_k) \vee \mathbf{v}_{k,0} \neq \mathbf{A}_{r,j,b} \mathbf{k} + \mathbf{e}_s \vee \mathbf{u}_{x,0} \neq \mathbf{C}_{x,j,b} \mathbf{k} + \mathbf{e}'_s)$; or
- $\text{NIZK}_2.V(\text{crs}_2, \pi_{s,1}, (\mathbf{u}_{x,1}, \mathbf{C}_{x,j,1-b}, \mathbf{c}_k, \mathbf{v}_{k,1}, \text{ck}_2, \mathbf{A}_{r,j,1-b})) = 1$, but $(\mathbf{c}_k \neq \text{COM.Commit}(\text{ck}_2, \mathbf{k}; \rho_k) \vee \mathbf{v}_{k,1} \neq \mathbf{A}_{r,j,1-b} \mathbf{k} + \mathbf{e}_s \vee \mathbf{u}_{x,1} \neq \mathbf{C}_{x,j,1-b} \mathbf{k} + \mathbf{e}'_s)$,

where $\mathbf{C}_{x,j}$ and $\mathbf{A}_{r,j}$ are values obtained from the j -th query to Request oracle. If BAD happens, then the challenger aborts, as depicted in Figure 17.

Clearly, we have that **Hybrid₂** and **Hybrid₃** are identical until the event BAD happens. Hence, we show that we can bound the probability $\Pr[\text{BAD}]$ by constructing a reduction \mathcal{B}_3 to the computational soundness of NIZK_2 argument system. We note that all queries are answered as in **Hybrid₂**. Let Q_F denote the number of Finalize queries that \mathcal{A} performs. \mathcal{B}_3 randomly chooses a $q^* \in [2Q_F]$ to serve as its guess for the the proof π_s that will be forged by the adversary \mathcal{A} (note that each Finalize query includes two proofs, hence the guess from $[2Q_F]$). Let $(j, \text{rep}_0 := (\mathbf{u}_{x,0}, \mathbf{v}_{k,0}, \pi_{s,0}), \text{rep}_1 := (\mathbf{u}_{x,1}, \mathbf{v}_{k,1}, \pi_{s,1}))$ be the j -th query to Finalize, where $j = \lceil q^*/2 \rceil$, then \mathcal{B}_3 outputs $\pi_{s,0}$ if q^* is odd and outputs $\pi_{s,1}$ otherwise. We note that \mathcal{B}_3 avoids recovering $\mathbf{k}, \mathbf{e}, \mathbf{e}_s$ or \mathbf{e}'_s (which are computationally infeasible to compute), and instead just guesses which of the proofs would have caused the event BAD to happen. Hence, we have that

$$|\text{Adv}_{\mathcal{A},2}(\lambda) - \text{Adv}_{\mathcal{A},3}(\lambda)| \leq 2Q_F \cdot \text{Adv}_{\text{NIZK}_2, \mathcal{B}_3}^{\text{CS}}(\lambda),$$

| | |
|--|---|
| F.Setup(1^λ) <hr/> 1 : $\forall i \in [2], \text{ck}_i \leftarrow \text{COM.Setup}(1^\lambda)$ 2 : $(\text{crs}_1^*, \tau) \leftarrow \text{NIZK}_1.\mathcal{S}_1(1^\lambda)$ 3 : $\text{crs}_2 \leftarrow \text{NIZK}_2.\text{Setup}(1^\lambda)$ 4 : $\text{pp} := (\text{ck}_1, \text{ck}_2, \text{crs}_1^*, \text{crs}_2)$ 5 : return pp | F.KeyGen(pp) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1^*, \text{crs}_2)$ 2 : $\mathbf{k} \xleftarrow{\$} \chi_k^m$ 3 : $\mathbf{c}_k \leftarrow \text{COM.Commit}(\text{ck}_2, \mathbf{k}; \rho_k)$ 4 : return $(\text{pk} := \mathbf{c}_k, \text{sk} := \mathbf{k})$ |
| F.Request(pp, pk, t, x) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1^*, \text{crs}_2)$ 2 : $\mathbf{R} \xleftarrow{\$} \chi_r^{h \times (\ell+m)}$ 3 : $\mathbf{c}_r^* \leftarrow \text{COM.Commit}(\text{ck}_1, \mathbf{0}_{h \cdot (\ell+m)+1}; \rho_r)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r^*) \in R_q^{(\ell+m) \times m}$ 5 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 6 : $\mathbf{C}_x := \mathbf{R}\mathbf{A}_r + \mathbf{B}_x \in R_q^{h \times m}$ 7 : $\text{stmt}_1 := (\mathbf{c}_r^*, \mathbf{C}_x, \text{ck}_1, \mathbf{A}_r, \mathcal{G}, t)$ 8 : $\pi_c^* \leftarrow \text{NIZK}_1.\mathcal{S}_2(\text{crs}_1^*, \tau, \text{stmt}_1)$ 9 : $\text{st} := (t, x, \mathbf{R}, \mathbf{C}_x, \text{pk}, \mathbf{A}_r)$ 10 : $\text{req} := (\mathbf{c}_r^*, \mathbf{C}_x, \pi_c^*)$ 11 : return (st, req) | F.BlindEval(pp, pk, sk, t, req) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1^*, \text{crs}_2)$ 2 : parse pk := $\mathbf{c}_k, \text{sk} := \mathbf{k}$ 3 : parse req := $(\mathbf{c}_r^*, \mathbf{C}_x, \pi_c^*)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r^*) \in R_q^{(\ell+m) \times m}$ 5 : if $\text{NIZK}_1.\mathcal{V}(\text{crs}_1^*, \pi_c^*, \text{stmt}_1) \neq 1$ then 6 : return \perp 7 : $\mathbf{e}_s \xleftarrow{\$} \chi^{\ell+m}, \mathbf{e}'_s \xleftarrow{\$} \chi_1^h$ 8 : $\mathbf{v}_k := \mathbf{A}_r \mathbf{k} + \mathbf{e}_s \in R_q^{\ell+m}$ 9 : $\mathbf{u}_x := \mathbf{C}_x \mathbf{k} + \mathbf{e}'_s \in R_q^h$ 10 : $\text{stmt}_2 := (\mathbf{u}_x, \mathbf{C}_x, \mathbf{c}_k, \mathbf{v}_k, \text{ck}_2, \mathbf{A}_r)$ 11 : $\text{wit}_2 := (\mathbf{k}, \mathbf{e}, \mathbf{e}_s, \mathbf{e}'_s, \rho_k)$ 12 : $\pi_s \leftarrow \text{NIZK}_2.\mathcal{P}(\text{crs}_2, \text{stmt}_2, \text{wit}_2)$ 13 : return rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s)$ |
| F.Finalize(pp, rep, st) <hr/> 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1^*, \text{crs}_2)$ 2 : parse rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s)$ 3 : parse st := $(t, x, \mathbf{R}, \mathbf{C}_x, \text{pk} := \mathbf{c}_k, \mathbf{A}_r)$ 4 : if $\text{NIZK}_2.\mathcal{V}(\text{crs}_2, \pi_s, \text{stmt}_2) \neq 1$ then 5 : return \perp 6 : $\mathbf{z} := \lfloor \mathbf{u}_x - \mathbf{R}\mathbf{v}_k \rfloor_p \in R_p^h \quad // = \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p$ 7 : $y := \text{RO}_z(t, x, \mathbf{z})$ 8 : return y | F.Eval(sk, t, x) <hr/> 1 : parse sk := \mathbf{k} 2 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 3 : $\mathbf{z} := \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p \in R_p^h$ 4 : $y := \text{RO}_z(t, x, \mathbf{z})$ 5 : return y |
| RO_r(t, c_r) <hr/> 1 : if $\mathcal{H}[t, \mathbf{c}_r] = \perp$ then 2 : $\mathcal{H}[t, \mathbf{c}_r] \xleftarrow{\$} R_q^{(\ell+m) \times m}$ 3 : return $\mathcal{H}[t, \mathbf{c}_r]$ | RO_z(t, x, z) <hr/> 1 : if $\mathcal{F}[t, x, \mathbf{z}] = \perp$ then 2 : $\mathcal{F}[t, x, \mathbf{z}] \xleftarrow{\$} \{0, 1\}^\lambda$ 3 : return $\mathcal{F}[t, x, \mathbf{z}]$ |

Fig. 16: **Hybrid₂** of request privacy proof.

| | |
|--|---|
| F.Setup(1^λ) 1 : $\forall i \in [2], \text{ck}_i \leftarrow \text{COM.Setup}(1^\lambda)$ 2 : $(\text{crs}_1^*, \tau) \leftarrow \text{NIZK}_1.\mathcal{S}_1(1^\lambda)$ 3 : $\text{crs}_2 \leftarrow \text{NIZK}_2.\text{Setup}(1^\lambda)$ 4 : $\text{pp} := (\text{ck}_1, \text{ck}_2, \text{crs}_1^*, \text{crs}_2)$ 5 : return pp | F.KeyGen(pp) 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1^*, \text{crs}_2)$ 2 : $\mathbf{k} \xleftarrow{\$} \chi_k^m$ 3 : $\mathbf{c}_k \leftarrow \text{COM.Commit}(\text{ck}_2, \mathbf{k}; \rho_k)$ 4 : return $(\text{pk} := \mathbf{c}_k, \text{sk} := \mathbf{k})$ |
| F.Request(pp, pk, t, x) 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1^*, \text{crs}_2)$ 2 : $\mathbf{R} \xleftarrow{\$} \chi_r^{h \times (\ell+m)}$ 3 : $\mathbf{c}_r^* \leftarrow \text{COM.Commit}(\text{ck}_1, \mathbf{0}_{h \cdot (\ell+m)+1}; \rho_r)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r^*) \in R_q^{(\ell+m) \times m}$ 5 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 6 : $\mathbf{C}_x := \mathbf{R}\mathbf{A}_r + \mathbf{B}_x \in R_q^{h \times m}$ 7 : $\text{stmt}_1 := (\mathbf{c}_r^*, \mathbf{C}_x, \text{ck}_1, \mathbf{A}_r, \mathcal{G}, t)$ 8 : $\pi_c^* \leftarrow \text{NIZK}_1.\mathcal{S}_2(\text{crs}_1^*, \tau, \text{stmt}_1)$ 9 : $\text{st} := (t, x, \mathbf{R}, \mathbf{C}_x, \text{pk}, \mathbf{A}_r)$ 10 : $\text{req} := (\mathbf{c}_r^*, \mathbf{C}_x, \pi_c^*)$ 11 : return (st, req) | F.BlindEval(pp, pk, sk, t, req) 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1^*, \text{crs}_2)$ 2 : parse pk := $\mathbf{c}_k, \text{sk} := \mathbf{k}$ 3 : parse req := $(\mathbf{c}_r^*, \mathbf{C}_x, \pi_c^*)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r^*) \in R_q^{(\ell+m) \times m}$ 5 : if $\text{NIZK}_1.\mathcal{V}(\text{crs}_1^*, \pi_c^*, \text{stmt}_1) \neq 1$ then 6 : return \perp 7 : $\mathbf{e}_s \xleftarrow{\$} \chi^{\ell+m}, \mathbf{e}'_s \xleftarrow{\$} \chi_1^h$ 8 : $\mathbf{v}_k := \mathbf{A}_r \mathbf{k} + \mathbf{e}_s \in R_q^{\ell+m}$ 9 : $\mathbf{u}_x := \mathbf{C}_x \mathbf{k} + \mathbf{e}'_s \in R_q^h$ 10 : $\text{stmt}_2 := (\mathbf{u}_x, \mathbf{C}_x, \mathbf{c}_k, \mathbf{v}_k, \text{ck}_2, \mathbf{A}_r)$ 11 : $\text{wit}_2 := (\mathbf{k}, \mathbf{e}, \mathbf{e}_s, \mathbf{e}'_s, \rho_k)$ 12 : $\pi_s \leftarrow \text{NIZK}_2.\mathcal{P}(\text{crs}_2, \text{stmt}_2, \text{wit}_2)$ 13 : return rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s)$ |
| F.Finalize(pp, rep, st) 1 : parse pp := $(\text{ck}_1, \text{ck}_2, \text{crs}_1^*, \text{crs}_2)$ 2 : parse rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s)$ 3 : parse st := $(t, x, \mathbf{R}, \mathbf{C}_x, \text{pk} := \mathbf{c}_k, \mathbf{A}_r)$ 4 : if $\text{NIZK}_2.\mathcal{V}(\text{crs}_2, \pi_s, \text{stmt}_2) = 1$ 5 : $\wedge (\mathbf{c}_k \neq \text{COM.Commit}(\text{ck}_2, \mathbf{k}; \rho_k))$ 6 : $\vee \mathbf{v}_k \neq \mathbf{A}_r \mathbf{k} + \mathbf{e}_s$ 7 : $\vee \mathbf{u}_x \neq \mathbf{C}_x \mathbf{k} + \mathbf{e}'_s$) then abort 8 : $\mathbf{z} := \lfloor \mathbf{u}_x - \mathbf{R}\mathbf{v}_k \rfloor_p \in R_p^h \quad // = \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p$ 9 : $y := \text{RO}_z(t, x, \mathbf{z})$ 10 : return y | F.Eval(sk, t, x) 1 : parse sk := \mathbf{k} 2 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 3 : $\mathbf{z} := \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p \in R_p^h$ 4 : $y := \text{RO}_z(t, x, \mathbf{z})$ 5 : return y |
| RO_r(t, c_r) 1 : if $\mathcal{H}[t, \mathbf{c}_r] = \perp$ then 2 : $\mathcal{H}[t, \mathbf{c}_r] \xleftarrow{\$} R_q^{(\ell+m) \times m}$ 3 : return $\mathcal{H}[t, \mathbf{c}_r]$ | RO_z(t, x, z) 1 : if $\mathcal{F}[t, x, \mathbf{z}] = \perp$ then 2 : $\mathcal{F}[t, x, \mathbf{z}] \xleftarrow{\$} \{0, 1\}^\lambda$ 3 : return $\mathcal{F}[t, x, \mathbf{z}]$ |

Fig. 17: **Hybrid₃** of request privacy proof.

and in particular $\mathbf{Hybrid}_2 \approx \mathbf{Hybrid}_3$.

Hybrid₄: In this hybrid, we replace the commitment key \mathbf{ck}_2 with a trapdoor variant $(\mathbf{ck}_2^*, \mathbf{td}) \leftarrow \text{COM}.\mathcal{E}_1(1^\lambda)$, and use the trapdoor \mathbf{td} to extract the secret key $\mathbf{sk} := \mathbf{k}$ from the public key $\mathbf{pk} := \mathbf{c}_k$, i.e., run $\mathbf{k} \leftarrow \text{COM}.\mathcal{E}_2(\mathbf{ck}_2^*, \mathbf{td}, \mathbf{c}_k)$, and store \mathbf{k} for later use. These changes are depicted in Figure 18.

In order to prove that $\mathbf{Hybrid}_3 \approx \mathbf{Hybrid}_4$ we construct an adversary \mathcal{B}_4 to the extractability property of COM. We note that all queries are answered as in **Hybrid₃**, and the only change here is that \mathcal{B}_4 samples a trapdoor commitment key $(\mathbf{ck}_2^*, \mathbf{td}) \leftarrow \text{COM}.\mathcal{E}_1(1^\lambda)$, instead of generating \mathbf{ck}_2 via the $\text{COM}.\text{Setup}$ algorithm. Let $Q_R^{\mathbf{pk}}$ denote the number of Request queries that the adversary \mathcal{A} makes with different \mathbf{pk} inputs. For each such query $(\mathbf{pk}, t, x_0, x_1)$, upon \mathcal{B}_4 receiving a Finalize query $(j, \text{rep}_0, \text{rep}_1)$, where j corresponds to one of the $Q_R^{\mathbf{pk}}$ queries made to the Request oracle, \mathcal{B}_4 computes $\mathbf{sk} := \mathbf{k} \leftarrow \text{COM}.\mathcal{E}_2(\mathbf{ck}_2^*, \mathbf{td}, \mathbf{pk} := \mathbf{c}_k)$, and stores it in the table $\mathcal{K}[\mathbf{pk}] := \mathbf{k}$ (for later use). The rest of the interaction is identical to **Hybrid₃**.

Due to the soundness of the server proof π_s we know that the adversarial inputs, and especially the public keys $\mathbf{pk} := \mathbf{c}_k$, are well-formed. Hence, if \mathcal{A} can distinguish between the two hybrids with non-negligible advantage, then \mathcal{B}_4 can break the extractability property of COM. Moreover, since the adversary \mathcal{A} makes at most $Q_R^{\mathbf{pk}}$ queries to Request oracle, it follows that

$$|\text{Adv}_{\mathcal{A},3}(\lambda) - \text{Adv}_{\mathcal{A},4}(\lambda)| \leq Q_R^{\mathbf{pk}} \cdot \text{Adv}_{\text{COM},\mathcal{B}_4}^{\text{Ext}}(\lambda),$$

and in particular $\mathbf{Hybrid}_3 \approx \mathbf{Hybrid}_4$.

Hybrid₅: In this hybrid, we replace $\mathbf{C}_x := \mathbf{R}\mathbf{A}_r + \mathbf{B}_x \in R_q^{h \times m}$ with $\mathbf{C}_x^* := \mathbf{U} + \mathbf{B}_x \in R_q^{h \times m}$ for a uniformly random matrix $\mathbf{U} \xleftarrow{\$} R_q^{h \times m}$, as shown in Figure 19.

In order to prove that $\mathbf{Hybrid}_4 \approx \mathbf{Hybrid}_5$ we construct an adversary \mathcal{B}_5 against the $\text{knMLWE}_{\ell+m,m,h,\chi}$ assumption. We note that the public parameters \mathbf{pp} and responses to all queries are computed as in **Hybrid₄** with the exception of the Request queries. For each Request query $(\mathbf{pk}, t, x_0, x_1)$, \mathcal{B}_5 receives from the $\text{knMLWE}_{\ell+m,m,h,\chi}$ challenger the pairs $(\mathbf{A}_i, \mathbf{U}_i)$, which it uses to compute $\mathbf{C}_{x_i}^* := \mathbf{U}_i + \mathcal{G}(t, x_i)$, for $i \in \{0, 1\}$. Note that $\mathbf{C}_{x_i}^*$ forms part of the request that \mathcal{B}_5 returns to \mathcal{A} as output of the Request query. Moreover, \mathcal{B}_5 programs the random oracle RO_r , such that on input the pair $(t, \mathbf{c}_{r,i}^*)$ it returns \mathbf{A}_i provided by the $\text{knMLWE}_{\ell+m,m,h,\chi}$ challenger (we note that this change is not highlighted in Figure 18, because the \mathbf{A}_i values returned by the $\text{knMLWE}_{\ell+m,m,h,\chi}$ challenger are also uniformly random values from $R_q^{(\ell+m) \times m}$). Upon receiving a Finalize query $(j, \text{rep}_0, \text{rep}_1)$ from \mathcal{A} , \mathcal{B}_5 proceeds as in the previous hybrid, to obtain $\mathbf{pk} := \mathbf{c}_k$ from the state st_j , and then extract $\mathbf{sk} := \mathbf{k}$ from \mathbf{pk} using the commitment trapdoor. Additionally, \mathcal{B}_5 uses the secret key \mathbf{k} to compute the value $\mathbf{z}^* := \lfloor \mathbf{u}_x - \mathbf{U}\mathbf{k} \rfloor_p$, which constitutes an alternative way of computing the correct \mathbf{z}^* value with the help of the secret key.

If the $\text{knMLWE}_{\ell+m,m,h,\chi}$ challenger provided a knapsack MLWE instance, i.e., $\mathbf{U}_i := \mathbf{R}_i \mathbf{A}_i$, for some $\mathbf{R}_i \in R_q^{h \times (\ell+m)}$, then the view is as in **Hybrid₄**, and if it provided a uniformly random matrix $\mathbf{U}_i \in R_q^{h \times m}$, then the view is as in **Hybrid₅**. Therefore, if \mathcal{A} can distinguish between the two hybrids with non-negligible advantage, then \mathcal{B}_5 can break the $\text{knMLWE}_{\ell+m,m,h,\chi}$ assumption. Since we consider here a multi-challenge variant of the $\text{knMLWE}_{\ell+m,m,h,\chi}$ problem (concretely, the 2-challenge variant) and the adversary \mathcal{A} makes at most Q_R queries to the Request oracle, it follows by a standard argument that

$$|\text{Adv}_{\mathcal{A},4}(\lambda) - \text{Adv}_{\mathcal{A},5}(\lambda)| \leq 2Q_R \cdot \text{Adv}_{\mathcal{B}_5}^{\text{knMLWE}_{\ell+m,m,h,\chi}}(\lambda),$$

and in particular $\mathbf{Hybrid}_4 \approx \mathbf{Hybrid}_5$.

We note that at this point the transcript observed by the adversary \mathcal{A} is independent of the challenge bit b , and hence $\text{Adv}_{\mathcal{A},5}(\lambda) = 1/2$. Putting everything together, we obtain

$$\begin{aligned} \text{Adv}_{\mathcal{F},\mathcal{A},\text{RO}_r,\text{RO}_z}^{\text{po-priv-2}}(\lambda) &\leq \text{Adv}_{\text{NIZK}_1,\mathcal{B}_1}^{\text{CZK}}(\lambda) + 2Q_R \cdot \text{Adv}_{\text{COM},\mathcal{B}_2}^{\text{CH}}(\lambda) + 2Q_F \cdot \text{Adv}_{\text{NIZK}_2,\mathcal{B}_3}^{\text{CS}}(\lambda) \\ &\quad + Q_R^{\mathbf{pk}} \cdot \text{Adv}_{\text{COM},\mathcal{B}_4}^{\text{Ext}}(\lambda) + 2Q_R \cdot \text{Adv}_{\mathcal{B}_5}^{\text{knMLWE}_{\ell+m,m,h,\chi}}(\lambda), \end{aligned}$$

as claimed. This completes the proof of Theorem 3. \square

| F.Setup(1^λ) | F.KeyGen(pp) |
|--|---|
| <pre> 1 : $\text{ck}_1 \leftarrow \text{COM.Setup}(1^\lambda)$ 2 : $(\text{ck}_2^*, \text{td}) \leftarrow \text{COM.E}_1(1^\lambda)$ 3 : $(\text{crs}_1^*, \tau) \leftarrow \text{NIZK}_1.\mathcal{S}_1(1^\lambda)$ 4 : $\text{crs}_2 \leftarrow \text{NIZK}_2.\text{Setup}(1^\lambda)$ 5 : $\text{pp} := (\text{ck}_1, \text{ck}_2^*, \text{crs}_1^*, \text{crs}_2)$ 6 : return pp </pre> | <pre> 1 : parse pp := $(\text{ck}_1, \text{ck}_2^*, \text{crs}_1^*, \text{crs}_2)$ 2 : $\mathbf{k} \xleftarrow{\\$} \chi_k^m$ 3 : $\mathbf{c}_k \leftarrow \text{COM.Commit}(\text{ck}_2^*, \mathbf{k}; \rho_k)$ 4 : return (pk := \mathbf{c}_k, sk := \mathbf{k}) </pre> |
| F.Request(pp, pk, t, x) | F.BlindEval(pp, pk, sk, t, req) |
| <pre> 1 : parse pp := $(\text{ck}_1, \text{ck}_2^*, \text{crs}_1^*, \text{crs}_2)$ 2 : $\mathbf{R} \xleftarrow{\\$} \chi_r^{h \times (\ell+m)}$ 3 : $\mathbf{c}_r^* \leftarrow \text{COM.Commit}(\text{ck}_1, \mathbf{0}_{h \cdot (\ell+m)+1}; \rho_r)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r^*) \in R_q^{(\ell+m) \times m}$ 5 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 6 : $\mathbf{C}_x := \mathbf{R}\mathbf{A}_r + \mathbf{B}_x \in R_q^{h \times m}$ 7 : $\text{stmt}_1 := (\mathbf{c}_r^*, \mathbf{C}_x, \text{ck}_1, \mathbf{A}_r, \mathcal{G}, t)$ 8 : $\pi_c^* \leftarrow \text{NIZK}_1.\mathcal{S}_2(\text{crs}_1^*, \tau, \text{stmt}_1)$ 9 : $\text{st} := (t, x, \mathbf{R}, \mathbf{C}_x, \text{pk}, \mathbf{A}_r)$ 10 : $\text{req} := (\mathbf{c}_r^*, \mathbf{C}_x, \pi_c^*)$ 11 : return (st, req) </pre> | <pre> 1 : parse pp := $(\text{ck}_1, \text{ck}_2^*, \text{crs}_1^*, \text{crs}_2)$ 2 : parse pk := \mathbf{c}_k, sk := \mathbf{k} 3 : parse req := $(\mathbf{c}_r^*, \mathbf{C}_x, \pi_c^*)$ 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r^*) \in R_q^{(\ell+m) \times m}$ 5 : if $\text{NIZK}_1.\mathcal{V}(\text{crs}_1^*, \pi_c^*, \text{stmt}_1) \neq 1$ then 6 : return \perp 7 : $\mathbf{e}_s \xleftarrow{\\$} \chi^{\ell+m}, \mathbf{e}'_s \xleftarrow{\\$} \chi_1^h$ 8 : $\mathbf{v}_k := \mathbf{A}_r \mathbf{k} + \mathbf{e}_s \in R_q^{\ell+m}$ 9 : $\mathbf{u}_x := \mathbf{C}_x \mathbf{k} + \mathbf{e}'_s \in R_q^h$ 10 : $\text{stmt}_2 := (\mathbf{u}_x, \mathbf{C}_x, \mathbf{c}_k, \mathbf{v}_k, \text{ck}_2^*, \mathbf{A}_r)$ 11 : $\text{wit}_2 := (\mathbf{k}, \mathbf{e}, \mathbf{e}_s, \mathbf{e}'_s, \rho_k)$ 12 : $\pi_s \leftarrow \text{NIZK}_2.\mathcal{P}(\text{crs}_2, \text{stmt}_2, \text{wit}_2)$ 13 : return rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s)$ </pre> |
| F.Finalize(pp, rep, st) | F.Eval(sk, t, x) |
| <pre> 1 : parse pp := $(\text{ck}_1, \text{ck}_2^*, \text{crs}_1^*, \text{crs}_2)$ 2 : parse rep := $(\mathbf{u}_x, \mathbf{v}_k, \pi_s)$ 3 : parse st := $(t, x, \mathbf{R}, \mathbf{C}_x, \text{pk} := \mathbf{c}_k, \mathbf{A}_r)$ 4 : if $\text{NIZK}_2.\mathcal{V}(\text{crs}_2, \pi_s, \text{stmt}_2) = 1$ 5 : $\wedge (\mathbf{c}_k \neq \text{COM.Commit}(\text{ck}_2^*, \mathbf{k}; \rho_k))$ 6 : $\vee \mathbf{v}_k \neq \mathbf{A}_r \mathbf{k} + \mathbf{e}_s$ 7 : $\vee \mathbf{u}_x \neq \mathbf{C}_x \mathbf{k} + \mathbf{e}'_s$) then abort 8 : if $\mathcal{K}[\text{pk}] = \perp$ then 9 : $\mathbf{k} \leftarrow \text{COM.E}_2(\text{ck}_2^*, \text{td}, \mathbf{c}_k)$ 10 : $\mathcal{K}[\text{pk}] := \mathbf{k}$ 11 : $\mathbf{z} := \lfloor \mathbf{u}_x - \mathbf{R}\mathbf{v}_k \rfloor_p \in R_p^h \quad // = \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p$ 12 : $y := \text{RO}_z(t, x, \mathbf{z})$ 13 : return y </pre> | <pre> 1 : parse sk := \mathbf{k} 2 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ 3 : $\mathbf{z} := \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p \in R_p^h$ 4 : $y := \text{RO}_z(t, x, \mathbf{z})$ 5 : return y </pre> <hr/> <pre> $\text{RO}_r(t, \mathbf{c}_r)$ 1 : if $\mathcal{H}[t, \mathbf{c}_r] = \perp$ then 2 : $\mathcal{H}[t, \mathbf{c}_r] \xleftarrow{\\$} R_q^{(\ell+m) \times m}$ 3 : return $\mathcal{H}[t, \mathbf{c}_r]$ </pre> <hr/> <pre> $\text{RO}_z(t, x, \mathbf{z})$ 1 : if $\mathcal{F}[t, x, \mathbf{z}] = \perp$ then 2 : $\mathcal{F}[t, x, \mathbf{z}] \xleftarrow{\\$} \{0, 1\}^\lambda$ 3 : return $\mathcal{F}[t, x, \mathbf{z}]$ </pre> |

Fig. 18: **Hybrid₄** of request privacy proof.

| F.Setup(1^λ) | F.KeyGen(pp) |
|---|---|
| 1 : $\text{ck}_1 \leftarrow \text{COM.Setup}(1^\lambda)$ | 1 : parse pp := ($\text{ck}_1, \text{ck}_2^*, \text{crs}_1^*, \text{crs}_2$) |
| 2 : $(\text{ck}_2^*, \text{td}) \leftarrow \text{COM.E}_1(1^\lambda)$ | 2 : $\mathbf{k} \xleftarrow{\$} \chi_k^m$ |
| 3 : $(\text{crs}_1^*, \tau) \leftarrow \text{NIZK}_1.\mathcal{S}_1(1^\lambda)$ | 3 : $\mathbf{c}_k \leftarrow \text{COM.Commit}(\text{ck}_2^*, \mathbf{k}; \rho_k)$ |
| 4 : $\text{crs}_2 \leftarrow \text{NIZK}_2.\text{Setup}(1^\lambda)$ | 4 : return (pk := \mathbf{c}_k , sk := \mathbf{k}) |
| 5 : pp := ($\text{ck}_1, \text{ck}_2^*, \text{crs}_1^*, \text{crs}_2$) | |
| 6 : return pp | |
| F.Request(pp, pk, t, x) | F.BlindEval(pp, pk, sk, t, req) |
| 1 : parse pp := ($\text{ck}_1, \text{ck}_2^*, \text{crs}_1^*, \text{crs}_2$) | 1 : parse pp := ($\text{ck}_1, \text{ck}_2^*, \text{crs}_1^*, \text{crs}_2$) |
| 2 : $\mathbf{R} \xleftarrow{\$} \chi_r^{h \times (\ell+m)}$ | 2 : parse pk := \mathbf{c}_k , sk := \mathbf{k} |
| 3 : $\mathbf{c}_r^* \leftarrow \text{COM.Commit}(\text{ck}_1, \mathbf{0}_{h \cdot (\ell+m)+1}; \rho_r)$ | 3 : parse req := ($\mathbf{c}_r^*, \mathbf{C}_x^*, \pi_c^*$) |
| 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r^*) \in R_q^{(\ell+m) \times m}$ | 4 : $\mathbf{A}_r := \text{RO}_r(t, \mathbf{c}_r^*) \in R_q^{(\ell+m) \times m}$ |
| 5 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ | 5 : if $\text{NIZK}_1.\text{V}(\text{crs}_1^*, \pi_c^*, \text{stmt}_1) \neq 1$ then |
| 6 : $\mathbf{U} \xleftarrow{\$} R_q^{h \times m}$ | 6 : return \perp |
| 7 : $\mathbf{C}_x^* := \mathbf{U} + \mathbf{B}_x \in R_q^{h \times m}$ | 7 : $\mathbf{e}_s \xleftarrow{\$} \chi^{\ell+m}, \mathbf{e}'_s \xleftarrow{\$} \chi_1^h$ |
| 8 : $\text{stmt}_1 := (\mathbf{c}_r^*, \mathbf{C}_x^*, \mathbf{ck}_1, \mathbf{A}_r, \mathcal{G}, t)$ | 8 : $\mathbf{v}_k := \mathbf{A}_r \mathbf{k} + \mathbf{e}_s \in R_q^{\ell+m}$ |
| 9 : $\pi_c^* \leftarrow \text{NIZK}_1.\mathcal{S}_2(\text{crs}_1^*, \tau, \text{stmt}_1)$ | 9 : $\mathbf{u}_x := \mathbf{C}_x^* \mathbf{k} + \mathbf{e}'_s \in R_q^h$ |
| 10 : st := ($t, x, \mathbf{R}, \mathbf{C}_x^*, \text{pk}, \mathbf{A}_r$) | 10 : $\text{stmt}_2 := (\mathbf{u}_x, \mathbf{C}_x^*, \mathbf{c}_k, \mathbf{v}_k, \text{ck}_2^*, \mathbf{A}_r)$ |
| 11 : req := ($\mathbf{c}_r^*, \mathbf{C}_x^*, \pi_c^*$) | 11 : $\text{wit}_2 := (\mathbf{k}, \mathbf{e}, \mathbf{e}_s, \mathbf{e}'_s, \rho_k)$ |
| 12 : return (st, req) | 12 : $\pi_s \leftarrow \text{NIZK}_2.\text{P}(\text{crs}_2, \text{stmt}_2, \text{wit}_2)$ |
| | 13 : return rep := ($\mathbf{u}_x, \mathbf{v}_k, \pi_s$) |
| F.Finalize(pp, rep, st) | F.Eval(sk, t, x) |
| 1 : parse pp := ($\text{ck}_1, \text{ck}_2^*, \text{crs}_1^*, \text{crs}_2$) | 1 : parse sk := \mathbf{k} |
| 2 : parse rep := ($\mathbf{u}_x, \mathbf{v}_k, \pi_s$) | 2 : $\mathbf{B}_x := \mathcal{G}(t, x) \in R_q^{h \times m}$ |
| 3 : parse st := ($t, x, \mathbf{R}, \mathbf{C}_x^*, \text{pk} := \mathbf{c}_k, \mathbf{A}_r$) | 3 : $\mathbf{z} := \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p \in R_p^h$ |
| 4 : if $\text{NIZK}_2.\text{V}(\text{crs}_2, \pi_s, \text{stmt}_2) = 1$ | 4 : $y := \text{RO}_z(t, x, \mathbf{z})$ |
| 5 : $\wedge (\mathbf{c}_k \neq \text{COM.Commit}(\text{ck}_2^*, \mathbf{k}; \rho_k))$ | 5 : return y |
| 6 : $\vee \mathbf{v}_k \neq \mathbf{A}_r \mathbf{k} + \mathbf{e}_s$ | |
| 7 : $\vee \mathbf{u}_x \neq \mathbf{C}_x^* \mathbf{k} + \mathbf{e}'_s$) then abort | |
| 8 : if $\mathcal{K}[\text{pk}] = \perp$ then | |
| 9 : $\mathbf{k} \leftarrow \text{COM.E}_2(\text{ck}_2^*, \text{td}, \mathbf{c}_k)$ | |
| 10 : $\mathcal{K}[\text{pk}] := \mathbf{k}$ | |
| 11 : $\mathbf{z}^* := \lfloor \mathbf{u}_x - \mathbf{U}\mathcal{K}[\text{pk}] \rfloor_p \in R_p^h \quad // = \lfloor \mathbf{B}_x \mathbf{k} \rfloor_p$ | |
| 12 : $y := \text{RO}_z(x, \mathbf{z}^*)$ | |
| 13 : return y | |
| | RO_r(t, c_r) |
| | 1 : if $\mathcal{H}[t, \mathbf{c}_r] = \perp$ then |
| | 2 : $\mathcal{H}[t, \mathbf{c}_r] \xleftarrow{\$} R_q^{(\ell+m) \times m}$ |
| | 3 : return $\mathcal{H}[t, \mathbf{c}_r]$ |
| | RO_z(t, x, z) |
| | 1 : if $\mathcal{F}[t, x, \mathbf{z}] = \perp$ then |
| | 2 : $\mathcal{F}[t, x, \mathbf{z}] \xleftarrow{\$} \{0, 1\}^\lambda$ |
| | 3 : return $\mathcal{F}[t, x, \mathbf{z}]$ |

Fig. 19: **Hybrid₅** of request privacy proof.

C Issues in Prior Constructions and Models

C.1 Issues in [ADDS21] and [ADDG24]

We describe here two flaws present in the security proofs of the existing lattice-based OPRF constructions. The first flaw is from the [ADDS21] paper, which aims to prove their construction secure in the simulation based setting (we refer the reader to [ADDS21] for details regarding the security model). The flaw appears during the Query phase of the malicious client security proof (given in [ADDS21, Lemma 8]), where for each query (\mathbf{c}_x, π) from the malicious client, the simulator needs to extract the client's secret input x from the NIZK argument π in order to complete the reduction. However, the authors make use of a NIZK argument system that is based on Fiat-Shamir transform [YAZ⁺19], and hence, is not straight-line extractable, i.e., cannot extract the witness without rewinding. This means that each query of the malicious client incurs a loss in the soundness of the NIZK argument system, and since a malicious client can make $Q = \text{poly}(\lambda)$ distinct queries, it means we will end up with an exponential decay in soundness. We note that this issue can be fixed by using the Katsumata transform [Kat21] to obtain a lattice-based straight-line extractable NIZK argument system, albeit by incurring a performance penalty.

The second flaw appears in the recent [ADDG24] paper. Specifically, during the security proof of request privacy against malicious servers (POPRIV2) (given in [ADDG24, Theorem 4]), one of the reductions requires access to the server's secret key sk . The authors attempt to solve this dilemma by requiring the server to give its sk to the reduction algorithm, however, this is in stark contrast to the request privacy game given in Definition 4 (also used in [ADDG24]). Since their construction already includes NIZK proofs, one potential way to patch this issue is to instead extract the sk from a NIZK proof. However, similar to the previously explained issue, one needs to consider here a straight-line extractable proof system, which degrades the performance of the scheme. An alternative solution is to use an extractable commitment scheme and include a commitment to sk as part of the public key pk , which is exactly what we do in our construction given in Section 2.1, and during our request privacy security proof given in Appendix B.3.

C.2 Uniqueness and Key Binding

In [TCR⁺22], the authors defined a property called *uniqueness*, which ensures that POPRF outputs are unique even in the case of a malicious server.

Definition 20 (Uniqueness [TCR⁺22]). *We say that a partial oblivious PRF (POPRF) F is unique against malicious servers, if for all PPT adversaries \mathcal{A} the following advantage is $\text{negl}(\lambda)$,*

$$\text{Adv}_{F, \mathcal{A}, \text{RO}}^{\text{po-uniq}}(\lambda) = \Pr [\text{POUNIQ}_{F, \mathcal{A}, \text{RO}}(\lambda) = 1],$$

where the experiment POUNIQ is defined as follows:

| $\text{POUNIQ}_{F, \mathcal{A}, \text{RO}}(\lambda)$ | $\text{Request}(\text{sk}, \text{aux}, \text{pk}, t, x)$ |
|--|--|
| 1 : $R := [], q := 0$ | 1 : assert $F.\text{Wellformed}(\text{sk}, \text{pk}, \text{aux})$ |
| 2 : $\text{pp} \leftarrow F.\text{Setup}(1^\lambda)$ | 2 : $(\text{st}, \text{req}) \leftarrow F.\text{Request}_{\text{pp}}^{\text{RO}}(\text{pk}, t, x)$ |
| 3 : $\text{st}_{\text{RO}} \leftarrow \text{RO}.\text{Init}(1^\lambda)$ | 3 : $q := q + 1$ |
| 4 : $(i, j, \text{rep}_i, \text{rep}_j) \leftarrow \mathcal{A}^{\text{Request}, \text{RO}}(\text{pp})$ | 4 : $R[q] := (\text{st}, \text{pk}, t, x)$ |
| 5 : assert $1 \leq i \leq j \leq q$ | 5 : return req |
| 6 : $(\text{st}_i, \text{pk}_i, t_i, x_i) := R[i]$ | |
| 7 : $(\text{st}_j, \text{pk}_j, t_j, x_j) := R[j]$ | |
| 8 : $y_i \leftarrow F.\text{Finalize}_{\text{pp}}^{\text{RO}}(\text{rep}_i, \text{st}_i)$ | |
| 9 : $y_j \leftarrow F.\text{Finalize}_{\text{pp}}^{\text{RO}}(\text{rep}_j, \text{st}_j)$ | |
| 10 : if $y_i = \perp \vee y_j = \perp$ then | |
| 11 : return 0 | |
| 12 : return $((\text{pk}_i, t_i, x_i) = (\text{pk}_j, t_j, x_j)) \wedge (y_i \neq y_j)$ | |

The highlighted part is included to address rogue key attacks using the knowledge of secret key model [Bol03]. If the Wellformed predicate does not require any auxiliary information to verify the secret/public key pair (sk, pk) , then we consider that $\text{aux} := \perp$.

Tyagi et al. [TCR+22] showed that uniqueness follows from the correctness and request privacy against malicious servers (POPRIV2) of the POPRF. However, since correctness only holds against well-formed public keys and uniqueness definition considers a malicious adversary, they proved this in the knowledge of secret key (KOSK) model [Bol03]. In KOSK, the adversary reveals its secret key and this is used to check the well-formedness of the secret/public key pair (e.g., by using the Wellformed predicate in Definition 20). We note that such a model can be instantiated by including an extractable proof of knowledge or commitment, from which the secret key can be extracted and the proof may proceed as in the KOSK model.

Nevertheless, the reduction from uniqueness to request privacy that was given by Tyagi et al. [TCR+22, Appendix H], only works when there is a bijection between the secret and public keys, i.e., for each public key pk there is a unique sk . This comes from the fact that during the reduction they consider that the output of a valid uniqueness adversary would result in two tuples $(\text{pk}_i, \text{sk}_i, t_i, x_i) = (\text{pk}_j, \text{sk}_j, t_j, x_j)$. However, the uniqueness definition (see Definition 20) only guarantees that $(\text{pk}_i, t_i, x_i) = (\text{pk}_j, t_j, x_j)$, i.e., we do not have the guarantee that $\text{sk}_i = \text{sk}_j$. Although, this holds in group-based OPRF constructions, such as in [TCR+22], this does not necessarily hold in lattice-based constructions. For example, in [ADDS21] the public key is of the form $\mathbf{c} := \mathbf{a} \cdot k + \mathbf{e} \bmod q$, such that for a fixed (non-zero) \mathbf{a} , there exist numerous pairs of secret keys (k, \mathbf{e}) that satisfy the public key equation.

In order to provide a more robust uniqueness guarantees, we define here an additional property for a POPRF called *key binding*, which ensures that a public key is bound to a particular secret key.

Definition 21 (Key Binding). We say that a partial oblivious PRF (POPRF) F is key binding with respect to a Wellformed predicate, if for all PPT adversaries \mathcal{A} the following advantage is $\text{negl}(\lambda)$,

$$\text{Adv}_{F, \mathcal{A}, \text{RO}}^{\text{po-key-bind}}(\lambda) = \Pr[\text{POKBIND}_{F, \mathcal{A}, \text{RO}}(\lambda) = 1],$$

where the experiment POKBIND is defined as follows:

$\text{POKBIND}_{F, \mathcal{A}, \text{RO}}(\lambda)$

1 : $\text{pp} \leftarrow F.\text{Setup}(1^\lambda)$
2 : $\text{st}_{\text{RO}} \leftarrow \text{RO}.\text{Init}(1^\lambda)$
3 : $(\text{pk}, \text{sk}, \text{sk}', \text{aux}, \text{aux}') \leftarrow \mathcal{A}^{\text{RO}}(\text{pp})$
4 : **assert** $\text{sk} \neq \text{sk}'$
5 : **return** $F.\text{Wellformed}(\text{sk}, \text{pk}, \text{aux}) \wedge F.\text{Wellformed}(\text{sk}', \text{pk}, \text{aux}')$

Next, we show that our construction from Figure 6 satisfies key binding.

Theorem 4. If COM is computationally binding, then the OPRF construction F from Figure 6 is key binding.

Proof. First we define a Wellformed predicate for our key binding definition. In our construction the secret key is of the form $\text{sk} := \mathbf{k}$, for $\mathbf{k} \xleftarrow{\$} \chi_k^m$, and the public key is computed as $\text{pk} := \mathbf{c}_k \leftarrow \text{COM}.\text{Commit}(\text{ck}_2, \mathbf{k})$, for some commitment key $\text{ck}_2 \leftarrow \text{COM}.\text{Setup}(1^\lambda)$. Hence, our Wellformed predicate can be defined as the verification algorithm of the commitment scheme COM , i.e., $F.\text{Wellformed}(\text{sk}, \text{pk}, \text{aux}) = \text{COM}.\text{Verify}(\text{ck}_2, \mathbf{c}_k, \rho_k, \mathbf{k})$, where $\text{aux} := \rho_k$ is the decommitment information.

In order to show that our construction satisfies key binding we construct a reduction \mathcal{B} to the computational binding property of COM . \mathcal{B} receives the commitment key ck^* from the computational binding challenger of COM , sets $\text{ck}_2 := \text{ck}^*$, generates the rest of the public parameters pp as in Figure 6, and sends pp to \mathcal{A} . Upon receiving the tuple $(\text{pk} := \mathbf{c}_k, \text{sk} := \mathbf{k}, \text{sk}' := \mathbf{k}', \text{aux} := \rho_k, \text{aux}' := \rho'_k)$ from \mathcal{A} , \mathcal{B} forwards the tuple $(\mathbf{c}_k, \rho_k, \rho'_k, \mathbf{k}, \mathbf{k}')$ to the computational binding challenger of COM .

We note that the winning conditions for the key binding of F (Definition 21) and computational binding of COM (Definition 16) are identical in this case. Hence, if \mathcal{A} can win the key binding experiment with non-negligible advantage, then \mathcal{B} can win the computational binding experiment with non-negligible advantage. \square

Theorem 5. *For every PPT adversary \mathcal{A} against the uniqueness of F , there exist PPT adversaries \mathcal{B}_1 and \mathcal{B}_2 against the key binding and request privacy of F , respectively, such that*

$$\text{Adv}_{F,\mathcal{A},\text{RO}}^{\text{po-uniq}}(\lambda) \leq \text{Adv}_{F,\mathcal{B}_1,\text{RO}}^{\text{po-key-bind}}(\lambda) + \text{Adv}_{F,\mathcal{B}_2,\text{RO}}^{\text{po-priv-2}}(\lambda).$$

Proof. We note that the adversary \mathcal{A} has access to the oracles **Request** and **RO**. We consider a single intermediate hybrid before reducing the uniqueness to the request privacy against malicious servers (POPRIV2). Let $\text{Adv}_{\mathcal{A},i}(\lambda)$ denote the advantage of \mathcal{A} in **Hybrid** _{i} and let **Hybrid** _{i} \approx **Hybrid** _{$i+1$} denote $|\Pr[\mathbf{Hybrid}_i = 1] - \Pr[\mathbf{Hybrid}_{i+1} = 1]| \leq \text{negl}(\lambda)$.

Hybrid₀: This corresponds to the uniqueness experiment $\text{POUNIQ}_{F,\mathcal{A},\text{RO}}$ (Definition 20). Hence, we have that

$$\text{Adv}_{F,\mathcal{A},\text{RO}}^{\text{po-uniq}}(\lambda) = \text{Adv}_{\mathcal{A},0}(\lambda).$$

Hybrid₁: Let **BAD** be the event that the adversary queries the **Request** oracle with two queries $(\text{sk}, \text{aux}, \text{pk}, t, x)$ and $(\text{sk}', \text{aux}', \text{pk}', t', x')$, such that $\text{pk} = \text{pk}'$ and $\text{sk} \neq \text{sk}'$. If **BAD** happens, then the challenger aborts.

Clearly, we have that **Hybrid**₀ and **Hybrid**₁ are identical until the event **BAD** happens. Hence, we show that we can bound the probability $\Pr[\mathbf{BAD}]$ by constructing a reduction \mathcal{B}_1 to the key binding of F . \mathcal{B}_1 receives the public parameters pp from the key binding challenger, and forwards pp to \mathcal{A} . When \mathcal{A} makes a **RO** query, then \mathcal{B}_1 forwards the same query to its own **RO** oracle and relays the answer back to \mathcal{A} . For answering the **Request** oracle queries, \mathcal{B}_1 proceeds exactly as in **Hybrid**₀, but when \mathcal{A} submits two queries $(\text{sk}, \text{aux}, \text{pk}, t, x)$ and $(\text{sk}', \text{aux}', \text{pk}', t', x')$ that satisfy $\text{pk} = \text{pk}'$ and $\text{sk} \neq \text{sk}'$, then \mathcal{B} submits the tuple $(\text{pk}, \text{sk}, \text{sk}', \text{aux}, \text{aux}')$ to the key binding challenger. Hence, we have that \mathcal{B}_1 wins the key binding experiment with the same probability that the event **BAD** happens, and therefore, we have that

$$|\text{Adv}_{\mathcal{A},0}(\lambda) - \text{Adv}_{\mathcal{A},1}(\lambda)| \leq \text{Adv}_{F,\mathcal{B}_1,\text{RO}}^{\text{po-key-bind}}(\lambda),$$

and in particular **Hybrid**₀ \approx **Hybrid**₁.

All that remains is to bound the advantage of the adversary \mathcal{A} against the uniqueness experiment in **Hybrid**₁, i.e., bound $\text{Adv}_{\mathcal{A},1}(\lambda)$. We can do so by constructing a reduction \mathcal{B}_2 to the request privacy of F . Analogous to the previous reduction, \mathcal{B}_2 receives the public parameters pp from the request privacy challenger, and forwards pp to \mathcal{A} . Moreover, **RO** queries of \mathcal{A} are forwarded to the **RO** oracle of \mathcal{B}_2 and the response relayed back to \mathcal{A} . Upon receiving a **Request** query $(\text{sk}, \text{aux}, \text{pk}, t, x)$ from \mathcal{A} , \mathcal{B}_2 makes a **Request** query to its oracle (provided by the request privacy challenger) with the input (pk, t, x, x) , received back a pair of request messages $(\text{req}_0, \text{req}_1)$ and sends req_0 to \mathcal{A} . Moreover, \mathcal{B}_2 stores the tuple $(\text{pk}, \text{sk}, t, x, \text{req}_1)$ associated with the **Request** query.

When \mathcal{A} outputs the responses $(i, j, \text{rep}_{i,0}, \text{rep}_{j,0})$, then \mathcal{B}_2 retrieves the tuples $(\text{pk}_i, \text{sk}_i, t_i, x_i, \text{req}_{i,1})$ and $(\text{pk}_j, \text{sk}_j, t_j, x_j, \text{req}_{j,1})$ that it previously stored. We consider the case where \mathcal{A} wins the uniqueness experiment, and hence, in this case we can drop the subscripts as $(\text{pk}_i, \text{sk}_i, t_i, x_i) = (\text{pk}_j, \text{sk}_j, t_j, x_j)$ (note that $\text{sk}_i = \text{sk}_j$ is guaranteed by the key binding property of F that we previously proved). \mathcal{B}_2 calls its **Finalize** oracle with $(i, \text{rep}_{i,0}, \text{rep}_{i,1})$, where $\text{rep}_{i,0}$ is provided by \mathcal{A} and $\text{rep}_{i,1}$ is generated honestly by \mathcal{B}_2 . By the correctness of F , we have that the value returned from the honest flow with $\text{rep}_{i,1}$ is equal to $y := F.\text{Eval}(\text{sk}, t, x)$. If **Finalize** oracle returns two different values, then by matching which position y is returned reveals the challenge bit and allows \mathcal{B}_2 to win the request privacy experiment. On the other hand, if **Finalize** returns two identical values, then \mathcal{B}_2 repeats the above for the j -th query. If \mathcal{A} wins the uniqueness experiment, then

we know that Finalize will not return identical values for the j -th query, and hence, the challenge bit will be revealed, allowing \mathcal{B}_2 to win the request privacy experiment. Therefore, \mathcal{B}_2 outputs the correct challenge bit in the request privacy experiment, whenever \mathcal{A} wins uniqueness. Hence, we have that

$$\text{Adv}_{\mathcal{A},1}(\lambda) \leq \text{Adv}_{\mathcal{F},\mathcal{B}_2,\text{RO}}^{\text{po-priv-2}}(\lambda).$$

Putting everything together, we obtain

$$\text{Adv}_{\mathcal{F},\mathcal{A},\text{RO}}^{\text{po-uniq}}(\lambda) \leq \text{Adv}_{\mathcal{F},\mathcal{B}_1,\text{RO}}^{\text{po-key-bind}}(\lambda) + \text{Adv}_{\mathcal{F},\mathcal{B}_2,\text{RO}}^{\text{po-priv-2}}(\lambda),$$

as claimed. This completes the proof of Theorem 5. \square

D More on the Message Mapping and NIZKs

D.1 Further Options for Message Mapping \mathcal{G}

Using the mapping from BLMR13 PRF [BLMR13] for \mathcal{G} . An other option is the BLMR13 PRF [BLMR13] (particularly its variant over module lattices as described in [SSS23]). In this setting, we have two square matrices $\mathbf{A}_0, \mathbf{A}_1 \in R_q^{m \times m}$ (with binary entries) published as public parameters (where the dimension parameter $h = m$ to match the secret key dimension in OPRF). To compute the PRF for an input $x = (x_0, \dots, x_{L-1})$ in binary and $\mathbf{k} \xleftarrow{\$} R_q^m$, we compute

$$F_{\mathbf{k}}^{\text{blmr}}(x) = \left[\prod_{i=0}^{L-1} \mathbf{A}_{x_i} \cdot \mathbf{k} \right]_p. \quad (28)$$

As a result, the message mapping $\mathcal{G}^{\text{blmr}}$ is defined as

$$\mathcal{G}^{\text{blmr}} : x \mapsto \mathbf{B}_x = \prod_{i=0}^{L-1} \mathbf{A}_{x_i}. \quad (29)$$

For the client's NIZK proof π_c to be discussed in Section 5.3, we will need to treat each \mathbf{A}_{x_i} as a set of different variables. It is easy to observe that there are m^2L variable polynomials in that case. As discussed in [SSS23], we require $m \geq 2$ as otherwise this module variant of the PRF scheme is insecure.

Using a (symmetric) PRG for \mathcal{G} . Another option one may consider is to use a (symmetric) PRG to instantiate \mathcal{G} . In this case, we may simply set $h = 1$ to minimize a dimension of certain matrices/vectors involved in communication and NIZK proofs for improved performance. In this case, the SampPRF oracle in iMLWER-RU would output samples (computationally) indistinguishable from MLWR samples (based on the PRG pseudorandomness property). One may then employ a suitable zk-SNARK system to instantiate the client's proof π_c . We do not see this option as competitive (in terms of efficiency) as the prior ones where fully lattice-based tools can be employed for improved compatibility and performance. Therefore, we do not discuss this option in further detail. However, as further progress is made in the development of more efficient zk-SNARK solutions, this option (just like the others) may become more efficient over time.

Modelling \mathcal{G} as a random oracle. We also note the option of modeling \mathcal{G} as a random oracle. In this case, we may again simply set $h = 1$ to minimize a dimension of certain matrices/vectors involved in communication and NIZK proofs for improved performance. In this case, the SampPRF oracle in iMLWER-RU would simply output MLWR samples. Assuming \mathcal{G} to be a random oracle would also reduce the control of an iMLWER-RU adversary on how \mathbf{C}_x is computed.

There are two (related) caveats with this choice. Firstly, we would need a NIZK proof that proves pre-image knowledge of a hash function (modelled as a random oracle). There are generic zk-SNARK systems like [COS20] that can achieve this for standard hash function such as SHA3, but they are currently not as efficient as we would like them to be. Alternatively, we may use a more algebraic hash function like Poseidon

[GKR⁺21] to help reduce the cost of the NIZK well-formedness proof. Overall, the first caveat is that the associated message-mapping well-formedness proof is not going to be as efficient.

The second caveat is that we would assume \mathcal{G} to be a random oracle, but then, as we need to prove message-mapping well-formedness via a NIZK proof, we would also need to assume that \mathcal{G} has a compact circuit representation. This is the same issue that has already been encountered in earlier works such as the recursive proof in [COS20], the blind signature in [BLNS23] and the aggregate signature in [ZSE⁺24]. Similar to earlier works, we can (heuristically) assume that the construction remains secure when the random oracle is instantiated with a hash function and then instantiate the NIZK accordingly.

Overall, while it does not currently appear as a very competitive option in terms of efficiency, we believe that the main advantage of modelling \mathcal{G} as a random oracle may arise in helping with the security reductions and reducing an adversary’s winning chance in iMLWER-RU. We emphasize that modelling \mathcal{G} as a random oracle will *not* be our main choice in this paper, and we merely present it as a possible option.

D.2 NIZK for BLMR13 PRF mapping

Here, for a known \mathbf{A}_r and \mathbf{C}_x , the prover (client) wants to prove knowledge of short \mathbf{R} and a bit-string $x = (x_0, x_1, \dots, x_{L-1}) \in \{0, 1\}^L$ such that $\mathbf{C}_x = \mathbf{R}\mathbf{A}_r + \mathbf{B}_x \bmod q$ where \mathbf{B}_x is defined in (29). To be more specific, $\mathbf{B}_x = \prod_{i=0}^{L-1} \mathbf{A}_{x_i}$, where $\mathbf{A}_0, \mathbf{A}_1$ are two public matrices. Now, define variables $\mathbf{B}_i \in R_q^{m \times m}$ for $i = L-1, \dots, 0$ as $\mathbf{B}_i := \prod_{j=0}^i \mathbf{A}_{x_j}$. Then, the statement being proved can be expressed as follows.

$$\begin{aligned} \mathbf{B}_0 &= \mathbf{A}_{x_0}, \\ \mathbf{B}_{i+1} &= \mathbf{B}_i \cdot \mathbf{A}_{x_{i+1}}, \quad i = 0, \dots, L-2, \\ \mathbf{C}_x &= \mathbf{R}\mathbf{A}_r + \mathbf{B}_{L-1}. \end{aligned}$$

This system of equations above is not linear due to the x_i and \mathbf{B}_{i+1} terms. We can represent the terms $\mathbf{B}_i \cdot \mathbf{A}_{x_{i+1}}$ as $\mathbf{B}_i \cdot \mathbf{A}_0 \cdot (1 - x_{i+1}) + \mathbf{B}_i \cdot \mathbf{A}_1 \cdot x_{i+1}$, then we obtain

$$\begin{aligned} \mathbf{B}_0 &= \mathbf{A}_0 \cdot (1 - x_0) + \mathbf{A}_1 \cdot x_0, \\ \mathbf{B}_1 &= \mathbf{B}_0 \cdot \mathbf{A}_0 \cdot (1 - x_1) + \mathbf{B}_0 \cdot \mathbf{A}_1 \cdot x_1, \\ &\vdots \\ \mathbf{B}_{L-1} &= \mathbf{B}_{L-2} \cdot \mathbf{A}_0 \cdot (1 - x_{L-1}) + \mathbf{B}_{L-2} \cdot \mathbf{A}_1 \cdot x_{L-1}, \\ \mathbf{C}_x &= \mathbf{R}\mathbf{A}_r + \mathbf{B}_{L-1}. \end{aligned}$$

Now the required proof boils down to proving that a sequence of quadratic equations holds, which can again be easily handled by LaBRADOR [BS23]. Observe that the witness dimension (over R_q) is again in the order of m^2L . Note also that when a tag is present, we will have a similar situation as described above for the BP14 PRF, where some public matrix \mathbf{B}_t will appear in the first expression describing \mathbf{B}_0 .

E Instantiation of the Commitment Scheme

From our analyses in Section 4.2, we can observe that the main properties we need from the commitment scheme are hiding the message (without trapdoor), binding, and enabling extraction of the message using the trapdoor. This is effectively an encryption scheme, and we will employ a variant of Regev’s encryption [Reg05] over module lattices for our purposes. Given this Regev-style instantiation is quite well-known by now, we do not delve into too many technical details here and refer the reader to the references provided below for more details.

For certain instantiations of the message mapping \mathcal{G} (such as BMLR13 PRF [BLMR13]), we need the server’s secret key \mathbf{k} to have large coefficients (i.e., unbounded mod q). As a result, we need to be able

to allow the decryption of large message inputs in the encryption scheme. For this, we can employ the technique (implicit in [GSW13]) described more explicitly in [LNPS21]. Suppose we want to commit to an N -dimensional vector over R_q , and define $M := N$ if the committed message has small coefficients or $M := 2N$ if the committed message has large coefficients. For readability, we write \sqrt{q} to mean $\lceil \sqrt{q} \rceil$ below.

- **Setup** samples $\mathbf{A} \xleftarrow{\$} R_q^{n \times (n+M+w)}$. In the case we want to generate the commitment key with a trapdoor, then we set $\mathbf{b}_{i,j}^\top = \mathbf{s}_{i,j}^\top \mathbf{A} + \mathbf{e}_{i,j}^\top$ for $j = 0, 1$ and $i = 1, \dots, N$, where $\mathbf{s}_{i,j}$'s are MLWE secret keys and $\mathbf{e}_{i,j}$'s are short error vectors where each entry is sampled from an error distribution χ_e . If we don't want the trapdoor, then we simply sample $\mathbf{b}_{i,j}$'s uniformly at random. The commitment key is then set as $\text{ck} = (\mathbf{A}, \mathbf{b}_{1,0}, \mathbf{b}_{1,1}, \dots, \mathbf{b}_{N,0}, \mathbf{b}_{N,1})$.
- To **Commit** to a message $\mathbf{m} = (m_1, \dots, m_N) \in R^N$, we proceed as follows:
 1. If \mathbf{m} has *small* coefficients compared to q with $\|m_i\|_\infty \leq q_m$, then we first sample a short randomness vector $\mathbf{r} \leftarrow \chi_e^{n+N+w}$ and set $\mathbf{c} = \mathbf{A}\mathbf{r}$. Then, we compute $c_i = q_m \mathbf{b}_{i,0}^\top \mathbf{r} + m_i$ for $i = 1, \dots, N$. The function outputs $(\mathbf{c}, c_1, \dots, c_N) \in R_q^{n+N}$.
 2. If \mathbf{m} has *large* coefficients compared to q (i.e., unbounded mod q), then we first sample a short randomness vector $\mathbf{r} \in \chi_e^{n+2N+w}$ and set $\mathbf{c} = \mathbf{A}\mathbf{r}$. Then, we compute $c_{i,0} = \mathbf{b}_{i,0}^\top \mathbf{r} + m_i$ and $c_{i,1} = \mathbf{b}_{i,1}^\top \mathbf{r} + \sqrt{q}m_i$. The function outputs $(\mathbf{c}, c_{1,0}, c_{1,1}, \dots, c_{N,0}, c_{N,1}) \in R_q^{n+2N}$.
- To **Extract** a message m_i from c_i for $i \in [N]$ using the trapdoor $\mathbf{s}_{i,j}$'s,
 1. If \mathbf{m} has *small* coefficients¹² compared to q , we will be given $(\mathbf{c}, c_1, \dots, c_N) \in R_q^{n+N}$. Then, we compute $u_i := c_i - q_m \mathbf{s}_{i,0}^\top \mathbf{c} = q_m \mathbf{e}_{i,0}^\top \mathbf{r} + m_i \bmod q$. Then, we round off the error term $\mathbf{e}_{i,0}^\top \mathbf{r}$ by computing $u_i \bmod q_m$ to recover each m_i .
 2. If \mathbf{m} has *large* coefficients¹³ compared to q (i.e., unbounded mod q), we will be given $(\mathbf{c}, c_{1,0}, c_{1,1}, \dots, c_{N,0}, c_{N,1}) \in R_q^{n+2N}$. Then, we compute $u_{i,0} = c_{i,0} - \mathbf{s}_{i,0}^\top \mathbf{c} = \mathbf{e}_{i,0}^\top \mathbf{r} + m_i$, and $u_{i,1} = c_{i,1} - \mathbf{s}_{i,1}^\top \mathbf{c} = \mathbf{e}_{i,1}^\top \mathbf{r} + \sqrt{q}m_i$. Now compute $e' := u_1 - \sqrt{q}u_0 \bmod \sqrt{q}$ and $m_i := (u_1 + e')/\sqrt{q}$.

We note that the server's commitment to the key \mathbf{k} can have small-coefficient message input when BP14 PRF is used; while for BLMR13 PRF, large-coefficient message support is needed. On the other hand, the client's commitment to (\mathbf{R}, x) has always small coefficients in the message (regardless of the PRF) when e.g. the bits of the input message are represented as the coefficients of a polynomial(s).

As already observed in earlier work such as [LNPS21, ESZ22a], we note that the above encryption scheme is effectively an extractable version of the BDLOP commitment scheme [BDL⁺18]. The (plain) BDLOP commitment is employed for instance in LNP22 proof [LNP22]. Therefore, we can optimize the performance by calculating a single BDLOP commitment (instead of multiple times) when both the underlying NIZK and our OPRF protocol require such a commitment.

As shown in [BDL⁺18], the commitment scheme is

1. (computationally) hiding¹⁴ if $\text{MLWE}_{w,n+M,\chi_e}$ is hard, and
2. (computationally) binding if $\text{MSIS}_{n,n+M+w,2\beta_r}$ is hard where β_r denotes the bound on $\|\mathbf{r}\|$ proven by the NIZK proof.

We note that in our case, we prove the commitment opening relations by the client ($\mathbf{c}_r = \text{COM.Commit}(\dots)$) and the server ($\mathbf{c}_k = \text{COM.Commit}(\dots)$) *without* a relaxation/approximation factor. Hence, the MSIS bound β_{SIS} is tighter (compared to what is provided in [BDL⁺18]) and does not involve terms depending on the size of the relaxation factor. We also note that commitment to \mathbf{R} is done by committing to the rows of \mathbf{R} (under one commitment).

¹² Note that well-formedness of the commitment along with having small coefficients in error, randomness, and message will be proven via a NIZK proof.

¹³ Note that well-formedness of the commitment along with “double-encryption” of $(m, \sqrt{q}m)$ will be proven via a NIZK proof. The double-encryption proof often comes for free in communication since it is a simple linear proof over R_q (see, e.g., [LNP22, Section 3]).

¹⁴ In fact, the commitment outputs are (computationally) indistinguishable from uniformly random vectors over R_q .

Extraction/decryption correctness. In our NIZK proofs employed in LeOPaRd, we prove that all error, randomness, and/or message (if small) coefficients are much smaller than the system modulus q . Therefore, the error terms described above to be removed will have a quite small infinity norm compared to the modulus sizes we need for our OPRF. Therefore, extraction/decryption correctness requirements are easily met for well-formed commitments (with probability 1) under typical parameters. Nevertheless, we explicitly state the required bounds below.

1. For *small* messages with $\|m_i\|_\infty \leq q_m$, we need

$$\|q_m \mathbf{e}_{i,0}^\top \mathbf{r} + m_i\|_\infty < q/2.$$

2. For *large* messages, we need

$$\|\mathbf{e}_{i,j}^\top \mathbf{r}\|_\infty < \sqrt{q}/4.$$

To argue that a commitment key with a trapdoor is indistinguishable from a regular commitment key, we require the hardness of MLWE_{n,χ_e} so that $\mathbf{b}_{i,j}^\top = \mathbf{s}_{i,j}^\top \mathbf{A} + \mathbf{e}_{i,j}^\top$ is indistinguishable from random. For the client's commitment \mathbf{c}_r , we assume the error entries are sampled from the same distribution χ_r as the entries of \mathbf{R} in LeOPaRd as the client security already relies on $\text{MLWE}_{\ell,\chi_r}$. Therefore, we get the dimension parameter n_c for the client's commitment as $n_c = \ell$ (provided the relevant $\text{MSIS}_{n_c,2\beta_r}$ problem is hard, which is easily satisfied for our parameter settings). For the server's commitment \mathbf{c}_k as its public key, we assume the error coefficients are sampled from a uniform distribution with standard deviation σ_0 as the server security already relies on MLWE with such errors and secret key dimension m . Therefore, we get the dimension parameter n_s for the server's commitment as $n_s = m$ (provided the relevant $\text{MSIS}_{n_s,2\beta_s}$ problem is hard, which is easily satisfied for our parameter settings).