# Optimizing Big Integer Multiplication on Bitcoin: Introducing *w*-windowed Approach

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**Abstract.** A crucial component of any zero-knowledge system is operations with finite fields. This, in turn, leads to the implementation of the fundamental operation: multiplying two big integers. In the realm of Bitcoin, this problem gets revisited, as Bitcoin utilizes its own stack-based and not Turing-complete scripting system called Bitcoin Script. Inspired by Elliptic Curve scalar multiplication, this paper introduces the *w*-windowed method for multiplying two numbers. We outperform state-of-the-art approaches, including BitVM's implementation. Finally, we also show how the windowed method can lead to optimizations not only in big integer arithmetic solely but in more general arithmetic problems.

**Keywords:** Bitcoin, Bitcoin Script, Fast Multiplication, Elliptic Curve Scalar Multiplication, BitVM

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## 1 Introduction

Introduced in 2009, Bitcoin has drastically changed the world of digital finance and led to the broad adoption of blockchain technology [Nak09]. Being the first cryptocurrency,

it put into action numerous novel concepts such as decentralization, digital security, and programmable conditions for operating with digital currency [Mun23, BCEM15]. However, by design, Bitcoin Smart Contract functionality is very limited. Essentially, one can only perform verifications on basic primitives such as ECDSA/Schnorr signatures, hashlocks, or timelocks. Despite such a limited set of tools, the Bitcoin community managed to come up with a multitude of exciting and complex protocols such as Atomic Swaps [TMMS21, Her18], Anonymized Taprootized Swaps [KZL<sup>+</sup>24], Lightning Network [PD16, SGNB20], RGB protocol [OTZ<sup>+</sup>23], LRC-20 [Mia24] etc.

Despite all the community's efforts, one of the most anticipated technologies yet to be fully developed is the L2 zero-knowledge (zk) rollup on top of Bitcoin. Currently, the adoption of L2 zk-rollups allows achieving much better scalability on Ethereum [W<sup>+</sup>14], resulting in lower fees and a higher number of transactions per second while maintaining the same security as in the L1 layer (that is, Ethereum blockchain itself). This is achieved through zero-knowledge technology, which allows for the formation of succinct validity proof for verification. One of the most widely used L2 zk-rollups are Aztec, Scroll, Polygon zkEVM, zkSync, Starknet, etc [CRTA<sup>+</sup>24]. The majority of them (besides Starknet), one way or another, rely on the Succinct Non-Interactive Argument of Knowledge (SNARK), allowing building proofs of certain statements with the size logarithmic in respect to the statement size<sup>1</sup> [CLKL23].

While there are many endeavors to achieve a similar SNARK-based zk-rollup on Bitcoin, currently, to the best of our knowledge, there is yet to be a production-ready system on top of Bitcoin Mainnet. For the most part, as we mentioned, the primary reason is the limitation of Bitcoin Script. In spite of all the limitations, there is significant progress in writing the full zero-knowledge SNARK verifier in Bitcoin Script. One of notable examples include BitVM [Lin23] and Alpen Labs with SNARKnado [GKSC23], but current implementations still require more optimizations of the underlying primitives.

#### 1.1 Our Contribution

A crucial component of any SNARK system is performing finite field arithmetic, which inherently involves the fundamental task of multiplying two large integers. Performing such arithmetic on Bitcoin is particularly challenging. Bitcoin Script is intentionally non-Turing complete and stack-based, designed with simplicity and security in mind. Hence, it lacks built-in support for complex arithmetic operations and has constraints on the size and number of stack elements. Implementing efficient big integer multiplication requires innovative techniques to work within these constraints.

Inspired by Elliptic Curve optimization tricks, this paper introduces the *w*-windowed method for multiplying two 254-bit prime (BN254 curve [DSD07]) integers, along with additional optimization techniques. Our approach improves upon the work done by the BitVM team, reducing script size for multiplication by roughly 3.2k opcodes. More notably, our approach can lead to even more optimizations for more general tasks such as multiple integer multiplication or fixed integer multiplication, so we expect that the methods considered are not limited to the multiplication of two integers solely.

All the code with implementation is available through the following link:

https://github.com/distributed-lab/bitcoin-window-mul

The paper is structured as follows: in Section 2 we will give a basic overview of Bitcoin Script and fast multiplication methods. In Section 3 we will list scripts to conduct the windowed multiplication (our primary proposed method). Finally, in Section 4 and Section 5, we will compare our performance with state-of-the-art and draw a conclusion.

<sup>&</sup>lt;sup>1</sup>More specifically, the proof's size is  $O(\log |C|)$  where |C| is the number of gates in the arithmetic circuit C, describing arbitrary logic that we want to prove and verify.

## 2 Preliminaries

### 2.1 Bitcoin Script

#### 2.1.1 Basic Structure

**Bitcoin Script** is a stack-based, not Turing-complete language used for specifying conditions on how UTXO can be spent [Ant14]. Informally, this condition is called scriptPubKey, while the data that must be provided to meet this condition is called scriptSig<sup>2</sup>. To verify that the condition is met based on scriptSig provided, one should first concatenate scriptSig || scriptPubKey, execute the script and verify that the resultant stack contains a non-false value (meaning, anything except for 0).

The stack consists of the values placed in the script and the so-called **opcodes** — keywords that operate with the elements in the stack. Let us consider some examples to introduce notation and describe how the script gets executed.

**Example 1.** The script  $\{ \langle a \rangle \langle b \rangle \text{ OP}\_ADD \langle c \rangle \text{ OP}\_EQUAL \}$  verifies whether given a, b, c satisfy a + b = c. We first push two integers a and b to the stack, then OP\\_ADD will consume a and b (meaning, they get removed) and output  $s \leftarrow a + b$ , so the stack becomes  $\{ \langle s \rangle \langle c \rangle \text{ OP}\_EQUAL \}$ . Finally, OP\\_EQUAL takes s and c and outputs OP\_TRUE if a + b = c, and OP\_FALSE, otherwise. Note that such notation is commonly called the *Reverse Polish Notation* in the literature [KS04].

**Example 2.** Suppose our condition on spending the coins is providing the pre-image of the given hash value h (that is, providing a message m such that h = H(m)), which is called the *Hashlock Script*. In this case, our scriptPubKey looks as follows<sup>3</sup>:

#### Stack: $OP_HASH160 \langle h \rangle OP_EQUAL$

Suppose we brought a message m, our scriptSig. Concatenating scriptSig and scriptPubKey would result in the following script:

#### Stack:

 $\langle m \rangle$  OP\_HASH160  $\langle h \rangle$  OP\_EQUAL

Execution in this case would proceed as follows:

- 1. First, m is added to the stack.
- 2. Next, OP\_HASH160 will hash the provided value  $h' \leftarrow H(m)$ , so the stack would become  $\{ \langle h' \rangle \langle h \rangle \text{ OP}_EQUAL \}$ .
- 3. Finally, after executing OP\_EQUAL, we will either get OP\_TRUE on the top of the stack if h = h', or OP\_FALSE otherwise.

Note that we get OP\_TRUE (meaning, we can spend the coins) only if h' = h or, equivalently, H(m) = h, what was needed from the start.

#### 2.1.2 Arithmetic in Bitcoin

To implement the SNARK verifier on Bitcoin, one must implement the finite field arithmetic over the elliptic curve scalar field  $\mathbb{F}_q$ . The bitsize of such scalar field is typically from 254 bits (as for *BN254* [DSD07]) to 381 bits and more (as for *BLS12-381* [KC22]). Currently, the common choice is the *BN254* based on 254-bit prime order q, which, for example, is currently used for elliptic curve precompiles in *Ethereum* [W<sup>+</sup>14]. Although further discussion is valid for any fairly large q, our implementation was focused on 254-bit q.

 $<sup>^{2}</sup>$ Formally, scriptSig might contain the logic as well, but we intentionally omit the details here.

<sup>&</sup>lt;sup>3</sup>It should be noted, though, that in the placeholder  $\langle h \rangle$  we should push 0x20 followed by 20 bytes of h.

Finite field arithmetic over N-bit q (where N = 254 for BN254, for example) includes implementing the widening multiplication of two N-bit numbers, resulting in a 2N-bit integer. Why is this a problem in Bitcoin at all? The main issue is that Bitcoin does not have a multiplication opcode<sup>4</sup>. To make matters worse, integers on the stack are 32-bit, meaning that representing large integers requires some additional workload. Therefore, we will use the **base**  $\beta$  representation of an integer.

**Definition 1.** Given positive integer  $x \in \mathbb{Z}_{>0}$ , base  $\beta$  representation is an expression

$$x = \sum_{k=0}^{\ell-1} x_k \times \beta^k,\tag{1}$$

where each **limb**  $x_k$  is between 0 and  $\beta - 1$ , and  $\ell$  is the length of such representation. We further denote such representation by  $(x_0, x_1, \ldots, x_{\ell-1})_{\beta}$ .

Empirically, it seems that using larger bases results in smaller scripts. The main reason is that larger bases result in the shorter representation of integers. However, this does not mean better methods with shorter integers will not produce shorter scripts in the future. Therefore, we pick  $\beta = 2^{30}$ : it is the power of two, which would come in handy later, and we will not run out of 32 bits when performing arithmetic (doublings, additions, etc.). Also, assume the limb size in bits is n = 30.

Moreover, Bitcoin does not have loops (recall that Bitcoin Script is not Turing complete!), meaning that the length of our representation must be fixed. It means that  $\ell = \lceil N/n \rceil$ , or,  $\ell = 9$  in our particular case.

All things combined, Algorithm 1 shows how to preprocess the given integer x and push the representation to the stack.

Algorithm 1: Pushing given integer to the stack
<b>Input</b> : Integer $x$ of bit size up to $N$
<b>Output</b> : Representation $(X_0, X_1, \ldots, X_{\ell-1})_{\beta}$ for $\beta = 2^n$ which can be inserted to
the stack (meaning $n \leq 32$ ).
<b>1</b> Decompose x to the binary form: $(x_0, x_1, \ldots, x_{N-1})_2$
<b>2</b> Split the form into chunks of size $n$ (the last chunk would be of size $N \mod n$ )
<b>3</b> For $k^{\text{th}}$ chunk with bits $(c_0, \ldots, c_{m-1})$ (where m is either n or $N \mod n$ ) set
$X_k \leftarrow \sum_{j=0}^{m-1} c_j 2^j$
$\mathbf{Return}: (X_0, X_1, \dots, X_{\ell-1})$

### 2.2 Multiplication Methods

#### 2.2.1 Karatsuba Algorithm

The **Karatsuba Algorithm** is a fast multiplication algorithm to multiply two integers using *divide and conquer* approach [WP06]. In contrast to naive  $O(N^2)$  complexity, the Karatsuba method allows to reduce the asymptotic complexity to  $O(N^{\log_2 3})$ .

Assume that we have integers x and y, represented in base  $\beta$  with  $\ell$  limbs. We divide each number into two halves: high bits  $x_H, y_H$  and low bits  $x_L, y_L$  as follows:

$$x = x_H \beta^{|\ell/2|} + y_L, \ y = y_H \beta^{|\ell/2|} + y_L \tag{2}$$

Then, a simple multiplication formula gives us:

$$xy = x_H y_H \beta^\ell + (x_H y_L + x_L y_H) \beta^{\lceil \ell/2 \rceil} + x_L y_L \tag{3}$$

<sup>&</sup>lt;sup>4</sup>At some point, Bitcoin did have OP\_MUL, but it was later disabled.

Which requires multiplying four times:  $x_H y_H, x_H y_L, x_L y_H, x_L y_L$ . Now, the Karatsuba algorithm consists in calculating these four expressions using only three multiplications. Indeed, calculate:  $c_0 = x_H y_H, c_1 = x_L y_L$ , then  $c_2 = (x_H + x_L)(y_H + y_L) - c_1 - c_0$ , and then

$$xy = c_0\beta^\ell + c_2\beta^{\lceil \ell/2 \rceil} + c_1 \tag{4}$$

The Karatsuba Algorithm is used in the current BitVM approach, where to represent the 254-bit number, one uses  $29 \times 9$  representation (that is,  $n = 29, \ell = 9$ ), resulting in roughly 74.9k opcodes [Lin23].

#### 2.2.2 Elliptic Curve Scalar Multiplication

Ideas from methods used for Elliptic curve scalar multiplication will be helpful in further optimizations. Subsequent methods will be primarily based on explanations from [HMV10].

Assume that  $(E(\mathbb{F}_q), \oplus)$  is the group of points on an elliptic curve under operation  $\oplus$ over some prime field  $\mathbb{F}_q$  of a prime order r. Suppose  $P \in E(\mathbb{F}_q)$  and  $k \in \mathbb{Z}_r$  and denote by [k]P adding P to itself k times (for k = 0 assume  $[0]P = \mathcal{O}$  where  $\mathcal{O}$  is the point at infinity). Also, assume that k is, again, N-bit sized for notation simplicity.

The basic classical approach of multiplying point P by k is specified in Algorithm 2.

#### Algorithm 2: Double-and-add method for scalar multiplication

**Input** :  $P \in E(\mathbb{F}_q)$  and  $k \in \mathbb{Z}_r$ **Output**: Result of scalar multiplication  $[k]P \in E(\mathbb{F}_q)$ 1 Decompose k to the binary form:  $(k_0, k_1, \ldots, k_{N-1})$ 2  $R \leftarrow O$ **3**  $T \leftarrow P$ 4 for  $i \in \{0, ..., N-1\}$  do  $\mathbf{5}$ if  $k_i = 1$  then  $R \leftarrow R \oplus T$ 6  $\mathbf{7}$ end  $T \leftarrow [2]T$ 8 9 end **Return** : Point R

As can be seen, the complexity of such an approach is  $O(\log_2 k)$ . Specifically, suppose A is the cost of addition while D is the cost of doubling<sup>5</sup>. In this case, the maximal total cost is roughly NA + ND. However, we can do better by using the *w*-width approach. The main idea is to decompose the scalar k into the *w*-width format.

**Definition 2.** The *w*-width form of a scalar  $k \in \mathbb{Z}_{\geq 0}$  is a base  $2^w$  representation, that is

$$k = \sum_{i=0}^{L-1} k_i \times 2^{wi}, \quad 0 \le k_i < 2^w \tag{5}$$

Let the **length** of such decomposition be  $L := \lceil N/w \rceil$ . We denote such decomposition by  $(k_0, k_1, \ldots, k_{L-1})_w$ .

Now, what does this form give us? Let us consider Algorithm 3. At first glance, the overall complexity is still  $O(\log_2 k)$ , but a closer inspection reveals that the number of additions is significantly lower for a suitable choice of w. Indeed, the number of doublings is still roughly N, but the number of additions is now approximately N/w. Of course, this comes at a cost of initializing the lookup table: to initialize  $2^w$  values we need roughly

 $<sup>^5\</sup>mathrm{Of}$  course, D is slightly easier to perform than A since doubling is a special case of addition.

 $2^{w-1}$  additions and  $2^{w-1}$  doublings (to calculate [2m]P we can always double [m]P, while for calculating [2m+1]P, add P to already precomputed [2m]P). So the overall cost is:

$$\left[2^{w-1}\mathsf{A} + 2^{w-1}\mathsf{D}\right] + \left[\frac{N}{w}\mathsf{A} + N\mathsf{D}\right] \tag{6}$$

Note that the cost of initializing the lookup table grows exponentially with respect to w, so typically, the best choice is w = 4. This way, instead of having roughly 254 additions maximum, we get 64 instead.

Algorithm 3: w-width windowed method for scalar multiplication			
$\mathbf{Input}  : P \in E(\mathbb{F}_q) \text{ and } k \in \mathbb{Z}_r$			
<b>Output</b> : Result of scalar multiplication $[k]P \in E(\mathbb{F}_q)$			
1 Decompose k to the w-width form: $(k_0, k_1, \ldots, k_{L-1})_w$			
<b>2</b> Precompute values $\{[0]P, [1]P, [2]P, \dots, [2^w - 1]P\}$ (in other words, implement the			
lookup table). Denote by $\mathcal{T}[j] = [j]P$ – referencing the lookup table at index $j$ .			
$\mathbf{s} \hspace{0.1cm} Q \leftarrow \mathcal{O}$			
4 for $i \in \{L - 1,, 0\}$ do			
5   for $\_ \in \{1, \ldots, w\}$ do			
$6     Q \leftarrow [2]Q$			
7 end			
$\mathbf{s}     Q \leftarrow Q \oplus \mathcal{T}[k_i]$			
9 end			
$\mathbf{Return}: Q$			

Yet another effective approach is w-width non-adjacent form (NAF). Let us introduce it first.

**Definition 3.** Again, assume  $w \ge 2$ . A width-w NAF of  $k \in \mathbb{Z}_{\ge 0}$  is an expression  $k = \sum_{i=0}^{L-1} k_i 2^i$  where each non-zero coefficient  $k_i$  is odd,  $|k_i| < 2^{w-1}$ , and at most one of any w consecutive digits is non-zero.

The main properties of width-w NAF are listed in the next theorem.

**Theorem 1.** Let  $k \in \mathbb{Z}_{\geq 0}$ . Then,

- 1. k has a unique width-w NAF, denoted by  $(k_0, \ldots, k_{L-1})_{w, NAF}$ .
- 2. The length of width-w NAF is at most one more than the binary representation of k.
- 3. The average density of non-zero digits in width-w NAF is approximately 1/(w+1).

Among the three listed properties, probably the most important is the third one. Indeed, if we take a random *L*-sized width-*w* NAF of some integer, most likely it would have only L/(w+1) non-zero digits, so the average number of additions would be L/(w+1)– this is slightly lower than L/w which we had before. The resultant algorithm is identical to Algorithm 3 except for the fact that it suffices to precompute only odd products  $\{[1]P, [3]P, \ldots, [2^{w-1}-1]P\}$  and their negatives (where negative is easily computed in case of  $E(\mathbb{F}_q)$  using relation  $\ominus P = \ominus(x_P, y_P) = (x_P, -y_P)$ ).

However, this method has not provided us with fewer opcodes for the reasons provided in subsequent sections.

## 3 Implementation

#### 3.1 Binary and Window Decomposition

First things first, we need to decompose our integer to the binary form using *Bitcoin Script*. Since we have chosen our base to be the power of two, it suffices to decompose the limbs to the binary form and then concatenate the result (this is the primary reason for using  $\beta = 2^n$  and not any other limb base). The implementation is specified in Algorithm 4.

```
Algorithm 4: Decomposing a limb to the binary form
    Input : A single n-bit integer x \ (n \le 32)
    Output: Bits (x_0, x_1, \ldots, x_{n-1}) in altstack
   { OP_TOALTSTACK } ;
                                                                      /* Moving limb to altstack */
 1
 2 for i \in \{0, \ldots, n-1\} do
   \left| \left\{ \left< 2 \ll i \right> \right\} \right|;
 3
                                                                         /* Pushing powers of two */
 4 end
 5 { OP_FROMALTSTACK } ;
                                                                          /* Getting element back */
 6 for \_ \in \{0, ..., n-1\} do
 \mathbf{7}
        { OP_2DUP OP_LESSTHANOREQUAL }
        {OP_IF }
 8
            \{ OP_SWAP OP_SUB \langle 1 \rangle \}
 9
        \{ OP\_ELSE \}
10
            \{ \mathsf{OP}\_\mathsf{NIP} \langle 0 \rangle \}
11
         OP_ENDIF }
12
        { OP_TOALTSTACK }
13
14 end
```

The idea here is quite straightforward: we first make the stack in a form

Stack: 
$$\langle 2^1 \rangle \langle 2^2 \rangle \langle 2^3 \rangle \dots \langle 2^n \rangle \langle x \rangle$$

Then, we duplicate top-stack elements to get  $\{\ldots \langle 2^n \rangle \langle x \rangle \langle 2^n \rangle \langle x \rangle \}$ , then checking whether  $2^n \leq x$ . If not, we remove  $2^n$  and push  $\langle 0 \rangle$  to the altstack, otherwise we modify x to be  $x - 2^n$ , push  $\langle 1 \rangle$  to the altstack and proceed.

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We then repeat this process for each limb  $(x_0, x_1, \ldots, x_{\ell-1})_{\beta}$ . This way, we have a script OP\_TOBEBITS\_TOALTSTACK which takes an N-bit integer in the main stack and pushes all bits to the altstack in the big-endian format.

Having this expansion, we can easily convert it to the *w*-width form using Algorithm 5. The idea is similar to one used in Algorithm 1 from Section 2.1.2: we split the binary expansion to the chunks of size w (except for, maybe, the last chunk, which might have a size less than w), suppose that the chunk is  $\{c_j\}_{j=0}^{m-1}$ , then the corresponding limb in *w*-width representation is  $\sum_{j=0}^{m-1} c_j 2^j$ . Then, having all limbs in the main stack, we can easily, if needed (which is the case), push it to the altstack.

All things considered, to get the *w*-width format, we simply call OP\_TOBEBITS\_TOALTSTACK and Algorithm 5 sequentially, and push resultant limbs to the altstack.

### 3.2 Addition and Doubling

To implement multiplication, we need to implement two additional "opcodes":  $OP\_ADD$ , which takes two N-bit integers and adds them up, and  $OP\_2MUL$ , which takes N-bit integer

Algorithm 5: Decomposing a limb to the *w*-width form

**Input** : Binary decomposition of a given limb x in the altstack **Output**: w-width decomposition  $(x_0, x_1, \ldots, x_{L-1})_w$  in the main stack 1 Prepare chunk sizes  $\{c_j\}_{j=0}^{L-1}$  where the last chunk is of size  $c_{L-1} := n - (L-1)w$ , while others are of size w. **2** for  $i \in \{0, \ldots, L-1\}$  do for  $j \in \{0, ..., c_i - 1\}$  do 3 { OP\_FROMALTSTACK } 4  $\{ \mathsf{OP}_{\mathsf{IF}} \langle 1 \ll j \rangle \mathsf{OP}_{\mathsf{ELSE}} \langle 0 \rangle \mathsf{OP}_{\mathsf{ENDIF}} \}$ 5 6 end for  $\_ \in \{0, ..., c_i - 2\}$  do  $\mathbf{7}$  $\{ OP\_ADD \}$ 8 end 9 10 end

and doubles it. In both cases, we assume no overflow occurs (which will be the case for our multiplication algorithm), meaning that the result is still an *N*-bit integer.

Addition. Let us start with addition. We will do addition limb-wise with handling the carry bit. For that reason, we need an intermediate opcode OP\_LIMB\_ADD\_CARRY, which takes  $\{ \langle a \rangle \langle b \rangle \langle \beta \rangle \}$  – two limbs a, b and base  $\beta$ , and outputs  $\{ \langle \beta \rangle \langle c \rangle \langle s \rangle \}$ , where c is the carry bit, while s is the sum  $(a + b \text{ if } c = 0 \text{ and } (a + b) - \beta \text{ if } c = 1)$ . We specify the algorithm in Algorithm 6.

<b>Algorithm 6:</b> Adding two limbs with carry bit			
<b>Input</b> : $\{\langle a \rangle \langle b \rangle \langle \beta \rangle\}$ – two limbs $a, b$ and base $\beta$			
<b>Output:</b> $\{ \langle \beta \rangle \langle c \rangle \rangle \}$ , where c is the carry bit, while s is the sum $(a + b \text{ if } a + b \text$			
$c = 0$ and $(a + b) - \beta$ if $c = 1$ )			
1 { OP_ROT OP_ROT }			
2 { OP_ADD OP_2DUP }			
3 { OP_LESSTHANOREQUAL }			
4 { OP_TUCK }			
5 { OP_IF }			
$6  \left\{ \langle 2 \rangle \text{ OP_PICK OP_SUB} \right\}$			
7 { OP_ENDIF }			

Now we are ready to add two integers: see Algorithm 7. Note that we use the helper opcode OP\_ZIP, which converts the stack

**Stack:** 
$$\langle x_{\ell-1} \rangle \langle x_{\ell-2} \rangle \dots \langle x_1 \rangle \langle x_0 \rangle \langle y_{\ell-1} \rangle \langle y_{\ell-2} \rangle \dots \langle y_1 \rangle \langle y_0 \rangle$$

to the following stack:

Stack: 
$$\langle x_{\ell-1} \rangle \langle y_{\ell-1} \rangle \langle x_{\ell-2} \rangle \langle y_{\ell-2} \rangle \dots \langle x_1 \rangle \langle y_1 \rangle \langle x_0 \rangle \langle y_0 \rangle$$

which makes it easy to perform subsequent element-wise operations. We do not concretize its implementation, but it is quite straightforward. Also, since we rely on the fact that x + y is still an N-bit integer (which, of course, is not always the case) when processing the last two limbs  $\{ \langle x_{\ell-1} \rangle \langle y_{\ell-1} \rangle \langle c \rangle \}$  with a carry bit c, we do not need to handle the case when  $x_{\ell-1} + y_{\ell-1} + c \geq \beta$ .

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Algorithm 7: Adding two integers assuming with no overflow **Input** : Two integers on the stack:  $\{ \langle x_{\ell-1} \rangle \dots \langle x_0 \rangle \langle y_{\ell-1} \rangle \dots \langle y_0 \rangle \}$ **Output**: Result of addition z = x + y in a form  $\{ \langle z_{\ell-1} \rangle \dots \langle z_0 \rangle \}$ 1  $\{ \mathsf{OP}\_\mathsf{ZIP} \}$ ; /\* Convert current stack  $\{ \langle x_{\ell-1} \rangle \dots \langle x_0 \rangle \langle y_{\ell-1} \rangle \dots \langle y_0 \rangle \}$  to the form  $\left\{ \left\langle x_{\ell-1} \right\rangle \left\langle y_{\ell-1} \right\rangle \dots \left\langle x_0 \right\rangle \left\langle y_0 \right\rangle \right\} * /$  $\mathbf{2} \{ \langle \beta \rangle \} ;$ /\* Push base to the stack \*/ { OP\_LIMB\_ADD\_CARRY OP\_TOALTSTACK } 3 4 for  $\_ \in \{0, \ldots, \ell - 3\}$  do /\* At this point, stack looks as  $\left\{ ig \langle x_n 
angle ig \langle y_n 
angle ig \langle \beta 
angle ig \langle c 
angle 
ight\}$  . We need to add carry cand call OP\_LIMB\_ADD\_CARRY \*/ { OP\_ROT } 5 OP\_ADD } 6 OP SWAP } 7 OP\_LIMB\_ADD\_CARRY OP\_TOALTSTACK } 8 9 end /\* At this point, again, stack looks as  $\left\{ raket{x_n}raket{y_n}raket{c}
ight\}$  . We need to drop the base, add carry, and conduct addition, assuming overflowing does not occur 10 { OP\_NIP OP\_ADD, OP\_ADD } /\* Return all limbs to the main stack \*/ 11 for  $\_ \in \{0, ..., \ell - 2\}$  do 12 { OP\_FROMALTSTACK } 13 end

**Doubling.** The doubling is performed similarly to addition, but we can avoid making the OP\_ZIP operation and simply duplicate the last limb in the stack at each step. In this particular case, we need an additional opcode OP\_LIMB\_DOUBLING\_STEP, which takes  $\{ \langle x \rangle \langle \beta \rangle \langle c \rangle \}$  – limb, base, and carry bit, and outputs  $\{ \langle \beta \rangle \langle c' \rangle \langle d \rangle \}$  – base, new carry bit c', and d = 2x + c. The implementation is specified in Algorithm 8. Additionally, we need the same version, but without c, which is executed at the beginning of the doubling, which we call OP\_LIMB\_DOUBLING\_INITIAL. The corresponding implementation is specified in Algorithm 9.

Algorithm 8: Doubling the limb with carry bit **Input** : {  $\langle x \rangle \langle \beta \rangle \langle c \rangle$  } – limb, base, and carry bit **Output:**  $\{ \langle \beta \rangle \langle c' \rangle \langle d \rangle \}$  – base, new carry bit c', and d = 2x + c1  $\{ OP_ROT \}$ { OP\_DUP OP\_ADD } ;  $\mathbf{2}$ /\* Multiplying a 32-bit integer by 2 \*/ { OP\_ADD } 3  $\{ OP_2 DUP \}$ 4 { OP\_LESSTHANOREQUAL }  $\mathbf{5}$ OP\_TUCK } 6 OP\_IF } 7  $\{ \langle 2 \rangle \text{ OP_PICK OP_SUB} \}$ 8 { OP ENDIF } 9

Now, all we are left to do is perform the algorithm similar to Algorithm 7, but with small optimizations, accounting for the fact that we do not need OP\_ZIP. The implementation is specified in Algorithm 10.

#### Algorithm 9: Doubling the limb without the carry bit

<b>Input</b> : $\{\langle x \rangle \langle \beta \rangle\}$ – limb and base				
<b>Output:</b> $\left\{ \langle \beta \rangle \langle c \rangle \langle d \rangle \right\}$ – base, new carry bit <i>c</i> , and limb doubled				
1 { OP_SWAP }				
$2 \{ OP_DUP OP_ADD \} ;$	/* Multiplying a 32-bit integer by 2 */			
3 { OP_2DUP }				
4 (OP_LESSTHANOREQUAL )				
5 { OP_TUCK }				
6 { OP_IF }				
7 $\{\langle 2 \rangle \text{ OP_PICK OP_SUB} \}$				
8 { OP_ENDIF }				

Algorithm 10: Doubling the integer without overflowing

**Input** : {  $\langle x_{\ell-1} \rangle \langle x_{\ell-2} \rangle \dots \langle x_1 \rangle \langle x_0 \rangle$  } - N-bit integer to be doubled **Output:**  $\{\langle z_{\ell-1} \rangle \langle z_{\ell-2} \rangle \dots \langle z_1 \rangle \langle z_0 \rangle\}$  – integer doubled z = 2x $1 \{ \langle \beta \rangle \} ;$ /\* Base  $\beta = 2^n$  \*/ /\* Double the limb, take the result to the altstack, and add initial carry \*/ 2 { OP\_LIMB\_DOUBLING\_INITIAL OP\_TOALTSTACK } **3** for  $\_ \in \{0, ..., \ell - 3\}$  do /\* Since we have  $\left\{ \left. \langle x \rangle \left. \langle \beta \rangle \left. \langle c \rangle \right. \right\} \right\}$  in the stack, we need to double the limb x and add an old carry c to it. \*/ { OP\_LIMB\_DOUBLING\_STEP OP\_TOALTSTACK }  $\mathbf{4}$ 5 end /\* At the end, we again get  $\left\{ \left. \langle x 
ight
angle \left\langle c 
ight
angle 
ight\} \right.$  where x is a limb in the stack. We drop the base and add the carry to the limb and double it without caring about overflowing. \*/ 6 { OP\_NIP OP\_SWAP }  $\tau \{ OP_DUP OP_ADD \} ;$ /\* Multiplying a 32-bit integer by 2 \*/ 8 { OP\_ADD } /\* Take all limbs from the altstack to the main stack \*/ 9 for  $\_ \in \{0, \ldots, \ell - 2\}$  do 10 | { OP\_FROMALTSTACK } 11 end

### 3.3 Binary Multiplication

Now comes the most interesting part: we will use methods from elliptic curve scalar multiplication to implement the product of two integers. Indeed: in Algorithm 2 and Algorithm 3 we might easily change  $E(\mathbb{F}_q)$  to any other set, equipped with the addition operation (for example, any abelian group). In our particular case, when implementing  $x \times y$ , we will interpret the y as a scalar, while x as an element to be added/doubled. So let us implement the Algorithm 2 in *Bitcoin Script* first. Note the following: although our initial number is N-bit, we expect the product  $x \times y$  to be 2N-bit, so in the intermediate steps, when performing additions and doublings, we should account for the fact that they can easily overflow N bits. The straightforward workaround is simply performing operations over the extended big integer of size 2N. This is, of course, not the best approach, and we will revisit it in Section 3.5 later on.

Since currently we have multiple various integers to work with, we will use notation  $\texttt{BigInt} < N > :: \{\texttt{OPCODE}\}\ to denote calling the OPCODE of an N-bit big integer. So, calling <math>\texttt{BigInt} < 2N > :: \{\texttt{OPCODE}\}\ would call the OPCODE of a 2N-bit integer. Additionally, assume OP_PICK, OP_ROLL and OP_DROP are implemented for integers of arbitrary bitlength. These methods are relatively trivial compared to OP_ADD and OP_2MUL, considered before: all one needs to do is to operate with integers "limbwise".$ 

So the implementation of Algorithm 2 in *Bitcoin Script* is specified in Algorithm 11. As can be seen, the cost (in opcodes) of conducting the double-and-add algorithm is NA+(N-1)D. Note that when analyzing the cost in Section 2.2, we specified the *maximal* number of additions that get performed, but here the situation is different: the number of additions is exactly N, despite the fact that the **OP\_IF** branch might be executed only a few times.

This is the primary reason why NAF methods did not significantly boost our performance: although additions might be called fewer times, we still need to include the logic in the script for each loop iteration. Therefore, we are interested in reducing the number of places where we need to place addition operations, not the number of times they get executed.

#### 3.4 Windowed Multiplication

Now, let us implement the windowed method from Algorithm 3. Again, similarly to how it was done in Section 3.3, we conduct the following steps:

- 1. Decompose y to the width-w form using opcode from Algorithm 5.
- 2. Push the resultant decomposition to the altstack. Call first and second steps as T::OP\_TOBEWINDOWEDFORM\_TOALTSTACK.
- 3. Extend x to be 2N-bit by appending zero limbs.
- 4. Precompute lookup table  $\{0, x, 2x, 3x, ..., (2^w 1)x\}$ .
- 5. Conduct the rest as described in Algorithm 3, assuming that additions and doublings never overflow (all intermediate are less than xy, which is a 2N-bit number at worst).

Steps 1-3 were already covered in our discussion, so let us discuss our strategy for implementing the lookup table. It looks as follows:

- 1. Push 0 and x to the stack.
- 2. On each step if we need to calculate  $2n \times x$ , simply BigInt<2N>::OP\_PICK the element  $n \times x$  and double it using {BigInt<2N>::OP\_DUP BigInt<2N>::OP\_ADD}

```
Algorithm 11: Double-and-add integer multiplication
    Input : Two N-bit integers on the stack: \{ \langle x_{\ell-1} \rangle \dots \langle x_0 \rangle \langle y_{\ell-1} \rangle \dots \langle y_0 \rangle \}
    Output: 2N-bit integer z = x \times y on the stack: \{ \langle z_{\ell'-1} \rangle \dots \langle z_1 \rangle \langle z_0 \rangle \}
 1 { BigInt<N>::OP_TOBEBITS_TOALTSTACK }
 2 { BigInt<N>::OP_EXTEND::<BigInt<2N>> } ;
                                                                                       /* Extend N-bit integer to
      2N\text{-}\mathrm{bit} integer by appending \ell'-\ell zero limbs */
 3 { BigInt<2N>::OP_0 } ;
                                                                      /* Pushing 2N\mbox{-bit} zero to the stack */
    { OP_FROMALTSTACK }
 \mathbf{4}
    \{ OP_{IF} \}
 \mathbf{5}
         \{ \langle 1 \rangle \texttt{BigInt} < 2N > ::: OP_PICK \}
 6
         \{ \texttt{BigInt} < 2N > : : \texttt{OP}\_\texttt{ADD} \}
 7
    { OP_ENDIF }
 8
 9 for \_ \in \{1, ..., N-2\} do
          \{ \langle 1 \rangle  BigInt<2N>::OP_ROLL \}
10
            BigInt < 2N > :: OP_2MUL \}
11
            \langle 1 \rangle BigInt<2N>::OP_ROLL \}
12
            OP_FROMALTSTACK }
13
          \{ OP_{IF} \}
\mathbf{14}
              \{ \langle 1 \rangle \text{ BigInt} < 2N > : : OP_PICK \}
15
              \{ \texttt{BigInt} < 2N > : : \texttt{OP}\_\texttt{ADD} \}
16
          {OP_ENDIF }
\mathbf{17}
18 end
19
     \{ \langle 1 \rangle \texttt{BigInt} < 2N > :: OP_ROLL \}
     \{ \texttt{BigInt} < 2N > : : OP_2MUL \}
\mathbf{20}
      OP_FROMALTSTACK }
\mathbf{21}
     ł
      OP_IF }
\mathbf{22}
         \{ \texttt{BigInt} < 2N > : : OP\_ADD \}
\mathbf{23}
     { OP_ELSE }
\mathbf{24}
         \{ \texttt{BigInt} < 2N > : : \texttt{OP}_DROP \}
\mathbf{25}
26 { OP_ENDIF }
```

3. If, instead, we need to calculate  $(2n + 1) \times x$ , copy the last element in the stack via BigInt<2N>::OP\_DUP (which is  $2n \times x$ ), then copy x and add them together via OP\_ADD.

The aforementioned strategy, as discussed before, costs  $(2^{w-1} - 1)A$  and  $(2^{w-1} - 1)D$ , which reduces to 7A and 7D for w = 4. Let us further encapsulate the logic of pushing  $\{0x, 1x, \ldots, (2^w - 1)x\}$  to the stack as BigInt<2N>::OP\_INITWINDOWEDTABLE(w).

Now we are ready to define the algorithm itself: see Algorithm 12.

Algorithm 12: Windowed integer multiplication				
<b>Input</b> : Parameter w; two N-bit integers on the stack:				
$\left\{ \left< x_{\ell-1} \right> \ldots \left< x_0 \right> \left< y_{\ell-1} \right> \ldots \left< y_0 \right> \right\}$				
<b>Output</b> : 2 <i>N</i> -bit integer $z = x \times y$ on the stack: $\{ \langle z_{\ell'-1} \rangle \dots \langle z_1 \rangle \langle z_0 \rangle \}$				
1 { BigInt <n>::OP_TOBEWINDOWEDFORM_TOALTSTACK }</n>				
2 { BigInt <n>::OP_EXTEND::<bigint<2n>&gt; } ; /* Extend N-bit integer to</bigint<2n></n>	0			
$2N$ -bit integer by appending $\ell'-\ell$ zero limbs */				
$\texttt{3} \left\{\texttt{BigInt} < 2N > :: \texttt{OP}\_\texttt{INITWINDOWEDTABLE}(w)\right\}; \qquad /* \texttt{Precomputing} = 0 \text{ for all } 1 \text{ forable } 1 \text{ for all } 1 \text{ for all } 1  for al$	g			
$\{0, x, \dots, ((1 \ll w) - 1)x\} */$				
4 { OP_FROMALTSTACK $(1)$ OP_ADD }; /* Picking first limb from the altstack +1 *	/			
5 $\{\langle 1 \ll w \rangle$ OP_SWAP OP_SUB BigInt<2N>::OP_PICKSTACK $\}$ ; /* Picking the	е			
corresponding value from the precomputed table */				
6 for $\_ \in \{1, \ldots, L-1\}$ do	,			
/* Double the result $w$ times *	/			
7 IOF $\_ \in \{0, \dots, w-1\}$ do				
a ord				
/* Dicking limb from the alterack and nicking the corresponding element from the				
lookup table After picking an element the stack would look like				
$\begin{cases} \langle 0 \rangle \langle x \rangle \dots \langle ((1 \ll w) - 1)x \rangle \langle r \rangle \langle y_i \rangle \end{cases}$ , where r is the temporary variable, being	g			
the final result, and $y_i$ is the limb at step $i$	:/			
$0  \{ \langle 1 \ll w \rangle \text{ OP}_{SWAP} \text{ OP}_{SUB} \}$				
$\{ \texttt{BigInt} < 2N > :: \texttt{OP}_\texttt{PICKSTACK} \texttt{BigInt} < 2N > :: \texttt{OP}_\texttt{ADD} \}$				
2 end				
<pre>/* Clearing the precomputed values from the stack.</pre>	1			
$3 \{ BigInt < 2N > : : OP_TOALTSTACK \}$				
4 for $\_ \in \{0,, ((1 \ll w) - 1)\}$ do				
5 $\left\{ \texttt{BigInt} < 2N > :: \texttt{OP} \_ \texttt{DROP} \right\}$				
6 end				
7 { $BigInt < 2N > :: OP_FROMALTSTACK$ }				

### 3.5 Gradual Bitsize Increase

Finally, notice that extending an integer from N bits to 2N bits from the very beginning is not optimal. For example, consider the first iteration of a loop in the windowed integer multiplication, where we multiply by  $2^w$  and then add the precomputed value. Notice that if we begin from the 256-bit number, for instance, multiplying by 16 and adding the 256-bit number would result in the 261-bit number maximum (in fact, 260-bit number, as we will see later). Similarly, when conducting the next iteration, we would not exceed 264 bits and so on. This motivates us to handle the size dynamically: when  $\ell$  limbs are insufficient to conduct the operations without overflowing, we push the zero limb (to extend an integer to  $\ell + 1$  limbs) and conduct the rest as usual. This would save tons of opcodes, as the number of useless additions of zero limbs is considerable.

Now, let us consider the following theorem.

**Theorem 2.** Suppose that Algorithm 12 is conducted using two N-bit integers, the window size of w with  $L = \lceil N/w \rceil$  limbs. For each  $k^{th}$  step, it suffices to extend the temporary variable q to  $\lambda + kw$  bits, resulting in  $\lceil (\lambda + kw)/n \rceil$  limbs for  $\lambda = 2N - w(L-1)$ .

**Proof.** Let us examine the first step. We decompose y to the width-w form, resulting in  $y = \sum_{i=0}^{L-1} y_i 2^{wi}$ , where each  $0 \le y_i < 2^w$ . Next, we initialize the lookup table, which involves calculating  $\{0, x, 2x, \ldots, (2^w - 1)x\}$ . Finally, we initialize the temporary variable  $q \leftarrow 0$  and set it to the value  $y_{L-1}x$  (since multiplication by  $2^w$  would leave q = 0 unchanged).

Now, x is N bits in size. An interesting question is the size of  $y_{L-1}$  in bits. Recall that  $y = y_{L-1}2^{w(L-1)} + y_{L-2}2^{w(L-2)} + \cdots + y_0$  is an N-bit number which means that  $y_{L-1}2^{w(L-1)}$  should also be N bits. If the size of  $y_{L-1}$  in bits is  $\lambda$ , then the size of  $y_{L-1}2^{w(L-1)}$  is  $\lambda + w(L-1)$  which is N maximum. Meaning,  $\lambda \leq N - w(L-1) = (N+w) - wL$ .

All in all, we conclude that the size of q in the beginning (call it  $\lambda$ ) is 2N - w(L-1). Then, suppose that we are at step k with a value  $q_k$ . In this case,

$$q_{k+1} = 2^w q_k + y_{L-k} x, \ q_0 = y_{L-1} x \tag{7}$$

This is a recurrence relation which is quite tough to solve generically as  $y_{L-k}$  term is different for each step. For that reason, assume the worst case: suppose  $y_{L-k} = 2^w - 1$  for each k > 1 and consider the recurrence relation

$$Q_{k+1} = 2^{w}Q_{k} + (2^{w} - 1)x, \ Q_{0} = q_{0} = y_{L-1}x$$
(8)

In this case,  $q_k < Q_k$  for each k > 1, so  $Q_k$  is our upper bound. Now, Equation (8) is an equation of form  $z_{k+1} = \alpha z_k + \beta$ , which has a closed solution  $z_n = \alpha^n z_0 + \frac{\alpha^n - 1}{\alpha - 1}\beta$ , so we get

$$Q_k = 2^{wk} Q_0 + (2^{wk} - 1)x \tag{9}$$

Notice that  $2^{wk}Q_0$  has a bitsize of  $wk + \lambda$ , while  $(2^w - 1)x$  is N + w bits in size. Notice this addition always results in the integer of bitsize  $wk + \lambda$ . Indeed:

$$Q_k < 2^{wk}(2^{\lambda} - 1) + (2^{wk} - 1)(2^N - 1) < 2^{wk+\lambda} + 2^{wk+N} < 2^{wk+\lambda+1},$$
(10)

so  $Q_k$  fits in  $\lambda + wk$  bits. Thus, as  $q_k < Q_k$ ,  $q_k$  also fits in  $\lambda + wk$  bits, concluding the proof.

With Theorem 2 in hand, we are ready to optimize the Algorithm 12 by introducing Algorithm 13.

### 4 Discussion

### 4.1 Window Width Choice

One of our key claims is that the width parameter w = 4 gives the best performance. In this section, we justify this claim. For that reason, we provide the following theorem.

**Theorem 3.** Suppose that Algorithm 12 is performed over two N-bit integers, and the cost of the addition of 2N-bit integers is  $C_{\mathsf{A}} \in \mathbb{N}$  and the cost of doubling is  $C_{\mathsf{D}} \in \mathbb{N}$ . Then, the optimal width parameter w is approximately  $\hat{w} \in \mathbb{R}$ , where  $\hat{w}$  satisfies:

$$\hat{w}^2 2^{\hat{w}} = \frac{2N}{\log 2} \cdot \frac{C_{\mathsf{A}}}{C_{\mathsf{A}} + C_{\mathsf{D}}}$$
(11)

In particular, if  $C_{\mathsf{A}} \approx C_{\mathsf{D}}$ , then this reduces to  $\hat{w}^2 2^{\hat{w}} = N/\log 2$ .

```
Algorithm 13: Windowed integer multiplication with gradual bitsize increase
              : Parameter w; two N-bit integers on the stack:
    Input
                 \{ \langle x_{\ell-1} \rangle \dots \langle x_0 \rangle \langle y_{\ell-1} \rangle \dots \langle y_0 \rangle \}
    Output: 2N-bit integer z = x \times y on the stack: \{ \langle z_{\ell'-1} \rangle \dots \langle z_1 \rangle \langle z_0 \rangle \}
 1 { BigInt<N>::OP_TOBEWINDOWEDFORM_TOALTSTACK }
    /* Important note: here, we assume that all precomputed values still fit in \ell limbs,
        so there is no need to extend an integer from N to \lambda bits. Yet, this can be
        easily accounted for if needed.
                                                                                                            */
 2 { BigInt<N>::OP_INITWINDOWEDTABLE(w) } ;
                                                                                           /* Precomputing
      \{0, x, \dots, ((1 \ll w) - 1)x\} */
 3 { OP_FROMALTSTACK \langle 1 \rangle OP_ADD }; /* Picking first limb from the altstack +1 */
 4 { \langle 1 \ll w \rangle OP_SWAP OP_SUB BigInt<N>::OP_PICKSTACK } ;
                                                                                            /* Picking the
     corresponding value from the precomputed table */
 5 for i \in \{1, ..., L-1\} do
        /* Extend the result from \lambda+(i-1)w bits to \lambda+iw
                                                                                                            */
        \{ \texttt{BigInt} < \lambda + (i-1)w > :: OP\_EXTEND: : < BigInt < \lambda + iw > \} \}
 6
        /* Double the result \boldsymbol{w} times
                                                                                                            */
        for \_ \in \{0, ..., w - 1\} do
 \mathbf{7}
         \{ \texttt{BigInt} < \lambda + iw > : : OP_2MUL \}
 8
        end
 9
        /* Picking limb from the altstack and picking the corresponding element from the
            lookup table. After picking an element, the stack would look like
            \left\{ egin{array}{ll} \langle 0 
angle \ \langle x 
angle \ \dots \ \langle ((1 \ll w) - 1) x 
angle \ \langle r 
angle \ \langle y_i 
angle 
ight\} , where r is the temporary variable, being
            the final result, and y_i is the limb at step i
                                                                                                            */
         \{ \langle 1 \ll w \rangle \text{ OP}\_SWAP \text{ OP}\_SUB \}
10
          BigInt < \lambda + iw > :: OP_PICKSTACK \}
11
         \{ \texttt{BigInt} < \lambda > :: \texttt{OP}\_\texttt{ADD} \} ; /* \texttt{Since we need to only care about the last limbs,}
\mathbf{12}
          we do not extend the result */
13 end
    /* Clearing the precomputed values from the stack.
                                                                                                            */
14 { BigInt<2N>::OP_TOALTSTACK }
15 for \_ \in \{0, \ldots, ((1 \ll w) - 1)\} do
    \{ \texttt{BigInt} < \lambda > : : OP_DROP \}
16
17 end
18 { BigInt<2N>::OP_FROMALTSTACK }
```

*Remark* 1. To simplify the analysis, we consider the Algorithm 12, which operates over extended integers. The analysis for optimized version Algorithm 13 would be ideologically similar but quite cumbersome, so let us stick to the simpler version.

**Proof.** The total cost C of width-w multiplication is, as mentioned in Section 2.2 is approximately (without accounting for operations not depending on the chosen w) given by the following formula:

$$C(w) = 2^{w-1}(C_{\mathsf{A}} + C_{\mathsf{D}}) + \frac{NC_{\mathsf{A}}}{w} + NC_{\mathsf{D}}$$
(12)

Therefore, it suffices to apply a simple calculus to find the optimal value of w. If  $\hat{w} \in \mathbb{R}$ is the optimal width, it should satisfy  $C'(\hat{w}) = 0$  which gives us:

$$C'(w) = (C_{\mathsf{A}} + C_{\mathsf{D}})2^{w-1}\log 2 - \frac{NC_{\mathsf{A}}}{w^2} \implies \hat{w}^2 2^{\hat{w}} = \frac{2N}{\log 2} \cdot \frac{C_{\mathsf{A}}}{C_{\mathsf{A}} + C_{\mathsf{D}}}$$
(13)

To see why this gives a minimum, compute the second derivative:

$$C''(w) = (C_{\mathsf{A}} + C_{\mathsf{D}})2^{w-1}\log^2 2 + \frac{2NC_{\mathsf{A}}}{w^3},$$
(14)

which is positive for any w > 0 (which is the case). The relation  $\hat{w}^2 2^{\hat{w}} = N/\log 2$ follows immediately after substituting  $C_{\mathsf{A}} = C_{\mathsf{D}}$ . 

So, now, let us substitute values corresponding to our implementation. We use N = 254, and the cost of the addition is 363 bytes (so we set  $C_A := 363$ ), while doubling takes 245 bytes (thus we set  $C_{\mathsf{D}} := 245)^6$ . Thus, approximately,  $\hat{w}^2 2^{\hat{w}} \approx 437.5$ , yielding  $\hat{w} \approx 4.45$ . After checking both w = 4 and w = 5, we conclude that w = 4 is the optimal choice.

Out of curiosity, we plot the dependence C(w) for different N's and w's. The result is depicted in Figure 1. Interestingly, for larger integers (in particular, for N = 512 or N = 1024, w = 4 most likely would no longer be the optimal choice.

#### 4.2 **Performance Comparison**

Now, we compare our multiplication implementation with the state-of-the-art approaches currently used.

- 1. BitVM "Overflow" Multiplication<sup>7</sup>: BitVM provides the default library to operate with big integers (therefore, called **bigint**) that implements the **mul** operation. The catch is that, based on two N-bit integers, this function also returns a N-bit integer, reduced modulo  $2^N$  (essentially, the lower limb in 2N-bit integer representation  $c_0 + c_1 \times 2^N$ ) — we call this "overflow multiplication". Therefore, for comparison, we adapted algorithm Algorithm 12 to have the same functionality, and also tweaked the BitVM's implementation to give the 2N-bit integer as the result.
- 2. Cmpeq's Implementation<sup>8</sup>: Quite recently, on Bitcoin Forum, cmpeq claimed to have roughly 100k opcodes in his multiplication of two 255-bit integers. The result is a 510-bit integer, compared to **bigint** multiplication from *BitVM*. Although it was claimed to have roughly 100k opcodes, after uploading the script, it appears that the real number of opcodes is, in fact, 200k. This probably happens because pushing a single integer to the stack does not always cost one opcode. For example, pushing  $10^3$  costs 3 opcodes while  $10^5$  costs 4.

 $<sup>^{6}</sup>$ In fact, it does not matter which units to use to represent  $C_{A}$  and  $C_{D}$  since at the end of the day, all that matters is the fraction  $\frac{C_{A}}{C_{A}+C_{D}}$ , which depends solely on ratio  $C_{D}/C_{A}$ . <sup>7</sup>https://github.com/BitVM/BitVM, Accessed: 25 July 2024

<sup>&</sup>lt;sup>8</sup>https://bitcointalk.org/index.php?topic=5477449.0, Accessed: 25 July 2024



Figure 1: Dependence of multiplication cost  $C_N(w)$  on the window size w for various integer bit-sizes (N). We plotted the dependence for four integers: 128 bits, 256 bits, 512 bits, 1024 bits. The dashed line in blue is most closely related to our case (N = 254). Here, we assumed that  $C_D/C_A \approx 0.675$ , corresponding to our multiplication.

3. BitVM  $29 \times 9$  Karatsuba Multiplication: This is the most recent version that BitVM mostly relies on that uses the Karatsuba multiplication (see Section 2.2.1) with  $(n = 29, \ell = 9)$  to represent a 254-bit integer.

The comparison results are depicted in Table 1.

**Table 1:** Comparison of our multiplication implementation with the current state-of-the-art. N/A means "non-applicable": that is, the algorithm is not adapted to the corresponding type of task.

Approach	<b>Overflowing Multiplication</b>	Widening Multiplication
Cmpeq	N/A	201,879
BitVM bigint	106,026	200,334
BitVM Karatsuba	N/A	74,907
Our w-width method	55,710	71,757

Most likely, our current version is not best-optimized. In particular, we list what can help to possibly reduce the number of opcodes even further:

- 1. Small polishes in gadgets used underneath (extending big integers to handle larger limbs, more effective addition or doubling, etc.).
- 2. We have not achieved any boost using NAF methods, but that does not mean these methods are not applicable: it is curious whether something can be achieved with them. In particular, w-NAF form might possibly decrease the number of additions from  $\frac{N}{w}$  to  $\frac{N}{w+1}$  and the cost of precomputing values. On the other hand, this would require implementing subtraction and sign handling, which might be troublesome.
- 3. Using different bases: we achieved the best results using 30-bit limbs to represent an integer, but maybe smaller limbs might result in something more effective.

### 4.3 Future Directions

Although this paper focuses on big integer arithmetic on Bitcoin, our proposed methods and tricks are not limited solely to that. In this section, we will present two more problems in which *w*-width decomposition might optimize Bitcoin scripts.

#### 4.3.1 Multiple Integer Multiplication

In the context of elliptic curves, quite frequently, one needs to calculate expressions in a form  $R \leftarrow [\alpha]P \oplus [\beta]Q$  for  $P, Q \in E(\mathbb{F}_q)$  and  $\alpha, \beta \in \mathbb{Z}_r^9$ . At first glance, it seems that such a problem is quite straightforward: simply calculate  $P_\alpha \leftarrow [\alpha]P$  first, then  $Q_\beta \leftarrow [\beta]Q$ , and finally  $R \leftarrow P_\alpha \oplus Q_\beta$  is the result.

However, as it turns out, this is not the case. In fact, we can decompose both  $\alpha$  and  $\beta$  into w-width form and, using the single loop iteration, compute R. We specify the implementation in Algorithm 14.

The specified algorithm shares similarities with Algorithm 3 for multiplication, particularly in the number of addition operations required. However, it offers savings on doubling costs. Instead of performing 2N doubling for two separate multiplications, this approach allows us to compute modular multiplication with just N doublings. The trade-off is an increased precomputation cost, as we need to maintain two lookup tables. That being said, the overall cost is:

$$2 \times \left[2^{w-1}\mathsf{A} + 2^{w-1}\mathsf{D}\right] + \left[\frac{2N}{w}\mathsf{A} + N\mathsf{D}\right]$$
(15)

#### Algorithm 14: w-width windowed method for multiple point multiplication

**Input** : Points  $P, Q \in E(\mathbb{F}_q)$  and scalars  $\alpha, \overline{\beta \in \mathbb{Z}_r}$ **Output**: Result of  $[\alpha]P \oplus [\beta]Q \in E(\mathbb{F}_q)$ 

1 Decompose  $\alpha$  to the *w*-width form:  $(\alpha_0, \alpha_1, \ldots, \alpha_{L-1})_w$ 

- **2** Decompose  $\beta$  to the *w*-width form:  $(\beta_0, \beta_1, \ldots, \beta_{L-1})_w$
- **3** Precompute values for two lookup tables:

$$\mathcal{T}_P : \{[0]P, [1]P, [2]P, \dots, [2^w - 1]P\}, \\ \mathcal{T}_Q : \{[0]Q, [1]Q, [2]Q, \dots, [2^w - 1]Q\}.$$

- 4 Denote by  $\mathcal{T}_P[j] = [j]P$  referencing the lookup table for P at index j.
- **5** Denote by  $\mathcal{T}_Q[j] = [j]Q$  referencing the lookup table for Q at index j.
- $6 \quad R \leftarrow \mathcal{O}$
- 7 for  $i \in \{L-1, \dots, 0\}$  do 8  $\mid R \leftarrow [2^w]R$
- 9  $R \leftarrow R \oplus (\mathcal{T}_P[\alpha_i] \oplus \mathcal{T}_Q[\beta_i])$

**Return** : Point  $R \in E(\mathbb{F}_q)$ 

Remark 2. Note that in Algorithm 14 we can alternatively store the single lookup table, say  $\mathcal{T}_{P,Q}[i,j]$ , which has all possible precomputed values  $[i]P \oplus [j]Q$  for  $i, j \in \{0, \ldots, 2^w - 1\}$ . However, that would make the lookup table of size  $2^{2w}$  (which, again, must be stored in the stack) compared to  $2 \times 2^w$ , making it impractical for Bitcoin Script.

 $<sup>^{9}</sup>$ For example, when calculating the scalar multiplication using efficiently computable endomorphisms and GLV decomposition, see [HMV10] for more details.

So, where might this be useful? Suppose you have the finite field  $\mathbb{F}_p$  with  $x, y \in \mathbb{F}_p$ , and you want to compute  $z \leftarrow xy \in \mathbb{F}_p$ . By definition, that means calculating  $xy \pmod{p}$ . Modulo operation is quite costly, so we can use a hint: notice that when we divide xy by p, we have: xy = pq + r where q is the quotient and  $0 \le r < p$  is the remainder. Essentially, r equals xy in  $\mathbb{F}_p$ , so it suffices to provide q to calculate the result as xy - pq.

That being said, Algorithm 14 can be used to optimize calculating xy - pq. Indeed, change  $\alpha$  and  $\beta$  to x and p, while P and Q to y and q, respectively, as we did in the previous sections. Moreover, since p is fixed, it can be decomposed beforehand, without needing to do that in run-time.

Most likely, there might be many other cases where multiple integer multiplication might come in handy.

#### 4.3.2 Fixed Integer Multiplication

Consider another problem typical for elliptic curve arithmetic: suppose  $P \in E(\mathbb{F}_q)$  is fixed, and we know it in compile-time. Then, we are given the scalar  $\alpha \in \mathbb{Z}_r$  and asked to compute  $[\alpha]P$ . Since P is fixed, we can precompute some data that depends solely on Pbeforehand. For example, in Algorithm 3, we can precompute expressions of a form  $[2^{wi}]P$ for  $i \in \{0, \ldots, L-1\}$  and apply the Algorithm 15.

Algorithm 15: w-width windowing method for fixed point multiplication			
<b>Input</b> : Point $P \in E(\mathbb{F}_q)$ known in compile-time and scalar $\alpha \in \mathbb{Z}_r$			
<b>Output</b> : Result of $[\alpha]P$			
1 Decompose $\alpha$ to the <i>w</i> -width form: $(\alpha_0, \alpha_1, \ldots, \alpha_{L-1})_w$			
2 Initialize the lookup table:			
$\mathcal{T}_P: \{P, [2^w]P, [2^{2w}]P, \dots, [2^{(L-1)w}]P\}$	(16)		
<b>3</b> Denote by $\mathcal{T}_P[j] = [2^{wj}]P$ – referencing the lookup table at index $j$ .			
$4 \ A \leftarrow \mathcal{O}, \ B \leftarrow \mathcal{O}$			
$-f_{} = (-1)w + (-1) + (-1$			

5 for  $j \in \{2^w - 1, \dots, 2, 1\}$  do 6 | for  $i : \alpha_i = j$  do 7 |  $B \leftarrow B \oplus \mathcal{T}_P[i]$ 8 | end 9 |  $A \leftarrow A \oplus B$ 10 end Return : Point  $A \in E(\mathbb{F}_q)$ 

The expected cost of performing such an algorithm is  $(2^w + L - 3)A$ , so we do not need to perform ND doublings anymore! Again, to use it for integer arithmetic (say, finding cx for c = const), simply change point P to the constant integer c, and  $\alpha$  to x. Needless to say, finding cx for constant c arises very frequently when implementing a complicated logic.

### 5 Conclusion

This paper introduced an innovative approach to performing big integer arithmetic within Bitcoin Script using the *w*-windowed method for multiplying 254-bit integers. Inspired by Elliptic Curve optimization techniques, our method reduces the BitVM's script size needed for multiplication, reducing approximately 3.2k opcodes. Moreover, we believe the applied approach opens the door to other optimizations involving multiple integer multiplication

or fixed integer multiplication, which are frequently used in the realm of Elliptic Curves arithmetic.

Our findings enable more efficient complex arithmetic operations. This advancement opens new possibilities for integrating advanced cryptographic protocols (and, in particular,  $L2 \ zk$ -rollup) within the Bitcoin ecosystem. For those interested in the technical details, our implementation code is available on GitHub through the provided link (see Section 1.1).

### References

- [Ant14] Andreas M. Antonopoulos. *Mastering Bitcoin: Unlocking Digital Crypto-Currencies.* O'Reilly Media, Inc., 1st edition, 2014.
- [BCEM15] Rainer Böhme, Nicolas Christin, Benjamin Edelman, and Tyler Moore. Bitcoin: Economics, technology, and governance. Journal of economic Perspectives, 29(2):213–238, 2015.
- [CLKL23] Thomas Chen, Hui Lu, Teeramet Kunpittaya, and Alan Luo. A review of zk-snarks, 2023.
- [CRTA<sup>+</sup>24] Stefanos Chaliasos, Itamar Reif, Adrià Torralba-Agell, Jens Ernstberger, Assimakis Kattis, and Benjamin Livshits. Analyzing and benchmarking ZKrollups. Cryptology ePrint Archive, Paper 2024/889, 2024. https://eprint. iacr.org/2024/889.
- [DSD07] Augusto Devegili, Michael Scott, and Ricardo Dahab. Implementing cryptographic pairings over barreto-naehrig curves. volume 4575, pages 197–207, 07 2007.
- [GKSC23] Simanta Gautam, Pramod Kandel, Chandan Sharma Subedi, and Abishkar Chhetri. Zk rollup on bitcoin: Technical whitepaper. https://github.com/ alpenlabs/Technical-Whitepaper, 2023.
- [Her18] Maurice Herlihy. Atomic cross-chain swaps. https://arxiv.org/abs/1801. 09515, 2018.
- [HMV10] Darrel Hankerson, Alfred J. Menezes, and Scott Vanstone. Guide to Elliptic Curve Cryptography. Springer Publishing Company, Incorporated, 1st edition, 2010.
- [KC22] Mahender Kumar and Satish Chand. Pairing-friendly elliptic curves: Revisited taxonomy, attacks and security concern. https://arxiv.org/abs/2212. 01855, 2022.
- [KS04] Predrag V. Krtolica and Predrag S. Stanimirović. Reverse polish notation method. International Journal of Computer Mathematics, 81(3):273–284, 2004.
- [KZL<sup>+</sup>24] Oleksandr Kurbatov, Dmytro Zakharov, Anton Levochko, Kyrylo Riabov, and Bohdan Skriabin. Multichain taprootized atomic swaps: Introducing untraceability through zero-knowledge proofs. *arXiv preprint*, 2024.
- [Lin23] Robin Linus. Bitvm: Compute anything on bitcoin. 2023.
- [Mia24] Akita Mia. Lrc-20: Scalable and fast tokenization on lightning. https: //github.com/akitamiabtc/LRC-20, 2024.

- [Mun23] Neelesh Mungoli. Deciphering the blockchain: A comprehensive analysis of bitcoin's evolution, adoption, and future implications. https://arxiv.org/ abs/2304.02655, 2023.
- [Nak09] Satoshi Nakamoto. Bitcoin: A peer-to-peer electronic cash system. May 2009.
- [OTZ<sup>+</sup>23] Maxim Orlovsky, Peter Todd, Giacomo Zucco, Federico Tenga, and Olga Ukolova. Rgb blackpaper. https://blackpaper.rgb.tech/, 2023. Accessed: 2024-08-02.
- [PD16] Joseph Poon and Thaddeus Dryja. The bitcoin lightning network: Scalable off-chain instant payments, 2016.
- [SGNB20] István András Seres, László Gulyás, Dániel A Nagy, and Péter Burcsi. Topological analysis of bitcoin's lightning network. In Mathematical Research for Blockchain Economy: 1st International Conference MARBLE 2019, Santorini, Greece, pages 1–12. Springer, 2020.
- [TMMS21] Sri AravindaKrishnan Thyagarajan, Giulio Malavolta, and Pedro Moreno-Sánchez. Universal atomic swaps: Secure exchange of coins across all blockchains. Cryptology ePrint Archive, Paper 2021/1612, 2021. https: //eprint.iacr.org/2021/1612.
- [W<sup>+</sup>14] Gavin Wood et al. Ethereum: A secure decentralised generalised transaction ledger. *Ethereum project yellow paper*, 151(2014):1–32, 2014.
- [WP06] Andre Weimerskirch and Christof Paar. Generalizations of the karatsuba algorithm for efficient implementations. *IACR Cryptology ePrint Archive*, 2006:224, 01 2006.