Efficient theta-based algorithms for computing (ℓ, ℓ)-isogenies on Kummer surfaces for arbitrary odd ℓ

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Abstract. Isogeny-based cryptography is one of the candidates for post-quantum cryptography. Recently, many isogeny-based cryptosystems using isogenies between Kummer surfaces were proposed. Most of those cryptosystems use $(2, 2)$-isogenies. However, to enhance the possibility of cryptosystems, higher degree isogenies, say (ℓ, ℓ)-isogenies for an odd ℓ, is also crucial. For an odd ℓ, the Lubicz-Robert gave a formula to compute $(\ell)^g$-isogenies in general dimension g. In this paper, we propose explicit and efficient algorithms to compute (ℓ, ℓ)-isogenies between Kummer surfaces, based on the Lubicz-Robert formula. In particular, we propose two algorithms for computing the codomain of the isogeny and two algorithms for evaluating the image of a point under the isogeny. Then, we count the number of arithmetic operations required for each of our proposed algorithms, and determine the most efficient algorithm in terms of the number of arithmetic operations from each of two types of algorithms for each ℓ. As an application, using the most efficient one, we implemented the SIDH attack on B-SIDH in SageMath. In setting that originally claimed 128-bit security, our implementation was able to recover that secret key in about 11 hours.

Keywords: Post-quantum cryptography · Isogeny-based cryptography · B-SIDH · Kummer surface· Theta function

1 Introduction

Isogeny-based cryptography is one of the candidates for post-quantum cryptography. Its advantage is that it has relatively small keys, ciphertexts, and signatures. On the other hand, its processing speed is slower than many of other candidates for post-quantum cryptography. This mainly comes from the computation of isogenies. Therefore, improving the computation of isogenies is important. Many researches have been done on this topic ([35],[15],[10],[2],[12],[32],[22]).

Vélu's formulas [35] give a method for calculating ℓ-isogenies between elliptic curves, where an *ℓ-isogeny* is defined to be an isogeny whose kernel is a cyclic group of order ℓ. The computational complexity of Vélu's formulas is $O(\ell)$ operations in the base field. Although the classical Vélu's formulas are formulas on the Weierstrass forms and use x- and y-coordinates, it is possible to obtain formulas using only x-coordinates, i.e., formulas on Kummer lines. In particular, formulas on Montgomery curves are well known. The first formulas on Montgomery curves were given by Jao and De Feo [15]. They showed a method to derive formulas for isogenies of arbitrary degree, however, explicit formulas for isogenies of degree greater than 4 were not given. Costello and Hisil [10] gave explicit formulas for isogenies of arbitrary odd degree on Montgomery curves. Their formulas are more efficient than ones derived from the method of Jao and De Feo. Based on their formulas, isogeny-based schemes such as CSIDH [6] and B-SIDH [9] were proposed. Later, the formula for the codomain curve of the isogeny was improved by [24]. The computational complexity of an ℓ-isogeny was reduced to $\tilde{O}(\sqrt{\ell})$ by [2].

A generalization of ℓ-isogenies to 2-dimensional isogenies is called *(ℓ,ℓ)-isogenies*. In recent years, many cryptosystems which combine isogenies between elliptic curves and isogenies between higher dimensional abelian varieties have been proposed ([1,7,11,29] for example). Many of these schemes use $(2,2)$-isogenies for the higher dimensional isogenies. The reason is that the computation of $(2,2)$-isogenies is relatively efficient compared to higher dimensional isogenies of other degrees. In particular, there is an efficient formula for $(2,2)$-isogenies on Kummer surfaces by [12]. For enhancing the variety of isogeny-based schemes, it is important to have efficient formulas for isogenies of higher degrees. Indeed, formulas for $(3,3)$-isogenies on Kummer surfaces were given by [32]. For a general prime number ℓ, formulas for (ℓ,ℓ)-isogenies were given by the Lubicz and Robert [22].

Lubicz-Robert formula. Let k be an algebraically colosed field of characteristic zero or odd prime number p. Let A be an abelian variety of dimension g over k, $\mathscr{L} = \mathscr{L}_0^n$ be a line bundle on A where $\mathscr{L}_0$ is a principal and n is even, and $\Theta_{\mathscr{L}}$ be a symmetric theta structure for $(A,\mathscr{L})$. For any odd prime number ℓ coprime to p and a maximal isotropic subgroup $K \subset A[\ell]$ with respect to the Weil pairing, the isogeny $f : A \to B = A/K$ called $(\ell)^g$-isogeny induces a line bundle $\mathscr{M}$ on B and a symmetric theta structure $\Theta_{\mathscr{M}}$ for $(B,\mathscr{M})$ of level n. The theta structure of level n gives a morphism $\varphi_n : A \to \mathbb{P}^{n^g-1}$, and for $x \in A$, the projective coordinate $\varphi_n(x) \in \mathbb{P}^{n^g-1}$ is called a *theta coordinate* of x. Especially, $\varphi_n(0)$ is called a *theta-null point*. We take a representation of ℓ as a sum of squares of integers: $\ell = \sum_{u=1}^{r} a_u^2$. Then, the Lubicz-Robert [22] gave a formula which gives a theta coordinate of $f(x) \in B$ for $x \in A$ up to multiplication by a constant from some theta coordinates on A in $O(\ell^g n^g)$ operations on k:

$$\theta_i^B(f(x)) = \sum_{e \in K} \prod_{u=1}^{r} \texttt{Mult}(a_u, \widetilde{x+e})_{a_u i} \ . \tag{1}$$

For a precise formula, see [22, Corollary 4.6] or Section 2.4.

When $n = 2$, if A is indecomposable, it is known that the above morphism φ_2 gives the embedding of the Kummer variety K_A to $\mathbb{P}^{2^g-1}$ where Kummer variety is the quotient $A/\langle \pm 1 \rangle$. Thus, the above formula for $n = 2$ gives an efficient way to calculate a morphism between Kummer varieties.

In [22], $(\ell)^g$-isogeny calculation algorithm based on (1) (for general dimension and general level) is given as [22, Algorithm 4]. However, we consider that there are the following points where improvements can be made:

1. By using (1), we can compute both the theta-null point of a codomain and the theta coordinate of the image under f for a given point. However, if we separate codomain and evaluation, are there improvements for each?
2. Which of the possible representations $\ell = \sum_{u=1}^{r} a_u^2$ makes the algorithm most efficient?
3. To use (1), we need to construct excellent lifts from given affine lifts, called normalization. Then, how can normalization be calculated efficiently?
4. In a cryptographic situation, calculating multiplicative inversion is expensive. Can we construct inversion-free algorithms?
5. What are explicit algorithms and their numbers of arithmetic operations on the base field k?

Our contribution. We propose some explicit algorithms of (ℓ, ℓ)-isogeny calculations between Kummer surfaces based on the Lubicz-Robert formula (1). Then, in our algorithms, we make the following contributions for the above listed points:

1. We consider codomain and evaluation separately, and propose algorithms for each. In particular, for codomain, we reduce some computation steps to half. See Section 3.4.
2. We separate two cases: representations $\ell = \sum_{u=1}^{r} a_u^2$ such that $r = O(1)$ and $\ell = 1^2 + \cdots + 1^2$. Then, for both codomain and evaluation, we provide two algorithms using these two representations. In addition, for the former case, we investigate in Section 3.5 in detail.
3. We provide a method to calculate normalization, which improves our isogeny calculation algorithms. In Section 3.3, we provide some necessary equations, and in Section 3.4, we propose the concrete method.
4. In our proposed algorithms, we avoid calculating multiplicative inversion on the base field k.
5. From the above items 1 and 2, we propose four algorithms. For them, we give explicit algorithms, complexities, and the numbers of arithmetic operations for small ℓ. For details, see Section 3.4 and Section 4.

About these algorithms, see Section 3.1 for overview, and see Section 3.4 for the concrete algorithms. Here, `CodSq` is $O(\ell^2)$ operations algorithms and `CodOne` is $O(\ell^2 \log(\ell))$ operations algorithm. Similarly, `EvalSq` is $O(\ell^2)$ operations algorithms and `EvalOne` is $O(\ell^2 \log(\ell))$ operations algorithm.

	$\ell = \sum_{u=1}^{r} a_u^2,\ r = O(1)$	$\ell = 1^2 + \cdots + 1^2$
Codomain	`CodSq`	`CodOne`
Evaluation	`EvalSq`	`EvalOne`

Table 1. Our proposed algorithms in Section 3.4

Moreover, we give implementations of these algorithms and count these operations on k for each $3 \leq \ell < 200$ (Table 6 in Section 4). As the result, we determine a more efficient algorithm for each ℓ: for codomain, for $3 \leq \ell \leq 11$ and $\ell = 19, 23$, `CodOne` is more efficient than `CodSq`, and for other ℓ, `CodSq` is more efficient. For evaluation, for all $3 \leq \ell < 200$, `EvalOne` is more efficient.

In addition, by using the most efficient algorithms selected above, we give SIDH attack on B-SIDH in about 11 hours (in Section 5).

Our implementation of (ℓ, ℓ)-isogeny counting and an attack on B-SIDH is written in computer algebra system SageMath [34] and is found at

`https://github.com/Yoshizumi-Ryo/ellell-isogeny_sage`.

Related works. Santos-Costello-Smith [32] proposed a method for computing $(3, 3)$-isogenies between Kummer surfaces. They implicitly utilize theta functions in their algorithm, but it should be noted that their algorithm is not derived from the Lubicz-Robert formula (i.e. our proposed algorithm is completely different from Santos-Costello-Smith's algotithm). As a result, their $(3, 3)$-isogeny computation algorithm is significantly more efficient than our algorithm (cf. [32, §4.3]).

Afterward, Santos-Flynn [33] generalized (ℓ, ℓ)-isogenies for any odd number ℓ. The asymptotic complexity of their algorithm with respect to ℓ is higher than that of theta-based algorithms, such as those based on the Lubicz-Robert formula and the Cosset-Robert formula [8]. However, as mentioned in [33, Section 6.3], for $\ell \leq 11$, their implementations outperform the `AVIsogenies v0.7` [4], which is an implementation of the algorithm based on the Cosset-Robert formula. In addition, their algorithm outputs the defining equations of the codomain Kummer surfaces and of the isogeny, unlike theta function-based algorithms.

On the other hand, our algorithms are based on the Lubicz-Robert formula. We will show that the algorithm based on the Lubicz-Robert formula is more efficient than that based on the Cosset-Robert formula (see Remark 5). Thus, for a sufficiently large ℓ, it can be said that our algorithm is more efficient than Santos-Flynn's algorithm.

Organizations. In Section 2, we recall some facts about theta functions and their addition algorithms, and the Lubicz-Robert formula. In Section 3, we describe the costs of arithmetic on Kummer surfaces and give relations for normalization. Then, we give explicit algorithms for codomain and evaluation, and give their asymptotic complexities. In Section 4, we count the number of the operations of their algorithms and decide which is efficient for each ℓ. In Section 5, we recall B-SIDH and SIDH attack briefly and show the result of the implementation. Finally, Section 6 gives the conclusion.

2 Preliminaries

In this section, we summarize some facts about abelian varieties and theta functions [3,26,27,30] as well as relevant algorithms [19,21,22,30] which are bases of our proposed method.

For simplicity, we consider our arguments over the complex number field $\mathbb{C}$. However, by using algebraic theta functions introduced by Mumford [26], these arguments are applicable even to the case of an algebraically closed field of characteristic p where p is coprime to 2ℓ. For more details, we refer to [30].

In addition, we only consider the case of dimension $g = 2$ although the arguments of this section hold for general $g \geq 1$.

2.1 Theta functions

Let $\mathbb{H}_2$ denote the Siegel upper half-space of degree two defined by

$$\mathbb{H}_2 = \{\Omega \in M(2, \mathbb{C}) \mid {}^t\Omega = \Omega, \mathrm{Im}(\Omega) > 0\}.$$

Then, an abelian surface A over $\mathbb{C}$ is isomorphic to $\mathbb{C}^2/\Lambda_\Omega$ where $\Lambda_\Omega = \Omega\mathbb{Z}^2 \oplus \mathbb{Z}^2$ for some $\Omega \in \mathbb{H}_2$. In addition, this Ω determines a principal line bundle $\mathscr{L}_0$ on A. For any $a, b \in \mathbb{Q}^2$, the theta function with characteristics (a, b) is an analytic function given by

$$\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](z, \Omega) := \sum_{m \in \mathbb{Z}^2} \exp(\pi \mathrm{i}\ {}^t(m+a)\Omega(m+a) + 2\pi \mathrm{i}\ {}^t(m+a)(z+b))$$

for any $z \in \mathbb{C}^2$. We say that an analytic function f on $\mathbb{C}^2$ is a Λ_Ω-periodic function of level n if $f(z+m) = f(z)$ and $f(z+\Omega m) = \exp(-\pi \mathrm{i} n\ {}^t m\Omega m - 2\pi \mathrm{i} n\ {}^t z m) f(z)$ for all $z \in \mathbb{C}^2$ and $m \in \mathbb{Z}^2$. Then, the set R_Ω^n of all Λ_Ω-periodic functions of level n is an n^2-dimensional $\mathbb{C}$-vector space. Moreover, the following n^2 functions $\theta\left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right](z, \frac{\Omega}{n})$ for $b \in \frac{1}{n}\mathbb{Z}^2/\mathbb{Z}^2$ form a basis of R_Ω^n [28]. Since $\theta\left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right](z, \frac{\Omega}{n}) = \theta\left[\begin{smallmatrix} 0 \\ b+\beta \end{smallmatrix}\right](z, \frac{\Omega}{n})$ for all $b \in \mathbb{Q}^2$ and $\beta \in \mathbb{Z}^2$, these functions do not depend on the representative of $b \in \frac{1}{n}\mathbb{Z}^2/\mathbb{Z}^2$. We can identify R_Ω^n with the vector space $\Gamma(A, \mathscr{L}_0^n)$ of global sections and thus the basis $\{\theta\left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right](z, \frac{\Omega}{n})\}_b$ of R_Ω^n gives the morphism

$$\begin{array}{rccc} \rho_n \colon & A = \mathbb{C}^2/\Lambda_\Omega & \longrightarrow & \mathbb{P}^{n^2-1} \\ & z & \longmapsto & (\theta\left[\begin{smallmatrix} 0 \\ b \end{smallmatrix}\right](z, \frac{\Omega}{n}))_b \end{array}.$$

We call $\rho_n(0) \in \mathbb{P}^{n^2-1}$ the *theta-null point* and call $\rho_n(x)$ the *theta coordinate* of $x \in A$. We write $\theta_i(z) := \theta\left[\begin{smallmatrix} 0 \\ i/n \end{smallmatrix}\right](z, \frac{\Omega}{n})$ for $i \in (\mathbb{Z}/n\mathbb{Z})^2$. Then, we have $\theta_i(-z) = \theta_{-i}(z)$. When $n \geq 3$, ρ_n is an embedding [3, Theorem 4.5.1]. When $n = 2$, since $\rho_2(-z) = \rho_2(z)$, $\rho_2 : A \to \mathbb{P}^3$ induces a morphism $K_A \to \mathbb{P}^3$ from a

Kummer surface K_A which is the quotient of A by automorphisms $\langle \pm 1_A \rangle$. If A is not a product of elliptic curves, this morphism $K_A \to \mathbb{P}^3$ is an embedding [3, Theorem 4.8.1].

Next, we recall Riemann relations [26], [19, Theorem 3.1] (Theorem 1). By using them, we derive some formulas for arithmetic operations on abelian surfaces later. To explain it, first we recall the notion of Riemann position [22, Definition 3.2].

Definition 1. *For any abelian group G, an 8-tuple $(g_1, g_2, g_3, g_4; g'_1, g'_2, g'_3, g'_4)$ of elements of G is said to be in* Riemann position *(on G) if there exists some element $h \in G$ such that $g'_i = g_i + h$ for $i = 1, \ldots, 4$ and $g_1 + g_2 + g_3 + g_4 = -2h$.*

Theorem 1 (Riemann relations [26], [19, Theorem 3.1]). *Let n be an even integer. For any 8-tuple $(z_1, z_2, z_3, z_4; z'_1, z'_2, z'_3, z'_4)$ of elements of $\mathbb{C}^2$ in Riemann position, any 8-tuple $(i_1, i_2, i_3, i_4; i'_1, i'_2, i'_3, i'_4)$ of elements of $(\mathbb{Z}/n\mathbb{Z})^2$ in Riemann position, and any character $\chi \in \widehat{(\mathbb{Z}/2\mathbb{Z})^2}$ of the group $(\mathbb{Z}/2\mathbb{Z})^2$, we have*

$$\left(\sum_{t \in (\mathbb{Z}/2\mathbb{Z})^2} \chi(t) \theta_{i_1+t}(z_1) \theta_{i_2+t}(z_2) \right) \left(\sum_{t \in (\mathbb{Z}/2\mathbb{Z})^2} \chi(t) \theta_{i_3+t}(z_3) \theta_{i_4+t}(z_4) \right)$$
$$= \left(\sum_{t \in (\mathbb{Z}/2\mathbb{Z})^2} \chi(t) \theta_{i'_1+t}(z'_1) \theta_{i'_2+t}(z'_2) \right) \left(\sum_{t \in (\mathbb{Z}/2\mathbb{Z})^2} \chi(t) \theta_{i'_3+t}(z'_3) \theta_{i'_4+t}(z'_4) \right)$$

where, for the indices of functions $\theta_i(z)$, we regard $(\mathbb{Z}/2\mathbb{Z})^2$ as a subgroup of $(\mathbb{Z}/n\mathbb{Z})^2$ via the embedding $\overline{a} \mapsto \overline{\frac{n}{2}a}$ ($a \in \mathbb{Z}^2$).

Now, theta coordinates are given as projective coordinates on $\mathbb{P}^{n^2-1}$. However, to treat each component as an element of k, we have to fix their theta coordinates as affine coordinates on $\mathbb{A}^{n^2} \backslash \{0\}$ by taking some representatives. Here, we define this content precisely [22]:

Definition 2. *Let $\kappa : \mathbb{A}^{n^2} \setminus \{0\} \to \mathbb{P}^{n^2-1}$ be the natural projection. For $x \in A$, we call any preimage of $\rho_n(x)$ for κ an* affine lift *of x. We write an affine lift of x as $\tilde{x}$ or $(\tilde{\theta}_i(x))_i$. For $i \in (\mathbb{Z}/n\mathbb{Z})^2$, we write the i^{th}-coordinate of $\tilde{x}$ by $(\tilde{x})_i$ or $\tilde{\theta}_i(x)$. For $\lambda \in \mathbb{C}^*$ and an affine lift $\tilde{x}$, we define $\lambda * \tilde{x}$ as $(\lambda \cdot (\tilde{x})_i)_i$.*

For later use, we extend the notion of Riemann relations to affine lifts.

Definition 3. *Let $(x_1, x_2, x_3, x_4; x'_1, x'_2, x'_3, x'_4)$ be in Riemann position on A and $(\tilde{x_1}, \ldots, \tilde{x'_4})$ be their affine lifts. Then, we say that $(\tilde{x_1}, \ldots, \tilde{x'_4})$ satisfy Riemann relations if for any $(i_1, i_2, i_3, i_4; i'_1, i'_2, i'_3, i'_4)$ in Riemann position on $(\mathbb{Z}/n\mathbb{Z})^2$ and* *any character $\chi \in \widehat{(\mathbb{Z}/2\mathbb{Z})^2}$, the following equation holds:*

$$\left(\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)(\tilde{x_1})_{i_1+t}(\tilde{x_2})_{i_2+t}\right)\left(\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)(\tilde{x_3})_{i_3+t}(\tilde{x_4})_{i_4+t}\right)$$
$$=\left(\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)(\tilde{x'_1})_{i'_1+t}(\tilde{x'_2})_{i'_2+t}\right)\left(\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)(\tilde{x'_3})_{i'_3+t}(\tilde{x'_4})_{i'_4+t}\right) .$$

Then, by Riemann relations (Theorem 1), we have the following lemma:

Lemma 1. *For given $(x_1, x_2, x_3, x_4; x'_1, x'_2, x'_3, x'_4)$ being in Riemann position on A and any affine lifts $\tilde{x_2}, \tilde{x_3}, \tilde{x_4}, \tilde{x'_1}, \tilde{x'_2}, \tilde{x'_3}, \tilde{x'_4}$, there exists an affine lift $\tilde{x_1}$ such that $(\tilde{x_1}, \dots, \tilde{x'_4})$ satisfies Riemann relations in the sense of Definition 3.*

2.2 Arithmetic on Kummer surfaces

In this subsection, we consider some arithmetic operations on Kummer surfaces using theta functions of level $n = 2$ [21]. As mentioned in the previous subsection, if A is not a product of ellipitc curves, level 2 theta functions give the embedding of the Kummer surface to the projective space $K_A \to \mathbb{P}^3$.

In the following, we introduce some known methods for arithmetic calculation on Kummer surfaces using theta coordinates [21, Section 5]. Here, we assume that $A = \mathbb{C}^2/\Lambda_\Omega$ is not isomorphic to a product of elliptic curves as a principally polarized abelian surface. In other words, all abelian surfaces in this subsection are Jacobians of some genus-2 hyperelliptic curves. Note that, if A is isomorphic to a product of elliptic curves as a polarized abelian surface, we can perform the arithmetic calculation by calculating on each elliptic curve.

The condition that A is the Jacobian of a genus-2 hyperellipctic curve is equivalent to that the following ten values called even theta-null points of level $(2, 2)$ are all non-zero:

$$\theta\left[\begin{smallmatrix} a/2 \\ b/2 \end{smallmatrix}\right](0, \Omega) \quad \text{for } a, b \in (\mathbb{Z}/2\mathbb{Z})^2 \text{ such that } {}^t a \cdot b = 0 \in \mathbb{Z}/2\mathbb{Z}.$$

For more details, see [18, Section 3.2]. Under this assumption, by the same argument as [18, Lemma 3], we have

$$\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\theta_{i+t}(0)\theta_t(0) \neq 0 \tag{2}$$

for all $i \in (\mathbb{Z}/2\mathbb{Z})^2$ and $\chi \in \widehat{(\mathbb{Z}/2\mathbb{Z})^2}$ such that $\chi(i) = 1 \in \langle \pm 1 \rangle$.

In the rest of this paper, we fix one affine lift $(\tilde{\theta}_i(0))_i$ of the theta-null point. Here, we summarize known methods for calculating the following arithmetic operations on Kummer surfaces:

Differential Addition: Given affine lifts $(\tilde{\theta}_i(x))_i, (\tilde{\theta}_i(y))_i, (\tilde{\theta}_i(x-y))_i$, output an affine lift $(\tilde{\theta}_i(x+y))_i$.

Doubling: Given an affine lift $(\tilde{\theta}_i(x))_i$, output an affine lift $(\tilde{\theta}_i(2x))_i$.

Three-way Addition: Given affine lifts $(\tilde{\theta}_i(x))_i, (\tilde{\theta}_i(y))_i, (\tilde{\theta}_i(z))_i$, $(\tilde{\theta}_i(x+y))_i$, $(\tilde{\theta}_i(y+z))_i$, $(\tilde{\theta}_i(z+x))_i$, output an affine lift $(\tilde{\theta}_i(x+y+z))_i$.

Scalar Multiplication: Given an affine lift $(\tilde{\theta}_i(x))_i$ and an integer N, output an affine lift $(\tilde{\theta}_i(Nx))_i$.

Normal Addition: Given affine lifts $(\tilde{\theta}_i(x))_i, (\tilde{\theta}_i(y))_i$, output a set of affine lifts $\{(\tilde{\theta}_i(x+y))_i, (\tilde{\theta}_i(x-y))_i\}$.

Compatible Addition: Given affine lifts $(\tilde{\theta}_i(y))_i, (\tilde{\theta}_i(z))_i, (\tilde{\theta}_i(x+y))_i$, $(\tilde{\theta}_i(x+z))_i$, output an affine lift $(\tilde{\theta}_i(y+z))_i$.

These concrete algorithms are written in Section B.2 and their costs are written in Section 3.2.

Remark 1. In our proposed isogeny algorithm in Section 3.4, we do not use Normal Addition and Compatible Addition since we give enough information as inputs not to need those algorithms. Thus, we do not introduce them in this subsection and they are written in Appendix A and we will not give those concrete algorithms and costs in Section 3.2. However, since these algorithms are needed when we construct attacks on B-SIDH in Section 5.2.

For Differential Addition, an affine lift $(\tilde{\theta}_i(x+y))_i$ could be obtained from given affine lifts $(\tilde{\theta}_i(x))_i, (\tilde{\theta}_i(y))_i, (\tilde{\theta}_i(x-y))_i$ by just applying Lemma 1 to $(x+y, x-y, 0, 0; y, -y, -x, -x)$ in Riemann position on A. But in fact, the computation can be made more efficient in the following manner. First we note that, for any $i \in (\mathbb{Z}/2\mathbb{Z})^2$, considering $(i, 0, i, 0; i, 0, i, 0)$ in Riemann position on $(\mathbb{Z}/2\mathbb{Z})^2$, we have

$$\begin{aligned}&\left(\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\tilde{\theta}_{i+t}(x+y)\tilde{\theta}_t(x-y)\right)\left(\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\tilde{\theta}_{i+t}(0)\tilde{\theta}_t(0)\right)\\ =&\left(\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\tilde{\theta}_{i+t}(x)\tilde{\theta}_t(x)\right)\left(\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\tilde{\theta}_{i+t}(y)\tilde{\theta}_t(y)\right)\end{aligned} \tag{3}$$

where we used $\theta_i(x) = \theta_i(-x)$ and $\theta_i(y) = \theta_i(-y)$. Secondly, we define certain values z_i^χ and κ_{ij} as follows. For any $(i, \chi) \in (\mathbb{Z}/2\mathbb{Z})^2 \times \widehat{(\mathbb{Z}/2\mathbb{Z})^2}$ such that $\chi(i) = 1$, we define

$$z_i^\chi := \frac{\left(\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\tilde{\theta}_{i+t}(x)\tilde{\theta}_t(x)\right)\left(\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\tilde{\theta}_{i+t}(y)\tilde{\theta}_t(y)\right)}{\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\tilde{\theta}_{i+t}(0)\tilde{\theta}_t(0)} \tag{4}$$

where the denominator of the right-hand side is not zero by (2). Here, these z_i^χ are computed from $(\tilde{\theta}_i(x))_i$ and $(\tilde{\theta}_i(y))_i$. Then, we define κ_{ij} for any $i, j \in (\mathbb{Z}/2\mathbb{Z})^2$ as follows:

$$\kappa_{ij} := \frac{1}{4} \sum_{\substack{\chi\in\widehat{(\mathbb{Z}/2\mathbb{Z})^2}\\ \text{s.t.}\,\chi(i+j)=1}} \frac{\chi(i)+\chi(j)}{2} z_{i+j}^\chi \;. \tag{5}$$

Thus, we can calculate all κ_{ij} from the values z_i^χ such as $\chi(i) = 1$ (note that κ_{ij} is symmetric with respect to i and j). Then, by the inverse Fourier transform, we have the following relations for $i, j \in (\mathbb{Z}/2\mathbb{Z})^2$:

$$\tilde{\theta}_i(x+y)\tilde{\theta}_j(x-y) + \tilde{\theta}_j(x+y)\tilde{\theta}_i(x-y) = 2\kappa_{ij} \quad . \tag{6}$$

Differential Addition and Doubling. By using equality (6), when $\tilde{\theta}_i(x-y) \neq 0$ for all i, we have

$$\tilde{\theta}_i(x+y) = \frac{\kappa_{ii}}{\tilde{\theta}_i(x-y)} \quad . \tag{7}$$

Thus, from affine lifts $(\tilde{\theta}_i(x))_i, (\tilde{\theta}_i(y))_i, (\tilde{\theta}_i(x-y))_i$, we can calculate an affine lift $(\tilde{\theta}_i(x+y))_i$ satisfying (3). We call this operation *Differential Addition.* When $x = y$, we call this *Doubling.*

Remark 2. Even if $\tilde{\theta}_i(x-y) = 0$ for some $i \in (\mathbb{Z}/2\mathbb{Z})^2$, we can still compute $(\tilde{\theta}_i(x+y))_i$. In fact, first we take $j \in (\mathbb{Z}/2\mathbb{Z})^2$ such that $\tilde{\theta}_j(x-y) \neq 0$ and compute $\tilde{\theta}_j(x+y)$ by using (7). Then, for $i \in (\mathbb{Z}/2\mathbb{Z})^2 \smallsetminus \{j\}$, by (6), we have

$$\tilde{\theta}_i(x+y) = \frac{2\kappa_{ij} - \tilde{\theta}_j(x+y)\tilde{\theta}_i(x-y)}{\tilde{\theta}_j(x-y)} \quad .$$

Three-way Addition. For given affine lifts $(\tilde{\theta}_i(x))_i, (\tilde{\theta}_i(y))_i, (\tilde{\theta}_i(z))_i$, $(\tilde{\theta}_i(x+y))_i, (\tilde{\theta}_i(y+z))_i, (\tilde{\theta}_i(z+x))_i$, we can calculate $(\tilde{\theta}_i(x+y+z))_i$ as follows. Note that this Three-way Addition algorithm does not always work on A but work on some Zariski dense subset of A. For details, we refer to [20, Section 3.6]. Here for simplicity, we assume $\tilde{\theta}_i(x) \neq 0, \tilde{\theta}_i(y) \neq 0$, and $\tilde{\theta}_i(z) \neq 0$ for all i, and in this condition, Three-way Addition algorithm works. First, for $\chi \in \widehat{(\mathbb{Z}/2\mathbb{Z})}^2$, we define

$$E^\chi := \frac{\left(\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\tilde{\theta}_t(0)\tilde{\theta}_t(y+z)\right) \cdot \left(\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\tilde{\theta}_t(z+x)\tilde{\theta}_t(x+y)\right)}{\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\tilde{\theta}_t(y)\tilde{\theta}_t(z)} \quad . \tag{8}$$

These E^χ are computed from the given affine lifts. Here, by applying Lemma 1 to points $(x+y+z, x, y, z; 0, -y-z, -z-x, -x-y)$ in Riemann position on A, and by focusing (among the resulting Riemann relations) on indices $(0,0,0,0;0,0,0,0)$ in Riemann position on $(\mathbb{Z}/2\mathbb{Z})^2$, for any $\chi \in \widehat{(\mathbb{Z}/2\mathbb{Z})}^2$, we have $\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\tilde{\theta}_t(x+y+z)\tilde{\theta}_t(x) = E^\chi$. Then, by the inverse Fourier transform, for any $i \in (\mathbb{Z}/2\mathbb{Z})^2$, we have

$$\tilde{\theta}_i(x+y+z) = \frac{\sum_{\chi\in\widehat{(\mathbb{Z}/2\mathbb{Z})}^2} \chi(i)E^\chi}{4\tilde{\theta}_i(x)} \quad . \tag{9}$$

Thus we have obtained the affine lift $(\tilde{\theta}_i(x+y+z))_i$. This operation is called *Three-way Addition* (or *Extended Addition*).

Scalar Multiplication. For a given affine lift $(\tilde{\theta}_i(x))_i$ and any integer $N \geq 3$, there are various ways of calculating $(\tilde{\theta}_i(Nx))_i$ and the result is denoted by $\texttt{Mult}(N, (\tilde{\theta}_i(x))_i)$. One way to compute it is using the Montgomery ladder [25]. Then, we require $n-1$ Doubling and n Differential Addition where n is the bit length of $N-1$. In our implementation of Section 4, we used this calculation way.

Remark 3. Let $x_1, \ldots, x_r \in A$ be any elements, $(\tilde{\theta}_i(x_j))_i, (\tilde{\theta}_i(x_{j_1}+x_{j_2}))_i$ be affine lifts for $1 \leq j \leq r$ and $1 \leq j_1 < j_2 \leq r$, and $m_1, \ldots, m_r \in \mathbb{Z}$ be any integers. Then, we can compute an affine lift $(\tilde{\theta}_i(m_1x_1 + \cdots + m_rx_r))_i$ in many different ways by using Differential Addition, Doubling, Three-way Addition, and Scalar Multiplication. Now the computation result does not depend on the order of these operations (cf. [19, Corollary 3.13]).

2.3 Excellent lifts

Here, we recall the notion of excellentness for some conditions (cf. [22, Definitions 3.6, 3.7, 3.10]). In the following definition, $\texttt{Multadd}(N, \tilde{x}, \tilde{y}, \widetilde{x+y})$ denotes the affine lift of $Nx + y \in A$ computed from affine lifts $\tilde{x}, \tilde{y}, \widetilde{x+y}$.

Definition 4. *Let ℓ be any odd prime number and $K \subset A[\ell]$ be a maximal isotropic subgroup with respect to the Weil pairing.*

1. *For any ℓ-torsion point $e \in A[\ell]$, an affine lift $\tilde{e}$ of e is said to be* excellent *if $\texttt{Mult}(\ell'+1, \tilde{e}) = \texttt{Mult}(\ell', \tilde{e})$ as affine lifts where $\ell' = \frac{\ell-1}{2}$.*
2. *A set of affine lifts $\tilde{K} = \{\tilde{e} \mid e \in K\}$ of K is said to be* excellent *if for any eight elements in Riemann position on K, their affine lifts in $\tilde{K}$ satisfy Riemann relations in the sense of Definition 3.*
3. *For any affine lift $\tilde{x}$ of $x \in A$ and an excellent lift $\tilde{e}$ of $e \in A[\ell]$, an affine lift $\widetilde{x+e}$ of $x+e$ is said to be* excellent *with respect to $\tilde{x}$ and $\tilde{e}$ if $\texttt{Multadd}(\ell, \tilde{e}, \tilde{x}, \widetilde{x+e}) = \tilde{x}$ as affine lifts.*
4. *For any excellent lift $\tilde{K}$ and any affine lift $\tilde{x}$ of $x \in A$, a set of affine lifts $\widetilde{x+K} = \{\widetilde{x+e} \mid e \in K\}$ is said to be* excellent *with respect to $\tilde{x}$ and $\tilde{K}$ if for any eight elements in Riemann position on A included in $K \cup (x+K)$, their affine lifts in $\tilde{K} \cup \widetilde{x+K}$ satisfy Riemann relations in the sense of Definition 3.*

Theorem 2 ([22, Theorems 3.8, 3.11]). *With the notation above, the following statements hold:*

(i) For any basis $\{e_1, e_2\}$ of $K \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$ and excellent lifts $\tilde{e_1}, \tilde{e_2}, \widetilde{e_1+e_2}$, a set $\tilde{K}$ of affine lifts of K computed from $\tilde{e_1}, \tilde{e_2}, \widetilde{e_1+e_2}$ is excellent.

(ii) Let $\tilde{K}$ be any excellent lift of K and $\tilde{x}$ be any affine lift of $x \in A$. In addition, let $\widetilde{x+e_1}, \widetilde{x+e_2}$ be excellent lifts. Then, a set $\widetilde{x+K}$ of affine lifts of $x+K$ computed from them is excellent.

Remark 4. For an excellent lift $\tilde{e}$ and $\lambda \in \mathbb{C}^*$, $\lambda * \tilde{e}$ is also excellent if and only if $\lambda^\ell = 1$ by Lemma 7 in Section 3.3. Therefore, excellent lifts of e are not necessarily unique and are at most finitely many.

2.4 Lubicz-Robert formula

In this subsection, we introduce an isogeny calculation formula given by the Lubicz-Robert [22]. In their paper, the formula is given for a general dimension and a general even level theta structure. Here we just use the formula in dimension 2 and level 2 theta structure, i.e., on Kummer surfaces. For a theta structure, we refer to [22,26].

Let k be an algebraically closed field of characteristic 0 or $p > 0$ where p is coprime to 2. Let $(A, \mathscr{L}_0)$ be a principal polarized abelian surface over k, $\mathscr{L} = \mathscr{L}_0^2$, and $\Theta_{\mathscr{L}}$ be a symmetric theta structure. In addition, ℓ be an odd prime number, and $K \subset A[\ell]$ be a maximal isotropic subgroup with respect to the Weil pairing. Then, the isogeny $f : A \to B = A/K$ induces a line bundle on B and symmetric theta structure. From an excellent lift $\tilde{K}$ of K, the formula gives the theta-null point $(\theta_i^B(0))_i$ of B with $O(\ell^2)$ operations on k. Moreover, for $x \in A$, an affine lift $\tilde{x}$, and excellent lifts $\widetilde{x+K}$, the formula gives a theta coordinate $(\theta_i^B(f(x)))_i$ of $f(x) \in B$ with $O(\ell^2)$ operations.

Theorem 3 ([22, Corollary 4.6]). *The notation is the same as above. Let $\tilde{K}$ be an excellent lift of K and $a_1, \dots, a_r$ be positive integers such that $\ell = \sum_{u=1}^r a_u^2$. For any $i \in (\mathbb{Z}/2\mathbb{Z})^2$, up to multiplication by a constant not depending on i, we have*

$$\theta_i^B(0) = \sum_{e \in K} \prod_{u=1}^{r} \mathtt{Mult}(a_u, \tilde{e})_{a_u i} \ . \tag{10}$$

For $x \in A$, let $\tilde{x}$ be any affine lift and $\widetilde{x+K}$ be an excellent lift with respect to $\tilde{x}$ and $\tilde{K}$. Then, for any $i \in (\mathbb{Z}/2\mathbb{Z})^2$, up to multiplication by a constant not depending on i, we have

$$\theta_i^B(f(x)) = \sum_{e \in K} \prod_{u=1}^{r} \mathtt{Mult}(a_u, \widetilde{x+e})_{a_u i} \ . \tag{11}$$

Note that we can take a representation $\ell = \sum_{u=1}^r a_u^2$ such that $r \leq 4$ by Lagrange's four-square theorem. Thus, if we take r such that $r = O(1)$, the complexities of both of the above formulas are $O(\ell^2)$ arithmetic operations on k by computing as follows. For (10), first, for a basis $\{e_1, e_2\}$ of K, we compute $\mathtt{Mult}(a_u, e_1), \mathtt{Mult}(a_u, e_2), \mathtt{Mult}(a_u, e_1 + e_2)$ for all $1 \leq u \leq r$. These computations require $O(\log(\ell))$ arithmetic operations. Next, we compute their linear combinations $\mathtt{Mult}(a_u, m_1 e_1 + m_2 e_2)$ for all $0 \leq m_1, m_2 < \ell$ and $1 \leq u \leq r$. These computations require $O(\ell^2)$ arithmetic operations. The case for (11) is similar (cf. [22, p.16]).

As a special case, in the formulas (10), (11), taking $\ell = 1^2 + \cdots + 1^2$, we obtain the following formulas:

$$\theta_i^B(0) = \sum_{e \in K} (\tilde{e})_i^{\ell} \ , \tag{12}$$

$$\theta_i^B(f(x)) = \sum_{e \in K} (\widetilde{x+e})_i^{\ell} \ . \tag{13}$$

For the formulas (12), (13), we need to calculate ℓ^{th} power on k for each $e \in K$. Thus, their complexities are $O(\ell^2 \log(\ell))$. Note that, despite of the asymptotically higher complexity than the previous method, the current method may still be more efficient than the previous one for some concrete choice of ℓ.

In [22], isogeny calculation algorithm based on (11) (for general dimension and general level) is given as [22, Algorithm 4]. Here, we consider the case of dimension 2 and level 2 the same as before. In the algorithm, first we compute excellent lift of $e_1, e_2, e_1 + e_2, x, x + e_1, x + e_2$ where $\{e_1, e_2\}$ is a basis of kernel K. Then, we compute excellent lifts $\tilde{K}$ and $\widetilde{x + K}$. At last, we compute the right-hand side of (11) for $i \in (\mathbb{Z}/2\mathbb{Z})^2$.

Remark 5. In [8], the Cosset-Robert gave another (ℓ, ℓ)-isogeny calculation algorithm based on Koizumi's formula [17] in $O(\ell^r)$ operations where $r = 2$ when $\ell \equiv 1 \pmod 4$ and $r = 4$ when $\ell \equiv 3 \pmod 4$. In the same way as above, we write $\ell = \sum_{u=1}^{r} a_u^2$ with $a_u \in \mathbb{N}$. Moreover, let $\tilde{K}$ be an excellent lift of the kernel $K \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$. Let F be an integer $(r \times r)$-matrix such that $F^tF = \ell\,\mathrm{id}_r$ and the first row is $(a_1, \ldots, a_r)$. Now, we define $F_K : K^r \to K^r$ as the $\mathbb{F}_\ell$-linear map induced by the matrix F. In [8, Equation (6)], the formula to compute the theta-null point of the codomain is

$$\theta_i^B(0) = \sum_{{}^t(e_1,\ldots,e_r)\in \mathrm{Ker}(F_K)} \prod_{u=1}^{r} (\tilde{e}_u)_{a_u i} \tag{14}$$

up to multiplication by a constant. Here, for any $e \in K$, we have ${}^t(a_1 e, \ldots, a_r e) \in \mathrm{Ker}(F_K)$. Hence, in (14), we need to compute and take the sum of the values $\prod_{u=1}^{r} \widetilde{(a_u e)}_{a_u i}$ for each $e \in K$. Thus, the complexity of (10) is the same as or more efficient than the complexity of (14). For more details and general arguments, we refer to [30, Section 4.4.3].

3 Proposed Algorithms

In this section, we propose some explicit algorithms of isogeny calculations between Kummer surfaces based on the Lubicz-Robert formula (Theorem 3). As the same notation as Section 2.4, k is an algebraically closed field of characteristic 0 or $p > 0$ where p is coprime to 2. As noted at the beginning of Section 2, the arguments of Sections 2.1 and 2.2 are applicable to the case of not only $\mathbb{C}$ but also the above k.

In our algorithms, calculations of multiplicative inverse on k are avoided as they are expensive especially in cryptographic situations. Hence we evaluate costs of algorithms by counting multiplication and square operations on k.

Throughout this section, A is an abelian surface over k, $\mathscr{L} = \mathscr{L}_0^2$ is line bundle where $\mathscr{L}_0$ is principal, and $\Theta_{\mathscr{L}}$ is symmetric theta structure of level 2 for $(A, \mathscr{L})$. In addition, ℓ is an odd prime number, and $K \subset A[\ell]$ is a maximal isotropic subgroup with respect to the Weil pairing. Then, $B := A/K$ is an induced level 2 symmetric theta structured abelian surface. Moreover, $f : A \to B$ is the isogeny with kernel K.

First, in Section 3.1, we give an overview of our algorithms. In Section 3.2, we give costs of arithmetic on Kummer surfaces given in Section 2.2. The results will be used in our isogeny calculation algorithms. Then, in Section 3.3, we give relations about excellent lifts. in Section 3.4, we give explicit algorithms of isogeny calculations. At last of this section, in Section 3.5, we consider a representation $\ell = \sum_{u=1}^{r} a_u^2$.

3.1 Overview of our proposed algorithms

In this subsection, we introduce an overview about our isogeny calculation algorithms. The explicit algorithms will be given in Section 3.4.

As we have seen in Theorem 3, the theta-null point of codomain $B := A/K$ can be computed from an excellent lift $\tilde{K}$ of the kernel K. In addition, for $x \in A$, the theta coordinate of the image $f(x)$ can be computed from an excellent lift $\widetilde{x+K}$. Here, we construct algorithms of codomain and evaluation with the following inputs and outputs. In this paper, we basically write an excellent lift by $\tilde{e}$ and any affine lift by $\overline{e}$ for $e \in A$.

Codomain:
- **Input:** Any affine lifts $\overline{e_1}, \overline{e_2}, \overline{e_1 + e_2}$ of $e_1, e_2, e_1 + e_2$ for a basis $\{e_1, e_2\}$ of K.
- **Output:** Theta-null point $(\theta_i^B(0))_i$ of B.

Evaluation:
- **Input:** Any affine lifts $\overline{e_1}, \overline{e_2}, \overline{e_1 + e_2}, \tilde{x}, \overline{x + e_1}, \overline{x + e_2}$ of $e_1, e_2, e_1+e_2, x, x+e_1, x + e_2$ for a basis $\{e_1, e_2\}$ of K and any point $x \in A$.
- **Output:** Theta coordinate $(\theta_i^B(f(x)))_i$ of $f(x) \in B$.

For both cases, we take a representation $\ell = \sum_{u=1}^{r} a_u^2$. We mainly have two cases; $r = O(1)$ (e.g., $r \leq 4$); and $r = \ell$ and $\ell = 1^2 + \cdots + 1^2$. For the former case, we will discuss more in Section 3.5. For the latter case, we use (12) and (13). Thus, for codomain, we consider two algorithms `CodSq` using $\ell = \sum_{u=1}^{r} a_u^2$ and `CodOne` using $\ell = 1^2 + \cdots + 1^2$. Similarly, for evaluation, we consider two algorithms `EvalSq`, `EvalOne`, see Table 1 in Section 1. As noted in Section 2.4, `CodSq` and `EvalSq` require $O(\ell^2)$ operations, while `CodOne` and `EvalOne` require $O(\ell^2 \log(\ell))$ operations.

For `CodSq`, First we compute affine lifts $\overline{s_1 e_2 + s_2 e_2}$ for $0 \leq s_1, s_2 < \ell$. Then instead of computing an affine lift $\overline{m_1 a_u e_1 + m_2 a_u e_2}$, we use $\overline{s_1 e_1 + s_2 e_2}$ where $0 \leq s_1, s_2 < \ell$ and $m_1 a_u \equiv s_1 \pmod{\ell}$, $m_2 a_u \equiv s_2 \pmod{\ell}$. Remark that the above $\overline{m_1 a_u e_1 + m_2 a_u e_2}$ and $\overline{s_1 e_1 + s_2 e_2}$ are in general different as affine lifts.

In addition, for `CodSq` and `CodOne`, we do not need to compute $\overline{m_1 e_1 + m_2 e_2}$ for all $0 \leq m_1, m_2 < \ell$. It is sufficient to compute them for half of $0 \leq m_1, m_2 < \ell$, since $m_1 e_1 + m_2 e_2 = (\ell - m_1)e_1 + (\ell - m_2)e_2$. For more detail, see Section 3.4.

We note that the inputs for our proposed algorithms are affine lifts such as $\overline{e_1}, \overline{e_2}, \overline{e_1 + e_2}$, while (10) and (11) require excellent lifts. Thus, we need to compute relations between affine lifts and excellent lifts. We will discuss it in Section 3.3.

3.2 Algorithms of arithmetic on Kummer surfaces

In this subsection, we describe explicit algorithms and their costs of calculation methods given in Section 2.2.

As well as Section 2.2, we use affine lifts $(\tilde{\theta}_i(x))_{i\in(\mathbb{Z}/2\mathbb{Z})^2}$ of level 2 theta coordinates, and algorithms based on Riemann relations of Definition 3, which take affine lifts as inputs and produce some affine lifts as outputs. Now, remark that we can choose and fix any affine lift of the theta-null point at the very beginning.

Notation. In our algorithms, in order to avoid calculating inverse elements on k, we often hold an element in k as a fraction, i.e., a pair of a numerator and a denominator. Then, if we hold an element $a \in k$ as a pair of $n \in k$ and $d \in k$ such that $a = \frac{n}{d}$, we write the data as (n, d) .

Moreover, we always hold any affine lift $\tilde{\theta}_{00}(x), \tilde{\theta}_{01}(x), \tilde{\theta}_{10}(x), \tilde{\theta}_{11}(x)$ as five elements $\theta'_{00}(x), \theta'_{01}(x)$, $\theta'_{10}(x)$, $\theta'_{11}(x)$, $d_x \in k$ such that $\tilde{\theta}_i(x) = \frac{\theta'_i(x)}{d_x}$ (with common denominator d_x) for all $i \in (\mathbb{Z}/2\mathbb{Z})^2$. In this case, we write the data as $(\theta'_i(x), d_x)_i$, and write $\theta'_i(x) = \texttt{Num}((\tilde{\theta}_i(x))_i, i)$ and $d_x = \texttt{Den}((\tilde{\theta}_i(x))_i)$. Since we can take any affine lift $(\tilde{\theta}_i(0))_i$ of the theta-null point, we select $(\tilde{\theta}_i(0))_i$ with denominator $d_0 = 1$. We omit the affine lift of the theta-null point from inputs for algorithms.

For counting the numbers of operations in the algorithms, we indicate a multiplication (resp. square) operation on the base field k by M (resp. S). Moreover, we indicate a multiplication (resp. square) operation computed only from the theta-null point by M_0 (resp. S_0). The values are reused after computed once. We do not count the numbers of addition on k and arithmetic operations on $\mathbb{Z}$. We note that since we hold some elements as a form of fraction, the results of our counting are not equal to the existing results such as [21] though the computation methods are similar.

Lemma 2. *For any integers $n, N \geq 1$, let $(b_{i,j}, a_i)$ for $0 \leq i \leq n-1$ and $0 \leq j \leq N-1$ be nN fractions in k. Then, the following statements hold:*

(i) We can reduce the fractions to

$$\texttt{Commondenom}\left((b_{i,j}, a_i)_{\substack{0\leq i\leq n-1\\0\leq j\leq N-1}}\right) := (b_{i,j}a_0\cdots a_{i-1}a_{i+1}\cdots a_{N-1}, \alpha)_{\substack{0\leq i\leq n-1\\0\leq j\leq N-1}}$$

with common denominator $\alpha = a_0 \cdots a_{N-1}$ in $C_{cd}(N, n) := ((n+3)N-5)\mathsf{M}$.

(ii) We can compute only the numerators of the result of (i):

$$\texttt{Projcommondenom}\left((b_{i,j}, a_i)_{\substack{0\leq i\leq n-1\\0\leq j\leq N-1}}\right) := (b_{i,j}a_0\cdots a_{i-1}a_{i+1}\cdots a_{N-1})_{\substack{0\leq i\leq n-1\\0\leq j\leq N-1}}$$

in $C_{pcd}(N, n) := ((n+3)N-6)\mathsf{M}$.

Proof. (i) For N elements $a_0, \dots, a_{N-1} \in k^*$, by Lemma 14 in Appendix B.1, we can compute N elements $a_0 \cdots a_{M-1} a_{M+1} \cdots a_{N-1}$ for $0 \le M \le N-1$ and a product $\alpha = a_0 \cdots a_{N-1}$ in $(3N-5)\mathsf{M}$. After that we multiply $b_{M,m}$ by numerators $a_0 \cdots a_{M-1} a_{M+1} \cdots a_{N-1}$ for $0 \le M \le N-1$ and $0 \le m \le n-1$ in $nN\mathsf{M}$. Thus, we can compute the fractions with a common denominator in $((n+3)N-5)\mathsf{M}$.

(ii) This is the same as (i) except for not computing α.

□

Costs of arithmetics on Kummer surfaces. Next, we evaluate costs of arithmetics on Kummer surfaces. First we give the cost to compute κ_{ii} defined by (5) as follows. First, we calculate z_0^χ using (4) for $i = 0$:

$$z_0^\chi = \frac{(\sum_{t \in (\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\tilde{\theta}_t(x)^2)(\sum_{t \in (\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\tilde{\theta}_t(y)^2)}{\sum_{t \in (\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\tilde{\theta}_t(0)^2} .$$

Then, we calculate κ_{ii} by using (5) for $i = j$, i.e., $\kappa_{ii} = \frac{1}{4} \sum_{\chi \in \widehat{(\mathbb{Z}/2\mathbb{Z})^2}} \chi(i) z_0^\chi$.

The next lemma is almost the same as [21, Lemma 5.1] except that we hold each affine lift as fractions with common denominator. From Algorithm 6 in Appendix B.2, we have the following number of arithmetic operations:

Lemma 3. *With the above notation, the following statements hold:*

(i) Computing κ_{ii} for all $i \in (\mathbb{Z}/2\mathbb{Z})^2$ requires $4\mathsf{S}_0 + 9\mathsf{S} + 17\mathsf{M}$.
(ii) When $x = y$, (i) reduces to $4\mathsf{S}_0 + 10\mathsf{S} + 12\mathsf{M}$.

Once we calculate $(\kappa_{ii})_i$ for some x_1 and y_1, we can reuse $\theta'_i(0)^2$ for other x_2 and y_2. Thus, we will count the $4\mathsf{S}_0$ only once.

The following lemma gives the number of arithmetic operations for Differential Addition and Doubling based on (7). From Algorithm 7 in Appendix B.2, we have the following number of arithmetic operations:

Lemma 4 (Differential Addition, Doubling). *For given any affine lifts $(\tilde{\theta}_i(x))_i$, $(\tilde{\theta}_i(y))_i$, $(\tilde{\theta}_i(x-y))_i$ with $\tilde{\theta}_i(x-y) \neq 0$ for all i, computing the affine lift $(\tilde{\theta}_i(x+y))_i$ requires $C_{dfa} := 4\mathsf{S}_0 + 9\mathsf{S} + 33\mathsf{M}$. When $x = y$, the cost reduces to $C_{dbl} := 4\mathsf{S}_0 + 10\mathsf{S} + 28\mathsf{M}$.*

Remark 6. As mentioned in Remark 2, we can compute $(\tilde{\theta}_i(x+y))_i$ if $\tilde{\theta}_i(x-y) = 0$ for some i. However, for simplicity, in our algorithms below, we always assume the condition $\tilde{\theta}_i(x-y) \neq 0$ for all i when we use Differential addition. Note that if $z \in A$ is 4-torsion point, this assumption $\tilde{\theta}_i(z) \neq 0$ often does not hold. Similarly, our implementation works only on this assumption. Unless we treat 4-torsion points, this assumption almost certainly holds experimentally.

Remark 7. After a calculation of $(\tilde{\theta}_i(x+y))_i$ once, the cost to calculate $(\tilde{\theta}_i(x+z))_i$ using Differential Addition reduces to $C_{rdfa} := 5\mathsf{S} + 33\mathsf{M}$ since we can reuse the data $\theta'_i(0)^2$, $\theta'_i(x)^2$.

The following lemma gives the number of arithmetic operations for Three-way Addition based on (8) and (9). From Algorithm 8 in Appendix B.2, we have the following number of arithmetic operations:

Lemma 5 (Three-way Addition). *For given affine lifts $(\tilde{\theta}_i(x))_i$, $(\tilde{\theta}_i(y))_i$, $(\tilde{\theta}_i(z))_i$, $(\tilde{\theta}_i(x+y))_i$, $(\tilde{\theta}_i(y+z))_i$, $(\tilde{\theta}_i(z+x))_i$ with $\tilde{\theta}_i(x) \neq 0, \tilde{\theta}_i(y) \neq 0, \tilde{\theta}_i(z) \neq 0$ for all i, computing $(\tilde{\theta}_i(x+y+z))_i$ using (8) and (9) requires* $48\mathsf{M}$.

The following lemma is used in isogeny calculations. Here, for any odd prime number ℓ and $\ell' := \frac{\ell-1}{2}$, we define a subset $H_\ell \subset \mathbb{Z}^2$ as

$$\begin{aligned} H_\ell :=& \{(m_1, 0) \in \mathbb{Z}^2 \mid 1 \leq m_1 \leq \ell'\} \sqcup \{(0, m_2) \in \mathbb{Z}^2 \mid 1 \leq m_2 \leq \ell'\} \\ &\sqcup \{(m_1, m_2) \in \mathbb{Z}^2 \mid 1 \leq m_1,\ 1 \leq m_2,\ m_1 + m_2 < \ell\} \\ &\sqcup \{(m_1, m_2) \in \mathbb{Z}^2 \mid \ell' < m_1 < \ell,\ m_1 + m_2 = \ell\}\ . \end{aligned} \tag{15}$$

If we define $\overline{H_\ell} := \{(\overline{m_1}, \overline{m_2}) \in (\mathbb{Z}/\ell\mathbb{Z})^2 \mid (m_1, m_2) \in H_\ell\}$, then for any $x \in (\mathbb{Z}/\ell\mathbb{Z})^2 \smallsetminus \{0\}$, we have $x \in \overline{H_\ell}$ if and only if $-x \notin \overline{H_\ell}$.

Lemma 6. *With the notation above, we have the following costs:*

(i) For given affine lifts $(\tilde{\theta}_i(e_1))_i, (\tilde{\theta}_i(e_2))_i$, and $(\tilde{\theta}_i(e_1 + e_2))_i$, computing all affine lifts $(\tilde{\theta}_i(m_1 e_1 + m_2 e_2))_i$ for $(m_1, m_2) \in H_\ell$ requires $C_{hlc}(\ell) := 2C_{dbl} + (\frac{\ell^2-1}{2} - 5)C_{rdfa}$ when $\ell \geq 5$. When $\ell = 3$, it requires once Differential Addition, thus, $C_{hlc}(3) := C_{dfa}$.

(ii) For given affine lifts $(\tilde{\theta}_i(e_1))_i, (\tilde{\theta}_i(e_2))_i, (\tilde{\theta}_i(e_1 + e_2))_i$, $(\tilde{\theta}_i(x))_i$, $(\tilde{\theta}_i(x + e_1))_i$, $(\tilde{\theta}_i(x + e_2))_i$, computing all affine lifts $(\tilde{\theta}_i(x + m_1 e_1 + m_2 e_2))_i$ for $0 \leq m_1, m_2 < \ell$ requires $C_{lc+}(\ell) := 48\mathsf{M} + 2C_{dfa} + (\ell^2 - 6)C_{rdfa}$ when $\ell \geq 3$.

Proof. For (i), since $\#H_\ell = \frac{\ell^2-1}{2}$ and we already have $(\tilde{\theta}_i(e_1))_i, (\tilde{\theta}_i(e_2))_i, (\tilde{\theta}_i(e_1 + e_2))_i$, the number of (m_1, m_2) not having $(\tilde{\theta}_i(m_1 e_1 + m_2 e_2))_i$ is $\frac{\ell^2-1}{2} - 3$. Among them, we can compute $(\tilde{\theta}_i(2e_1))_i$, $(\tilde{\theta}_i(2e_2))_i$ by Doubling. After that we reuse some values to compute other affine lifts, see Remark 7. For (ii), first we compute $(\tilde{\theta}_i(x + e_1 + e_2))_i$ by Three-way Addition in $48\mathsf{M}$. For remaining (m_1, m_2), it is similar to (i). □

In the above lemma, asymptotically, we have $C_{hlc}(\ell) = \frac{5}{2}\ell^2\mathsf{S} + \frac{33}{2}\ell^2\mathsf{M} + O(1)\mathsf{M}$ and $C_{lc+}(\ell) = 5\ell^2\mathsf{S} + 33\ell^2\mathsf{M} + O(1)\mathsf{M}$.

3.3 Normalization

In this subsection, as noted in Section 3.1, we give relations of affine lifts and excellent lifts.

First, we give a fundamental equality used later. This lemma is a generalization of [19, Lemma 3.10] and [20, Lemma 2].

Lemma 7. *Let $x_1, \ldots, x_r \in A$ and let $\overline{x_i}, \overline{x_i + x_j}$ be any affine lifts for $1 \leq i \leq r, 1 \leq i < j \leq r$. Let $\overline{\sum_{i=1}^{r} m_i x_i}$ for $m_i \in \mathbb{Z}$ be the affine lifts computed from $\overline{x_i}, \overline{x_i + x_j}$ by using computation of Section 2.2. In addition, we take any $\lambda_i, \lambda_{ij} \in k^*$, and we put $\tilde{x_i} := \lambda_i * \overline{x_i}$ and $\widetilde{x_i + x_j} := \lambda_{ij} * \overline{x_i + x_j}$. Let $\widetilde{\sum_{i=1}^{r} m_i x_i}$ for $m_i \in \mathbb{Z}$ be the affine lift computed from $\tilde{x_i}, \widetilde{x_i + x_j}$. Then, we have*

$$\widetilde{\sum_{i=1}^{r} m_i x_i} = \left(\left(\prod_{1 \leq i \leq r} \lambda_i^{m_i^2} \right) \cdot \left(\prod_{1 \leq i < j \leq r} \left(\frac{\lambda_{ij}}{\lambda_i \lambda_j} \right)^{m_i m_j} \right) \right) * \overline{\sum_{i=1}^{r} m_i x_i} \ .$$

Proof. We show the claim by induction for $r \geq 1$. The case of $r = 1$ is just Equation (17) of [19, Lemma 3.10]. Next, we consider the case of $r = 2$. The case of $m_2 = 1$ is just Equation (16) of [19, Lemma 3.10]. For a general integer m_2, we have

$$\begin{aligned}
\widetilde{m_1 x_1 + m_2 e_2} &= \texttt{Multadd}(m_2, \tilde{x_2}, \widetilde{m_1 x_1}, \widetilde{m_1 x_1 + x_2}) \\
&= \texttt{Multadd}\left(m_2, \lambda_2 * \overline{x_2}, \lambda_1^{m_1^2} * \overline{m_1 x_1}, \left(\lambda_1^{m_1^2} \lambda_2 \left(\frac{\lambda_{12}}{\lambda_1 \lambda_2} \right)^{m_1} \right) * \overline{m_1 x_1 + x_2}\right) \\
&= \left(\lambda_1^{m_1^2} \lambda_2^{m_2^2} \left(\frac{\lambda_{12}}{\lambda_1 \lambda_2} \right)^{m_1 m_2} \right) * \overline{m_1 x_1 + m_2 x_2} \ .
\end{aligned}$$

Thus, we obtained the result for $r = 2$. Next, we assume that the result holds for r. Here, $\widetilde{m_1 e_1 + \cdots + m_{r+1} e_{r+1}}$ is the result of Three-way Addition of $\widetilde{m_1 e_1 + \cdots + m_{r-1} e_{r-1}}$ and $\widetilde{m_r e_r}$ and $\widetilde{m_{r+1} e_{r+1}}$. $\overline{m_1 e_1 + \cdots + m_{r+1} e_{r+1}}$ is similar. Thus, from [20, Lemma 2] and the induction hypothesis, we obtain the result for $r + 1$. $\square$

Codomain. When we use the Lubicz-Robert formula, we need excellent lifts of the kernel. $(A, \mathscr{L}, \Theta_{\mathscr{L}})$ and $K \subset A[\ell]$ are thesame notations as earlier.

For $e \in K$, let $\overline{e}$ be any affine lift and $\tilde{e}$ be an excellent lift with $\tilde{e} = \lambda * \overline{e}$ for $\lambda \in k^*$. Then since $\texttt{Mult}(m, \tilde{e}) = \lambda^{m^2} * \texttt{Mult}(m, \overline{e})$ for $m \in \mathbb{Z}$ by Lemma 7, we have

$$\lambda^{\ell} = \frac{\texttt{Mult}(\ell', \overline{e})_i}{\texttt{Mult}(\ell' + 1, \overline{e})_i} \tag{16}$$

where $\ell' = \frac{\ell - 1}{2}$ for $i \in (\mathbb{Z}/2\mathbb{Z})^2$.

Let $\{e_1, e_2\}$ be a basis of $K \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$ and $\tilde{e_1}, \tilde{e_2}, \widetilde{e_1 + e_2}$ be excellent lifts. Then, the set $\tilde{K} = \{\widetilde{m_1 e_2 + m_2 e_2} \mid 0 \leq m_1, m_2 < \ell\}$ computed from $\tilde{e_1}, \tilde{e_2}, \widetilde{e_1 + e_2}$ is excellent by Theorem 2. In addition, for any affine lifts $\overline{e_1}, \overline{e_2}, \overline{e_1 + e_2}$, we write the affine lift of $m_1 e_2 + m_2 e_2$ computed from $\overline{e_1}, \overline{e_2}, \overline{e_1 + e_2}$ by $\overline{m_1 e_2 + m_2 e_2}$. If $\tilde{e_1} = \lambda_1 * \overline{e_1}, \tilde{e_2} = \lambda_2 * \overline{e_2}, \widetilde{e_1 + e_2} = \lambda_{12} * \overline{e_1 + e_2}$ for $\lambda_1, \lambda_2, \lambda_{12} \in k^*$, we have the following some relational expressions. Here, $\prod \tilde{e}$ means $(\prod \tilde{e_i})_i$.

Lemma 8. *The notation is the same as above. Let $a_1, \ldots, a_r$ be positive integers such that $\ell = \sum_{u=1}^{r} a_u^2$. Let m_1, m_2 be integers such that $0 \leq m_1, m_2 < \ell$. For each $1 \leq u \leq r$, we divide $a_u m_1$ and $a_u m_2$ by ℓ, i.e., $a_u m_1 = t_{1,u} \ell + s_{1,u}$ and*

$a_u m_2 = t_{2,u}\ell + s_{2,u}$ *where* $t_{1,u}, t_{2,u}, s_{1,u}, s_{2,u}$ *are integers with* $0 \le s_{1,u}, s_{2,u} < \ell$. *Then, we have*

$$\overline{m_1e_1} = \left(\lambda_1^{\ell}\right)^{\ell-2m_1} * \overline{(\ell-m_1)e_1}, \qquad \overline{m_2e_2} = \left(\lambda_2^{\ell}\right)^{\ell-2m_2} * \overline{(\ell-m_2)e_2}\ ,$$
$$\overline{m_1e_1+m_2e_2} = \left(\left(\lambda_1^{\ell}\right)^{\ell-2m_1}\left(\lambda_2^{\ell}\right)^{\ell-2m_2}\left(\frac{\lambda_{12}^{\ell}}{\lambda_1^{\ell}\lambda_2^{\ell}}\right)^{\ell-m_1-m_2}\right) * \overline{(\ell-m_1)e_1+(\ell-m_2)e_2}\ , \tag{17}$$

$$\prod_{u=1}^{r}(\widetilde{m_1a_ue_1+m_2a_ue_2}) = \left(\left(\lambda_1^{\ell}\right)^{h_1}\left(\lambda_2^{\ell}\right)^{h_2}\left(\frac{\lambda_{12}^{\ell}}{\lambda_1^{\ell}\lambda_2^{\ell}}\right)^{h_{12}}\right) * \prod_{u=1}^{r}\overline{s_{1,u}e_1+s_{2,u}e_2} \tag{18}$$

where

$$h_1 := m_1^2 + \ell\sum_{u=1}^{r} t_{1,u}^2 - 2m_1\sum_{u=1}^{r} a_u t_{1,u}, \quad h_2 := m_2^2 + \ell\sum_{u=1}^{r} t_{2,u}^2 - 2m_2\sum_{u=1}^{r} a_u t_{2,u},$$
$$h_{12} := m_1m_2 + \ell\sum_{u=1}^{r} t_{1,u}t_{2,u} - m_1\sum_{u=1}^{r} a_u t_{2,u} - m_2\sum_{u=1}^{r} a_u t_{1,u}$$

and they satisfy $0 \le h_1, h_2, h_{12} \le r(\ell-1)$. *In addition, we have*

$$(\widetilde{m_1e_1+m_2e_2})^{\ell} = \left(\left(\lambda_1^{\ell}\right)^{m_1^2}\left(\lambda_2^{\ell}\right)^{m_2^2}\left(\frac{\lambda_{12}^{\ell}}{\lambda_1^{\ell}\lambda_2^{\ell}}\right)^{m_1m_2}\right) * (\overline{m_1e_1+m_2e_2})^{\ell}\ . \tag{19}$$

Proof. By Lemma 7, we have

$$\widetilde{m_1e_1+m_2e_2} = \left(\lambda_1^{m_1^2}\lambda_2^{m_2^2}\left(\frac{\lambda_{12}}{\lambda_1\lambda_2}\right)^{m_1m_2}\right) * \overline{m_1e_1+m_2e_2}\ .$$

Then, by raising the both sides to the ℓ^{th} power, we have (19). Now, by the excellentness, we have $\widetilde{m_1e_1} = \widetilde{(\ell-m_1)e_1}$. In addition, by Lemma 7, we have $\widetilde{m_1e_1} = \lambda_1^{m_1^2} * \overline{m_1e_1}$ and $\widetilde{(\ell-m_1)e_1} = \lambda_1^{(\ell-m_1)^2} * \overline{(\ell-m_1)e_1}$. Hence, we have $\overline{m_1e_1} = \lambda_1^{\ell^2-2\ell m_1} * \overline{(\ell-m_1)e_1}$. Now, $\widetilde{m_2e_2}$ and $\widetilde{m_1e_1+m_2e_2}$ are similar, thus we have (17).

Similarly by the excellentness, we have $\widetilde{m_1a_ue_1+m_2a_ue_2} = \widetilde{s_{1,u}e_1+s_{2,u}e_2}$. Applying Lemma 7 to the right-hand side, we have

$$\widetilde{m_1a_ue_1+m_2a_ue_2} = \left(\lambda_1^{s_{1,u}^2}\lambda_2^{s_{2,u}^2}\left(\frac{\lambda_{12}}{\lambda_1\lambda_2}\right)^{s_{1,u}s_{2,u}}\right) * \overline{s_{1,u}e_1+s_{2,u}e_2}\ .$$

At last, taking the product for $1 \le u \le r$, we can show (18). Then, we have $h_1\ell = \sum_{u=1}^{r} s_{1,u}^2$ and $h_2\ell = \sum_{u=1}^{r} s_{2,u}^2$ and $h_{12}\ell = \sum_{u=1}^{r} s_{1,u}s_{2,u}$. Since $0 \le s_{1,u}, s_{2,u} \le \ell - 1$, we have $0 \le h_1, h_2, h_{12} \le r(\ell-1)$. □

Evaluation. Let $\tilde{K}$ be any excellent lift, $\tilde{x}$ be any affine lift of $x \in A$, and $\widetilde{x+e}$ be an excellent lift with respect to $\tilde{K}$ and $\tilde{x}$. For any affine lifts $\overline{e}$ and $\overline{x+e}$, we put $\tilde{e} = \lambda * \overline{e}$ and $\widetilde{x+e} = \mu * \overline{x+e}$ for $\lambda, \mu \in k^*$. Since $\texttt{Multadd}(\ell, \tilde{e}, \tilde{x}, \widetilde{x+e}) = (\lambda^{\ell^2} \cdot (\frac{\mu}{\lambda})^{\ell}) * \texttt{Multadd}(\ell, \overline{e}, \tilde{x}, \overline{x+e})$ by Lemma 7, we have

$$\left(\frac{\mu}{\lambda}\right)^{\ell} = \frac{\tilde{x}_i}{(\lambda^{\ell})^{\ell} \cdot \texttt{Multadd}(\ell, \overline{e}, \tilde{x}, \overline{x+e})_i} \tag{20}$$

for $i \in (\mathbb{Z}/2\mathbb{Z})^2$.

Lemma 9. *The notation is the same as above. Let $\lambda_1, \lambda_2, \lambda_{12}$ be the same as Lemma 8 and $\widetilde{x+e_1} = \mu_1 * \overline{x+e_1}$, $\widetilde{x+e_2} = \mu_2 * \overline{x+e_2}$ for $\mu_1, \mu_2 \in k^*$. Moreover, let $x + \widetilde{m_1 e_1 +} m_2 e_2$ be the set of affine lifts computed from $\tilde{e_1}$, $\tilde{e_2}$, $\widetilde{e_1+e_2}$, $\tilde{x}$, $\widetilde{x+e_1}$,$\widetilde{x+e_2}$. Similarly, let $\overline{x+m_1e_1+m_2e_2}$ be the set of affine lifts computed from $\overline{e_1}, \overline{e_2}, \overline{e_1+e_2}$, $\tilde{x}$, $\overline{x+e_1}$, $\overline{x+e_2}$. Then, we have the following two equalities:*

$$\begin{aligned}
&\prod_{u=1}^{r} (a_u x + m_1 \widetilde{a_u e_1} + m_2 a_u e_2) \\
&= \left(\left(\lambda_1^{\ell}\right)^{m_1^2} \left(\lambda_2^{\ell}\right)^{m_2^2} \left(\frac{\lambda_{12}^{\ell}}{\lambda_1^{\ell}\lambda_2^{\ell}}\right)^{m_1 m_2} \left(\frac{\mu_1^{\ell}}{\lambda_1^{\ell}}\right)^{m_1} \left(\frac{\mu_2^{\ell}}{\lambda_2^{\ell}}\right)^{m_2} \right) * \prod_{u=1}^{r} \overline{(a_u x + m_1 a_u e_1 + m_2 a_u e_2)} \ .
\end{aligned} \tag{21}$$

$$\begin{aligned}
&(x + m_1 \widetilde{e_1 +} m_2 e_2)^{\ell} \\
&= \left(\left(\lambda_1^{\ell}\right)^{m_1^2} \left(\lambda_2^{\ell}\right)^{m_2^2} \left(\frac{\lambda_{12}^{\ell}}{\lambda_1^{\ell}\lambda_2^{\ell}}\right)^{m_1 m_2} \left(\frac{\mu_1^{\ell}}{\lambda_1^{\ell}}\right)^{m_1} \left(\frac{\mu_2^{\ell}}{\lambda_2^{\ell}}\right)^{m_2} \right) * \overline{(x + m_1 e_1 + m_2 e_2)}^{\ell} \ .
\end{aligned} \tag{22}$$

Proof. (21) is obtained by Lemma 7. By applying $\ell = 1^2 + \cdots + 1^2$ to (21), we have (22). □

3.4 Explicit algorithms of the Lubicz-Robert formula

In this subsection, we propose explicit algorithms computing the theta-null point of the codomain B and computing the theta coordinate of the image of $x \in A$ under f based on the Lubicz-Robert formula (Theorem 3).

In the rest of this paper, logarithm always has base 2. Recall that M (resp. S) means the cost of a multiplication (resp. square) operation on k. In addition, $P(N)$ for any positive integer N means the cost of computing N^{th} power of an element in k. For integers $N_1, \ldots, N_m$ and $\lambda \in k$, $P(\{N_1, \ldots, N_m\})$ means the cost of computing N_i^{th} powers $\lambda^{N_1}, \ldots, \lambda^{N_m}$ all.

Codomain. Here, we calculate the theta-null point of the codomain using the Lubicz-Robert formula. As noted in Section 3.1, we give two algorithms $\texttt{CodSq}$, $\texttt{CodOne}$ for computation of the theta-null point of a codomain.

`CodSq` is based on (10) using $\ell = \sum_{u=1}^{r} a_u^2$ and `CodOne` is based on (12) using $\ell = 1^2 + \cdots + 1^2$.

On the other hand, in `CodSq`, we use equalities $m_1 a_u e_1 + m_2 a_u e_2 = s_{1,u} e_1 + s_{2,u} e_2$ if $m_1 a_u \equiv s_{1,u} \pmod{\ell}$ and $m_2 a_u \equiv s_{2,u} \pmod{\ell}$ as follows. The affine lift $\overline{s_1 e_1 + s_2 e_2}$ and $\overline{m_1 a_u e_1 + m_2 a_u e_2}$ correspond to the same projective coordinate but are not equal as affine lifts. Thus, by multiplying by an appropriate constant, we can compute $\overline{m_1 a_u e_1 + m_2 a_u e_2}$ from $\overline{s_{1,u} e_1 + s_{2,u} e_2}$. By this way, we avoid computing linear combinations many times.

We summarize these two options `CodSq` and `CodOne` in Table 2.

	Formula	Normalization	Algorithm	Complexity
`CodSq`	(10)	(17) and (18)	1	$O(\ell^2)$
`CodOne`	(12)	(19)	2	$O(\ell^2 \log(\ell))$

Table 2. Two calculation methods of the *codomain*

In any case, since (level 2) a projective theta coordinate of an element $e \in K$ is the same as one of the inverse element $-e$, we can reduce the complexity to half. To explain that, we use a subset $H_\ell \subseteq \mathbb{Z}^2$ of (15) in Section 3.2. Then, for a basis $\{e_1, e_2\}$ of K, we have $\{m_1 e_1 + m_2 e_2 \in K \mid (m_1, m_2) \in H_\ell\} \sqcup \{-(m_1 e_1 + m_2 e_2) \in K \mid (m_1, m_2) \in H_\ell\} = K \smallsetminus \{0\}$.

For `CodSq`, remark that affine lifts of $m_1 e_1 + m_2 e_2$ for $(m_1, m_2) \in H_\ell$ are not sufficient since it is not necessarily $(s_{1,u}, s_{2,u}) \in H_\ell$. Thus, we have to extend affine lifts $\overline{m_1 e_1 + m_2 e_2}$ for $(m_1, m_2) \in H_\ell$ to $0 \leq m_1, m_2 < \ell$ using (17) of Lemma 8. Especially, it is not clear whether `CodSq` is more efficient.

Explicit algorithms of `CodSq` and `CodOne` are Algorithm 1 and 2, respectively.

Remark 8. For `CodSq` and `CodOne`, since we compute $\overline{m_1 e_1 + m_2 e_2}$ for $(m_1, m_2) \in H_\ell$ first, we have $\overline{\ell' e_1}$ and $\overline{\ell' e_2}$ and $\overline{\ell' e_1 + \ell' e_2}$. Thus, when we compute $\lambda_1^\ell, \lambda_2^\ell, \lambda_{12}^\ell$ using (16), we only need $\overline{(\ell'+1) e_1}$ and $\overline{(\ell'+1) e_2}$ and $\overline{(\ell'+1) e_1 + (\ell'+1) e_2}$. They are computed from $\overline{m_1 e_1 + m_2 e_2}$ for $(m_1, m_2) \in H_\ell$ by Differential Addition. Moreover, we only need these the i^{th}-coordinate for one $i \in (\mathbb{Z}/2\mathbb{Z})^2$. Especially, we can compute $\overline{\ell' e_1}$, $\overline{\ell' e_2}$, $\overline{\ell' e_1 + \ell' e_2}$ in $O(1)\mathsf{M}$.

Here, we give complexities of `CodSq` and `CodOne`.

1. (Alg. 1, lines 3-4) When we calculate $\alpha^{m_1^2}$ for $0 \leq m_1 < \ell$, we compute individually. Thus, we can approximate $P(\{m_1^2 \mid 0 \leq m_1 < \ell\}) = O(\ell \log(\ell))\mathsf{M}$. The case for $P(\{m_2^2 \mid 0 \leq m_2 < \ell\})$ is similar.
2. (Alg. 1, lines 5-6) When we calculate $\beta^{m_1 m_2}$ for $(m_1, m_2) \in H_\ell$, since $\max\{m_1 m_2 \mid (m_1, m_2) \in H_\ell\} = \frac{\ell^2 - 1}{4}$, we calculate $\beta^2, \beta^3, \beta^4, \cdots, \beta^{\frac{\ell^2-1}{4}}$ straightforwardly. Thus, we approximate $P(\{m_1 m_2 \mid (m_1, m_2) \in H_\ell\}) = \frac{1}{4}\ell^2\mathsf{M}$. Similarly, we approximate $P(\{m_1^2 + m_2^2 + m_1 m_2 \mid (m_1, m_2) \in H_\ell\}) = \frac{3}{4}\ell^2\mathsf{M}$.

Lemma 10. *From Algorithms 1 and 2, these costs are as follows:*

Algorithm 1 `CodSq`

```
Input: Affine lifts $\overline{e_1}, \overline{e_2}, \overline{e_1+e_2}$ of the basis $\{e_1, e_2\}$ of the kernel.
Output: A projective theta-null point of the codomain.
 1: Compute $\overline{m_1e_1+m_2e_2}$ for $(m_1, m_2) \in H_\ell$.                                  $(\triangleright) C_{hlc}(\ell)$
 2: Compute $(\alpha_1, d_1)$ such that $\frac{\alpha_1}{d_1} = \lambda_1^\ell$ using (16).                     $(\triangleright) O(\ell)\mathsf{M}$
 3: Compute $(\alpha_2, d_2)$ such that $\frac{\alpha_2}{d_2} = \lambda_2^\ell$ using (16).                     $(\triangleright) O(\ell)\mathsf{M}$
 4: Compute $(\alpha_{12}, d_{12})$ such that $\frac{\alpha_{12}}{d_{12}} = \lambda_{12}^\ell$ using (16).               $(\triangleright) O(\ell)\mathsf{M}$
 5: $(\beta, d') := (\alpha_{12} \cdot d_1 \cdot d_2, d_{12} \cdot \alpha_1 \cdot \alpha_2)$ where $\frac{\beta}{d'} = \frac{\lambda_{12}^\ell}{\lambda_1^\ell \lambda_2^\ell}$.                     $(\triangleright) 4\mathsf{M}$
 6: $((\alpha_1, d), (\alpha_2, d), (\beta, d)) :=$ Commondenom$((\alpha_1, d_1), (\alpha_2, d_2), (\beta, d'))$.$(\triangleright) C_{cd}(3,1) = 7\mathsf{M}$
 7: Take a representation $\ell = \sum_{u=1}^r a_u^2$.
 8: Calculate $\alpha_1^s, \alpha_2^s, \beta^s, d^s$ for needed $s$ in lines 9, 13.                      $(\triangleright) 6r\ell\mathsf{M}$
 9: Extend $\overline{m_1e_1+m_2e_2}$ from $H_\ell$ to $0 \le m_1, m_2 < \ell$ using (17).     $(\triangleright)(3\ell^2 - 4\ell + 3)\mathsf{M}$
10: $(\theta_i'(f(0)), d_i)_i := (\prod_{u=1}^r$ Num$((\tilde{\theta}_i(0))_i, a_u i), 1)_i$ for $i \in (\mathbb{Z}/2\mathbb{Z})^2$.       $(\triangleright) 4(r-1)\mathsf{M}$
11: for $(m_1, m_2) \in H_\ell$ do
12:     $a_u m_1 = \ell t_{1,u} + s_{1,u}, a_u m_2 = \ell t_{2,u} + s_{2,u}$ for $1 \le u \le r$.
13:     $(c_n, c_d) := (\alpha_1^{h_1} \cdot \alpha_2^{h_2} \cdot \beta^{h_{12}}, d^{h_1+h_2+h_{12}})$ where $h_1, h_2, h_{12}$ are of (18).     $(\triangleright) 2\mathsf{M}$
14:     $t_d := c_d \cdot \prod_{u=1}^r$ Den$(\overline{s_{1,u}e_1 + s_{2,u}e_2})$.                           $(\triangleright) r\mathsf{M}$
15:     for $i \in (\mathbb{Z}/2\mathbb{Z})^2$ do
16:         $t_n := 2c_n \cdot \prod_{u=1}^r$ Num$(\overline{s_{1,u}e_1 + s_{2,u}e_2}, a_u i)$.                   $(\triangleright) r\mathsf{M}$
17:         $(\theta_i'(f(0)), d_i) := (\theta_i'(f(0)) \cdot t_d + d_i \cdot t_n, d_i \cdot t_d)$.                  $(\triangleright) 3\mathsf{M}$
18:     end for
19: end for
20: $(\theta_i'(f(0)))_{i\in(\mathbb{Z}/2\mathbb{Z})^2} :=$ Projcommondenom$((\theta_i'(f(0)), d_i)_{i\in(\mathbb{Z}/2\mathbb{Z})^2})$.
                                                                  $(\triangleright) C_{pcd}(4,1) = 10\mathsf{M}$
21: return $(\theta_i(f(0)), 1)_i$.
```

`CodSq`: $(\frac{5}{2}r + \frac{53}{2})\ell^2\mathsf{M} + \frac{5}{2}\ell^2\mathsf{S} + O(\ell)\mathsf{M}$.

`CodOne`: $\frac{49}{2}\ell^2\mathsf{M} + \frac{5}{2}\ell^2 P(\ell) + \frac{5}{2}\ell^2\mathsf{S} + O(\ell\log(\ell))\mathsf{M}$.

Especially, the complexity of `CodSq` is $O(\ell^2)\mathsf{M}$ and that of `CodOne` is $O(\ell^2\log(\ell))\mathsf{M}$ since $P(\ell) = O(\log(\ell))$. Concrete counts of operations for each ℓ are written in Section 4.

Evaluation. We give similar algorithms for general points, i.e., for $x \in A$, we compute the theta coordinate of $f(x) \in B$ from some theta coordinates of $e_1, e_2, e_1+e_2, x, x+e_1, x+e_2$.

Remark that we need to compute all linear combinations $x + m_1e_1 + m_2e_2$ for $0 \le m_1, m_2 < \ell$, not only for $(m_1, m_2) \in H_\ell$.

In the same notations as codomain, let $\overline{e_1}, \overline{e_2}$ and $\overline{e_1+e_2}$ be affine lifts of e_1, e_2 and e_1+e_2 for a basis $\{e_1, e_2\}$ of K. For given any affine lifts $\tilde{x}, \overline{x+e_1}$, and $\overline{x+e_2}$ of $x, x+e_1$, and $x+e_2$, we give a projective theta coordinate of the image $f(x)$.

Now, we give two concrete algorithms `EvalSq` and `EvalOne`. `EvalSq` is based on (11) and `EvalOne` is based on (13). We summarize in Table 3.

Algorithm 2 `CodOne`

```
Input: Affine lifts $\overline{e_1}, \overline{e_2}, \overline{e_1+e_2}$ of the basis $\{e_1, e_2\}$ of the kernel.
Output: A projective theta-null point of the codomain.
 1: Compute $\overline{m_1e_1+m_2e_2}$ for $(m_1, m_2) \in H_\ell$.    $(\triangleright) C_{hlc}(\ell)$
 2: Here is the same as lines 2 to 6 of Algorithm 1.
 3: Calculate $\alpha_1^{m_1^2}$ for $0 \le m_1 < \ell$    $(\triangleright) P(\{m_1^2 \mid 0 \le m_1 < \ell\})$
 4: Calculate $\alpha_2^{m_2^2}$ for $0 \le m_2 < \ell$    $(\triangleright) P(\{m_2^2 \mid 0 \le m_2 < \ell\})$
 5: Calculate $\beta^{m_1m_2}$ for $(m_1, m_2) \in H_\ell$    $(\triangleright) P(\{m_1m_2 \mid (m_1, m_2) \in H_\ell\})$
 6: Calculate $d^{m_1^2+m_2^2+m_1m_2}$ for $(m_1, m_2) \in H_\ell$.
                        $(\triangleright) P(\{m_1^2+m_2^2+m_1m_2 \mid (m_1, m_2) \in H_\ell\})$
 7: for $(m_1, m_2) \in H_\ell$ do
 8:     $(c_n, c_d) := (\alpha_1^{m_1^2} \cdot \alpha_2^{m_2^2} \cdot \beta^{m_1m_2}, d^{m_1^2+m_2^2+m_1m_2})$.    $(\triangleright)2\mathsf{M}$
 9:     Compute Num$(\overline{m_1e_1+m_2e_2}, i)^\ell$ for $i \in (\mathbb{Z}/2\mathbb{Z})^2$.    $(\triangleright)4P(\ell)$
10:     Compute Den$(\overline{m_1e_1+m_2e_2})^\ell$.    $(\triangleright)P(\ell)$
11:     Num$(Excl(m_1, m_2), i) := c_n \cdot$ Num$(\overline{m_1e_1+m_2e_2}, i)^\ell$ for $i \in (\mathbb{Z}/2\mathbb{Z})^2$.    $(\triangleright)4\mathsf{M}$
12:     Den$(Excl(m_1, m_2)) := c_d \cdot$ Den$(\overline{m_1e_1+m_2e_2})^\ell$.    $(\triangleright)1\mathsf{M}$
13: end for
14: Calculate Num$(Excl(0,0)) :=$ Num$(\theta_i(0), i)^\ell$ for $i \in (\mathbb{Z}/2\mathbb{Z})^2$.    $(\triangleright)4P(\ell)$
15: Calculate Den$(Excl(0,0)) := 1$.
16: $(Excl(m_1, m_2))_{(m_1,m_2)} :=$ Projcommondenom$((Excl(m_1, m_2))_{(m_1,m_2)})$
    w.r.t. $i \in (\mathbb{Z}/2\mathbb{Z})^2$ and $(m_1, m_2) \in H_\ell \sqcup \{(0,0)\}$. $(\triangleright) C_{pcd}(\frac{\ell^2+1}{2}, 4) = (\frac{7}{2}\ell^2 + O(1))\mathsf{M}$
17: $\theta'_i(f(0)) := Excl(m_1, m_2)_i$ for $i \in (\mathbb{Z}/2\mathbb{Z})^2$.
18: for $(m_1, m_2) \in H_\ell$ do
19:     for $i \in (\mathbb{Z}/2\mathbb{Z})^2$ do
20:         $\theta'_i(f(0)) := \theta'_i(f(0)) + 2Excl(m_1, m_2)_i$.
21:     end for
22: end for
23: return $(\theta'_i(f(0)), 1)_i$.
```

In advance, we calculate $(\lambda_1^\ell)^{m_1^2}(\lambda_2^\ell)^{m_2^2}(\frac{\lambda_{12}^\ell}{\lambda_1^\ell\lambda_2^\ell})^{m_1m_2}$ for all $0 \le m_1, m_2 < \ell$ and $\lambda_1^{\ell^2}, \lambda_2^{\ell^2}$ which are independent on x.

Their explicit algorithms of `EvalSq`,`EvalOne` are Algorithms 3, 4 respectively.

Remark 9. For `EvalOne`, we can use the similar optimization as Remark 8. Concretely, since we compute $\overline{x+m_1e_1+m_2e_2}$ for $0 \le m_1, m_2 < \ell$ first, we have $\overline{x+(\ell-1)e_1}$ and $\overline{x+(\ell-1)e_2}$. Hence, when we compute $(\frac{\mu_1}{\lambda_1})^\ell$ and $(\frac{\mu_2}{\lambda_2})^\ell$, we only need $\overline{x+\ell e_1}$ and $\overline{x+\ell e_2}$ which are computed from $\overline{x+m_1e_1+m_2e_2}$ by Differential Addition. Moreover, we only need the i^{th}-coordinate for one $i \in (\mathbb{Z}/2\mathbb{Z})^2$. Especially, we can compute $(\frac{\mu_1}{\lambda_1})^\ell$ and $(\frac{\mu_2}{\lambda_2})^\ell$ in $O(1)\mathsf{M}$. This is also valid for `EvalSq` if $a_u = 1$ for some u.

Lemma 11. *By Algorithms 3 and 4, we give concrete costs of* `EvalSq` *and* `EvalOne` *as follows. Here,* $r' := \#\{a_1, \ldots, a_r\} \le r$ *for a representation* $\ell = \sum_{u=1}^r a_u^2$.

`EvalSq`: $(5r + 33r' + 15)\ell^2\mathsf{M} + 5r'\ell^2\mathsf{S} + O(\ell)\mathsf{M}$.

	Formula	Normalization	Algorithm	Complexity
`EvalSq`	(11)	(21)	3	$O(\ell^2)$
`EvalOne`	(13)	(22)	4	$O(\ell^2 \log(\ell))$

Table 3. Two calculation methods of the *evaluation*

`EvalOne`: $51\ell^2\mathsf{M} + 5\ell^2 P(\ell) + 5\ell^2\mathsf{S} + O(\ell)\mathsf{M}$.

Especially, the complexity of `EvalSq` is $O(\ell^2)$ and that of `EvalOne` is $O(\ell^2 \log(\ell))$. Concrete counts of operations for each ℓ are written in Section 4.

3.5 Representation $\ell = \sum_{u=1}^{r} a_u^2$

When we use Algorithm `CodSq` or `EvalSq`, we take a representation of ℓ by the sum of squares of positive integers: $\ell = \sum_{u=1}^{r} a_u^2$. If $\ell = 3$, such a representation is only $3 = 1^2+1^2+1^2$. Otherwise, i.e. $\ell \geq 5$, what kind of representation is efficient for each algorithms? In the following, for $\ell \geq 5$, we except the representation $\ell = 1^2 + \cdots + 1^2$ since the case is just `CodOne` and `EvalOne`.

For `CodSq`, by the asymptotic complexity in Lemma 10, we should take a representation such that r is minimized. On the other hand, for `EvalSq`, by their asymptotic complexities in Lemmas 10, 11, we should take a representation such that $(5r + 33r')\mathsf{M} + 5r'\mathsf{S}$ is minimized where $r' := \#\{a_1, \ldots, a_r\} (\geq 2)$.

Lemma 12. *Let ℓ be a prime number such that $\ell \geq 5$. For each of `CodSq` and `EvalSq`, and for each ℓ, we should take a representation $\ell = \sum_{u=1}^{r} a_u^2$ (instead of $\ell = 1^2 + \cdots + 1^2$) as follows:*

1. *If $\ell \not\equiv -1 \pmod{24}$, the minimum value r_{min} of r for each ℓ is as follows. When $\ell \equiv 1 \pmod 4$, $r_{min} = 2$, when $\ell \equiv 3 \pmod 8$, $r_{min} = 3$, and when $\ell \equiv 7 \pmod{24}$, $r_{min} = 4$. Moreover, there exists a representation satisfying $r = r_{min}$ and $r' = 2$. Thus, for any `CodSq` and `EvalSq`, we should take any such representation with $r = r_{min}$ and $r' = 2$.*
2. *If $\ell \equiv -1 \pmod{24}$, the minimum value r_{min} of r is 4. Thus, for `CodSq`, we take a representation with $r = 4$. For `EvalSq`, for each $\ell < 200$, under the assumption $\mathsf{M} : \mathsf{S} = 3 : 2$, we should take a representation in Table 4 which minimizes $(5r + 33r')\mathsf{M} + 5r'\mathsf{S}$.*

Proof. 1. By Fermat's theorem on sums of two squares, there exists a representation such that $r = 2$, if and only if $\ell \equiv 1 \pmod 4$. In this case, clearly, $r' = 2$. Next, we consider the other case, $\ell \equiv 3 \pmod 4$. By Legendre's three-square theorem, there exists a representation such that $r = 3$, if and only if $\ell \equiv 3 \pmod 8$. In addition, in this case, it is known that there exists a representation such that $r = 3$ and $r' = 2$. When $\ell \equiv 7 \pmod 8$, by Lagrange's four-square theorem, there exists a representation such that $r = 4$. Then, there exists a representation such that $r = 4, r' = 2$ if and only if $\ell \equiv 1 \pmod 3$. This condition is equivalent to $\ell \equiv 7 \pmod{24}$. In any case, since $r' = 2$, r and $(5r + 33r')\mathsf{M} + 5r'\mathsf{S}$ are minimized.

Algorithm 3 `EvalSq`

```
Input: Affine lifts $\overline{e_1}, \overline{e_2}, \overline{e_1+e_2}$, $\tilde{x}, \overline{x+e_1}, \overline{x+e_2}$ and $(\lambda_1^\ell)^{m_1^2}(\lambda_2^\ell)^{m_2^2}(\frac{\lambda_{12}^\ell}{\lambda_1^\ell\lambda_2^\ell})^{m_1m_2}$.
Output: A projective theta coordinate of $f(x)$.
 1: Take a representation $\ell = \sum_{u=1}^r a_u^2$.
 2: Compute $\overline{a_ue_1}, \overline{a_ue_2}, \overline{a_u(e_1+e_2)}, \overline{a_ux}, \overline{a_u(e_1+x)}, \overline{a_u(e_2+x)}$ for $1 \le u \le r$.
                                                  $(\triangleright) 6C_{mlt}(\{a_u \mid 1 \le u \le r\})$
 3: Compute $\overline{a_ux + m_1a_ue_1 + m_2a_ue_2}$ for $0 \le m_1, m_2 < \ell$ and for $1 \le u \le r$.
    Here, $r' := \#\{a_1, \dots, a_r\}$.                          $(\triangleright) r'C_{lc+}(\ell)$
 4: Compute $(\gamma_1, d_1)$ such that $\frac{\gamma_1}{d_1} = (\frac{\mu_1}{\lambda_1})^\ell$ using (20).        $(\triangleright) O(\ell)\mathsf{M}$
 5: Compute $(\gamma_2, d_2)$ such that $\frac{\gamma_2}{d_2} = (\frac{\mu_2}{\lambda_2})^\ell$ using (20).        $(\triangleright) O(\ell)\mathsf{M}$
 6: $((\gamma_1, d), (\gamma_2, d)) :=$ Commondenom$((\gamma_1, d_1), (\gamma_2, d_2))$.   $(\triangleright) C_{cd}(2,1) = 3\mathsf{M}$
 7: Calculate numerators of $\gamma_1^m, \gamma_2^m$ for $0 \le m \le \ell - 1$.         $(\triangleright) 2(\ell-2)\mathsf{M}$
 8: Calculate $d^m$ for $0 \le m \le 2(\ell-1)$.                        $(\triangleright) (2\ell-3)\mathsf{M}$
 9: Take a representation $\ell = \sum_{u=1}^r a_u^2$.
10: $(\theta_i'(f(x)), d_i) := (0, 1)$ for $i \in (\mathbb{Z}/2\mathbb{Z})^2$.
11: for $0 \le m_1, m_2 < \ell$ do
12:     $c_n :=$ Num$((\lambda_1^\ell)^{m_1^2}(\lambda_2^\ell)^{m_2^2}(\frac{\lambda_{12}^\ell}{\lambda_1^\ell\lambda_2^\ell})^{m_1m_2}) \cdot \gamma_1^{m_1} \cdot \gamma_2^{m_2}$      $(\triangleright) 2\mathsf{M}$
13:     $c_d :=$ Den$((\lambda_1^\ell)^{m_1^2}(\lambda_2^\ell)^{m_2^2}(\frac{\lambda_{12}^\ell}{\lambda_1^\ell\lambda_2^\ell})^{m_1m_2}) \cdot d^{m_1+m_2}$.      $(\triangleright) 1\mathsf{M}$
14:     $t_d := c_d \cdot \prod_{u=1}^r$ Den$(\overline{a_ux + m_1a_ue_1 + m_2a_ue_2})$.          $(\triangleright) r\mathsf{M}$
15:     for $i \in (\mathbb{Z}/2\mathbb{Z})^2$ do
16:         $t_n := c_n \cdot \prod_{u=1}^r$ Num$(\overline{a_ux + m_1a_ue_1 + m_2a_ue_2}, a_ui)$.     $(\triangleright) r\mathsf{M}$
17:         $(\theta_i'(f(x)), d_i) := (\theta_i'(f(x)) \cdot t_d + d_i \cdot t_n, d_i \cdot t_d)$.       $(\triangleright) 3\mathsf{M}$
18:     end for
19: end for
20: $(\theta_i'(f(x)))_{i \in (\mathbb{Z}/2\mathbb{Z})^2} :=$ Projcommondenom$((\theta_i'(f(x)), d_i)_{i \in (\mathbb{Z}/2\mathbb{Z})^2})$.
                                                  $(\triangleright) C_{pcd}(4,1) = 10\mathsf{M}$
21: return $(\theta_i'(f(x)), 1)_i$.
```

2. By Lagrange's four-square theorem, we have $r_{min} = 4$. Now, $(5r + 33r')\mathsf{M} + 5r'\mathsf{S}$ is minimized if and only if $(5r+33r') \cdot 3 + 5r' \cdot 2 = 15r + 109r'$ is minimized. For each $\ell < 200$, by comparing $15r + 109r'$ for all representations, we have the result of Table 4. □

Table 5 summarises Lemma 12.

4 Counting the Number of Operations

In this section, we count the number of operation on k of algorithms `CodSq`, `CodOne`, `EvalSq`, and `EvalOne` of Section 3.4. Here, we consider that the base field is $\mathbb{F}_{p^2}$ for the sake of application to isogeny-based cryptography. Remark that characteristic p does not affect the number of operation. Here, we consider the cost as $\mathsf{M} : \mathsf{S} = 3 : 2$ and thus we compare an integer $3m + 2s$ for $m\mathsf{M} + s\mathsf{S}$. Table 6 shows the values of $3m + 2s$ for each algorithm and for each ℓ. The underlined values in red font are the minimum ones for each ℓ.

ℓ	$\sum_{u=1}^{r} a_r^2$	r	r'
23	$5 \cdot 1^2 + 2 \cdot 3^2$	7	2
47	$2 \cdot 1^2 + 5 \cdot 3^2$	5	2
71	$2^2 + 2 \cdot 3^2 + 7^2$	4	3
167	$5 \cdot 1^2 + 2 \cdot 9^2$	7	2
191	$4^2 + 7 \cdot 5^2$	8	2

Table 4. The most efficient representation for ℓ such that $\ell \equiv -1 \pmod{24}$ for `EvalSq`.

ℓ (≥ 5)	r	r'	example
$\ell \equiv 1 \pmod 4$	2	2	$5 = 1^2 + 2^2$
$\ell \equiv 3 \pmod 8$	3	2	$11 = 1^2 + 1^2 + 3^2$
$\ell \equiv 7 \pmod{24}$	4	2	$7 = 1^2 + 1^2 + 1^2 + 2^2$
$\ell \equiv -1 \pmod{24}$	$r \geq 4$	–	See Table 4

Table 5. An efficient representation $\ell = \sum_{u=1}^{r} a_u^2$ where $r' = \#\{a_1, \ldots, a_r\}$.

ℓ	`CodSq`	`CodOne`	`EvalSq`	`EvalOne`
	$O(\ell^2)$	$O(\ell^2 \log \ell)$	$O(\ell^2)$	$O(\ell^2 \log \ell)$
3	1071	771	2118	1823
5	2711	2452	8270	5164
7	6740	5282	18034	10619
11	14924	13876	41573	27053
13	18579	19466	54745	37749
17	31829	32740	91496	63336
19	44376	43676	121861	83431
23	71692	68158	228845	128574
29	92627	108580	265217	204366
31	129896	131354	347335	245053
37	150807	173612	428398	327161
41	185105	213334	526691	401713
43	226500	248600	623287	464048
47	297904	313726	952613	580903
53	309335	378028	878633	704968
59	426020	494794	1169783	915391
61	409683	528968	1161502	978503
67	549240	593390	1509055	1108635
71	678412	704272	2363562	1305458
73	586713	704588	1663126	1316091
79	839528	919016	2244289	1691121
83	842600	962854	2311751	1784042
89	871961	1107298	2469014	2051306
97	1035741	1244930	2932321	2323755
101	1122983	1426324	3176858	2641778
103	1426148	1563008	3812926	2874753
107	1399436	1686874	3840095	3102377
109	1307811	1750586	3701596	3219441
113	1405469	1785742	3976406	3306842
127	2166776	2618972	5796664	4757679
131	2096960	2357542	5754983	4375653
137	2065769	2578540	5844425	4785669
139	2360796	2799332	6479773	5158276
149	2443367	3216862	6910958	5927196
151	3061844	3474890	8190025	6361003
157	2712747	3756698	7672825	6876571
163	3245712	3850136	8903431	7093396
167	3293819	4250704	12022895	7780491
173	3744484	4561804	9315983	8349627
179	3913640	4883890	10739111	8938851
181	3605295	4993700	10194670	9139723
191	4896904	6108136	16601459	11053113
193	4099125	5119244	11591572	9497845
197	4270919	5624758	12075206	10361340
199	5315588	6036608	14222314	11048011

Table 6. Values of $3m + 2s$ where $m\mathsf{M} + s\mathsf{S}$ is the count of operations of (ℓ, ℓ)-isogeny

Algorithm 4 `EvalOne`

```
Input: Affine lifts $\overline{e_1}, \overline{e_2}, \overline{e_1+e_2}$, $\tilde{x}, \overline{x+e_1}, \overline{x+e_2}$ and $(\lambda_1^\ell)^{m_1^2}(\lambda_2^\ell)^{m_2^2}(\frac{\lambda_{12}^\ell}{\lambda_1^\ell\lambda_2^\ell})^{m_1m_2}$ and
    $(\theta_i(x))_i$.
Output: A projective theta coordinate of $f(x)$.
 1: Compute $\overline{x+m_1e_1+m_2e_2}$ for $0 \le m_1, m_2 < \ell$.                    $(\triangleright)C_{lc+}(\ell)$
 2: Here is the same as lines 4 to 8 of Algorithm 3.
 3: for $0 \le m_1, m_2 < \ell$ do
 4:     $c_n :=$ Num$((\lambda_1^\ell)^{m_1^2}(\lambda_2^\ell)^{m_2^2}(\frac{\lambda_{12}^\ell}{\lambda_1^\ell\lambda_2^\ell})^{m_1m_2}) \cdot \gamma_1^{m_1} \cdot \gamma_2^{m_2}$                $(\triangleright)$2M
 5:     $c_d :=$ Den$((\lambda_1^\ell)^{m_1^2}(\lambda_2^\ell)^{m_2^2}(\frac{\lambda_{12}^\ell}{\lambda_1^\ell\lambda_2^\ell})^{m_1m_2}) \cdot d^{m_1+m_2}$.                 $(\triangleright)$1M
 6:     Compute Num$(\overline{x+m_1e_1+m_2e_2}, i)^\ell$ for $i \in (\mathbb{Z}/2\mathbb{Z})^2$.          $(\triangleright)4P(\ell)$
 7:     Compute Den$(\overline{x+m_1e_1+m_2e_2})^\ell$.                                $(\triangleright)P(\ell)$
 8:     Num$(Excl(m_1,m_2), i) := c_n \cdot$ Num$(\overline{x+m_1e_1+m_2e_2}, i)^\ell$ for $i \in (\mathbb{Z}/2\mathbb{Z})^2$.   $(\triangleright)$4M
 9:     Den$(Excl(m_1,m_2), i) := c_d \cdot$ Den$(\overline{x+m_1e_1+m_2e_2}, i)^\ell$.                $(\triangleright)$4M
10: end for
11: $(Excl(m_1,m_2))_{(m_1,m_2)} :=$ Projcommondenom$((Excl(m_1,m_2))_{(m_1,m_2)})$
    w.r.t. $i \in (\mathbb{Z}/2\mathbb{Z})^2$ and $0 \le m_1, m_2 < \ell$.          $(\triangleright)C_{pcd}(\ell^2, 4) = (7\ell^2 + O(1))$M
12: $\theta'_i(f(x)) := 0$ for $i \in (\mathbb{Z}/2\mathbb{Z})^2$.
13: for $0 \le m_1, m_2 < \ell$ do
14:     for $i \in (\mathbb{Z}/2\mathbb{Z})^2$ do
15:         $\theta'_i(f(x)) := \theta'_i(f(x)) + Excl(m_1,m_2)_i$.
16:     end for
17: end for
18: return $(\theta'_i(f(x)), 1)_i$.
```

Codomain. For $3 \le \ell \le 11$ and $\ell = 19, 23$, `CodOne` is the most efficient, and for $\ell = 13, 17$ and $\ell \ge 29$, `CodSq` is the most efficient. Indeed, the asymptotic complexity of `CodOne` is $O(\ell^2 \log(\ell))$, but that of `CodSq` is $O(\ell^2)$.

The cost of `CodSq` depends on r which is determined by $\ell \pmod 8$.

The cost of `CodOne` depends on the Hamming weight of ℓ, since we calculate ℓ^{th} power many times in the algorithm. In fact, for example, the cost of `CodOne` is large when $\ell = 127 = (1111111)_2$ and $\ell = 191 = (10111111)_2$.

Evaluation. For $3 \le \ell < 200$, `EvalOne` is more efficient, even though the asymptotic complexity of `EvalOne` is $O(\ell^2 \log(\ell))$ and that of `EvalSq` is $O(\ell^2)$. For sufficiently large ℓ, `EvalSq` would be more efficient. The smallest ℓ that `EvalSq` is more efficient is 509.

As well as codomain, the cost of `EvalSq` depends on r and r' which are determined by $\ell \pmod{24}$. Concretely, see Table 5 and Table 4. The cost of `EvalOne` depends on the Hamming weight of ℓ.

5 Application to Attack on B-SIDH

In this section, we implement SIDH attack on key exchange protocol B-SIDH. In the attack, we calculate (ℓ, ℓ)-isogenies between Kummer surfaces. Then, we will use the result of Section 3 and Section 4.

5.1 SIDH (B-SIDH) attacks

In this subsection, we explain SIDH (B-SIDH) attacks briefly.

B-SIDH is key exchange protocol given by Costello [9] which is based on the same problem as SIDH [15], but by using quadratic twist of elliptic curve we can use smaller characteristic than one of SIDH.

The security of both SIDH and B-SIDH is guaranteed by the hardness of *Supersingular Isogney with Torsion Problem* below. Here, p is a prime number and k is a finite field of characteristic p.

Problem 1 (Supersingular Isogney with Torsion). Let N_A and N_B be coprime integers, E_0/k and E_B/k be elliptic curves, $\varphi_B : E_0 \to E_B$ be N_B-isogeny, and $\{P_A, Q_A\}$ be a basis of $E_0[N_A]$.

Then, given $N_A, N_B, E_0, E_B, P_A, Q_A, \varphi_B(P_A), \varphi_B(P_B)$, construct φ_B.

SIDH attacks. However, in 2022, Castryck, Decru [5] and Maino, Martindale, Panny, Pope, Wesolowski [23] and Robert [31] gave a polynomial-time attack on SIDH by solving the above problem. Thus, as noted in [5], the security of B-SIDH was also broken.

In the attack, the following lemma based on a criterion by Kani [16] is essential. Here, we consider the case of dimension one, even though it holds for a general dimension, see [31, Lemma 3.4].

Lemma 13 ([31, Lemma 3.4]). *Let E, E_1, E_2, and E' be elliptic curves. For coprime d_1, d_2, let f_1, g_1 be d_1-isogenies and f_2, g_2 be d_2-isogenies such that the following diagram is commutative:*

$$\begin{tikzcd} E \arrow[r, "f_1"] \arrow[d, "f_2"'] & E_1 \arrow[d, "g_2"] \\ E_2 \arrow[r, "g_1"'] & E' \end{tikzcd}$$

Then, an isogeny $F : E \times E' \to E_1 \times E_2$ defined by a matrix $\begin{pmatrix} f_1 & \hat{g_2} \\ -f_2 & \hat{g_1} \end{pmatrix}$ is (d, d)-isogeny where $d := d_1 + d_2$ with respect to the natural product polarizations on $E \times E'$ and $E_1 \times E_2$. In addition, the kernel of F is represented by

$$\operatorname{Ker} F = \{(\hat{f}_1(P), g_2(P)) \in E \times E' \mid P \in E_1[d]\} \ .$$

Now, we construct an attack on SIDH for the following case: $p \equiv 3 \pmod 4$ and E_0 is a supersingular elliptic curve $E_0/\mathbb{F}_{p^2} : y^2 = x^3 + x$. In addition, we can assume $N_A > N_B$, if necessary by changing Alice and Bob of SIDH protocol. Then, for $a := N_A - N_B$, as given in Section 5.2 below, there exists a way to construct $\alpha(P_A), \alpha(Q_A)$ for some a-isogeny $\alpha : E_0 \to E'$ by using the information of $\mathrm{End}(E_0)$, Then, we have the left-hand side diagram below by taking the pushout of φ_B and α. Hence, we have the right-hand side commutative diagram below:

$$\begin{tikzcd} E_0 \arrow[r, "\varphi_B"] \arrow[d, "\alpha"'] & E_B \arrow[d, "\alpha'"] \\ E' \arrow[r, "\varphi'_B"'] & E'_B. \end{tikzcd} \qquad \begin{tikzcd} E' \arrow[r, "\hat{\alpha}"] \arrow[d, "\varphi'_B"'] & E_0 \arrow[d, "\varphi_B"] \\ E'_B \arrow[r, "\hat{\alpha'}"'] & E_B. \end{tikzcd}$$

We apply Lemma 13 to the above right-hand side diagram, namely, let $F : E' \times E_B \to E_0 \times E'_B$ be the (N_A, N_A)-isogeny given by a matrix $\begin{pmatrix} \hat{\alpha} & \hat{\varphi_B} \\ -\varphi'_B & \alpha' \end{pmatrix}$. Then, we have

$$\mathrm{Ker}\, F = \{(\alpha(P), \varphi_B(P)) \in E' \times E_B \mid P \in E_0[N_A]\} \ .$$

Since the attacker has $(\alpha(P_A), \varphi_B(P_A)), (\alpha(Q_A), \varphi_B(Q_A))$ which generate $\mathrm{Ker}\, F$, the attacker can calculate F. Then, the attacker takes a basis $\{S_1, S_2\}$ of $E_B[N_B]$ and computes $F((0, S_i)) = (\hat{\varphi_B}(S_i), \alpha'(S_i))$ for $i = 1, 2$. Since $\mathrm{Ker}\, \varphi_B = \langle \hat{\varphi_B}(S_1), \hat{\varphi_B}(S_2) \rangle$, the attacker gets the generator of $\mathrm{Ker}\, \varphi_B$.

Difference between attacks on SIDH and attacks on B-SIDH. One of the differences between SIDH and B-SIDH is the number of prime factors of N_A and N_B. In SIDH, N_A and N_B are the form of 2^a or 3^b, on the other hand, in B-SIDH, N_A and N_B have a lot of prime factors. At the point of attacks, since attackers need to compute to (N_A, N_A)-isogeny F, for SIDH they compute the composition of $(2, 2)$-isogenies or $(3, 3)$-isogenies. On the other hand, for B-SIDH they compute the composition high digree isogenies, i.e., if $N_A = \ell_1 \cdots \ell_m$ is the prime factorization, they compute the composition of (ℓ_i, ℓ_i)-isogenies. In fact, although an implementation of attack on SIDH is given by Castryck-Decru [5, Section 9], for B-SIDH it is not given.

5.2 Concrete construction of attack on B-SIDH

Computation of images of some a-isogeny. The notation is the same as Section 5.1, i.e., p is a prime number such that $p \equiv 3 \pmod 4$, E_0 is a supersingular elliptic curve $E_0/\mathbb{F}_{p^2} : y^2 = x^3 + x$, $N_A > N_B$ are coprime integers, and $\{P_A, Q_A\}$ is a basis of $E_0[N_A]$. As noted in the previous section, we can construct $\alpha(P_A), \alpha(Q_A)$ for some a-isogeny $\alpha : E_0 \to E'$ where $a := N_A - N_B$. Here, we give the construction.

We use some theory about quaternion algebra and refer to [13, Section 2]. The endomorphism ring $\mathrm{End}(E_0)$ is isomorphic to the maximal order $\mathcal{O}_0 = \langle 1, i, \frac{i+j}{2}, \frac{1+k}{2} \rangle$ with $i^2 = -1, j^2 = -p, k = ij$ of $H(-1,-p)$. Concretely, we have the isomorpshim by $\iota \mapsto i$ and $\pi \mapsto j$ where $\iota : E_0 \to E_0$ is $(x,y) \mapsto (-x, \sqrt{-1}y)$ and $\pi : E_0 \to E_0$ is $(x,y) \mapsto (x^p, y^p)$. In addition, we use $\texttt{FullRepresentInteger}_{\mathcal{O}_0}(M)$ of [14, Algorithm 1] which gives an element of $\mathcal{O}_0$ of norm M for a given integer $M > p$.

First, by applying $\texttt{FullRepresentInteger}_{\mathcal{O}_0}(aN_B)$, we obtain an (aN_B)-isogeny $\gamma : E_0 \to E_0$. Then, we decompose γ to an a-isogeny $\alpha : E_0 \to E'$ and an N_B-isogeny $\delta : E' \to E_0$ with $\delta \circ \alpha = \gamma$. Then, since $\hat{\delta} \circ \delta = [N_B]_{E'}$, the left-hand side diagram below is commutative. Since $[N_B]_{E'} \circ \alpha = \alpha \circ [N_B]_{E_0}$, we have the right-hand side commutative diagram. Here, since $\gcd(a, N_B) = 1$ and we have $\mathrm{Ker}\,\hat{\delta} = \gamma(E_0[N_B])$, we can calculate $\hat{\delta} : E_0 \to E'$. Then, by the right-hand side commutative diagram, we have $\alpha(P_A) = \hat{\delta}(\gamma(\frac{P_A}{N_B}))$ and $\alpha(Q_A) = \hat{\delta}(\gamma(\frac{Q_A}{N_B}))$.

$$\begin{tikzcd}
E_0 \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & E' \arrow[d, "{[N_B]_{E'}}"] \arrow[dl, "\delta"'] \\
E_0 \arrow[r, "\hat{\delta}"'] & E'
\end{tikzcd}
\qquad
\begin{tikzcd}
E_0 \arrow[r, "{[N_B]_{E_0}}"] \arrow[d, "\gamma"'] & E_0 \arrow[d, "\alpha"] \\
E_0 \arrow[r, "\hat{\delta}"'] & E'
\end{tikzcd}$$

Composition of isogeny. As we have just seen, when we attack on B-SIDH, we calculate isogeny of high degree. Thus, we decompose the isogeny to prime-degree isogenies. We generalize the situation slightly.

Let $K \subset A[N]$ be a maximal isotropic subgroup and $F : A \to B$ be the (N,N)-isogeny. When $N = \ell_1 \cdots \ell_m$ is the prime factorization, we have a decomposition $F = \varphi_m \circ \cdots \circ \varphi_1$ where $\varphi_i : A_i \to A_{i+1}$ is an (ℓ_i, ℓ_i)-isogeny with $A_1 = A$ and $A_{m+1} = B$.

For a basis $\{f_1, f_2\}$ of $K \simeq (\mathbb{Z}/N\mathbb{Z})^2$, from theta coordinates $\overline{f_1}, \overline{f_2}$, we will calculate a theta-null point of B. To do this, first we calculate $\overline{f_1 + f_2}$ by Normal Addition. Then, we multiply $\overline{f_1}, \overline{f_2}, \overline{f_1 + f_2}$ by $\texttt{Mult}(\ell_2 \cdots \ell_m, *)$ and call them $\overline{e_1}, \overline{e_2}, \overline{e_1 + e_2}$. Since $\overline{e_1}, \overline{e_2}$ are affine lifts of a basis of $\mathrm{Ker}\,\varphi_1$, we can calculate theta null point of A_2. Then, we compute affine lifts $\texttt{Mult}(\ell_2 \cdots \ell_m + 1, f_1)$ of $f_1 + e_1$ and $\texttt{Multadd}(\ell_2 \cdots \ell_m, f_2, f_1, f_1 + f_2)$ of $f_1 + e_2$. From them we calculate a theta coordinate of the image $\varphi_1(f_1) \in A_2$. Similarly, we calculate one of $\varphi_2(f_2) \in A_2$. By iterating this calculation m times, we have a theta-null point of B. Thus, the total cost is $5mC_{mlt}$ and m times Normal Addition and m times codomain calculation and $2m$ times evaluations. Remark that since we use Normal Addition we need once a calculating square root for one step.

For $x \in A$, from an affine lift $\tilde{x}$, we will calculate a theta coordinate of $F(x)$. We calculate $\overline{x + e_1}$ by Normal Addition. Then, we calculate $\overline{x + e_2}$ by Compatible Addition. Thus, we can calculate theta coordinate of the image of $\varphi_1(x) \in A_2$. In one step, we need three times calculations of square roots.

Remark 10. For applying this argument to the attack on B-SIDH, since the domain A is a product of elliptic curves, A does not satisfy the assumption of non-zeroness of even theta-null points. Thus, on A we prepare needed affine lifts by using additions of elliptic curves. Since for $A_2, \dots, A_m$ the probability that each A_i is a product of elliptic curves is $O(\frac{10}{p})$, we consider that does not happen for sufficiently large p such as the parameter of B-SIDH.

5.3 Implementation of the attack

We implemented the attack on B-SIDH for the following parameter based on [13, Appendix.C]:

$$p = \texttt{0x1E409D8D53CF3BEB65B5F41FB53B25EBEAF37761CD8BA996684150A40FFFFFFFF},$$

$$N_A = 3^{56} \cdot 31 \cdot 43 \cdot 59 \cdot 271 \cdot 311 \cdot 353 \cdot 461 \cdot 593 \cdot 607 \cdot 647 \cdot 691 \cdot 743 \cdot 769 \cdot 877 \cdot 1549,$$

$$N_B = 2^{32} \cdot 5^{21} \cdot 7 \cdot 11 \cdot 163 \cdot 1181 \cdot 2389 \cdot 5233 \cdot 8353 \cdot 10139 \cdot 11939 \cdot 22003 \cdot 25391 \cdot 41843.$$

Here, p is 257-bit and N_A is 216-bit and N_B is 213-bit. In addition, $p \equiv 3 \pmod 4$, $N_A \mid (p-1)$, $N_B \mid (p+1)$, and $N_A > N_B$. We used this parameter since (2,2)-isogeny is not main point of this paper and $2 \nmid N_A$.

As we have just seen in Section 5.1, we calculate the image of two points for (N_A, N_A)-isogeny $F : E' \times E_B \to E_0 \times E'_B$.

We implemented this attack using our algorithms in Computer algebra system SageMath [34]. Then, we done the attack in about 40500 seconds (11.25 hours) on an Apple M1 3200MHz CPU. The implementation can be found in

https://github.com/Yoshizumi-Ryo/ellell-isogeny_sage.

6 Conclusion

In this paper, we gave explicit inversion-free algorithms of (ℓ, ℓ)-isogeny between Kummer surfaces based on the Lubicz-Robert formula for an odd prime number ℓ.

Specifically, we proposed two algorithms using two representations $\ell = \sum_{u=1}^{r} a_u^2$ with $r = O(1)$ and $\ell = 1^2 + \cdots + 1^2$ for codomain and evaluation each. Then, we made several improvements. First, for codomain, we reduced the complexity of computing affine lifts to half. Second, for representations $\ell = \sum_{u=1}^{r} a_u^2$, we determined the most efficient representation for each ℓ. Third, we constructed relations to compute excellent lifts from affine lifts using in the Lubicz-Robert formula. Then, we provided some improvements based on the relations. Fourth, in our algorithms, we avoided computing multiplicative inversions, which are expensive for cryptrographic situations. Finally, by counting and comparing the number of arithmetic operations, we determined the most efficient algorithm for each ℓ from each of two algorithms.

In addition, using the most efficient one, we implemented the SIDH attack on B-SIDH in SageMath. In setting that originally claimed 128-bit security, we were able to recover 128-bit secure B-SIDH in about 11 hours.

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A Additional arithmetic on Kummer surfaces of Section 2.2

As mentioned in Remark 1 in Section 2.2, in this section, we introduce Normal Addition and Compatible Addition. They are used attack on B-SIDH.

Normal Addition. Next, we consider the case that $(\tilde{\theta}_i(x-y))_i$ is not given as a part of input. Then, remark that we cannot distinguish $(\tilde{\theta}_i(x+y))_i$ from $(\tilde{\theta}_i(x-y))_i$ by using $(\tilde{\theta}_i(x))_i, (\tilde{\theta}_i(y))_i$ only, since $(\tilde{\theta}_i(y))_i = (\tilde{\theta}_i(-y))_i$. Nonetheless, we can compute a set (unordered pair) $\{(X_i)_i, (Y_i)_i\} := \{(\tilde{\theta}_i(x+y))_i, (\tilde{\theta}_i(x-y))_i\}$ as follows. First, remark that for any pair of $(\tilde{\theta}_i(x+y))_i$ and $(\tilde{\theta}_i(x-y))_i$ which satisfies (3) and for any $\lambda \in \mathbb{C}^*$, a pair of $\lambda * (\tilde{\theta}_i(x+y))_i$ and $\frac{1}{\lambda} * (\tilde{\theta}_i(x-y))_i$ also satisfies (3). Thus, we may fix $X_0 := 1$, and then we have $Y_0 = \kappa_{00}$ by (7). Here, we assume that $\kappa_{00} \neq 0$. If necessary, we replace by another i with $\kappa_{ii} \neq 0$.

In the above notation, equality (6) becomes $\kappa_{ij} = \frac{1}{2}(X_iY_j + X_jY_i)$. Thus, for all $i \in (\mathbb{Z}/2\mathbb{Z})^2$, we have $\frac{X_i}{X_0} \cdot \frac{Y_i}{Y_0} = \frac{\kappa_{ii}}{\kappa_{00}}$ and $\frac{X_i}{X_0} + \frac{Y_i}{Y_0} = \frac{2\kappa_{i0}}{\kappa_{00}}$. Hence, $\frac{X_i}{X_0}$ and $\frac{Y_i}{Y_0}$ are solutions of the following quadratic equation:

$$\kappa_{00}t^2 - 2\kappa_{i0}t + \kappa_{ii} = 0 \ . \tag{23}$$

If x or y is a 2-torsion point, then we have $x - y \in \{\pm(x+y)\}$ and hence $(\theta_i(x+y))_i = (\theta_i(x-y))_i$ as projective theta coordinates, therefore $\frac{X_i}{X_0} = \frac{Y_i}{Y_0} = \frac{\kappa_{i0}}{\kappa_{00}}$. Thus, in this case, we can compute the set $\{(X_i)_i, (Y_i)_i\}$ from κ_{ij}.

Otherwise, we have $x - y \notin \{\pm(x+y)\}$ and hence $(X_i)_i \neq (Y_i)_i$ as projective theta coordinates. Therefore, there exists $\alpha \in (\mathbb{Z}/2\mathbb{Z})^2$ such that $\frac{X_\alpha}{X_0} \neq \frac{Y_\alpha}{Y_0}$. Then the quadratic equation (23) with $i = \alpha$ has two distinct solutions; among them, we can set $\frac{X_\alpha}{X_0} = \frac{\kappa_{\alpha 0} + \sqrt{D_\alpha}}{\kappa_{00}}$ by symmetry where $D_\alpha := \kappa_{\alpha 0}^2 - \kappa_{\alpha\alpha}\kappa_{00}$ (note that now $D_\alpha \neq 0$). Since we fixed $X_0 = 1$, we have

$$X_\alpha = \frac{\kappa_{\alpha 0} + \sqrt{D_\alpha}}{\kappa_{00}} \ .$$

Moreover, for the remaining $i \in (\mathbb{Z}/2\mathbb{Z})^2 \setminus \{0, \alpha\}$, we have the following linear equation:

$$\begin{pmatrix} 1 & 1 \\ \frac{X_\alpha}{X_0} & \frac{Y_\alpha}{Y_0} \end{pmatrix} \begin{pmatrix} \frac{Y_i}{Y_0} \\ \frac{X_i}{X_0} \end{pmatrix} = \begin{pmatrix} \frac{2\kappa_{i0}}{\kappa_{00}} \\ \frac{2\kappa_{i\alpha}}{\kappa_{00}} \end{pmatrix} \ .$$

Since $\det \begin{pmatrix} 1 & 1 \\ \frac{X_\alpha}{X_0} & \frac{Y_\alpha}{Y_0} \end{pmatrix} = \frac{-2\sqrt{D_\alpha}}{\kappa_{00}} \neq 0$, by solving the above linear equation, we can calculate X_i as follows:

$$X_i = \frac{X_\alpha \kappa_{i0} - \kappa_{i\alpha}}{\sqrt{D_\alpha}} \ .$$

Moreover, since $\kappa_{ii} = X_iY_i$, we have $Y_i = \frac{\kappa_{ii}}{X_i}$ if $X_i \neq 0$. Even if $X_i = 0$, we can compute Y_i in the same way as we compute X_i.

Thus, from affine lifts $(\tilde{\theta}_i(x))_i, (\tilde{\theta}_i(y))_i$, we have obtained the set $\{(\tilde{\theta}_i(x+y))_i, (\tilde{\theta}_i(x-y))_i\}$. We call this algorithm *Normal Addition* (cf. [21, Section 5.2]). Remark that this operation requires one square root computation.

Compatible Addition. For given $(\tilde{\theta}_i(y))_i, (\tilde{\theta}_i(z))_i, (\tilde{\theta}_i(x+y))_i, (\tilde{\theta}_i(x+z))_i$, we can compute $(\tilde{\theta}_i(y+z))_i$ as follows. If y or z is a 2-torsion point, since $(\tilde{\theta}_i(y+z))_i = (\tilde{\theta}_i(y-z))_i$ as projective theta coordinates, it suffices to compute the Normal Addition of $(\tilde{\theta}_i(y))_i$ and $(\tilde{\theta}_i(z))_i$. Otherwise, we can compute $(\tilde{\theta}_i(y+z))_i$ by using Normal Addition twice as follows. Firstly, we calculate the set $\{Y, Z\} := \{(\tilde{\theta}_i(y+z))_i, (\tilde{\theta}_i(y-z))_i\}$ from $(\tilde{\theta}_i(y))_i, (\tilde{\theta}_i(z))_i$ using Normal Addition. Then, we compute the set S of Normal Addition of Y and $(\tilde{\theta}_i(x+y))_i$. If $Y = (\tilde{\theta}_i(y+z))_i$, we have $S = \{(\tilde{\theta}_i(x+2y+z))_i, (\tilde{\theta}_i(x-z))_i\}$. Thus, in this case, $(\tilde{\theta}_i(x+z))_i$ is not contained in S since neither y nor z is a 2-torsion point. On the other hand, if $Y = (\tilde{\theta}_i(y-z))_i$, we have $S = \{(\tilde{\theta}_i(x+2y-z))_i, (\tilde{\theta}_i(x+z))_i\}$. Thus, if S contains the projective theta coordinate $(\tilde{\theta}_i(x+z))_i$, we have $Z = (\tilde{\theta}_i(y+z))_i$. Otherwise, we have $Y = (\tilde{\theta}_i(y+z))_i$. We call this algorithm *Compatible Addition* (cf. [19, Section 3.2.1]).

B Explicit algorithms of Section 3.2

In this appendix, as mentioned in Section 3.2, we give some concrete algorithms of arithmetic on Kummer surfaces.

B.1 Batch inversion

First, in order to unify the denominators of some given fractions (Lemma 2), we give an algorithm to compute some products from given elements of k (Algorithm 5) and an evaluation of its cost (Lemma 14). For any integer $M \geq 0$, we write the binary expansion as $M = (d_{n-1}, \dots, d_0)_2$ where $M = \sum_{i=0}^{n-1} d_i 2^i$ for $d_i \in \{0, 1\}$. Here, we do *not* require $d_{n-1} = 1$.

Lemma 14. *Let $N \geq 2$ and $a_0, \dots, a_{N-1} \in k$. Then the output of Algorithm 5 satisfies that $\widetilde{\alpha} = \alpha := a_0 \cdots a_{N-1}$ and $\widetilde{\alpha}_M = \alpha_M := a_0 \cdots a_{M-1} a_{M+1} \cdots a_{N-1}$ for any $M = 0, \dots, N-1$, and Algorithm 5 requires $(3N-5)\mathsf{M}$. If the part α of the output is not needed, then the cost reduces to $(3N-6)\mathsf{M}$.*

Proof. As for line 2 in the algorithm, let L denote the set of all leaves of the binary tree T, and for each node v of T, let $L(v)$ denote the set of all $v' \in L$ that is covered by v, i.e., the upward path from v' to the root of T involves the node v. Then we have $\alpha = \prod_{v \in L} \mathsf{a}[v]$ and $\alpha_M = \prod_{v \in L \setminus \{v[n;M]\}} \mathsf{a}[v]$ for any $M \in \{0, \dots, N-1\}$. Now a recursive argument implies that $\mathsf{a}[v] = \prod_{w \in L(v)} \mathsf{a}[w]$ for any node v of T; this follows from the fact that for each non-leaf node v, $L(v)$ is the disjoint union of $L(v'_1)$ and $L(v'_2)$ if v has two child nodes v'_1 and v'_2, and $L(v) = L(v')$ otherwise where v' is the unique child node of v. The former case occurs $N-1$ times in total by the argument of "counting losers in knockout tournament", therefore $N-1$ multiplications on k are performed during this process. Finally, for the root $v[0;0]$ of T, we have

$$\widetilde{\alpha} = \mathsf{a}[v[0;0]] = \prod_{w \in L(v[0;0])} \mathsf{a}[w] = \prod_{w \in L} \mathsf{a}[w] = \alpha$$

since $L(v[0;0]) = L$. Hence the part $\widetilde{\alpha}$ of the output is correct.

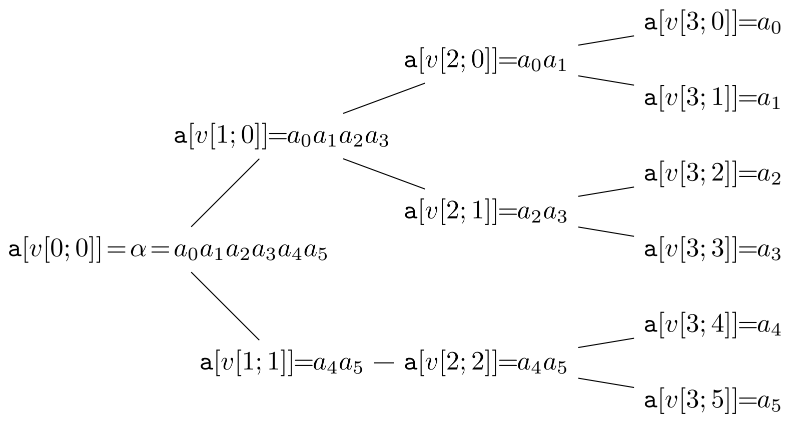
Secondly, a recursive argument also implies that $\mathtt{b}[v] = \prod_{w \in L \setminus L(v)} \mathtt{a}[w]$ for any node v of T. Indeed, this follows from the fact that for each non-leaf node v, if v has two child nodes v'_1 and v'_2, then $L \setminus L(v'_1)$ is the disjoint union of $L \setminus L(v)$ and $L(v'_2)$ and vice versa; while if v has a single child node v', then $L \setminus L(v') = L \setminus L(v)$. (Note that the relations $\mathtt{b}[v[1;0]] = \mathtt{b}[v[0;0]] \cdot \mathtt{a}[v[1;1]]$ and $\mathtt{b}[v[1;1]] = \mathtt{b}[v[0;0]] \cdot \mathtt{a}[v[1;0]]$ also hold for nodes at level 1.) Multiplication on k occurs only when a non-leaf node v is of the former type (except for the case of the root $v = v[0;0]$); now two multiplications on k are performed in the calculation at the two child nodes. By the argument at the previous paragraph, there are $(N-1) - 1 = N-2$ such nodes v, hence there are $2(N-2)$ multiplications in total. Finally, for each leaf $v[n;M]$ with $0 \le M \le N-1$, we have

$$\begin{aligned}\widetilde{\alpha}_M = \mathtt{b}[v[n;M]] = \prod_{w \in L \setminus L(v[n;M])} \mathtt{a}[w] &= \prod_{w \in L \setminus \{v[n;M]\}} \mathtt{a}[w] \\ &= a_0 \cdots a_{M-1} a_{M+1} \cdots a_{N-1} = \alpha_M\end{aligned}$$

since $L(v[n;M]) = \{v[n;M]\}$. Hence the part $\widetilde{\alpha}_M$ of the output is correct.

The total number of multiplication is $(N-1) + 2(N-2) = 3N-5$. Now if the part α of the output is not needed, then the calculation of $\mathtt{a}[v[0;0]]$ in the algorithm can be removed, decreasing the number of multiplications by one; i.e., $3N-6$ multiplications in total. $\square$

Example 1. We give an example of Lemma 14 in the case of $N = 6$. Then, we use the following two binary trees:


$\mathtt{a}[v[0;0]] = \alpha = a_0a_1a_2a_3a_4a_5$
$\mathtt{a}[v[1;0]] = a_0a_1a_2a_3$
$\mathtt{a}[v[1;1]] = a_4a_5$
$\mathtt{a}[v[2;0]] = a_0a_1$
$\mathtt{a}[v[2;1]] = a_2a_3$
$\mathtt{a}[v[2;2]] = a_4a_5$
$\mathtt{a}[v[3;0]] = a_0$
$\mathtt{a}[v[3;1]] = a_1$
$\mathtt{a}[v[3;2]] = a_2$
$\mathtt{a}[v[3;3]] = a_3$
$\mathtt{a}[v[3;4]] = a_4$
$\mathtt{a}[v[3;5]] = a_5$

$$\begin{tikzcd}
 & & & \mathtt{b}[v[3;0]]{=}a_1a_2a_3a_4a_5 \\
 & & \mathtt{b}[v[2;0]]{=}a_2a_3a_4a_5 \arrow[ur, -] \arrow[dr, -] & \\
 & & & \mathtt{b}[v[3;1]]{=}a_0a_2a_3a_4a_5 \\
 & \mathtt{b}[v[1;0]]{=}a_4a_5 \arrow[uur, -] \arrow[ddr, -] & & \\
 & & & \mathtt{b}[v[3;2]]{=}a_0a_1a_3a_4a_5 \\
 & & \mathtt{b}[v[2;1]]{=}a_0a_1a_4a_5 \arrow[ur, -] \arrow[dr, -] & \\
\mathtt{b}[v[0;0]] = 1 \arrow[uuur, -] \arrow[ddr, -] & & & \mathtt{b}[v[3;3]]{=}a_0a_1a_2a_4a_5 \\
 & & & \mathtt{b}[v[3;4]]{=}a_0a_1a_2a_3a_5 \\
 & \mathtt{b}[v[1;1]]{=}a_0a_1a_2a_3 \arrow[r, -] & \mathtt{b}[v[2;2]]{=}a_0a_1a_2a_3 \arrow[ur, -] \arrow[dr, -] & \\
 & & & \mathtt{b}[v[3;5]]{=}a_0a_1a_2a_3a_4
\end{tikzcd}$$

B.2 Concrete algorithms

Here, we write concrete algorithms used in Lemma 3, Lemma 4, and Lemma 5, see the following table:

Content	Lemma	Algorithm
Computing κ_{ii}	Lemma 3	Algorithm 6
Differential Addition, Doubling	Lemma 4	Algorithm 7
Three-way Addition	Lemma 5	Algorithm 8

Table 7. Correspondence between lemmas and algorithms

Algorithm 5 Algorithm to compute some products

```
Input: $N$ elements $a_0, \dots, a_{N-1} \in k$ ($N \geq 2$)
Output: $N$ products $\alpha_M := a_0 \cdots a_{M-1} a_{M+1} \cdots a_{N-1}$ for $0 \leq M \leq N-1$ and a
    product $\alpha := a_0 \cdots a_{N-1}$
 1: Write $N - 1 = (d_{n-1}, \dots, d_0)_2$ where $d_{n-1} = 1$    # possible since $N \geq 2$
 2: Generate a binary tree $T$ with 0-th level (root) to $n$-th level (leaves), where
    - for $\ell = 0, \dots, n$, the $\ell$-th level consists of nodes $v[\ell; 0], v[\ell; 1], \dots, v[\ell; N'_\ell]$ where
      $N'_\ell = \lfloor (N-1)/2^{n-\ell} \rfloor = (d_{n-1}, \dots, d_{n-\ell})_2$ (now $N'_0 = 0$);
    - for $\ell = 0, \dots, n-1$, node $v[\ell; c]$ has child node(s) $v[\ell+1; 2c]$ and $v[\ell+1; 2c+1]$
      (if it exists); we call $v[\ell+1; 2c+1]$ *the sibling node* of $v[\ell+1; 2c]$ and vice versa
 3: # Multiply from leaves to the root
 4: for each leaf $v := v[n; c]$ do
 5:     $\mathtt{a}[v] := a_c$
 6: end for
 7: for $\ell = n-1$ downto 0 do
 8:     for each node $v := v[\ell; c]$ do
 9:         if $v$ has two child nodes $v'_1$ and $v'_2$ then
10:             $\mathtt{a}[v] := \mathtt{a}[v'_1] \cdot \mathtt{a}[v'_2]$                                  (▷) 1M
11:         else
12:             $\mathtt{a}[v] := \mathtt{a}[v']$ for the unique child node $v'$ of $v$
13:         end if
14:     end for
15: end for
16: # Multiply from root to leaves
17: $\mathtt{b}[v[0; 0]] := 1 \in k$    # $v[0; 0]$ is the root of $T$
18: $\mathtt{b}[v[1; 0]] := \mathtt{a}[v[1; 1]]$ and $\mathtt{b}[v[1; 1]] := \mathtt{a}[v[1; 0]]$    # $N'_1 = 1$ since $N \geq 2$
19: for $\ell = 2$ to $n$ do
20:     for each node $v := v[\ell; c]$ (with its parent node $\hat{v}$) do
21:         if $v$ has the sibling node $v'$ then
22:             $\mathtt{b}[v] := \mathtt{b}[\hat{v}] \cdot \mathtt{a}[v']$                                  (▷) 1M
23:         else
24:             $\mathtt{b}[v] := \mathtt{b}[\hat{v}]$
25:         end if
26:     end for
27: end for
28: return $\tilde{\alpha}_M := \mathtt{b}[v[n; M]]$ for $M = 0, \dots, N-1$ (as $\alpha_M$) and $\tilde{\alpha} := \mathtt{a}[v[0; 0]]$ (as $\alpha$)
                                                                         (▷) total: $(3N-5)$M
```

Algorithm 6 Algorithm to calculate κ_{ii} in Lemma 3 (i) (resp. (ii))

```
Input: Affine lifts $(\theta'_i(x), d_x)_i$, $(\theta'_i(y), d_y)_i$.
Output: $\kappa_{ii}$ for all $i \in (\mathbb{Z}/2\mathbb{Z})^2$.
1: Calculate $\theta'_i(0)^2, \theta'_i(x)^2, \theta'_i(y)^2$ for all $i \in (\mathbb{Z}/2\mathbb{Z})^2$.     $(\triangleright)4\mathsf{S}_0 + 8\mathsf{S}$ (resp. $4\mathsf{S}_0 + 4\mathsf{S}$)
2: for $\chi \in \widehat{(\mathbb{Z}/2\mathbb{Z})^2}$ (4 elements in total) do
3:     $z'^{\chi}_0 := (\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\theta'_t(x)^2) \cdot (\sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\theta'_t(y)^2)$.     $(\triangleright)1\mathsf{M}$ (resp. $1\mathsf{S}$)
4:     $d_\chi := \sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\theta'_t(0)^2$.
5: end for
6: $(z'^{\chi}_0, d)_{\chi\in\widehat{(\mathbb{Z}/2\mathbb{Z})^2}} :=$ Commondenom$((z'^{\chi}_0, d_\chi)_{\chi\in\widehat{(\mathbb{Z}/2\mathbb{Z})^2}})$.     $(\triangleright)C_{cd}(4,1) = 11\mathsf{M}$
7: $\kappa'_{ii} := \sum_{\chi\in\widehat{(\mathbb{Z}/2\mathbb{Z})^2}} \chi(i)z'^{\chi}_0$ for $i \in (\mathbb{Z}/2\mathbb{Z})^2$.
8: $d_\kappa := 4d \cdot (d_x \cdot d_y)^2$.     $(\triangleright)1\mathsf{S} + 2\mathsf{M}$ (resp. $2\mathsf{S} + 1\mathsf{M}$)
9: return $(\kappa'_{ii}, d_\kappa)_i$.     $(\triangleright)$ total : $4\mathsf{S}_0 + 9\mathsf{S} + 17\mathsf{M}$ (resp. $4\mathsf{S}_0 + 10\mathsf{S} + 12\mathsf{M}$)
```

Algorithm 7 Differential Addition (resp. Doubling)

```
Input: Affine lifts $(\theta'_i(x), d_x)_i$, $(\theta'_i(y), d_y)_i$, and $(\theta'_i(x-y), d_{x-y})_i$.
Output: The affine lift $(\tilde{\theta}_i(x+y))_i$.
1: Compute $(\kappa'_{ii}, d_\kappa)_i$ for $i \in (\mathbb{Z}/2\mathbb{Z})^2$ by Algorithm 6.
                                   $(\triangleright)4\mathsf{S}_0 + 9\mathsf{S} + 17\mathsf{M}$ (resp. $4\mathsf{S}_0 + 10\mathsf{S} + 12\mathsf{M}$)
2: $(\theta'_i(x+y), d_{x+y})_{i\in(\mathbb{Z}/2\mathbb{Z})^2} :=$ Commondenom$((\kappa'_{ii}, \theta_i(x-y))_{i\in(\mathbb{Z}/2\mathbb{Z})^2})$.
                                   $(\triangleright)C_{cd}(4,1) = 11\mathsf{M}$
3: $\theta'_i(x+y) := d_{x-y} \cdot \theta'_i(x+y)$ for all $i \in (\mathbb{Z}/2\mathbb{Z})^2$.     $(\triangleright)4\mathsf{M}$
4: $d_{x+y} := d_\kappa \cdot d_{x+y}$.     $(\triangleright)1\mathsf{M}$
5: return $(\theta'_i(x+y), d_{x+y})_i$.     $(\triangleright)$total: $4\mathsf{S}_0 + 9\mathsf{S} + 33\mathsf{M}$ (resp. $4\mathsf{S}_0 + 10\mathsf{S} + 28\mathsf{M}$)
```

Algorithm 8 Three-way Addition

```
Input: Affine lifts $(\theta'_i(x), d_x)_i$, $(\theta'_i(y), d_y)_i$, $(\theta'_i(z), d_z)_i$, $(\theta'_i(x+y), d_{x+y})_i$,
    $(\theta'_i(y+z), d_{y+z})_i$, and $(\theta'_i(z+x), d_{z+x})_i$.
Output: The affine lift $(\tilde{\theta}_i(x+y+z))_i$.
 1: $R_1^\chi := \sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\theta'_t(0)\theta'_t(y+z)$ for all $\chi \in \widehat{(\mathbb{Z}/2\mathbb{Z})^2}$.    (▷)4M
 2: $R_2^\chi := \sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\theta'_t(z+x)\theta'_t(x+y)$ for all $\chi \in \widehat{(\mathbb{Z}/2\mathbb{Z})^2}$.    (▷)4M
 3: $L_2^\chi := \sum_{t\in(\mathbb{Z}/2\mathbb{Z})^2} \chi(t)\theta'_t(y)\theta'_t(z)$ for all $\chi \in \widehat{(\mathbb{Z}/2\mathbb{Z})^2}$.    (▷)4M
 4: for $\chi \in \widehat{(\mathbb{Z}/2\mathbb{Z})^2}$ do
 5:     $E'^\chi := R_1^\chi \cdot R_2^\chi$.    (▷)1M
 6:     $d_\chi := L_2^\chi$.
 7: end for
 8: $(\tilde{E}^\chi, d)_{\chi\in\widehat{(\mathbb{Z}/2\mathbb{Z})^2}} :=$ Commondenom$((E'^\chi, d_\chi)_{\chi\in\widehat{(\mathbb{Z}/2\mathbb{Z})^2}})$.    (▷)$C_{cd}(4,1) = 11$M
 9: for $i \in (\mathbb{Z}/2\mathbb{Z})^2$ do
10:     $\theta'_i(x+y+z) := \sum_\chi \chi(i)\tilde{E}^\chi$.
11:     $d_i := 4\theta'_i(x)$.
12: end for
13: $(\theta'_i(x+y+z), d_{x+y+z})_{i\in(\mathbb{Z}/2\mathbb{Z})^2} :=$ Commondenom$((\theta'_i(x+y+z), d_i)_{i\in(\mathbb{Z}/2\mathbb{Z})^2})$.
                                                                  (▷)$C_{cd}(4,1) = 11$M
14: $d_{x+y+z} := d_{x+y+z} \cdot d_{y+z} \cdot d_{z+x} \cdot d_{x+y} \cdot d$.    (▷)4M
15: $d_{xyz} := d_x \cdot d_y \cdot d_z$.    (▷)2M
16: Calculate $\theta'_i(x+y+z) := \theta'_i(x+y+z) \cdot d_{xyz}$ for all $i \in (\mathbb{Z}/2\mathbb{Z})^2$.    (▷)4M
17: return $(\theta'_i(x+y+z), d_{x+y+z})_i$.    (▷) total: 48M
```