ON DEGREES OF CARRY AND SCHOLZ'S CONJECTURE

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Abstract. Exploiting the notion of carries, we obtain improved upper bounds for the length of the shortest addition chains $\iota(2^n-1)$ producing 2^n-1. Most notably, we show that if 2^n-1 has carries of degree at most

$$\kappa(2^n-1)=\frac{1}{2}(\iota(n)-\lfloor\frac{\log n}{\log 2}\rfloor+\sum_{j=1}^{\lfloor\frac{\log n}{\log 2}\rfloor}\{\frac{n}{2^j}\})$$

then the inequality

$$\iota(2^n-1)\leq n+1+\sum_{j=1}^{\lfloor\frac{\log n}{\log 2}\rfloor}\bigg(\{\frac{n}{2^j}\}-\xi(n,j)\bigg)+\iota(n)$$

holds for all $n\in\mathbb{N}$ with $n\geq 4$, where $\iota(\cdot)$ denotes the length of the shortest addition chain producing $\cdot$, $\{\cdot\}$ denotes the fractional part of $\cdot$ and where $\xi(n,1):=\{\frac{n}{2}\}$ with $\xi(n,2)=\{\frac{1}{2}\lfloor\frac{n}{2}\rfloor\}$ and so on.

1. Introduction

An addition chain producing $n\geq 3$, roughly speaking, is a sequence of numbers of the form $1,2,s_3,s_4,\ldots,s_{k-1},s_k=n$ where each term is the sum of two earlier terms- not necessarily distinct - in the sequence, obtained by adding each sum generated to an earlier term in the sequence. The length of the chain is determined by the number of entries in the sequence excluding the mandatory first term 1, since it is the only term which cannot be expressed as the sum of two previous terms in the sequence. There are numerous addition chains that result in a fixed number n; In other words, it is always possible to construct as many addition chains producing a fixed number positive integer n as n grows in magnitude. The shortest among these possible chains producing n is regarded as the optimal or the shortest addition chain producing n. There is currently no efficient method for getting the shortest addition yielding a given number, thus reducing an addition chain might be a difficult task, thereby making addition chain theory a fascinating subject to study. By letting $\iota(n)$ denotes the length of the shortest addition chain producing n, then Arnold Scholz conjectured and alfred Braurer proved the following inequalities

Theorem 1.1 (Braurer)**.** *The inequality*

$$m+1\leq\iota(n)\leq 2m$$

for $2^m+1\leq n\leq 2^{m+1}$ *holds for* $m\geq 1$.

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Conjecture 1.1 (Scholz)**.** The inequality

$$\iota(2^n - 1) \leq n - 1 + \iota(n)$$

holds for all $n \geq 2$.

It has been shown computationally by Neill Clift, that the conjecture holds for all $n \leq 5784688$ and in fact it is an equality for all exponents $n \leq 64$ [2]. Alfred Brauer proved the Scholz conjecture for the star addition chain, a special type of addition chain where each term in the sequence obtained by summing uses the immediately subsequent number in the chain. By denoting with $\iota^*(n)$ as the length of the shortest star addition chain producing n, it is shown that (See [1])

Theorem 1.2. *The inequality*

$$\iota^*(2^n - 1) \leq n - 1 + \iota^*(n)$$

holds for all $n \geq 2$*.*

In relation to Conjecture 1.1, Arnold Scholz postulated that Conjecture 1.1 can be improved in general. In particular, Alfred Braurer [1] proved the inequality

$$\iota(n) < \frac{\log n}{\log 2}\left(1 + \frac{1}{\log \log n} + \frac{2 \log 2}{(\log n)^{1-\log 2}}\right)$$

for $2^m \leq n < 2^{m+1}$ for all sufficiently large n.
Quite a particular special cases of the conjecture has also be studied by many authors in the past. For instance, it is shown in [4] that the Scholz conjecture holds for all numbers of the form $2^n - 1$ with $n = 2^q$ and $n = 2^s(2^q + 1)$ for $s, q \geq 0$. If we let $\nu(n)$ denotes the number of $1'$s in the binary expansion of n for $m = 2^n - 1$, then it is shown in [3] that the Scholz conjecture holds in the case $\nu(n) = 5$.

2. **Sub-addition chains**

In this section we introduce the notion of sub-addition chains.

Definition 2.1. Let $n \geq 3$, then by the addition chain of length $k - 1$ producing n we mean the sequence

$$1, 2, \ldots, s_{k-1}, s_k$$

where each term s_j $(j \geq 3)$ in the sequence is the sum of two earlier terms, with the corresponding sequence of partition

$$2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n$$

with $a_{i+1} = a_i + r_i$ and $a_{i+1} = s_i$ for $2 \leq i \leq k$. We call the partition $a_i + r_i$ the i th **generator** of the chain for $2 \leq i \leq k$. We call a_i the **determiners** and r_i the **regulator** of the i th generator of the chain. We call the sequence (r_i) the regulators of the addition chain and (a_i) the determiners of the chain for $2 \leq i \leq k$.

Definition 2.2. Let the sequence $1, 2, \ldots, s_{k-1}, s_k = n$ be an addition chain producing n with the corresponding sequence of partition

$$2 = 1 + 1, \ldots, s_{k-1} = a_{k-1} + r_{k-1}, s_k = a_k + r_k = n.$$

Then we call the sub-sequence (s_{j_m}) for $1 \leq j \leq k$ and $1 \leq m \leq t \leq k$ a **sub-addition** chain of the addition chain producing n. We say it is **complete** sub-addition chain of the addition chain producing n if it contains exactly the first t terms of the addition chain. Otherwise we say it is an **incomplete** sub-addition chain.

Lemma 2.3. *Let $\iota(n)$ denotes the shortest addition chain producing n. If $a, b \in \mathbb{N}$ then*

$$\iota(ab) \leq \iota(a) + \iota(b).$$

Proof. The proof of this Lemma can be found in [1]. $\square$

3. **The notion of carries**

We devote this section to the study of the notion of carries and its number theoretic properties. It turns out that this notion plays an important role in controlling the length of an addition for numbers of the form $2^n - 1$. Short addition chains with small carries almost satisfy the Scholz conjecture . We launch the following languages.

Definition 3.1. Consider the decomposition

$$2^n - 1 = (2^{\lfloor \frac{n}{2} \rfloor} - 1)(2^{\lfloor \frac{n}{2} \rfloor} + 1) + \frac{(1 - (-1)^n)}{2}(2^{n - (1-(-1)^n)\frac{1}{2}})$$

for $n \geq 2$. Then the non-zero remainder

$$\eta(2^n - 1) := \frac{(1 - (-1)^n)}{2}(2^{n - (1-(-1)^n)\frac{1}{2}})$$

is the level one carry of $2^n - 1$. We say that $2^n - 1$ is free of level one carries if $\eta(2^n - 1) = 0$. By letting

$$m = \left\lfloor \frac{n}{2} \right\rfloor$$

then we obtain the decomposition

$$2^m - 1 = (2^{\lfloor \frac{m}{2} \rfloor} - 1)(2^{\lfloor \frac{m}{2} \rfloor} + 1) + \frac{(1 - (-1)^m)}{2}(2^{m - (1-(-1)^m)\frac{1}{2}})$$

and we denote the carry with

$$\eta(2^m - 1) = \frac{(1 - (-1)^m)}{2}(2^{m - (1-(-1)^m)\frac{1}{2}})$$

and we say it is the level two carry of $2^n - 1$ if $\eta(2^m - 1) \neq 0$. In general, we denote the level k carry of $2^n - 1$ as the remainder

$$\eta(2^r - 1) = \frac{(1 - (-1)^r)}{2}(2^{r - (1-(-1)^r)\frac{1}{2}})$$

with

$$r = \lfloor \frac{n}{2^k} \rfloor.$$

We say that $2^n - 1$ is free of level k carries if $\eta(2^r - 1) = 0$. The number of non-zero levels of carry of $2^n - 1$ for all $1 \leq k \leq \lfloor \frac{\log n}{\log 2} \rfloor$ is the **degree** of carry of $2^n - 1$.

Proposition 3.1. *The number* $2^n - 1 \quad (n \geq 2)$ *is free of level one carry if and only if* $n \equiv 0 \pmod 2$.

Proof. Suppose that $2^n - 1$ is free of level one carry, then

$$\eta(2^n - 1) = \frac{(1-(-1)^n)}{2}(2^{n-(1-(-1)^n)\frac{1}{2}}) = 0.$$

This is only possible with $(1-(-1)^n) = 0$ and when $n \equiv 0 \pmod 2$. Conversely, suppose that $n \equiv 0 \pmod 2$ then $\frac{n}{2} \in \mathbb{N}$ and we can write

$$2^n - 1 = (2^{\frac{n}{2}} - 1)(2^{\frac{n}{2}} + 1)$$

and we see that

$$\eta(2^n - 1) = 0.$$

□

Integers of the form $2^n - 1$ with high degrees of carry serve as an obstruction to achieving the inequality

$$\iota(2^n - 1) \leq n - 1 + \iota(n)$$

using our current method. At best, avoiding them can yield progress on the conjecture using the current method but only for a specialized set of integers of the form $2^n - 1$ with low degrees of carry. It turns out that the nature of the exponents in large part characterizes integers with high degree (resp. low degree) carries. Encountering integers of the form $2^n - 1$ with exponents giving rise to high degree carries can be controlled in a way to minimize the corresponding length of the addition chain. At the moment we prove that we can obtain a chain of small length for numbers $2^n - 1$ with exponents giving rise to low degree carries.

4. Improved inequality using the method of carries

In this section, we prove an explicit upper bound for the length of the shortest addition chain producing numbers of the form $2^n - 1$. We begin with the following important but fundamental result.

Theorem 4.1. *if* $2^n - 1$ *has carries of degree at most*

$$\kappa(2^n - 1) = \frac{1}{2}(\iota(n) - \lfloor \frac{\log n}{\log 2} \rfloor + \sum_{j=1}^{\lfloor \frac{\log n}{\log 2} \rfloor} \{\frac{n}{2^j}\})$$

then the inequality

$$\iota(2^n - 1) \leq n + 1 + \sum_{j=1}^{\lfloor \frac{\log n}{\log 2} \rfloor} \left(\{\frac{n}{2^j}\} - \xi(n,j) \right) + \iota(n)$$

holds for all $n \in \mathbb{N}$ *with* $n \geq 4$*, where* $\iota(\cdot)$ *denotes the length of the shortest addition chain producing* $\cdot$*,* $\{\cdot\}$ *denotes the fractional part of* $\cdot$ *and where* $\xi(n,1) := \{\frac{n}{2}\}$ *with* $\xi(n,2) = \{\frac{1}{2}\lfloor \frac{n}{2} \rfloor\}$ *and so on*

Proof. Suppose that 2^n-1 has at most
$$\frac{1}{2}(\iota(n)-\lfloor \frac{\log n}{\log 2}\rfloor+\sum_{j=1}^{\lfloor \frac{\log n}{\log 2}\rfloor}\{\frac{n}{2^j}\})$$
degrees of carries. Next decompose the number 2^n-1 and obtain the decomposition
$$2^n-1=(2^{\lfloor \frac{n}{2}\rfloor}-1)(2^{\lfloor \frac{n}{2}\rfloor}+1)+\eta(2^n-1)$$
where
$$\eta(2^n-1):=\frac{(1-(-1)^n)}{2}(2^{n-(1-(-1)^n)\frac{1}{2}})$$
is the level one carry of 2^n-1. It is easy to see that we can recover the general factorization of 2^n-1 from this identity according to the parity of the exponent n. In particular, if $n\equiv 0 \pmod 2$, then we have
$$2^n-1=(2^{\frac{n}{2}}-1)(2^{\frac{n}{2}}+1)$$
and
$$2^n-1=(2^{\frac{n-1}{2}}-1)(2^{\frac{n-1}{2}}+1)+2^{n-1}$$
if $n\equiv 1 \pmod 2$. By combining both cases, we obtain the inequality
$$\iota(2^n-1)\leq \iota\Big((2^{\lfloor \frac{n}{2}\rfloor}-1)(2^{\lfloor \frac{n}{2}\rfloor}+1)\Big)+\eta(2^n-1).$$
Applying Lemma 2.3, we obtain further the inequality
$$\iota(2^n-1)\leq \iota(2^{\lfloor \frac{n}{2}\rfloor}-1)+\iota(2^{\lfloor \frac{n}{2}\rfloor}+1)+\eta(2^n-1) \tag{4.1}$$
Again let us set $\lfloor \frac{n}{2}\rfloor=k$ in (4.1), then we obtain the general decomposition
$$2^k-1=(2^{\lfloor \frac{k}{2}\rfloor}-1)(2^{\lfloor \frac{k}{2}\rfloor}+1)+\eta(2^k-1)$$
where
$$\eta(2^k-1)=\frac{(1-(-1)^k)}{2}(2^{k-(1-(-1)^k)\frac{1}{2}})$$
is the carry of 2^k-1. It is easy to see that we can recover the general factorization of 2^k-1 from this identity according to the parity of the exponent k. In particular, if $k\equiv 0 \pmod 2$, then we have
$$2^k-1=(2^{\frac{k}{2}}-1)(2^{\frac{k}{2}}+1)$$
and
$$2^k-1=(2^{\frac{k-1}{2}}-1)(2^{\frac{k-1}{2}}+1)+2^{k-1}$$
if $k\equiv 1 \pmod 2$. By combining both cases, we obtain the inequality
$$\iota(2^k-1)\leq \iota((2^{\lfloor \frac{k}{2}\rfloor}-1)(2^{\lfloor \frac{k}{2}\rfloor}+1))+\eta(2^k-1).$$
Applying Lemma 2.3, we obtain further the inequality
$$\begin{aligned}\iota(2^k-1)&\leq \iota(2^{\lfloor \frac{k}{2}\rfloor}-1)+\iota(2^{\lfloor \frac{k}{2}\rfloor}+1)+\eta(2^k-1)\\ &=\iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2}\rfloor\rfloor}-1)+\iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2}\rfloor\rfloor}+1)+\eta(2^{\lfloor \frac{n}{2}\rfloor}-1)\end{aligned} \tag{4.2}$$
so that by inserting (4.2) into (4.1), we obtain the inequality
$$\begin{aligned}\iota(2^n-1)&\leq \iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2}\rfloor\rfloor}-1)+\iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2}\rfloor\rfloor}+1)+\eta(2^{\lfloor \frac{n}{2}\rfloor}-1)\\ &\quad+\iota(2^{\lfloor \frac{n}{2}\rfloor}+1)+\eta(2^n-1).\end{aligned} \tag{4.3}$$

Next, we iterate the factorization up to frequency s to obtain

$$\iota(2^n-1) \leq \iota(2^{\lfloor \frac{n}{2}\rfloor}+1)+\eta(2^n-1)+\iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2}\rfloor\rfloor}-1)+\iota(2^{\lfloor \frac{1}{2}\lfloor \frac{n}{2}\rfloor\rfloor}+1)+\eta(2^{\lfloor \frac{n}{2}\rfloor}-1)$$
$$+\cdots+\iota(2^{\frac{n}{2^s}-\xi(n,s)}-1)+\iota(2^{\frac{n}{2^s}-\xi(n,s)}+1)+\eta(2^{\lfloor \frac{n}{2^{s-1}}\rfloor}-1) \tag{4.4}$$

where $0 \leq \xi(n,s) < 1$ for an integer $2 \leq s := s(n)$ fixed to be chosen later. For instance,

$$\xi(n,1)=(1-(-1)^n)\frac{1}{4}<1$$

and

$$\xi(n,2)=(1-(-1)^n)\frac{1}{8}+(1-(-1)^k)\frac{1}{4}<1$$

with

$$k:=\lfloor \frac{n}{2}\rfloor$$

and so on. Indeed the function $\xi(n,s)$ for values of $s \geq 3$ can be read from exponents of the terms arising from the iteration process. It follows from (4.4) the inequality

$$\iota(2^n-1) \leq \sum_{v=1}^{s}\frac{n}{2^v}+s+2\sum_{j=1}^{s}\sum_{\substack{\eta(2^m-1)\neq 0\\ m=\lfloor \frac{n}{2^{j-1}}\rfloor}}1-\theta(n,s)+\iota(2^{\frac{n}{2^s}-\xi(n,s)}-1)$$
$$=n(1-\frac{1}{2^s})+s+2\sum_{j=1}^{s}\sum_{\substack{\eta(2^m-1)\neq 0\\ m=\lfloor \frac{n}{2^{j-1}}\rfloor}}1-\theta(n,s)+\iota(2^{\frac{n}{2^s}-\xi(n,s)}-1) \tag{4.5}$$

where the term

$$\sum_{j=1}^{s}\sum_{\substack{\kappa(2^m-1)\neq 0\\ m=\lfloor \frac{n}{2^{j-1}}\rfloor}}1$$

counts the number of all non-zero carry of 2^n-1 up to level s and $0 \leq \theta(n,s) := \sum\limits_{j=1}^{s}\xi(n,j)$ and $2 \leq s := s(n)$ fixed, an integer to be chosen later. It is worth noting that

$$\theta(n,s):=\sum_{j=1}^{s}\xi(n,j)=0$$

if $n=2^r$ for some $r \in \mathbb{N}$, since $\xi(n,j)=0$ for each $1 \leq j \leq s$ for all n which are powers of 2. It is also important to note that the $2s$ term is obtained by noting that there are at most s terms with odd exponents under the iteration process and each term with odd exponent contributes 2, and the other s term comes from summing 1 with frequency s finding the total length of the short addition chains producing numbers of the form 2^v+1. Now, we set $k=\frac{n}{2^s}-\xi(n,s)$ and construct the addition chain producing 2^k as $1,2,2^2,\ldots,2^{k-1},2^k$ with corresponding sequence of partition

$$2=1+1, 2+2=2^2, 2^2+2^2=2^3\ldots, 2^{k-1}=2^{k-2}+2^{k-2}, 2^k=2^{k-1}+2^{k-1}$$

with $a_i=2^{i-2}=r_i$ for $2 \leq i \leq k+1$, where a_i and r_i denotes the determiner and the regulator of the i^{th} generator of the chain. Let us consider only the complete sub-addition chain

$$2=1+1, 2+2=2^2, 2^2+2^2=2^3, \ldots, 2^{k-1}=2^{k-2}+2^{k-2}.$$

Next we extend this complete sub-addition chain by adjoining the sequence

$$2^{k-1}+2^{\lfloor \frac{k-1}{2}\rfloor},2^{k-1}+2^{\lfloor \frac{k-1}{2}\rfloor}+2^{\lfloor \frac{k-1}{2^2}\rfloor},\ldots,2^{k-1}+2^{\lfloor \frac{k-1}{2}\rfloor}+2^{\lfloor \frac{k-1}{2^2}\rfloor}+\cdots+2^1.$$

Since $\xi(n,s)=0$ if $n=2^r$ and $0\leq \xi(n,s)<1$ if $n\neq 2^r$, we note that the adjoined sequence contributes at most

$$\lfloor \frac{\log k}{\log 2}\rfloor=\lfloor \frac{\log(\frac{n}{2^s}-\xi(n,s))}{\log 2}\rfloor=\lfloor \frac{\log n-s\log 2}{\log 2}\rfloor=\lfloor \frac{\log n}{\log 2}\rfloor-s$$

terms to the original complete sub-addition chain. Since the inequality holds

$$\begin{aligned}2^{k-1}+2^{\lfloor \frac{k-1}{2}\rfloor}+2^{\lfloor \frac{k-1}{2^2}\rfloor}+\cdots+2^1&<\sum_{i=1}^{k-1}2^i\\&=2^k-2\end{aligned}$$

we insert terms into the sum

$$2^{k-1}+2^{\lfloor \frac{k-1}{2}\rfloor}+2^{\lfloor \frac{k-1}{2^2}\rfloor}+\cdots+2^1 \tag{4.6}$$

so that we have

$$\sum_{i=1}^{k-1}2^i=2^k-2.$$

Let us now analyze the cost of filling in the missing terms of the underlying sum. We note that we have to insert $2^{k-2}+2^{k-3}+\cdots+2^{\lfloor \frac{k-1}{2}\rfloor+1}$ into (4.6) and this comes at the cost of adjoining

$$k-2-\lfloor \frac{k-1}{2}\rfloor$$

terms to the term in (4.6). The last term of the adjoined sequence is given by

$$2^{k-1}+(2^{k-2}+2^{k-3}+\cdots+2^{\lfloor \frac{k-1}{2}\rfloor+1})+2^{\lfloor \frac{k-1}{2}\rfloor}+2^{\lfloor \frac{k-1}{2^2}\rfloor}+\cdots+2^1. \tag{4.7}$$

Again we have to insert $2^{\lfloor \frac{k-1}{2}\rfloor-1}+\cdots+2^{\lfloor \frac{k-1}{2^2}\rfloor+1}$ into (4.7) and this comes at the cost of adjoining

$$\lfloor \frac{k-1}{2}\rfloor-\lfloor \frac{k-1}{2^2}\rfloor-1$$

terms to the term in (4.7). The last term of the adjoined sequence is given by

$$2^{k-1}+(2^{k-2}+2^{k-3}+\cdots+2^{\lfloor \frac{k-1}{2}\rfloor+1})+2^{\lfloor \frac{k-1}{2}\rfloor}+(2^{\lfloor \frac{k-1}{2}\rfloor-1}+\cdots+2^{\lfloor \frac{k-1}{2^2}\rfloor+1})+2^{\lfloor \frac{k-1}{2^2}\rfloor}+\cdots+2^1. \tag{4.8}$$

By iterating the process, it follows that we have to insert into the immediately previous term by inserting into (4.8) and this comes at the cost of adjoining

$$\lfloor \frac{k-1}{2^j}\rfloor-\lfloor \frac{k-1}{2^{j+1}}\rfloor-1$$

terms to the term in (4.8) for $j\leq \lfloor \frac{\log n}{\log 2}\rfloor-s$, since we are filling in at most $\lfloor \frac{\log k}{\log 2}\rfloor$ blocks with $k=\frac{n}{2^s}-\xi(n,s)$. It follows that the contribution of these new terms is at most

$$k-1-\left\lfloor \frac{k-1}{2^{\lfloor \frac{\log k}{\log 2}\rfloor}}\right\rfloor-\lfloor \frac{\log k}{\log 2}\rfloor \tag{4.9}$$

obtained by adding the numbers in the chain

$$k-1-\lfloor \frac{k-1}{2}\rfloor -1$$

$$\lfloor \frac{k-1}{2}\rfloor - \lfloor \frac{k-1}{2^2}\rfloor -1$$

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$$\lfloor \frac{k-1}{2^{\lfloor \frac{\log k}{\log 2}\rfloor}}\rfloor - \lfloor \frac{k-1}{2^{\lfloor \frac{\log k}{\log 2}\rfloor+1}}\rfloor -1.$$

By undertaking a quick book-keeping, it follows that the total number of terms in the constructed addition chain producing 2^k-1 with $k=\frac{n}{2^s}-\xi(n,s)$ is

$$\begin{aligned}\delta(2^k-1) &\leq k+k-1-\left\lfloor \frac{k-1}{2^{\lfloor \frac{\log k}{\log 2}\rfloor+1}}\right\rfloor - \lfloor \frac{\log k}{\log 2}\rfloor + \lfloor \frac{\log n}{\log 2}\rfloor - s\\ &\leq \frac{n}{2^{s-1}}-1-\left\lfloor \frac{\frac{n}{2^s}-\xi(n,s)-1}{2^{\lfloor \frac{\log n}{\log 2}\rfloor+1-s}}\right\rfloor - \lfloor \frac{\log n}{\log 2}\rfloor + s + \lfloor \frac{\log n}{\log 2}\rfloor - s\\ &= \frac{n}{2^{s-1}}-1-\left\lfloor \frac{\frac{n}{2^s}-\xi(n,s)-1}{2^{\lfloor \frac{\log n}{\log 2}\rfloor+1-s}}\right\rfloor. \end{aligned} \tag{4.10}$$

By plugging the inequality (4.10) into the inequalities in (4.5) and noting that $\iota(\cdot)\leq \delta(\cdot)$, we obtain the inequality

$$\iota(2^n-1)\leq \sum_{v=1}^{s}\frac{n}{2^v}+s+2\sum_{j=1}^{s}\sum_{\substack{\eta(2^m-1)\neq 0\\ m=\lfloor \frac{n}{2^{j-1}}\rfloor}}1-\theta(n,s)+\iota(2^{\frac{n}{2^s}-\xi(n,s)}-1)$$

(4.11)

$$\begin{aligned}&= n(1-\frac{1}{2^s})+\frac{n}{2^{s-1}}-1+s+2\sum_{j=1}^{s}\sum_{\substack{\eta(2^m-1)\neq 0\\ m=\lfloor \frac{n}{2^{j-1}}\rfloor}}1-\theta(n,s)-\left\lfloor \frac{\frac{n}{2^s}-\xi(n,s)-1}{2^{\lfloor \frac{\log n}{\log 2}\rfloor+1-s}}\right\rfloor\\ &= n-1+\frac{n}{2^s}+s+2\sum_{j=1}^{s}\sum_{\substack{\eta(2^m-1)\neq 0\\ m=\lfloor \frac{n}{2^{j-1}}\rfloor}}1-\theta(n,s)-\left\lfloor \frac{\frac{n}{2^s}-\xi(n,s)-1}{2^{\lfloor \frac{\log n}{\log 2}\rfloor+1-s}}\right\rfloor\end{aligned}$$

where we note that

$$\sum_{j=1}^{s}\sum_{\substack{\eta(2^m-1)\neq 0\\ m=\lfloor \frac{n}{2^{j-1}}\rfloor}}1$$

counts the number of non-zero carries up to the s level for the number 2^n-1. By taking $2\leq s:=s(n)$ such that $s=\lfloor \frac{\log n}{\log 2}\rfloor$ which is the maximum frequency of the iteration, then

$$\left\lfloor \frac{\frac{n}{2^s}-\xi(n,s)-1}{2^{\lfloor \frac{\log n}{\log 2}\rfloor+1-s}}\right\rfloor = 0$$

and we obtained that

$$\sum_{j=1}^{s}\sum_{\substack{\eta(2^m-1)\neq 0\\ m=\lfloor \frac{n}{2^{j-1}}\rfloor}}1 \leq \frac{1}{2}(\iota(n)-\lfloor \frac{\log n}{\log 2}\rfloor+\sum_{j=1}^{\lfloor \frac{\log n}{\log 2}\rfloor}\{\frac{n}{2^j}\})$$

and the inequality

$$\iota(2^n-1)\leq n-1-\theta(n,\lfloor \frac{\log n}{\log 2}\rfloor)+\lfloor \frac{\log n}{\log 2}\rfloor+2+\iota(n)-\lfloor \frac{\log n}{\log 2}\rfloor+\sum_{j=1}^{\lfloor \frac{\log n}{\log 2}\rfloor}\{\frac{n}{2^j}\}$$

for $\theta(n,\lfloor \frac{\log n}{\log 2}\rfloor):=\sum\limits_{j=1}^{\lfloor \frac{\log n}{\log 2}\rfloor}\xi(n,j)>0$ with $n\geq 4$ and $0\leq \xi(n,j)<1$, where $\{\cdot\}$ denotes the fractional part of a real number. The claimed inequality follows as a consequence. □

It turns out that proving integers of the form 2^n-1 has carries of degree at most

$$\kappa(2^n-1)=\frac{1}{2}(\iota(n)-\lfloor \frac{\log n}{\log 2}\rfloor+\sum_{j=1}^{\lfloor \frac{\log n}{\log 2}\rfloor}\{\frac{n}{2^j}\})$$

would yield the inequality

$$\iota(2^n-1)\leq n+1+\sum_{j=1}^{\lfloor \frac{\log n}{\log 2}\rfloor}\left(\{\frac{n}{2^j}\}-\xi(n,j)\right)+\iota(n)$$

for all $n\in\mathbb{N}$ as shown in the preceding proof, which is slightly short of the original conjecture. Indeed, we expect the degree of carries of all integers to be of the above form since for integers of the form $n=2^k$, it matches exactly with degree given by the formula. In particular for $2^{2^k}-1$ is always free of carries and we see that

$$\kappa(2^{2^k}-1)=\frac{1}{2}(\iota(2^k)-\lfloor \frac{\log 2^k}{\log 2}\rfloor+\sum_{j=1}^{\lfloor \frac{\log n}{\log 2}\rfloor}\{\frac{2^k}{2^j}\})=0.$$

We make the following conjecture

Conjecture 4.1. (The carry-addition chain conjecture) Let $n\geq 2$ and let $\kappa(\cdot)$ denotes the degree of carry of $\cdot$. Then we have

$$\kappa(2^n-1)=\left\lceil \frac{1}{2}(\iota(n)-\lfloor \frac{\log n}{\log 2}\rfloor+\sum_{j=1}^{\lfloor \frac{\log n}{\log 2}\rfloor}\{\frac{n}{2^j}\})\right\rceil$$

where $\lceil\cdot\rceil$ denotes the ceiling of $\cdot$.

[1].

[1] .

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