Network Agnostic MPC with Statistical Security

Ananya Appan$^1$ and Ashish Choudhury$^2$

$^1$ SAP Labs, Bangalore India
ananaya.appan@iiitb.ac.in

$^2$ IIIT Bangalore, India
ashish.choudhury@iiitb.ac.in

Abstract. In this work, we initiate the study of network agnostic MPC protocols with statistical security. Network agnostic MPC protocols give the best possible security guarantees, irrespective of the underlying network type. While network agnostic MPC protocols have been designed earlier with perfect and computational security, nothing is known in the literature regarding the possibility of network agnostic MPC protocols with statistical security. We consider the general-adversary model, where the adversary is characterized by an adversary structure, which enumerates all possible candidate subsets of corrupt parties. Given an unconditionally-secure PKI setup (a.k.a pseudo-signature setup), known statistically-secure synchronous MPC (SMPC) protocols are secure against adversary structures satisfying the $Q^{(2)}$ condition, meaning that the union of any two subsets from the adversary structure does not cover the entire set of parties. On the other hand, known statistically-secure asynchronous MPC (AMPC) protocols can tolerate $Q^{(3)}$ adversary structures where the union of any three subsets from the adversary structure does not cover the entire set of parties.

Fix a set of $n$ parties $\mathcal{P} = \{P_1, \ldots, P_n\}$ and adversary structures $\mathcal{Z}_s$ and $\mathcal{Z}_a$, satisfying the $Q^{(2)}$ and $Q^{(3)}$ conditions respectively, where $\mathcal{Z}_a \subset \mathcal{Z}_s$. Then given an unconditionally-secure PKI, we ask whether it is possible to design a statistically-secure MPC protocol, which is resilient against $\mathcal{Z}_s$ and $\mathcal{Z}_a$ in a synchronous and an asynchronous network respectively, even if the parties in $\mathcal{P}$ are unaware of the network type. We show that it is possible iff $\mathcal{Z}_s$ and $\mathcal{Z}_a$ satisfy the $Q^{(2,1)}$ condition, meaning that the union of any two subsets from $\mathcal{Z}_s$ and any one subset from $\mathcal{Z}_a$ is a proper subset of $\mathcal{P}$. Enroute our MPC protocol, we design several important network agnostic building blocks with the $Q^{(2,1)}$ condition, such as Byzantine broadcast, Byzantine agreement (BA), information checking protocol (ICP), verifiable secret-sharing (VSS) and secure multiplication protocol, whose complexity is polynomial in $n$ and $|\mathcal{Z}_s|$.

1 Introduction

A secure multiparty computation (MPC) protocol $[53,36,13,52]$ allows a set of $n$ mutually distrusting parties $\mathcal{P} = \{P_1, \ldots, P_n\}$ with private inputs to securely compute any known function $f$ of their inputs. This is achieved even if a subset

* Work done as a student at IIIT Bangalore.
of the parties are under the control of a centralized adversary and behave maliciously in a Byzantine fashion during the protocol execution. In any MPC protocol, the parties need to interact over the underlying communication network. Two types of networks have been predominantly considered. The more popular synchronous MPC (SMPC) protocols operate over a synchronous network, where every message sent is delivered within a known $\Delta$ time. Hence, if a receiving party does not receive an expected message within $\Delta$ time, then it knows that the corresponding sender party is corrupt. The synchronous model does not capture real world networks like the Internet appropriately, where messages can be arbitrarily delayed. Such networks are better modelled by the asynchronous communication model [19]. In any asynchronous MPC (AMPC) protocol [12,14], there are no timing assumptions on message delays and messages can be arbitrarily, yet finitely delayed. The only guarantee is that every message sent will be eventually delivered. The major challenge here is that no participant will know how long it has to wait for an expected message and cannot distinguish a “slow” sender party from a corrupt sender party. Consequently, in any AMPC protocol, a party cannot afford to receive messages from all the parties, to avoid an endless wait. Hence, to make “progress”, as soon a party receives messages from a “subset” of the parties, it has to process them as per the protocol, thus ignoring messages from a subset of potentially non-faulty but slow parties.

SMPC protocols are relatively simpler and enjoy better fault-tolerance (which is the maximum number of faults tolerable) compared to AMPC protocols. However, SMPC protocols become completely insecure even if a single message (from a non-faulty party) gets delayed. AMPC protocols do not suffer from this shortcoming. On the negative side, AMPC protocols are far more complex than SMPC protocols and enjoy poor fault-tolerance. Moreover, every AMPC protocol suffers from input deprivation [10] where, to avoid an endless wait, inputs of all non-faulty parties may not be considered for the computation of $f$.

Network Agnostic MPC Protocols. There is a third category of protocols called network agnostic MPC protocols, where the parties will not be knowing the network type and the protocol should provide the best possible security guarantees depending upon the network type. Such protocols are practically motivated, since the parties need not have to worry about the network type.

1.1 Our Motivation and Results

One of the earliest demarcations made in the literature is to categorize MPC protocols based on the computing power of the underlying adversary. The two main categories are unconditionally-secure protocols, which remain secure even against computationally-unbounded adversaries, and conditionally-secure MPC protocols (also called cryptographically-secure), which remain secure only against computationally-bounded adversaries [53,36]. Unconditionally-secure protocols can be further categorized as perfectly-secure [13,12] or statistically-secure [52,14], depending upon whether the security guarantees are error-free or achieved except with a negligible probability. The fault-tolerance of statistically-secure MPC
protocols are significantly better compared to perfectly-secure protocols. The above demarcation carries over even for network agnostic MPC protocols. While perfectly-secure and cryptographically-secure network agnostic MPC protocols have been investigated earlier, nothing is known regarding network agnostic statistically-secure MPC protocols. In this work we derive necessary and sufficient condition for such protocols for the first time.

Existing Results for Statistically-Secure MPC. Consider the threshold setting, where the maximum number of corrupt parties under the adversary’s control is upper bounded by a given threshold. In this model, it is known that statistically-secure SMPC tolerating up to \( t_s \) faulty parties is possible iff \( t_s < n/2 \) [52], provided the parties are given some unconditionally-secure PKI (a.k.a pseudo-signature setup) [50,34]. On the other hand, statistically-secure AMPC tolerating up to \( t_a \) faulty parties is possible iff \( t_a < n/3 \) [14,2].

A more generalized form of corruption is the general adversary model (also called non-threshold model) [38]. Here, the adversary is specified through a publicly known adversary structure \( Z \subset 2^P \), which is the set of all subsets of potentially corruptible parties during the protocol execution. The adversary is allowed to choose any one subset from \( Z \) for corruption. There are several “merits” of studying the general adversary model. For example, it provides more flexibility to model corruption in a fine-grained fashion. A threshold adversary is always a “special” type of non-threshold adversary. Consequently, a protocol in the non-threshold setting always implies a protocol in the threshold setting. Also, the protocols in this model are relatively simpler and based on simpler primitives, compared to protocols against threshold adversaries based on complex properties of bivariate polynomials. The downside is that the complexity of the protocols in the non-threshold model is polynomial in \( n \) and \(|Z|\), where the latter could be \( \mathcal{O}(2^n) \) in the worst case. In fact, as noted in [38,39], this is unavoidable.

Following [38], given a subset of parties \( P' \subseteq P \) and \( Z \), we say that \( Z \) satisfies the \( Q^{(k)}(P', Z) \) condition, if the union of any \( k \) subsets from \( Z \) does not “cover” \( P' \). That is, for any subsets \( Z_{i_1}, \ldots, Z_{i_k} \in Z \), the condition \( (Z_{i_1} \cup \ldots \cup Z_{i_k}) \subseteq P' \) holds. In the non-threshold model, statistically-secure SMPC is possible if the underlying adversary structure \( Z_s \) satisfies the \( Q^{(2)}(P, Z_s) \) condition, provided the parties are given an unconditionally-secure PKI setup [40], while statistically-secure AMPC requires the underlying adversary structure \( Z_a \) to satisfy the \( Q^{(3)}(P, Z_a) \) condition [40,5].

Our Results for Network Agnostic Statistically-Secure MPC. We consider the most generic form of corruption and ask the following question:

Given an unconditionally-secure PKI, a synchronous adversary structure \( Z_s \) and an asynchronous adversary structure \( Z_a \) satisfying the \( Q^{(2)}(P, Z_s) \) and \( Q^{(3)}(P, Z_a) \) conditions respectively, where \( Z_a \subseteq Z_s \), does there exist a statistically-secure MPC protocol, which remains secure against \( Z_s \) and \( Z_a \) in a synchronous and an asynchronous network respectively?

---

3 The setup realizes unconditionally-secure Byzantine agreement [49] with \( t_s < n/2 \).
We answer the above question affirmatively, iff $Z_s$ and $Z_a$ satisfy the $Q^{(2,1)}(P, Z_s, Z_a)$ condition, where by $Q^{(k,k)}(P, Z_s, Z_a)$ condition, we mean that for any $Z_{i_1}, \ldots, Z_{i_k} \in Z_s$ and $Z_{j_1}, \ldots, Z_{j_k'} \in Z_a$, the following holds:

$$(Z_{i_1} \cup \ldots \cup Z_{i_k} \cup Z_{j_1} \cup \ldots \cup Z_{j_k'}) \subset P.$$ 

Our results when applied against threshold adversaries imply that given an unconditionally-secure PKI, and thresholds $0 < t_a < \frac{n}{3} < t_s < \frac{n}{3}$, network agnostic statistically-secure MPC tolerating $t_a$ and $t_a$ corruptions in the synchronous and asynchronous network is possible, iff $2t_s + t_a < n$ holds. Our results in the context of relevant literature are summarized in Table 1.

<table>
<thead>
<tr>
<th>Network Type</th>
<th>Corruption Scenario</th>
<th>Security</th>
<th>Condition</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Synchronous</td>
<td>Threshold (t)</td>
<td>Perfect</td>
<td>$t &lt; n/3$</td>
<td>[13]</td>
</tr>
<tr>
<td>Synchronous</td>
<td>Non-threshold (Z)</td>
<td>Perfect</td>
<td>$Q^{(1)}(P, Z)$</td>
<td>[38]</td>
</tr>
<tr>
<td>Synchronous</td>
<td>Threshold (t)</td>
<td>Statistical</td>
<td>$t &lt; n/2$</td>
<td>[52]</td>
</tr>
<tr>
<td>Synchronous</td>
<td>Non-threshold (Z)</td>
<td>Statistical</td>
<td>$Q^{(1)}(P, Z)$</td>
<td>[40]</td>
</tr>
<tr>
<td>Asynchronous</td>
<td>Threshold (t)</td>
<td>Perfect</td>
<td>$t &lt; n/4$</td>
<td>[12]</td>
</tr>
<tr>
<td>Asynchronous</td>
<td>Non-threshold (Z)</td>
<td>Perfect</td>
<td>$Q^{(1)}(P, Z)$</td>
<td>[42]</td>
</tr>
<tr>
<td>Asynchronous</td>
<td>Threshold (t)</td>
<td>Statistical</td>
<td>$t &lt; n/3$</td>
<td>[14,2]</td>
</tr>
<tr>
<td>Asynchronous</td>
<td>Non-threshold (Z)</td>
<td>Statistical</td>
<td>$Q^{(1)}(P, Z)$</td>
<td>[5]</td>
</tr>
<tr>
<td>Network Agnostic</td>
<td>Threshold $(t_s, t_a)$</td>
<td>Perfect</td>
<td>$0 &lt; t_a &lt; n/4 &lt; t_s &lt; n/3$ and $3t_a + t_a &lt; n$</td>
<td>[4]</td>
</tr>
<tr>
<td>Network Agnostic</td>
<td>Non-threshold $(Z_s, Z_a)$</td>
<td>Perfect</td>
<td>$Z_s \subset Z_s, Q^{(1)}(P, Z_s), Q^{(1)}(P, Z_s)$ and $Q^{(1)}(P, Z_s, Z_a)$</td>
<td>[4]</td>
</tr>
<tr>
<td>Network Agnostic</td>
<td>Threshold $(t_s, t_a)$</td>
<td>Computational</td>
<td>$0 &lt; t_a &lt; n/3 &lt; t_s &lt; n/2$ and $2t_a + t_a &lt; n$</td>
<td>[17]</td>
</tr>
<tr>
<td>Network Agnostic</td>
<td>Non-threshold $(Z_s, Z_a)$</td>
<td>Statistical</td>
<td>$Z_s \subset Z_s, Q^{(1)}(P, Z_s), Q^{(1)}(P, Z_s)$ and $Q^{(1)}(P, Z_s, Z_a)$</td>
<td>This work</td>
</tr>
<tr>
<td>Network Agnostic</td>
<td>Threshold $(t_s, t_a)$</td>
<td>Statistical</td>
<td>$0 &lt; t_a &lt; n/3 &lt; t_s &lt; n/2$ and $2t_a + t_a &lt; n$</td>
<td>This work</td>
</tr>
</tbody>
</table>

Table 1: Various conditions for MPC in different settings

### 1.2 Detailed Technical Overview

We perform shared circuit-evaluation [13, 52], where $f$ is abstracted as an arithmetic circuit $ckt$ over a finite field $F$ and the goal is to securely evaluate each gate in $ckt$ in a secret-shared fashion. For every value during the circuit-evaluation, each party holds a share, such that the shares of the corrupt parties do not reveal any additional information. Once the function output is secret-shared, it is publicly reconstructed. We deploy a linear secret-sharing scheme, which enables the parties to evaluate linear gates in $ckt$ in a non-interactive fashion. Non-linear gates are evaluated using Beaver’s method [8] by deploying secret-shared random multiplication-triples which are generated beforehand.

To instantiate the above approach with statistical security, we need the following ingredients: a Byzantine agreement (BA) protocol [49], an information checking protocol (ICP) [52], a verifiable secret sharing (VSS) protocol [22], a
reconstruction protocol and finally, a secure multiplication protocol. All existing statistically-secure SMPC \cite{52,26,9,37,40} and AMPC \cite{14,48,24,25,5} protocols have instantiations of the above building blocks, either in the synchronous or asynchronous setting. However, in a network agnostic setting, we face several challenges to instantiate the above building blocks. We now take the reader through a detailed tour of the technical challenges and how we deal with them.

1.2.1 Network Agnostic BA with $Q^{(2,1)}(P, Z_s, Z_a)$ Condition

A BA protocol \cite{49} allows the parties in $P$ with private input bits to agree on a common output bit (consistency), which is the input of the non-faulty parties, if they have the same input bit (validity). Given an unconditionally-secure PKI, synchronous BA (SBA) is possible iff the underlying adversary structure $Z_s$ satisfies the $Q^{(2)}(P, Z_s)$ condition \cite{50,33,34}, while asynchronous BA (ABA) requires the underlying adversary structure $Z_a$ to satisfy the $Q^{(3)}(P, Z_a)$ condition \cite{23}. Existing SBA protocols become completely insecure in an asynchronous network. On the other hand, any ABA protocol becomes insecure when executed in a synchronous network, since $Z_s$ need not satisfy the $Q^{(3)}(P, Z_a)$ condition. Hence, we design a network agnostic BA protocol with $Q^{(2,1)}(P, Z_s, Z_a)$ condition. The protocol is obtained by generalizing the blueprint for network agnostic BA against threshold adversaries, first proposed in \cite{15} and later used in \cite{3}. While \cite{15} proposed it for computational security with conditions $0 < t_a < \frac{n}{3} < t_s < \frac{n}{2}$ and $2t_s + t_a < n$ in the presence of a computationally-secure PKI, later, \cite{3} modified it for perfect security and used it with conditions $t_a, t_s < n/3$.\footnote{Unlike computationally-secure BA, the necessary condition for perfectly-secure BA is $t < n/3$ for both SBA as well as ABA, where $t$ is the maximum number of faults.} We replace the computationally-secure PKI with an unconditionally-secure PKI and generalize the building blocks of \cite{15} against non-threshold adversaries and upgrade their security to unconditional-security. Additionally, we also generalize certain primitives from \cite{3} and adapt them to work with the $Q^{(2,1)}(P, Z_s, Z_a)$ condition. Since this part mostly follows the existing works, we refer to Section 3 for full details.

1.2.2 Network Agnostic ICP with $Q^{(2,1)}(P, Z_s, Z_a)$ Condition

An ICP \cite{52,26} is used for authenticating data in the presence of a computationally-unbounded adversary. In an ICP, there are four entities, a signer $S \in P$, an intermediary $I \in P$, a receiver $R \in P$ and all the parties in $P$ acting as verifiers (note that $S$, $I$ and $R$ also act as verifiers). An ICP has two sub-protocols, one for the authentication phase and one for the revelation phase.

In the authentication phase, $S$ has a private input $s \in F$, which it distributes to $I$ along with some authentication information. Each verifier is provided with some verification information, followed by the parties verifying whether $S$ has distributed “consistent” information. If the verification is “successful”, then the data held by $I$ at the end of this phase is called $S$’s IC-Signature on $s$ for intermediary $I$ and receiver $R$, denoted by $ICSig(S, I, R, s)$. Later, during the revelation
phase, \( I \) reveals \( \text{ICSig}(S, I, R, s) \) to \( R \), who “verifies” it with respect to the verification information provided by the verifiers and either accepts or rejects \( s \). We require the same security guarantees from ICP as expected from cryptographic signatures, namely correctness (if \( S, I \) and \( R \) are all honest, then \( R \) should accept \( s \)), unforgeability (a corrupt \( I \) should fail to reveal an honest \( S \)'s signature on \( s' \neq s \)) and non-repudiation (if an honest \( I \) holds some \( \text{ICSig}(S, I, R, s) \), then later an honest \( R \) should accept \( s \), even if \( S \) is corrupt). Additionally, we need privacy, guaranteeing that if \( S, I \) and \( R \) are all honest, then \( \text{Adv} \) does not learn \( s \).\(^5\)

The only known instantiation of ICP in the synchronous network [40] is secure against \( \mathbb{Q}^{(2)} \) adversary structures and becomes insecure in the asynchronous setting. On the other hand, the only known instantiation of ICP in the asynchronous setting [5] can tolerate only \( \mathbb{Q}^{(3)} \) adversary structures. Our network agnostic ICP is a careful adaptation of the asynchronous ICP of [5]. We first try to naively adapt the ICP to deal with the network agnostic setting, followed by the technical problems in the naive adaptation and the modifications needed.

During authentication phase, \( S \) embeds \( s \) in a random \( t \)-degree polynomial \( F(x) \) at \( x = 0 \), where \( t \) is the cardinality of the maximum-sized subset in \( \mathbb{Z}_s \), and gives \( F(x) \) to \( I \). In addition, each verifier \( P_i \) is given a random verification-point \( (\alpha_i, v_i) \) on \( F(x) \). To let the parties securely verify that it has distributed consistent information, \( S \) additionally distributes a random \( t \)-degree polynomial \( M(x) \) to \( I \), while each verifier \( P_i \) is given a point on \( M(x) \) at \( \alpha_i \). Each verifier, upon receiving its verification-points, publicly confirms the same. Upon receiving these confirmations, \( I \) identifies a a subset of supporting verifiers \( SV \) which have confirmed the receipt of their verification-points. To avoid an endless wait, \( I \) waits until \( P \setminus SV \in \mathbb{Z}_s \). After this, the parties publicly check the consistency of the \( F(x), M(x) \) polynomials and the points distributed to \( SV \), with respect to a random linear combination of these polynomials and points, where the linear combiner is selected by \( I \). This ensures that \( S \) has no knowledge beforehand about the random combiner and hence, any “inconsistency” will be detected with a high probability. If no inconsistency is detected, the parties proceed to the revelation phase, where \( I \) reveals \( F(x) \) to \( R \), while each verifier in \( SV \) reveals its verification-point to \( R \), who accepts \( F(x) \) (and hence \( F(0) \)) if it sure that the verification point of at least one non-faulty verifier in \( SV \) is “consistent” with the revealed \( F(x) \). This would ensure that the revealed \( F(x) \) is indeed correct with a high probability, since a corrupt \( I \) will have no information about the verification point of any non-faulty verifier in \( SV \), provided \( S \) is non-faulty. To avoid an endless wait, once \( R \) finds a subset of verifiers \( SV' \subseteq SV \), where \( SV \setminus SV' \in \mathbb{Z}_s \), whose verification-points are found to be “consistent” with \( F(x) \), it outputs \( F(0) \).

\(^5\) IC-signatures are different from pseudo-signatures. Pseudo-signatures are “transferable”, where a party can transfer a signed message to other parties for verification (depending upon the allowed level of transferability), while IC-signatures can be verified only by the designated \( R \) and cannot be further transferred. Due to the same reason, IC-signatures satisfy the privacy property unlike pseudo-signatures. Most importantly, IC-signatures are generated from the scratch, assuming a pseudo-signature setup which is used to instantiate the instances of broadcast in the ICP.
A Technical Problem and Way-out. The protocol outlined above will achieve all the properties in an asynchronous network, due to the $Q^{(3)}(P, Z_s)$ condition. However, it fails to satisfy the unforgeability property in a synchronous network. Namely, a corrupt $I$ may not include all the non-faulty verifiers in $SV$ and may purposely exclude a subset of non-faulty verifiers belonging to $Z_s$. Let $H_{SV}$ be the set of non-faulty verifiers in $SV$ and let $C_{SV}$ be the set of corrupt verifiers in $SV$. Due to the above strategy, the condition $Q^{(1)}(H_{SV}, Z_s)$ may not be satisfied and $SV \setminus C_{SV} = H_{SV} \in Z_s$ may hold. As a result, during the revelation phase, I may produce an incorrect $F'(x) \neq F(x)$ and the verifiers in $C_{SV}$ may change their verification points to “match” $F'(x)$, while only the verification points of the verifiers in $H_{SV}$ may turn out to be inconsistent with $F'(x)$. Consequently, $SV' = C_{SV}$ and if $H_{SV} \in Z_s$, then clearly $s' = F'(0)$ will be the output of $R$, thus breaking the unforgeability property.

To deal with the above issue, we let $S$ identify and announce $SV$. This ensures that all honest verifiers are present in $SV$, if $S$ is honest and the network is synchronous, provided $S$ waits for “sufficient” time to let the verifiers announce the receipt of their verification points. Consequently, now the condition $Q^{(1)}(H_{SV}, Z_s)$ will be satisfied. Hence, if a corrupt $I$ reveals an incorrect $F(x)$, then it will not be accepted, as the condition $SV \setminus C_{SV} \in Z_s$ no longer holds.

Linearity of ICP. Our ICP satisfies the linearity property (which will be useful later in our VSS), provided “special care” is taken while generating the ICP-signatures. Consider a fixed $S, I$ and $R$ and let $s_a$ and $s_b$ be two values, such that $I$ holds $ICSig(S, I, R, s_a)$ and $ICSig(S, I, R, s_b)$, where all the following conditions are satisfied during the underlying instances of the authentication phase.

- The set of supporting verifiers $SV$ are the same during both the instances.
- For $i = 1, \ldots, n$, corresponding to the verifier $P_i$, signer $S$ uses the same $\alpha_i$, to compute the verification points, during both the instances.
- $I$ uses the same linear combiner to verify the consistency of the distributed data in both the instances.

Let $s \triangleq c_1 \cdot s_a + c_2 \cdot s_b$, where $c_1, c_2$ are publicly known constants from $\mathbb{F}$. It then follows that if all the above conditions are satisfied, then $I$ can locally compute $ICSig(S, I, R, s)$ from $ICSig(S, I, R, s_a)$ and $ICSig(S, I, R, s_b)$, while each verifier in $SV$ can locally compute their corresponding verification information.

1.2.3 Network Agnostic VSS and Reconstruction

In the network agnostic setting, to ensure privacy, all the values during the circuit evaluation need to be secret-shared “with respect” to $Z_s$ irrespective of the network type. We follow the notion of additive secret-sharing [41], also used in the earlier MPC protocols [44, 40, 5]. Given $Z_s = \{Z_1, \ldots, Z_{|Z_s|}\}$, we consider the sharing specification $S_{Z_s} = \{S_1, \ldots, S_{|Z_s|}\}$, where each $S_q = P \setminus Z_q$. Hence there exists at least one subset $S_q \in S_{|Z_s|}$ which does not contain any faulty party, irrespective of the network type (since $Z_q \subset Z_s$). A value $s \in \mathbb{F}$ is said to be secret-shared, if there exist shares $s_1, \ldots, s_{|Z_s|}$ which sum up to $s$, such that all (non-faulty) parties in $S_q$ have the share $s_q$. We denote a secret-sharing of $s$ by $[s]$. 

7
We also note that the above secret-sharing is linear since, given secret-sharing \([a]\) and \([b]\) and publicly known constants \(c_1, c_2 \in \mathbb{F}\), the condition \(c_1 \cdot [a] + c_2 \cdot [b] = \{c_1 \cdot a + c_2 \cdot b\}\) holds. Consequently, the parties can non-interactively compute any publicly known linear function of secret-shared values. Unfortunately, the above secret-sharing does not allow for the robust reconstruction of a secret-shared value. This is because the corrupt parties may produce incorrect shares at the time of reconstruction. To deal with this, we “augment” the above secret-sharing.

As part of secret-sharing \(s\), we also have publicly known core-sets \(W_1, \ldots, W_{|Z_s|}\), where each \(W_q \subseteq S_q\) such that \(Z_s\) satisfies the \(Q^{(1)}(W_q, Z_s)\) condition (ensuring \(W_q\) has at least one non-faulty party). Moreover, each (non-faulty) \(P_i \in W_q\) will have the IC-signature \(\text{ICSig}(P_j, P_i, P_k, [s]_q)\) of every \(P_j \in W_q\), for every \(P_k \not\in S_q\), such that the underlying IC-signatures satisfy the linearity property.

We call this augmented secret sharing as linear secret-sharing with IC-signatures, which is still denoted as \([s]\). Now to robustly reconstruct a secret-shared \(s\), we ask the parties in \(W_q\) to make public the share \([s]_q\), along with the IC-signatures of all the parties in \(W_q\) on \([s]_q\). Any party \(P_k\) can then verify whether \([s]_q\) revealed by \(P_i\) is correct by verifying the IC-signatures. If \(P_i\) is corrupt then, due to the unforgeability if ICP, it will fail to forge IC-signature of a non-faulty \(P_j\) on an incorrect \([s]_q\). On the other hand, a non-faulty \(P_i\) will be able to reveal the correct \([s]_q\) and the IC-signature of every \(P_j \in W_q\) on \([s]_q\), which are accepted even if \(P_j\) is corrupt (follows from non-repudiation of ICP).

We design a network agnostic VSS protocol \(\Pi_{\text{VSS}}\), which allows a designated dealer \(D \in \mathcal{P}\) with input \(s \in \mathbb{F}\) to verifiably generate \([s]\), where \(s\) remains private for a non-faulty \(s\). If \(D\) is faulty then either no non-faulty party obtains any output (if \(D\) does not invoke the protocol) or there exists some \(s^* \in \mathbb{F}\) such that the parties output \([s^*]\). To design \(\Pi_{\text{VSS}}\), we use certain ideas from the statistically-secure synchronous VSS (SVSS) and asynchronous VSS (AVSS) of [40] and [5] respectively, along with some new counter-intuitive ideas. In the sequel, we first give a brief outline of the SVSS and AVSS of [40,5], followed by the technical challenges arising in the network agnostic setting and how we deal with them.

**Statistically-Secure SVSS of [40] with \(Q^{(2)}(\mathcal{P}, Z_s)\) Condition.** The SVSS of [40] proceeds as a sequence of synchronized phases. During the first phase, \(D\) picks random shares \(s_1, \ldots, s_{|Z_s|}\) which sum up to \(s\) and sends \(s_q\) to the parties in \(S_q\). To verify whether \(D\) has distributed consistent shares to the parties in \(S_q\), during the second phase, every pair of parties \(P_i, P_j \in S_q\) exchange the supposedly common shares received from \(D\), along with their respective IC-signatures. That is, \(P_i\) upon receiving \(s_{qi}\) from \(D\), gives \(\{\text{ICSig}(P_j, P_i, P_k, s_{qi})\}_{P_k \in \mathcal{P}}\) to \(P_j\) while \(P_j\), upon receiving \(s_{qj}\) from \(D\), gives \(\{\text{ICSig}(P_j, P_i, P_k, s_{qj})\}_{P_k \not\in \mathcal{P}}\) to \(P_i\). Then during the third phase, the parties in \(S_q\) publicly complain about any “inconsistency”, in response to which \(D\) makes public the share \(s_q\) corresponding to \(S_q\) during the fourth phase. Hence, by the end of fourth phase it is ensured that, for every \(S_q\), either the share \(s_q\) is publicly known (if any complaint was
reported for $S_q$) or all (non-faulty) parties in $S_q$ have the same share (along with the respective IC-signatures of each other on it). The privacy of $s$ is maintained for a non-faulty $D$, since the share $s_q$ corresponding to the set $S_q$ consisting of only non-faulty parties is never made public.

**Statistically-Secure AVSS of [5] with $Q^{(3)}(P, Z_a)$ Condition.** Let $Z_a = \{Z_1, \ldots, Z_{|Z_a|}\}$ and $S_{Z_a} = \{S_1, \ldots, S_{|Z_a|}\}$, where each $S_q = P \setminus Z_q$. The AVSS protocol of [5] also follows an idea similar to the SVSS of [40]. However, now the parties cannot afford to wait for all the parties in $S_q$ to report the statuses of pairwise consistency tests, as the corrupt parties in $S_q$ may never turn up. Hence instead of looking for inconsistencies in $S_q$, the parties rather check how many parties in $S_q$ are reporting the pairwise consistency of their supposedly common share. The idea is that if $D$ has not cheated, then a subset of parties $W_q$ where $S_q \setminus W_q \in Z_a$ should eventually confirm the receipt of a common share from $D$. Hence, the parties check for core-sets $W_1, \ldots, W_{|Z_a|}$, where each $S_q \setminus W_q \in Z_a$, such that the parties in $W_q$ have confirmed the receipt of a common share from $D$. Note that irrespective of $D$, each $W_q$ is bound to have at least one non-faulty party, since $Z_a$ will satisfy the $Q^{(1)}(W_q, Z_a)$ condition.

The existence of $W_1, \ldots, W_{|Z_a|}$ does not imply that all non-faulty parties in $S_q$ have received a common share, even if $D$ is non-faulty, since there might be non-faulty parties outside $W_q$. Hence, after the confirmation of the sets $W_1, \ldots, W_{|Z_a|}$, the goal is to ensure that every (non-faulty) party in $S_q \setminus W_q$ also gets the common share held by the (non-faulty) parties in $W_q$. For this, the parties in $W_q$ reveal their shares to these “outsider” parties, along with the required IC-signatures. The outsider parties then “filter” out the correctly revealed shares. The existence of at least one non-faulty party in each $W_q$ guarantees that the shares filtered by the outsider parties are indeed correct.

**Technical Challenges for Network Agnostic VSS and Way Out.** Since, in our context, the parties will not be knowing the network type, our approach will be to follow the AVSS of [5], where we look for pairwise consistency of supposedly the common share in each group. Namely, $D$ on having the input $s$, picks random shares $s_1, \ldots, s_{|Z_a|}$ which sum up to $s$ and distributes $s_q$ to each $S_q \in S_{|Z_a|}$. The parties in $S_q$ then exchange IC-signed versions of their supposedly common share. To avoid an endless wait, the parties can only afford to wait till a subset of parties $W_q \subseteq S_q$ have confirmed the receipt of a common share from $D$, where $S_q \setminus W_q \in Z_a$ holds. Unfortunately, $S_q \setminus W_q \in Z_a$ need not guarantee that $W_q$ has at least one non-faulty party, since $Z_a$ need not satisfy the $Q^{(1)}(W_q, Z_a)$ condition, which is desired as per our semantics of linear secret-sharing with IC-signatures.

To deal with the above problem, we note that if $D$ has distributed the shares consistently, then the subset of parties $S \in S_{Z_a}$ which consists of only non-faulty parties will publicly report the pairwise consistency of their supposedly common share. Hence, we now let $D$ search for a candidate set $S_p$ of parties from

---

6 Recall that we need $[s]$ with respect to $Z_a$, irrespective of the network type.
\(S_Z\), which have publicly confirmed the pairwise consistency of their supposedly common share. Once \(D\) finds such a candidate \(S_p\), it computes and make public the core-sets \(W_q\) as per the following rules, for \(q = 1, \ldots, |Z_s|\).

- If all the parties in \(S_p\) have confirmed the pairwise consistency of their supposedly common share, then set \(W_q = S_q\). \((A)\)
- Else if \(Z_s\) satisfies the \(Q^{(1)}(S_p \cap S_q, Z_s)\) condition and the parties in \((S_p \cap S_q)\) have confirmed the consistency of their supposedly common share, then set \(W_q = (S_p \cap S_q)\). \((B)\)
- Else set \(W_q = S_q\) and make public the share \(s_q\). \((C)\)

The parties wait till they see \(D\) making public some set \(S_p \in S_{Z_s}\) along with sets \(W_1, \ldots, W_{|Z_s|}\). Upon receiving, the parties verify and “approve” these sets as valid, provided all parties in \(S_p\) have confirmed the pairwise consistency of their supposedly common share and if each \(W_q\) is computed as per the rule \((A), (B)\) or \((C)\). If \(W_1, \ldots, W_{|Z_s|}\) are approved, then they indeed satisfy the requirements of core-sets as per our semantics of linear secret-sharing with IC-signatures. While this is trivially true if any \(W_q\) is computed using rule \((A)\) or rule \((B)\), the same holds even if \(W_q\) is computed using rule \((C)\). This is because, in this case, the parties publicly set \(|s_q| = s_q\). Moreover, the parties take a “default” (linear) IC-signature of \(s_q\) on the behalf of \(S_q\), where the IC-signature as well as verification points are all set to \(s_q\).

If \(D\) is non-faulty, then irrespective of the network type, it will always find a candidate \(S_p\) and hence, compute and make public \(W_1, \ldots, W_{|Z_s|}\) as per the above rules. This is because the set \(S\) always constitutes a candidate \(S_p\). Surprisingly we can show that even if the core-sets are computed with respect to some different candidate \(S_p \neq S\), a non-faulty \(D\) will never make public the share corresponding to \(S\), since the rule \((C)\) will not be applicable over \(S\), implying the privacy of \(s\). If the network is synchronous, then the parties in \(S_p\) as well as \(S\) would report the pairwise consistency of their respective supposedly common share at the same time. This is ensured by maintaining sufficient “timeouts” in the protocol to report pairwise consistency of supposedly common shares. Consequently, rule \((A)\) will be applied on \(S\). For an asynchronous network, rule \((B)\) will be applicable for \(S\), as \(Z_s\) will satisfy the \(Q^{(1)}(S_p \cap S, Z_s)\) condition, due to the \(Q^{(2,1)}(P, Z_s, Z_a)\) condition and the fact that \(S = P \setminus Z\) for some \(Z \in Z_a\) in the asynchronous network.

### 1.2.4 Network Agnostic VSS for Multiple Dealers with Linearity

**Technical Challenge in \(II_{\text{VSS}}\) for Multiple Dealers.** If different dealers invoke instances of \(II_{\text{VSS}}\) to secret-share their inputs, then the linearity property of \([\cdot]\)-sharing need not hold, since the underlying core-sets might be different. In more detail, let \(D_a\) and \(D_b\) be two different dealers which invoke instances \(II_{\text{VSS}}^{(a)}\) and \(II_{\text{VSS}}^{(b)}\) to generate \([a]\) and \([b]\) respectively. Let \(W_1^{(a)}, \ldots, W_{|Z_s|}^{(a)}\) and \(W_1^{(b)}, \ldots, W_{|Z_s|}^{(b)}\) be the underlying core-sets for \(II_{\text{VSS}}^{(a)}\) and \(II_{\text{VSS}}^{(b)}\) respectively. Now consider a scenario where, for some \(q \in \{1, \ldots, |Z_s|\}\), the core-sets \(W_q^{(a)}\) and \(W_q^{(b)}\) are different, even though \(Z_s\) satisfies the \(Q^{(1)}(W_q^{(a)}, Z_s)\) and
signature

{pad from the OTP-encryption. This results in 
and then take the “default” secret-sharing of the OTP-encryption with IC-signatures one-time pad (OTP) and makes public an OTP-encryption of $x$.

parties during the circuit-evaluation. We show that $L$ is the maximum number of values which need to be secret-shared by different parties during the circuit-evaluation. Namely, for each $\mathcal{GW}_q$, $Z_s$ satisfies the $\mathbb{Q}^{(1)}(\mathcal{GW}_q, Z_s)$ condition. Maintaining common core-sets is challenging, especially in an asynchronous network and $\Pi_{\text{VSS}}$ alone is not sufficient to achieve this goal. Rather we use a different approach. We generate a “bunch” of linearly secret-shared random values with IC-signatures and common core-sets $\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z_s|}$ in advance through another protocol called $\Pi_{\text{Rand}}$ (discussed in the next section). Later, if any party $P_i$ needs to secret-share some $x$, then one of these random values is reconstructed only towards $P_i$, which uses it as a one-time pad (OTP) and makes public an OTP-encryption of $x$. The parties can then take the “default” secret-sharing of the OTP-encryption with IC-signatures and $\mathcal{GW}_{1}, \ldots, \mathcal{GW}_{|Z_s|}$ as the core-sets and then non-interactively “remove” the pad from the OTP-encryption. This results in $[x]$, with $\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z_s|}$ as core-sets. To ensure privacy, we need to generate $L$ random values through $\Pi_{\text{Rand}}$, if $L$ is the maximum number of values which need to be secret-shared by different parties during the circuit-evaluation. We show that $L \leq n^3 \cdot c_M + 4n^2 \cdot c_M + n^2 + n$ where $c_M$ is the number of multiplication gates in ckt.

1.2.5 Secret-Shared Random Values with Global Core Sets

Protocol $\Pi_{\text{Rand}}$ generates linearly secret-shared random values with IC-signatures and common core-sets. We explain the idea behind the protocol for generating one random value. The “standard” way will be to let each $P_i$ pick a random value $r^{(i)}$ and generate $[r^{(i)}]$ by invoking an instance of $\Pi_{\text{VSS}}$. To avoid an endless wait, the parties only wait for the completion of $\Pi_{\text{VSS}}$ instances invoked by a set of dealers $\mathcal{P} \setminus Z$ for some $Z \in Z_s$. To identify the common subset
of dealers for which the corresponding $\Pi_{VSS}$ instances have completed, the parties run an instance of agreement on a common subset (ACS) primitive [14,19]. This involves invoking $n$ instances of our network agnostic BA, one on behalf of each dealer, to decide the $\Pi_{VSS}$ instances of which dealers have completed. Let $C$ be the set of common dealers identified through ACS, where $\mathcal{P} \setminus C \in \mathbb{Z}$. The set $C$ has at least one non-faulty party who has shared a random value. Hence, the sum of the values shared by the dealers in $C$ will be random for the adversary.

**Technical Challenges.** The above approach fails in our context due to the following two “problems” in the protocol $\Pi_{VSS}$, when executed by different dealers.

**Problem I:** The first challenge is to maintain the linearity of underlying IC-signatures. To understand the issue, consider a triplet of parties $P_i, P_j, P_k$, acting as $S, l, R$ respectively in various instances of $\Pi_{VSS}$ invoked by different dealers. Recall that, to maintain the linearity of IC-signatures, it is necessary that $P_i$ selects the same set of supporting-verifiers $SV$ in all the instances of authentication phase involving $P_j$ and $P_k$. This is possible only if $P_i$ knows all the values on which it wants to generate the IC-signature for $P_j$ and $P_k$ and starts invoking all the instances of authentication phase. Instead, if $P_i$ invokes instances of authentication phase as and when it has some data to be authenticated for $P_j$ and $P_k$, then it may not be possible to have the same $SV$ in all the instances of authentication phase, involving $P_i, P_j$ and $P_k$ in the above roles, especially in an asynchronous network. Since, in $\Pi_{VSS}$, IC-signatures are generated on the supposedly common shares (after receiving them from the underlying dealer) and multiple instances of $\Pi_{VSS}$ are invoked (by different dealers), this means that $P_i$ should first have the data from all the dealers for the various instances of $\Pi_{VSS}$ and before invoking instances of authentication phase to generate IC-signatures on these values for $P_j$. This may not be possible, since $P_i$ need not know beforehand which dealers it will be receiving shares from as part of $\Pi_{VSS}$.

**Way Out.** To deal with the above issue, we now let the dealers publicly commit their shares for the $\Pi_{VSS}$ instances through secure verifiable multicast (SVM). The primitive allows a designated sender $Sen \in \mathcal{P}$ with input $v$ to “verifiably” send $v$ to a designated set of receivers $R \subseteq \mathcal{P}$, without leaking any additional information. The verifiability guarantees that even if $Sen$ is corrupt, if the non-faulty parties in $R$ get any value from $Sen$, then it will be common and all the (non-faulty) parties in $\mathcal{P}$ will “know” that $Sen$ has sent some value to $R$. Our instantiation of SVM is very simple: $Sen$ acts as a dealer and generates $[v]$ through $\Pi_{VSS}$. Once $[v]$ is generated, the parties know that $Sen$ has “committed” to some unknown value. The next step is to let only the parties in $R$ reconstruct $v$.

Using SVM, we now let the various dealers distribute the shares during the underlying instances of $\Pi_{VSS}$ (for $\Pi_{Rand}$) as follows. Consider the dealer $P_i$ who has invoked an instance of $\Pi_{VSS}$ with input $r^{(t)}$. For this, it picks random shares $r_1^{(t)}, \ldots, r_{|Z_i|}^{(t)}$ which sum up to $r^{(t)}$. Now instead of directly sending send $r^{(t)}$ to the parties in $S_i$, it invokes $|Z_i|$ instances of SVM with input $r_1^{(t)}, \ldots, r_{|Z_i|}^{(t)}$ and $S_1, \ldots, S_{|Z_i|}$ as the designated set of receivers respectively. This serves two
purposes. First, it guarantees that all the parties in $S_q$ receive a common share from $P_t$. Second and more importantly, all the parties in $\mathcal{P}$ will now know that $P_t$ has distributed shares to each set from $S_Z$. The parties then run an instance of ACS to identify a common subset of committed dealers $CD \subseteq \mathcal{P}$, where $\mathcal{P} \setminus CD \in Z_s$, which have invoked the desired instances of SVM and delivered the required shares to each group $S_q \in S_{|Z_s|}$. The way timeouts are maintained as part of the ACS, it will be ensured that in a synchronous network, all non-faulty dealers are present in $CD$. Once the set $CD$ is identified, it is guaranteed that every non-faulty party $P_t$ will have the shares from all the dealers in $CD$. And once it has the shares from all the dealers in $CD$, it starts generating the IC-signatures on these shares for the designated parties as part of the $\Pi_{\text{VSS}}$ instances corresponding to the dealers in $CD$ and ensures that all the pre-requisites are satisfied to guarantee the linearity of the underlying IC-signatures. Now instead of selecting the set of dealers $C$ (for $\Pi_{\text{Rand}}$) from $\mathcal{P}$, the parties run an instance of ACS over the set of committed dealers $CD$ to select $C$ where $CD \setminus C \in Z_s$ holds. We stress that irrespective of the network type, the set $C$ is still guaranteed to have at least one non-faulty party. While this is trivially true in an asynchronous network where $Z_a$ satisfies the $Q^{(1)}(C, Z_a)$ condition, the same is true in the synchronous network because $CD$ will have all non-faulty dealers.

**Problem II:** The second problem (in the proposed $\Pi_{\text{Rand}}$) is that the underlying core-sets might be different for the values shared by the dealers in $CD$ (and hence $C$). Instead, we require every dealer in $CD$ to secret-share random values with common underlying core-sets. Only then will it be ensured that the random values generated through $\Pi_{\text{Rand}}$ are secret-shared with common core-sets.

**Way Out.** Getting rid of the above problem is not possible if we let every dealer in $CD$ compute individual core-sets during their respective instances of $\Pi_{\text{VSS}}$, as per the steps of $\Pi_{\text{VSS}}$. Recall that in $\Pi_{\text{VSS}}$, the dealer $D$ computes the underlying core-sets with respect to the “first” set of parties $S_p$ from $S_{|Z_s|}$ which confirm the pairwise consistency of their supposedly common share after exchanging IC-signatures on these values. As a result, different dealers (in $\Pi_{\text{Rand}}$) may end up computing different core-sets in their instances of $\Pi_{\text{VSS}}$ with respect to different candidate $S_p$ sets. To deal with this issue, we instead let each dealer in $CD$ continue computing and publishing different “legitimate” core-sets with respect to various “eligible” candidate $S_p$ sets from $S_{Z_s}$. The parties run an instance of ACS to identify a common subset of dealers $C \in CD$ where $CD \setminus C \in Z_s$, such that all the dealers have computed and published “valid” core-sets, computed with the respect to the same $S_p \in S_{Z_s}$. The idea here is that there always exists a set $S \in S_{Z_s}$ consisting of only non-faulty parties. So if the set of non-faulty dealers $\mathcal{H}$ in $CD$ keep computing and publishing all possible candidate core-sets in their $\Pi_{\text{VSS}}$ instances, then they will publish core-sets with respect to $S$. Hence, $\mathcal{H}$ and $S$ always constitute the candidate $CD$ and the common $S_p$ set.

Note that identifying $C$ out of $CD$ through ACS satisfying the above requirements is non-trivial and requires carefully executing the underlying instances of BA in “two-dimensions”. We first run $|Z_s|$ instances of $\Pi_{\text{BA}}$, one on the behalf of
each set in $S_{Z_a}$, where the $q$th instance is executed to decide whether a subset of
dealers in $CD \setminus Z$ for some $Z \in Z_a$ have published valid core-sets with respect to
the set $S_q \in S_{Z_a}$. This enables the parties to identify a common set $S_{q_{core}} \in S_{Z_a}$,
such that it is guaranteed that a subset of dealers in $CD \setminus Z$ for some $Z \in Z_a$
have indeed published valid core-sets with respect to the set $S_{q_{core}}$. Once the set
$S_{q_{core}}$ is identified, the parties then run $|CD|$ instances of BA to decide which
dealers in $CD$ have published core-sets with respect to $S_{q_{core}}$.

1.2.6 Network Agnostic Secure Multiplication

To generate secret-shared random multiplication-triples for evaluating the mul-
tiplication gates in ckt (using Beaver’s trick), we need a network agnostic secure
multiplication protocol which securely generates a secret-sharing of the product
of two secret-shared values. The key subprotocol behind our multiplication
protocol is a non-robust multiplication protocol $\Pi_{\text{BasicMult}}$ (standing for basic
multiplication), which takes inputs $[a]$ and $[b]$ and an existing set of globally
discarded parties $GD$, which contains only corrupt parties. The protocol securely
generates $[c]$ without revealing any additional information about $a$, $b$ (and $c$). If
no party in $P \setminus GD$ cheats, then $c = a \cdot b$ holds. The idea behind the protocol
is to let each summand $[a]_p \cdot [b]_q$ be secret-shared by a summand-sharing party.
Then $[a \cdot b]$ can be computed from the secret-sharing of each summand, owing to
the linearity property. Existing multiplication protocols in the synchronous and
asynchronous setting [40,5] also use an instantiation of $\Pi_{\text{BasicMult}}$, based on the
above idea. In the sequel, we recall them, followed by the technical challenges
faced in the network agnostic setting and how we deal with them.

$\Pi_{\text{BasicMult}}$ in the Synchronous Setting with $Q^{(2)}(P, Z_a)$ Condition [40].
In [40], each summand $[a]_p \cdot [b]_q$ is statically assigned to a designated summand-
sharing party through some deterministic assignment, which is possible since
$[a]_p$ and $[b]_q$ are held by the parties in $(S_p \cap S_q)$. This is non-empty, since the
$Q^{(2)}(P, Z_a)$ condition holds. Since the parties in $GD$ are already known to be
corrupted, all the shares $[a]_p, [b]_q$ held by the parties in $GD$ are publicly reconstructed
and instead of letting the parties in $GD$ secret-share their assigned summands,
the parties take the “default” secret-sharing of these summands.

$\Pi_{\text{BasicMult}}$ in the Asynchronous Setting with $Q^{(3)}(P, Z_a)$ Condition [5].
The idea of statically designating each summand $[a]_p \cdot [b]_q$ to a unique party in
$P \setminus GD$ need not work in the asynchronous setting, since the designated party
may be corrupt and need not secret-share any summand, thus resulting in an
endless wait. To deal with this challenge, [5] dynamically selects the summand-
sharing parties for each summand. In more detail, let $Z_a = \{Z_1, \ldots, Z_{|Z_a|}\}$ and
$S_{Z_a} = \{S_1, \ldots, S_{|Z_a|}\}$, where each $S_r = P \setminus Z_r$. Since the $Q^{(3)}(P, Z_a)$ condition
is satisfied and $GD \in Z_a$, it follows that $(S_p \cap S_q) \setminus GD \neq \emptyset$. This implies that
there exists at least one non-faulty party in $(S_p \cap S_q)$ who can secret-share the
summand $[a]_p \cdot [b]_q$. Hence, every party in $P \setminus GD$ is allowed to secret-share
all the summands it is “capable” of, with special care taken to ensure that
each summand \([a]_p \cdot [b]_q\) is considered exactly once. For this, the protocol now proceeds in “hops”, where in each hop all the parties in \(\mathcal{P} \setminus \mathcal{GD}\) secret-share all the summands they are capable of, but a single summand sharing party is finally selected for the hop through ACS. Then, all the summands which have been shared by the elected summand-sharing party are “marked” as shared and not considered for sharing in the future hops. Moreover, a party who has already served as a summand-sharing party is not selected in the future hops.

**Technical Challenges in the Network Agnostic Setting.** The asynchronous \(\Pi_{\text{BasicMult}}^{\text{BasicMult}}\) based on dynamically selecting summand-sharing parties will fail in the synchronous network, since the \(Q^{(3)}\) condition need not be satisfied. On the other hand, synchronous \(\Pi_{\text{BasicMult}}^{\text{BasicMult}}\) based on statically selecting summand-sharing parties will fail if a designated summand-sharing party does not secret-share the required summands, resulting in an endless wait. The way out is to select summand-sharing parties in three phases. We first select summand-sharing parties dynamically in hops, following the approach of [5], till we find a subset of parties from \(S_{Z^*}\) which have shared all the summands they are capable of. Then in the second phase, the remaining summands which are not yet secret-shared are statically assigned and shared by the respective designated summand-sharing parties. To avoid an endless wait in this phase, the parties wait only for a “fixed” time required for the parties to secret-share the assigned summands (corresponding to the time taken in a synchronous network) and run instances of BA to identify which of the designated summand-sharing parties have shared their summands up during the second phase. During the third phase, any “leftover” summand which is not yet shared is publicly reconstructed by reconstructing the corresponding shares and a default sharing is taken for such summands.

The idea here is the following: all non-faulty parties will share the summands which are assigned to them, either statically or dynamically, irrespective of the network type. Consequently, the first phase will be always over, since the set consisting of only non-faulty parties always constitutes a candidate set of summand-sharing parties which the parties look for to complete of the first phase. Once the first phase is over, the second phase is bound to be over since the parties wait only for a fixed time. The third phase is always bound to be over, once the first two phases are over, since it involves publicly reconstructing the leftover summands. The way summands are assigned across the three phases, it will be always guaranteed that every summand is considered for sharing once in exactly one of the three phases and no summand will be left out. The crucial point here is that the shares held only by the non-faulty parties never get publicly reconstructed, thus guaranteeing that the adversary does not learn any additional information about \(a\) and \(b\). This is obviously true in a synchronous network because we always have the second phase where every non-faulty party who is not selected as a summand-sharing party during the first phase will get the opportunity to secret-share its assigned summands. On the other hand, in an asynchronous network, it can be shown that all the summands which involve any share held by the non-faulty parties would have been secret-shared during the first phase itself. In more detail, let \(Z^* \in \mathcal{Z}_a\) be the set of corrupt parties and
let $H = \mathcal{P} \setminus \mathbb{Z}^*$ be the set of honest parties. Moreover, let $S_h \in \mathbb{S}_{\mathbb{Z}_s}$ be the group consisting of only non-faulty parties which hold the shares $[a]_h$ and $[b]_h$. Consider an arbitrary summand $[a]_h \cdot [b]_q$. Suppose the first phase gets over because every party in $S_\ell \in \mathbb{S}_{\mathbb{Z}_s}$ has been selected as a summand-sharing party during the first phase. Then consider the set $(S_\ell \cap H \cap S_q)$, which is not empty due to the $Q(2,1)(\mathcal{P}, \mathbb{Z}_s, \mathbb{Z}_a)$ condition. Hence, there exists some $P_j \in (H \cap S_\ell \cap S_q)$, who would have shared $[a]_h \cdot [b]_q$ when selected as a summand-sharing party during some hop in the first phase. Due to a similar reason, any summand of the form $[a]_q \cdot [b]_h$ would have been secret-shared during the first phase itself.

1.3 Other Related Works

The domain of network agnostic cryptographic protocols is relatively new and almost all the existing works have considered threshold adversaries. The work of [16] presents a network agnostic cryptographically-secure atomic broadcast protocol. The work of [45] studies Byzantine fault tolerance and state machine replication protocols for multiple thresholds, including $t_s$ and $t_a$. The work of [35] presents a network agnostic protocol for the task of approximate agreement using the condition $2t_s + t_a < n$. The same condition has been used to design a network agnostic distributed key-generation (DKG) protocol in [6]. A recent work [28] has studied the problem of network agnostic perfectly-secure message transmission (PSMT) [29] over incomplete graphs.

1.4 Open Problems

There are several interesting directions to explore for network agnostic MPC protocols. Here we mention few of them. It is not known whether the condition $3t_s + t_a < n$ (resp. $Q(3,1)(\mathcal{P}, \mathbb{Z}_s, \mathbb{Z}_a)$) is necessary for the network agnostic MPC with perfect security against threshold (resp. non-threshold) adversary. An abundant amount of research effort has been spent to improve both the theoretical as well as practical efficiency of (unconditionally-secure) SMPC and AMPC protocols. The works of [3,4] and this work just focus on the possibility of unconditionally-secure network agnostic MPC. Upgrading the efficiency of these protocols to those of state of the art SMPC and AMPC protocols seems to require a significant research effort. Even though the complexity of our MPC protocol is polynomial in $n$ and $|\mathbb{Z}_s|$, when instantiated for threshold adversaries (where $|\mathbb{Z}_s|$ has all subsets of $\mathcal{P}$ of size up to $t_s$), it may require an exponential (in $n$) amount of computation and communication. This is unlike the case for perfect security, where we have a network agnostic MPC protocol against threshold adversaries with a complexity polynomial (in $n$) [3]. Hence, designing network agnostic MPC protocol against threshold adversaries with statistical security and polynomial complexity is left as a challenging open problem.
2 Preliminaries and Definitions

We assume the pair-wise secure channel model, where the parties in $\mathcal{P}$ are assumed to be connected by pair-wise secure channels. The underlying communication network can be either synchronous or asynchronous, with parties being unaware about the exact network type. In a synchronous network, every message sent is delivered within a known time $\Delta$. In an asynchronous network, messages can be delayed arbitrarily, but finitely, with every message sent being delivered eventually. The distrust among $\mathcal{P}$ is modelled by a malicious (Byzantine) adversary $\text{Adv}$, who can corrupt a subset of the parties in $\mathcal{P}$ and force them to behave in any arbitrary fashion during the execution of a protocol. The parties not under the control of $\text{Adv}$ are called honest. We assume the adversary to be static, who decides the set of corrupt parties at the beginning of the protocol execution. As our main goal is to show the possibility of statistically-secure network agnostic MPC, we keep the formalities to a bare minimum and prove the security of our protocols using the property-based definition, by listing the security properties achieved by our protocols. However, our protocols can also be proven to be secure using the more rigorous Universal Composability (UC) definitional framework [20], without affecting their efficiency.

Adversary $\text{Adv}$ can corrupt any one subset of parties from $Z_s$ and $Z_a$ in synchronous and asynchronous network respectively. The adversary structures are monotone, implying that if $Z \in Z_s$ ($Z \in Z_a$ resp.), then every subset of $Z$ also belongs to $Z_s$ (resp. $Z_a$). We assume that $Z_s$ and $Z_a$ satisfy the conditions $Q^{(2)}(\mathcal{P}, Z_s)$ and $Q^{(3)}(\mathcal{P}, Z_a)$ respectively, which are necessary for statistically-secure MPC in the synchronous and asynchronous network respectively. Additionally, we assume that $Z_a \subset Z_s$. Moreover, $Z_s$ and $Z_a$ satisfy the $Q^{(2,1)}(\mathcal{P}, Z_s, Z_a)$ condition.

In our protocols, all computations are done over a finite field $F$, where $|F| > n^5 \cdot 2^{s_{\text{sec}}}$ and $s_{\text{sec}}$ is the underlying statistical security parameter. Looking ahead, this will ensure that the error probability in our MPC protocol is upper bounded by $2^{-s_{\text{sec}}}$. Without loss of generality, we assume that each $P_i$ has an input $x_i \in F$, and the parties want to securely compute a function $f : F^n \rightarrow F$, represented by an arithmetic circuit $\text{ckt}$ over $F$, consisting of linear and non-linear (multiplication) gates, where $\text{ckt}$ has $c_M$ multiplication gates and a multiplicative depth of $D_M$.

We assume the existence of an unconditionally-secure public-key infrastructure (PKI), for an unconditionally-secure signature scheme, also called pseudo-signature [50,34]. We briefly explain the requirements from such a setup and refer to [34] for complete formal details. There exists a publicly known vector of public keys $(pk_1, \ldots, pk_n)$, where each honest $P_i$ holds the generated secret key $sk_i$, associated with $pk_i$. A valid signature $\tau$ on message $m$ from $P_i$ is one for which $\text{Verify}_{pk_i}(m, \tau) = 1$, where $\text{Verify}$ is the verification function of the underlying signature scheme. For simplicity, we make the standard convention of treating signatures as idealized objects during our protocol analysis; i.e., we assume that

$\text{Corrupt parties may choose their keys arbitrarily.}$
the signatures are perfectly unforgeable and hence Adv will fail to forge signature of an honest party on any message, which is not signed by the party. We also assume that the signatures are transferable and any party upon receiving a valid signature from a party can send and get it verified by any other party. However, unlike the standard digital signatures which are computationally-secure and offer arbitrary number of transfers, pseudo-signatures offer a “limited” number of transfers, which is typically bounded as a function of the number of parties in the protocol where pseudo-signature is used as a primitive. We assume that the given setup supports the required number of transfers demanded by our protocols. We use |σ| to represent the size of a pseudo-signature in bits. If \( P_i \) signs a message \( m \), then we denote the resultant signed message as \( \langle m \rangle_i \).

**Termination Guarantees of Our Sub-Protocols:** As done in [3,5], for simplicity, we will not be specifying any termination criteria for our sub-protocols. The parties will keep on participating in these sub-protocol instances even after computing their outputs. The termination criteria of our MPC protocol will ensure the termination of all underlying sub-protocol instances. We will be using an existing randomized ABA protocol [23] which ensures that the honest parties (eventually) obtain their respective output almost-surely with probability 1. This means that the probability that an honest party obtains its output after participating for infinitely many rounds approaches 1 asymptotically [1,47,7]. That is:

\[
\lim_{T \to \infty} \Pr[\text{An honest } P_i \text{ obtains its output by local time } T] = 1,
\]
where the probability is over the random coins of the honest parties and the adversary in the protocol. The property of almost-surely obtaining the output carries over to the “higher” level protocols, where ABA is used as a building block. We will say that the “honest parties obtain some output almost-surely from protocol \( \Pi \)” to mean that every honest \( P_i \) asymptotically obtains its output in \( \Pi \) with probability 1, in the above sense.

3 Network Agnostic Unconditionally Secure Byzantine Agreement

We recall the definition of BA from [4], which is adapted from [15,3].

**Definition 3.1 (BA).** Let \( \Pi \) be a protocol for \( P \), where every party \( P_i \) has an input \( b_i \in \{0, 1\} \) and a possible output from \( \{0, 1, \perp\} \). Moreover, let \( \text{Adv} \) be a computationally-unbounded adversary, characterized by adversary structure \( Z \), where \( \text{Adv} \) can corrupt any subset of parties from \( Z \) during the execution of \( \Pi \).

- **\( Z \)-Guaranteed Liveness:** \( \Pi \) has \( Z \)-guaranteed liveness if all honest parties obtain an output.
- **\( Z \)-Almost-Surely Liveness:** \( \Pi \) has \( Z \)-almost-surely liveness if, almost-surely, all honest parties obtain some output.
- **\( Z \)-Validity:** \( \Pi \) has \( Z \)-validity if the following holds: If all honest parties have input \( b \), then every honest party with an output, outputs \( b \).
Z-Weak Validity: \( \Pi \) has Z-weak validity if the following holds: If all honest parties have input \( b \), then every honest party with an output, outputs \( b \) or \( \perp \).

Z-Consistency: \( \Pi \) has Z-consistency if all honest parties with an output, output the same value (which can be \( \perp \)).

Z-Weak Consistency: \( \Pi \) has Z-weak consistency if all honest parties with an output, outputs either a common \( v \in \{0, 1\} \) or \( \perp \).

\( \Pi \) is called a Z-secure synchronous BA (SBA) if, in a synchronous network, it achieves Z-guaranteed liveness, Z-validity, and Z-consistency. \( \Pi \) is called a Z-secure asynchronous BA (ABA) if, in an asynchronous network it has Z-almost-surely liveness, Z-validity and Z-consistency.\(^8\)

To design our network agnostic BA protocol, we will be using a special type of broadcast protocol. We next review the definition of broadcast from [4] which is further adapted from [15,3].

Definition 3.2 (Broadcast). Let \( \Pi \) be a protocol, where a designated sender \( \text{Sen} \in \mathcal{P} \) has input \( m \in \{0, 1\}^l \), and parties obtain a possible output, including \( \perp \). Moreover, let \( \text{Adv} \) be a computationally-unbounded adversary, characterized by an adversary structure \( \mathcal{Z} \), where \( \text{Adv} \) can corrupt any subset from \( \mathcal{Z} \) during \( \Pi \).

Z-Liveness: \( \Pi \) has Z-liveness if all honest parties obtain some output.

Z-Validity: \( \Pi \) has Z-validity if the following holds: if \( \text{Sen} \) is honest, then every honest party with an output, outputs \( m \).

Z-Weak Validity: \( \Pi \) has Z-weak validity if the following holds: if \( \text{Sen} \) is honest, then every honest party with an output, outputs either \( m \) or \( \perp \).

Z-Consistency: \( \Pi \) has Z-consistency if the following holds: if \( \text{Sen} \) is corrupt, then every honest party with an output, outputs a common value.

Z-Weak Consistency: \( \Pi \) has Z-weak consistency if the following holds: if \( \text{Sen} \) is corrupt, then every honest party with an output, outputs a common \( m^* \in \{0, 1\}^l \) or \( \perp \).

\( \Pi \) is called a Z-secure broadcast protocol if it has Z-Liveness, Z-Validity, and Z-Consistency.

We next recall a blueprint for network agnostic BA from [15,3].

3.1 A Blueprint for Network Agnostic BA [15,3]

To design our network agnostic BA, we assume the existence of the following sub-protocols.

Synchronous BA with Asynchronous Guarantees: we assume the existence of a protocol \( \Pi_{\text{SBA}} \), which is a Z\(_a\)-secure SBA and which has Z\(_a\)-weak validity and Z\(_a\)-guaranteed liveness in the asynchronous network. At (local) time

\(^8\) The seminal FLP impossibility result [32] rules out the possibility of any deterministic ABA, where there always exists a “bad” execution in which the honest parties may keep on running the protocol forever, without obtaining any output. To circumvent this, one can opt for randomized ABA protocols and hope that the bad executions occur asymptotically with probability 0. 

19
T_{SBA}, all honest parties will have an output (which could be ⊥), irrespective of the network type.

- **Asynchronous BA with Synchronous Guarantees**: we assume the existence of a protocol Π_{ABA}, which is a Z_a-secure ABA. Moreover, in a synchronous network, the protocol has Z_s-validity, with all honest parties computing their output within time T_{ABA} = c \cdot \Delta in this case, for some known constant c.\(^9\)

Based on Π_{SBA} and Π_{ABA}, one can design a network agnostic BA protocol Π_{BA} as follows, following the blueprint of [15,3]. The parties first invoke an instance of Π_{SBA}, assuming a synchronous network. If the network is indeed synchronous, then the (honest) parties should have a binary output at time T_{SBA}. The parties check the same and either switch their input to the output of Π_{SBA}, if it is not ⊥, or stick to their original input. The parties then invoke an instance of Π_{ABA} with “updated” inputs and the output of Π_{ABA} is set to be the overall output.

The description of Π_{BA}, taken from [3], is presented in Fig 1.

<table>
<thead>
<tr>
<th>Protocol Π_{BA}</th>
</tr>
</thead>
<tbody>
<tr>
<td>- On having input b_i, participate in an instance of Π_{SBA} with input b_i and wait till the local time becomes T_{SBA}. Let u_i be the output from Π_{SBA} at time T_{SBA}. If u_i ≠ ⊥, then set v_i^* = u_i. Else set v_i^* = b_i.</td>
</tr>
<tr>
<td>- Participate in an instance of Π_{ABA} with input v_i^*. Output the result of Π_{ABA}, when it is available.</td>
</tr>
</tbody>
</table>

Fig. 1: Network agnostic BA protocol from Π_{SBA} and Π_{ABA}. The above code is executed by P_i.

Theorem 3.1, follows from [3], given that Π_{SBA} and Π_{ABA} achieve the stated properties. For completeness, the theorem is proved in Appendix A.

**Theorem 3.1.** Let Z_a ⊂ Z_s such that Z_s and Z_a satisfy the conditions Q^{(2)}(P, Z_s) and Q^{(3)}(P, Z_a) respectively. Moreover, let Z_s and Z_a satisfy the Q^{(2,1)}(P, Z_s, Z_a) condition. Then protocol Π_{BA} achieves the following.\(^10\)

- **Synchronous Network**: the protocol is a Z_s-secure SBA, where all honest parties get their output at time T_{BA} = T_{SBA} + T_{ABA}.
- **Asynchronous Network**: the protocol is a Z_a-secure ABA.

We now proceed to instantiate protocols Π_{SBA} and Π_{ABA}.

### 3.2 Π_{SBA}: Synchronous BA with Asynchronous Weak Validity and Guaranteed Liveness

To design protocol Π_{SBA}, we again follow the blueprint of [3], which design Π_{SBA} based on three components.

\(^9\) Thus Π_{ABA} has Z_a-guaranteed liveness in the synchronous network, if all honest parties have the same input. However, if the honest parties have different inputs, then Π_{ABA} need not provide guaranteed liveness or consistency guarantees in the synchronous network.

\(^10\) As the number of invocations of Π_{BA} in our MPC protocol will be independent of |ckt|, we do not focus on its exact complexity. However, we confirm that it will be polynomial in n and |Z_s|.
3.2.1 SBA with Asynchronous Guaranteed Liveness

The first component for designing $\Pi_{\text{SBA}}$ is an SBA protocol, which has just guaranteed liveness in an asynchronous network. In [50], the authors have presented an SBA protocol against threshold adversaries, tolerating up to $t_s < n/2$ faults. The protocol which we denote as $\Pi_{\text{PW}}$, modifies the Dolev-Strong BA protocol [30], by replacing digital signatures with pseudo-signatures. We note that $\Pi_{\text{PW}}$ can be easily generalized, if $\mathcal{Z}_s$ satisfies the $Q^{(2)}(\mathcal{P}, \mathcal{Z}_s)$ condition. To achieve guaranteed liveness in an asynchronous network, the parties run the protocol till the supposed timeout in the synchronous network and check if any output is computed and in case no output is computed, $\bot$ is taken as the output. Since the protocol is an easy generalization of the existing protocol against threshold adversaries, protocol $\Pi_{\text{PW}}$ and proof of Lemma 3.1 are available in Appendix A.

Lemma 3.1. Protocol $\Pi_{\text{PW}}$ achieves the following, where $t$ is the cardinality of the maximum-sized subset in $\mathcal{Z}_s$.

- **Synchronous Network**: The protocol is a $\mathcal{Z}_s$-secure SBA protocol, where all honest parties compute their output at time $T_{\text{PW}} \overset{\text{def}}{=} (t + 1) \cdot \Delta$.
- **Asynchronous Network**: The protocol achieves $\mathcal{Z}_s$-guaranteed liveness, where all honest parties compute their output at time $T_{\text{PW}}$.
- **Communication Complexity**: $O(n^4 \cdot \ell \cdot |\sigma|)$ bits are sent by the honest parties, if the inputs of the parties are of size $\ell$ bits.

3.2.2 Asynchronous Broadcast with Synchronous Guarantees

The second component for designing $\Pi_{\text{SBA}}$ is an asynchronous broadcast protocol $\Pi_{\text{Acast}}$ (also called Acast), which provides liveness, validity and a “variant” of consistency in a synchronous network. The variant guarantees that if $\text{Sen}$ is corrupt and the honest parties compute an output in a synchronous network, then they may not do it at the same time and there might be a gap in the time at which the honest parties compute an output. In [43,4], an instantiation of $\Pi_{\text{Acast}}$ is provided against $Q^{(3)}$ adversary structures. Unfortunately, the protocol fails to provide any security guarantees in a synchronous network against $Q^{(2)}$ adversary structures. So we provide a different instantiation of Acast for our setting (see Fig 2). The protocol is obtained by generalizing the ideas used in the broadcast protocol of [46]. The protocol of [46] uses a computational PKI, which we replace with an unconditional PKI. The protocol consists of three phases and each (honest) party executes a phase at most once.

Protocol $\Pi_{\text{Acast}}(\text{Sen}, m, \mathcal{Z}_s, \mathcal{Z}_a)$

Each $P_i \in \mathcal{P}$ executes each of the following phases at most once.

- **(Propose)**: If $P_i = \text{Sen}$, then on having the input $m$, send $\langle (\text{propose}, m) \rangle_{\text{Sen}}$ to all the parties.

The protocol is a generalization of the classic Bracha’s threshold Acast protocol [18], tolerating $t < n/3$ corruptions.
(Vote): Upon receiving the first propose message \(\langle \text{propose}, m \rangle_{\text{Sen}}\) with valid signature, send \(\langle \text{propose}, m \rangle\) to all the parties and wait till the local time increases by \(\Delta\). If \(\langle \text{propose}, m' \rangle\) is not received from any party where \(m' \neq m\), then send \(\langle \text{vote}, m \rangle\) to all the parties.

(Output): Upon receiving a vote message \(\langle \text{vote}, m \rangle\) with valid signature corresponding to every \(P_j \in P \setminus Z\) for some \(Z \in Z_s\), do the following:

- Let \(C(m)\) denote the collection of signed \(\langle \text{vote}, m \rangle\) messages.
- Send \(C(m)\) to all the parties and output \(m\).

Fig. 2: Asynchronous broadcast with synchronous guarantees. The above code is executed by each \(P_i \in P\) including the designated sender \(\text{Sen}\).

The proof of Lemma 3.2 is available in Appendix A.

Lemma 3.2. Protocol \(\Pi_{Acast}\) achieves the following properties.
- Asynchronous Network: The protocol is a \(Z_a\)-secure broadcast protocol.
- Synchronous Network: (a) \(Z_s\)-Liveness: If \(\text{Sen}\) is honest, then all honest parties obtain an output within time \(3\Delta\). (b) \(Z_s\)-Validity: If \(\text{Sen}\) is honest, then every honest party with an output, outputs \(m\). (c) \(Z_s\)-Consistency: If \(\text{Sen}\) is corrupt and some honest party outputs \(m^*\) at time \(T\), then every honest \(P_i\) outputs \(m^*\) by the end of time \(T + \Delta\).
- Communication Complexity: \(O(n^3 \cdot \ell \cdot |\sigma|)\) bits are communicated by the honest parties, where \(\ell\) is the size of \(\text{Sen}\)'s input.

Terminologies for Using \(\Pi_{Acast}\). In the protocol \(\Pi_{Acast}\), any party from \(P\) can be designated as \(\text{Sen}\). In the rest of the paper we will say that “\(P_i\) Acasts \(m\)” to mean that \(P_i\) acts as \(\text{Sen}\) and invokes an instance of \(\Pi_{Acast}\) with input \(m\), and the parties participate in this instance. Similarly, “\(P_j\) receives \(m\) from the Acast of \(P_i\)” means that \(P_j\) outputs \(m\) in the corresponding instance of \(\Pi_{Acast}\).

3.2.3 Synchronous Broadcast with Asynchronous Guarantees

The third component for designing \(\Pi_{SBA}\) is a broadcast protocol \(\Pi_{BC}\), which is secure in a synchronous network and which also provides liveness, weak validity and weak consistency in an asynchronous network. Note that the guarantees of \(\Pi_{BC}\) are different from that of \(\Pi_{Acast}\). The design of \(\Pi_{BC}\) is based on the idea from [4], by carefully combining protocols \(\Pi_{Acast}\) and \(\Pi_{PW}\). In the protocol, \(\text{Sen}\) first Acasts its message. If the network is synchronous, then at time \(3\Delta\), all honest parties should have an output. To confirm this, the parties start participating in an instance of \(\Pi_{PW}\), with whatever output has been obtained from the \(\Pi_{Acast}\) instance at time \(3\Delta\); in case no output is obtained, then the input is \(\bot\). Finally, at time \(3\Delta + T_{PW}\), the parties output an \(m^*\), if it is the output of the \(\Pi_{Acast}\) instance as well as the output of \(\Pi_{PW}\), else the output of the parties will be \(\bot\). We recall the description of \(\Pi_{BC}\) from [4] and present it in Fig 3.

Protocol \(\Pi_{BC}\) as a Network Agnostic Secure Broadcast. Protocol \(\Pi_{BC}\) only guarantees weak validity and weak consistency in an asynchronous network, since only a subset of honest parties may receive \(\text{Sen}\)'s message from the Acast.
of Sen within time $3\Delta + T_{PW}$. Note that maintaining the time-out is essential, as we need liveness from $\Pi_{BC}$ (irrespective of the network type) when used later in protocol $\Pi_{SBA}$. Looking ahead, we will use $\Pi_{BC}$ in our VSS protocol for broadcasting values. The weak validity and consistency may lead to a situation where, in an asynchronous network, one subset of honest parties may output a value different from $\perp$ at the end of the time-out $3\Delta + T_{PW}$, while others may output $\perp$. For the security of the VSS protocol, we would require the latter category of parties to eventually output the common non-$\perp$ value if the parties continue participating in $\Pi_{BC}$. Following [3,4], we make a provision for this in $\Pi_{BC}$. Namely, each $P_i$ who outputs $\perp$ at time $3\Delta + T_{PW}$ “switches” its output to $m^*$, if $P_i$ eventually receives $m^*$ from Sen’s Acast. We stress that this switching is only for the parties who obtained $\perp$ at time $3\Delta + T_{PW}$. To differentiate between the two ways of obtaining output, we use the terms regular-mode and fallback-mode. Regular-mode is the process of deciding the output at time $3\Delta + T_{PW}$, while fallback-mode is the process of deciding the output beyond time $3\Delta + T_{PW}$.

**Protocol** $\Pi_{BC}(\text{Sen, } m, Z_\triangle, Z_\triangle)$

(Regular Mode)
- Sender Sen on having the input $m$, Acasts $m$.
- At time $3\Delta$, each $P_i \in \mathcal{P}$ participates in an instance of $\Pi_{PW}$, where the input of $P_i$ is $m^*$ if $m^* \in \{0,1\}^\ell$ is received from the Acast of Sen, else the input is $\perp$.
- *(Local Computation):* At time $3\Delta + T_{PW}$, each $P_i \in \mathcal{P}$ does the following.
  - If some $m^* \in \{0,1\}^\ell$ is received from the Acast of Sen and $m^*$ is computed as the output during the instance of $\Pi_{PW}$, then output $m^*$. Else output $\perp$.

(Fallback Mode)
- Each $P_i \in \mathcal{P}$ who outputs $\perp$ at time $3\Delta + T_{PW}$, changes it to $m^*$, if $m^*$ is received from Acast of Sen.

**Fig. 3:** Synchronous broadcast with asynchronous guarantees.

Theorem 3.2 follows from [4] and is proved in Appendix A.

**Theorem 3.2.** Protocol $\Pi_{BC}$ achieves the following, with a communication complexity of $O(n^4 \cdot \ell \cdot |\sigma|)$ bits, where $T_{BC} = 3\Delta + T_{PW}$.

- Synchronous network: (a) $Z_\triangle$-Liveness: At time $T_{BC}$, each honest party has an output. (b) $Z_\triangle$-Validity: If Sen is honest, then at time $T_{BC}$, each honest party outputs $m$. (c) $Z_\triangle$-Consistency: If Sen is corrupt, then the output of every honest party is the same at time $T_{BC}$. (d) $Z_\triangle$-Fallback Consistency: If Sen is corrupt, and some honest party outputs $m^* \neq \perp$ at time $T$ through fallback-mode, then every honest party outputs $m^*$ by time $T + \Delta$.

- Asynchronous Network: (a) $Z_\triangle$-Liveness: At time $T_{BC}$, each honest party has an output. (b) $Z_\triangle$-Weak Validity: If Sen is honest, then at time $T_{BC}$, each honest party outputs $m$ or $\perp$. (c) $Z_\triangle$-Fallback Validity: If Sen is honest, then each honest party with output $\perp$ at time $T_{BC}$, eventually outputs $m$ through fallback-mode. (d) $Z_\triangle$-Weak Consistency: If Sen is corrupt, then there exists an $m^* \neq \perp$, such that at time $T_{BC}$, each honest party outputs $m^*$ or $\perp$. (e) $Z_\triangle$-Fallback Consistency: If Sen is corrupt, and

23
some honest party outputs $m^* \neq \bot$ at time $T \geq T_{BC}$, then each honest party eventually outputs $m^*$.

In the rest of the paper, we use the following terminologies while using $\Pi_{BC}$.

**Terminologies for $\Pi_{BC}$:** We say that $P_i$ broadcasts $m$ to mean that $P_i$ invokes an instance of $\Pi_{BC}$ as $\text{Sen}$ with input $m$, and the parties participate in this instance. Similarly, we say that $P_j$ receives $m$ from the broadcast of $P_i$ through regular-mode (resp. fallback-mode), to mean that $P_j$ has the output $m$ at time $T_{BC}$ (resp. after time $T_{BC}$) during the instance of $\Pi_{BC}$.

### 3.2.4 $\Pi_{BC} \rightarrow \Pi_{SBA}$

Finally, using $\Pi_{BC}$ we instantiate protocol $\Pi_{SBA}$, following the blueprint of [4]. In the protocol, every party broadcasts its input bit (for $\Pi_{SBA}$) through an instance of $\Pi_{BC}$. At time $T_{BC}$, the parties check if “sufficiently many” instances of $\Pi_{BC}$ have produced a binary output (which should have happened in a synchronous network) and if so, they output the “majority” of those values. Otherwise, the network is asynchronous, in which case the parties output $\bot$. The description of $\Pi_{SBA}$ is recalled from [4] and presented in Fig 4.

#### Protocol $\Pi_{SBA}(Z_s, Z_a)$

- On having input $b_i \in \{0, 1\}$, broadcast $b_i$.
- For $j = 1, \ldots, n$, let $b_j^{(i)} \in \{0, 1, \bot\}$ be received from the broadcast of $P_j$ through regular-mode. Include $P_j$ to a set $\mathcal{SV}$ if $b_j^{(i)} \neq \bot$.
- If $P \setminus \mathcal{SV} \in Z_s$, then compute the output as follows.
  - If there exists a subset of parties $\mathcal{SV}_i \subseteq \mathcal{SV}$, such that $\mathcal{SV} \setminus \mathcal{SV}_i \in Z_s$ and $b_j^{(i)} = b$ for all the parties $P_j \in \mathcal{SV}_i$, then output $b$.
  - Else output 1.
- Else output $\bot$.

*a* If there are multiple such $\mathcal{SV}_i$, then break the tie using some pre-determined rule.

Fig. 4: Synchronous BA with asynchronous guaranteed liveness and weak validity. The above code is executed by every $P_i \in \mathcal{P}$.

Theorem 3.3 follows from [4] and is proved in Appendix A.

**Theorem 3.3.** Protocol $\Pi_{SBA}$ achieves the following where $T_{SBA} = T_{BC}$, incurring a communication of $O(n^5 \cdot |\sigma|)$ bits.

- Synchronous Network: the protocol is a $Z_s$-secure SBA protocol where honest parties have an output, different from $\bot$, at time $T_{SBA}$.
- Asynchronous Network: the protocol achieves $Z_a$-guaranteed liveness and $Z_a$-weak validity, such that all honest parties have an output at (local) time $T_{SBA}$.

### 3.3 $\Pi_{ABA}$: Asynchronous BA with Synchronous Validity

To the best of our knowledge, the only known unconditionally-secure ABA protocol is due to [23], which generalizes the framework of randomized ABA
against general adversaries. The protocol of \cite{23} uses a graded agreement (GA) protocol (also known as the vote protocol), along with a secure coin-flip protocol. Unfortunately, both these primitives provide security (in an asynchronous network) against $Q^{(3)}$ adversary structures and fail to provide any security guarantees against $Q^{(2)}$ adversary structures in a synchronous network. Consequently, the ABA protocol of \cite{23} fails to provide any security guarantees in a synchronous network against $Q^{(2)}$ adversary structures.\footnote{Recall that we need validity, coupled with guaranteed liveness from $\Pi_{ABA}$, when used in our network agnostic BA protocol $\Pi_{BA}$.}

We give a different instantiation of $\Pi_{ABA}$ by generalizing a few ideas used in \cite{15}. Our instantiation of $\Pi_{ABA}$ is based on following two components.

**Component I: Asynchronous Graded Agreement with Synchronous Validity.** We assume the existence of a GA protocol $\Pi_{GA}$, where each party has a binary input. The output for each party is a value from \{0, 1, ⊥\}, along with a grade from \{0, 1, 2\}. The protocol achieves the following properties.

- **Asynchronous Network** — The following properties are achieved, even if the adversary corrupts any subset from $Z_a$: (a) $Z_a$-Liveness: If all honest parties participate in the protocol, then each honest party eventually obtains an output. (b) $Z_a$-Graded Validity: If every honest party’s input is $b$, then all honest parties with an output, output (b, 2). (c) $Z_a$-Graded Consistency: If two honest parties output grades $g, g'$, then $|g - g'| \leq 1$ holds; moreover, if two honest parties output $(v, g)$ and $(v', g')$ with $g, g' \geq 1$, then $v = v'$.

- **Synchronous Network** — The following properties are achieved, even if the adversary corrupts any subset from $Z_s$: (a) $Z_s$-Liveness: If all honest parties participate in the protocol with the same input, then after some fixed time $T_{GA}$, all honest parties obtain an output. (b) $Z_s$-Graded Validity: If every honest party’s input is $b$, then all honest parties with an output, output (b, 2).

Note that we do not require any form of consistency from $\Pi_{GA}$ in the synchronous network. We give an instantiation of $\Pi_{GA}$ with the $Q^{(2,1)}(P, Z_s, Z_a)$ condition, by generalizing the threshold GA protocol of \cite{15} with condition $2t_s + t_a < n$, such that $T_{GA} = 4 \cdot \Delta$. The protocol of \cite{15} uses digital signatures (hence, is computationally secure), which we replace with pseudo-signatures. For the description of $\Pi_{GA}$ and its properties, see Appendix A.

**Component II: Asynchronous Coin-Flipping with Synchronous Liveness.** We assume the existence of a $p$-coin-flipping protocol $\Pi_{CoinFlip}$, where $0 < p < 1$ is a parameter. In the protocol, the parties participate with random inputs and the output of each party is a bit satisfying the following properties.

- **Asynchronous Network**: The following properties are achieved even if the adversary corrupts any subset from $Z_a$: (a) $Z_a$-Almost-Surely Liveness: If all honest parties participate in the protocol, then almost-surely, all honest parties eventually get an output. (b) $(Z_a, p)$-Commonness: With probability $p$, the output of all honest parties is a random bit $b \in \{0, 1\}$. 


Synchronous Network: The following property is achieved even if the adversary corrupts any subset from $Z_s$: (a): $Z_s$-Guaranteed Liveness: If all honest parties participate in the protocol, then all honest parties get an output, after some fixed time $T_{CoinFlip}$.

In [23], the authors presented an instantiation of $Π_{CoinFlip}$, which achieves $Z_a$-Almost-Surely Liveness as well as $(Z_a, p)$-Commonness in an asynchronous network, where $p = \frac{1}{n}$, provided $Z_a$ satisfies the $Q^{(3)}(P, Z_a)$ condition. The protocol incurs an expected communication of $O(poly(n, |Z_s|, \log |F|))$ bits. Interestingly, the protocol also achieves $Z_s$-Guaranteed Liveness in a synchronous network, irrespective of $Z_s$. Namely, after time $T_{CoinFlip} = 20 \cdot \Delta$, all honest parties will have an output, with (honest) parties communicating $O(poly(n, |Z_s|, \log |F|))$ bits.

### 3.3.1 $Π_{GA} + Π_{CoinFlip} → Π_{ABA}$

Once we have instantiations of $Π_{GA}$ and $Π_{CoinFlip}$, we can easily combine it using the framework of [51, 11, 21, 31] to get the protocol $Π_{ABA}$. The protocol consists of several iterations, where in each iteration, the parties run two instances of $Π_{GA}$, along with an instance of $Π_{CoinFlip}$. Using the first instance of $Π_{GA}$, the parties check if they all have the same input. Independent of this finding, they then run an instance of $Π_{CoinFlip}$. Finally, they again run an instance of $Π_{GA}$, with inputs being carefully chosen. Namely, if a party obtained an output with the highest grade from the first instance of $Π_{GA}$, then it participates with this input, else it participates with the coin-output. Finally, based on the output received from the second instance of $Π_{GA}$, the parties update their input for the next iteration as follows: if a bit with a non-zero grade is obtained, then it is set as the updated input, else the updated input is set to the input of the second instance of $Π_{GA}$. During each iteration, the parties keep a tab on whether they have received an output bit with the highest grade from the second instance of $Π_{GA}$, in which case, they indicate it to the others by sending a signed ready message and the bit. Once “sufficiently many” parties send the same signed ready bit, it is taken as the output of the protocol.

The idea here is that if the honest parties start an iteration with the same input bit $b$, then the output of $Π_{CoinFlip}$ is not considered (irrespective of the network type) and all instances of $Π_{GA}$ output $(b, 2)$. Thus, all honest parties will send a signed ready message for $b$. Consequently, all honest parties will output $b$. This ensures validity, coupled with guaranteed liveness, both in synchronous and asynchronous networks. On the other hand, if the honest parties start an iteration with different inputs, then with probability at least $p \cdot \frac{1}{2} = \frac{1}{n} \cdot \frac{1}{2}$, all of them will have the same input for the second instance of $Π_{GA}$. And consequently, all honest parties will have the same input from the next iteration onward and consistency is achieved (in the asynchronous network). The description of $Π_{ABA}$ based on $Π_{GA}$ and $Π_{CoinFlip}$ is recalled from [23] and presented in Fig 5.
• Initialization: Set $b = b_i$, committed = false and $k = 1$. Then do the following.
  1. Participate in an instance of $\Pi_{GA}$ protocol with input $b$ and wait for time $T_{GA}$.
  2. Once an output $(b, g)$ is received from the instance of $\Pi_{GA}$, participate in an
     instance of $\Pi_{CoinFlip}$ and wait for time $T_{CoinFlip}$. Let $Coin_k$ denote the output
     received from $\Pi_{CoinFlip}$.
  3. If $g < 2$, then set $b = Coin_k$.
  4. Participate in an instance of $\Pi_{GA}$ protocol with input $b$ and wait for time $T_{GA}$.
     Let $(b', g')$ be the output received. If $g' > 0$, then set $b = b'$.
  5. If $g' = 2$ and committed = false, then set committed = true and send
     $\langle (\text{ready}, b) \rangle_i$ to all the parties.
  6. Set $k = k + 1$ and repeat from 1.

• Output Computation:
  - Upon receiving $\langle (\text{ready}, b) \rangle_i$ messages with valid signatures corresponding to
    every $P_j \in P \setminus Z$ for some $Z \in Z_s$, send $(b, C(b))$ to all the parties and
    output $b$. Here $C(b)$ denotes the collection of signed $\langle (\text{ready}, b) \rangle_i$ messages.
  - Upon receiving $(b, C(b))$ where $C(b)$ has valid signatures corresponding to
    every $P_j \in P \setminus Z$ for some $Z \in Z_s$, send $(b, C(b))$ to all the parties and
    output $b$.

Fig. 5: The ABA protocol. The above code is executed by every $P_i \in P$ with input $b_i$.

Theorem 3.4 follows from [15] and is proved in Appendix A.

Theorem 3.4. Protocol $\Pi_{ABA}$ achieves the following where $T_{ABA} = T_{CoinFlip} + 2T_{GA} + \Delta$.
  - Synchronous Network: If all honest parties have the same input $b \in \{0, 1\}$, then all honest parties output $b$, at time $T_{ABA}$. Moreover, $O(\text{poly}(n, |Z_s|, \log |F|))$ bits are communicated by the honest parties.
  - Asynchronous Network: the protocol is a $Z_a$-secure ABA, incurring an expected communication of $O(\text{poly}(n, |Z_s|, \log |F|))$ bits.

4 Network Agnostic Information Checking Protocol

In this section, we present our network agnostic ICP protocol (Fig 6). A detailed overview of the protocol has been already presented in Section 1.2.2. The protocol consists of two subprotocols $\Pi_{Auth}$ and $\Pi_{Reveal}$, implementing the authentication and revelation phase respectively, where the parties participate in the revelation phase only upon completing the authentication phase. During the authentication phase, $S$ distributes the authentication and verification information, followed by parties publicly verifying the consistency of distributed information and once the consistency is established, the authentication phase is over. During the revelation phase, $I$ reveals the IC-signature which is verified by $R$ with respect to the verification information revealed by a “selected” subset of the verifiers.
Protocol $\Pi_{ICP}(P, Z, Z_0, S, I, R)$

Protocol $\Pi_{auth}(P, Z, Z_0, S, I, R, s)$: $t \triangleq \max\{|Z| : Z \in Z_s\}$

- **Distributing Data**: $S$ executes the following steps.
  - Randomly select $t$-degree signing-polynomial $F(x)$ and $t$-degree masking-polynomial $M(x)$, where $F(0) = s$. For $i = 1, \ldots, n$, randomly select $\alpha_i \in \mathbb{F} \setminus \{0\}$, and compute $v_i = F(\alpha_i)$ and $m_i = M(\alpha_i)$.
  - Send $(F(x), M(x))$ to $I$. For $i = 1, \ldots, n$, send $(\alpha_i, v_i, m_i)$ to party $P_i$.

- **Confirming Receipt of Verification Points**: Each party $P_i$ (including $S$, $I$, and $R$), upon receiving $(\alpha_i, v_i, m_i)$ from $S$, broadcasts (Received, $i$).

- **Announcing Set of Supporting Verifiers**: only $S$ does the following.
  - Initialize the set of supporting verifiers $SV$ to $\emptyset$, and wait till the local time is $\Delta + T_{BC}$. Upon receiving (Received, $i$) from the broadcast of $P_i$, add $P_i$ to $SV$. Once $P \setminus SV \in \mathbb{Z}_s$, broadcast the set $SV$.

- **Announcing Masked Polynomial**: only $I$ does the following.
  - Wait till the local time is $\Delta + 2T_{BC}$. Upon receiving $SV$ from the broadcast of $S$ such that $P \setminus SV \in \mathbb{Z}_s$, wait till (Received, $i$) is received from the broadcast of every $P_i \in SV$. Then randomly pick $d \in \mathbb{F} \setminus \{0\}$ and broadcast $(d, B(x))$, where $B(x) \triangleq dF(x) + M(x)$.

- **Announcing Validity of Masked Polynomial**: only $S$ does the following.
  - Wait till the local time is $\Delta + 3T_{BC}$. Upon receiving $(d, B(x))$ from the broadcast of $I$, broadcast OK, if $B(x)$ is a $t$-degree polynomial and if $dv_j + m_i = B(\alpha_j)$ holds for every $P_j \in SV$.

- **Deciding Whether Authentication is Successful**: each $P_i \in P$ (including $S$, $I$, and $R$) waits till the local time is $\Delta + 4T_{BC}$. Upon receiving $SV$ and $(d, B(x))$ from the broadcast of $S$ and $I$ respectively, where $P \setminus SV \in \mathbb{Z}_s$, it set the variable authCompleted$_{(S, I, R)}$ to 1 if OK is received from the broadcast of $S$. Upon setting authCompleted$_{(S, I, R)}$ to 1, $I$ sets $ICSig(S, I, R, s) = F(x)$.

- **Revealing Signing Polynomial and Verification Points**: Each party $P_i$ (including $S$, $I$, and $R$) does the following, if authCompleted$_{(S, I, R)}$ is set to 1.
  - If $P_i = I$ then send $F(x)$ to $R$, if $ICSig(S, I, R, s)$ is set to $F(x)$ during $\Pi_{auth}$.
  - If $P_i \in SV$, then send $(\alpha_i, v_i, m_i)$ to $R$.

- **Accepting the IC-Signature**: The following steps are executed only by $R$, if authCompleted$_{(S, I, R)}$ is set to 1 during the protocol $\Pi_{auth}$.
  - Wait till the local time becomes a multiple of $\Delta$. Upon receiving $F(x)$ from $I$, where $F(x)$ is a $t$-degree polynomial, proceed as follows.
  1. If $(\alpha_i, v_i, m_i)$ is received from $P_i \in SV$, then accept $(\alpha_i, v_i, m_i)$ if either $v_i = F(\alpha_i)$ or $B(\alpha_i) \neq dv_i + m_i$, where $B(x)$ is received from the broadcast of $I$ during $\Pi_{auth}$. Otherwise, reject $(\alpha_i, v_i, m_i)$.
  2. Wait till a subset of parties $SV' \subseteq SV$ is found, such that $SV \setminus SV' \in \mathbb{Z}_s$, and for every $P_i \in SV'$, the corresponding revealed point $(\alpha_i, v_i, m_i)$ is accepted. Then, output $s = F(0)$.

Fig. 6: The network-agnostic ICP

The proof of Theorem 4.1 is available in Appendix B.
Theorem 4.1. Protocols $(\Pi_{\text{Auth}}, \Pi_{\text{Reveal}})$ satisfy the following properties, except with probability at most $\epsilon_{\text{ICP}} \overset{\text{def}}{=} \frac{nt}{|\mathbb{F}| - 1}$, where $t = \max\{|Z| : Z \in \mathbb{Z}_s\}$.

- If $S, I$ and $R$ are honest, then the following hold.
  - $Z_s$-Correctness: In a synchronous network, each honest party sets $\text{authCompleted}_{(S,I,R)}$ to 1 during $\Pi_{\text{Auth}}$ at time $T_{\text{Auth}} = \Delta + 4T_{\text{BC}}$. Moreover, if $R$ outputs $s$ during $\Pi_{\text{Reveal}}$ which takes $T_{\text{Reveal}} = \Delta$ time.
  - $Z_s$-Correctness: In an asynchronous network, each honest party eventually sets $\text{authCompleted}_{(S,I,R)}$ to 1 during $\Pi_{\text{Auth}}$ and $R$ eventually outputs $s$ during $\Pi_{\text{Reveal}}$.

- Privacy: The view of Adv is independent of $s$, irrespective of the network.

- Unforgeability: If $S, R$ are honest, $I$ is corrupt and if $R$ outputs $s' \in \mathbb{F}$ during $\Pi_{\text{Reveal}}$, then $s' = s$ holds, irrespective of the network type.

- If $S$ is corrupt, $I, R$ are honest and if $I$ sets $\text{ICSig}(S, I, R, s) = F(x)$ during $\Pi_{\text{Auth}}$, then the following holds.
  - $Z_s$-Non-Repudiation: In a synchronous network, $R$ outputs $s = F(0)$ during during $\Pi_{\text{Reveal}}$, which takes $T_{\text{Reveal}} = \Delta$ time.
  - $Z_s$-Non-Repudiation: In an asynchronous network, $R$ eventually outputs $s = F(0)$ during during $\Pi_{\text{Reveal}}$.

- Communication Complexity: Irrespective of the network type, $\Pi_{\text{Auth}}$ incurs a communication of $O(n^5 \cdot \log |\mathbb{F}| \cdot |\sigma|)$ bits, while $\Pi_{\text{Reveal}}$ incurs a communication of $O(n \cdot \log |\mathbb{F}|)$ bits.

Looking ahead, in our VSS protocols, there will be several instances of ICP running, with different parties playing the role of $S, I$ and $R$. It will be convenient to use the following notations while invoking instances of ICP.

Notation 4.2 (for ICP) While using $(\Pi_{\text{Auth}}, \Pi_{\text{Reveal}})$, we will say that:

- “$P_i$ gives $\text{ICSig}(P_i, P_j, P_k, s)$ to $P_j$” to mean that $P_i$ acts as $S$ and invokes an instance of $\Pi_{\text{Auth}}$ with input $s$, where $P_j$ and $P_k$ play the role of $I$ and $R$ respectively.

- “$P_j$ receives $\text{ICSig}(P_i, P_j, P_k, s)$ from $P_i$” to mean that $P_j$, as $I$, has set $\text{authCompleted}_{(P_i, P_j, P_k)}$ to 1 and $\text{ICSig}(P_i, P_j, P_k, s)$ to some $t$-degree polynomial with $s$ as the constant term during the instance of $\Pi_{\text{Auth}}$, where $P_i$ and $P_k$ play the role of $S$ and $R$ respectively.

- “$P_j$ reveals $\text{ICSig}(P_i, P_j, P_k, s)$ to $P_k$” to mean that $P_j$, as $I$, invokes an instance of $\Pi_{\text{Reveal}}$, with $P_i$ and $P_k$ playing the role of $S$ and $R$ respectively.

- “$P_k$ accepts $\text{ICSig}(P_i, P_j, P_k, s)$” to mean that $P_k$, as $R$, outputs $s$ during the instance of $\Pi_{\text{Reveal}}$, invoked by $P_j$ as $I$, with $P_i$ playing the role of $S$.

4.1 Linearity of IC Signature

Our ICP satisfies the linearity property, provided “special care” is taken while generating the IC-signatures. In more detail, consider a fixed $S, I$ and $R$ and let $s_a$ and $s_b$ be two values, such that $I$ has received $\text{ICSig}(S, I, R, s_a)$ and $\text{ICSig}(S, I, R, s_b)$ from $S$, through instances $\Pi_{\text{Auth}}^{(a)}$ and $\Pi_{\text{Auth}}^{(b)}$ of $\Pi_{\text{Auth}}$ respectively, where all the following conditions are satisfied.

29
- Supporting verifiers $SV_a$ and $SV_b$, during $\Pi^{(a)}_{Auth}$ and $\Pi^{(b)}_{Auth}$, are the same.
- For $i = 1, \ldots, n$, corresponding to the verifier $P_i$, $S$ has used the same $\alpha_i \in \mathbb{F} \setminus \{0\}$, to compute the verification points, during $\Pi^{(a)}_{Auth}$ and $\Pi^{(b)}_{Auth}$.
- $I$ has used the same linear combiner $d \in \mathbb{F} \setminus \{0\}$ during the instances $\Pi^{(a)}_{Auth}$ and $\Pi^{(b)}_{Auth}$, to compute the linearly-combined masked polynomials.

Let $s \stackrel{\text{def}}{=} c_1 \cdot s_a + c_2 \cdot s_b$, where $c_1, c_2$ are publicly known constants from $\mathbb{F}$. It then follows that if all the above conditions are satisfied, then $I$ can locally compute $ICSig(S, I, R, s)$ from $ICSig(S, I, R, s_a)$ and $ICSig(S, I, R, s_b)$. Namely, $I$ can set $ICSig(S, I, R, s) = c_1 \cdot ICSig(S, I, R, s_a) + c_2 \cdot ICSig(S, I, R, s_b)$. On the other hand, let the verifier $P_i \in SV_a$ hold the verification points $(\alpha_i, v_{a,i}, m_{a,i})$ and $(\alpha_i, v_{b,i}, m_{b,i})$, corresponding to $ICSig(S, I, R, s_a)$ and $ICSig(S, I, R, s_b)$ respectively. Then $P_i$ can locally compute $(\alpha_i, v_i, m_i)$ as its verification point corresponding to $ICSig(S, I, R, s)$, where $v_i = c_1 \cdot v_{a,i} + c_2 \cdot v_{b,i}$ and $m_i = c_1 \cdot m_{a,i} + c_2 \cdot m_{b,i}$.

During the protocol $\Pi_{Reveal}$, to reveal $ICSig(S, I, R, s)$, the intermediary $I$ can reveal $ICSig(S, I, R, s)$ to $R$, while each verifier $P_i \in SV_a$ can reveal the verification information $(\alpha_i, v_i, m_i)$. To accept $(\alpha_i, v_i, m_i)$, the receiver $R$ either checks for the “consistency” of $(\alpha_i, v_i)$ with $ICSig(S, I, R, s)$, or the “inconsistency” of masked polynomial $B(x) \stackrel{\text{def}}{=} c_1 \cdot B_a(x) + c_2 \cdot B_b(x)$ with $(d, v_i, m_i)$; here $B_a(x)$ and $B_b(x)$ denote the masked polynomials, made public by $I$, during the instances $\Pi^{(a)}_{Auth}$ and $\Pi^{(b)}_{Auth}$ respectively, both computed with respect to the linear combiner $d$.

Looking ahead, we will require the linearity property from ICP, when used in our VSS protocols, where there will be multiple instances of $\Pi_{Auth}$ running, involving the same $(S, I, R)$ triplet. To achieve this, we will ensure that in all the $\Pi_{Auth}$ instances invoked during VSS involving the same triplet $(S, I, R)$, the signer uses the same non-zero evaluation point $\alpha_{S,I,R,i}$ for the verifier $P_i$, while distributing verification information to $P_i$, as part of the respective $\Pi_{Auth}$ instances. Similarly, $S$ should find and make public a common set of supporting verifiers $SV$, on behalf of all the instances of $\Pi_{Auth}$. And finally, $I$ should use the same non-zero random linear combiner $d$, to compute the masked polynomials for all the instances of $\Pi_{Auth}$ and once computed, it should together make public $d$ and the masked polynomials for all the instances of $\Pi_{Auth}$.

In the rest of the paper, we will use the term “parties follow linearity principle while generating IC-signatures”, to mean that the underlying instances of $\Pi_{Auth}$ are invoked as above.

### 4.2 Default IC Signature

In our VSS protocols, we will also encounter situations where some publicly known value $s$ and a triplet $(S, I, R)$ exist. Then $I$ can locally compute $ICSig(S, I, R, s)$ by setting $ICSig(S, I, R, s)$ to the constant polynomial $F(x) = s$. Each verifier $P_i \in \mathcal{P}$ locally sets $(\alpha_{S,I,R,i}, v_i, m_i)$ as its verification information, where $v_i = m_i = s$. Moreover, the set of supporting verifiers $SV$ is set as $\mathcal{P}$. Notice that the way in which $ICSig(S, I, R, s)$ and the verification information is
set guarantees that, later, if an honest I reveals $\text{ICSig}(S, I, R, s)$ to an honest R during $\Pi_{\text{Reveal}}$, then R always outputs $s$.

In the rest of the paper, we will use the term "parties set $\text{ICSig}(S, I, R, s)$ to the default value", to mean the above.

5 Network Agnostic Verifiable Secret Sharing (VSS)

This section presents our network-agnostic VSS protocol, which allows a designated dealer to generate a linear secret-sharing with IC-signatures for its input. We first define the notion of linear secret-sharing with IC-signatures.

**Definition 5.1 (Linear Secret Sharing with IC-Signatures).** A value $s \in \mathbb{F}$ is said to be linearly secret-shared with IC-signatures, if there exist shares $s_1, \ldots, s_{|Z|} \in \mathbb{F}$ where $s = s_1 + \ldots + s_{|Z|}$. Moreover, for $q = 1, \ldots, |Z|$, there exists some publicly-known core-set $W_q \subseteq S_q$, such that all the following hold.

- $Z_s$ satisfies the $Q^{(1)}(W_q, Z_s)$ condition and all (honest) parties in the set $S_q$ have the share $s_q$.
- Every honest $P_i \in W_q$ has the IC-signature $\text{ICSig}(P_j, P_i, P_k, s_q)$ of every $P_j \in W_q$ for every $P_k \notin S_q$. Moreover, if any corrupt $P_j \in W_q$ has $\text{ICSig}(P_j, P_i, P_k, s_q')$ of any honest $P_i \in W_q$ for any $P_k \notin S_q$, then $s_q' = s_q$ holds. Furthermore, all the underlying IC-signatures satisfy the linearity property.

The vector of information corresponding to a linear secret-sharing with IC-signature of $s$ is denoted by $[s]$, which includes the share $s_1, \ldots, s_{|Z|}$, the core sets $W_1, \ldots, W_{|Z|}$ and IC-signatures $\{\text{ICSig}(P_j, P_i, P_k, s_q)\}_{P_j, P_i, P_k \in W_q, P_k \notin S_q}$. For convenience, we denote the $q^{th}$ share of $s$, corresponding to $S_q$, by $[s]_q$.

A vector of values $\vec{s} = (s^{(1)}, \ldots, s^{(L)})$ where $L \geq 1$ is said to be linearly secret-shared with IC-signatures, if each $s^{(\ell)} \in \vec{s}$ is linearly secret-shared with IC-signatures and if there exist common core sets $W_1, \ldots, W_{|Z|}$, corresponding to the secret-sharings $[s^{(1)}], \ldots, [s^{(L)}]$.

If $\vec{s} = (s^{(1)}, \ldots, s^{(L)})$ are linearly secret-shared with IC-signatures, then the parties can locally compute any publicly-known function of these secret-shared values. In more detail, let $c_1, \ldots, c_L \in \mathbb{F}$ be publicly-known constants and let $s \overset{\text{def}}{=} c_1 \cdot s^{(1)} + \ldots + c_L \cdot s^{(L)}$. Then the following holds:

$$c_1 \cdot [s^{(1)}] + \ldots + c_L \cdot [s^{(L)}] = [s],$$

where the core-sets corresponding to $[s]$ are $W_1, \ldots, W_{|Z|}$. And corresponding to each $S_q \in S_{Z_q}$, the share $[s]_q$ for each (honest) party in $S_q$ can be computed locally as $c_1 \cdot [s^{(1)}]_q + \ldots + c_L \cdot [s^{(L)}]_q$. Moreover, every (honest) $P_i \in W_q$ can compute the IC-signature $\text{ICSig}(P_j, P_i, P_k, [s]_q)$ of every $P_j \in W_q$ for every $P_k \notin S_q$, from $\text{ICSig}(P_j, P_i, P_k, [s^{(1)}]_q), \ldots, \text{ICSig}(P_j, P_i, P_k, [s^{(L)}]_q)$. In the rest of the paper, we will say that the "parties in $\mathcal{P}$ locally compute $c_1 \cdot s^{(1)} + \ldots + c_L \cdot s^{(L)}$ from $[s^{(1)}], \ldots, [s^{(L)}]$" to mean the above.
5.1 The VSS Protocol

We present a network-agnostic VSS protocol $\Pi_{\text{VSS}}$ (Fig 7). In the protocol, there exists a designated dealer $D \in \mathcal{P}$ with input $s$ (the protocol can be easily generalized if $D$ has $L$ inputs). The protocol allows $D$ to "verifiably" generate a linear secret-sharing of $s$ with IC-signatures. In a synchronous network, the (honest) parties output $[s]$ after a "fixed" time, while in an asynchronous network, they do so eventually, such that $s$ remains private. The verifiability here guarantees that if $D$ is corrupt and some honest party gets an output, then there exists some value, say $s^\ast$ (which could be different from $s$), such that $s^\ast$ is linearly secret-shared with IC-signatures. Note that in this case, we cannot bound the time within which $s^\ast$ will be secret-shared, since a potentially corrupt $D$ may delay sending the required messages and the parties will not be knowing the exact network type. A detailed overview of the protocol has been already presented in Section 1.2.3 and so we directly present the protocol.

**Protocol $\Pi_{\text{VSS}}(D, Z_s, Z_a, s, S_{Z_s})$**

- **Distribution of Shares**: $D$, on having input $s$, randomly chooses $s_1, \ldots, s_{|Z_s|} \in \mathbb{F}$, such that $s = s_1 + \cdots + s_{|Z_s|}$. It then sends $s_q$ to all $P_i \in S_q$, for $q = 1, \ldots, |Z_s|$.
- **Exchanging IC-Signed Values**: Each $P_i \in \mathcal{P}$ (including $D$), waits till the local time becomes $\Delta$. Then, for each $S_q \in S_{Z_s}$, such that $P_i \in S_q$, upon receiving $s_q$ from $D$, give $\text{ICSig}(P_i, P_k, P_h, s_q)$ to every $P_j \in S_q$, for every $P_k \in \mathcal{P}$, such that the parties follow the linearity principle while generating IC-signatures (see Section 4.1).
- **Announcing Results of Pairwise Consistency Tests**: Each $P_i \in \mathcal{P}$ (including $D$) waits till the local time becomes $\Delta + T_{\text{Auth}}$ and then does the following.
  - Upon receiving $\text{ICSig}(P_j, P_k, s_q)$ from $P_j$ for each $S_q \in S_s$ such that $P_j, P_k \in S_q$, corresponding to every $P_k \in \mathcal{P}$, broadcast $\text{OK(i,j)}$, if $s_qi = s_qj$ holds.
  - Corresponding to every $P_j \in \mathcal{P}$, participate in any instance of $\Pi_{\text{BC}}$ initiated by $P_j$ as a sender, to broadcast any $\text{OK}(P_j, \ast)$ message.
- **Constructing Consistency Graph**: Each $P_i \in \mathcal{P}$ (including $D$) waits till the local time becomes $\Delta + T_{\text{Auth}} + T_{\text{BC}}$ and then constructs an undirected consistency graph $G(i)$ with $\mathcal{P}$ as the vertex set, where the edge $(P_j, P_k)$ is added to $G(i)$, provided $\text{OK}(j, k)$ and $\text{OK}(k, j)$ is received from the broadcast of $P_j$ and $P_k$ respectively (through any mode).
- **Identification of Core Sets and Public Announcement by the Dealer**: $D$ waits till its local time is $\Delta + T_{\text{Auth}} + T_{\text{BC}}$, and then executes the following steps to compute core sets.
  - Once any $S_p \in S_{Z_s}$ forms a clique in the graph $G(D)$, then for $q = 1, \ldots, |Z_s|$, compute core-set $W_q$ and broadcast-set $BS$ with respect to $S_q$, as follows, followed by broadcasting $(\text{CanCS.D,} S_p, \{W_q\}_{q=1,\ldots,|Z_s|}, BS, \{s_q\}_{q\in BS})$.
    - If $S_q$ constitutes a clique in the graph $G(D)$, then set $W_q = S_q$.
    - Else if $(S_p \cap S_q)$ constitutes a clique in $G(D)$ and $Z_s$ satisfies the $Q(1)(S_p \cap S_q, Z_s)$ condition, then set $W_q = (S_p \cap S_q)$.
    - Else set $W_q = S_q$ and include $q$ to $BS$.
- **Identifying Valid Core Sets**: Each $P_i \in \mathcal{P}$ waits till its local time is $\Delta + T_{\text{Auth}} + 2T_{\text{BC}}$ and then initializes a set $C_i = \emptyset$. For $p = 1, \ldots, |Z_s|$, party $P_i$ includes $(D, S_p)$ to $C_i$ (initialized to $\emptyset$), provided all the following hold.
Theorem 5.1. Protocol $\PiVSS$ achieves the following, except with a probability of $O(|S_Z| \cdot n^2 \cdot \epsilon_{CP})$, where D has input $s \in F$ for $\PiVSS$, and where $T_{VSS} = \Delta + T_{Auth} + 2T_{BC} + T_{Reveal}$.

- If D is honest, then the following hold.
  - $Z_s$-correctness: In a synchronous network, the honest parties output $[s]$ at time $T_{VSS}$.
  - $Z_a$-correctness: In an asynchronous network, the honest parties eventually output $[s]$.
  - Privacy: Adversary’s view remains independent of $s$ in any network.

- If D is corrupt, then the following hold.
  - $Z_s$-commitment: In a synchronous network, either no honest party computes any output or there exists some $s^* \in F$, such that the honest parties output $[s^*]$. Moreover, if any honest party computes its output at time $T$, then all honest parties compute their required output by time $T + \Delta$. 

---

Fig. 7: The network agnostic VSS protocol

---

If there are multiple $S_q$ from $S_Z$, which constitute a clique in $G^{(D)}$, then consider the one with the smallest index.
- $Z_a$-commitment: In an asynchronous network, either no honest party computes any output or there exists some $s^* \in \mathbb{F}$, such that the honest parties eventually output $[s^*]$.
- Communication Complexity: $O(|Z_a| \cdot n^5 \cdot \log \|\mathbb{F}\| \cdot |\sigma|)$ bits are communicated by the honest parties.

6 Network Agnostic Reconstruction Protocols and Secure Verifiable Multicast

Let $s$ be a value which is linearly secret-shared with IC-signatures and let $S_q \in \mathbb{S}_Z$.
Moreover, let $R \subseteq \mathcal{P}$ be a designated set. Then protocol $\Pi_{RecShare}([s], S_q, R)$ allows all the (honest) parties in $R$ to reconstruct the share $[s]_q$ without disclosing any additional information. For this, every $P_j \in W_q$ reveals $[s]_q$ to all the parties outside $W_q$, who are in $R$ (the parties in $W_q$ who are in $R$ already have $[s]_q$). To ensure that $P_j$ does not cheat, $P_j$ actually reveals the IC-signature of every party in $W_q$ on the revealed $[s]_q$. The idea here is that since $W_q$ has at least one honest party (irrespective of the network type), a potentially corrupt $P_j$ will fail to reveal the signature of an honest party from $W_q$ on an incorrect $[s]_q$.

On the other hand, an honest $P_j$ will be able to reveal the signature of all the parties in $W_q$ on $[s]_q$.

Based on $\Pi_{RecShare}$, we design another protocol $\Pi_{Rec}([s], R)$, which allows all the (honest) parties in $R$ to reconstruct $s$. The idea is to run an instance of $\Pi_{RecShare}$ for every $S_q \in \mathbb{S}_Z$. Since the protocols are standard, we present them and prove their properties (Lemma 6.1 and Lemma 6.2) in Appendix D.

Lemma 6.1. Let $s$ be a value which is linearly secret-shared with IC-signatures, let $S_q \in \mathbb{S}_Z$, be a designated set and let $R \subseteq \mathcal{P}$ be a designated set of receivers. Then protocol $\Pi_{RecShare}$ achieves the following.

- $Z_a$-correctness: In a synchronous network, all honest parties in $R$ output $[s]_q$ after time $T_{RecShare} = T_{Reveal}$, except with a probability of $O(n^2 \cdot \epsilon_{ICP})$.
- $Z_a$-correctness: In an asynchronous network, all honest parties in $R$ eventually output $[s]_q$, except with a probability of $O(n^2 \cdot \epsilon_{ICP})$.
- Privacy: If $R$ consists of only honest parties, then the view of the adversary remains independent of $[s]_q$.
- Communication Complexity: $O(|R| \cdot n^3 \cdot \log \|\mathbb{F}\|)$ bits are communicated.

Lemma 6.2. Let $s$ be a value which is linearly secret-shared with IC-signatures and let $R \subseteq \mathcal{P}$ be a set of designated receivers. Then protocol $\Pi_{Rec}$ achieves the following.

- $Z_a$-correctness: In a synchronous network, all honest parties in $R$ output $s$ after time $T_{Rec} = T_{RecShare}$, except with probability $O(|\mathbb{S}_Z| \cdot n^2 \cdot \epsilon_{ICP})$.
- $Z_a$-correctness: In an asynchronous network, all honest parties in $R$ eventually output $s$, except with probability $O(|\mathbb{S}_Z| \cdot n^2 \cdot \epsilon_{ICP})$.
- Privacy: If $R$ consists of only honest parties, then the view of the adversary remains independent of $s$.
- Communication Complexity: $O(|Z_a| \cdot |R| \cdot n^3 \cdot \log \|\mathbb{F}\|)$ bits are communicated.

34
6.1 $\Pi_{VSS}$ and Reconstruction Protocol for Superpolynomial $|Z_s|$ 

From Theorem 5.1 and Lemma 6.2, the error probability of $\Pi_{VSS}$ and $\Pi_{Rec}$ depend linearly on $|S_{Z_s}|$, which is the same as $|Z_s|$. This is because there are $\Omega(|Z_s|)$ instances of $\Pi_{Auth}/\Pi_{Reveal}$ in which the unforgeability/non-repudiation properties might get violated with probability $\epsilon_{ICP}$. This might be problematic for a “large-sized” $Z_s$. To avoid this, we use the idea of local dispute control used in [40,4], which ensures that irrespective of the number of instances of $\Pi_{Auth}/\Pi_{Reveal}$, the overall error probability is only $O(n^3 \cdot \epsilon_{ICP})$. This is done by ensuring that the unforgeability/non-repudiation properties get violated only $O(n^3)$ times across all these instances. The idea here is that the parties start locally discarding corrupt parties the “moment” they are caught cheating during any instance of $\Pi_{Auth}$ or $\Pi_{Reveal}$. Once a party $P_j$ is locally discarded by any party $P_i$, then $P_i$ “behaves” as if $P_i$ has certainly behaved maliciously in all the “future” instances of $\Pi_{Auth}$ or $\Pi_{Reveal}$, even if this is not the case. This restricts the number of attempts of cheating for the adversary to a “fixed” number and consequently, the total error probability of arbitrary many instances of $\Pi_{Auth}/\Pi_{Reveal}$ will no longer depend on $|Z_s|$. To incorporate the above idea, each party $P_i$ now maintains a list of locally discarded parties $\mathcal{LD}^{(i)}$, which it keeps populating across all the instances of $\Pi_{Auth}$ and $\Pi_{Reveal}$, as soon as $P_i$ identifies any party cheating. It will be ensured that an honest $P_i$ never includes an honest $P_j$ to $\mathcal{LD}^{(i)}$. We next discuss the modifications in $\Pi_{Auth}$ and $\Pi_{Reveal}$ and how the parties populate their $\mathcal{LD}$ sets across instances of $\Pi_{Auth}$ and $\Pi_{Reveal}$.

Populating $\mathcal{LD}$ Sets During Instances of $\Pi_{Auth}$. In any instance of $\Pi_{Auth}$, if $P_i$ is present in the corresponding set of supporting verifiers $S_V$ (i.e. $P_i \in S_V$), then $P_i$ includes the corresponding signer $P_j$ of the $\Pi_{Auth}$ instance to $\mathcal{LD}^{(i)}$ if the following condition holds during $\Pi_{Auth}$:

\[(P_j \text{ broadcasts } \text{OK}) \land (B(\alpha_i) \neq dv_i + m_i),\]

where $B(x)$ is the masked polynomial broadcasted by the corresponding $l$ of the $\Pi_{Auth}$ instance. The idea here is that if $P_i$ is honest and if the above condition holds, then clearly the signer $P_j$ is corrupt and is trying to break the non-repudiation property. Once $P_j \in \mathcal{LD}^{(i)}$, then in any pair of ($\Pi_{Auth}$, $\Pi_{Reveal}$) instances involving $P_j$ as the signer, if $P_i$ is present in the corresponding set $S_V$, then in the $\Pi_{Reveal}$ instance, $P_i$ reveals $\bot$ as its verification information to the corresponding receiver $R$.\footnote{This serves as an indicator for $R$ that $P_i$ is in conflict with the signer $P_j$.} Upon receiving $\bot$ as the verification information, the strategy for $R$ is to always accept it without doing any verification, irrespective of the polynomial $F(x)$ revealed as $\text{ICSig}$ by the corresponding $l$.

The above modification ensures that if in any instance of $\Pi_{Auth}$ involving a corrupt signer $P_j$ and an honest $P_i$, $P_j$ distributes an inconsistent verification point to an honest verifier $P_i$ from the corresponding $S_V$ set and still broadcasts an OK message during $\Pi_{Auth}$, then $P_i$ will locally be discarded by the verifier $P_i$, except with probability $\epsilon_{ICP}$ (follows from the non-repudiation property of ICP). From
then onwards, in all the instances of \((\Pi_{\text{Auth}}, \Pi_{\text{Reveal}})\), involving \(P_j\) as the signer, if the verifier \(P_i\) is added to the \(SV\) set, then the “special” verification information revealed by \(P_i\) during \(\Pi_{\text{Reveal}}\) will always be considered as accepted, irrespective of what verification information it actually receives from \(P_j\) during \(\Pi_{\text{Auth}}\). Hence, \(P_j\) will not have any chance of cheating the verifier \(P_i\) in any \(\Pi_{\text{Auth}}\) instance. By considering all possibilities for a corrupt \(S\) and an honest verifier \(P_i\), along with an honest \(I\), it follows that except with probability at most \(O(n^3 \cdot \epsilon_{\text{ICP}})\), the verification-points of all honest verifiers from corresponding \(SV\), will be accepted by every honest \(R\), during all the instances of \(\Pi_{\text{Reveal}}\), in any instance of \(\Pi_{\text{VSS}}\) or \(\Pi_{\text{Rec}}\). Consequently, except with probability at most \(O(n^3 \cdot \epsilon_{\text{ICP}})\), the signatures revealed by all honest \(I\) will be always accepted.

We stress that the above modification does not help a corrupt \(I\) to break the unforgeability property for an honest \(S\) and an honest \(R\), with the help of potentially corrupt verifiers.

**Populating LD Sets During Instances of \(\Pi_{\text{Reveal}}\).** Consider an instance of \(\Pi_{\text{Reveal}}\) involving \(P_i\) as \(R\) and \(P_j\) as \(I\). If \(P_i\) finds that \(P_j\) has tried to forge a signature on an incorrect value, then \(P_i\) adds \(P_j\) to \(\mathcal{LD}^{(i)}\). To achieve this goal, during \(\Pi_{\text{Reveal}}\), \(P_i\) (as \(R\)) now additionally checks if there exists a subset of verifiers \(SV'' \subseteq SV\), where \(SV \setminus SV'' \in Z\), such that the verification-points of all the parties in \(SV''\) are rejected. If such a subset \(SV''\) exists, then clearly \(P_j\) (as \(I\)) has cheated and tried to break the unforgeability property, since \(SV''\) is bound to contain at least one honest verifier. If the verification point of an honest verifier is rejected, then clearly \(P_j\) is corrupt. Once \(P_j \in \mathcal{LD}^{(i)}\), from then onwards, in any instance of \(\Pi_{\text{Reveal}}\) involving \(P_j\) as \(I\) and \(P_i\) as \(R\), party \(P_i\) always rejects any IC-signature revealed by \(P_j\).

The above modification ensures that if in any instance of \(\Pi_{\text{Reveal}}\) involving an honest signer, a corrupt intermediary \(P_j\) and an honest receiver \(P_i\), \(P_j\) tries to reveal an incorrect signature during \(\Pi_{\text{Reveal}}\), then except with probability \(\epsilon_{\text{ICP}}\), the intermediary \(P_j\) will be locally discarded by the receiver \(P_i\) (follows from the unforgeability property of ICP). From then onwards, in all the instances of \((\Pi_{\text{Auth}}, \Pi_{\text{Reveal}})\), involving \(P_j\) as the intermediary and \(P_i\) as the receiver, the signature revealed by \(P_j\) during \(\Pi_{\text{Reveal}}\) will always be rejected, irrespective of what data is actually revealed by \(P_j\). Hence, by considering all possibilities for a corrupt \(I\), honest \(S\) and honest \(R\), it follows that except with probability at most \(O(n^3 \cdot \epsilon_{\text{ICP}})\), no corrupt \(I\) will be able to forge an honest \(S\)’s signature to any honest \(R\), in any instance of \(\Pi_{\text{Reveal}}\), during any instance of \(\Pi_{\text{VSS}}\) or \(\Pi_{\text{Rec}}\).

### 6.2 Network Agnostic Secure Multicast

Based on protocols \(\Pi_{\text{VSS}}\) and \(\Pi_{\text{Rec}}\), we design a secure verifiable multicast protocol \(\Pi_{\text{SVM}}\). In the protocol, there exists a designed sender \(\text{Sen} \in \mathcal{P}\) with input \(v \in \mathcal{F}\) and a designated set of receivers \(\mathcal{R}\). The goal is to let every party in \(\mathcal{R}\) receive \(v\), without revealing any additional information to the adversary.\(^\text{14}\) While

\(^\text{14}\) Note that the requirements here are different from broadcast since we need the privacy of \(v\) if \(\text{Sen}\) is honest and if \(\mathcal{R}\) consists of only honest parties.
in a synchronous network, the (honest) parties in $R$ get $v$ after a “fixed” time, in an asynchronous network, they do so eventually. Note that if $Sen$ is corrupt, then the parties need not obtain any output, as $Sen$ may not invoke the protocol. However, if any honest party in $R$ computes an output $v^*$ (which could be different from $v$), then all honest parties in $R$ will also output $v^*$. The “verifiability” here guarantees that in case the honest parties in $R$ get any output, then all the (honest) parties in $P$ will be “aware” of this; namely there will be a Boolean variable $\text{flag}^{(Sen,R)}$, which all the honest parties will set to 1.

The idea behind $\Pi_{SVM}$ is very simple. The parties participate in an instance of $\Pi_{VSS}$, where $Sen$ plays the role of the dealer with input $v$. Once any (honest) party computes an output during $\Pi_{VSS}$ (implying that $Sen$ is committed to some value $v^*$ which is the same as $v$ for an honest $Sen$), then it turns $\text{flag}^{(Sen,R)}$ to 1. Once $\text{flag}^{(Sen,R)}$ is turned to 1, the parties invoke an instance of $\Pi_{Rec}$ to let only the parties in $R$ reconstruct the committed value. Protocol $\Pi_{SVM}$ and proof of its properties (stated in Lemma 6.3), are available in Appendix D.

Lemma 6.3. Protocol $\Pi_{SVM}$ achieves the following, where $Sen$ participates with input $v$ and where each honest party initializes $\text{flag}^{(Sen,R)}$ to 0.

- **Synchronous Network**: If $Sen$ is honest, then all honest parties set $\text{flag}^{(Sen,R)}$ to 1 at time $T_{VSS}$ and except with probability $\mathcal{O}(n^3 \cdot \epsilon_{ICP})$, all honest parties in $R$ output $v$, after time $T_{SVM} = T_{VSS} + T_{Rec}$. Moreover, if $R$ consists of only honest parties, then the view of $Adv$ remains independent of $v$. If $Sen$ is corrupt and some honest party sets $\text{flag}^{(Sen,R)}$ to 1, then there exists some $v^*$ such that, except with probability $\mathcal{O}(n^3 \cdot \epsilon_{ICP})$, all honest parties in $R$ output $v^*$. Moreover, if any honest party sets $\text{flag}^{(Sen,R)}$ to 1 at time $T$, then all honest parties in $R$ output $v^*$ by time $T + 2\Delta$.

- **Asynchronous Network**: If $Sen$ is honest, then all honest parties eventually set $\text{flag}^{(Sen,R)}$ to 1 and except with probability $\mathcal{O}(n^3 \cdot \epsilon_{ICP})$, all honest parties in $R$ eventually output $v$. Moreover, if $R$ consists of only honest parties, then the view of the adversary remains independent of $v$. If $Sen$ is corrupt and some honest party sets $\text{flag}^{(Sen,R)}$ to 1, then there exists some $v^*$ such that, except with probability $\mathcal{O}(n^3 \cdot \epsilon_{ICP})$, all honest parties in $R$ eventually output $v^*$.

- **Communication Complexity**: $\mathcal{O}(|Z_s| \cdot n^8 \cdot \log |\mathbb{F}| \cdot |\sigma|)$ bits are communicated.

7 Network Agnostic Protocol for Generating Linearly Secret-Shared Random Values with IC-Signatures

In this section, we present a network agnostic protocol $\Pi_{Rand}$, which allows the parties to jointly generate linear secret-sharing of random values with IC-signatures. To design the protocol $\Pi_{Rand}$, we first design a subprotocol $\Pi_{MDVSS}$.
7.1 Network Agnostic VSS for Multiple Dealers

Protocol $\Pi_{\text{MDVSS}}$ (Fig 8) is a multi-dealer VSS. In the protocol, each party $P_t \in \mathcal{P}$ participates as a dealer with some input $s^{(t)}$. Then, irrespective of the network type, the protocol outputs a common subset of dealers $\mathbf{CORE} \subseteq \mathcal{P}$, which is guaranteed to have at least one honest dealer. Moreover, corresponding to every dealer $P_t \in \mathbf{CORE}$, there will be some value, say $s^*_{(t)}$, which will be the same as $s^{(t)}$ for an honest $P_t$, such that the values $\{s^*_{(t)}\}_{P_t \in \mathbf{CORE}}$ are linearly secret-shared with IC-signatures. While in a synchronous network, $\{s^*_{(t)}\}_{P_t \in \mathbf{CORE}}$ is generated after a “fixed” time, in an asynchronous network, $\{s^*_{(t)}\}_{P_t \in \mathbf{CORE}}$ is generated eventually.

The high level overview of $\Pi_{\text{MDVSS}}$ has been already discussed in detail in Section 1.2.5. The idea is to let every dealer $P_t$ to invoke an instance of $\Pi_{\text{VSS}}$ to secret-share its input. However, we need to take special care to ensure that the inputs of all the dealers in $\mathbf{CORE}$ are secret-shared with common core-sets. For this, each individual dealer in its instance of $\Pi_{\text{VSS}}$ computes and publishes as many “legitimate” core-sets as possible and the parties run instances of agreement on common subset (ACS) to identify whether “sufficiently many” dealers have published the same legitimate core-sets in their respective instances of $\Pi_{\text{VSS}}$. Moreover, to ensure that all the underlying IC-signatures satisfy the linearity property, we first need to identify the dealers who distribute shares as part of their respective $\Pi_{\text{VSS}}$ instances. For this, we let each dealer distribute shares in its instance of $\Pi_{\text{VSS}}$ through instances of $\Pi_{\text{SVM}}$. This enables the parties to identify a set of committed dealers $\mathbf{CD}$ who have indeed distributed shares as part of their $\Pi_{\text{VSS}}$ instances through instances of $\Pi_{\text{SVM}}$.

**Protocol $\Pi_{\text{MDVSS}}(P, Z_s, Z_w, (s^{(1)}, \ldots, s^{(n)}), S_{Z_s})$**

- **Committing Shares:** Each $P_t \in \mathcal{P}$ executes the following steps.
  - On having input $s^{(t)}$, randomly choose $s^{(t)}_1, \ldots, s^{(t)}_{|Z_s|}$, such that $s^{(t)} = s^{(t)}_1 + \cdots + s^{(t)}_{|Z_s|}$. Act as Sen and invoke instances $\Pi_{\text{SVM}}(P_t, s^{(t)}_1, S_1), \ldots, \Pi_{\text{SVM}}(P_t, s^{(t)}_{|Z_s|}, S_{Z_s})$ of $\Pi_{\text{SVM}}$.
  - Corresponding to every dealer $P_t \in \mathcal{P}$, participate in the instances of $\Pi_{\text{SVM}}$, invoked by $P_t$ as a Sen and wait till the local time becomes $T_{\text{SVM}}$. For $q = 1, \ldots, |Z_s|$, let $\text{flag}^{(P_t, S_q)}$ be the Boolean flag, corresponding to the instance $H_{\text{SVM}}(P_t, s^{(t)}_q, S_q)$, invoked by $P_t$.

- **Identifying the Set of Committed Dealers Through ACS:** Each $P_t \in \mathcal{P}$ does the following.
  - For $\ell = 1, \ldots, n$, participate in an instance $\Pi_{\text{BA}}^{(\ell)}$ of $\Pi_{\text{BA}}$ with input 1, provided $P_t$ has set $\text{flag}^{(P_t, S_q)} = 1$, for $q = 1, \ldots, |Z_s|$.
  - Once there exists a subset of dealers $\mathbf{CD}_t$ where $P \setminus \mathbf{CD}_t \subseteq Z_s$, such that corresponding to every dealer $P_t \in \mathbf{CD}_t$, the instance $\Pi_{\text{BA}}^{(\ell)}$ has produced output 1, then participate with input 0 in all the BA instances $\Pi_{\text{BA}}^{(\ell)}$, for which no input is provided yet.

15 Actually, the overview was for the protocol $\Pi_{\text{Rand}}$, but the same idea is also used in the protocol $\Pi_{\text{MDVSS}}$. 

38
Identifying Valid Core Sets

Constructing Consistency Graphs

Exchanging IC-Signed Values

Announcing Results of Pairwise Consistency Tests

\[ \forall p \in \mathcal{P}, \text{ each dealer } P_i \text{ executes the following steps to compute core sets.} \]

\[ \text{Construct an undirected consistency graph } G^{(t,i)} \text{ with } \mathcal{P} \text{ as the vertex set, where the edge } (P_j, P_k) \text{ is added to } G^{(t,i)}, \text{ provided } \text{OK}^{(t)}(j,k) \text{ and } \text{OK}^{(t)}(k,j) \text{ is received from the broadcast of } P_j \text{ and } P_k \text{ respectively (through any mode).} \]

Public Announcement of Core Sets by the Committed Dealers: Each dealer \( P_i \in \mathcal{C} \) waits till its local time is \( T_{\text{SVM}} + 2T_{\text{BA}} + T_{\text{Auth}} + T_{\text{BC}} \), and then executes the following steps to compute core sets.

- Once all the \( n \) instances of \( \Pi_{\text{BA}}^{(t)} \) have produced a binary output, set \( C_D \) to be the set of dealers \( P_i \), such that \( \Pi_{\text{BA}}^{(t)} \) has produced output 1.

- Exchanging IC-Signed Values: Each \( P_i \in \mathcal{P} \) waits till the local time becomes \( T_{\text{SVM}} + 2T_{\text{BA}} \). Then corresponding to each dealer \( P_i \in C_D \), does the following.

  - For each \( S_q \in \mathbb{S}_z \), such that \( P_i \in S_q \), upon computing an output \( s_q^{(t)} \) during \( \Pi_{\text{SVM}}(P_i, s_q^{(t)}, S_q) \), give \( \text{ICSig}(P_i, P_j, P_k, s_q^{(t)}) \) to every \( P_j \in S_q \), for every \( P_k \in \mathcal{P} \), where the parties follow the linearity principle while generating IC-signatures.

- Announcing Results of Pairwise Consistency Tests: Each \( P_i \in \mathcal{P} \) waits till the local time becomes \( T_{\text{SVM}} + 2T_{\text{BA}} + T_{\text{Auth}} \), and then does the following, corresponding to each dealer \( P_i \in C_D \).

  - Upon receiving \( \text{ICSig}(P_j, P_i, P_k, s_q^{(t)}) \) from \( P_j \) for each \( S_q \in \mathbb{S} \) such that \( P_j, P_i \in S_q \), corresponding to every \( P_k \in \mathcal{P} \), broadcast \( \text{OK}^{(t)}(i,j) \), if \( s_q^{(t)} = s_q^{(t)} \) holds.

  - Corresponding to every \( P_j \in \mathcal{P} \), participate in any instance of \( \Pi_{\text{BC}} \) initiated by \( P_j \) as a sender, to broadcast any \( \text{OK}^{(t)}(P_j, \star) \) message.

- Constructing Consistency Graphs: Each \( P_i \in \mathcal{P} \) waits till the local time becomes \( T_{\text{SVM}} + 2T_{\text{BA}} + T_{\text{Auth}} + T_{\text{BC}} \), and then does the following, corresponding to each dealer \( P_i \in C_D \).

  - \( \forall p \in \mathbb{S}_z \), once \( S_p \) forms a clique in \( G^{(t,i)} \), then for \( q = 1, \ldots, |\mathbb{S}_z| \), compute core-set \( W_{p,q}^{(t)} \) and broadcast-set \( \text{BS}_{p,q}^{(t)} \) with respect to \( S_p \) as follows, followed by broadcasting \( \text{CanCS}(P_i, S_p, W_{p,q}^{(t)}) \).

  - If \( S_q \) constitutes a clique in the graph \( G^{(t,i)} \), then set \( W_{p,q}^{(t)} = S_q \).

  - Else if \( (S_p \cap S_q) \) constitutes a clique in \( G^{(t,i)} \) and \( S_z \) satisfies the \( Q^{(t)}(S_p \cap S_q, Z) \) condition, then set \( W_{p,q}^{(t)} = (S_p \cap S_q) \).

  - Else set \( W_{p,q}^{(t)} = S_p \) and include \( q \) to \( \text{BS}_{p}^{(t)} \).

- Identifying Valid Core Sets: Each \( P_i \in \mathcal{P} \) waits for time \( T_{\text{SVM}} + 2T_{\text{BA}} + T_{\text{Auth}} + T_{\text{BC}} \), and then initializes a set \( C_i = \emptyset \). Corresponding to \( P_i \in \mathcal{C}_D \), and then initializes a set \( C_i = \emptyset \). Party \( P_i \) includes \( (P_i, S_p) \) to \( C_i \), provided all the following hold.

  - \( \text{CanCS}(P_i, S_p, W_{p,q}^{(t)}) \) is received from the broadcast of \( P_i \), such that for \( q = 1, \ldots, |\mathbb{S}_z| \), the following hold.

  - If \( q \in \text{BS}_{p}^{(t)} \), then set \( W_{p,q}^{(t)} = S_p \).

  - If \( (q \notin \text{BS}_{p}^{(t)}) \), then \( W_{p,q}^{(t)} \) is either \( S_p \) or \( (S_p \cap S_q) \), such that:

    - If \( W_{p,q}^{(t)} = S_p \), then \( S_p \) constitutes a clique in \( G^{(t,i)} \).

    - Else \( W_{p,q}^{(t)} = (S_p \cap S_q) \), then \( (S_p \cap S_q) \) constitutes a clique in \( G^{(t,i)} \).

- Selecting the Common Committed Dealers and Core Sets through ACS: Each party \( P_i \in \mathcal{P} \) does the following.
Theorem 7.1. Protocol $\Pi^\text{MDVSS}$ achieves the following where each $P_i$ participates with input $s^{(i)}$ and where $T^\text{MDVSS} = T^\text{SVM} + T^\text{Auth} + 2T^\text{BC} + 6T^\text{BA}$. 

The properties of $\Pi^\text{MDVSS}$, stated in Theorem 7.1 are proved in Appendix E.
- **Z$_a$-Correctness&Commitment:** If the network is synchronous, then except with probability $O(n^3 \cdot n_{GP})$, at time $T_{MDVSS}$, all honest parties output a common set $\text{CORE} \subseteq \mathcal{P}$ such that at least one honest party will be present in $\text{CORE}$. Moreover, corresponding to every $P_i \in \text{CORE}$, there exists some $s^{(i)}$, where $s^{(i)} = s^{(j)}$ for an honest $P_j$, such that the values $\{s^{(i)}\}_{P_i \in \text{CORE}}$ are linearly secret-shared with IC-signatures.

- **Z$_a$-Correctness&Commitment:** If the network is asynchronous, then except with probability $O(n^3 \cdot n_{GP})$, almost-surely all honest parties output a common set $\text{CORE} \subseteq \mathcal{P}$ eventually such that at least one honest party will be present in $\text{CORE}$. Moreover, corresponding to every $P_i \in \text{CORE}$, there exists some $s^{(i)}$, where $s^{(i)} = s^{(j)}$ for an honest $P_j$, such that the values $\{s^{(i)}\}_{P_i \in \text{CORE}}$ are eventually linearly secret-shared with IC-signatures.

- **Privacy:** Irrespective of the network type, the view of the adversary remains independent of $s^{(i)}$, corresponding to every honest $P_i \in \text{CORE}$.

- **Communication Complexity:** $O(|Z_s|^2 \cdot n^3 \cdot \log |F| \cdot |σ|)$ bits are communicated by the honest parties. In addition, $O(|Z_s|+n)$ instances of $Π_{BA}$ are invoked.

**Protocol $Π_{MDVSS}$ with $L$ Values for Each Dealer.** In protocol $Π_{MDVSS}$, each dealer $P_i \in \mathcal{P}$ participates with a single input. Consider a scenario where each $P_i$ participates with $L$ inputs $S^{(i)} = (s^{(i,1)}, \ldots, s^{(i,L)})$, where $L \geq 1$. The goal is to identify a common subset of dealers $\text{CORE} \subseteq \mathcal{P}$ which is guaranteed to have at least one honest dealer, irrespective of the network type. Corresponding to every dealer $P_i \in \text{CORE}$, there exist $L$ values, say $S^{(i)} = (s^{(i,1)}, \ldots, s^{(i,L)})$, which will be the same as $S^{(j)}$ for an honest $P_j$, where all the values in $\{S^{(i)}\}_{P_i \in \text{CORE}}$ are linearly secret-shared with IC-signatures. To achieve this, we run the protocol $Π_{MDVSS}$ with the following modifications, so that the number of instances of $Π_{BA}$ in the protocol still remains to be $O(|Z_s|+n)$, which is independent of $L$.

Corresponding to each $s^{(i)} \in S^{(i)}$, the dealer $P_i$ will pick $|Z_s|$ random shares (which sum up to $s^{(i)}$) and the shares corresponding to the group $S_q \in S_{|Z_s|}$ are communicated through an instance of $Π_{SVM}$; hence $|Z_s| \cdot L$ instances are invoked by $P_i$ as a Sen. Then, while identifying the set of committed dealers $CD$, parties vote 1 for $P_i$ in the instance $Π_{BA}^{(i)}$ provided the underlying flag variable is set to 1 in all the $|Z_s| \cdot L$ instances of $Π_{SVM}$ invoked by $P_i$. The rest of the steps for identifying $CD$ remains the same. This way, by executing only $O(n)$ instances of $Π_{BA}$, we identify the set $CD$.

Next, the parties exchange IC-signatures on their supposedly common shares for each group, corresponding to all the $L$ values shared by each dealer from $CD$. However, each $P_i$ now broadcasts a single $OK^{(i)}(i,j)$ message, corresponding to each $P_j \in S_q$, provided $P_i$ receives IC-signed common share from $P_j$ on the behalf of all the $L$ values, shared by $P_i$. This ensures that, for each $P_i$, every $P_j$ constructs a single consistency graph. Next, each dealer $P_i$ computes and broadcasts the candidate core-sets and broadcast-sets, as and when they are ready. The parties identify the CORE set by running $O(|Z_s|+n)$ instances of $Π_{BA}$.
To avoid repetition, we do not present the formal steps of the modified $\Pi_{MDVSS}$ protocol. The protocol incurs a communication of $O(|Z_s|^2 \cdot L \cdot n^9 \cdot \log |F| \cdot |\sigma|)$ bits, apart from $O(|Z_s| + n)$ instances of $\Pi_{BA}$.

### 7.2 Protocol for Generating Secret-Shared Random Values

Protocol $\Pi_{Rand}$ (Fig 9) allows the parties to jointly generate linear secret-sharings of $L$ values with IC-signatures, where $L \geq 1$, which are random for the adversary. For this, the parties invoke an instance of the (modified) $\Pi_{MDVSS}$ where each dealer $P_i \in P$ participates with a random vector of $L$ values. Let $\text{CORE}$ be the set of common dealers identified during the instance of $\Pi_{MDVSS}$. Then for $i = 1, \ldots, L$, the parties output the sum of $i^{th}$ value shared by all the dealers in $\text{CORE}$. Since there will be at least one honest dealer in $\text{CORE}$ whose shared values will be random for the adversary, it follows that the resultant values also remain random for the adversary.

In the rest of the paper, we will refer to the core sets $W_1, \ldots, W_{|Z_s|}$ obtained during $\Pi_{Rand}$ as global core-sets and denote them by $GW_{W_1}, \ldots, GW_{W_{|Z_s|}}$. From now onwards, all the secret-shared values will be generated with respect to these global core-sets.

![Protocol $\Pi_{Rand}$](image)

**Protocol $\Pi_{Rand}(P, Z_s, Z_a, S_{Z_s}, L)$**

- **Secret-Sharing Random Values:** Each $P_i \in P$ picks $L$ random values $R^{(i)} = (r_1^{(i)}, \ldots, r_L^{(i)})$ and participates in an instance of $\Pi_{MDVSS}$ with input $R^{(i)}$ and waits for time $T_{MDVSS}$.
- **Computing Output:** Let $(\text{CORE}, W_1, \ldots, W_{|Z_s|}, \{(r^{(1)}_1, \ldots, r^{(1)}_L), \ldots, (r^{(L)}_1, \ldots, r^{(L)}_L)\})_{P_i \in \text{CORE}}$ be the output from the instance of $\Pi_{MDVSS}$. For $i = 1, \ldots, L$, the parties locally compute $\sum_{P_i \in \text{CORE}} [r^{(i)}] = \sum_{P_i \in \text{CORE}} [r^{(i)}_1] \ldots [r^{(i)}_L]$ from $\{(r^{(i)}_1, \ldots, r^{(i)}_L)\}_{P_i \in \text{CORE}}$. The parties then output $(GW_1, \ldots, GW_{|Z_s|}, \{[r^{(i)}]_{i=1,\ldots,L}\})$, where $GW_q = W_q$ for $q = 1, \ldots, |Z_s|$.

Fig. 9: Protocol for generating linearly secret-shared random values with IC-signatures

**Theorem 7.2.** Protocol $\Pi_{Rand}$ achieves the following where $T_{Rand} = T_{MDVSS} = T_{SVM} + T_{Auth} + 2T_{BC} + 6T_{BA}$ and $L \geq 1$.

- **$Z_s$-correctness:** If the network is synchronous, then except with probability $O(n^3 \cdot \epsilon_{CP})$, at the time $T_{Rand}$, there exist values $r^{(1)}, \ldots, r^{(L)}$, which are linearly secret-shared with IC-signatures, where the core-sets are $GW_{W_1}, \ldots, GW_{W_{|Z_s|}}$.
- **$Z_a$-correctness:** If the network is asynchronous, then except with probability $O(n^3 \cdot \epsilon_{CP})$, there exist values $r^{(1)}, \ldots, r^{(L)}$, which are almost-surely linearly secret-shared with IC-signatures, where the core-sets are $GW_{W_1}, \ldots, GW_{W_{|Z_s|}}$.
- **Privacy:** Irrespective of the network type, the view of the adversary remains independent of $r^{(1)}, \ldots, r^{(L)}$.
- **Communication Complexity:** The protocol incurs a communication of $O(|Z_s|^2 \cdot L \cdot n^9 \cdot \log |F| \cdot |\sigma|)$ bits, apart from $O(|Z_s| + n)$ instances of $\Pi_{BA}$. 

42
8 Network Agnostic Protocol for Generating Random Multiplication Triples

In this section, we present our network-agnostic triple-generation protocol, which generates random and private multiplication-triples which are linearly secret-shared with IC-signatures. The protocol is based on several sub-protocols which we present next. Throughout this section, we will assume the existence of global core-sets $GW_1, \ldots, GW_{|Z_s|}$, where $Z_s$ satisfies the $Q^{(1)}(GW_q, Z_s)$ condition for $q = 1, \ldots, |Z_s|$. Looking ahead, these core-sets will be generated by first running the protocol $\Pi_{Rand}$, using an appropriate value of $L$, which will be determined across all the sub-protocols which we will be discussing next. All the secret-shared values in the various sub-protocols in the sequel will have $GW_1, \ldots, GW_{|Z_s|}$ as underlying core-sets.

8.1 Verifiably Generating Linear Secret Sharing of a Value with IC-signatures

In protocol $\Pi_{LSh}$ (Fig 10), there exists a designated dealer $D \in P$ with private input $s$. In addition, there is a random value $r \in \mathbb{F}$, which is linearly secret-shared with IC-signatures, such that the underlying core-sets are $GW_1, \ldots, GW_{|Z_s|}$ (the value $r$ will not be known to $D$ at the beginning of the protocol). The protocol allows the parties to let $D$ verifiably generate a linear secret-sharing of $s$ with IC-signatures, such that the underlying core-sets are $GW_1, \ldots, GW_{|Z_s|}$, where $s$ remains private for an honest $D$. The verifiability guarantees that even if $D$ is corrupt, if any (honest) party computes an output, then there exists some value, say $s^*$, which is linearly secret-shared with IC-signatures, such that the underlying core-sets are $GW_1, \ldots, GW_{|Z_s|}$.

The protocol idea is very simple and standard. We first let $D$ reconstruct the value $r$, which is then used as a one-time pad (OTP) by $D$ to make public an OTP-encryption of $s$. Then, using the linearity property of secret-sharing, the parties locally remove the OTP from the OTP-encryption.

\begin{protocol}
\textbf{Protocol $\Pi_{LSh}(D, s, Z_s, s_a, S_{Z_s}, [r], GW_1, \ldots, GW_{|Z_s|})$}

- **Reconstructing the OTP Towards the Dealer:** The parties in $P$ invoke an instance $\Pi_{Rec}([r], \{D\})$ of $\Pi_{Rec}$ to let $D$ reconstruct $r$ and wait for time $T_{Rec}$.
- **Making the OTP-encryption Public:** $D$, upon computing the output $r$ from the instance of $\Pi_{Rec}$, broadcasts $s = s + r$.
- **Computing the Output:** The parties in $P$ wait till the local time becomes $T_{Rec} + T_{BC}$. Then upon receiving $s$ from the broadcast of $D$, the parties in $P$ locally compute $[s - r]$ from $[s]$ and $[r]$. Here $[s]$ denotes the default linear secret-sharing of $s$ with IC-signatures and core-sets $GW_1, \ldots, GW_{|Z_s|}$, where $[s]_1 = s$ and $[s]_2 = \ldots = [s]_{|Z_s|} = 0$, and where the parties set $\text{ICSig}(P_i, P, P_k, [s]) \rightarrow P_j \in GW_q, P_k \in P$ to the default value. The parties then output $(GW_1, \ldots, GW_{|Z_s|}, [s - r])$.
\end{protocol}

Fig. 10: VSS for verifiably generating a linear secret-sharing of a value with IC-signatures with respect to given global core-sets
The properties of the protocol \( \Pi_{LSh} \) stated in Lemma 8.1 are proved in Appendix F.

**Lemma 8.1.** Let \( r \) be a random value which is linearly secret-shared with IC-signatures with \( GW_1, \ldots, GW_{|Z_s|} \) being the underlying core-sets. Then protocol \( \Pi_{LSh} \) achieves the following where \( D \) participates with the input \( s \).

1. If \( D \) is honest, then the following hold, where \( T_{LSh} = T_{Rec} + T_{BC} \).
   - **Z_s-Correctness:** If the network is synchronous, then except with probability \( O(n^3 \cdot \epsilon_{ICP}) \), the honest parties output \( [s] \) at the time \( T_{LSh} \), with \( GW_1, \ldots, GW_{|Z_s|} \) being the underlying core-sets.

2. **Z_s-Commitment:** If the network is asynchronous, then except with probability \( O(n^3 \cdot \epsilon_{ICP}) \), the honest parties eventually output \( [s] \), with \( GW_1, \ldots, GW_{|Z_s|} \) being the underlying core-sets.

3. **Z_a-Commitment:** If the network is asynchronous, then except with probability \( O(n^3 \cdot \epsilon_{ICP}) \), the honest parties eventually output \( [s^*] \), with \( GW_1, \ldots, GW_{|Z_s|} \) being the underlying core-sets.

4. **Communication Complexity:** \( O(|Z_s| \cdot n^3 \cdot \log |\mathbb{F}| + n^4 \cdot \log |\mathbb{F}| \cdot |\sigma|) \) bits are communicated by the honest parties.

We end this section with some notations which we use while invoking the protocol \( \Pi_{LSh} \) in the rest of the paper.

**Notation 8.1 (Notations for Using Protocol \( \Pi_{LSh} \))** Let \( P_i \in P \). In the rest of the paper we will say that “\( P_i \) invokes an instance of \( \Pi_{LSh} \) with input \( s \)” to mean that \( P_i \) acts as \( D \) and invokes an instance \( \Pi_{LSh}(D, s, Z_s, Z_a, S_{Z_s}, [r], GW_1, \ldots, GW_{|Z_s|}) \) of \( \Pi_{LSh} \). Here, \( r \) will be the corresponding random “pad” for this instance of \( \Pi_{LSh} \), which will already be linearly secret-shared with IC-signatures, with \( GW_1, \ldots, GW_{|Z_s|} \) being the underlying core-sets. If there are multiple instances of \( \Pi_{LSh} \) invoked by \( D \), then corresponding to each instance, there will be a random secret-shared pad available to the parties beforehand. The parties will be knowing which secret-shared pad is associated with which instance of \( \Pi_{LSh} \). This will be ensured by upper-bounding the maximum number of \( \Pi_{LSh} \) instances \( L_{max} \) invoked across all our protocols. The parties then generate \( L_{max} \) number of linearly secret-shared random values with IC-signatures, with \( GW_1, \ldots, GW_{|Z_s|} \) being the underlying core-sets, by running the protocol \( \Pi_{Rand} \) beforehand.
8.2 Non-Robust Multiplication Protocol

Protocol $\Pi_{\text{BasicMult}}$ (Fig 11) takes input $a$ and $b$, which are linearly secret-shared with IC-signatures, with $GW_1, \ldots, GW_{\mid Z_s \mid}$ being the underlying core-sets and a publicly known subset $GD \subset P$, consisting of only corrupt parties. The parties output a linear secret-sharing of $c$ with IC-signatures, with $GW_1, \ldots, GW_{\mid Z_s \mid}$ being the underlying core-sets. If all the parties in $P \setminus GD$ behave honestly, then $c = a \cdot b$, else $c = a \cdot b + \delta$, where $\delta \neq 0$. Moreover, the adversary does not learn anything additional about $a$ and $b$ in the protocol. The protocol also takes input an iteration number $\text{iter}$ and all the sets computed in the protocol are tagged with $\text{iter}$. Looking ahead, our robust triple-generation protocol will be executed iteratively, with each iteration invoking instances of $\Pi_{\text{BasicMult}}$.

A detailed overview of the protocol $\Pi_{\text{BasicMult}}$ has been already presented in Section 1.2.6. The idea is to let each summand $[a]_p \cdot [b]_q$ be linearly secret-shared by exactly one summand-sharing party. A secret-sharing of $a \cdot b$ then follows from the secret-sharing of each summand $[a]_p \cdot [b]_q$ owing to the linearity property of the secret-sharing. To deal with the network agnostic condition, the summand-sharing parties are selected in two phases: first, we select them dynamically, without pre-assigning any summand to any designated party. Once there exists a subset of parties from $S_{Z_s}$ who have served the role of summand-sharing parties, we go to the second phase, where each remaining summand is designated to the left-over parties through some publicly known assignment. Strict timeouts are maintained to ensure that we don’t stuck forever during the second phase. Finally, if there are still any remaining summands which are not yet secret-shared, they are publicly reconstructed and the default sharing is taken on their behalf. Throughout, the parties in $GD$ are not let to secret-share any summand, since they are already known to be corrupt and at the same time, it is ensured that the shares of the honest parties are never publicly reconstructed.

- **Initialization**: The parties in $P$ do the following.
  - Initialize the summand-index-set of indices of all summands:
    \[
    \text{SIS}_{\text{iter}} = \{(p, q)\}_{p, q=1, \ldots, \mid S_{Z_s} \mid}.
    \]
  - Initialize the summand-index-set corresponding to each $P_j \in P \setminus GD$:
    \[
    \text{SIS}^{(j)}_{\text{iter}} = \{(p, q)\}_{p_j \in S_{p_j} \cap S_q}.
    \]
  - Initialize the summand-index-set corresponding to each $S_q \in S_{Z_s}$:
    \[
    \text{SIS}^{(S_q)}_{\text{iter}} = \cup_{P_j \in S_q} \text{SIS}^{(j)}_{\text{iter}}.
    \]
  - Initialize the set of summands-sharing parties:
    \[
    \text{Selected}_{\text{iter}} = \emptyset.
    \]
  - Initialize the hop number:
    \[
    \text{hop} = 1.
    \]
Phase I: Sharing Summands Through Dynamic Assignment

- While there exists no \( S_q \in \mathbb{S}_S \), where \( \text{SIS}_{iter}(S_q) = \emptyset \), the parties do the following:
  - **Sharing Sum of Eligible Summands**: Every \( P_i \notin (\text{Selected}_{iter} \cup GD) \) invokes an instance \( P_{iter}^{(\phi \text{h}, hop,i)} \) of \( P_{iter}^{\phi \text{h}} \) with input \( c_{iter}^{(i)} \), where
    \[
    c_{iter}^{(i)} = \sum_{(p,q) \in \text{SIS}_{iter}^{(i)} } [a_p]_q b_q.
    \]
    Corresponding to every \( P_i \notin (\text{Selected}_{iter} \cup GD) \), the parties in \( P \) participate in the instance \( P_{iter}^{(\phi \text{h}, hop,i)} \), if invoked by \( P_j \).
  - **Selecting Summand-Sharing Party for the Hop Through ACS**: The parties in \( P \) wait for time \( T_{LS} \) and then do the following.
    - For \( j = 1, \ldots, n \), participate in an instance \( P_{iter}^{(\phi \text{h}, hop,j)} \) of \( P_{iter}^{\phi \text{h}} \) corresponding to \( P_i \in P \) with input 1 if all the following hold:
      - \( P_j \notin (\text{Selected}_{iter} \cup GD) \);
      - An output \((c_{iter}^{(j)})\) is computed during the instance \( P_{iter}^{(\phi \text{h}, hop,j)} \).
    - Upon computing an output 1 during the instance \( P_{iter}^{(\phi \text{h}, hop,j)} \) corresponding to some \( P_j \in P \), participate with input 0 in the instances \( P_{iter}^{(\phi \text{h}, hop,k)} \) corresponding to parties \( P_k \notin (\text{Selected}_{iter} \cup GD) \), for which no input has been provided yet.
    - Upon computing outputs during the instances \( P_{iter}^{(\phi \text{h}, hop,i)} \) corresponding to each \( P_i \notin (\text{Selected}_{iter} \cup GD) \), let \( P_j \) be the least-indexed party, such that the output 1 is computed during the instance \( P_{iter}^{(\phi \text{h}, hop,j)} \). Then update the following.
      - \( \text{Selected}_{iter} = \text{Selected}_{iter} \cup \{ P_j \} \).
      - \( \text{SIS}_{iter} = \text{SIS}_{iter} \setminus \text{SIS}_{iter}^{(j)} \).
      - \( \forall P_k \in P \setminus \{GD \cup \text{Selected}_{iter}\}: \text{SIS}_{iter}^{(k)} = \text{SIS}_{iter}^{(k)} \setminus \text{SIS}_{iter}^{(j)} \).
      - For each \( S_q \in \mathbb{S}_S \), \( \text{SIS}_{iter}^{(S_q)} = \text{SIS}_{iter}^{(S_q)} \setminus \text{SIS}_{iter}^{(j)} \).
      - Set \( hop = hop + 1 \).

Phase II: Sharing Remaining Summands Through Static Assignment

- **Re-assigning the Summand-Index-Set of Each Party**: Corresponding to each \( P_i \in P \setminus \text{Selected}_{iter} \), the parties in \( P \) set \( \text{SIS}_{iter}^{(i)} \) as
  \[
  \text{SIS}_{iter}^{(i)} = \text{SIS}_{iter} \setminus \{ (p,q) \} P_j = \text{min}(S_p \cap S_q),
  \]
  where \( \text{min}(S_p \cap S_q) \) denotes the minimum indexed party in \((S_p \cap S_q)\).
  - **Sharing Sum of Assigned Summands**: Every party \( P_i \notin (\text{Selected}_{iter} \cup GD) \) invokes an instance \( P_{iter}^{(\phi S), hop,i} \) of \( P_{iter}^{\phi S} \) with input \( c_{iter}^{(i)} \), where
    \[
    c_{iter}^{(i)} = \sum_{(p,q) \in \text{SIS}_{iter}^{(i)} } [a_p]_q b_q.
    \]
    Corresponding to every \( P_j \in P \setminus (\text{Selected}_{iter} \cup GD) \), the parties in \( P \) participate in the instance \( P_{iter}^{(\phi S, hop,j)} \), if invoked by \( P_j \).
  - **Agreeing on the Summand-Sharing parties of the Second Phase**: The parties in \( P \) wait for \( T_{LS} \) time after the beginning of the second phase. Then for
Lemma 8.2. During any instance $\PiBA^{\text{G(W,j)}}$ of $\PiBA$ with input 1, if all the following hold, otherwise participate with input 0.
- $P_j \notin \text{Selected}_{\text{iter}} \cup \text{GD}$;
- An output $[c^{(j)}_{\text{iter}}]$ is computed during the instance $\PiBA^{\text{G(W,j)}}$.

**Updating the Sets for the Second Phase:** Corresponding to each $P_j \notin \text{Selected}_{\text{iter}} \cup \text{GD}$, such that 1 is computed as the output during $\PiBA^{\text{G(W,j)}}$, update
- $\text{SIS}_{\text{iter}} = \text{SIS}_{\text{iter}} \setminus \text{SIS}_{\text{iter}}^{(j)}$;
- $\text{Selected}_{\text{iter}} = \text{Selected}_{\text{iter}} \cup \{P_j\}$.

**Phase III: Reconstructing the Remaining Summands**

- **Reconstructing the Remaining Summands and Taking the Default Sharing:** The parties in $\mathcal{P}$ do the following:
  - Corresponding to each $[a]_p$ such that $(p, \star) \in \text{SIS}_{\text{iter}}$, participate in the instance $\Pi_{\text{RecShare}}([a], S_p, \mathcal{P})$ of $\Pi_{\text{RecShare}}$ to publicly reconstruct $[a]_p$;
  - Corresponding to each $[b]_q$ such that $(\star, q) \in \text{SIS}_{\text{iter}}$, participate in the instance $\Pi_{\text{RecShare}}([b], S_q, \mathcal{P})$ of $\Pi_{\text{RecShare}}$ to publicly reconstruct $[b]_q$;
  - Corresponding to every $P_j \in \mathcal{P} \setminus \text{Selected}_{\text{iter}}$, take the default linear secret-sharing of the public input $c^{(j)}_{\text{iter}}$ with IC-signatures and core-sets $\mathcal{G}W_1, \ldots, \mathcal{G}W_{|Z_j|}$, where
    \[ c^{(j)}_{\text{iter}} = \sum_{(p,q) \in \text{SIS}_{\text{iter}}^{(j)}} [a]_p [b]_q. \]

- **Output Computation:** The parties output $(\mathcal{G}W_1, \ldots, \mathcal{G}W_{|Z_j|}, [c^{(1)}_{\text{iter}}], \ldots, [c^{(n)}_{\text{iter}}], [c_{\text{iter}}])$, where $c_{\text{iter}} \overset{\text{def}}{=} c^{(1)}_{\text{iter}} + \ldots + c^{(n)}_{\text{iter}}$.

\[ ^{\text{a The default linear secret-sharing of } c^{(j)}_{\text{iter}} \text{ is computed in a similar way as done in the protocol } \Pi_{\text{Rand}} \text{ for } a \text{ (see Fig 9).}} \]

The properties of the protocol $\Pi_{\text{BasicMult}}$ are claimed in the following lemmas, which are proved in Appendix F.

**Lemma 8.3.** Suppose that no honest party is present in $\text{GD}$. If the honest parties start participating during hop number $\text{hop}$ of Phase I of $\Pi_{\text{BasicMult}}$ with iteration number $\text{iter}$, then except with probability $O(n^3 \cdot c_{\text{CP}})$, the hop takes $T_{\text{LSh}} + 2T_{\text{BA}}$ time to complete in a synchronous network, or almost-surely completes eventually in an asynchronous network.

**Lemma 8.4.** If no honest party is present in $\text{GD}$, then in protocol $\Pi_{\text{BasicMult}}$, except with probability $O(n^3 \cdot c_{\text{CP}})$, all honest parties compute some output by the time $T_{\text{BasicMult}} = (2n + 1) \cdot T_{\text{BA}} + (n + 1) \cdot T_{\text{LSh}} + T_{\text{Rec}}$ in a synchronous network, or almost-surely, eventually in an asynchronous network.
Lemma 8.5. If no honest party is present in $\mathcal{GD}$, then the view of the adversary remains independent of $a$ and $b$ throughout the protocol, irrespective of the network type.

Lemma 8.6. If no honest party is present in $\mathcal{GD}$ and if all parties in $\mathcal{P} \setminus \mathcal{GD}$ behave honestly, then in protocol $\Pi_{BasicMult}$, the honest parties output a linear secret-sharing of $a \cdot b$ with IC-signatures, with $\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z_s|}$ being the underlying core-sets, irrespective of the network type.

Lemma 8.7. Protocol $\Pi_{BasicMult}$ incurs a communication of $O(|Z_s| \cdot n^5 \cdot \log |F| + n^6 \cdot \log |F| \cdot |\sigma|)$ bits and makes $O(n^2)$ calls to $\Pi_{BA}$.

As a corollary of Lemma 8.7, we can derive the following corollary, which determines the maximum number of instances of $\Pi_{LSH}$ which are invoked during an instance of $\Pi_{BasicMult}$. Looking ahead, this will be useful to later calculate the maximum number of instances of $\Pi_{LSH}$ which need to be invoked as part of our final multiplication protocol. This will be further useful to determine the number of linearly secret-shared values with IC-signatures and core-sets $\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z_s|}$, which need to be generated through the protocol $\Pi_{Rand}$ beforehand.

Corollary 8.1. During any instance of $\Pi_{BasicMult}$, there can be at most $n^2 + n$ instances of $\Pi_{LSH}$ invoked.

Protocol $\Pi_{BasicMult}$ for $L$ Pairs of Inputs. Protocol $\Pi_{BasicMult}$ can be easily generalized, if there are $L$ pairs of inputs $\{(a^t, b^t)\}_{t=1,\ldots,L}$, all of which are linearly secret-shared with IC-signatures, with $\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z_s|}$ being the underlying core-sets. However, with a slight modification during Phase I and Phase II, we can ensure that the number of instances of $\Pi_{BA}$ remain only $O(n^2)$, which is independent of $L$. Consider Phase I. During hop number hop, every party $P_i \notin (\text{Selected}_{iter} \cup \mathcal{GD})$ invokes $L$ instances of $\Pi_{LSH}$ to linearly secret-share $L$ candidate summand-sums. Now while selecting the summand-sharing party through ACS for this hop, the parties vote for a candidate $P_j \notin (\text{Selected}_{iter} \cup \mathcal{GD})$, provided an output is computed in all the $L$ instances of $\Pi_{LSH}$ invoked by $P_j$. Consequently, the number of instances of $\Pi_{BA}$ during Phase I will be $O(n^2)$. Similarly during Phase II, each party outside $\text{Selected}_{iter} \cup \mathcal{GD}$ invokes $L$ instances of $\Pi_{LSH}$ to linearly secret-share $L$ candidate reassigned summand-sums. And then the parties vote for a candidate $P_j$ as a summand-sharing party, if an output is computed in all the $L$ instances of $\Pi_{LSH}$ invoked by $P_j$. Finally, during Phase III, the default linear secret-sharing with IC-signatures is taken for the sum of all the summands, which are not yet secret-shared by any party, by making public all these summands. The resultant protocol incurs a communication of $O(|Z_s|^3 \cdot n^4 \cdot L \cdot \log |F| + |Z_s| \cdot n^5 \cdot L \cdot \log |F| + n^6 \cdot L \cdot \log |F| \cdot |\sigma|)$ bits and makes $O(n^2)$ calls to $\Pi_{BA}$. We also note that there will be at most $n^2 \cdot L + n \cdot L$ instances of $\Pi_{LSH}$ invoked in the generalized protocol. To avoid repetition, we do not present the steps of the generalized protocol here.
8.3 Network Agnostic Random Triple Generation with Cheater Identification

The network-agnostic protocol \( \Pi_{\text{RandMultCI}} \) (Fig 12) takes an iteration number \( \text{iter} \) and a publicly known subset of parties \( GD \), who are guaranteed to be corrupt. If all the parties in \( P \setminus GD \) behave honestly, then the protocol outputs a random linearly secret-shared multiplication-triple with IC-signatures, with \( GW_1, \ldots, GW_{|Z|} \) being the underlying core sets. Otherwise, with a high probability, the honest parties identify a new corrupt party, which is added to \( GD \).

Protocol \( \Pi_{\text{RandMultCI}} \) is based on [40] and consists of two stages: during the first stage, the parties jointly generate a pair of random values, which are linearly secret-shared with IC-signatures, with \( GW_1, \ldots, GW_{|Z|} \) being the underlying core sets. During the second stage, the parties run an instance of \( \Pi_{\text{BasicMult}} \) to compute the product of the pair of secret-shared random values from the first stage. To check whether any cheating has occurred during the instance of \( \Pi_{\text{BasicMult}} \), the parties then run a probabilistic test, namely the “sacrificing trick” [27], for which the parties need additional secret-shared random values, which are generated during the first stage itself.

\[ \text{Protocol } \Pi_{\text{RandMultCI}}(P, Z, Z_a, S_Z, GW_1, \ldots, GW_{|Z|}, GD, \text{iter}) \]

- **Generating Linear Secret Sharing of Random Values with IC-signatures:** Each \( P_i \in P \) does the following.
  - Invoke instances of \( \Pi_{\text{LSB}} \) with randomly chosen inputs \( a_{\text{iter}}^{(i)}, b_{\text{iter}}^{(i)}, r_{\text{iter}}^{(i)} \in \mathbb{F} \).
  - Corresponding to every \( P_j \in P \), participate in the instances of \( \Pi_{\text{LSB}} \) invoked by \( P_i \) (if any) and wait for \( T_{\text{LSB}} \). Initialize a set \( C_i = \emptyset \) after local time \( T_{\text{LSB}} \) and include \( P_j \) in \( C_i \), if any output is computed in all the instances of \( \Pi_{\text{LSB}} \) invoked by \( P_j \).
  - Corresponding to every \( P_j \in P \), participate in an instance of \( \Pi_{\text{BA}}^{(j)} \) of \( \Pi_{\text{BA}} \) with input 1, if \( P_j \in C_i \).
  - Once 1 has been computed as the output from instances of \( \Pi_{\text{BA}} \) corresponding to a set of parties in \( P \setminus Z \) for some \( Z \in Z_a \), participate with input 0 in all the \( \Pi_{\text{BA}} \) instances \( \Pi_{\text{BA}}^{(j)} \), such that \( P_j \notin C_i \).
  - Once a binary output is computed in all the instances of \( \Pi_{\text{BA}} \) corresponding to the parties in \( P \), compute \( CS \), which is the set of parties \( P_j \in P \), such that 1 is computed as the output in the instance \( \Pi_{\text{BA}}^{(j)} \).

Once \( CS \) is computed, the parties in \( P \) locally compute \( [a_{\text{iter}}], [b_{\text{iter}}], [b'_{\text{iter}}] \) and \( [r_{\text{iter}}] \) from \([a_{\text{iter}}], [b_{\text{iter}}], [b'_{\text{iter}}]) \) and \([r_{\text{iter}}] \) respectively, such as follows:

- **Generating Linear Secret Sharing of Random Values with IC-signatures:**
  - \( a_{\text{iter}} = \sum_{P_j \in CS} a_{\text{iter}}^{(j)} \), \( b_{\text{iter}} = \sum_{P_j \in CS} b_{\text{iter}}^{(j)} \), \( b'_{\text{iter}} = \sum_{P_j \in CS} b'_{\text{iter}}^{(j)} \), \( r_{\text{iter}} = \sum_{P_j \in CS} r_{\text{iter}}^{(j)} \).

- **Computing Secret-Shared Products:** The parties in \( P \) do the following.
  - Participate in instances \( \Pi_{\text{BasicMult}}(Z, Z_a, S_Z, a, [a_{\text{iter}}], [b_{\text{iter}}], GW_1, \ldots, GW_{|Z|}, GD, \text{iter}) \) and \( \Pi_{\text{BasicMult}}(Z, Z_a, S_Z, [a], [b_{\text{iter}}], GW_1, \ldots, GW_{|Z|}, GD, \text{iter}) \) of \( \Pi_{\text{BasicMult}} \) to compute the outputs \( (GW_1, \ldots, GW_{|Z|}, [a_{\text{iter}}], [b_{\text{iter}}], [r_{\text{iter}}]) \) and \( (GW_1, \ldots, GW_{|Z|}, [a_{\text{iter}}], [b_{\text{iter}}], [r_{\text{iter}}]) \) respectively. Let \( \text{Selected}_{\text{iter}, c} \) and \( \text{Selected}_{\text{iter}, c'} \) be the summand-sharing parties for the two in-
stances respectively. Moreover, for each \( P_j \in \text{Selected}_{\text{iter},c} \), let \( \text{SIS}_{\text{iter},c}^{(j)} \) be the set of ordered pairs of indices corresponding to the summands whose sum has been shared by \( P_j \) during the instance \( \Pi_{\text{BasicMult}}(Z_a, Z_b, Z_c, [a]_\text{iter}, [b]_\text{iter}, [c]_\text{iter}, [a]_\text{iter}, [b]_\text{iter}, [c]_\text{iter}, [a]_\text{iter}, [b]_\text{iter}, [c]_\text{iter}, [a]_\text{iter}, [b]_\text{iter}, [c]_\text{iter}) \). And similarly, for each \( P_j \in \text{Selected}_{\text{iter},c'} \), let \( \text{SIS}_{\text{iter},c'}^{(j)} \) be the set of ordered pairs of indices corresponding to the summands whose sum has been shared by \( P_j \) during the instance \( \Pi_{\text{BasicMult}}(Z_a, Z_b, Z_c, [a]_\text{iter}, [b]_\text{iter}, [c]_\text{iter}, [a]_\text{iter}, [b]_\text{iter}, [c]_\text{iter}, [a]_\text{iter}, [b]_\text{iter}, [c]_\text{iter}, [a]_\text{iter}, [b]_\text{iter}, [c]_\text{iter}) \).

- **Error Detection in the Instances of \( \Pi_{\text{BasicMult}} \):** The parties in \( \mathcal{P} \) do the following.
  - Upon computing outputs from the instances of \( \Pi_{\text{BasicMult}} \), participate in an instance \( \Pi_{\text{Rec}}([r]_\text{iter}, \mathcal{P}) \) of \( \Pi_{\text{Rec}} \) to publicly reconstruct \( r_{\text{iter}} \).
  - Locally compute \( e_{\text{iter}} \) \( \equiv r_{\text{iter}}[b]_\text{iter} + ([b]_\text{iter}) \) from \([b]_\text{iter}\) and \([b]_\text{iter}\). Participate in an instance \( \Pi_{\text{Rec}}([e]_\text{iter}, \mathcal{P}) \) of \( \Pi_{\text{Rec}} \) to publicly reconstruct \( e_{\text{iter}} \).
  - Locally compute \( d_{\text{iter}} \) \( \equiv e_{\text{iter}}[a]_\text{iter} - [c]_\text{iter} - [c']_\text{iter} \) from \([a]_\text{iter}\), \([c]_\text{iter}\) and \([c']_\text{iter}\). Participate in an instance \( \Pi_{\text{Rec}}([d]_\text{iter}, \mathcal{P}) \) of \( \Pi_{\text{Rec}} \) to publicly reconstruct \( d_{\text{iter}} \).

- **Output Computation in Case of Success:** If \( d_{\text{iter}} = 0 \), then set the boolean variable \( \text{flag}_{\text{iter}} = 0 \) and output \((G\mathcal{W}_1, \ldots, G\mathcal{W}_{|Z|}; [a]_\text{iter}, [b]_\text{iter}, [c]_\text{iter}) \).

- **Cheater Identification in Case of Failure:** If \( d_{\text{iter}} \neq 0 \), then set the boolean variable \( \text{flag}_{\text{iter}} = 1 \) and proceed as follows.
  - For each \( S_q \in S_{Z_q} \), participate in instances \( \Pi_{\text{RecShare}}([a]_\text{iter}, S_q, \mathcal{P}), \Pi_{\text{RecShare}}([b]_\text{iter}, S_q, \mathcal{P}) \) and \( \Pi_{\text{RecShare}}([c]_\text{iter}, S_q, \mathcal{P}) \) of \( \Pi_{\text{RecShare}} \) to publicly reconstruct the shares \( ([a]_\text{iter}, [b]_\text{iter}, [c]_\text{iter}) \) for each \( S_q \in S_{Z_q} \). In addition, for \( i = 1, \ldots, n \), participate in instances \( \Pi_{\text{Rec}}(\mathcal{S}_i^{(i)}), \mathcal{P}) \) and \( \Pi_{\text{Rec}}(\mathcal{S}_i^{(i)}), \mathcal{P}) \) to publicly reconstruct \( \mathcal{S}_i^{(i)} \) and \( \mathcal{S}_i^{(i)} \).

- Set \( \mathcal{G} = \mathcal{G} \cup \{P_j\} \), if \( P_j \in \text{Selected}_{\text{iter},c} \cup \text{Selected}_{\text{iter},c'} \) and the following holds for \( P_j \):
  \[
  r_{\text{iter}}, c^{(j)}_{\text{iter}} + c^{(j)}_{\text{iter}} \neq r_{\text{iter}}, \sum_{(p,q) \in \text{Sig}_{\text{iter},c}^{(j)}} [a]_\text{iter} [b]_\text{iter} + \sum_{(p,q) \in \text{Sig}_{\text{iter},c'}^{(j)}} [a]_\text{iter} [b']_\text{iter}.
  \]

Fig. 12: Network-agnostic protocol for generating secret-shared random multiplication-triple with cheater identification.

The properties of the protocol \( \Pi_{\text{RandMultCI}} \) are claimed in the following lemmas, which are proved in Appendix F.

**Lemma 8.8.** In protocol \( \Pi_{\text{RandMultCI}} \), the following hold.

- **Synchronous Network:** Except with probability \( O(n^3 \cdot \epsilon_{\text{CP}}) \), honest parties will have linearly secret-shared \( a_{\text{iter}}, b_{\text{iter}}, b'_{\text{iter}} \) and \( r_{\text{iter}} \) with IC-signatures, with \( G\mathcal{W}_1, \ldots, G\mathcal{W}_{|Z|} \) being the underlying core-sets, by the time \( T_{\text{LSh}} + 2T_{\text{BA}} \). Moreover, adversary’s view is independent of \( a_{\text{iter}}, b_{\text{iter}}, b'_{\text{iter}} \) and \( r_{\text{iter}} \).

- **Asynchronous Network:** Except with probability \( O(n^3 \cdot \epsilon_{\text{CP}}) \), almost-surely, honest parties will eventually have linearly secret-shared \( a_{\text{iter}}, b_{\text{iter}}, b'_{\text{iter}} \) and \( r_{\text{iter}} \) with IC-signatures, with \( G\mathcal{W}_1, \ldots, G\mathcal{W}_{|Z|} \) being the underlying core-sets. Moreover, adversary’s view is independent of \( a_{\text{iter}}, b_{\text{iter}}, b'_{\text{iter}} \) and \( r_{\text{iter}} \).

**Lemma 8.9.** Consider an arbitrary \( \text{iter} \), such that all honest parties participate in the instance \( \Pi_{\text{RandMultCI}}(\mathcal{P}, Z_s, Z_a, Z_b, Z_c, [a]_\text{iter}, [b]_\text{iter}, [c]_\text{iter}, [a]_\text{iter}, [b]_\text{iter}, [c]_\text{iter}, [a]_\text{iter}, [b]_\text{iter}, [c]_\text{iter}, [a]_\text{iter}, [b]_\text{iter}, [c]_\text{iter}, [a]_\text{iter}, [b]_\text{iter}, [c]_\text{iter}) \), where no
honest party is present in $\mathcal{GD}$. Then except with probability $O(n^3\epsilon_{\text{ICP}})$, all honest parties reconstruct a (common) value $d_{\text{iter}}$ and set $\text{flag}_{\text{iter}}$ to a common Boolean value, at the time $T_{\text{LSH}} + 2T_{\text{BA}} + T_{\text{BasicMult}} + 3T_{\text{Rec}}$ in a synchronous network, or eventually in an asynchronous network.

Lemma 8.10. Consider an arbitrary $\text{iter}$, such that all honest parties participate in the instance $\Pi_{\text{RandMultCI}}(\mathcal{P}, Z_a, Z_a, S_{Z_a}, \mathcal{GW}_{1}, \ldots, \mathcal{GW}_{|Z_a|}, \mathcal{GD}, \text{iter})$, where no honest party is present in $\mathcal{GD}$. If no party in $\mathcal{P} \setminus \mathcal{GD}$ behaves maliciously, then $d_{\text{iter}} = 0$ and the honest parties output $(|a_{\text{iter}}|, |b_{\text{iter}}|, |c_{\text{iter}}|)$ at the time $T_{\text{LSH}} + 2T_{\text{BA}} + T_{\text{BasicMult}} + 3T_{\text{Rec}}$ in a synchronous network or eventually in an asynchronous network, where $c_{\text{iter}} = a_{\text{iter}} \cdot b_{\text{iter}}$ and where $\mathcal{GW}_{1}, \ldots, \mathcal{GW}_{|Z_a|}$ are the underlying core-sets.

Lemma 8.11. Consider an arbitrary $\text{iter}$, such that all honest parties participate in the instance $\Pi_{\text{RandMultCI}}(\mathcal{P}, Z_a, Z_a, S_{Z_a}, \mathcal{GW}_{1}, \ldots, \mathcal{GW}_{|Z_a|}, \mathcal{GD}, \text{iter})$, where no honest party is present in $\mathcal{GD}$. If $d_{\text{iter}} \neq 0$, then except with probability $O(n^3\epsilon_{\text{ICP}})$, the honest parties update $\mathcal{GD}$ by adding a new maliciously-corrupt party in $\mathcal{GD}$, either at the time $T_{\text{RandMultCI}} = T_{\text{LSH}} + 2T_{\text{BA}} + T_{\text{BasicMult}} + 4T_{\text{Rec}}$ in a synchronous network or eventually in an asynchronous network.

Lemma 8.12. Consider an arbitrary $\text{iter}$, such that all honest parties participate in the instance $\Pi_{\text{RandMultCI}}(\mathcal{P}, Z_a, Z_a, S_{Z_a}, \mathcal{GW}_{1}, \ldots, \mathcal{GW}_{|Z_a|}, \mathcal{GD}, \text{iter})$, where no honest party is present in $\mathcal{GD}$. If $d_{\text{iter}} = 0$, then the honest parties output linearly secret-shared $(a_{\text{iter}}, b_{\text{iter}}, c_{\text{iter}})$ with IC-signatures with $\mathcal{GW}_{1}, \ldots, \mathcal{GW}_{|Z_a|}$ being the underlying core-sets, at the time $T_{\text{LSH}} + 2T_{\text{BA}} + T_{\text{BasicMult}} + 3T_{\text{Rec}}$ in a synchronous network or eventually in an asynchronous network where, except with probability $\frac{1}{|\mathcal{P}|}$, the condition $c_{\text{iter}} = a_{\text{iter}} \cdot b_{\text{iter}}$ holds. Moreover, the view of $\text{Adv}$ will be independent of $(a_{\text{iter}}, b_{\text{iter}}, c_{\text{iter}})$.

Lemma 8.13. Protocol $\Pi_{\text{RandMultCI}}$ incurs a communication of $O(|Z_a| \cdot n^5 \cdot \log |F| + n^6 \cdot \log |F| \cdot |\sigma|)$ bits and makes $O(n^2)$ calls to $\Pi_{\text{BA}}$.

Protocol $\Pi_{\text{RandMultCI}}$ for $L$ Triples. Protocol $\Pi_{\text{RandMultCI}}$ can be easily generalized to generate $L$ triples with cheater identification, such that the number of instances of $\Pi_{\text{BA}}$ is independent of $L$. To begin with, every party $P_j$ now picks $3L+1$ random values $(r_1^{(i)}, \{a_{\text{iter}}^{(i)} \cdot b_{\text{iter}}^{(i)} \cdot c_{\text{iter}}^{(i)}\}_{i=1}, \ldots, L)$ and invokes $3L+1$ instances of $\Pi_{\text{LSH}}$ to generate linear secret-sharing of these values with IC-signatures, with $\mathcal{GW}_{1}, \ldots, \mathcal{GW}_{|Z_a|}$ being the underlying core-sets. Next, to determine $CS$, the parties invoke only $n$ instances of $\Pi_{\text{BA}}$, where the $j^{th}$ instance is used to decide whether $P_j$ should be included in $CS$, the criteria being whether some output is computed in all the $3L+1$ instances of $\Pi_{\text{LSH}}$ invoked by $P_j$. Once $CS$ is decided, the parties locally compute $\{a_{\text{iter}}^{(i)}\}_{i=1}, \ldots, L$, $\{b_{\text{iter}}^{(i)}\}_{i=1}, \ldots, L$, $\{r_{\text{iter}}^{(i)}\}_{i=1}, \ldots, L$ and $\{c_{\text{iter}}^{(i)}\}_{i=1}, \ldots, L$ respectively. The parties then invoke two instances of (generalized) $\Pi_{\text{BasicMult}}$ protocol with $L$ pairs of secret-shared inputs to compute $\{c_{\text{iter}}^{(i)}\}_{i=1}, \ldots, L$ respectively. Note that this requires only
\(O(n^2)\) instances of \(\Pi_{BA}\), apart from \(n^2 \cdot L + n \cdot L\) instances of \(\Pi_{LSh}\). The rest of the protocol steps are then generalized to deal with \(L\) inputs. The resultant protocol incurs a communication of \(O(|Z_4| \cdot n_5 \cdot L \cdot \log |F| + n^6 \cdot L \cdot \log |F| \cdot |\sigma|)\) bits and makes \(O(n^2)\) calls to \(\Pi_{BA}\). We also note that there will be at most \(n^2 \cdot L + 4n \cdot L + n\) instances of \(\Pi_{LSh}\) invoked overall in the generalized protocol. To avoid repetition, we do not present the steps of the generalized protocol here.

### 8.4 The Multiplication-Triple Generation Protocol

Protocol \(\Pi_{TripGen}\) for generating a single secret-shared multiplication-triple is presented in Fig 13. The idea of the protocol is very simple and based on [40]. The parties iteratively run instances of \(\Pi_{RandMultCI}\), till they hit upon an instance when no cheating is detected. Corresponding to each “failed” instance of \(\Pi_{RandMultCI}\), the parties keep updating the set \(GD\). Since after each failed instance the set \(GD\) is updated with one new corrupt party, there will be at most \((t + 1)\) iterations, where \(t\) is the cardinality of the largest-sized subset in \(Z_4\).

**Protocol \(\Pi_{TripGen}(P, Z_4, S_{Z_4}, GW_1, \ldots, GW_{|Z_4|})\)**

- **Initialization**: The parties in \(\mathcal{P}\) initialize \(GD = \emptyset\) and \(\text{iter} = 1\).
- **Triple Generation with Cheater Identification**: The parties in \(\mathcal{P}\) participate in an instance \(\Pi_{RandMultCI}(P, Z_4, S_{Z_4}, GW_1, \ldots, GW_{|Z_4|}, GD, \text{iter})\) of \(\Pi_{RandMultCI}\) and wait for its completion. Upon computing output from the instance, the parties proceed as follows.
  - **Positive Output**: If the Boolean variable \(\text{flag}_{\text{iter}}\) is set to 0 during the instance of \(\Pi_{RandMultCI}\), then output \((GW_1, \ldots, GW_{|Z_4|}, [a_{\text{iter}}], [b_{\text{iter}}], [c_{\text{iter}}])\), computed during the instance of \(\Pi_{RandMultCI}\).
  - **Negative Output**: Else set \(\text{iter} = \text{iter} + 1\) and go to the step labelled **Triple Generation with Cheater Identification**.

Fig. 13: Network-agnostic protocol to generate a linear secret sharing with IC-signature of a random multiplication-triple.

The properties of the protocol \(\Pi_{TripGen}\) are claimed in the following lemmas, which are proved in Appendix F.

**Lemma 8.14.** Let \(t\) be the size of the largest set in \(Z_4\). Then except with probability \(O(n^3 \cdot \epsilon_{CP})\), the honest parties compute an output during \(\Pi_{TripGen}\), by the time \(T_{\Pi_{TripGen}} = (t + 1) \cdot T_{\Pi_{RandMultCI}}\) in a synchronous network, or almost surely, eventually in an asynchronous network, where \(T_{\Pi_{RandMultCI}} = T_{LSh} + 2T_{BA} + T_{BasicMult} + 4T_{Rec}\).

**Lemma 8.15.** If the honest parties output \((GW_1, \ldots, GW_{|Z_4|}, [a_{\text{iter}}], [b_{\text{iter}}], [c_{\text{iter}}])\) during the protocol \(\Pi_{TripGen}\), then \(a_{\text{iter}}, b_{\text{iter}}\) and \(c_{\text{iter}}\) are linearly secret-shared with IC-signatures, with \(GW_1, \ldots, GW_{|Z_4|}\) being the underlying core-sets. Moreover, \(c_{\text{iter}} = a_{\text{iter}}b_{\text{iter}}\) holds, except with probability \(\frac{1}{|\Gamma|}\). Furthermore, the view of the adversary remains independent of \(a_{\text{iter}}, b_{\text{iter}}\) and \(c_{\text{iter}}\).

**Lemma 8.16.** Protocol \(\Pi_{TripGen}\) incurs a communication of \(O(|Z_4| \cdot n^6 \cdot \log |F| + n^7 \cdot \log |F| \cdot |\sigma|)\) bits and makes \(O(n^2)\) calls to \(\Pi_{BA}\).
Protocol $\Pi_{\text{TripGen}}$ for Generating $L$ Multiplication-Triples. To generate $L$ multiplication-triples, the parties now need to invoke an instance of the generalized (modified) $\Pi_{\text{RandMultCl}}$ protocol in each iteration, which generates $L$ triples with cheater identification. The rest of the protocol steps remain the same. To avoid repetition, we do not present the formal details here. The protocol incurs a communication of $O(|Z_s| \cdot n^6 \cdot L \cdot \log |F| + n^7 \cdot L \cdot \log |F| \cdot |\sigma|)$ bits and makes $O(n^3)$ calls to $\Pi_{\text{BA}}$.

On the Maximum Number of Calls of $\Pi_{\text{LSh}}$ in $\Pi_{\text{TripGen}}$. As discussed in the previous section, each instance of $\Pi_{\text{RandMultCl}}$ for $L$ triples requires at most $n^2 \cdot L + 4n \cdot L + n$ instances of $\Pi_{\text{LSh}}$. Now as there can be up to $t + 1 \approx n$ such instances of $\Pi_{\text{RandMultCl}}$ in the protocol $\Pi_{\text{TripGen}}$, it follows that at most $n^3 \cdot L + 4n^2 \cdot L + n^2$ instances of $\Pi_{\text{LSh}}$ are invoked in the protocol $\Pi_{\text{TripGen}}$ for generating $L$ multiplication-triples.

9 Network Agnostic Circuit-Evaluation Protocol

The network-agnostic circuit-evaluation protocol $\Pi_{\text{cktEval}}$ is presented in Fig 14. The idea behind the protocol is to perform shared circuit-evaluation, where each value remains linearly secret-shared with IC-signatures and common core-sets. Once the function-output is secret-shared, it is publicly reconstructed. To achieve this goal, the parties first secret-share their respective inputs for the function $f$ through instances of $\Pi_{\text{LSh}}$. The parties then agree on a common subset of parties $\mathcal{CS}$, where $\mathcal{P} \setminus \mathcal{CS} \in Z_s$, such that the inputs of the parties in $\mathcal{CS}$ are linearly secret-shared with IC-signatures. If the network is synchronous then it will be ensured that all honest parties are present in $\mathcal{CS}$ and hence the inputs of all honest parties are considered for the circuit-evaluation. The linearity of secret-sharing ensures that the linear gates in $\text{ckt}$ are evaluated non-interactively, while Beaver’s trick is deployed for evaluating multiplication gates in $\text{ckt}$. For the latter, the parties need to have $c_M$ number of random multiplication-triples apriori, which are linearly secret-shared with IC-signatures. This is achieved by apriori calling the protocol $\Pi_{\text{TripGen}}$ with $L = c_M$, which in turn will require at most $n^3 \cdot c_M + 4n^2 \cdot c_M + n^2$ number of instances of $\Pi_{\text{LSh}}$. As the total number of instances of $\Pi_{\text{LSh}}$ across $\Pi_{\text{TripGen}}$ and the input-phase is at most $n^3 \cdot c_M + 4n^2 \cdot c_M + n^2 + n$, the parties first invoke an instance of $\Pi_{\text{Rand}}$ by setting $L = n^3 \cdot c_M + 4n^2 \cdot c_M + n^2 + n$, to generate these many linearly secret-shared random values with IC-signatures, with $\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z_s|}$ being the underlying core-sets. This will ensure that all the values during the circuit-evaluation are linearly secret-shared with IC-signatures, with $\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z_s|}$ being the underlying core-sets.

Notice that if the network is asynchronous then different parties may be in the different phases of the protocol. And consequently, a party upon reconstructing the function output cannot afford to immediately terminate, as its presence may be required in the other parts of the protocol. Hence there is also a termination phase, which is executed concurrently, where the parties check if it is “safe” to terminate the protocol.
Protocol $\Pi_{\text{ctEval}}(P, Z_\ast, Z_a, S_{Z_\ast}, \text{Ct}, c_M)$

Pre-Processing Phase

- Generating Linearly Secret-Shared Random Values with IC-signatures: The parties invoke an instance $\Pi_{\text{Rand}}(P, Z_\ast, Z_a, S_{Z_\ast}, L)$ where $L = n^3 \cdot c_M + 4n^2 \cdot c_M + n^2 + n$ and compute output $(GW_1, \ldots, GW_{|Z_\ast|}, \{[r^{(i)}]\}_{i=1,\ldots,L})$.
- Generating Linearly Secret-Shared Random Multiplication-Triples with IC-signatures: Upon computing an output in the instance of $\Pi_{\text{Rand}}$, the parties invoke an instance $\Pi_{\text{TripGen}}(P, Z_\ast, Z_a, S_{Z_\ast}, GW_1, \ldots, GW_{|Z_\ast|})$ with $L = c_M$ and compute output $(GW_1, \ldots, GW_{|Z_\ast|}, \{[a^{(i)}], [b^{(i)}], [c^{(i)}]\}_{i=1,\ldots,c_M})$. During the instance of $\Pi_{\text{TripGen}}$, the secret-shared values $\{[r^{(i)}]\}_{i=1,\ldots,n^3 \cdot c_M + 4n^2 \cdot c_M + n^2}$ are used as the corresponding pads in the underlying instances of $\Pi_{\text{LSK}}$, invoked as part of $\Pi_{\text{TripGen}}$.

Input Phase

Upon computing an output during the instance of $\Pi_{\text{TripGen}}$, each $P_i \in P$ does the following:

- On having the input $x^{(i)}$, invoke an instance of $\Pi_{\text{LSK}}$ with input $x^{(i)}$.
- Corresponding to every $P_j \in P$, participate in the instance of $\Pi_{\text{LSK}}$ invoked by $P_j$ and wait for local time $T_{\text{LSK}}$ after starting the input phase. Then initialize a set $C_i = \emptyset$ and include $P_j$ in $C_i$, if any output is computed in the instance of $\Pi_{\text{LSK}}$ invoked by $P_j$.
- Corresponding to every $P_j \in P$, participate in an instance of $\Pi_{\text{BA}}^{(i)}$ of $\Pi_{\text{BA}}$ with input 1, if $P_j \in C_i$.
- Once 1 has been computed as the output from instances of $\Pi_{\text{BA}}$ corresponding to a set of parties in $P \setminus Z_\ast$ for some $Z \in Z_\ast$, participate with input 0 in all the $\Pi_{\text{BA}}$ instances $\Pi_{\text{BA}}^{(i)}$, such that $P_j \notin C_i$.
- Once a binary output is computed in all the instances of $\Pi_{\text{BA}}$ corresponding to the parties in $P$, compute $\mathcal{C}_S$, which is the set of parties $P_j \in P$, such that 1 is computed as the output in the instance $\Pi_{\text{BA}}^{(i)}$.

Once $\mathcal{C}_S$ is computed, corresponding to every $P_j \notin \mathcal{C}_S$, the parties set $x^{(j)} = 0$ and take the default linear secret-sharing of 0 with IC-signatures, with $GW_1, \ldots, GW_{|Z_\ast|}$ being the underlying core-sets.

Circuit-Evaluation Phase

- Evaluate each gate $g$ in the circuit according to the topological ordering as follows, depending upon the type:
  - Addition Gate: If $g$ is an addition gate with inputs $x, y$ and output $z$, then the parties in $P$ locally compute $z = x + y$ from $[x]$ and $[y]$.
  - Multiplication Gate: If $g$ is the $\ell^{th}$ multiplication gate with inputs $x, y$ and output $z$, then the parties in $P$ do the following:
    - Locally compute $[d^{(\ell)}] = [x - a^{(\ell)}]$ from $[x]$ and $[a^{(\ell)}]$ and $[e^{(\ell)}] = [y - b^{(\ell)}]$ from $[y]$ and $[b^{(\ell)}]$.
    - Participate in instances $\Pi_{\text{Rec}}([d^{(\ell)}], P)$ and $\Pi_{\text{Rec}}([e^{(\ell)}], P)$ of $\Pi_{\text{Rec}}$ to publicly reconstruct $d^{(\ell)}$ and $e^{(\ell)}$, where $d^{(\ell)} \overset{\text{def}}{=} x - a^{(\ell)}$ and $e^{(\ell)} \overset{\text{def}}{=} y - b^{(\ell)}$.
    - Upon reconstructing $d^{(\ell)}$ and $e^{(\ell)}$, take the default linear secret-sharing of $d^{(\ell)}, e^{(\ell)}$ with IC-signatures, with $GW_1, \ldots, GW_{|Z_\ast|}$ being the underlying
core-sets. Then locally compute \( [z] = [d^{(f)}] + [e^{(f)}] \) from \( [d^{(f)}] = [a^{(f)}] + [c^{(f)}] \) and \( [e^{(f)}] = [b^{(f)}] \).

- **Output Gate:** If \( g \) is the output gate with output \( y \), then participate in an instance \( \Pi_{\text{Rec}}(y, \mathcal{P}) \) of \( \Pi_{\text{Rec}} \) to publicly reconstruct \( y \).

**Termination Phase**

Every \( P_i \in \mathcal{P} \) concurrently executes the following steps during the protocol:

- Upon computing the circuit-output \( y \), send the message (\( \text{ready}, P_i, y \)) to every party in \( \mathcal{P} \).
- Upon receiving the message (\( \text{ready}, P_j, y \)) from a set of parties \( \mathcal{A} \) such that \( Z_s \) satisfies \( Q^{(1)}(A, Z_s) \), condition, send (\( \text{ready}, P_i, y \)) to every party in \( \mathcal{P} \), provided no (\( \text{ready}, P_i, * \)) message has been sent yet.
- Upon receiving the message (\( \text{ready}, P_j, y \)) from a set of parties \( \mathcal{W} \) such that \( \mathcal{P} \setminus \mathcal{W} \in Z_s \), output \( y \) and terminate.

*The secret-shared random value \( r^{(n^3 \cdot c_M + 4n^2 \cdot c_M + n^2 + j)} \) serves as the pad for the instance of \( \Pi_{\text{Sec}} \) invoked by \( P_j \) in this phase.*

Fig. 14: Network agnostic secure circuit-evaluation protocol.

The properties of the protocol \( \Pi_{\text{cktEval}} \) stated in Theorem 9.1, are proved in Appendix G.

**Theorem 9.1.** Let \( Z_s \) and \( Z_a \) be monotone adversary structures where \( Z_a \subset Z_s \), the set \( Z_s \) satisfy the condition \( Q^{(2)}(\mathcal{P}, Z_s) \), the set \( Z_a \) satisfy the condition \( Q^{(3)}(\mathcal{P}, Z_s) \) and \( Z_s, Z_a \) together satisfy the condition \( Q^{(2,1)}(\mathcal{P}, Z_s, Z_a) \). Let \( F \) be a finite field such that \( |F| \geq n^5 \cdot 2^{ssec} \) where \( ssec \) is the statistical security parameter. Moreover, let \( y = f(x^{(1)}, \ldots, x^{(n)}) \) be a publicly known function over \( F \) represented by an arithmetic circuit \( \text{ckt} \) over \( F \), where each \( P_i \in \mathcal{P} \) has the input \( x^{(i)} \). Furthermore, let \( c_M \) and \( D_M \) be the number of multiplication gates and the multiplicative depth of \( \text{ckt} \) respectively. Then given an unconditionally-secure PKI, protocol \( \Pi_{\text{cktEval}} \) achieves the following, where every honest \( P_j \) participates with the input \( x^{(j)} \) and where \( t \) denotes the cardinality of the maximum sized subset in \( Z_s \).

- **Synchronous Network:** Except with probability \( 2^{-ssec} \), all honest parties output \( y = f(x^{(1)}, \ldots, x^{(n)}) \) at the time \( (|3n + 5|t^2 + (74n + 140) t + 69n + D_M + 438) \cdot \Delta \), where \( x^{(j)} = 0 \), for every \( P_j \not\in \mathcal{CS} \) and where every honest party is present in \( \mathcal{CS} \), such that \( \mathcal{P} \setminus \mathcal{CS} \subset Z_s \).
- **Asynchronous Network:** Except with probability \( 2^{-ssec} \), almost-surely, all honest parties eventually output \( y = f(x^{(1)}, \ldots, x^{(n)}) \) where \( x^{(j)} = 0 \), for every \( P_j \not\in \text{CoreSet} \) and where \( \mathcal{P} \setminus \mathcal{CS} \subset Z_s \).

The protocol incurs a communication of \( \mathcal{O}(|Z_s|^2 \cdot n^{12} \cdot \log |F|) \) bits and makes \( \mathcal{O}(n^3) \) calls to \( \Pi_{\text{BA}} \). Moreover, irrespective of the network type, the view of the adversary remains independent of the inputs of the honest parties in \( \mathcal{CS} \).
10 Impossibility Result

Here we show the necessity of the \( Q^{(2.1)}(P, Z_s, Z_a) \) condition for network agnostic MPC. In fact we show that the condition is even necessary for network agnostic BA. For this, we generalize the impossibility proof of [15] which shows the impossibility of network agnostic BA against threshold adversaries if \( 2t_s + t_a \geq n \).

**Theorem 10.1.** Let \( Z_s \) and \( Z_a \) satisfy the \( Q^{(2)}(P, Z_s) \) and \( Q^{(3)}(P, Z_a) \) conditions respectively, where \( Z_a \subset Z_s \).

\(^{16}\) Moreover, let the parties have access to the setup of an unconditional PKI. Furthermore, let \( P \) be an \( n \)-party protocol, which is a \( Z_s \)-secure SBA protocol in the synchronous network and which is a \( Z_a \)-secure ABA protocol in the asynchronous network (as per Definition 3.1). Then \( P \) exists only if \( Z_s \) and \( Z_a \) satisfy the \( Q^{(2.1)}(P, Z_s, Z_a) \) condition.

**Proof.** The proof is by contradiction. Let \( P \) exist, even if \( Z_s \) and \( Z_a \) do not satisfy the \( Q^{(2.1)}(P, Z_s, Z_a) \) condition. Then, there exist sets, say \( Z_0, Z_1 \in Z_s \) and \( Z_2 \in Z_a \) such that \( Z_0 \cup Z_1 \cup Z_2 \supseteq P \) holds. For simplicity and without loss of generality, assume that \( Z_0, Z_1 \) and \( Z_2 \) are disjoint. Now consider the following executions of \( P \). In all these executions, parties in \( Z_0 \) participate with input 0, and parties in \( Z_1 \) participate with input 1.

- **Execution \( E_1 \):** In this execution, the network is synchronous. All the parties in \( Z_0 \) are corrupted by the adversary and simply abort, and all parties in \( Z_2 \) participate with input 1. Since all the honest parties (namely the parties in \( Z_1 \cup Z_2 \)) have input 1, from the \( Z_s \)-validity of \( P \) in the synchronous network, the parties in \( Z_1 \cup Z_2 \) should output 1 after some fixed time, say \( T_1 \).

- **Execution \( E_2 \):** In this execution, the network is synchronous. All the parties in \( Z_1 \) are corrupted by the adversary and simply abort, and all parties in \( Z_2 \) participate with input 0. Since all the honest parties (namely the parties in \( Z_0 \cup Z_2 \)) have input 0, from the \( Z_s \)-validity of \( P \) in the synchronous network, the parties in \( Z_0 \cup Z_2 \) should output 0 after some fixed time, say \( T_2 \).

- **Execution \( E_3 \):** In this execution, the network is asynchronous, the adversary corrupts all the parties in \( Z_2 \) and behaves as follows: the communication between the parties in \( Z_0 \) and \( Z_1 \) is delayed by at least time \( \max(T_1, T_2) \). The adversary communicates with parties in \( Z_0 \) and \( Z_1 \), such that the views of the parties in \( Z_0 \) and \( Z_1 \) are identical to \( E_1 \) and \( E_2 \) respectively. For this, the adversary runs \( P \) with input 0 when interacting with the parties in \( Z_0 \) and runs \( P \) with input 1 when interacting with the parties in \( Z_1 \). Hence, the parties in \( Z_0 \) output 0, while the parties in \( Z_1 \) output 1, which violates the \( Z_a \)-consistency of \( P \) in the asynchronous network. This is a contradiction and hence, \( P \) does not exist.

\(^{16}\) The necessity of the \( Q^{(2)}(P, Z_s) \) and \( Q^{(3)}(P, Z_a) \) conditions follow from the existing results on the impossibility of unconditionally-secure SBA and ABA respectively, without these conditions. The condition \( Z_a \subset Z_s \) is also necessary since any potential corrupt subset which is tolerable in an asynchronous network should also be tolerable if the network is synchronous.
11 Acknowledgements

We would like to thank Anirudh Chandramouli for several helpful discussions during the early stages of this research.

References


A Properties of the Network Agnostic BA Protocol

In this section, we prove the properties of the network agnostic BA protocol (see Fig 1 for the protocol). We first formally present the sub-protocol $\Pi_{PW}$ and prove its properties.

A.1 Protocol $\Pi_{PW}$: Synchronous BA with Asynchronous Guaranteed Liveness

Protocol $\Pi_{PW}$ is presented in Fig 15, where for simplicity we assume that the input of each party is a bit. The protocol is very simple. Each party $P_i$ uses a Dolev-Strong (DS) style protocol [30] to broadcast its input $b_i$. The protocol runs for $t + 1$ “rounds”, where $t$ is the size of the largest set in $Z_s$. Each party $P_i$ accumulates values on the behalf of every party $P_j$ in a set $\text{ACC}_{ij}$. A bit $b$ is added to $\text{ACC}_{ij}$ during round $r$ only if $P_i$ receives signatures on $b$ from $r$ distinct parties including $P_j$. Party $P_i$ computes the final output by taking the “majority” among the accumulated values. This is done by computing a final set $\text{FIN}_i$ of values based on each $\text{ACC}_{ij}$ set. Since the DS protocol is designed for the synchronous network, for convenience, we present the protocol $\Pi_{PW}$ in a round-based fashion, where the parties set the duration of each round to $\Delta$ and will know the beginning and end of each round.

```
Protocol $\Pi_{PW}(P, Z_s, Z_a)$

- Initialization: Initialize the following sets.
  - For $j = 1, \ldots, n$: the set of values accumulated on the behalf of $P_j$, $\text{ACC}_{ij} = \emptyset$.
  - $t$: size of the largest set in $Z_s$.
  - The final set of values to be considered while computing the output $\text{FIN}_i = \emptyset$.
- Round 0: On having input bit $b_i$, sign $b_i$ to obtain the signature $\sigma_{ii}$. Set $\text{SET}_i = \{\sigma_{ii}\}$ and send $(b_i, i, \text{SET}_i)$ to every party $P_j \in P$.
- Round $r = 0$ to $t + 1$: On receiving $(b, j, \text{SET}_j)$ in round $r$, check if all the following hold.
  - $\text{SET}_j$ contains valid signatures on $b$ from $r$ distinct parties, including $P_j$;
  - $b \notin \text{ACC}_{ij}$.
  - If all the above hold, then do the following.
    - Add $b$ to $\text{ACC}_{ij}$.
    - If $r \neq (t + 1)$, then compute a signature $\sigma_{ij}$ on $b$ and send $(b, j, \text{SET}_j \cup \{\sigma_{ij}\})$ to every party $P_k \in P$.
- Output Computation: At time $(t + 1) \cdot \Delta$, do the following.
  - If $\text{ACC}_{ij}$ contains exactly one value $b$, then add $(j, b)$ to $\text{FIN}_i$. Else, add $(j, \perp)$ to $\text{FIN}_i$.
  - If there exists a set $Z \in Z_s$ and a value $b \neq \perp$ such that, for every $P_j \in P \setminus Z$, $(j, b)$ belongs to $\text{FIN}_i$, then output $b$. Else, output $\perp$.
```

Fig. 15: Synchronous BA with asynchronous guaranteed liveness. The above code is executed by every party $P_i$ with input $b_i$.

We next prove the properties of the protocol $\Pi_{PW}$, which are a straightforward generalization of the properties of the DS protocol.
Lemma A.1. Protocol $\Pi_{PW}$ achieves $Z_s$-Consistency in a synchronous network.

Proof. We claim that each honest party $P_i$ computes the same set $\text{ACC}_{ij}$ corresponding to every party $P_j$, by time $(t + 1) \cdot \Delta$. Assuming the claim is true, the proof then follows from the fact that $\text{FIN}_i$ is computed deterministically at the time $(t + 1) \cdot \Delta$, based on the sets $\text{ACC}_{ij}$. To prove the claim, consider an arbitrary honest party $P_i$ and an arbitrary $P_j$. If $P_i$ includes $b$ to $\text{ACC}_{ij}$, then we show that by the time $(t + 1) \cdot \Delta$, the value $b$ will be present in the set $\text{ACC}_{kj}$ of every honest party $P_k$. For this, we consider the following two cases.

- Case 1 - $P_i$ added $b$ to $\text{ACC}_{ij}$ during round $r$ where $r \leq t$: In this case, $P_i$ must have received $(b, j, \text{SET}_j)$ in round $r$, where $\text{SET}_j$ contained valid signatures on $b$ from $r - 1$ distinct parties apart from $P_j$. Party $P_i$ then computes $\sigma_{ij}$, adds this to $\text{SET}_j$, and sends $(b, j, \text{SET}_j)$ to every party. When $P_k$ receives this during round $r + 1$, it will find that $\text{SET}_j$ contains $r$ valid signatures on $b$ apart from party $P_j$’s, including $\sigma_{ij}$. Hence, $P_k$ will add $b$ to $\text{ACC}_{kj}$. Since $r + 1 \leq t + 1$, this happens by time $(t + 1) \cdot \Delta$.

- Case 2 - $P_i$ added $b$ to $\text{ACC}_{ij}$ during round $r = t + 1$: In this case $P_i$ must have received $(b, j, \text{SET}_j)$ in round $t + 1$, where $\text{SET}_j$ contained valid signatures on $b$ from $t$ distinct parties apart from $P_j$. This means that, in total, $P_i$ has received valid signatures on $b$ from $t + 1$ distinct parties. Among these, at least one party, say $P_h$, must be honest, as there can be at most $t$ corrupt parties. This means that $P_h$ must have added $b$ to $\text{ACC}_{kj}$ during some round $r'$, where $r' \leq t$. Thus, as argued in the previous case, each party $P_k$ also adds $b$ to $\text{ACC}_{kj}$ by round $t + 1$ and hence time $(t + 1) \cdot \Delta$.

Lemma A.2. Protocol $\Pi_{PW}$ achieves $Z_s$-Validity in a synchronous network.

Proof. Suppose that all honest parties have the same input $b$. Corresponding to each honest party $P_j$, every honest $P_i$ sets $\text{ACC}_{ij} = \{b\}$. This is because $P_i$ would receive a valid signature on $b$ from $P_j$ during round 0, and adds $b$ to $\text{ACC}_{ij}$. Further, $P_i$ will not add any $b' \neq b$ to $\text{ACC}_{ij}$ during any of the rounds, since the adversary cannot forge a signature on $b'$ on the behalf of $P_j$. Thus, for each honest $P_j$, party $P_i$ adds $(j, b)$ to $\text{FIN}_i$. Let $Z^* \in Z_s$ be the set of corrupt parties and let $\mathcal{H} = \mathcal{P} \setminus Z^*$ be the set of honest parties. Hence corresponding to every $P_j \in \mathcal{H}$, then value $(j, b)$ is added to $\text{FIN}_i$ of every $P_i \in \mathcal{H}$ by time $(t + 1) \cdot \Delta$. Moreover, since $Q^{(2)}(\mathcal{P}, Z_s)$ conditions holds, $\mathcal{P} \setminus Z^* \notin Z_s$. Consequently, as per the “majority” rule, every party in $\mathcal{H}$ outputs $b$.

Lemma A.3. In protocol $\Pi_{PW}$, irrespective of the network type, all honest parties obtain an output at the time $(t + 1) \cdot \Delta$.

Proof. The proof follows from the fact that irrespective of the network type, the parties compute an output (which could be $\bot$) at the local time $(t + 1) \cdot \Delta$.

If the inputs of the parties are of size $\ell$ bits, then we invoke $\ell$ instances of $\Pi_{PW}$. The following lemma describes the communication cost incurred while doing this.
**Lemma A.4.** If the inputs of the parties are of size $\ell$ bits, then protocol $\Pi_{\text{PW}}$ incurs a communication of $O(n^4 \cdot \ell \cdot |\sigma|)$ bits from the honest parties.

**Proof.** During round 0, each party signs its input and sends this to every other party, incurring a total communication of $O(\ell \cdot n^2 \cdot |\sigma|)$ bits. During the next $t$ rounds, each party $P_i$ sends $(b, j, \text{SET}_j)$ to every other party $P_k$ at most once. This is because $P_i$ sends this only if $b \notin \text{ACC}_{ij}$ holds, and does not do this once it adds $b$ to $\text{ACC}_{ij}$. Considering all possibilities for $b, i, j$ and $k$, and taking into account that $\text{SET}_j$ will contain $O(n)$ signatures, the communication cost of this will be $O(\ell \cdot n^4 \cdot |\sigma|)$ bits.

The proof of Lemma 3.1 now follows from Lemma A.1-A.4.

### A.2 Protocol $\Pi_{\text{Acast}}$: Asynchronous Broadcast with Synchronous Guarantees

In this section, we prove the properties of the protocol $\Pi_{\text{Acast}}$ (see Fig 2 for the protocol description).

**Lemma 3.2.** Protocol $\Pi_{\text{Acast}}$ achieves the following properties.
- **Asynchronous Network:** The protocol is a $Z_s$-secure broadcast protocol.
- **Synchronous Network:** (a) $Z_s$-Liveness: If Sen is honest, then all honest parties obtain an output within time $3\Delta$. (b) $Z_s$-Validity: If Sen is honest, then every honest party with an output, outputs $m$. (c) $Z_s$-Consistency: If Sen is corrupt and some honest party outputs $m^*$ at time $T$, then every honest $P_i$ outputs $m^*$ by the end of time $T + \Delta$.
- **Communication Complexity:** $O(n^3 \cdot \ell \cdot |\sigma|)$ bits are communicated by the honest parties, where $\ell$ is the size of Sen’s input.

**Proof.** We first consider a synchronous network, followed by an asynchronous network.

**Properties in the Synchronous Network.** Let $Z^* \in Z_s$ be the set of corrupt parties and let $H = P \setminus Z^*$ be the set of honest parties. Suppose Sen is honest. Then by time $\Delta$, each party in $H$ receives $<(\text{propose}, m)>_{\text{Sen}}$ from Sen and no honest party ever receives $<(\text{propose}, m')>_{\text{Sen}}$ from any party by time $2\Delta$, for any $m' \neq m$, since signature of an honest Sen cannot be forged. Thus, every $P_j \in H$ sends $<(\text{vote}, m)>_j$ by time $2\Delta$. Consequently, by time $3\Delta$, every $P_i \in H$ will have a quorum $C(m)$ of legitimately signed $<(\text{vote}, m)>_j$ messages corresponding to every $P_j \in H$. The parties $P_k \in Z^*$ may send signed $<(\text{vote}, m')>_k$ messages where $m' \neq m$ and consequently the parties in $H$ may also have a quorum $C(m')$ of legitimately signed $<(\text{vote}, m')>_k$ messages corresponding to every $P_k \in Z^*$. However, since $Z_s$ satisfies the $Q^{(2)}(P, Z_s)$ condition, $P \setminus Z^* = H \notin Z_s$.

Consequently, the parties in $H$ outputs $m$ by time $3\Delta$, as the condition for putting an $m' \neq m$ will be never satisfied for the parties in $H$. This proves the $Z_s$-liveness and $Z_s$-validity.
We now consider $\text{Sen}$ to be corrupt. We first show that no two parties in $H$ can vote for different messages. On the contrary, let $P_i \in H$ sends $\langle \text{vote}, m' \rangle_i$ at time $T_i$, and let $P_j \in H$ sends $\langle \text{vote}, m'' \rangle_j$ at time $T_j$, where $T_j \geq T_i$. This implies that $P_i$ must have received $\langle \text{propose}, m' \rangle_{\text{Sen}}$ within time $T_i - \Delta$, and would have sent $\langle \text{propose}, m' \rangle_{\text{Sen}}$ to $P_j$. And $P_j$ would have received $\langle \text{propose}, m'' \rangle_{\text{Sen}}$ from $P_i$ within time $T_i$. Now since $T_i \leq T_j$, it implies that $P_j$ would not have sent $\langle \text{vote}, m'' \rangle_j$ at time $T_j$, and this is a contradiction.

Now based on the above fact, we proceed to prove that $Z_a$-consistency holds. Let $P_h \in H$ outputs $m^*$ at time $T$. This implies that at time $T$, there exists a subset $Z_a \in Z_a$, such that $P_h$ has a quorum $\mathcal{C}(m^*)$ of legitimately signed $\langle \text{vote}, m \rangle_j$ messages, corresponding to every $P_j \in P \setminus Z_a$. Now since $Z_a$ satisfies the $Q^2(P, Z_a)$ condition, it follows that $H \cap (P \setminus Z_a) \neq \emptyset$. This implies that there exists at least one party in $H$, say $P_k$, who has voted for $m^*$ by sending a $\langle \text{vote}, m \rangle_k$ message. Consequently, no other party in $H$ ever votes for $m^{**} \neq m^*$. The parties in $Z^*$ may vote for $m^{**} \neq m^*$. But since $P \setminus Z^* \notin Z_a$, it follows that no party in $H$ will ever have a sufficiently large quorum of legitimately signed vote messages for $m^{**}$ to output $m^{**}$. Since $P_h$ sends $\mathcal{C}(m^*)$ to all parties at time $T$, every other party in $H$ will receive $\mathcal{C}(m^*)$ by time $T + \Delta$. Consequently, all the parties in $H$ will output $m^*$, latest by time $T + \Delta$.

Properties in the Asynchronous Network. We now consider an asynchronous network. Let $Z^* \subseteq Z_a$ be the set of corrupt parties and let $H = P \setminus Z^*$ be the set of honest parties. We first consider an honest $\text{Sen}$. Each party in $H$ eventually receives $\langle \text{propose}, m \rangle_{\text{Sen}}$ from $\text{Sen}$. Furthermore, no party in $H$ ever receives $\langle \text{propose}, m \rangle_{\text{Sen}}$ from any party, for any $m' \neq m$, since the signature of an honest $\text{Sen}$ cannot be forged. Hence, each party in $H$ eventually sends a signed vote message for $m$, which is eventually delivered to every party in $H$. The parties in $Z^*$ may send signed vote messages for $m' \neq m$. However, since $Z_a$ satisfies the $Q^3(P, Z_a)$ condition, it follows that each party in $H$ eventually outputs $m$ and the conditions for outputting $m' \neq m$ will be never satisfied for any party in $H$. This proves the $Z_a$-liveness and $Z_a$-consistency.

Next, consider a corrupt $\text{Sen}$. Let $P_h \in H$ outputs $m^*$. This implies there exists a subset $Z_a \subseteq Z_a$, such that $P_h$ has a quorum $\mathcal{C}(m^*)$ of legitimately signed vote messages for $m^*$, corresponding to every party in $P \setminus Z_a$. Now consider an arbitrary $P_i \in H$, where $P_i \neq P_h$. We claim that for any $Z \subseteq Z_a$, party $P_i$ will never have a quorum $\mathcal{C}(m^{**})$ of legitimately signed vote messages for $m^{**}$, corresponding to the parties in $P \setminus Z$. On the contrary, let $P_i$ eventually have a quorum $\mathcal{C}(m^{**})$ of legitimately signed vote messages for $m^{**}$, corresponding to every party in $P \setminus Z_a$, for some $Z_a \subseteq Z_a$. Now since the $Q^{2,3}(P, Z_a)$ condition is satisfied, it follows that $H \cap (P \setminus Z_a) \cap (P \setminus Z_a) \neq \emptyset$. This implies that there exists at least one party from $H$, say $P_k$, such that $P_k$ has voted both for $m^*$, as well as $m^{**}$, which is a contradiction. Consequently, $P_i$ will never output $m^{**}$. Now, since $P_h$ sends $\mathcal{C}(m^*)$ to all the parties, party $P_i$ eventually receives $\mathcal{C}(m^*)$ and outputs $m^*$. This proves $Z_a$-consistency.
Finally, the communication complexity follows from the fact that irrespective of the type of network, every party may have to send a quorum of up to $O(n)$ signed vote messages, to every other party.

### A.3 Properties of the Protocol $\Pi_{BC}$

In this section, we prove the properties of the protocol $\Pi_{BC}$ (see Fig 3 for the formal description).

**Theorem 3.2.** Protocol $\Pi_{BC}$ achieves the following, with a communication complexity of $O(n^4 \cdot \ell \cdot |\sigma|)$ bits, where $T_{BC} = 3\Delta + T_{PW}$.

- **Synchronous network:**
  - (a) $Z_s$-Liveness: At time $T_{BC}$, each honest party has an output.
  - (b) $Z_s$-Validity: If $\text{Sen}$ is honest, then at time $T_{BC}$, each honest party outputs $m$.
  - (c) $Z_s$-Consistency: If $\text{Sen}$ is corrupt, then the output of every honest party is the same at time $T_{BC}$.
  - (d) $Z_s$-Fallback Consistency: If $\text{Sen}$ is corrupt and some honest party outputs $m^* \neq \bot$ at time $T$ through fallback-mode, then every honest party outputs $m^*$ by time $T + \Delta$.

- **Asynchronous Network:**
  - (a) $Z_a$-Liveness: At time $T_{BC}$, each honest party has an output.
  - (b) $Z_a$-Weak Validity: If $\text{Sen}$ is honest, then at time $T_{BC}$, each honest party outputs $m$ or $\bot$.
  - (c) $Z_a$-Fallback Validity: If $\text{Sen}$ is honest, then each honest party with output $\bot$ at time $T_{BC}$, eventually outputs $m$ through fallback-mode.
  - (d) $Z_a$-Weak Consistency: If $\text{Sen}$ is corrupt, then there exists some $m^* \neq \bot$, such that at time $T_{BC}$, each honest party outputs either $m^*$ or $\bot$.
  - (e) $Z_a$-Fallback Consistency: If $\text{Sen}$ is corrupt, and some honest party outputs $m^* \neq \bot$ at time $T$ where $T \geq T_{BC}$, then each honest party eventually outputs $m^*$.

**Proof.** The $Z_s$-liveness and $Z_a$-liveness properties follow from the fact that every honest party outputs something (including $\bot$) at (local) time $T_{BC}$, irrespective of the type of the network. We next prove the rest of the properties of the protocol in the synchronous network.

**Properties in the Synchronous Network.** If $\text{Sen}$ is honest, then due to the $Z_s$-liveness and $Z_s$-validity properties of $\Pi_{Acast}$ in the synchronous network (see Lemma 3.2), all honest parties receive $m$ from the Acast of $\text{Sen}$ at time $3\Delta$. Consequently, all honest parties participate with input $m$ in the instance of $\Pi_{PW}$. The $Z_s$-guaranteed liveness and $Z_s$-validity properties of $\Pi_{PW}$ in the synchronous network (see Lemma 3.1) guarantees that at time $3\Delta + T_{PW}$, all honest parties will have $m$ as the output from the instance of $\Pi_{PW}$. As a result, all honest parties output $m$ at time $T_{BC}$, thus proving the $Z_s$-validity property.
To prove the $Z_s$-consistency property, we consider a corrupt $\Sen$. From the $Z_s$-consistency property of $\Pi_{PW}$ in the synchronous network (see Lemma 3.1), all honest parties will have the same output from the instance of $\Pi_{PW}$ at time $T_{BC}$. If all honest parties have the output $\perp$ for $\Pi_{BC}$ at time $T_{BC}$, then $Z_s$-consistency holds trivially. So, consider the case when some honest party, say $P_i$, has the output $m^* \neq \perp$ for $\Pi_{BC}$ at time $T_{BC}$. This implies that all honest parties have the output $m^*$ from the instance of $\Pi_{PW}$. Moreover, at time $3\Delta$, at least one honest party, say $P_h$, has received $m^*$ from the Acast of $\Sen$. If the latter does not hold, then all honest parties would have participated with input $\perp$ in the instance of $\Pi_{PW}$, and from the $Z_s$-validity of $\Pi_{PW}$ in the synchronous network (see Lemma 3.1), all honest parties would compute $\perp$ as the output during the instance of $\Pi_{PW}$, which is a contradiction. Since $P_h$ has received $m^*$ from $\Sen$’s Acast at time $3\Delta$, it follows from the $Z_s$-consistency property of $\Pi_{Acast}$ in the synchronous network (see Lemma 3.2) that all honest parties will receive $m^*$ from $\Sen$’s Acast by time $4\Delta$. Moreover, $4\Delta < 3\Delta + T_{PW}$ holds. Consequently, at time $3\Delta + T_{BC}$, all honest parties will have $m^*$ from $\Sen$’s Acast and as the output of $\Pi_{PW}$, implying that all honest parties output $m^*$ for $\Pi_{BC}$.

We next prove the $Z_s$-fallback consistency property for which we again consider a corrupt $\Sen$. Let $P_i$ be an honest party who outputs $m^* \neq \perp$ at time $T$ through fallback-mode. Note that $T > T_{BC}$, as the output during the fallback-mode is computed only after time $T_{BC}$. We also note that each honest party has output $\perp$ at time $T_{BC}$. This is because, from the proof of the $Z_s$-consistency property of $\Pi_{BC}$ (see above), if any honest party has an output $m' \neq \perp$ at time $T_{BC}$, then all honest parties (including $P_h$) must have computed the output $m'$ at time $T_{BC}$. Hence, $P_h$ will never change its output to $m^*$.\footnote{17} Now since $P_h$ has obtained the output $m^*$, it implies that at time $T$, it has received $m^*$ from the Acast of $\Sen$. It then follows from the $Z_s$-consistency of $\Pi_{Acast}$ in the synchronous network that every honest party will also receive $m^*$ from the Acast of $\Sen$, latest by time $T + \Delta$ and output $m^*$. This completes the proof of all the properties in the synchronous network.

**Properties in the Asynchronous Network.** The $Z_s$-weak validity property follows from the $Z_s$-validity property of $\Pi_{Acast}$ in the asynchronous network (see Lemma 3.2), which ensures that no honest party ever receives an $m'$ from the Acast of $\Sen$, where $m' \neq m$. So, if at any honest party outputs a value different from $\perp$ at time $T_{BC}$, it has to be $m$. The $Z_s$-weak consistency property follows using similar arguments as used to prove $Z_s$-consistency in the synchronous network; however, we now rely on the $Z_s$-validity and $Z_s$-consistency properties of $\Pi_{Acast}$ in the asynchronous network (see Lemma 3.2). The latter property ensures that for a corrupt $\Sen$, two different honest parties never end up receiving $m_1$ and $m_2$ from the Acast of $\Sen$, where $m_1 \neq m_2$.

For the $Z_s$-fallback validity property, consider an honest $\Sen$, and let $P_i$ be an arbitrary honest party who outputs $\perp$ at (local) time $T_{BC}$. Since the parties keep...
on participating in the protocol beyond time $T_{BC}$, it follows from the $Z_a$-liveness and $Z_a$-validity properties of $\Pi_{Acast}$ in the asynchronous network (see Lemma 3.2) that party $P_i$ will eventually receive $m$ from the Acast of Sen, by executing the steps of the fallback-mode of $\Pi_{BC}$. Consequently, party $P_i$ eventually changes its output from $\bot$ to $m$.

For the $Z_a$-fallback consistency property, we consider a corrupt Sen. Let $P_j$ be an honest party who outputs some $m^\star$ different from $\bot$ at time $T$, where $T \geq T_{BC}$. This implies that $P_j$ has obtained $m^\star$ from the Acast of Sen. Now, consider an arbitrary honest $P_i$. From the $Z_a$-liveness and $Z_a$-weak consistency properties of $\Pi_{BC}$ in asynchronous network proved above, it follows that $P_i$ outputs either $m^\star$ or $\bot$ at local time $T_{BC}$. If $P_i$ has output $\bot$, then from the $Z_a$-consistency property of $\Pi_{Acast}$ in the asynchronous network (see Lemma 3.2), it follows that $P_i$ will also eventually obtain $m^\star$ from the Acast of Sen, by executing the steps of the fallback-mode of $\Pi_{BC}$. Consequently, party $P_i$ eventually changes its output from $\bot$ to $m^\star$.

The communication complexity (both in the synchronous as well as asynchronous network) follows from the communication complexity of $\Pi_{PW}$ and $\Pi_{Acast}$.

### A.4 Properties of the Protocol $\Pi_{SBA}$

In this section, we prove the properties of the protocol $\Pi_{SBA}$ (see Fig 4 for the formal description).

**Theorem 3.3.** Protocol $\Pi_{SBA}$ achieves the following where $T_{SBA} = T_{BC}$, incurring a communication of $O(n^5 \cdot |\sigma|)$ bits.

- **Synchronous Network:** the protocol is a $Z_s$-secure SBA protocol where honest parties have an output, different from $\bot$, at time $T_{SBA}$.
- **Asynchronous Network:** the protocol achieves $Z_a$-guaranteed liveness and $Z_a$-weak validity, such that all honest parties have an output at (local) time $T_{SBA}$.

**Proof.** The communication complexity simply follows from the fact that $n$ instances of $\Pi_{BC}$ are invoked in the protocol. The guaranteed liveness, both in the synchronous and asynchronous network trivially follows from the $Z_s$-liveness and $Z_a$-liveness of $\Pi_{BC}$ (see Theorem 3.2), which ensures that all the $n$ instances of $\Pi_{BC}$ produce some output within (local) time $T_{BC}$, both in a synchronous as well as an asynchronous network, through regular mode. Hence, at local time $T_{SBA} = T_{BC}$, all honest parties will have some output. We next prove the rest of the properties in a synchronous network.

**Properties in the Synchronous Network.** Let $Z^\star \in Z_s$ be the set of corrupt parties and let $\mathcal{H} = \mathcal{P} \setminus Z^\star$ be the set of honest parties. In a synchronous network, the instances of $\Pi_{BC}$ corresponding to the senders in $\mathcal{H}$, result in an output different from $\bot$ for all honest parties (follows from the $Z_a$-validity property of $\Pi_{BC}$ in the synchronous network, Theorem 3.2). Hence $\mathcal{H} \subseteq SV$ will hold.
Moreover, from the $Z_s$-consistency property of $\Pi_{BC}$ in the synchronous network (Theorem 3.2), all the parties in $\mathcal{H}$ obtain a common output from the $i^{th}$ instance of $\Pi_{BC}$, for $i = 1, \ldots, n$, at time $T_{BC}$. Hence, all honest parties output the same value, different from $\bot$, at time $T_{BC}$, proving the $Z_s$-consistency of $\Pi_{SBA}$. Finally, if all parties in $\mathcal{H}$ have the same input bit $b$, then only the instances of $\Pi_{BC}$ corresponding to the parties in $\mathcal{S}\mathcal{V}\setminus\mathcal{H}$ may output $b = 1 - b$. However, $\mathcal{S}\mathcal{V}\setminus\mathcal{H} \in Z_s$. Moreover, $\mathcal{S}\mathcal{V}\setminus Z^* \not\in Z_s$ (since $Z_s$ satisfies the $Q^{(2)}(\mathcal{P}, Z_s)$ condition). It then follows that all honest parties output $b$, proving $Z_s$-validity of $\Pi_{SBA}$.

Properties in the Asynchronous Network. Let $Z^* \in Z_a$ be the set of corrupt parties and let $\mathcal{H} = \mathcal{P} \setminus Z^*$ be the set of honest parties. Suppose all the parties in $\mathcal{H}$ have the same input bit $b$. Let $P_i \in \mathcal{H}$ be an arbitrary party, that obtains an output $c$, different from $\bot$, at time $T_{BC}$. This implies that there exists a subset of parties $\mathcal{S}\mathcal{V}$ for $P_i$, where $\mathcal{P} \setminus \mathcal{S}\mathcal{V} \in Z_s$, such that $P_i$ has obtained a Boolean output $b^{(j)}_i$ from the $\Pi_{BC}$ instances, corresponding to every $P_j \in \mathcal{S}\mathcal{V}$. Moreover, there also exists a subset of parties $\mathcal{S}\mathcal{V}_i \subseteq \mathcal{S}\mathcal{V}$, where $\mathcal{S}\mathcal{V} \setminus \mathcal{S}\mathcal{V}_i \in Z_s$, such that the output $b^{(j)}_i = c$, corresponding to every $P_j \in \mathcal{S}\mathcal{V}_i$. Now since the $Q^{(2,1)}(\mathcal{P}, Z_s, Z_a)$ condition is satisfied, it follows that $Z_a$ satisfies the $Q^{(1)}(\mathcal{S}\mathcal{V}_i, Z_a)$ condition and hence $\mathcal{S}\mathcal{V}_i \cap \mathcal{H} \neq \emptyset$. Consequently, $c = b$ holds. This proves the $Z_a$-weak validity in the asynchronous network.

A.5 Protocol $\Pi_{GA}$: Asynchronous Graded Agreement with Synchronous Validity

To design protocol $\Pi_{GA}$, we first design a sub-protocol $\Pi_{Prop}$ for proposing values.

A.5.1 $\Pi_{Prop}$: A Network Agnostic Protocol for Proposing Values

Protocol $\Pi_{Prop}$ takes an input value from each party from the set $\{0, 1, \lambda\}$ and outputs a set $\text{prop}$ of proposed values for each party. Liveness is ensured in an asynchronous network as long as each honest party holds one of two inputs. In an asynchronous network, it will be ensured that each value in the output $\text{prop}$ must be the input of some honest party. Moreover, if any two honest parties output a singleton set for $\text{prop}$, then they must output the same set. In a synchronous network, validity and liveness are ensured as long as each honest party participates with the same input. Protocol $\Pi_{Prop}$ is presented in Figure 16.

<table>
<thead>
<tr>
<th>Protocol $\Pi_{Prop}(\mathcal{P}, Z_s, Z_a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Set $\text{vals} = \text{prop} = \emptyset$.</td>
</tr>
<tr>
<td>2. On having the input $v \in {0, 1, \lambda}$, send (prepare, $v$) to every party $P_j \in \mathcal{P}$.</td>
</tr>
<tr>
<td>3. On receiving (prepare, $b$) for some $b \in {0, 1, \lambda}$ from a set of parties $\mathcal{S}^{(b)}$ satisfying $Q^{(1)}(\mathcal{S}^{(b)}, Z_s)$ condition, send (prepare, $b$) to all $P_j \in \mathcal{P}$, if not sent earlier.</td>
</tr>
<tr>
<td>4. Upon receiving the message (prepare, $b$) for some $b \in {0, 1, \lambda}$ from parties in set $\text{PrepareSet}^{(b)} = \mathcal{P} \setminus Z$ for some $Z \in Z_s$, set $\text{vals} = \text{vals} \cup {b}$.</td>
</tr>
<tr>
<td>5. Upon adding the first value $b$ to $\text{vals}$, send (propose, $b$) to every party $P_j \in \mathcal{P}$.</td>
</tr>
</tbody>
</table>
Upon receiving \((\text{propose}, b)\) messages from a set of parties \(\text{ProposeSet} = \mathcal{P} \setminus Z\) for some \(Z \in \mathcal{Z}\), on values \(b \in \text{vals}\), let \(\text{prop} \subseteq \text{vals}\) be the set of values carried by those messages. Output \(\text{prop}\).

Fig. 16: Sub-protocol to propose values. The above code is executed by every party \(P_i\) with input \(v \in \{0, 1, \lambda\}\)

The guarantees provided by \(\Pi_{\text{Prop}}\) are proven in a series of lemmas below. In the below proofs, we assume that \(Z^*\) is the set of corrupt parties.

**Lemma A.5.** Suppose that the network is asynchronous. If two honest parties \(P_i\) and \(P_j\) output \(\{b\}\) and \(\{b'\}\) respectively, then \(b = b'\).

*Proof.* Since \(P_i\) outputs \(\{b\}\), it must have received \((\text{propose}, b)\) from a set of parties \(\text{ProposeSet} = \mathcal{P} \setminus Z\) for some \(Z \in \mathcal{Z}\). Similarly, since \(P_j\) outputs \(\{b'\}\), it must have received \((\text{propose}, b')\) from a set of parties \(\text{ProposeSet}' = \mathcal{P} \setminus Z'\) for some \(Z' \in \mathcal{Z}\). Let \(\text{ProposeSet}_H\) and \(\text{ProposeSet}_H'\) be the set of honest parties in \(\text{ProposeSet}\) and \(\text{ProposeSet}'\) respectively. Since \(\mathcal{Z}_s, \mathcal{Z}_a\) satisfy the \(Q^{(2,1)}(\mathcal{P}, \mathcal{Z}_s, \mathcal{Z}_a)\) condition and \(Z^* \in \mathcal{Z}_a\), it follows that \(\text{ProposeSet}_H \cap \text{ProposeSet}_H' \neq \emptyset\). Let \(P_h \in \text{ProposeSet}_H \cap \text{ProposeSet}_H'\). If \(b \neq b'\), this would mean that \(P_h\) has sent both \((\text{propose}, b)\) and \((\text{propose}, b')\), which is a contradiction, since an honest party sends at most one \text{propose} message as per the protocol.

**Lemma A.6.** Suppose that the network is asynchronous. If no honest party has input \(v\), then no honest party outputs \(\text{prop}\) containing \(v\).

*Proof.* If \(v\) was not input by any honest party, then no honest party sends \((\text{prepare}, v)\) in step 2. Hence, no honest party receives \((\text{prepare}, v)\) from a set of parties \(S^{(v)}\) which satisfies the \(Q^{1}(S^{(v)}, \mathcal{Z}_s)\) condition during step 3, since such a set must contain at least one honest party. Consequently, no honest party sends \((\text{propose}, v)\). Thus, no honest party adds \(v\) to \text{vals} and no honest party outputs \(\text{prop}\) containing \(v\).

The following lemmas help prove liveness.

**Lemma A.7.** Suppose that the network is asynchronous. If all honest parties hold one of two inputs, say \(v_0\) and \(v_1\), then all honest parties eventually compute an output.

*Proof.* We first show that every honest party eventually sends a \text{propose} message. Let \(\mathcal{H}_0\) and \(\mathcal{H}_1\) be the sets of honest parties holding inputs \(v_0\) and \(v_1\) respectively and let \(\mathcal{H}\) be the set of honest parties. We know that due to the \(Q^{(2,1)}(\mathcal{P}, \mathcal{Z}_s, \mathcal{Z}_a)\) condition, \(\mathcal{Z}_s\) satisfies the \(Q^{(2)}(\mathcal{H}, \mathcal{Z}_s)\) condition. Now, consider the following cases.

- **Case 1 - \(\mathcal{H}_0\) satisfies the \(Q^{1}(\mathcal{H}_0, \mathcal{Z}_s)\) condition:** In this case, \(\mathcal{H}_0\) is a candidate for the set \(S^{(v_0)}\), since all the parties in \(\mathcal{H}_0\) send \((\text{prepare}, v_0)\) in step 2.
Case 2 - $\mathcal{H}_0$ does not satisfy the $Q^1(\mathcal{H}_0, Z_s)$ condition: In this case, $Z_s$ must satisfy the $Q^1(\mathcal{H}_1, Z_s)$ condition, since $Z_s$ satisfies the $Q^2(\mathcal{H}, Z_s)$ condition. Then similar to what was argued in the previous case, $\mathcal{H}_1$ is a candidate for the set $S^{(v_1)}$.

Let $v_b$ be the input corresponding to the candidate set $S^{(v_b)}$. Every honest party will now eventually send ($\text{prepare}, v_b$) in step 3. This would mean that the set of honest parties $\mathcal{H} = \mathcal{P} \setminus Z^*$ is a candidate for the set $\text{PrepareSet}^{(v_b)}$. Thus, every honest party eventually adds $v_b$ to $\text{vals}$ and sends a $\text{propose}$ message for some value $v_b$. This way, every honest party eventually receives $\text{propose}$ messages from every other honest party for $v_b$ and thus, the set $\mathcal{H}$ also forms a candidate for the set $\text{ProposeSet}$. Thus, all honest parties eventually compute an output.

Lemma A.8. If the network is synchronous and if all honest parties participate with input $v$, then all honest parties output $\text{prop} = \{v\}$ at time $2\Delta$.

Proof. Let $\mathcal{H} = \mathcal{P} \setminus Z^*$ be the set of honest parties. Then in step 2, every party in $\mathcal{H}$ sends ($\text{prepare}, v$) to every party, which is delivered within $\Delta$ time. Thus, at time $\Delta$, all the parties in $\mathcal{H}$ receive ($\text{prepare}, v$) from the parties in $\mathcal{H}$ and add $v$ to $\text{vals}$. Further, no other value $v^*$ will be added to $\text{vals}$, since only the parties in $Z^*$ may send a $\text{prepare}$ message for $v^*$ and $Z_s$ does not satisfy the $Q^1(Z^*, Z_s)$ condition. Thus, all the parties in $\mathcal{H}$ send ($\text{propose}, v$) in step 5, which gets delivered to all the parties in $\mathcal{H}$ at time $2\Delta$. Consequently, all the parties in $\mathcal{H}$ output $\text{prop} = \{v\}$.

Lemma A.9. Protocol $\Pi_{\text{Prop}}$ incurs a communication of $O(n^2)$ bits.

Proof. The proof follows from the fact that each party sends ($\text{prepare}, b$) to every other party at most once for any value of $b \in \{0, 1, \lambda\}$.

A.5.2 The Graded Agreement Protocol

We now present protocol $\Pi_{\text{GA}}$ (Figure 17) based on protocol $\Pi_{\text{Prop}}$. The protocol cleverly “stitches” together two instances of $\Pi_{\text{Prop}}$, by defining the input for the second instance based on the output from the first instance. Each party, with input either 0 or 1, participates in the first instance of $\Pi_{\text{Prop}}$ with their input. Since the parties participate with one of two inputs, this instance will eventually complete (in an asynchronous network), and the parties obtain an output, say $\text{prop}_1$. Only if $\text{prop}_1$ is a singleton set, say $\{b\}$, for some party, then that party participates in the second instance of $\Pi_{\text{Prop}}$ with the input $b$. Otherwise, it participates in the second instance with a default input of $\lambda$.

Since no two honest parties can output different singleton sets for $\text{prop}_1$, this ensures that each honest party participates with an input of either $b$ or $\lambda$ (in an asynchronous network). Thus, the second instance of $\Pi_{\text{Prop}}$ also eventually completed with an output, say $\text{prop}_2$. This also ensures that $\text{prop}_2$ can contain only values $b$ and $\lambda$. If $\text{prop}_2$ contains only $b$ for some honest party, then that party outputs $b$ with a grade of 2. If $\text{prop}_2$ contains $b$ along with $\lambda$, then the party outputs $b$ with a grade of 1. Else, if $\text{prop}_2$ contains only $\lambda$, then the party
outputs ⊥ with a grade of 0. If the network is synchronous and all honest parties start the protocol with the same input, then both \(\text{prop}_1\) as well as \(\text{prop}_2\) will be a singleton set containing that value and hence all honest parties will output that value with the highest grade.

**Protocol \(\Pi_{GA}(P, Z_s, Z_a)\)**

1. On having the input \(v \in \{0, 1\}\), set \(b_1 = v\). Participate in an instance of the protocol \(\Pi_{prop}\) with input \(b_1\) and wait for its completion. Let \(\text{prop}_1\) be the output computed during the instance of \(\text{prop}\).
2. If \(\text{prop}_1 = \{b\}\) for some \(b \in \{0, 1\}\), then set \(b_2 = b\). Else, set \(b_2 = \lambda\). Then participate in an instance of the protocol \(\Pi_{prop}\) with input \(b_2\) and wait for its completion. Let \(\text{prop}_2\) be the output computed from this instance of \(\text{prop}\).
3. If \(\text{prop}_2 = \{b'\}\) and \(b' \neq \lambda\), then output \((b', 2)\). If \(\text{prop}_2 = \{b', \lambda\}\) where \(b' \in \{0, 1\}\), then output \((b', 1)\). Else, if \(\text{prop}_2 = \{\lambda\}\), then output \((\perp, 0)\).

Fig. 17: Asynchronous graded agreement with synchronous validity. The above code is executed by every party \(P\) with input \(v \in \{0, 1\}\).

We now proceed to prove the properties of the protocol \(\Pi_{GA}\).

**Lemma A.10.** Protocol \(\Pi_{GA}\) achieves the following in a synchronous network, where \(T_{GA} = 4\Delta\).

- (a) \(Z_s\)-Liveness: If all honest parties participate in the protocol with the same input, then at time \(T_{GA}\), all honest parties obtain an output.
- (b) \(Z_s\)-Graded Validity: If every honest party’s input is \(b\), then all honest parties with an output, output \((b, 2)\).

**Proof.** If all honest parties participate in the protocol \(\Pi_{GA}\) with the same input \(b\), then from Lemma A.8, all honest parties output \(\text{prop}_1 = \{b\}\) at time \(2\Delta\). Thus, all honest parties participate with input \(b_2 = b\) in the second instance of \(\Pi_{GA}\) and, once again from Lemma A.8, output \(\text{prop}_2 = \{b\}\) at time \(4\Delta\). Thus, all honest parties output \((b, 2)\).

**Lemma A.11.** Protocol \(\Pi_{GA}\) achieves the following in an asynchronous network.

- (a) \(Z_a\)-Liveness: If all honest parties participate in the protocol with a binary input, then each honest party eventually obtains an output.
- (b) \(Z_a\)-Graded Validity: If every honest party’s input is \(b\), then all honest parties with an output, output \((b, 2)\).
- (c) \(Z_a\)-Graded Consistency: If two honest parties output grades \(g, g'\), then \(|g - g'| \leq 1\) holds; moreover, if two honest parties output \((v, g)\) and \((v', g')\) with \(g, g' \geq 1\), then \(v = v'\).

**Proof.** Since each honest party participates with a binary input, from Lemma A.7, each party eventually outputs some value for \(\text{prop}_1\) during the first instance of \(\text{prop}\). Now there are two possible cases.

- Case 1 - Some honest party outputs \(\{b\}\) as its value for \(\text{prop}_1\) where \(b \in \{0, 1\}\): From Lemma A.5, no honest party can output \(\{b'\}\), where \(b \neq b'\),
as $\text{prop}_1$. Thus, each honest party participates with input either $b$ or $\lambda$ for the second instance of $\text{prop}$.

- **Case 2 - No honest party outputs $\{b\}$ as its value for $\text{prop}_1$ for any $b \in \{0, 1\}$:** In this case, all honest parties participate with input $\lambda$ in the second instance of $\text{prop}$.

In either case, the honest parties participate in the second instance of $\Pi_{\text{Prop}}$ with no more than two different inputs. Thus, from Lemma A.7, all parties eventually compute some value for $\text{prop}_2$ during the second instance of $\text{prop}$ and hence compute some output for protocol $\Pi_{\text{GA}}$. This proves the $Z_a$-Liveness.

We next prove the $Z_a$-Graded Consistency. We first show that the grades output by any two parties differ by at most 1. For this, suppose that some honest party $P_i$ outputs $(b, 2)$. We show that no other honest party $P_j$ can output $(\bot, 0)$. Since $P_i$ output $(b, 2)$, from Lemma A.5, $P_j$ cannot output $\text{prop}_2 = \{\lambda\}$. Thus, $P_j$ cannot output $(\bot, 0)$. Next, we show that any two honest parties which output non-zero grades must output the same value. Similar to what was argued for the proof of $Z_a$-Liveness, there exists a bit $b$ such that each honest party participates in $\Pi_{\text{Prop}}$ with input $b$ or $\lambda$ during step 2. Thus, $\text{prop}_2 \subseteq \{b, \lambda\}$ for every honest party. This means that any honest party which outputs a non-zero grade must output it along with the bit $b$.

We finally prove the $Z_a$-Graded Validity. Suppose that each honest party participates with the same input bit $b$. From Lemma A.7, we know that all honest parties output some value for $\text{prop}_1$. From Lemma A.6, all honest parties must output $\text{prop}_1 = \{b\}$. Hence, all honest parties participate in $\Pi_{\text{Prop}}$ in step 2 with input $b$. By the same argument, all honest parties output $\text{prop}_2 = \{b\}$. Hence, all honest parties output $(b, 2)$.

**Lemma A.12.** Protocol $\Pi_{\text{GA}}$ incurs a communication of $O(n^2)$ bits.

*Proof.* The proof follows from Lemma A.9, since $\Pi_{\text{Prop}}$ is invoked twice in the protocol.

### A.6 Properties of the Protocol $\Pi_{\text{ABA}}$

In this section, we prove the properties of the protocol $\Pi_{\text{ABA}}$ (see Fig 5 for the protocol steps). We start with the properties in the asynchronous network first, which mostly follows from [23] and are recalled from [23]. We start with the validity property.

**Lemma A.13.** In protocol $\Pi_{\text{ABA}}$, if the network is asynchronous and all honest parties have the same input bit $b$, then all honest parties eventually output $b$.

*Proof.* Let $Z^* \in Z_a$ be the set of corrupt parties. If every honest party has the same input bit $b$, then from the $Z_a$-Graded Validity of $\Pi_{\text{GA}}$ in the asynchronous network (Lemma A.11), all honest parties eventually output $(b, 2)$ at the end of the first as well as the second instance of the $\Pi_{\text{GA}}$ protocol during the first iteration. Consequently, every honest party eventually sends a signed $(\text{ready}, b)$ message to all the parties and only the parties in $Z^*$ may send a signed $(\text{ready}, \bar{b})$
message. It now follows easily from the steps of the output computation stage that no honest party ever sends a signed \( (\text{ready}, \tilde{b}) \) message and all hence honest parties eventually output \( b \).

We next prove the consistency property.

**Lemma A.14.** In protocol \( \Pi_{ABA} \), if the network is asynchronous and if any honest party outputs \( b \), then every other honest party eventually outputs \( b \).

**Proof.** Let \( P_i \) be the first honest party who sends a signed \( \text{ready} \) message for some bit \( b \in \{0, 1\} \), during some iteration, say iteration \( r \). We show that no honest party ever sends a signed \( (\text{ready}, b) \) message during iteration \( r \) or in the subsequent iterations. Since \( P_i \) has sent a signed \( \text{ready} \) message for \( b \), it implies that \( P_i \) outputs \((b, 2)\) in the second instance of the \( \Pi_{GA} \) protocol during iteration \( r \) and sets committed to true. Then, from the \( Z_a \)-Graded Consistency of \( \Pi_{GA} \) in the asynchronous network (Lemma A.11), every other honest party outputs either \((b, 2)\) or \((b, 1)\) in the second instance of the \( \Pi_{GA} \) protocol during iteration \( r \). Consequently, no other honest party sends the signed \( (\text{ready}, b) \) message during iteration \( r \). Also, from the protocol steps, all honest parties update their input to \( b \) for the next iteration. This further implies that all honest parties will continue to input \( b \) to each subsequent invocation of \( \Pi_{GA} \), ignoring the output of \( \Pi_{CoinFlip} \), for as long as they continue running. Consequently, no honest party ever sends a signed \( (\text{ready}, b) \) message.

Now let some honest party, say \( P_h \), computes the output \( b \) during iteration \( k \). This implies that \( P_h \) receives the signed \( (\text{ready}, b) \) message from a set of parties, say \( T \), such that \( P \setminus T \subseteq Z_a \). The set \( T \) is bound to have at least one honest party, due to the \( Q^{(2,1)}(P, Z_a, Z_a) \) condition, implying that at least one honest party has sent a signed \( (\text{ready}, b) \) message, either during the iteration \( k \) or some previous iteration. From the protocol steps, \( P_h \) sends \( C(b) \), the set of signed \( (\text{ready}, b) \) messages of the parties in \( T \) to all other parties, which get eventually delivered. Moreover, as shown above, no honest party will ever send a signed \( (\text{ready}, b) \) message. Consequently, every honest party eventually receives sufficiently many numbers of signed \( (\text{ready}, b) \) messages and outputs \( b \).

We next prove that at the end of each iteration, the updated value of all honest parties will be the same with the probability at least \( \frac{1}{2^n} \).

**Lemma A.15.** In protocol \( \Pi_{ABA} \), if the network is asynchronous and if all honest parties participate during iteration \( k \), then with probability at least \( \frac{1}{2^n} \), all honest parties have the same updated bit \( b \) at the end of iteration \( k \).

**Proof.** To prove the lemma statement, we consider an event \( \text{Agree} \), which denotes that all honest parties have the same input for the second instance of \( \Pi_{GA} \) during iteration \( k \). If the event \( \text{Agree} \) occurs, then from the \( Z_a \)-Graded Validity of \( \Pi_{GA} \) in the asynchronous network (Lemma A.11), all honest parties will have the same updated bit at the end of iteration \( k \). We show that the event \( \text{Agree} \) occurs during iteration \( k \) with a probability of at least \( \frac{1}{2^n} \). For this, we consider two different possible cases with respect to the output from the first instance of \( \Pi_{GA} \) during iteration \( k \).
– Case I: No honest party obtains an output \((b, 2)\) for any \(b \in \{0, 1\}\) during the first instance of \(\Pi_GA\). In this case, all honest parties set the output from the instance of \(\Pi_{CoinFlip}\) during iteration \(k\) as the input for the second instance of \(\Pi_GA\). From the \((Z_a, p)\)-commonness of \(\Pi_{CoinFlip}\) in asynchronous network \([23]\), all honest parties will have the same output bit \(Coin_k\) from the instance of \(\Pi_{CoinFlip}\) with a probability of at least \(p = \frac{1}{n} > \frac{1}{2n}\).

– Case II: Some honest party obtains an output \((b, 2)\) during the first instance of \(\Pi_GA\). In this case, the \(Z_a\)-Graded Consistency of \(\Pi_GA\) in the asynchronous network (Lemma A.11) ensure that all honest parties obtain the output \((b, 2)\) or \((b, 1)\) from the first instance of \(\Pi_GA\). Moreover, from the protocol steps, the output of the instance of \(\Pi_{CoinFlip}\) during iteration \(k\) is not revealed, until the first honest party generates an output from the first instance of \(\Pi_GA\) during iteration \(k\). Consequently, the output bit \(b\) from the first instance of \(\Pi_GA\) is independent of the output of \(\Pi_{CoinFlip}\). From the \((Z_a, p)\)-commonness of \(\Pi_{CoinFlip}\) in asynchronous network \([23]\), all honest parties will have the same output bit \(Coin_k\) from the instance of \(\Pi_{CoinFlip}\) with a probability of at least \(p = \frac{1}{n}\). Then the probability that \(Coin_k = b\) holds is at least \(\frac{1}{2} \cdot \frac{1}{n} = \frac{1}{2n}\) and with this probability, all honest parties will have the same input for the second instance of \(\Pi_GA\).

We next derive the expected number of iterations required in the protocol \(\Pi_{ABA}\) for the honest parties to produce an output. This automatically gives the expected running time in an asynchronous network, since each iteration takes a constant time.

**Lemma A.16.** If the network is asynchronous, then in protocol \(\Pi_{ABA}\), it requires expected \(O(n^2)\) iterations for the honest parties to compute an output.

**Proof.** To prove the lemma, we need to derive the expected number of iterations, until all the honest parties have the same input during the second instance of \(\Pi_GA\) of an iteration. This is because once all the honest parties have the same input during the second instance of \(\Pi_GA\) of an iteration, then all honest parties will set committed to true at the end of that iteration and start sending signed ready messages, followed by computing an output. Let \(\tau\) be the random variable which counts the number of iterations until all honest parties have the same input during the second instance of \(\Pi_GA\) in an iteration. Then the probability that \(\tau = k\) is given as:

\[
Pr(\tau = k) = Pr(\tau \neq 1) \cdot Pr(\tau \neq 2 \mid \tau \neq 1) \cdot \ldots \cdot Pr(\tau \neq (k - 1) \mid \tau \neq 1 \cap \ldots \cap \\
\tau \neq (k - 2)) \cdot Pr(\tau = k \mid \tau \neq 1 \cap \ldots \cap \tau \neq (k - 1)).
\]

From Lemma A.15, every multiplicand on the right-hand side in the above equation, except the last one, is upper bounded by \((1 - \frac{1}{2n})\) and the last multiplicand is upper bounded by \(\frac{1}{2n}\). Hence, we get

\[
Pr(\tau = k) \leq (1 - \frac{1}{2n})^{k-1}(\frac{1}{2n}).
\]
Now the expected value \( E(\tau) \) of \( \tau \) is computed as follows:

\[
E(\tau) = \sum_{k=0}^{\infty} \tau \cdot \Pr(\tau = k)
\]

\[
\leq \sum_{k=0}^{\infty} k(1 - \frac{1}{2n})^{k-1}(\frac{1}{2n})
\]

\[
= \frac{1}{2n} \sum_{k=0}^{\infty} k(1 - \frac{1}{2n})^{k-1}
\]

\[
= \frac{1}{1 - (1 - \frac{1}{2n})} + \frac{1 - \frac{1}{2n}}{(1 - (1 - \frac{1}{2n}))^2}
\]

\[
= 2n + 4n^2 - 2n = 4n^2
\]

The expression for \( E(\tau) \) is a sum of AGP up to infinite terms, which is given by

\[
a + \frac{dr}{(1 - r)^2},
\]

where \( a = 1 \), \( r = 1 - \frac{1}{2n} \) and \( d = 1 \). Hence, we have \( E(\tau) \leq 4n^2 \).

We finally prove the properties of the protocol \( \Pi_{ABA} \) in a synchronous network.

**Lemma A.17.** If the network is synchronous and if all honest parties have the same input \( b \in \{0, 1\} \) during \( \Pi_{ABA} \), then all honest parties output \( b \), at time \( T_{ABA} = T_{CoinFlip} + 2T_{GA} + \Delta \).

**Proof.** Let \( Z^* \in Z_s \) be the set of corrupt parties and let \( H = \mathcal{P} \setminus Z^* \) be the set of honest parties. If all the parties in \( H \) participate with input \( b \), then from the \( Z_s \)-liveness and \( Z_s \)-Graded Validity of \( \Pi_{GA} \) in the synchronous network (Lemma A.10), all the parties in \( H \) output \( (b, 2) \) during the first instance of \( \Pi_{GA} \) at time \( T_{GA} \) in the first iteration. The \( Z_s \)-Guaranteed Liveness of \( \Pi_{GA} \) in the synchronous network [23] ensures that all honest parties compute some output from the instance of \( \Pi_{CoinFlip} \) during the first instance at the time \( T_{GA} + T_{CoinFlip} \). Since the parties in \( H \) output \( (b, 2) \) during the first instance of \( \Pi_{GA} \), they participate with input \( b \) during the second instance of \( \Pi_{GA} \). Consequently, from the \( Z_s \)-liveness and \( Z_s \)-Graded Validity of \( \Pi_{GA} \) in the synchronous network, all the parties in \( H \) compute the output \( (b, 2) \) during the second instance of \( \Pi_{GA} \) in the first iteration at the time \( 2T_{GA} + T_{CoinFlip} \). Hence every party in \( H \) sends a signed \textit{ready} message for \( b \) at the time \( 2T_{GA} + T_{CoinFlip} \), which gets delivered at the time \( T_{ABA} \). Moreover, only the parties in \( Z^* \) may send a signed \textit{ready} message for \( b \). Since \( \mathcal{P} \setminus H = Z^* \in Z_s \) and since \( H \notin Z_s \) (due to the \( Q^{(2)}(\mathcal{P}, Z_s) \) condition), it follows that all the parties in \( H \) will have sufficiently many signed \textit{ready} messages for \( b \) at the time \( T_{ABA} \) to output \( b \).

The proof of Theorem 3.4 now follows from Lemma A.13-A.17. The communication complexity follows from the communication complexity of \( \Pi_{CoinFlip} \) [23] and the communication complexity of \( \Pi_{GA} \) (Lemma A.12) and the fact that in a synchronous network, only a constant number of invocations of \( \Pi_{GA} \) and \( \Pi_{CoinFlip} \) are involved, while in an asynchronous network, there are poly(\( n \)) invocations of \( \Pi_{GA} \) and \( \Pi_{CoinFlip} \) in expectation.
B Properties of Our Network Agnostic ICP

In this section, we prove the properties of our network-agnostic ICP (see Fig 6 for the formal details). Throughout this section, we assume that $Z_s$ and $Z_a$ satisfy the conditions $Z_a \subseteq Z_s$, $Q(2)(P, Z_s)$, $Q(3)(P, Z_a)$, and $Q(2,1)(P, Z_s, Z_a)$.

Lemma B.1. If $S, I$ and $R$ are honest, then the following hold during protocol $\Pi_{\text{Auth}}$ and $\Pi_{\text{Reveal}}$:

- $Z_s$-Correctness: In a synchronous network, each honest party sets $\mathsf{authCompleted}(S, I, R)$ to 1 during $\Pi_{\text{Auth}}$ at time $T_{\text{Auth}} = \Delta + 4T_{\text{BC}}$. Moreover, $R$ outputs $s$ during $\Pi_{\text{Reveal}}$ which takes $T_{\text{Reveal}} = \Delta$ time.

- $Z_a$-Correctness: In an asynchronous network, each honest party eventually sets $\mathsf{authCompleted}(S, I, R)$ to 1 during $\Pi_{\text{Auth}}$ and $R$ eventually outputs $s$ during $\Pi_{\text{Reveal}}$.

Proof. We first start with the synchronous network. Let $Z^* \subseteq Z_s$ be set of corrupt parties and let $\mathcal{H} = P \setminus Z^*$ be the set of honest parties. During $\Pi_{\text{Auth}}$, $S$ chooses a random $t$-degree signing-polynomial $F(x)$ such that $s = F(0)$ holds, a random $t$-degree masking-polynomial $M(x)$, and computes verification points $(\alpha_i, v_i, m_i)$ such that $v_i = F(\alpha_i)$ and $m_i = M(\alpha_i)$ hold. $S$ then sends the signing-polynomial $F(x)$ and masking-polynomial $M(x)$ to $I$, and the corresponding verification-point $(\alpha_i, v_i, m_i)$ to each verifier $P_i$. Consequently, each verifier in $\mathcal{H}$ receives its verification-point by time $\Delta$, and indicates this by broadcasting $(\mathsf{Received}, i)$. Since $P \setminus \mathcal{H} = Z^* \subseteq Z_s$, from the $Z_s$-validity of $\Pi_{\text{BC}}$ in the synchronous network (see Theorem 3.2), it follows that at time $\Delta + T_{\text{BC}}$, $S$ will find a set $\mathcal{SV}$, such that $P \setminus \mathcal{SV} \subseteq Z_s$, where each verifier in $\mathcal{SV}$ has indicated that it has received its verification-point. Consequently, $S$ will broadcast $\mathcal{SV}$ at time $\Delta + T_{\text{BC}}$. From the $Z_s$-validity of $\Pi_{\text{BC}}$ in the synchronous network, I will receive $\mathcal{SV}$ at time $\Delta + 2T_{\text{BC}}$. Moreover, due to the $Z_s$-Consistency and $Z_s$-Validity of $\Pi_{\text{BC}}$ in the synchronous network, party $I$ would have gotten $(\mathsf{Received}, i)$, corresponding to every verifier $P_i \in \mathcal{SV}$, by time $\Delta + 2T_{\text{BC}}$. Furthermore, $P \setminus \mathcal{SV} \subseteq Z_s$ will hold. Hence, I will randomly select $d \in F$, compute $B(x) = dF(x) + M(x)$, and broadcast $(d, B(x))$. From the $Z_s$-validity of $\Pi_{\text{BC}}$ in the synchronous network, this will be delivered to every honest party, including $S$, by time $\Delta + 3T_{\text{BC}}$. Moreover, $S$ will find that $B(\alpha_i) = dv_i + m_i$ holds for all the verifiers $P_j \in \mathcal{SV}$. Consequently, $S$ will broadcast an $\mathsf{OK}$ message, which is received by every $P_i \in \mathcal{H}$ at time $\Delta + 4T_{\text{BC}}$, due to the $Z_s$-validity of $\Pi_{\text{BC}}$ in the synchronous network. Thus, each $P_i \in \mathcal{H}$ sets $\mathsf{authCompleted}(S, I, R)$ to 1, while I additionally sets $\mathsf{ICSig}(S, I, R, s)$ to $F(x)$ at time $\Delta + 4T_{\text{BC}}$.

During $\Pi_{\text{Reveal}}$, I will send $F(x)$ to $R$, and each verifier $P_i \in \mathcal{H} \cap \mathcal{SV}$ will send its verification point $(\alpha_i, v_i, m_i)$ to $R$. These points and the polynomial $F(x)$ are received by $R$ within $\Delta$ time. Moreover, the condition $v_i = F(\alpha_i)$ will hold true for these points, and consequently, these points will be accepted. Since $\mathcal{SV} \setminus (\mathcal{H} \cap \mathcal{SV}) \subseteq Z^* \subseteq Z_s$, it follows that at time $\Delta$, receiver $R$ will find a subset $\mathcal{SV}' \subseteq \mathcal{SV}$ where $\mathcal{SV} \setminus \mathcal{SV}' \subseteq Z_s$, such that the points corresponding to all the parties in $\mathcal{SV}'$ are accepted. This implies that $R$ will output $s = F(0)$ within time $\Delta$. 

75
The proof for the asynchronous case is similar as above, except that each “favourable” event occurs eventually, and follows from the fact that every set in $Z_a$ is a subset of some set in $Z_s$. Moreover, we rely on the properties of $\Pi_{\text{BC}}$ in the asynchronous network.

We next prove the privacy property, for which we again need to consider an honest $S$, $I$, and $R$.

**Lemma B.2.** If $S$, $I$, and $R$ are honest, then the view of Adv remains independent of $s$ during $\Pi_{\text{Auth}}$ and $\Pi_{\text{Reveal}}$, irrespective of the network type.

**Proof.** We prove privacy in a synchronous network. The privacy in an asynchronous network automatically follows, since $Z_a$ is a subset of $Z_s$. Let $t = \max\{|Z| : Z \in Z_s\}$ and let $Z^* \subseteq Z_s$ be the set of corrupt parties. For simplicity and without loss of generality, let $|Z^*| = t$. During $\Pi_{\text{Auth}}$, the adversary Adv learns $t$ verification-points $\{(\alpha_i, v_i, m_i)\}_{P_i \in Z^*}$. However, since $F(x)$ is a random $t$-degree polynomial with $F(0) = s$, the points $\{(\alpha_i, v_i)\}_{P_i \in Z^*}$ are distributed independently of $s$. That is, for every candidate $s \in \mathbb{F}$ from the point of view of Adv, there is a corresponding unique $t$-degree polynomial $F(x)$, such that $F(\alpha_i) = v_i$ holds corresponding to every $P_i \in Z^*$.

During $\Pi_{\text{Auth}}$, the adversary Adv also learns $d$ and the blinded-polynomial $B(x) = dF(x) + M(x)$. However, this does not add any new information about $s$ to the view of the adversary. This is because $M(x)$ is a random $t$-degree polynomial and Adv learns $t$ points on $M(x)$, corresponding to the parties in $Z^*$. Hence, for every candidate $M(x)$ polynomial from the point of view of Adv where $M(\alpha_i) = m_i$ holds for every $P_i \in Z^*$, there is a corresponding unique $t$-degree polynomial $F(x)$, such that $F(\alpha_i) = v_i$ holds corresponding to every $P_i \in Z^*$, and where $dF(x) + M(x) = B(x)$. Finally, Adv does not learn anything new about $s$ during $\Pi_{\text{Reveal}}$, since the verification-points and the signing-polynomial are sent only to $R$, who is honest as per the lemma conditions.

We next prove the unforgeability property, for which we have to consider a corrupt $I$.

**Lemma B.3.** If $S$, $R$ are honest, $I$ is corrupt and if $R$ outputs $s' \in \mathbb{F}$ during $\Pi_{\text{Reveal}}$, then $s' = s$ holds except with probability at most $\epsilon_{\text{ICP}} \stackrel{\text{def}}{=} \frac{nt}{|\mathcal{P}| - 1}$, where $t = \max\{|Z| : Z \in Z_s\}$, irrespective of the network type.

**Proof.** Let $\mathcal{H}$ be the set of honest parties in $\mathcal{P}$ and let $Z^* = \mathcal{P} \setminus \mathcal{H}$ be the set of corrupt parties. Since $R$ outputs $s'$ during $\Pi_{\text{Reveal}}$, it implies that during $\Pi_{\text{Auth}}$, the variable authCompleted$_{(S, I, R)}$ is set to 1 by $R$. This further implies that $S$ has broadcasted an OK message during $\Pi_{\text{Auth}}$, which also implies that during $\Pi_{\text{Auth}}, I$ had broadcasted a $t$-degree blinded-polynomial $B(x)$, and $S$ broadcasted the set $\mathcal{SV}$. Furthermore, $S$ has verified that $B(\alpha_i) = dv_i + m_i$ holds for every verifier $P_i \in \mathcal{SV}$. Now during $\Pi_{\text{Reveal}}, I$ sends $F(x)$ as ICSig$(S, I, R, s)$ to $R$, then $s' = s$ holds with probability 1. So, consider the case when $I$ sends $F'(x)$ as ICSig$(S, I, R, s)$ to $R$, where $F'(x)$ is a $t$-degree polynomial such that $F'(x) \neq F(x)$.
and where $F'(0) = s'$. In this case, we claim that except with probability at most $\frac{n}{|\mathcal{H}| - 1}$, the verification-point of no honest verifier from $\mathcal{SV}$ will get accepted by $R$ during $\Pi_{Reveal}$, with respect to $F'(x)$. Now, assuming that the claim is true, the proof follows using the following arguments, depending upon the network type.

**Synchronous Network**: In this case, all the verifiers in $\mathcal{H}$ will be present in $\mathcal{SV}$. This is because, each verifier $P_i \in \mathcal{H}$ would have received its verification-point from $S$ during $\Pi_{Auth}$, within time $\Delta$ and indicates this by broadcasting ($\text{Received}, i$), which is received by $S$ at time $\Delta + T_{BC}$. Let $\mathcal{SV}'$ be the set of verifiers from which $R$ receives verification points which it accepts. Since the verification point of none of the honest verifier will be accepted. Hence $(\mathcal{H} \cap \mathcal{SV}') = \emptyset$ and so $\mathcal{H} \subseteq \mathcal{SV} \setminus \mathcal{SV}'$ must hold. Since $\mathcal{H}$ satisfies the $Q^{(1)}(P, Z_s)$ condition, $\mathcal{SV} \setminus \mathcal{SV}' \in Z_s$ will never hold true. Hence, $R$ will not output $s' \neq s$.

**Asynchronous Network**: In this case, we first note that $Z_s$ and $Z_a$ satisfy the $Q^{(1,1)}(\mathcal{SV}, Z_s, Z_a)$ condition. This is because $P \setminus \mathcal{SV} \in Z_s$ and $Z_s$ and $Z_a$ satisfy the $Q^{(2,1)}(P, Z_s, Z_a)$ condition. From the steps of $\Pi_{Reveal}$, it follow that for $R$ to output $F'(0)$, $R$ should find a subset of verifiers $\mathcal{SV}' \subseteq \mathcal{SV}$, where $\mathcal{SV} \setminus \mathcal{SV}' \in Z_s$, such that the verification-points of all the verifiers in $\mathcal{SV}'$ are accepted by $R$. This further implies that $\mathcal{SV} \cap \mathcal{H} \neq \emptyset$, as $Z_a$ satisfies the $Q^{(1)}(\mathcal{SV}', Z_a)$ condition. And hence $\mathcal{SV}'$ has at least one honest verifier, whose verification-point is accepted with respect to $F'(x)$. However, from the above claim, it is not possible and hence $R$ will not output $s' \neq s$.

We now prove the claimed statement. So consider an arbitrary verifier $P_i \in \mathcal{H} \cap \mathcal{SV}$ from whom $R$ receives the verification-point $(\alpha_i, v_i, m_i)$ during $\Pi_{Reveal}$. This point can be accepted with respect to $F'(x)$, only if either of the following holds.

- $v_i = F'(\alpha_i)$: This is possible with probability at most $\frac{t}{|\mathcal{H}| - 1}$. This is because $F'(x)$ and $F(x)$, being distinct $t$-degree polynomials, can have at most $t$ points in common. And the evaluation-point $\alpha_i$ corresponding to $P_i$, being randomly selected from $\mathbb{F} \setminus \{0\}$, will not be known to $i$.

- $dv_i + m_i \neq B(\alpha_i)$: This is impossible, as otherwise $S$ would have not broadcasted $\text{OK}$ during $\Pi_{Auth}$, which is a contradiction.

As there could be up to $n - 1$ honest verifiers in $\mathcal{SV}$, it follows from the union bound that except with probability at most $\frac{n}{|\mathcal{H}| - 1}$, the verification-point of no honest verifier from $\mathcal{SV}$ will get accepted by $R$ during $\Pi_{Reveal}$, with respect to $F'(x)$.

We next prove the non-repudiation property, for which we have to consider a corrupt $S$.

**Lemma B.4.** If $S$ is corrupt, $I, R$ are honest and if $I$ sets $\text{ICSig}(S, I, R, s)$ during $\Pi_{Auth}$, then the following hold, except with probability at most $\frac{n}{|\mathcal{H}| - 1}$.

- $Z_s$-Non-Repudiation: In a synchronous network, $R$ outputs $s$ during $\Pi_{Reveal}$, which takes $T_{Reveal} = \Delta$ time.

- $Z_s$-Non-Repudiation: In an asynchronous network, $R$ eventually outputs $s$ during $\Pi_{Reveal}$.
Proof. Let \( \mathcal{H} \) be the set of honest parties in \( \mathcal{P} \) and \( Z^* \) be the set of corrupt parties, where \( \mathcal{H} = \mathcal{P} \setminus Z^* \). Since I has set IC\text{Sig}(S, I, R, s) during \( \Pi_{\text{Auth}} \), it implies that I has set the variable \text{authCompleted}_{(S, I, R)} \) to 1. This further implies that I has broadcasted \( (d, B(x)) \), where \( B(x) = dF(x) + M(x) \), and where \( F(x) \) and \( M(x) \) are the t-degree signing and masking-polynomials received by I from S. Moreover, I also received the set of supporting verifiers \( SV \) from the broadcast of S, and verified that \( \mathcal{P} \setminus SV \in Z^* \) holds. Furthermore, S has broadcasted an OK message. Consequently, from the consistency properties of \( \Pi_{\text{BC}} \) (see Theorem 3.2), irrespective of the network type, all honest parties including R eventually set \text{authCompleted}_{(S, I, R)} \) to 1. Moreover, I sets IC\text{Sig}(S, I, R, s) to \( F(x) \), where \( s = F(0) \). During \( \Pi_{\text{Reveal}} \), I sends \( F(x) \) to R. Moreover, every verifier \( P_i \in \mathcal{H} \cap SV \) sends its verification-point \((\alpha_i, v_i, m_i)\) to R. In a synchronous network, these will be received by R within time \( \Delta \), while in an asynchronous network, these will be eventually received by R. We claim that except with probability at most \( \frac{n}{|F| - 1} \), all these verification-points are accepted by R. Now, assuming that the claim is true, the proof follows from the fact that \( \mathcal{H} \cap SV = SV \setminus Z^* \), and \( Z^* \in Z_s \) holds, irrespective of the network type (since \( Z_s \subset Z_s \)). Consequently, R accepts the verification-points from a subset of the verifiers \( SV' \subset SV \) where \( SV \setminus SV' \in Z_s \). And hence it outputs s, either within time \( \Delta \) in a synchronous network, or eventually, in an asynchronous network.

We now proceed to prove the claim. So consider an arbitrary verifier \( P_i \in \mathcal{H} \cap SV \) whose verification-point \((\alpha_i, v_i, m_i)\) is received by R during \( \Pi_{\text{Reveal}} \). Now, there are two possible cases, depending upon the relationship that holds between \( F(\alpha_i) \) and \( v_i \) during \( \Pi_{\text{Auth}} \).

- \( v_i = F(\alpha_i) \) holds: In this case, according to the protocol steps of \( \Pi_{\text{Reveal}} \), the point \((\alpha_i, v_i, m_i)\) is accepted by R.
- \( v_i \neq F(\alpha_i) \) holds: In this case, we claim that except with probability at most \( \frac{1}{|F| - 1} \), the condition \( dv_i + m_i \neq B(\alpha_i) \) will hold, implying that the point \((\alpha_i, v_i, m_i)\) is accepted by R. This is because the only way \( dv_i + m_i \neq B(\alpha_i) \) holds is when S distributes \((\alpha_i, v_i, m_i)\) to \( P_i \) where \( v_i \neq F(\alpha_i) \) and \( m_i \neq M(\alpha_i) \) holds, and I selects \( d = (M(\alpha_i) - m_i) \cdot (v_i - F(\alpha_i))^{-1} \). However, S will not be knowing the random \( d \) from \( F \setminus \{0\} \) which I is going to pick, while distributing \( F(x), M(x) \) to I, and \((\alpha_i, v_i, m_i)\) to \( P_i \). Hence, the probability that I indeed selects \( d = (M(\alpha_i) - m_i) \cdot (v_i - F(\alpha_i))^{-1} \) is \( \frac{1}{|F| - 1} \).

As there can be up to \( n - 1 \) honest verifiers in \( SV \), from the union bound, it follows that except with probability at most \( \frac{n}{|F| - 1} \), the verification-point of all honest verifiers in \( SV \) are accepted by R.

We finally derive the communication complexity.

**Lemma B.5.** Protocol \( \Pi_{\text{Auth}} \) incurs a communication of \( O(n^5 \cdot \log |F| \cdot |\sigma|) \) bits. Protocol \( \Pi_{\text{Reveal}} \) incurs a communication of \( O(n \cdot \log |F|) \) bits.

**Proof.** During \( \Pi_{\text{Auth}} \), signer S sends t-degree polynomials \( F(x) \) and \( M(x) \) to I, and verification-points to each verifier. This requires a communication of \( O(n \cdot \log |F|) \) bits. Intermediary I needs to broadcast \( B(x) \) and \( d \) using protocol \( \Pi_{\text{BC}}, \)
while $S$ needs to broadcast the set $\mathcal{SV}$ using $\Pi_{BC}$. Moreover, $S$ may need to broadcast $s$ using $\Pi_{BC}$. By substituting the communication cost of $\Pi_{BC}$ (see Theorem 3.2), the overall communication cost of $\Pi_{Auth}$ turns out to be $O(n^5 \cdot \log |\mathcal{F}| \cdot |\sigma|)$ bits. During $\Pi_{Reveal}$, I may send $F(x)$ to $R$, and each verifier may send its verification-point to $R$. This incurs a communication of $O(n \cdot \log |\mathcal{F}|)$ bits.

The proof of Theorem 4.1 now follows easily from Lemma B.1-B.5.

C Properties of Network Agnostic VSS

In this section, we prove the properties of the protocol $\Pi_{VSS}$ (see Fig 7). Throughout this section, we assume that $Z_s$ and $Z_a$ satisfy the conditions $Z_a \subset Z_s$, $Q^{(2)}(P, Z_a)$, $Q^{(3)}(P, Z_s)$ and $Q^{(2,1)}(P, Z_s, Z_a)$. We start with the properties in a synchronous network and first consider an honest $D$. We first show that an honest $D$ will broadcast some candidate core-sets, which will be accepted by all honest parties. Moreover, adversary will not learn any additional information about $s$.

**Lemma C.1.** If the network is synchronous and $D$ is honest, participating in $\Pi_{VSS}$ with input $s$, then all the following hold, where $H$ is the set of honest parties.

- There exists some $S_p \in S_{Z_s}$, such that $D$ broadcasts a message $(\text{CanCS}, D, S_p, \{W_q\}_{q=1, \ldots, |Z_s|}, BS, \{s_q\}_{q \in BS})$ at time $\Delta + T_{Auth} + T_{BC}$, and every $P_i \in H$ includes $(D, S_p)$ to the set $C_i$ at time $\Delta + T_{Auth} + 2T_{BC}$. Moreover, all the following hold for $q = 1, \ldots, |Z_s|$. - If $S_q = H$, then $q \not\in BS$.
- $W_q$ will be either $S_q$ or $(S_p \cap S_q)$. Moreover, $Z_s$ will satisfy the $Q^{(1)}(W_q, Z_s)$ condition.
- Corresponding to every $S_q \in S_{Z_s}$, every honest $P_i \in W_q$ will have the IC-signature $\text{ICSig}(P_j, P_i, P_k, s_q)$ of every $P_j \in W_q$ for every $P_k \not\in S_q$, such that the underlying signatures will satisfy the linearity principle. Furthermore, if any corrupt $P_j \in W_q$ has the IC-signature $\text{ICSig}(P_i, P_j, P_k, s'_q)$ of any honest $P_i \in W_q$ for any $P_k \in P$, then $s'_q = s_q$ and the underlying signatures will satisfy the linearity principle.
- Corresponding to every $S_q \in S_{Z_s}$, every honest $P_i \in S_q$ will have the share $s_q$, except with a probability $O(|S_{Z_s}| \cdot n^2 \cdot \varepsilon_{ICP})$, at time $\Delta + T_{Auth} + 2T_{BC} + T_{Reveal}$, where $s_q = s_1 + \cdots + s_{|Z_s|}$.
- The view of the adversary will be independent of $s$.

**Proof.** Let $Z^* \in Z_s$ be the set of corrupt parties and let $H = P \setminus Z^*$ be the set of honest parties. We note that $H \in S_{Z_s}$. Since $D$ is honest, it picks the shares $s_1, \ldots, s_{|Z_s|}$ such that $s = s_1 + \cdots + s_{|Z_s|}$ and sends $s_q$ to each party $P_i \in S_q$, corresponding to every $S_q \in S_{Z_s}$. These shares are delivered within time $\Delta$. Now consider an arbitrary $S_q \in S_{Z_s}$. At time $\Delta$, each party $P_i \in (S_q \cap H)$ starts giving $\text{ICSig}(P_i, P_j, P_k, s_q)$ to every $P_j \in S_q$, for every $P_k \in P$, where $s_{qi} = s_q$. 

79
holds. Moreover, the linearity principle is followed while generating these IC-signatures. Then from the $Z_s$-correctness of $\Pi_{\text{Auth}}$ in the synchronous network (Theorem 4.1), it follows that at time $\Delta + T_{\text{Auth}}$, each party $P_i \in (S_q \cap H)$ will receive $\text{ICSig}(P_j, P_l, P_k, s_{ij})$ from every $P_j \in (S_q \cap H)$, for every $P_k \in P$, such that $s_{ij} = s_{kl} = s_q$ holds. Since $S_q$ is arbitrary, it follows that at time $\Delta + T_{\text{Auth}}$, every party $P_i \in H$ broadcasts an $\text{OK}(i, j)$ message, corresponding to every $P_j \in H$. From the $Z_s$-validity of $\Pi_{\text{BC}}$ in the synchronous network (Theorem 3.2), it follows that these $\text{OK}(i, j)$ messages are received by every party in $H$ through regular-mode at time $\Delta + T_{\text{Auth}} + T_{\text{BC}}$. Consequently, the set $H$ constitutes a clique in the consistency graph of every party in $H$ at time $\Delta + T_{\text{Auth}} + T_{\text{BC}}$. Now since the set $H \in S_{Z_s}$, it follows that at time $\Delta + T_{\text{Auth}} + T_{\text{BC}}$, there exists some $S_p \in S_{Z_s}$, such that $D$ computes the core-sets $\{W_q\}_{q=1,\ldots,|Z_s|}$ and broadcast-set $\mathcal{BS}$, followed by broadcasting (CanCS, D, $S_p$, $\{W_q\}_{q=1,\ldots,|Z_s|}$, $\mathcal{BS}$, $\{s_q\}_{q \in \mathcal{BS}}$) at time $\Delta + T_{\text{Auth}} + T_{\text{BC}}$. Moreover, since $D$ is honest, it computes the sets $\{W_q\}_{q=1,\ldots,|Z_s|}$ and $\mathcal{BS}$ honestly, satisfying the following conditions, for $q = 1, \ldots, |Z_s|$.

- If $S_q$ constitutes a clique in the graph $G^{(D)}$, then $W_q = S_q$.
- Else if $(S_p \cap S_q)$ constitutes a clique in $G^{(D)}$ and $Z_s$ satisfies the $Q^{(1)}(S_p \cap S_q, Z_s)$ condition, then $W_q = (S_p \cap S_q)$.
- Else $W_q = S_q$ and $q \in \mathcal{BS}$.

Note that for each $W_q$, the condition $Q^{(1)}(W_q, Z_s)$ holds. This is obviously true if $W_q = (S_p \cap S_q)$, since in this case $D$ also checks that $Q^{(1)}(S_p \cap S_q, Z_s)$ condition holds. On the other hand, even if $W_q = S_q$, the condition $Q^{(1)}(W_q, Z_s)$ holds, as $Q^{(1)}(S_q, Z_s)$ holds due to the $Q^{(2)}(P, Z_s)$ condition. We also note that if $S_q = H$, then $q \notin \mathcal{BS}$ and consequently, $D$ does not make public the share $s_q$. This is because as shown above, at time $\Delta + T_{\text{Auth}} + T_{\text{BC}}$, the parties in $S_q$ constitute a clique in the graph $G^{(D)}$.

Since $D$ broadcasts (CanCS, D, $S_p$, $\{W_q\}_{q=1,\ldots,|Z_s|}$, $\mathcal{BS}$, $\{s_q\}_{q \in \mathcal{BS}}$) at time $\Delta + T_{\text{Auth}} + T_{\text{BC}}$, from the $Z_s$-validity of $\Pi_{\text{BC}}$ in the synchronous network, it follows that all the parties in $H$ will receive (CanCS, D, $S_p$, $\{W_q\}_{q=1,\ldots,|Z_s|}$, $\mathcal{BS}$, $\{s_q\}_{q \in \mathcal{BS}}$) from the broadcast of $D$, at time $\Delta + T_{\text{Auth}} + 2T_{\text{BC}}$. To show that every $P_i \in H$ will include $(D, S_p)$ to the set $\mathcal{C}_i$, we need to show that all the conditions which hold for $D$ in its graph $G^{(D)}$ at time $\Delta + T_{\text{Auth}} + T_{\text{BC}}$, are bound to hold for every $P_i \in H$, at the time $\Delta + T_{\text{Auth}} + 2T_{\text{BC}}$. Namely, all the edges which are present in $G^{(D)}$ at time $\Delta + T_{\text{Auth}} + T_{\text{BC}}$, are bound to be present in the graph $G^{(1)}$ every $P_i \in H$, at the time $\Delta + T_{\text{Auth}} + 2T_{\text{BC}}$. However, this simply follows from the $Z_s$-validity, $Z_s$-consistency and $Z_s$-fallback consistency of $\Pi_{\text{BC}}$ in the synchronous network (see Theorem 3.2) and the fact that edges are added to consistency graphs, based on the receipt of $\text{OK}(\ast, \ast)$ messages, which are broadcasted through various $\Pi_{\text{BC}}$ instances. Consequently, any edge $(i, j)$ which is included in $G^{(D)}$ at the time $\Delta + T_{\text{Auth}} + T_{\text{BC}}$, is bound to be included in the graph $G^{(i)}$ of every $P_i \in H$, latest by time $\Delta + T_{\text{Auth}} + 2T_{\text{BC}}$.

We next note that corresponding to every $S_q \in S_{Z_s}$, every honest $P_i \in W_q$ will have the share $s_q$, which is either made public by $D$ as part of the CanCS message or received from $D$. Each honest $P_i \in W_q$ will thus set $|s_q|$ to $s_q$ at time $\Delta + T_{\text{Auth}} + 2T_{\text{BC}}$. We next show that each $P_i \in H$ will have $\text{ICSig}(P_j, P_i, P_k, s_q)$
corresponding to every $S_q \in \mathbb{S}_z$, where $P_i \in \mathcal{W}_q$, for every $P_j \in \mathcal{W}_q$ and every $P_k \notin S_q$. This also is set at time $\Delta + T_{\text{Auth}} + 2T_{\text{BC}}$. This is because there are two possible cases with respect to $q$. If $q \in \mathcal{B}_s$, then from the protocol steps, $S_q$ is publicly set to $s_q$ and $\text{ICSig}(P_j, P_i, P_k, s_q)$ is set to the default value. On the other hand, if $q \notin \mathcal{B}_s$, then also $P_i$ will possess $\text{ICSig}(P_j, P_i, P_k, s_q)$. This is because as per the protocol steps, since $P_i, P_j \in \mathcal{W}_q$, it follows that $P_i$ must have verified that the edge $(P_i, P_j) \in G^{(i)}$, which further implies that $P_i$ has received $\text{ICSig}(P_j, P_i, P_k, s_{ij})$ from $P_j$, where $s_{ij} = s_q$ holds. And since $D$ is honest, $s_{q_i} = s_q$ holds, implying that $\text{ICSig}(P_j, P_i, P_k, s_{q_j})$ is the same as $\text{ICSig}(P_j, P_i, P_k, s_q)$. On the other hand, in the protocol, $P_i$ gives $\text{ICSig}(P_i, P_j, P_k, s_q)$ to every $P_j \in S_q$ for every $P_k \in \mathcal{P}$, where $s_{q_i} = s_q$ holds. Hence if any corrupt $P_j \in \mathcal{W}_q$ has $\text{ICSig}(P_j, P_i, P_k, s_{q})$ of any honest $P_i \in \mathcal{W}_q$ for any $P_k \in \mathcal{P}$, then $s_q = s_q$ holds.

We now show that, corresponding to each $S_q \in \mathbb{S}_z$, every honest party $P_i \in S_q \setminus \mathcal{W}_q$ sets $[s]_q$ to $s_q$, except with a probability of $O(|\mathbb{S}_z| \cdot n^2 \cdot \epsilon_{\text{ICP}})$, at time $\Delta + T_{\text{Auth}} + 2T_{\text{BC}} + T_{\text{Reveal}}$. For this, we first show that $P_i$ sets $[s]_q$ to some value. Since $Z_q$ satisfies the $Q^{(1)}(\mathcal{W}_q, Z_q)$ condition, the set $\mathcal{W}_q$ contains at least one honest party, say $P_j$. Since $P_j$ follows the protocol steps honestly, it reveals $\text{ICSig}(P_k, P_j, P_i, [s]_q)$ of every $P_k \in \mathcal{W}_q$ to $P_i$, at time $\Delta + T_{\text{Auth}} + 2T_{\text{BC}}$. From the $Z_q$-correctness of ICP in the synchronous network (see Theorem 4.1), it follows that $P_i$ will accept these signatures after time $T_{\text{Reveal}}$. On the other hand, even if $P_k \in \mathcal{W}_q$ is corrupt, then also from the $Z_q$-non-repudiation property of ICP in the synchronous network (see Theorem 4.1), it follows that $P_i$ accepts $\text{ICSig}(P_k, P_j, P_i, [s]_q)$, except with a probability $\epsilon_{\text{ICP}}$, after time $T_{\text{Reveal}}$. As there can be $O(n)$ corrupt parties in $\mathcal{W}_q$, from the union bound, it follows that except with a probability $O(n \cdot \epsilon_{\text{ICP}})$, every honest party in $\mathcal{W}_q \setminus \mathcal{W}_q$ will find a candidate party from $\mathcal{W}_q$, who reveals $[s]_q$, along with the IC-signature of all the parties in $\mathcal{W}_q$, after time $T_{\text{Reveal}}$. Now as there can be $O(n)$ parties in $S_q \setminus \mathcal{W}_q$, it follows that except with probability $O(n^2 \cdot \epsilon_{\text{ICP}})$, every honest party $P_i \in S_q \setminus \mathcal{W}_q$ will find a candidate party from $\mathcal{W}_q$, who reveals $[s]_q$ along with the IC-signature of all the parties in $\mathcal{W}_q$ at time $\Delta + T_{\text{Auth}} + 2T_{\text{BC}} + T_{\text{Reveal}}$.

We next show that $P_i \in S_q \setminus \mathcal{W}_q$ indeed sets $[s]_q$ to $s_q$. Suppose that $P_i$ sets $[s]_q$ to some value $s'$. From the protocols steps, this implies that there exists some $P_j \in \mathcal{W}_q$, such that $P_i$ has accepted $\text{ICSig}(P_k, P_j, P_i, s')$ of every $P_k \in \mathcal{W}_q$, revealed by $P_j$. If $P_j$ is honest, then indeed $s' = [s]_q$, as one of the IC-signatures $\text{ICSig}(P_k, P_j, P_i, s')$ is the same as $\text{ICSig}(P_k, P_j, P_i, [s]_q)$, corresponding to the honest $P_k \in \mathcal{W}_q$, which is guaranteed to exist. So consider the case when $P_j$ is corrupt. Moreover, let $P_k \in \mathcal{W}_q$ be an honest party (which is guaranteed to exist). In order that $s' \neq [s]_q$, it must be the case that $P_i$ accepts $\text{ICSig}(P_k, P_j, P_i, s')$, revealed by $P_j$. However, from the unforgeability property of ICP (see Theorem 4.1), this can happen only with probability $\epsilon_{\text{ICP}}$. Now as there can be up to $O(n)$ corrupt parties in $\mathcal{W}_q$, from the union bound, it follows that the probability that $P_i$ outputs $s' \neq [s]_q$ is at most $O(n \cdot \epsilon_{\text{ICP}})$. Since there can be up to $O(n)$ parties in $S_q \setminus \mathcal{W}_q$, it follows that except with probability at most $O(n^2 \cdot \epsilon_{\text{ICP}})$, the output of every honest party in $S_q$ is indeed $s_q$. Now, there can be $|\mathbb{S}_z|$ possibilities for $S_q$. From the union bound, it follows that corresponding to each $S_q \in \mathbb{S}_z$,
every honest party \( P_i \in S_q \setminus W_q \) sets \([s]_q\) to \( s_q\), except with a probability of \( O(|S_{Z_s}| \cdot n^2 \cdot \epsilon_{ICP})\).

Finally, the privacy for \( s \) follows from the fact that throughout the protocol, the view of the adversary remains independent of the share \( s_q \), corresponding to the group \( S_q \), where \( S_q = \mathcal{H} \). This is because as shown above, \( q \notin BS \) and consequently, \( D \) does not make public the share \( s_q \). Moreover, during the pairwise consistency tests, the view of the adversary remains independent of \( s_q \), when the parties in \( \mathcal{H} \) exchange IC-signed \( s_q \), which follows from the privacy property of ICP (see Theorem 4.1). Further, while computing the output, ICSig\((P_k, P_j, P_i, s_q)\) is revealed by party \( P_j \in W_q \) only to each party \( P_i \in S_q \setminus W_q \). Hence, the adversary does not learn \( s_q \).

An immediate corollary of Lemma C.1 is that if \( D \) is honest, then the parties output \([s] \) at time \( \Delta + T_{Auth} + 2T_{BC} + T_{Reveal} \), which follows from the definition of \([\cdot]\)-sharing.

**Corollary C.1.** If the network is synchronous and \( D \) is honest and participates in \( \Pi_{VSS} \) with input \( s \), then the parties output \([s] \) at time \( \Delta + T_{Auth} + 2T_{BC} + T_{Reveal} \), except with a probability of \( O(|S_{Z_s}| \cdot n^2 \cdot \epsilon_{ICP})\).

We next consider a corrupt \( D \) in the synchronous network and show that if any honest party computes an output at time \( T \), then there exists some \( s^* \in \mathbb{F} \) such that the (honest) parties output \([s^*] \) by time \( T + \Delta \), except with probability \( O(|S_{Z_s}| \cdot n^2 \cdot \epsilon_{ICP})\).

**Lemma C.2.** If the network is synchronous and \( D \) is corrupt and if any honest party computes an output at time \( T \), then there exists some \( s^* \in \mathbb{F} \), such that the honest parties output \([s^*] \) by time \( T + \Delta \), except with a probability of \( O(|S_{Z_s}| \cdot n^2 \cdot \epsilon_{ICP})\).

**Proof.** Let \( Z^* \in Z_s \) be the set of corrupt parties and let \( \mathcal{H} = \mathcal{P} \setminus Z^* \) be the set of honest parties. Let \( P \in \mathcal{H} \) be the first honest party which computes an output at time \( T \). The way in which the core sets are defined ensures that each set in \( \{W_q\}_{q=1,...,|Z_s|} \) must contain at least one honest party. Since \( H_{Reveal} \) takes at least \( T_{Reveal} \) time to complete, this means that there exists \( S_p \in S_{Z_s} \) such that some honest party \( P_m \) receives a message \((\text{CanCS, } D, S_p, \{W_q\}_{q=1,...,|Z_s|}, BS, \{s_q\}_{q\in BS})\) from the broadcast of \( D \) by the time \( T - T_{Reveal} \) and includes \((D, S_p)\) to \( \mathcal{C}_q \). This further implies that \( P_m \) has verified the following hold, for \( q = 1, \ldots, |Z_s| \), by time \( T - T_{Reveal} \):

- If \( q \notin BS \), then \( W_q \) is either \( S_q \) or \((S_p \cap S_q)\). Moreover, \( Z_s \) satisfies the \( Q^{(1)}(W_q, Z_s) \) condition and the parties in \( W_q \) constitute a clique in the graph \( G^{(m)} \).
- If \( q \in BS \), then \( D \) has made public \( s_q \), as part of the CanCS message. Moreover, \( W_q \) is set to \( S_q \).

From the \( Z_s \)-consistency and \( Z_s \)-fallback consistency of \( \Pi_{BC} \) in the synchronous network (see Theorem 3.2), it follows that every party in \( \mathcal{H} \) will receive \((\text{CanCS, } D, S_p, \{W_q\}_{q=1,...,|Z_s|}, BS, \{s_q\}_{q\in BS})\) from the broadcast of \( D \), latest by time \( T - T_{Reveal} + \Delta \). We next show that each \( P_i \in \mathcal{H} \) will include \((D, S_p)\) to
the parties follow the linearity principle while generating IC-signatures.

We next show that by time $T - T_{\text{Reveal}} + \Delta$, corresponding to every $S_q \in \mathbb{S}_{Z_s}$, every $P_i \in W_q$ will have a common share, say $s_i^*$. If $q \in \mathcal{BS}$, this is trivially true, since in this case, $D$ makes public the share $s_q$ and hence $s_q^* = s_q$. On the other hand, consider the case when $q \notin \mathcal{BS}$ and consider arbitrary $P_i, P_j \in (W_q \cap \mathcal{H})$. Since $P_i$ and $P_j$ are part of a clique, it follows that $P_i$ and $P_j$ have broadcasted the messages $\text{OK}(i, j)$ and $\text{OK}(j, i)$ respectively. Moreover, these messages were broadcasted, latest by time $T - T_{\text{Reveal}} - T_{\text{BC}}$, since it takes at least $T_{\text{BC}}$ time to compute any output in an instance of $\Pi_{\text{BC}}$ in the synchronous network (see Theorem 3.2). Now since $P_i$ and $P_j$ have broadcasted $\text{OK}(i, j)$ and $\text{OK}(j, i)$ messages, it implies that they have verified that $s_{qi} = s_{qj}$ holds, where $s_{qi}$ and $s_{qj}$ are the shares, received by $P_i$ and $P_j$ respectively, from $D$. Let $s_{qi} = s_{qj} = s^*$. We define

$$s^* \overset{def}{=} \sum_{q=1, \ldots, |Z_s|} s_q^*.$$

Till now we have shown that there exists some $s^* \in \mathbb{F}$, such that by time $T - T_{\text{Reveal}} + \Delta$, all the parties in $\mathcal{H}$ will have the core-sets $W_1, \ldots, W_{|Z_s|}$, where $S_q \cap W_q \in \mathbb{Z}_{\mathcal{H}}$, for $q = 1, \ldots, |Z_s|$ and where each $P_i \in (W_q \cap \mathcal{H})$ will have a common share $[s^*]_q$. We need to show that by the time $T - T_{\text{Reveal}} + \Delta$, the parties in $\mathcal{H}$ will have the required IC-signatures, as part of $[s^*]$, satisfying the linearity property. Namely, each $P_i \in \mathcal{H}$ will have $\text{ICSig}(P_j, P_i, P_k, [s^*]_q)$, corresponding to every $S_q \in \mathbb{S}_{Z_s}$, where $P_i \in W_q$, of every $P_j \in W_q$ and for every $P_k \in \mathcal{P}$. There are two possible cases with respect to $q$. If $q \in \mathcal{BS}$, then from the protocol steps, $W_q$ is publicly set to $S_q$ and $\text{ICSig}(P_j, P_i, P_k, [s^*]_q)$ is set to the default value of $\text{ICSig}(P_j, P_i, P_k, q)$, where $s_q$ is made public by $D$. As per our notations, $s_q = s^*_q$ for every $q \in \mathcal{BS}$. On the other hand, if $q \notin \mathcal{BS}$, then also $P_i$ will possess $\text{ICSig}(P_j, P_i, P_k, [s^*]_q)$. This is because as per the protocol steps, since $P_i, P_j \in W_q$, it follows that $P_i$ must have verified that the edge $(P_i, P_j) \in G^{(i)}$, which further implies that $P_i$ has received $\text{ICSig}(P_j, P_i, P_k, s_{qj})$ from $P_j$, where $s_{qj} = s_{qi}$ holds. Here $s_{qi}$ is the share received by $P_i$ from $D$ and as per our notation, $s_{qi} = s^*_q$. Hence $\text{ICSig}(P_j, P_i, P_k, s_{qj})$ is the same as $\text{ICSig}(P_j, P_i, P_k, [s^*]_q)$. On the other hand, in the protocol, $P_i$ gives $\text{ICSig}(P_i, P_j, P_k, s_{qi})$ to every $P_j \in S_q$ for every $P_k \in \mathcal{P}$, where $s_{qi} = [s^*]_q$ holds. Hence if any corrupt $P_j \in W_q$ has $\text{ICSig}(P_i, P_j, P_k, s_q^*)$, of any honest $P_i \in W_q$ for any $P_k \in \mathcal{P}$, then $s_q^* = [s^*]_q$ holds. It is easy to see that all the underlying IC-signatures will be linear since the parties follow the linearity principle while generating IC-signatures.
Finally, we now show that every honest party \( P_i \in S_q \setminus W_q \) gets \([s^*]_q\), except with a probability of \( \mathcal{O}(|S_Z| \cdot n^2 \cdot \epsilon_{\text{ICP}}) \), at time \( T + \Delta \). For this, we first show that \( P_i \) computes some share on the behalf of \( S_q \). Since \( Z_s \) satisfies the \( Q^{(1)}(W_q, Z_s) \) condition, the set \( W_q \) contains at least one honest party, say \( P_j \). Since \( P_j \) follows the protocol steps honestly, it reveals \( \text{ICSig}(P_k, P_j, P_j, [s^*]_q) \) of every \( P_k \in W_q \) to \( P_i \), at time \( T - T_{\text{Reveal}} + \Delta \). From the \( Z_s \)-correctness of ICP in the synchronous network (see Theorem 4.1), it follows that \( P_i \) will accept the IC-signatures \( \text{ICSig}(P_k, P_j, P_j, [s^*]_q) \), revealed by \( P_j \), after time \( T_{\text{Reveal}} \). On the other hand, even if \( P_k \in W_q \) is corrupt, then also from the \( Z_s \)-non-repudiation property of ICP in the synchronous network (see Theorem 4.1), it follows that \( P_i \) accepts \( \text{ICSig}(P_k, P_j, P_j, [s^*]_q) \), except with a probability of \( \mathcal{O}(n \cdot \epsilon_{\text{ICP}}) \), after time \( T_{\text{Reveal}} \). As there can be \( \mathcal{O}(n) \) corrupt parties in \( W_q \), from the union bound, it follows that except with a probability of \( \mathcal{O}(n^2 \cdot \epsilon_{\text{ICP}}) \), every honest party \( P_i \in S_q \setminus W_q \) will find a candidate party from \( W_q \), who reveals \([s^*]_q\), along with the IC-signature of all the parties in \( W_q \), after time \( T_{\text{Reveal}} \). Now as there can be \( \mathcal{O}(n) \) parties in \( S_q \setminus W_q \), it follows that except with a probability of \( \mathcal{O}(n^2 \cdot \epsilon_{\text{ICP}}) \), every honest party \( P_i \in S_q \setminus W_q \) will find a candidate party from \( W_q \), who reveals \([s^*]_q\) along with the IC-signature of all the parties in \( W_q \) at time \( T + \Delta \).

We next show that \( P_i \in S_q \setminus W_q \) indeed sets \([s^*]_q\) as the share corresponding to \( S_q \). Suppose that \( P_i \) sets the share to some value \( s' \). From the protocol steps, this implies that there exists some \( P_j \in W_q \), such that \( P_i \) has accepted the IC-signatures \( \text{ICSig}(P_k, P_j, P_j, s') \) of every \( P_k \in W_q \), revealed by \( P_j \). If \( P_j \) is honest, then indeed \( s' = [s^*]_q \), as one of the IC-signatures \( \text{ICSig}(P_k, P_j, P_j, [s^*]_q) \), corresponding to the honest \( P_k \in W_q \), which is guaranteed to exist. So consider the case when \( P_j \) is corrupt. Moreover, let \( P_k \in W_q \) be an honest party (which is guaranteed to exist). In order that \( s' \neq [s^*]_q \), it must be the case that \( P_i \) accepts \( \text{ICSig}(P_k, P_j, P_j, s') \), revealed by \( P_j \). However, from the unforgeability property of ICP (see Theorem 4.1), this can happen only with probability \( \epsilon_{\text{ICP}} \). Now as there can be up to \( \mathcal{O}(n) \) corrupt parties in \( W_q \), from the union bound, it follows that the probability that \( P_i \) outputs \( s' \neq [s^*]_q \) is at most \( \mathcal{O}(n \cdot \epsilon_{\text{ICP}}) \). Since there can be up to \( \mathcal{O}(n) \) parties in \( S_q \setminus W_q \), it follows that except with probability at most \( \mathcal{O}(n^2 \cdot \epsilon_{\text{ICP}}) \), the output of every honest party in \( S_q \) is indeed \([s^*]_q\). Now, there can be \(|S_Z| \) possibilities for \( S_q \). From the union bound, it follows that corresponding to each \( S_q \in S_Z \), every honest party in \( S_q \) outputs \([s^*]_q\), except with a probability of \( \mathcal{O}(|S_Z| \cdot n^2 \cdot \epsilon_{\text{ICP}}) \).

We next consider an asynchronous network and prove the analogue of Lemma C.1 by showing that if \( D \) is honest, then the parties eventually output \([s]\) except with a probability of \( \mathcal{O}(|S_Z| \cdot n^2 \cdot \epsilon_{\text{ICP}}) \). Further, the adversary learns no additional information about \( s \). The proof of the lemma follows closely the proof of Lemma C.1, except that we now rely on the properties of \( H_{\text{GC}} \) and ICP in the asynchronous network.

**Lemma C.3.** If the network is asynchronous and \( D \) is honest, participating in \( \Pi_{\text{VSS}} \) with input \( s \), then the honest parties eventually output \([s]\) except with a probability of \( \mathcal{O}(|S_Z| \cdot n^2 \cdot \epsilon_{\text{ICP}}) \), with the view of the adversary remaining independent of \( s \).
Proof. Let \( Z^* \in Z_s \) be the set of corrupt parties and let \( \mathcal{H} = \mathcal{P} \setminus Z^* \) be the set of honest parties. We note that \( \mathcal{H} \subseteq S_{Z_s} \), since \( Z^* \subseteq Z_s \) as \( Z_s \subseteq Z_s \). Since \( D \) is honest, it picks the shares \( s_1, \ldots, s_{|Z_s|} \) such that \( s = s_1 + \cdots + s_{|Z_s|} \) and sends \( s \) to each party \( P_i \in S_q \), corresponding to every \( S_q \in S_{Z_s} \). These shares are eventually delivered. Now consider an arbitrary \( S_q \in S_{Z_s} \). After receiving the share \( s_{q_i} \) from \( D \), each party \( P_i \in (S_q \cap \mathcal{H}) \) starts giving \( \text{ICSig}(P_i, P_j, P_k, s_{q_j}) \) to every \( P_j \in S_q \), for every \( P_k \in \mathcal{P} \), where \( s_{q_j} = s_q \) holds, such that the linearity principle is followed while generating these IC-signatures. Then from the \( Z_s \)-correctness of \( \Pi_{\text{Auth}} \) in the asynchronous network (Theorem 4.1), it follows that each party \( P_i \in (S_q \cap \mathcal{H}) \) will eventually receive \( \text{ICSig}(P_j, P_k, s_{q_j}) \) from every \( P_j \in (S_q \cap \mathcal{H}) \), for every \( P_k \in \mathcal{P} \), such that \( s_{q_j} = s_q \) holds. Since \( S_q \) is arbitrary, it follows that eventually, every party \( P_i \in \mathcal{H} \) broadcasts an \( \text{OK}(i,j) \) message, corresponding to every \( P_j \in \mathcal{H} \). From the \( Z_s \)-weak validity and \( Z_s \)-fallback validity of \( \Pi_{\text{BC}} \) in the asynchronous network (Theorem 3.2), it follows that these \( \text{OK}(i,j) \) messages are eventually received by every party in \( \mathcal{H} \). Consequently, the set \( \mathcal{H} \) eventually constitutes a clique in the consistency graph of every party in \( \mathcal{H} \). Now since the set \( \mathcal{H} \subseteq S_{Z_s} \), it follows that eventually, there exists some \( S_p \in S_{Z_s} \), such that \( D \) computes the core-sets \( \{W_q\}_{q=1,\ldots,|Z_s|} \) and broadcast-set \( BS \), followed by broadcasting \( (\text{CanCS}, D, S_p, \{W_q\}_{q=1,\ldots,|Z_s|}, BS, \{s_q\}_{q\in BS}) \). Moreover, since \( D \) is honest, it computes the sets \( \{W_q\}_{q=1,\ldots,|Z_s|} \) and \( BS \) honestly, satisfying the following conditions, for \( q = 1, \ldots, |Z_s| \).

- If \( S_q \) constitutes a clique in the graph \( G^{(D)} \), then \( W_q = S_q \).
- Else if \( (S_p \cap S_q) \) constitutes a clique in \( G^{(D)} \) and \( Z_s \) satisfies the \( Q^{(1)}(S_p \cap S_q, Z_s) \) condition, then \( W_q = (S_p \cap S_q) \).
- Else \( W_q = S_q \) and \( q \in BS \).

Note that for each \( W_q \), the condition \( Q^{(1)}(W_q, Z_s) \) holds. This is obviously true if \( W_q = (S_p \cap S_q) \), since in this case \( D \) also checks that \( Q^{(1)}(S_p \cap S_q, Z_s) \) holds. On the other hand, even if \( W_q = S_q \), the condition \( Q^{(1)}(W_q, Z_s) \) holds, since \( Q^{(1)}(S_q, Z_s) \) holds. We also note that if \( S_q = \mathcal{H} \), then \( q \notin BS \) and consequently, \( D \) does not make public the share \( s_q \). This is because the parties in \( S_p \cap S_q \) will constitute a clique and the \( Q^{(1)}(S_p \cap S_q, Z_s) \) condition will be satisfied due to the \( Q^{(2,1)}(P, Z_s, Z_a) \) condition. Hence \( W_q \) will be set to \( S_p \cap S_q \).

Since \( D \) eventually broadcasts \( (\text{CanCS}, D, S_p, \{W_q\}_{q=1,\ldots,|Z_s|}, BS, \{s_q\}_{q\in BS}) \), from the \( Z_s \)-weak validity and \( Z_s \)-fallback validity of \( \Pi_{\text{BC}} \) in the asynchronous network, it follows that all the parties in \( \mathcal{H} \) will eventually receive \( (\text{CanCS}, D, S_p, \{W_q\}_{q=1,\ldots,|Z_s|}, BS, \{s_q\}_{q\in BS}) \) from the broadcast of \( D \). We next show that every \( P_i \in \mathcal{H} \) will eventually include \( (D, S_p) \) to the set \( C_i \). For this, we need to show that all the conditions which hold for \( D \) in its graph \( G^{(D)} \) when it broadcasts the \( \text{CanCS} \) message, are bound to eventually hold for every \( P_i \in \mathcal{H} \). However, this simply follows from the \( Z_s \)-weak validity, \( Z_s \)-fallback validity, \( Z_s \)-weak consistency and \( Z_s \)-fallback consistency of \( \Pi_{\text{BC}} \) in the asynchronous network (see Theorem 3.2) and the fact that edges are added to consistency graphs, based on the receipt of \( \text{OK}(\cdot, \cdot) \) messages, which are broadcasted through various \( \Pi_{\text{BC}} \) instances. Consequently, any edge \((i,j)\) which is included in \( G^{(D)} \), is bound to be eventually included in the graph \( G^{(i)} \) of every \( P_i \in \mathcal{H} \).
Note that corresponding to every $S_q \in \mathbb{S}_{Z_q}$, every honest $P_i \in W_q$ will have the share $s_q$, which is either made public by $D$ as part of the CanCS message or received from $D$. Next, we note that each $P_i \in \mathcal{H}$ will have $ICSig(P_i, P_s, P_k, s_q)$, corresponding to every $S_q \in \mathbb{S}_{Z_q}$, where $P_i \in W_q$, from every $P_j \in W_q$ and for every $P_k \in \mathcal{P}$. On the other hand, if any corrupt $P_j \in W_q$ has $ICSig(P_i, P_j, P_k, s'_q)$ of any honest $P_i \in W_q$ for any $P_k \in \mathcal{P}$, then $s'_q = s_q$ holds. The proof for this is exactly the same as that of Lemma C.1.

We now show that every honest party $P_i \in S_q \setminus W_q$ eventually sets $[s]_q$ to $s_q$, except with a probability of $O(|\mathbb{S}_{Z_q}|\cdot n^2 \cdot \epsilon_{ICP})$. We first show that $P_i$ sets $[s]_q$ to some value. Since $Z_s$ satisfies the $\mathbb{Q}^{(1)}(W_q, Z_s)$ condition, this means that $W_q$ contains at least one honest party, say $P_j$. Since $P_j$ follows the protocol steps honestly, it reveals $ICSig(P_k, P_j, P_i, [s]_q)$ of every $P_k \in W_q$ to $P_i$. From the $Z_a$-correctness property of ICP in the asynchronous network (see Theorem 4.1), it follows that $P_i$ will eventually accept these signatures. On the other hand, even if $P_k \in W_q$ is corrupt, then from the $Z_a$-non-repudiation property of ICP in the asynchronous network (see Theorem 4.1), it follows that $P_i$ eventually accepts $ICSig(P_k, P_j, P_i, [s]_q)$, except with a probability $\epsilon_{ICP}$. From the union bound, it follows that except with probability $O(n^2 \cdot \epsilon_{ICP})$, each party $P_i \in S_q$ outputs some value for $[s]_q$. Further, this value must be $s_q$. The proof of this follows from what was shown in C.1. Now, there can be $|\mathbb{S}_{Z_q}|$ possibilities for $s_q$. From the union bound, it follows that corresponding to each $S_q \in \mathbb{S}_{Z_q}$, every honest party in $S_q$ eventually outputs $[s]_q$, except with a probability of $O(|\mathbb{S}_{Z_q}|\cdot n^2 \cdot \epsilon_{ICP})$.

Finally, the privacy for $s$ follows from the fact that throughout the protocol, the view of the adversary remains independent of the share $s_q$, corresponding to the group $S_q$, where $S_q = \mathcal{H}$. This is because as shown above, $q \notin BS$ and consequently, $D$ does not make public the share $s_q$. Moreover, during the pairwise consistency tests, the view of the adversary remains independent of $s_q$, when the parties in $\mathcal{H}$ exchange IC-signed $s_q$, which follows from the privacy property of ICP (see Theorem 4.1). Further, while computing the output, $ICSig(P_k, P_j, P_i, s_q)$ is revealed by party $P_j \in W_q$ only to each party $P_i \in S_q \setminus W_q$. Hence, the adversary does not learn $s_q$.

Finally, we consider a corrupt $D$ in the asynchronous network and prove the analogue of Lemma C.2, whose proof is very similar to that of Lemma C.2.

**Lemma C.4.** If the network is asynchronous and $D$ is corrupt and if any honest party computes an output, then there exists some $s^* \in \mathbb{F}$, such that the honest parties eventually output $[s^*]$, except with a probability of $O(|\mathbb{S}_{Z_0}|\cdot n^2 \cdot \epsilon_{ICP})$.

**Proof.** Let $Z^* \in Z_a$ be the set of corrupt parties and let $\mathcal{H} = \mathcal{P} \setminus Z^*$ be the set of honest parties. Let $P_m \in \mathcal{H}$ be the first honest party, who computes an output in the protocol $\Pi_{VSS}$. This implies that there exists some $S_p \in \mathbb{S}_{Z_q}$, such that $P_m$ receives a message $(CanCS, D, S_p, \{W_q\}_{q=1, \ldots, |Z_s|}, BS, \{s_q\}_{q \in BS})$ from the broadcast of $D$, and includes $(D, S_p)$ to $C_m$. This further implies that $P_m$ has verified that the following hold, for $q = 1, \ldots, |Z_s|$.

- If $q \notin BS$, then $W_q$ is either $S_q$ or $(S_q \cap S_q)$. Moreover, $Z_s$ satisfies the $\mathbb{Q}^{(1)}(W_q, Z_s)$ condition and the parties in $W_q$ constitute a clique in the graph $G^{(m)}$. 86
If \( q \in \mathcal{BS} \), then \( \mathcal{D} \) has made public \( s_q \), as part of the \( \text{CanCS} \) message. Moreover, \( W_q \) is set to \( S_q \).

From the \( Z_a \)-weak consistency and \( Z_a \)-fallback consistency of \( \Pi_{BC} \) in the asynchronous network (see Theorem 3.2), it follows that every party in \( \mathcal{H} \) will eventually receive (\( \text{CanCS}, \mathcal{D}, S_p, \{W_q\}_{q=1,\ldots,|Z_s|}, \mathcal{BS}, \{s_q\}_{q \in \mathcal{BS}} \)) from the broadcast of \( \mathcal{D} \). We next show that each \( P_i \in \mathcal{H} \) will eventually include (\( \mathcal{D}, S_p \)) to \( C_i \). For this, it is enough to show that all the edges which are present in \( G^{(i)} \) will eventually receive (\( \text{CanCS}, \mathcal{D}, S_p, \{W_q\}_{q=1,\ldots,|Z_s|}, \mathcal{BS}, \{s_q\}_{q \in \mathcal{BS}} \)) from the broadcast of \( \mathcal{D} \). We define \( (\mathcal{H} \cap \mathcal{W}_q) \) will eventually have the required IC-signatures on \( [s^*]_q \) as part of \( [s^*] \) and will reveal these to parties in \( S_q \setminus W_q \). Consequently, each party in \( \mathcal{H} \) will eventually set \( [s^*]_q \) as the share corresponding to \( S_q \) and hence, \( s^* \) will eventually be secret-shared, except with a probability of \( O(|S_{Z_s}| \cdot n^2 \cdot \epsilon_{ICP}) \).

Next, it can be shown that corresponding to every \( S_q \in S_{Z_s} \), every \( P_i \in W_q \) will have a common share, say \( s^*_q \). The proof for this is the same as Lemma C.2.

We define
\[
s^*_q \overset{def}{=} \sum_{q=1,\ldots,|Z_s|} s^*_q.
\]

Now similar to the proof of Lemma C.2, it can be shown that each party \( P_i \) in \( (\mathcal{H} \cap \mathcal{W}_q) \) will eventually have the required IC-signatures on \( [s^*]_q \) as part of \( [s^*] \) and will reveal these to parties in \( S_q \setminus W_q \). Consequently, each party in \( \mathcal{H} \) will eventually set \( [s^*]_q \) as the share corresponding to \( S_q \) and hence, \( s^* \) will eventually be secret-shared, except with a probability of \( O(|S_{Z_s}| \cdot n^2 \cdot \epsilon_{ICP}) \).

We next derive the communication complexity of the protocol \( \Pi_{VSS} \).

**Lemma C.5.** Protocol \( \Pi_{VSS} \) incurs a communication of \( O(|Z_s| \cdot n^8 \cdot \log |\mathcal{F}| \cdot |\sigma|) \) bits.

**Proof.** In the protocol, \( \mathcal{D} \) needs to send the share \( s_q \) to all the parties in \( S_q \). This incurs a total communication of \( O(|Z_s| \cdot n \cdot \log |\mathcal{F}|) \) bits. There are \( O(|Z_s| \cdot n^3) \) instances of \( \Pi_{Auth} \) invoked, to exchange IC-signed values, during the pairwise consistency tests. From Theorem 4.1, this incurs a total communication of \( O(|Z_s| \cdot n^8 \cdot \log |\mathcal{F}| \cdot |\sigma|) \) bits. There are \( O(n^2) \) OK messages which need to be broadcasted, which from Theorem 3.2, incurs a total communication of \( O(n^6 \cdot \log n \cdot |\sigma|) \) bits, since each OK message encodes the identity of two parties, requiring \( \log n \) bits. Finally, \( \mathcal{D} \) needs to broadcast a candidate \( \langle \text{CanCS}, \mathcal{D}, S_p, \{W_q\}_{q=1,\ldots,|Z_s|}, \mathcal{BS}, \{s_q\}_{q \in \mathcal{BS}} \rangle \) message, where \( S_p, \mathcal{BS} \) and each \( W_q \) can be represented by \( O(n) \) bits. And corresponding to the indices in \( \mathcal{BS} \), the dealer \( \mathcal{D} \) may end up broadcasting \( O(|Z_s|) \) shares. From Theorem 3.2, this incurs a total communication of \( O(|Z_s| \cdot (n^5 \cdot |\sigma| + n^4 \cdot \log |\mathcal{F}| \cdot |\sigma|)) \) bits. While computing the output, \( O(n^3 \cdot |Z_s|) \) instances of \( \Pi_{Reveal} \) are involved, which incur a communication of \( O(|Z_s| \cdot n^4 \cdot \log |\mathcal{F}|) \) bits.

Theorem 5.1 now follows from Lemma C.1-C.5.
Theorem 5.1. Protocol $\Pi_{VSS}$ achieves the following, except with a probability of $O(|S_z| \cdot n^2 \cdot \epsilon_{ICP})$, where $D$ has input $s \in \mathbb{F}$ for $\Pi_{VSS}$ and where $T_{VSS} = \Delta + T_{Auth} + 2T_{BC} + T_{Reveal}$.

- If $D$ is honest, then the following hold.
  - $Z_s$-correctness: In a synchronous network, the honest parties output $[s]$ at time $T_{VSS}$.
  - $Z_a$-correctness: In an asynchronous network, the honest parties eventually output $[s]$.
  - Privacy: Adversary’s view remains independent of $s$ in any network.

- If $D$ is corrupt, then the following hold.
  - $Z_s$-commitment: In a synchronous network, either no honest party obtains any output or there exists some $s^* \in \mathbb{F}$, such that the parties output $[s^*]$. Moreover, if any honest party computes its output corresponding to $[s^*]$ at time $T$, then all honest parties compute their output corresponding to $[s^*]$ by time $T + \Delta$.
  - $Z_a$-commitment: In an asynchronous network, either no honest party obtains any output or there exists some $s^* \in \mathbb{F}$, such that the honest parties eventually output $[s^*]$.
  - Communication Complexity: $O(|Z_s| \cdot n^8 \cdot \log |\mathbb{F}| \cdot |\sigma|)$ bits are communicated by the honest parties.

D Network Agnostic Reconstruction Protocols and Secure Multicast

This section presents our network-agnostic reconstruction protocols and secure multicast protocol, along with their properties. We start with the protocol $\Pi_{RecShare}$ for reconstructing a designated share, presented in Fig 18.

```
Protocol $\Pi_{RecShare}([s],S_q,\mathcal{R})$

| Sending IC-signed Share to the Parties: If $P_i \in W_q$, then reveal $\text{ICSig}(P_j, P_i, [s]_q)$ of every $P_j \in W_q$ to every $P_k \in \mathcal{R} \setminus W_q$. Here $W_q$ denotes the publicly known core-set corresponding to $S_q \in S_z$, as part of $[s]$. |
| Computing Output: If $P_i \in (\mathcal{R} \cap W_q)$, then output $[s]_q$. Else if $P_i \in \mathcal{R} \setminus W_q$, then check if there exists any $P_j \in W_q$ and a value $s_{qj}$, such that $P_i$ has accepted $\text{ICSig}(P_k, P_j, P_i, s_{qj})$ of every $P_k \in W_q$. Upon finding such a $P_j$, output $[s]_q = s_{qj}$. |

```

Fig. 18: Network agnostic reconstruction protocol to reconstruct a designated share $[s]_q$. The above code is executed by each $P_i \in \mathcal{P}$.

We next prove the properties of the protocol $\Pi_{RecShare}$.

Lemma 6.1. Let $s$ be a value which is linearly secret-shared with IC sig-
natures, let $S_q \in \mathbb{S}_Z$, be a designated set and let $R \subseteq P$ be a designated set of receivers. Then protocol $\Pi_{RecShare}$ achieves the following.

- **$Z_q$-correctness:** In a synchronous network, all honest parties in $R$ output $[s]_q$ at time $T_{RecShare} = T_{Reveal}$, except with a probability of $O(n^2 \cdot \epsilon_{ICP})$.
- **$Z_q$-correctness:** In an asynchronous network, all honest parties in $R$ eventually output $[s]_q$, except with a probability of $O(n^2 \cdot \epsilon_{ICP})$.
- **Privacy:** If $R$ consists of only honest parties, then the view of the adversary remains independent of $[s]_q$.
- **Communication Complexity:** $O(|R| \cdot n^3 \cdot \log |\mathbb{F}|)$ bits are communicated.

Proof. We first note that all honest parties in $(R \cap W_q)$ output $[s]_q$ correctly. So consider an arbitrary honest $P_i \in R \setminus W_q$. We first show that $P_i$ indeed computes an output in the protocol, irrespective of the network type.

Since $Z_q$ satisfies the $\Omega^{(1)}(W_q, Z_q)$ condition, it contains at least one honest party, say $P_j$. Since $P_j$ follows the protocol steps honestly, it reveals $\text{ICSig}(P_k, P_j, P_i, [s]_q)$ of every $P_k \in W_q$ to $P_i$. From the correctness properties of ICP (see Theorem 4.1), it follows that $P_i$ will accept the IC-signatures $\text{ICSig}(P_k, P_j, P_i, [s]_q)$, revealed by $P_j$, after time $T_{Reveal}$ in a synchronous network, or eventually in an asynchronous network. On the other hand, even if $P_k \in W_q$ is corrupt, then also from the non-repudiation properties of ICP, it follows that $P_i$ accepts $\text{ICSig}(P_k, P_j, P_i, [s]_q)$, except with a probability $\epsilon_{ICP}$, after time $T_{Reveal}$ in a synchronous network, or eventually in an asynchronous network.

As there can be $O(n)$ corrupt parties in $W_q$, from the union bound, it follows that except with a probability $O(n \cdot \epsilon_{ICP})$, party $P_i$ will find a candidate party from $W_q$, who reveals $[s]_q$, along with the IC-signature of all the parties in $W_q$, after time $T_{Reveal}$ in a synchronous network, or eventually in an asynchronous network. This is because the honest party in $W_q$ always constitutes a candidate party. Now as there can be $O(n)$ parties in $R \setminus W_q$, it follows that except with probability $O(n^2 \cdot \epsilon_{ICP})$, every honest party in $R \setminus W_q$ will find a candidate party from $W_q$, who reveals $[s]_q$, along with the IC-signature of all the parties in $W_q$, after time $T_{Reveal}$ in a synchronous network, or eventually in an asynchronous network. Hence all honest parties in $R$ compute an output, after time $T_{Reveal}$ in a synchronous network, or eventually in an asynchronous network, except with a probability $O(n^2 \cdot \epsilon_{ICP})$.

We next show that the output computed by all the honest parties is indeed correct. While this is trivially true for the parties in $(R \cap W_q)$, consider an arbitrary honest party $P_i \in R \setminus W_q$. The above argument shows that $P_i$ computes an output in the protocol, irrespective of the network type. So let $P_i$ output $s'$. We wish to show that $s' = [s]_q$. From the protocol steps, since $P_i$ outputs $s'$, it implies that there exists some $P_j \in W_q$, such that $P_i$ has accepted $\text{ICSig}(P_k, P_j, P_i, s')$ of every $P_k \in W_q$, revealed by $P_j$. If $P_j$ is honest, then indeed $s' = [s]_q$, as one of the IC-signatures $\text{ICSig}(P_k, P_j, P_i, s')$ is the same as $\text{ICSig}(P_k, P_j, P_i, [s]_q)$, corresponding to the honest $P_k \in W_q$, which is guaranteed to exist. So consider the case when $P_j$ is corrupt. Moreover, let $P_k \in W_q$ be an honest party (which is guaranteed to exist). In order that $s' \neq [s]_q$, it must be the case that $P_i$ accepts $\text{ICSig}(P_k, P_j, P_i, s')$, revealed by $P_j$. However, from the unforgeability property of
ICP (see Theorem 4.1), this can happen only with probability $\epsilon_{\text{ICP}}$. Now as there can be up to $O(n)$ corrupt parties in $W_q$, from the union bound, it follows that the probability that $P_i$ outputs $s' \neq [s]_q$ is at most $O(n \cdot \epsilon_{\text{ICP}})$. And since there can be up to $O(n)$ parties in $W_q$, it follows that except with probability at most $O(n^2 \cdot \epsilon_{\text{ICP}})$, the output of every honest party in $R$ is indeed $[s]_q$.

Communication complexity follows from the communication complexity of $\Pi_{\text{Reveal}}$ (Theorem 4.1) and the fact that $O(|R| \cdot n^2)$ instances of $\Pi_{\text{Reveal}}$ are involved. And privacy follows from the privacy of ICP.

Protocol $\Pi_{\text{Rec}}$ for reconstructing $s$ by a designated set of receivers is presented in Fig 19.

**Protocol $\Pi_{\text{Rec}}([s], R)$**

- **Reconstructing Individual Shares**: Corresponding to each $S_q \in S_{Z_s}$, participate in an instance $\Pi_{\text{RecShare}}([s], S_q, R)$ of $\Pi_{\text{RecShare}}$, to let the parties in $R$ reconstruct $[s]_q$.
- **Computing Output**: If $P_i \in R$, then output $s = \sum_{S_q \in S} [s]_q$.

Fig. 19: Network agnostic reconstruction protocol to reconstruct a secret-shared value with IC signatures. The above code is executed by each party $P_i \in P$.

The properties of the protocol $\Pi_{\text{Rec}}$ are stated in Lemma 6.2.

**Lemma 6.2.** Let $s$ be a value which is linearly secret-shared with IC signatures and let $R \subseteq P$ be a set of designated receivers. Then protocol $\Pi_{\text{Rec}}$ achieves the following.

- **$Z_x$-correctness**: In a synchronous network, all honest parties in $R$ output $s$ at time $T_{\text{Rec}} = T_{\text{RecShare}}$, except with probability $O(|S_{Z_s}| \cdot n^2 \cdot \epsilon_{\text{ICP}})$.
- **$Z_x$-correctness**: In an asynchronous network, all honest parties in $R$ eventually output $s$, except with probability $O(|S_{Z_s}| \cdot n^2 \cdot \epsilon_{\text{ICP}})$.
- **Privacy**: If $R$ consists of only honest parties, then the view of the adversary remains independent of $s$.
- **Communication Complexity**: $O(|Z_s| \cdot |R| \cdot n^3 \cdot \log |F|)$ bits are communicated.

**Proof.** The proof follows from Lemma 6.1, and the fact that $|S_{Z_s}|$ instances of $\Pi_{\text{RecShare}}$ are invoked.

**D.1 Network Agnostic Secure Multicast**

Protocol $\Pi_{\text{SVM}}$ is presented in Fig 20.

**Protocol $\Pi_{\text{SVM}}(\text{Sen}, v, R)$**

- **Sending the Value to the Parties**: $\text{Sen}$ on having the input $v$, invokes an instance of $\Pi_{\text{VSS}}$ with input $v$ and the parties in $P$ participates in this instance.
- Verifying if Sender has Committed Any Value: Each \( P_i \in \mathcal{P} \) waits till its local time becomes \( T_{\text{VSS}} \), initializes a Boolean variable \( \text{flag}^{(\text{Sen}, \mathcal{R})} \) to 0 and then do the following.
- Upon computing an output in the \( \Pi_{\text{VSS}} \) instance, set \( \text{flag}^{(\text{Sen}, \mathcal{R})} \) to 1.
- Upon setting \( \text{flag}^{(\text{Sen}, \mathcal{R})} \) to 1, participate in an instance \( \Pi_{\text{Rec}}([v], \mathcal{R}) \) of \( \Pi_{\text{Rec}} \) to let the parties in \( \mathcal{R} \) reconstruct \( v \).

Computing Output: Each \( P_i \in \mathcal{R} \) upon computing an output \( v \) during the instance \( \Pi_{\text{Rec}}([v], \mathcal{R}) \), outputs \( v \).

Fig. 20: The network agnostic SVM protocol

Lemma 6.3. Protocol \( \Pi_{\text{SVM}} \) achieves the following, where \( \text{Sen} \) participates with input \( v \) and where each honest party initializes \( \text{flag}^{(\text{Sen}, \mathcal{R})} \) to 0.

- Synchronous Network: If \( \text{Sen} \) is honest, then all honest parties set \( \text{flag}^{(\text{Sen}, \mathcal{R})} \) to 1 at time \( T_{\text{VSS}} \) and except with probability \( O(n^3 \cdot \epsilon_{\text{ICP}}) \), all honest parties in \( \mathcal{R} \) output \( v \), after time \( T_{\text{SVM}} = T_{\text{VSS}} + T_{\text{Rec}} \). Moreover, if \( \mathcal{R} \) consists of only honest parties, then the view of \( \text{Adv} \) remains independent of \( v \). If \( \text{Sen} \) is corrupt and some honest party sets \( \text{flag}^{(\text{Sen}, \mathcal{R})} \) to 1, then there exists some \( v^* \) such that, except with probability \( O(n^3 \cdot \epsilon_{\text{ICP}}) \), all honest parties in \( \mathcal{R} \) output \( v^* \). Moreover, if any honest party sets \( \text{flag}^{(\text{Sen}, \mathcal{R})} \) to 1 at time \( T \), then all honest parties in \( \mathcal{R} \) output \( v^* \) by time \( T + 2 \Delta \).

- Asynchronous Network: If \( \text{Sen} \) is honest, then except with probability \( O(n^3 \cdot \epsilon_{\text{ICP}}) \), all honest parties in \( \mathcal{R} \) eventually output \( v \). Moreover, if \( \mathcal{R} \) consists of only honest parties, then the view of the adversary remains independent of \( v \). If \( \text{Sen} \) is corrupt and some honest party sets \( \text{flag}^{(\text{Sen}, \mathcal{R})} \) to 1, then there exists some \( v^* \) such that, except with probability \( O(n^3 \cdot \epsilon_{\text{ICP}}) \), all honest parties in \( \mathcal{R} \) eventually output \( v^* \).

- Communication Complexity: \( O(|\mathcal{Z}_s| \cdot n^8 \cdot \log |\mathcal{F}| \cdot |\sigma|) \) bits are communicated.

Proof. Let us first consider an honest \( \text{Sen} \). If the network is synchronous, then from the \( \mathcal{Z}_s \)-correctness of \( \Pi_{\text{VSS}} \) in the synchronous network (Theorem 5.1), at time \( T_{\text{VSS}} \), all honest parties will output \([v]\). Consequently, each honest party will set \( \text{flag}^{(\text{Sen}, \mathcal{R})} \) to 1 and start participating in the instance of \( \Pi_{\text{Rec}} \). Hence, from the \( \mathcal{Z}_s \)-correctness of \( \Pi_{\text{Rec}} \) in the synchronous network (Lemma 6.2), coupled with the modifications presented in Section 6.1, it follows that all honest parties in \( \mathcal{R} \) output \( v \), except with probability \( O(n^3 \cdot \epsilon_{\text{ICP}}) \), after time \( T_{\text{SVM}} = T_{\text{VSS}} + T_{\text{Rec}} \). The privacy of \( v \) follows from the privacy of \( \Pi_{\text{VSS}} \) (Theorem 5.1) and privacy of \( \Pi_{\text{Rec}} \) (Lemma 6.2). The proof for the case of honest \( \text{Sen} \) in an asynchronous network is the same as above, except that we now rely on the \( \mathcal{Z}_a \)-correctness of \( \Pi_{\text{VSS}} \) in the asynchronous network (Theorem 5.1) and the \( \mathcal{Z}_a \)-correctness of \( \Pi_{\text{Rec}} \) in the asynchronous network (Lemma 6.2).

Next consider a corrupt \( \text{Sen} \). Let us first consider a synchronous network. Let \( P_i \) be the first honest party who sets \( \text{flag}^{(\text{Sen}, \mathcal{R})} \) to 1. This implies that there exists some \( v^* \), such that \( P_i \) outputs \([v^*]\). Let \( T \) be the time when \( P_i \) outputs \([v^*]\) during the instance of \( \Pi_{\text{VSS}} \) (and hence sets \( \text{flag}^{(\text{Sen}, \mathcal{R})} \) to 1). From the
Zs-commitment of \( \Pi_{VSS} \) in the synchronous network, it follows that all honest parties will output \([v^*]\) (and hence set flag\(^{(\text{Sen}, R)}\) to 1), latest by time \( T + \Delta \). Hence all honest parties will start participating in the instance of \( \Pi_{\text{Rec}} \), latest by time \( T + \Delta \). Hence, from the \( Z_s \)-correctness of \( \Pi_{\text{Rec}} \) in the synchronous network (Lemma 6.2), coupled with the modifications presented in Section 6.1, it follows that all honest parties in \( \mathcal{R} \) output \( v^* \), except with probability \( O(n^3 \cdot \epsilon_{ICP}) \), by time \( T + 2\Delta \).

The proof for the case of a corrupt \( \text{Sen} \) in an asynchronous network is the same as above, except that we now rely on the \( Z_s \)-commitment of \( \Pi_{VSS} \) in the asynchronous network (Theorem 5.1) and the \( Z_s \)-correctness of \( \Pi_{\text{Rec}} \) in the asynchronous network (Lemma 6.2).

The communication complexity follows from the communication complexity of \( \Pi_{VSS} \) and \( \Pi_{\text{Rec}} \).}

### E Properties of the Protocol \( \Pi_{MDVSS} \)

In this section, we prove the properties of the protocol \( \Pi_{MDVSS} \) (see Fig 8 for the formal description).

We first show that if the network is synchronous, then all honest parties will compute a common candidate set of committed dealers \( CD \) set by time \( T_{\text{SVM}} + 2T_{\text{BA}} \), such that all honest dealers are guaranteed to be present in \( CD \).

**Lemma E.1.** If the network is synchronous and \( P_i \in \mathcal{P} \) is an honest dealer participating with input \( s^{(t)} \), then all the following hold in \( \Pi_{MDVSS} \), where \( \mathcal{H} \) is the set of honest parties.

- Except with probability \( O(n^3 \cdot \epsilon_{ICP}) \), all the parties in \( \mathcal{H} \) will have a common \( CD \) set by time \( T_{\text{SVM}} + 2T_{\text{BA}} \), where \( \mathcal{H} \subseteq CD \).
- Except with probability \( O(n^3 \cdot \epsilon_{ICP}) \), corresponding to every dealer \( P_i \in CD \) and every \( S_q \in [\mathbb{S}_{Z_s}] \), every party in \( (\mathcal{H} \cap S_q) \) will have a common share, say \( s_q^{(t)} \), which is the same as \( s_q^{(t)} \), for an honest \( P_i \).

**Proof.** Let \( Z^* \in Z_s \) be the set of corrupt parties and let \( \mathcal{H} = \mathcal{P} \setminus Z^* \) be the set of honest parties. From the properties of \( \Pi_{\text{SVM}} \) in the synchronous network (Lemma 6.3), it follows that every \( P_i \in \mathcal{H} \) will set flag\(^{(P_i,S_q)}\) to 1 at time \( T_{\text{VSS}} \), corresponding to every \( P_i \in \mathcal{H} \) and every \( S_q \in [\mathbb{S}_{Z_s}] \). Moreover, at time \( T_{\text{SVM}} \), corresponding to every \( P_i \in \mathcal{H} \) and every \( S_q \in [\mathbb{S}_{Z_s}] \), each \( P_i \in (\mathcal{H} \cap S_q) \) computes the output \( s_q^{(t)} \) during the instance \( \Pi_{\text{SVM}}^{(P_i,S_q)} \). Furthermore, except with probability \( O(n^3 \cdot \epsilon_{ICP}) \), the value \( s_q^{(t)} \) will be the same as \( s_q^{(t)} \). Hence, corresponding to every \( P_i \in \mathcal{H} \), every \( P_i \in \mathcal{H} \) will start participating with input 1 during the instance \( \Pi_{\text{BA}}^{(t)} \) at time \( T_{\text{SVM}} \). Hence from the \( Z_s \)-validity of \( \Pi_{\text{BA}} \) in the synchronous network (Theorem 3.1), at time \( T_{\text{SVM}} + T_{\text{BA}} \), all the parties in \( \mathcal{H} \) will obtain the output 1 from the \( \Pi_{\text{BA}}^{(t)} \) instances, corresponding to each \( P_i \in \mathcal{H} \). Since \( \mathcal{P} \setminus \mathcal{H} \in Z_s \), it follows that at time \( T_{\text{SVM}} + T_{\text{BA}} \), all the parties in \( \mathcal{H} \) start

---

\(^{18}\) This automatically implies that \( \mathcal{P} \setminus CD \in Z_s \).
participating with input 0 in any remaining instance \( \pi_{BA}^{(s)} \) of \( \pi_{BA} \), for which no input is provided yet. Hence from the \( Z_s \)-security of \( \Pi_{BA} \) in the synchronous network (Theorem 3.1), all the parties in \( \mathcal{H} \) will compute some output in all the \( n \) instances of \( \pi_{BA}^{(s)} \) by time \( T_{SVM} + 2T_{BA} \). Moreover, the outputs will be common for the parties in \( \mathcal{H} \). Consequently, all the parties in \( \mathcal{H} \) will have a common CD set at the time \( T_{SVM} + 2T_{BA} \). Moreover, \( \mathcal{H} \subseteq CD \), since CD includes all the dealers \( P_t \) such that \( \pi_{BA}^{(t)} \) outputs 1. And as shown above the \( \pi_{BA}^{(t)} \) instances corresponding to \( P_t \in \mathcal{H} \) outputs 1.

Next, consider an arbitrary \( P_t \in CD \). This implies that at time \( T_{SVM} + T_{BA} \), at least one party from \( \mathcal{H} \), say \( P_k \), has participated with input 1 during the instance \( \pi_{BA}^{(t)} \). If not, then from the \( Z_s \)-validity of \( \pi_{BA} \) in the synchronous network (Theorem 3.1), all the parties in \( \mathcal{H} \) would have obtained the output 0 from the instance \( \pi_{BA}^{(t)} \) at the time \( T_{SVM} + 2T_{BA} \) and hence \( P_t \notin CD \), which is a contradiction. This implies that by the time \( T_{SVM} + T_{BA} \), party \( P_k \) has set \( \text{flag}^{(P_t, s_q)} \) to 1 during the instance \( \pi_{SVM}(P_t, s_q^{(t)}, s_q) \), for \( q = 1, \ldots, |Z_s| \). So consider an arbitrary \( S_q \in S_Z \). From the properties of \( \pi_{SVM} \) in the synchronous network (Lemma 6.3), it follows that there exists some value \( s_q^{* (t)} \), which is the same as \( s_q \) for an honest \( P_t \), such that except with probability \( O(n^3 \cdot \epsilon_{ICP}) \), all the parties in \( \mathcal{H} \) output \( s_q^{* (t)} \) during the instance \( \pi_{SVM}^{(P_t, S_q)} \) by time \( T_{SVM} + T_{BA} + 2\Delta < T_{SVM} + 2T_{BA} \).

We next show that if the network is synchronous and if an honest dealer from CD broadcasts any set of candidate core-sets, then all honest parties will “accept” the core-sets. Moreover, the dealer will never make public the share corresponding to the group from \( S_Z \), consisting of only honest parties while making public these core-sets. Furthermore, each honest dealer in CD will start making public at least one candidate set of core-sets, namely the one computed with respect to the group from \( S_Z \), consisting of only honest parties. A consequence of all these properties is that if the dealer is honest, the adversary will not learn any information about the dealer’s input.

**Lemma E.2.** If the network is synchronous and \( P_t \in CD \) is an honest dealer participating with input \( s^{(t)} \), then all the following hold in \( \Pi_{MDVSS} \) except with probability \( O(n^3 \cdot \epsilon_{ICP}) \), where \( \mathcal{H} \) is the set of honest parties.

- If \( S_p = \mathcal{H} \), then \( P_t \) will broadcast \( (\text{CanCS}, P_t, S_p, \{\mathcal{W}_{P,q}^{(t)}\}_{q=1,\ldots,|Z_s|}, BS_p^{(t)}, \{s_q^{(t)}\}_{q \in BS_p^{(t)}}) \) at time \( T_{SVM} + 2T_{BA} + T_{Auth} + T_{BC} \).
- If \( P_t \) broadcasts any \( (\text{CanCS}, P_t, S_p, \{\mathcal{W}_{P,q}^{(t)}\}_{q=1,\ldots,|Z_s|}, BS_p^{(t)}, \{s_q^{(t)}\}_{q \in BS_p^{(t)}}) \) at time \( T \), then every honest \( P_t \in \mathcal{P} \) will include \( (P_t, S_p) \) to \( \mathcal{C}_t \) at time \( T + T_{BC} \). Moreover, the following will hold.
  - If \( S_q = \mathcal{H} \), then \( q \notin BS_p^{(t)} \).
  - For \( q = 1, \ldots, |Z_s| \), each \( \mathcal{W}_{P, q}^{(t)} \) will be either \( S_q \) or \( S_p \cap S_q \). Moreover, \( Z_s \) will satisfy the \( Q^{(t)}(W_{P,q}, Z_s) \) condition.
  - If \( q \notin BS_p^{(t)} \), then every honest \( P_t \in S_q \) will have the share \( s_q^{(t)} \). Moreover, every honest \( P_t \in \mathcal{W}_{P,q}^{(t)} \) will have \( \text{ICSig}(P_j, P_t, P_k, s_q^{(t)}) \) of every \( P_j \in \mathcal{P} \).
follows. Moreover, if any corrupt\( P_j \in W_{p,q}^{(l)} \) have
\( \text{ICSig}(P_i, P_j, P_k, s_{q}^{(l)}) \) of any honest \( P_i \in W_{p,q}^{(l)} \) for any \( P_k \in \mathcal{P} \), then
\( s_{q}^{(l)} = s_{q}^{(l)} \) holds. Also, all the underlying IC-signatures will satisfy the linearity property.

- The view of the adversary will be independent of \( s^{(l)} \).

Proof. Let \( Z^{*} \in Z_s \) be the set of corrupt parties and let \( \mathcal{H} = \mathcal{P} \setminus Z^{*} \) be the set of honest parties. Since the dealer \( P_i \) is honest, from Lemma E.1 it follows that corresponding to every \( S_q \in S_{Z_s} \), every party in \( (\mathcal{H} \setminus S_q) \) will have the share \( s_{q}^{(l)} \), by time \( T_{\text{SVM}} + 2T_{\text{BA}} \), except with probability \( O(n^{3}, \epsilon_{\text{ICP}}) \), where \( s_{q}^{(l)} = s_{q}^{(l)} \). Now consider an arbitrary \( S_q \in S_{Z_s} \). At time \( T_{\text{SVM}} + 2T_{\text{BA}} \), each party \( P_i \in (S_q \cap \mathcal{H}) \) starts giving \( \text{ICSig}(P_i, P_j, P_k, s_{q}^{(l)}) \) to every \( P_j \in S_q \), for every \( P_k \in \mathcal{P} \), where \( s_{q}^{(l)} = s_{q}^{(l)} \) holds. Then from the \( Z_s \)-correctness of \( \mathcal{H}_{\text{Auth}} \) in the synchronous network (Theorem 4.1), it follows that at time \( T_{\text{SVM}} + 2T_{\text{BA}} + T_{\text{Auth}} \), each party \( P_i \in (S_q \cap \mathcal{H}) \) will receive \( \text{ICSig}(P_j, P_k, s_{q}^{(l)}) \) from every \( P_j \in (S_q \cap \mathcal{H}) \), for every \( P_k \in \mathcal{P} \), such that \( s_{q}^{(l)} = s_{q}^{(l)} = s_{q}^{(l)} \) holds. Since \( S_q \) is arbitrary, it follows that at time \( T_{\text{SVM}} + 2T_{\text{BA}} + T_{\text{Auth}} \), every party \( P_i \in \mathcal{H} \) broadcasts an \( \text{OK}^{(l)}(i, j) \) message, corresponding to every \( P_j \in \mathcal{H} \). From the \( Z_s \)-validity of \( \Pi_{\text{BC}} \) in the synchronous network (Theorem 3.2), it follows that these \( \text{OK}^{(l)}(i, j) \) messages are received by every party in \( \mathcal{H} \) through regular-mode at time \( T_{\text{SVM}} + 2T_{\text{BA}} + T_{\text{Auth}} + T_{\text{BC}} \).

Consequently, the set \( \mathcal{H} \) constitutes a clique in the consistency graph \( G^{(l, i)} \) of every party \( P_i \in \mathcal{H} \) at time \( T_{\text{SVM}} + 2T_{\text{BA}} + T_{\text{Auth}} + T_{\text{BC}} \). Note that the set \( \mathcal{H} \subseteq S_{Z_s} \). Let \( S_p \) be the set from \( S_{Z_s} \), such that \( S_p = \mathcal{H} \). From the protocol steps, it then follows that at time \( T_{\text{SVM}} + 2T_{\text{BA}} + T_{\text{Auth}} + T_{\text{BC}} \), the dealer \( P_i \) will compute core-sets \( \mathcal{W}_{p,q}^{(l)} \) for \( q = 1, \ldots, |Z_s| \) and broadcast-set \( \mathcal{B}_{S_p}^{(l)} \) with respect to \( S_p \) as follows.

- If \( S_q \) constitutes a clique in the graph \( G^{(l, l)} \), then \( \mathcal{W}_{p,q}^{(l)} \) is set as \( S_q \).
- Else if \( (S_p \cap S_q) \) constitutes a clique in \( G^{(l, l)} \) and \( Z_s \) satisfies the \( Q^{(l)}(S_p \cap S_q, Z_s) \) condition, then \( \mathcal{W}_{p,q}^{(l)} \) is set as \( (S_p \cap S_q) \).
- Else \( \mathcal{W}_{p,q}^{(l)} \) is set to \( S_q \) and \( q \) is included to \( \mathcal{B}_{S_p}^{(l)} \).

After computing the core-sets and broadcast-set, \( P_i \) will broadcast \( \{\text{CanCS}, P_i, S_p, \{\mathcal{W}_{p,q}^{(l)}\}_{q=1, \ldots, |Z_s|}, \mathcal{B}_{S_p}^{(l)}, \{s_{q}^{(l)}\}_{q \in \mathcal{B}_{S_p}^{(l)}}\} \) at time \( T_{\text{SVM}} + 2T_{\text{BA}} + T_{\text{Auth}} + T_{\text{BC}} \). This proves the first part of the lemma.

We next proceed to prove the second part of the lemma. So consider an arbitrary \( S_p \in S_{Z_s} \), such that \( P_i \) compute core-sets \( \mathcal{W}_{p,q}^{(l)} \) for \( q = 1, \ldots, |Z_s| \) and broadcast-set \( \mathcal{B}_{S_p}^{(l)} \) with respect to \( S_p \) and broadcasts \( \{\text{CanCS}, P_i, S_p, \{\mathcal{W}_{p,q}^{(l)}\}_{q=1, \ldots, |Z_s|}, \mathcal{B}_{S_p}^{(l)}, \{s_{q}^{(l)}\}_{q \in \mathcal{B}_{S_p}^{(l)}}\} \) at time \( T \). This means at time \( T \), the parties in \( S_p \) constitute a clique in the graph \( G^{(l, l)} \). We also note that \( T \geq T_{\text{SVM}} + 2T_{\text{BA}} + T_{\text{Auth}} + T_{\text{BC}} \). This is because any instance of \( \Pi_{\text{BC}} \) takes at least \( T_{\text{BC}} \) time in a synchronous network to generate an output. And the parties in \( \mathcal{H} \) start participating in any \( \Pi_{\text{BC}} \) instance invoked for broadcasting any \( \text{OK}^{(l)}(\ast, \ast) \) message, only after time \( T_{\text{SVM}} + 2T_{\text{BA}} + T_{\text{Auth}} \). Consequently, any
We also note that any edge \((P, Z)\) of time \(T\) at time \(T\) is bound to be present in the graph \(G_{\ell}(\cdot, \cdot)\) of every \(P_i \in \mathcal{H}\), latest by time \(T + \Delta\). This is because the edge \((P, P)\) is added to \(G_{\ell}(\cdot, \cdot)\) upon the receipt of \(OK_{\ell}(\cdot, \cdot)\) and \(OK_{\ell}(\cdot, \cdot)\) messages from the broadcast of \(P_j\) and \(P_k\) respectively. And from the \(Z_s\)-validity, \(Z_s\)-consistency and \(Z_s\)-fallback consistency of \(\Pi_{BC}\) in the synchronous network, these \(OK_{\ell}(\cdot, \cdot)\) messages will be received by every party \(P_i \in \mathcal{H}\), latest by time \(T + \Delta\). Since \(P_i\) is assumed to be honest, it follows that the sets \(\{W_{p,q}\}_{q=1, \ldots, \ell, \ell}\) and \(BS_{p}^{(\ell)}\) satisfy the following properties.

- If \(S_q\) constitutes a clique in the graph \(G_{\ell}(\cdot, \cdot)\), then \(W_{p,q}^{(\ell)}\) is set as \(S_q\).
- Else if \((S_p \cap S_q)\) constitutes a clique in \(G_{\ell}(\cdot, \cdot)\) and \(Z_s\) satisfies the \(Q^{(1)}(S_p \cap S_q, Z_s)\) condition, then \(W_{p,q}^{(\ell)}\) is set as \((S_p \cap S_q)\).
- Else \(W_{p,q}^{(\ell)}\) is set to \(S_q\) and \(q\) is included to \(BS_{p}^{(\ell)}\).

We also note that if \(S_q = \mathcal{H}\), then \(q \notin BS_{p}^{(\ell)}\) and consequently, \(P_i\) will not make the share \(s_q^{(\ell)}\) public. This is because \(T \geq T_{\text{SVM}} + 2T_{\text{BA}} + T_{\text{Auth}} + T_{\text{BC}}\). And as shown in the proof of the first part, the set \(\mathcal{H}\) will constitute a clique in the graph \(G_{\ell}(\cdot, \cdot)\) at time \(T_{\text{SVM}} + 2T_{\text{BA}} + T_{\text{Auth}} + T_{\text{BC}}\). Since \(P_i\) is honest, from the \(Z_{s}\)-validity of \(\Pi_{BC}\) in the synchronous network, it follows that all the parties in \(\mathcal{H}\) will receive \((\text{CanCS}, P, S_p, \{W_{p,q}^{(\ell)}\}_{q=1, \ldots, \ell, \ell}, BS_{p}^{(\ell)}, \{s_q^{(\ell)}\}_{q \in BS_{p}^{(\ell)}})\) through the regular-mode at time \(T + T_{\text{BC}}\). Moreover, each party \(P_i \in \mathcal{H}\) will include \((P_i, S_p)\) to the set \(C_i\) at time \(T + T_{\text{BC}}\). This is because since \(P_i\) has computed the sets \(\{W_{p,q}^{(\ell)}\}_{q=1, \ldots, \ell, \ell}\) and \(BS_{p}^{(\ell)}\) honestly, these sets will pass all the verifications for each \(P_i \in \mathcal{H}\) at time \(T + \Delta\).

Next consider an arbitrary \(q \notin BS_{p}^{(\ell)}\). This implies that \(P_i\) has set \(W_{p,q}^{(\ell)}\) as \((S_p \cap S_q)\) because the parties in \((S_p \cap S_q)\) constitutes a clique in the graph \(G_{\ell}(\cdot, \cdot)\). Now consider an arbitrary \(P_i \in (\mathcal{H} \cap S_q)\). This implies that \(P_i\) has computed \(s_q^{(\ell)}\) during the instance \(\Pi_{\text{SVM}}(P_i, s_q^{(\ell)}, S_q)\) at time \(T_{\text{SVM}}\), which will be the same as \(s_q^{(\ell)}\), since \(P_i\) is honest. Next consider arbitrary \(P_i, P_j \in W_{p,q}^{(\ell)}\), such that \(P_j \neq P_i\). This implies that the edge \((i, j)\) is present in the graph \(G_{\ell}(\cdot, \cdot)\), which further implies that \(P_i\) has broadcasted the message \(OK_{\ell}(i, j)\). This further implies that \(P_i\) must have received \(\text{ICSig}(P_j, P, P_k, s_q^{(\ell)})\) from \(P_j\), for every \(P_k \in \mathcal{P}\), such that \(s_q^{(\ell)} = s_q^{(\ell)}\) holds. Since \(s_q^{(\ell)} = s_q^{(\ell)}\), it follows that \(\text{ICSig}(P_j, P, P_k, s_q^{(\ell)})\) is the same as \(\text{ICSig}(P_j, P, P_k, s_q^{(\ell)})\). On the other hand, consider an arbitrary \(P_j \in W_{p,q}^{(\ell)}\), such that \(P_j\) is corrupt and where \(P_j\) has received \(\text{ICSig}(P, P, P_k, s_q^{(\ell)})\) from \(P_i\), for any \(P_k \in \mathcal{P}\). Then from the protocol steps, it follows that \(s_q^{(\ell)} = s_q^{(\ell)}\), since \(P_i\) gives the IC-signature on the share \(s_q^{(\ell)}\), received from \(P_k\). And since \(s_q^{(\ell)} = s_q^{(\ell)}\), it follows that \(\text{ICSig}(P, P, P_k, s_q^{(\ell)})\) is the same as \(\text{ICSig}(P, P, P_k, s_q^{(\ell)})\). The linearity of the underlying IC-signatures follow from the fact the parties follow the linearizer principle while generating IC-signatures.

Finally, the privacy of \(s^{(\ell)}\) follows from the fact that throughout the protocol, adversary does not learn anything about the share \(s_q^{(\ell)}\), provided \(S_q = \mathcal{H}\).
Namely, during the instance $H_{\text{SVM}}(P_t, s_q^{(t)}, S_q)$ where $(S_q \cap Z^*) = \emptyset$, the view of the adversary remains independent of $s_q^{(t)}$, which follows from the privacy of $H_{\text{SVM}}$ (Lemma 6.3). Moreover, as shown above, $P_t$ never makes public the share $s_q^{(t)}$, as $q \notin BS_p^{(t)}$. Furthermore, since the set $H$ will consists of only honest parties, from the privacy of ICP (see Theorem 4.1), it follows that the adversary does not learn any additional information about $s_q^{(t)}$, when the parties in $H$ exchange IC-signed $s_q^{(t)}$ during the pairwise consistency tests.

We now next that if the network is synchronous, then any candidate set of core-sets “accepted” on the behalf of a corrupt dealer by any honest party at the time $T$, is bound to be accepted by all honest parties, latest by time $T + \Delta$.

**Lemma E.3.** If the network is synchronous and if in $H_{\text{MDVSS}}$ any honest party $P_i$ receives $(\text{CanCS}, P_t, S_p, \{W_p^{(t)}\}_{q=1,...,|Z_s|}, BS_p^{(t)}, \{s_q^{(t)}\}_{q \in BS_p^{(t)}})$ from the broadcast of any corrupt dealer $P_t \in CD$ and includes $(P_t, S_p)$ to $C_i$ at time $T$, then all honest parties $P_j$ will receive $(\text{CanCS}, P_t, S_p, \{W_p^{(t)}\}_{q=1,...,|Z_s|}, BS_p^{(t)}, \{s_q^{(t)}\}_{q \in BS_p^{(t)}})$ from the broadcast of $P_t$ and include $(P_t, S_p)$ to $C_j$ by time $T + \Delta$. Moreover, for $q = 1, \ldots, |Z_s|$, the following holds, except with probability $O(n^3 \cdot \alpha_{\text{ICP}})$.

- $W_{p,q}^{(t)}$ is either $S_q$ or $(S_p \cap S_q)$. Moreover, $Z_s$ satisfies the $Q^{(1)}(W_{p,q}^{(t)}, Z_s)$ condition.

- If $q \notin BS_p^{(t)}$, then every honest $P_i \in S_q$ will have a common share, say $s_q^{(t)}$. Moreover, every honest $P_i \in W_{p,q}^{(t)}$ will have $\text{ICSig}(P_j, P_i, P_k, s_q^{(t)})$ of every $P_j \in W_{p,q}^{(t)}$ and for every $P_k \in P$. Furthermore, if any corrupt $P_j \in W_{p,q}^{(t)}$ has $\text{ICSig}(P_i, P_j, P_k, s_q^{(t)})$ of any honest $P_i \in W_{p,q}^{(t)}$ for any $P_k \in P$, then $s_q^{(t)} = s_q^{(t)}$. Also, all the underlying IC-signatures will satisfy the linearity principle.

**Proof.** The proof follows very closely the proof of Lemma E.2. Let $Z^* \in Z_s$ be the set of corrupt parties and let $H = P \setminus Z^*$ be the set of honest parties. Now consider an arbitrary corrupt dealer $P_t \in CD$ and an arbitrary $P_i \in H$, such that $P_t$ receives $(\text{CanCS}, P_t, S_p, \{W_p^{(t)}\}_{q=1,...,|Z_s|}, BS_p^{(t)}, \{s_q^{(t)}\}_{q \in BS_p^{(t)}})$ from the broadcast of $P_t$ and includes $(P_t, S_p)$ to $C_i$ at time $T$. Now consider another arbitrary $P_j \in H$, such that $P_j \neq P_i$. From the $Z_s$-consistency and $Z_s$-fallback consistency of $H_{\text{BC}}$ in the synchronous network, it follows that $P_j$ is bound to receive $(\text{CanCS}, P_t, S_p, \{W_p^{(t)}\}_{q=1,...,|Z_s|}, BS_p^{(t)}, \{s_q^{(t)}\}_{q \in BS_p^{(t)}})$ from the broadcast of $P_t$, latest by time $T + \Delta$. We wish to show that $P_j$ will include $(P_t, S_p)$ to $C_j$, by time $T + \Delta$. For this, we note that since $P_t$ has included $(P_t, S_p)$ to $C_i$ at time $T$, all the following conditions hold for $P_t$ at time $T$, for $q = 1, \ldots, |Z_s|$.

- If $q \in BS_p^{(t)}$, then the set $W_{p,q}^{(t)} = S_q$.

- If $(q \notin BS_p^{(t)})$, then $W_{p,q}^{(t)}$ is either $S_q$ or $(S_p \cap S_q)$, such that:

  - If $W_{p,q}^{(t)} = S_q$, then $S_q$ constitutes a clique in $G^{(t)}$. 

96
- Else if \( \mathcal{W}_{p,q}^{(t)} = (S_p \cap S_q) \), then \((S_p \cap S_q)\) constitutes a clique in \( G^{(t,i)} \) and \( Z_s \) satisfies the \( Q^{(1)}(S_p \cap S_q, Z_s) \) condition.

We claim that all the above conditions will hold even for \( P_j \) by time \( T + \Delta \). This is because all the edges which are present in the consistency graph \( G^{(t,i)} \) at time \( T \) are bound to be present in the consistency graph \( G^{(t,i)} \) by time \( T + \Delta \). This follows from the \( Z_s \)-validity, \( Z_s \)-consistency and \( Z_s \)-fallback consistency of \( \Pi_{BC} \) in the synchronous network (see Theorem 3.2) and the fact that the edges in the graph \( G^{(t,i)} \) are based on \( \text{OK}^{(t)}(s,*) \) messages, which are received through various \( \Pi_{BC} \) instances.

Next consider an arbitrary \( q \notin B_S^{(t)} \). This implies that \( \mathcal{W}_{p,q}^{(t)} \) is set as \((S_p \cap S_q)\) and all the parties in \((S_p \cap S_q)\) constitute a clique in the consistency graph of every party in \( \mathcal{H} \). From the properties of \( \Pi_{SVM} \) in the synchronous network, all honest parties in \( S_q \) compute a common output, say \( s_q^{(t)} \), during the instance \( \Pi_{SVM}^{(l)}(P_i, s_q^{(t)} , S_q) \). Next consider an arbitrary \( P_i \in \mathcal{H} \cap \mathcal{W}_{p,q}^{(t)} \) and any arbitrary \( P_j \in \mathcal{W}_{p,q}^{(t)} \). This implies that \( P_j \) has broadcasted the message \( \text{OK}^{(t)}(i,j) \), after receiving \( \text{ICSig}(P_j, P_i, P_k, s_q^{(t)}) \) from \( P_j \), for every \( P_k \in \mathcal{P} \), and verifying that the share \( s_q^{(t)} \) is the same as the one, computed during the instance \( \Pi_{SVM}^{(l)}(P_i, s_q^{(t)}, S_q) \).

Since the share computed by \( P_i \) during \( \Pi_{SVM}^{(l)}(P_i, s_q^{(t)}, S_q) \) is \( s_q^{(t)} \), it follows that \( \text{ICSig}(P_j, P_i, P_k, s_q^{(t)}) \) is the same as \( \text{ICSig}(P_j, P_i, P_k, s_q^{(t)} \) ). On the other hand, since \( P_i \) gives its IC-signature on \( s_q^{(t)} \) to every \( P_j \in S_q \), it follows that if any corrupt \( P_j \in \mathcal{W}_{p,q}^{(t)} \) has \( \text{ICSig}(P_i, P_j, P_k, s_q^{(t)}) \) from \( P_i \), for any \( P_k \in \mathcal{P} \), then \( s_q^{(t)} = s_q^{(t)} \) holds. The linearity of the underlying IC-signatures follow from the fact that the parties follow the linearity principle while generating the IC-signatures.

Now based on the previous two lemmas, we show that in a synchronous network, all honest parties will output a “legitimate” set of parties \( \text{CORE} \) after time \( T_{SVM} + T_{Auth} + 2T_{BC} + 6T_{BA} \), such that at least one honest party is present in \( \text{CORE} \). And corresponding to every party in \( \text{CORE} \), there exists some value, which is linearly secret-shared with IC-signatures. Moreover, the values corresponding to the honest parties remain private.

**Lemma E.4.** If the network is synchronous, then in \( \Pi_{MDVSS} \) except with probability \( O(n^3 \cdot \epsilon_{ICP}) \), at the time \( T_{MDVSS} = T_{SVM} + T_{Auth} + 2T_{BC} + 6T_{BA} \), all honest parties output a common set \( \text{CORE} \), such that at least one honest party will be present in \( \text{CORE} \). Moreover, corresponding to every \( P_i \in \text{CORE} \), there exists some \( s_i^{(t)} \), where \( s_i^{(t)} = \frac{s_i^{(t)}}{s_q^{(t)}} \) for an honest \( P_i \), which is the input of \( P_i \) for \( \Pi_{MDVSS} \), such that the values \( \{s_i^{(t)}\}_{P_i \in \text{CORE}} \) are linearly secret-shared with IC-signatures. Furthermore, if \( P_i \) is honest, then adversary’s view is independent of \( s_i^{(t)} \).

**Proof.** Let \( Z^* \in Z_s \) be the set of corrupt parties and let \( \mathcal{H} = \mathcal{P} \setminus Z^* \) be the set of honest parties. From Lemma E.1, except with probability \( O(n^3 \cdot \epsilon_{ICP}) \), all the parties in \( \mathcal{H} \) will have a common set of committed dealers \( CD \) by the time.
Consider the set \( S_p \in S_z, \) such that \( S_p = \mathcal{H}. \) Then corresponding to each \( P_i \in (\mathcal{H} \cap CD), \) every \( P_i \in \mathcal{H} \) will receive (CanCS, \( P_i, S_p, \{ W_{p,q}^{(i)}, q = 1, \ldots, |Z_i|, B_{S_p}^{(i)}, \{ s_{p,q}^{(i)} \}_{q \in B_{S_p}^{(i)}} \) ) from the broadcast of \( P_i \) and includes \( (P_i, S_p) \) to \( C_i \) at time \( T_{BA} + T_{BA} + T_{Auth} + 2T_{BC} \) (see Lemma E.2). Since \( P \cap \mathcal{H} \in Z_s \) and \( \mathcal{H} \subseteq CD, \) it follows that at the time \( T_{BA} + T_{BA} + T_{Auth} + 2T_{BC} \), every party \( P_i \in \mathcal{H} \) will have a set \( A_{p,i} \) (namely \( A_{p,i} = \mathcal{H} \)), where \( CD \setminus A_{p,i} \in Z_s \) and where \( (P_i, S_p) \in C_i \) for every \( P_i \in A_{p,i}. \) Consequently, each \( P_i \in \mathcal{H} \) starts participating in the instance \( \Pi_{BA}^{(1,p)} \) with input 1, at the time \( T_{BA} + T_{BA} + T_{Auth} + 2T_{BC}. \) From the \( Z_s \)-security of \( \Pi_{BA} \) in the synchronous network (see Theorem 3.1), it follows that at the time \( T_{BA} + T_{BA} + T_{BC} + 3T_{BA}, \) every \( P_i \in \mathcal{H} \) obtains the output 1 from the instance \( \Pi_{BA}^{(1,p)}. \) Consequently, at the time \( T_{BA} + T_{Auth} + 2T_{BC} + 3T_{BA}, \) every party in \( \mathcal{H} \) will start participating in the remaining \( \Pi_{BA}^{(1,\ast)} \) instances for which no input has been provided yet (if there are any), with input 0. And from the \( Z_s \)-security of \( \Pi_{BA} \) in the synchronous network, these \( \Pi_{BA}^{(1,\ast)} \) instances will produce common outputs, for every party in \( \mathcal{H}, \) at the time \( T_{BA} + T_{BA} + 2T_{BC} + 4T_{BA}. \) As a result, at the time \( T_{BA} + T_{BA} + 2T_{BC} + 4T_{BA}, \) all the parties in \( \mathcal{H} \) will compute a \( q_{core}. \) Moreover, \( q_{core} \) will be common for all the parties in \( \mathcal{H}, \) since it corresponds to the least-indexed \( \Pi_{BA}^{(1,\ast)} \) instance among \( \Pi_{BA}^{(1,1)}, \ldots, \Pi_{BA}^{(1,|Z_s|)}, \) which produces output 1.

And from the \( Z_s \)-security of \( \Pi_{BA} \) in the synchronous network, each \( \Pi_{BA}^{(1,\ast)} \) instance produces a common output for every party in \( \mathcal{H}. \) We also note that \( q_{core} \) will be indeed set to some value from the set \( \{1, \ldots, |Z_s|\}. \) This is because as shown above, the index \( p \) where \( S_p = \mathcal{H} \) always constitute a candidate \( q_{core}. \)

We next claim that corresponding to \( S_{gram}, \) there exists a subset of parties \( B_{gram}, \) where \( CD \setminus B_{gram} \in Z_s, \) such that corresponding to every \( P_j \in B_{gram}, \) the ordered pair \( (P_j, S_{gram}) \) is present in the set \( C_i \) of every \( P_i \in \mathcal{H}, \) at the time \( T_{BA} + T_{BA} + 2T_{BC} + 4T_{BA}. \) Assuming that the claim is true, it implies that all the parties in \( \mathcal{H} \) will participate with input 1 in the instances \( \Pi_{BA}^{(2,j)} \), corresponding to every \( P_j \in B_{gram} \) at the time \( T_{BA} + T_{BA} + 2T_{BC} + 4T_{BA}. \) And hence from the \( Z_s \)-security of the \( \Pi_{BA} \) in the synchronous network (Theorem 3.1), all the parties will obtain the output 1 in the \( \Pi_{BA}^{(2,j)} \) instances, corresponding to every \( P_j \in B_{gram}, \) at the time \( T_{BA} + T_{BA} + 2T_{BC} + 5T_{BA}. \) As a result, at time \( T_{BA} + T_{BA} + 2T_{BC} + 5T_{BA}, \) all the parties in \( \mathcal{H} \) will start participating in the remaining \( \Pi_{BA}^{(2,\ast)} \) instances for which no input has been provided yet (if there are any), with input 0. Consequently, from the \( Z_s \)-security of the \( \Pi_{BA} \) in the synchronous network (Theorem 3.1), at the time \( T_{MDVSS}, \) all the parties in \( \mathcal{H} \) will have some output from all the \( |CD| \) instances of \( \Pi_{BA}^{(2,\ast)}. \) Moreover, the outputs
will be common for all the parties in \( \mathcal{H} \). Furthermore, the parties in \( \mathcal{H} \) will have a subset \( \text{CORE} \), which corresponds to all the \( I_{\text{BA}}^{(j)} \) instances, which have produced output 1. Note that \( (\mathcal{H} \cap \text{CORE}) \neq \emptyset \). This is because \( B_{\text{qores}} \subseteq \text{CORE} \) and \( \mathcal{H} \subseteq CD \). Consequently, \( (\mathcal{H} \cap B_{\text{qores}}) \neq \emptyset \), as otherwise \( Z_s \) does not satisfy the \( \mathcal{Q}^{(j)}(P, Z_s) \) condition, which is a contradiction.

We next proceed to prove our claim. Since the instance \( I_{\text{BA}}^{(1, qores)} \) has produced output 1, it follows that at least one party from \( \mathcal{H} \), say \( P_k \), have participated with input 1 in the instance \( I_{\text{BA}}^{(1, qores)} \). This is because if all the parties in \( \mathcal{H} \) participates with input 0 in the instance \( I_{\text{BA}}^{(1, qores)} \), then from the \( Z_s \)-validity of \( I_{\text{BA}} \) in the synchronous network (Theorem 3.1), all the parties in \( \mathcal{H} \) would have obtained the output 0 from the instance \( I_{\text{BA}}^{(1, qores)} \), which is a contradiction. We also note that \( P_k \) would have started participating with input 1 in the instance \( I_{\text{BA}}^{(1, qores)} \), latest by time \( T_{\text{SVM}} + T_{\text{Auth}} + 2T_{\text{BC}} + 3T_{\text{BA}} \). This is because as argued above, by time \( T_{\text{SVM}} + T_{\text{Auth}} + 2T_{\text{BC}} + 3T_{\text{BA}} \), all the parties in \( \mathcal{H} \) would have started participating in all the \( |Z_s| \) instances of \( I_{\text{BA}}^{(1, *)} \), with some input. Now since \( P_k \) has participated with input 1 in the instance \( I_{\text{BA}}^{(1, qores)} \), it follows that at the time \( T_{\text{SVM}} + T_{\text{Auth}} + 2T_{\text{BC}} + 3T_{\text{BA}} \), there exists a subset of parties \( A_{\text{qores}, k} \), where \( CD \setminus A_{\text{qores}, k} \subseteq Z_s \), such that \( (P_i, S_{\text{qores}}) \) is present in the set \( \mathcal{C}_k \), corresponding to every \( P_i \in A_{\text{qores}, k} \). We show that the set \( A_{\text{qores}, k} \) constitutes the candidate \( B_{\text{qores}} \). For this, note that for any \( P_i \in A_{\text{qores}, k} \), party \( P_k \) includes \((P_i, S_{\text{qores}})\) to \( \mathcal{C}_k \), only after receiving a message \((\text{CanCS}, P_i, \{W_{\text{qores}}, q\} q=1,...,|Z_s|, BS_{\text{qores}}^{(i)}, \{s_q\}_{q \in BS_{\text{qores}}^{(i)}})\) from the broadcast of \( P_i \) and verifying it. Moreover, \( P_k \) must have received the message \((\text{CanCS}, P_i, \{W_{\text{qores}}, q\} q=1,...,|Z_s|, BS_{\text{qores}}^{(i)}, \{s_q\}_{q \in BS_{\text{qores}}^{(i)}})\) from each \( P_i \in A_{\text{qores}, k} \), latest by time \( T_{\text{SVM}} + T_{\text{Auth}} + 2T_{\text{BC}} + 3T_{\text{BA}} \). It then follows from Lemma E.4 that by time \( T_{\text{SVM}} + T_{\text{Auth}} + 2T_{\text{BC}} + 3T_{\text{BA}} + \Delta < T_{\text{SVM}} + T_{\text{Auth}} + 2T_{\text{BC}} + 4T_{\text{BA}} \), every party in \( \mathcal{H} \) would have received \((\text{CanCS}, P_i, \{W_{\text{qores}}, q\} q=1,...,|Z_s|, BS_{\text{qores}}^{(i)}, \{s_q\}_{q \in BS_{\text{qores}}^{(i)}})\) from each \( P_i \in A_{\text{qores}, k} \). And hence each party \( P_i \in \mathcal{H} \) would include \((P_i, S_{\text{qores}})\) to the set \( \mathcal{C}_i \), corresponding to every \( P_i \in A_{\text{qores}, k} \), by time \( T_{\text{SVM}} + T_{\text{Auth}} + 2T_{\text{BC}} + 4T_{\text{BA}} \). This proves our claim.

We next claim that at the time \( T_{\text{MDVSS}} \), corresponding to every \( P_i \in \text{CORE} \), every \( P_i \in \mathcal{H} \) would have received a message \((\text{CanCS}, P_i, \{W_{\text{qores}}, q\} q=1,...,|Z_s|, BS_{\text{qores}}^{(i)}, \{s_q\}_{q \in BS_{\text{qores}}^{(i)}})\) from the broadcast of \( P_i \). The proof for this is very similar to the proof of the previous claim and relies on the properties of \( I_{\text{BA}} \). So consider an arbitrary \( P_i \in \text{CORE} \). This implies that the instance \( I_{\text{BA}}^{(2, \ell)} \) has produced output 1 for all the parties in \( \mathcal{H} \), which further implies that at least one party from \( \mathcal{H} \), say \( P_m \), has participated with input 1 during the instance \( I_{\text{BA}}^{(2, \ell)} \). If not, then from the \( Z_s \)-validity of \( I_{\text{BA}} \) in the synchronous network (Theorem 3.1), the instance \( I_{\text{BA}}^{(2, \ell)} \) would have produced output 0 for all the parties in \( \mathcal{H} \), which is a contradiction. We also note that \( P_m \) must have started participating in the instance \( I_{\text{BA}}^{(2, \ell)} \), latest by time \( T_{\text{SVM}} + T_{\text{Auth}} + 2T_{\text{BC}} + 5T_{\text{BA}} \). This is because as shown
above, by time $T_{\text{SVM}} + T_{\text{Auth}} + 2T_{\text{BC}} + 5T_{\text{BA}}$, all the parties in $\mathcal{H}$ would have started participating in all the $|\mathcal{C}|$ instances of $\Pi_{\text{BA}}(2,*)$, with some input. Now since $P_m$ participates with input 1 in the instance $\Pi_{\text{BA}}(2,*)$, it follows that by time $T_{\text{SVM}} + T_{\text{Auth}} + 2T_{\text{BC}} + 5T_{\text{BA}}$, party $P_m$ must have received a message $(\text{CanCS}, P_i, S_{\text{qcore}}, \{W_{\text{qcore},q}\}_{q=1,\ldots,|Z_s|}, BS_{\text{qcore}}^{(\ell)}, \{s_q^{(\ell)}\}_{q \in BS_{\text{qcore}}^{(\ell)}})$ from the broadcast of $P_i$ and included $(P_i, S_{\text{qcore}})$ to $C_m$. It then follows from Lemma E.4 that by time $T_{\text{SVM}} + T_{\text{Auth}} + 2T_{\text{BC}} + 5T_{\text{BA}} + \Delta < T_{\text{SVM}} + T_{\text{Auth}} + 2T_{\text{BC}} + 6T_{\text{BA}}$, every party $P_i$ in $\mathcal{H}$ would have received $(\text{CanCS}, P_i, S_{\text{qcore}}, \{W_{\text{qcore},q}\}_{q=1,\ldots,|Z_s|}, BS_{\text{qcore}}^{(\ell)}, \{s_q^{(\ell)}\}_{q \in BS_{\text{qcore}}^{(\ell)}})$ from $P_i$ and would include $(P_i, S_{\text{qcore}})$ to $C_i$.

Till now we have shown that all the all the $|Z_s|$ instances $\Pi_{\text{BA}}(1,*)$ of $\Pi_{\text{BA}}$ and then all the $|\mathcal{C}|$ instances $\Pi_{\text{BA}}(2,*)$ of $\Pi_{\text{BA}}$ will produce some output, for all the parties in $\mathcal{H}$, by time $T_{\text{MDVSS}}$. Moreover, at the time $T_{\text{MDVSS}}$, all the parties in $\mathcal{H}$ will have a common $q_{\text{core}} \in \{1,\ldots,|Z_s|\}$ and a common set $\text{CORE} \subseteq \mathcal{P}$, where $\text{CORE}$ has at least one honest party. Furthermore, corresponding to every $P_i \in \text{CORE}$, each $P_i \in \mathcal{H}$ would have received a message $(\text{CanCS}, P_i, S_{\text{qcore}}, \{W_{\text{qcore},q}\}_{q=1,\ldots,|Z_s|}, BS_{\text{qcore}}^{(\ell)}, \{s_q^{(\ell)}\}_{q \in BS_{\text{qcore}}^{(\ell)}})$ from the broadcast of $P_i$. Furthermore, from Lemma E.4, corresponding to each $P_i \in \text{CORE}$, the set $W_{\text{qcore},q}$ will be either the set $S_q$ or $(S_{\text{qcore} \cap S_q})$, for $q = 1,\ldots,|Z_s|$. If $W_{\text{qcore},q} = S_q$ for every $P_i \in \text{CORE}$, then all the parties in $\mathcal{H}$ will set $W_q$ to $S_q$. On the other hand, if $W_{\text{qcore},q} = (S_{\text{qcore} \cap S_q})$ for any $P_i \in \text{CORE}$, then all the parties in $\mathcal{H}$ will set $W_q$ to $(S_{\text{qcore} \cap S_q})$. Irrespective of the case, all the parties in $\mathcal{H}$ would set $W_q$ to a common subset. We also note that irrespective of the case, $S_q \setminus W_q$ holds. This is because from Lemma E.4 and Lemma E.3, the condition $S_q \setminus W_{\text{qcore},q} \in Z_q$ holds, corresponding to every $P_i \in \text{CORE}$.

Finally consider an arbitrary $P_i \in \text{CORE}$ and an arbitrary $S_q \in \mathcal{S}_{Z_s}$. We claim that at the time $T_{\text{MDVSS}}$, all the parties in $(\mathcal{H} \cap S_q)$ will have a common share, say $s_{\text{qcore}}^{(\ell)}$, where $s_{\text{qcore}}^{(\ell)} = s_q^{(\ell)}$ for an honest $P_i$. For this, we consider two possible cases. If $q \in BS_{\text{qcore}}^{(\ell)}$, then each $P_i \in (\mathcal{H} \cap S_q)$ would have received $s_q^{(\ell)}$ from the broadcast of $P_i$, as part of the $(\text{CanCS}, P_i, S_{\text{qcore}}, \{W_{\text{qcore},q}\}_{q=1,\ldots,|Z_s|}, BS_{\text{qcore}}^{(\ell)}, \{s_q^{(\ell)}\}_{q \in BS_{\text{qcore}}^{(\ell)}})$ message. Consequently, in this case $s_{\text{qcore}}^{(\ell)}$ is the same as $s_q^{(\ell)}$, received from the broadcast of $P_i$. On the other hand, if $q \not\in BS_{\text{qcore}}^{(\ell)}$, then from Lemma E.3, each $P_i \in (\mathcal{H} \cap S_q)$ would have a common share, say $s_{\text{qcore}}^{(\ell)}$, moreover, from Lemma E.2, if $P_i$ is honest, then $s_{\text{qcore}}^{(\ell)} = s_q^{(\ell)}$ holds. We define

$$s_{\text{qcore}}^{(\ell)} \triangleq \sum_{q=1,\ldots,|Z_s|} s_q^{(\ell)}$$

where $s_{\text{qcore}}^{(\ell)} = s_q^{(\ell)}$ for an honest $P_i$. Hence at time $T_{\text{MDVSS}}$, each party in $(\mathcal{H} \cap S_q)$ has $[s_{\text{qcore}}^{(\ell)}]_q$. We also note that if $q \in BS_{\text{qcore}}^{(\ell)}$, then every $P_i \in W_q$ sets $I\text{CSig}(P_j, P_i, P_k, [s_{\text{qcore}}^{(\ell)}]_q)$ to the default value, corresponding to every $P_j \in W_q$ and every $P_k \in \mathcal{P}$. On the other hand, if $q \not\in BS_{\text{qcore}}^{(\ell)}$, then every $P_i \in (\mathcal{H} \cap S_q)$ will
have $\text{ICSig}(P_j, P_k, P, s_q^{(t)})$ of every $P_j \in \mathcal{W}_q$ and for every $P_k \in \mathcal{P}$. Furthermore, if any corrupt $P_j \in \mathcal{W}_q$ has $\text{ICSig}(P_j, P_k, P, s_q^{(t)})$ of any $P_i \in (\mathcal{H} \cap S_q)$ for any $P_k \in \mathcal{P}$, then $s_q^{(t)} = s_q^{(t)}$ holds. Moreover, from Lemma E.3, if $P_t$ is honest, then $s_q^{(t)}$ in the IC-signatures mentioned above will be the same as $s_q^{(t)}$. It then follows that $s_q^{(t)}$ will be linearly secret-shared; the linearity of the underlying IC-signatures follows since the (honest) parties follow the linearity principle, while generating the IC-signatures.

The privacy of $s_q^{(t)}$ for an honest $P_t$ follows from Lemma E.3.

We next consider an asynchronous network. We first prove an analogue of Lemma E.1 in the asynchronous network.

**Lemma E.5.** If the network is asynchronous and $P_t \in \mathcal{P}$ is an honest dealer participating with input $s_q^{(t)}$, then all the following hold in $\Pi_{MDVSS}$, where $\mathcal{H}$ is the set of honest parties.

- Except with probability $O(n^3 \cdot \epsilon_{\text{CP}})$, almost-surely, all the parties in $\mathcal{H}$ will eventually have a common $\mathcal{C} \mathcal{D}$ set, where $\mathcal{P} \setminus \mathcal{C} \mathcal{D} \in \mathcal{Z}_a$.
- Except with probability $O(n^3 \cdot \epsilon_{\text{CP}})$, corresponding to every dealer $P_t \in \mathcal{C} \mathcal{D}$ and every $S_q \in \mathcal{S}_{[Z_a]}$, every party in $(\mathcal{H} \cap S_q)$ will eventually have a common share, say $s_q^{(t)}$, which is the same as $s_q^{(t)}$, for an honest $P_t$.

**Proof.** Let $Z^* \in \mathcal{Z}_a$ be the set of corrupt parties and let $\mathcal{H} = \mathcal{P} \setminus Z^*$ be the set of honest parties. Note that $\mathcal{H} \in \mathcal{S}_{Z_a}$, since $\mathcal{Z}_a \subset \mathcal{Z}_a$. From the properties of $\Pi_{SVM}$ in the asynchronous network (Lemma 6.3), it follows that every $P_t \in \mathcal{H}$ will eventually set $\text{flag}^{(P_t, S_q)}$ to 1 during the instance $\Pi_{SVM}(P_t, s_q^{(t)}, S_q)$, corresponding to every $P_t \in \mathcal{H}$ and every $S_q \in \mathcal{S}_{Z_a}$. Moreover, corresponding to every $P_t \in \mathcal{H}$ and every $S_q \in \mathcal{S}_{Z_a}$, each $P_i \in (\mathcal{H} \cap S_q)$ eventually computes an output $s_q^{(t)}$ during the instance $\Pi_{SVM}(P_i, s_q^{(t)}, S_q)$. Furthermore, except with probability $O(n^3 \cdot \epsilon_{\text{CP}})$, the value $s_q^{(t)}$ will be the same as $s_q^{(t)}$.

We first claim that there always exists a subset of parties $\mathcal{D}$, where $\mathcal{P} \setminus \mathcal{D} \in \mathcal{Z}_a$, such that the $\Pi_{BA}$ instance $\Pi_{BA}^{(t)}$ eventually produces output 1 for all the parties in $\mathcal{H}$, corresponding to every $P_t \in \mathcal{D}$. Assuming that the claim is true, it implies that all the parties in $\mathcal{H}$ will eventually participate with some input in the $\Pi_{BA}$ instances $\Pi_{BA}^{(1)}, \ldots, \Pi_{BA}^{(n)}$. This is because from the protocol steps, once the $\Pi_{BA}^{(s)}$ instances corresponding to the parties in $\mathcal{D}$ produce output 1, all the parties in $\mathcal{H}$ will start participating with input 0 in the remaining instances $\Pi_{BA}^{(s)}$ of $\Pi_{BA}$ (if any), for which no input has been provided yet. And hence from the $\mathcal{Z}_a$-security of $\Pi_{BA}$ in the asynchronous network, it follows that almost-surely, all these $\Pi_{BA}$ instances will eventually produce some output for all the parties in $\mathcal{H}$. Moreover, the outputs will be the same for all the parties in $\mathcal{H}$. Consequently, all the parties in $\mathcal{H}$ will eventually obtain a common $\mathcal{C} \mathcal{D}$ set. Moreover, $\mathcal{P} \setminus \mathcal{C} \mathcal{D} \in \mathcal{Z}_a$, since $\mathcal{C} \mathcal{D}$ consists of all those parties $P_t$, such that the instance $\Pi_{BA}^{(t)}$ produces output 1. And according to our claim, $\mathcal{D} \subseteq \mathcal{C} \mathcal{D}$ holds. We now proceed to prove our claim.

There are two possible cases. Consider the case when some $P_i \in \mathcal{H}$ starts participating with input 0 in any $\Pi_{BA}^{(s)}$ instance. This implies that for $P_t$, there
exists a subset of parties $CD_i$ where $\mathcal{P} \setminus CD_i \in Z_s$, such that corresponding to every $P_t \in CD_i$, the instance $\Pi_{BA}^{(t)}$ has produced output 1 for $P_t$. In this case, the set $CD_i$ is the candidate $D$ set, whose existence we want to prove. Next, consider the case when no party in $H$ has started participating with input 0 in any of the $\Pi_{BA}^{(t)}$ instances. In this case, the set $H$ constitutes the candidate $D$ set. This is because as shown above, every $P_t \in H$ will eventually set $\text{flag}_{P_t,S_q}$ to 1, corresponding to every $P_t \in H$ and every $S_q \in S_{Z_s}$. And hence every $P_t \in H$ will eventually start participating with input 1 in the $\Pi_{BA}^{(t)}$ instances, corresponding to $P_t \in H$. Consequently, the $Z_s$-validity of $\Pi_{BA}$ in the asynchronous network (Theorem 3.1) will guarantee that the $\Pi_{BA}^{(t)}$ instances, corresponding to $P_t \in H$ eventually produce output 1 for all the parties in $H$.

Next, consider an arbitrary $P_t \in CD$. This implies that at least one party from $H$, say $P_k$, has participated with input 1 during the instance $\Pi_{BA}^{(t)}$. If not, then from the $Z_s$-validity of $\Pi_{BA}$ in the asynchronous network (Theorem 3.1), all the parties in $H$ would have obtained the output 0 from the instance $\Pi_{BA}^{(t)}$ and hence $P_t \notin CD$, which is a contradiction. This implies that party $P_k$ has set $\text{flag}_{P_t,S_q}$ to 1 during the instance $\Pi_{SVM}(P_t,s_q^{(t)},S_q)$, for $q = 1, \ldots, |Z_s|$. So consider an arbitrary $S_q \in S_{Z_s}$. From the properties of $\Pi_{SVM}$ in the asynchronous network (Lemma 6.3), it follows that there exists some value $s_q^{*}$, which is the same as $s_q^{(t)}$ for an honest $P_t$, such that except with probability $O(n^3 \cdot \epsilon_{ICP})$, all the parties in $H$ eventually output $s_q^{*}$ during the instance $\Pi_{SVM}(P_t,s_q^{(t)},S_q)$.

We next prove the analogue of Lemma E.2 in the asynchronous network.

**Lemma E.6.** If the network is asynchronous and $P_t \in CD$ is an honest dealer participating with input $s(t)$, then all the following hold in $\Pi_{MDVSS}$ except with probability $O(n^3 \cdot \epsilon_{ICP})$, where $H$ is the set of honest parties.

- If $S_p = H$, then $P_t$ will eventually broadcast $(\text{CanCS}, P_t, S_p, \{W_{p,q}^{(t)} \mid q = 1, \ldots, |Z_s|\}, B_{S_p}^{(t)}, \{s_q^{(t)} \mid q \in B_{S_p}^{(t)}\})$.
- If $P_t$ broadcasts any $(\text{CanCS}, P_t, S_p, \{W_{p,q}^{(t)} \mid q = 1, \ldots, |Z_s|\}, B_{S_p}^{(t)}, \{s_q^{(t)} \mid q \in B_{S_p}^{(t)}\})$ then every honest $P_t \in \mathcal{P}$ will eventually include $(P_t, S_p)$ to $C_i$. Moreover, the following will hold.
  - If $S_q = H$, then $q \notin B_{S_p}^{(t)}$.
  - For $q = 1, \ldots, |Z_s|$, each $W_{p,q}^{(t)}$ will be either $S_q$ or $(S_p \cap S_q)$ such that $Z_s$ satisfies the $Q^{(1)}(W_{p,q}, Z_s)$ condition.
  - If $q \notin B_{S_p}^{(t)}$, then every honest $P_t \in S_q$ will have the share $s_q^{(t)}$. Moreover, every honest $P_t \in W_{p,q}^{(t)}$ will have $\text{ICSig}(P_j, P_t, P_k, s_q^{(t)})$ of every $P_j \in W_{p,q}^{(t)}$ for every $P_k \in \mathcal{P}$. Furthermore, if any corrupt $P_j \in W_{p,q}^{(t)}$ have $\text{ICSig}(P_j, P_t, P_k, s_q^{(t)})$ of any honest $P_t \in W_{p,q}^{(t)}$ for any $P_k \in \mathcal{P}$, then $s_q^{(t)} = s_q^{(t)}$ holds. Also, all the underlying IC-signatures will satisfy the linearity property.
  - The view of the adversary will be independent of $s(t)$.

102
Proof. Let \( Z^* \in Z_a \) be the set of corrupt parties and let \( \mathcal{H} = \mathcal{P} \setminus Z^* \) be the set of honest parties. We first note that \( \mathcal{H} \in \mathcal{S}_Z \), since \( Z^* \in Z_a \) as \( Z_a \subset Z_s \). The proof for the first part of the lemma is similar to the proof of the first part of Lemma E.2, except that all the “favourable” conditions hold for an honest \( P_i \) eventually. In more detail, consider an arbitrary \( S_q \in \mathcal{S}_Z \). Then each \( P_i \in (S_q \cap \mathcal{H}) \) eventually computes the share \( s_q^{(i)} \) during the instance \( I_{\Sigma M}(P_r, s_q^{(i)}, S_q) \), where \( s_q^{(i)} = s_q^{(i)} \) holds, except with probability \( O(n^3 \cdot \epsilon_{I_CP}) \). Consequently, \( P_i \) starts giving \( ICSig(P_i, P_j, P_k, s_q^{(i)}) \) to every \( P_j \in S_q \), for every \( P_k \in \mathcal{P} \). Then from the \( \mathcal{Z}_a \)-correctness of \( I_{\text{Auth}} \) in the asynchronous network (Theorem 4.1), it follows that each party \( P_i \in (S_q \cap \mathcal{H}) \) will eventually receive \( ICSig(P_i, P_j, P_k, s_q^{(i)}) \) from every \( P_j \in (S_q \cap \mathcal{H}) \), for every \( P_k \in \mathcal{P} \), such that \( s_q^{(i)} = s_q^{(i)} = s_q^{(i)} \) holds. Since \( S_q \) is arbitrary, it follows that eventually, every party \( P_i \in \mathcal{H} \) broadcasts a \( OK^{(i)}(i, j) \) message, corresponding to every \( P_j \in \mathcal{H} \). From the \( \mathcal{Z}_a \)-weak validity and \( \mathcal{Z}_a \)-fallback validity of \( I_{\text{BC}} \) in the asynchronous network (Theorem 3.2), it follows that these \( OK^{(i)}(i, j) \) messages are eventually received by every party in \( \mathcal{H} \). Consequently, the set \( \mathcal{H} \) eventually becomes a clique in the consistency graph \( G^{(\ell)} \) of every party \( P_i \in \mathcal{H} \). Let \( S_p \) be the set from \( \mathcal{S}_Z \), such that \( S_p = \mathcal{H} \). From the protocol steps, it then follows that the dealer \( P_\ell \) will eventually compute core-sets \( W_{\ell,q} \) for \( q = 1, \ldots, |Z_a| \) and broadcast-set \( BS^{(\ell)}_p \) with respect to \( S_p \) as follows,

- If \( S_q \) constitutes a clique in the graph \( G^{(\ell)} \), then \( W_{\ell,q} \) is set as \( S_q \).
- Else if \( (S_p \cap S_q) \) constitutes a clique in \( G^{(\ell)} \) and \( S_a \) satisfies the \( Q^{(1)}(S_p \cap S_q) \) condition, then \( W_{\ell,q} \) is set as \( (S_p \cap S_q) \).
- Else \( W_{\ell,q} \) is set to \( S_q \) and \( q \) is included to \( BS^{(\ell)}_p \).

After computing the core-sets and broadcast-set, \( P_\ell \) will eventually broadcast \((\text{CanCS}, P_\ell, S_p, (W_{\ell,q})_{q=1,\ldots,|Z_a|}, BS^{(\ell)}_p, (s_q^{(\ell)})_{q=1,\ldots,|Z_a|})\).

We next proceed to prove the second part of the lemma, whose proof is again similar to the proof of the second part of the Lemma E.2, except that all the “favourable” conditions which hold for \( P_i \), are guaranteed to hold eventually for all the parties in \( \mathcal{H} \). In more detail, consider an arbitrary \( S_p \in \mathcal{S}_Z \), such that \( P_i \) compute core-sets \( W_{\ell,q} \) for \( q = 1, \ldots, |Z_a| \) and broadcast-set \( BS^{(\ell)}_p \) with respect to \( S_p \) and broadcasts \((\text{CanCS}, P_i, S_p, (W_{\ell,q})_{q=1,\ldots,|Z_a|}, BS^{(\ell)}_p, (s_q^{(\ell)})_{q=1,\ldots,|Z_a|})\). This means the parties in \( S_p \) constitute a clique in the graph \( G^{(\ell)} \). We note that all the edges which are present in the graph \( G^{(\ell)} \) when \( S_p \) constitute a clique in \( G^{(\ell)} \) are bound to be eventually included in the graph \( G^{(\ell)} \) of every party \( P_i \in \mathcal{H} \). This is because the edges are included by \( P_i \) based on various \( OK^{(i)}(i, j) \) messages, which are received by \( P_i \) through various \( I_{\text{BC}} \) instances. Consequently, due to the various properties of \( I_{\text{BC}} \) in the asynchronous network, these \( OK^{(i)}(i, j) \) messages are bound to be eventually delivered to every party in \( \mathcal{H} \). As a result, all the properties which hold for \( P_i \) in the graph \( G^{(\ell)} \) when \( S_p \) constitute a clique in \( G^{(\ell)} \) are bound to hold eventually for every
$P_t \in \mathcal{H}$ in the graph $G^{(t,\ell)}$. Since $P_t$ is assumed to be honest, it computes the sets $\{W_{p,q}^{(t)}\}_{q=1,\ldots,|Z|}$ and $\mathcal{BS}_p^{(t)}$, satisfying the following properties.

- If $S_q$ constitutes a clique in the graph $G^{(t,\ell)}$, then $W_{p,q}^{(t)}$ is set as $S_q$.
- Else if $(S_p \cap S_q)$ constitutes a clique in $G^{(t,\ell)}$ and $Z_s$ satisfies the $Q^{(1)}(S_p \cap S_q, Z_s)$ condition, then $W_{p,q}^{(t)}$ is set as $(S_p \cap S_q)$.
- Else $W_{p,q}^{(t)}$ is set to $S_q$ and $q$ is included to $\mathcal{BS}_p^{(t)}$.

We also note that if $S_q = \mathcal{H}$, then $q \notin \mathcal{BS}_p^{(t)}$ and consequently, $P_t$ will not make the share $s_q^{(t)}$ public. This is because $P_t$ will set $W_{p,q}^{(t)}$ to $(S_p \cap S_q)$. In more detail, the parties in $(S_p \cap S_q)$ will constitute a clique in $G^{(t,\ell)}$, since $S_p$ constitutes a clique in $G^{(t,\ell)}$, when $P_t$ starts computing the core-sets $\{W_{p,q}^{(t)}\}_{q=1,\ldots,|Z|}$. Moreover, $Z_s$ will satisfy the $Q^{(1)}(S_p \cap S_q, Z_s)$ condition, due to the $Q^{(2,1)}(P, Z_s, Z_a)$ condition.

Since $P_t$ is honest, from the $Z_a$-weak validity and $Z_a$-fallback validity of $\Pi_{BC}$ in the asynchronous network (see Theorem 3.2), it follows that all the parties in $\mathcal{H}$ will eventually receive $\{\text{CanCS}, P_t, S_p, \{W_{p,q}^{(t)}\}_{q=1,\ldots,|Z|}, \mathcal{BS}_p^{(t)}, \{s_q^{(t)}\}_{q \in \mathcal{BS}_p^{(t)}}\}$ from the broadcast of $P_t$. Moreover, each party $P_t \in \mathcal{H}$ will eventually include $(P_t, S_p)$ to the set $C_t$. This is because since $P_t$ has computed the sets $\{W_{p,q}^{(t)}\}_{q=1,\ldots,|Z|}$ and $\mathcal{BS}_p^{(t)}$ honestly, these sets will eventually pass all the verifications for each $P_t \in \mathcal{H}$. The last statement is true because as shown above, all the properties which hold for $P_t$ in the graph $G^{(t,\ell)}$ when $S_p$ constitute a clique in $G^{(t,\ell)}$ are bound to hold eventually for every $P_t \in \mathcal{H}$ in the graph $G^{(t,\ell)}$.

The proof for the rest of the properties stated in the lemma is similar to that of Lemma E.2, except that we now rely on the security properties of ICP in the asynchronous network; to avoid repetition we do not produce the details here. The linearity of the underlying IC-signatures is ensured since the parties follow the linearity principle while generating the IC-signatures.

We next prove an analogue of Lemma E.3 in the asynchronous network.

**Lemma E.7.** If the network is asynchronous and if in $\Pi_{MDVSS}$ any honest party $P_i$ receives $\{\text{CanCS}, P_t, S_p, \{W_{p,q}^{(t)}\}_{q=1,\ldots,|Z|}, \mathcal{BS}_p^{(t)}, \{s_q^{(t)}\}_{q \in \mathcal{BS}_p^{(t)}}\}$ from the broadcast of any corrupt dealer $P_t \in CD$ and includes $(P_t, S_p)$ to $C_t$, then all honest parties $P_j$ will eventually receive $\{\text{CanCS}, P_t, S_p, \{W_{p,q}^{(t)}\}_{q=1,\ldots,|Z|}, \mathcal{BS}_p^{(t)}, \{s_q^{(t)}\}_{q \in \mathcal{BS}_p^{(t)}}\}$ from the broadcast of $P_t$ and include $(P_t, S_p)$ to $C_j$. Moreover, for $q = 1, \ldots, |Z_s|$, the following holds, except with probability $O(n^3 \cdot \epsilon_{CP})$.

- $W_{p,q}^{(t)}$ is either $S_q$ or $(S_p \cap S_q)$. Moreover, $Z_s$ satisfies the $Q^{(1)}(W_{p,q}^{(t)}, Z_s)$ condition.
- If $q \notin \mathcal{BS}_p^{(t)}$, then every honest $P_i \in S_q$ will have a common share, say $s_q^{(t)}$. Moreover, every honest $P_i \in W_{p,q}^{(t)}$ will have $\text{ICSig}(P_j, P_i, P_k, s_q^{(t)})$ of every $P_j \in W_{p,q}^{(t)}$ for every $P_k \in P$. Furthermore, if any corrupt $P_j \in W_{p,q}^{(t)}$ has $\text{ICSig}(P_j, P_j, P_k, s_q^{(t)})$ of any honest $P_i \in W_{p,q}^{(t)}$ for any $P_k \in P$, then
\[ s_q^{(t)} = s_q^{(t)} * a_q \] holds. Also, all the underlying IC-signatures will satisfy the linearity property.

**Proof.** The proof is very similar to the proof of Lemma E.3, except that we now rely on the properties of \( \Pi_{BC} \) in the **asynchronous** network. Let \( Z^* \subseteq Z_a \) be the set of corrupt parties and let \( H = P \setminus Z^* \) be the set of honest parties. Now consider an arbitrary corrupt dealer \( P_t \in CD \) and an arbitrary \( P_i \in H \), such that \( P_i \) receives \((\text{CanCS}, P_t, S_p, \{\mathcal{W}_{p,q}^{(t)}\}_{q=1,\ldots,|Z_s|}, \mathcal{BS}_p^{(t)}, \{s_q^{(t)}\}_{q \in \mathcal{BS}_p^{(t)}})\) from the broadcast of \( P_t \) and includes \((P_t, S_p)\) to \( C_i \). Now consider another arbitrary \( P_j \in H \), such that \( P_j \neq P_i \). From the \( Z_a \)-weak consistency and \( Z_a \)-fallback consistency of \( \Pi_{BC} \) in the **asynchronous** network, it follows that \( P_j \) is bound to eventually receive \((\text{CanCS}, P_t, S_p, \{\mathcal{W}_{p,q}^{(t)}\}_{q=1,\ldots,|Z_s|}, \mathcal{BS}_p^{(t)}, \{s_q^{(t)}\}_{q \in \mathcal{BS}_p^{(t)}})\) from the broadcast of \( P_t \). We wish to show that \( P_j \) will eventually include \((P_t, S_p)\) to \( C_j \). For this, we note that since \( P_j \) has included \((P_t, S_p)\) to \( C_i \), all the following conditions hold for \( P_i \), for \( q = 1, \ldots, |Z_s| \).

- If \( q \in \mathcal{BS}_p^{(t)} \), then the set \( \mathcal{W}_{p,q}^{(t)} = S_q \).
- If \( q \notin \mathcal{BS}_p^{(t)} \), then \( \mathcal{W}_{p,q}^{(t)} \) is either \( S_q \) or \((S_p \cap S_q)\), such that:
  - If \( \mathcal{W}_{p,q}^{(t)} = S_q \), then \( S_q \) constitutes a clique in \( G^{(t,i)} \).
  - Else if \( \mathcal{W}_{p,q}^{(t)} = (S_p \cap S_q) \), then \((S_p \cap S_q)\) constitutes a clique in \( G^{(t,i)} \) and \( Z_s \) satisfies the \( O^{(1)}(S_p \cap S_q, Z_a) \) condition.

We claim that all the above conditions will hold eventually even for \( P_j \). This is because all the edges which are present in the consistency graph \( G^{(t,i)} \) when \( P_j \) includes \((P_t, S_p)\) to \( C_i \) are bound to be eventually present in the consistency graph \( G^{(t,j)} \). This follows from the \( Z_a \)-weak validity, \( Z_a \)-fallback validity, \( Z_a \)-weak consistency and \( Z_a \)-fallback consistency of \( \Pi_{BC} \) in the asynchronous network (see Theorem 3.2) and the fact that the edges in the graph \( G^{(t,j)} \) are based on \( OK^{(t)}(s,*) \) messages, which are received through various \( \Pi_{BC} \) instances.

The proof regarding the IC-signatures is exactly the same as Lemma E.3 and to avoid repetition, we do not produce the formal details here.

Finally, based on the previous two lemmas, we prove an analogue of Lemma E.4 and show that in an **asynchronous** network, all honest parties will eventually output a "legitimate" set of parties \( \text{CORE} \).

**Lemma E.8.** If the network is asynchronous, then in \( \Pi_{MDVSS} \), except with probability \( O(n^3 \cdot \epsilon_{ICP}) \), almost-surely all honest parties eventually output a common set \( \text{CORE} \), such that at least one honest party will be present in \( \text{CORE} \). Moreover, corresponding to every \( P_t \in \text{CORE} \), there exists some \( s^{(t)} \), where \( s^{(t)} = s^{(t)} \) for an honest \( P_t \), which is the input of \( P_t \) for \( \Pi_{MDVSS} \), such that the values \( \{s^{(t)}\}_{p_t \in \text{CORE}} \) are linearly secret-shared with IC-signatures. Furthermore, if \( P_t \) is honest, then the adversary’s view is independent of \( s^{(t)} \).

**Proof.** The proof structure is very similar to that of Lemma E.4, except that we now rely on the properties of \( \Pi_{BA} \) and \( \Pi_{BC} \) in the **asynchronous** network and Lemma E.5-E.7. Let \( Z^* \subseteq Z_a \) be the set of corrupt parties and let \( H = P \setminus Z^* \) be
the set of honest parties. From Lemma E.5, except with probability $O(n^3 \cdot \epsilon_{CD})$, almost-surely all the parties in $\mathcal{H}$ will eventually have a common set of committed 
dealers $CD$, where $P \setminus CD \in Z_s$. Moreover, corresponding to every dealer $P_i \in CD$ and every $S_q \in S_{|Z_s|}$, every party in $(\mathcal{H} \cap S_q)$ will eventually have a common 
share, say $s_q^{(t)}$, which is the same as $s_q^{(t)}$, for an $P_i$. We begin by showing that 
one the set of committed dealers $CD$ is decided, then almost-surely, all the $|Z_s|$ instances $\Pi_{\text{BA}}^{(1,s)}$ of $\Pi_{\text{BA}}$ and then all the $|CD|$ instances $\Pi_{\text{BA}}^{(2,*)}$ of $\Pi_{\text{BA}}$ will 
evolve produce some output, for all the parties in $\mathcal{H}$.

We first claim that irrespective of way messages are scheduled and the order in 
which the parties in $\mathcal{H}$ participate in various $\Pi_{\text{BA}}^{(1,s)}$ instances, there will be 
some instance $\Pi_{\text{BA}}^{(1,p)}$ corresponding to some $S_p \in S_{|Z_s|}$, which will eventually 
produce output 1 for all the parties in $\mathcal{H}$. For this, consider the set $S_p \in S_{|Z_s|}$, such 
that $S_p = \mathcal{H}$ (such an $S_p$ is bound to exist since $Z^* \in Z_s$ also holds, as $Z_a \subset Z_s$).
If there exists some $P_i \in \mathcal{H}$ which starts participating with input 0 in the 
instance $\Pi_{\text{BA}}^{(1,p)}$, then the claim is true, because $P_i$ participates with input 0 during 
$\Pi_{\text{BA}}^{(1,p)}$, only after receiving the output 1 from some other instance of $\Pi_{\text{BA}}^{(1,*)}$, say 
$\Pi_{\text{BA}}^{(1,q)}$. And hence from the $Z_a$-consistency of $\Pi_{\text{BA}}$ in the asynchronous 
network (Theorem 3.1), all the parties in $\mathcal{H}$ will eventually obtain the output 1 from 
the instance $\Pi_{\text{BA}}^{(1,q)}$, thus proving our claim. On the other hand, consider the 
case when no party has yet started participating with any input in the instance 
$\Pi_{\text{BA}}^{(1,p)}$. Then corresponding to each $P_i \in (\mathcal{H} \cap CD)$, every $P_i \in \mathcal{H}$ will eventually 
receive $(\text{CanCS}, P_i, S_p, \{W_{p,q}^{(t)}\}_{q=1,...,|Z_s|}, BS_{p}^{(t)}, \{s_q^{(t)}\}_{q \in BS_{p}^{(t)}})$ from the broadcast 
of $P_i$ and includes $(P_i, S_p)$ to $C_i$ (see Lemma E.6). Since $CD \setminus \mathcal{H} \subset Z^*$ and 
$Z^* \in Z_s$ (due to condition $Z_s \subset Z_a$), it follows that every party $P_i \in \mathcal{H}$ will 
be able to have a set $A_{p,i}$ (namely $A_{p,i} = \mathcal{H}$), where $CD \setminus A_{p,i} \in Z_s$ and where 
$(P_i, S_p) \in C_i$ for every $P_i \in A_{p,i}$. Consequently, each $P_i \in \mathcal{H}$ will eventually start 
participating in the instance $\Pi_{\text{BA}}^{(1,p)}$ with input 1, if they have not done so. And from 
the $Z_a$-validity of $\Pi_{\text{BA}}$ in the asynchronous network (see Theorem 3.1), it 
follows that all the parties in $\mathcal{H}$ eventually obtain the output 1 from the instance 
$\Pi_{\text{BA}}^{(1,p)}$, thus proving our claim in this case as well.

Now from the above claim, it follows that every party in $\mathcal{H}$ will eventually 
start participating in the remaining $\Pi_{\text{BA}}^{(1,*)}$ instances for which no input has been 
provided yet (if there are any), with input 0. And from the $Z_a$-security of $\Pi_{\text{BA}}$ in the 
asynchronous network, almost-surely, these $\Pi_{\text{BA}}^{(1,*)}$ instances will eventually 
produce common outputs, for every party in $\mathcal{H}$. As a result, all the parties in $\mathcal{H}$ will 
evolve compute a $q_{\text{core}} \in \{1, \ldots, |Z_s|\}$. Moreover, $q_{\text{core}}$ will be common 
for all the parties in $\mathcal{H}$, since it corresponds to the least-indexed $\Pi_{\text{BA}}^{(1,*)}$ instance 
among $\Pi_{\text{BA}}^{(1,1)}, \ldots, \Pi_{\text{BA}}^{(1,|Z_s|)}$, which produces output 1. And from the $Z_a$-security 
of $\Pi_{\text{BA}}$ in the asynchronous network, each $\Pi_{\text{BA}}^{(1,*)}$ instance produces a common 
output for every party in $\mathcal{H}$. We also note that $q_{\text{core}}$ will be indeed set to some 
value from the set $\{1, \ldots, |Z_s|\}$. This is because as shown above, the index $p$ 
where $S_p = \mathcal{H}$ always constitute a candidate $q_{\text{core}}$. 

106
We next claim that corresponding to \( S_{aq} \in \mathcal{S}_z \), there exists a subset of parties \( B_{qa} \), where \( CD \setminus B_{qa} \in \mathcal{Z}_s \), such that corresponding to every \( P_j \in B_{qa} \), the ordered pair \((P_j, S_{qa})\) is eventually included in the set \( C_\ell \) of every \( P_j \in \mathcal{H} \). Assuming that the claim is true, we next show that there always exists a set of parties \( B \) where \( CD \setminus B \in \mathcal{Z}_s \), such that the \( I_{2,j}^{(2,j)} \) instance eventually produce output 1 for all the parties in \( \mathcal{H} \), corresponding to every \( P_j \in B \). For this, we consider two possible cases. If any \( P_j \in \mathcal{H} \) has started participating with input 0 in any instance \( I_{2,i}^{(2,i)} \), then as per the claim, all the parties in \( \mathcal{H} \) will eventually start participating with input 0. Consequently, from the \( Z_a \)-consistency of \( BA \) in the synchronous network (see Theorem 3.1), all these \( I_{2,i}^{(2,i)} \) instances will eventually produce output 1 for all the parties in \( \mathcal{H} \). On the other hand, consider the case when no party in \( \mathcal{H} \) has started participating with input 0 in any instance \( I_{2,i}^{(2,i)} \). Then as per the claim, all the parties in \( \mathcal{H} \) will eventually start participating with input 1 in the instances \( I_{2,j}^{(2,j)} \), corresponding to every \( P_j \in B_{qa} \). Hence from the \( Z_a \)-security of the \( BA \) in the synchronous network (Theorem 3.1), all the parties will eventually obtain the output 1 in the \( I_{2,j}^{(2,j)} \) instances, corresponding to every \( P_j \in B_{qa} \). Thus irrespective of the case, the set \( B \) is guaranteed. As a result, all the parties in \( \mathcal{H} \) will eventually start participating in the remaining \( I_{2,i}^{(2,i)} \) instances for which no input has been provided yet (if there are any), with input 0. Consequently, from the \( Z_a \)-security of the \( BA \) in the asynchronous network (Theorem 3.1), almost-surely, all the parties in \( \mathcal{H} \) will eventually have some output from all the \( |CD| \) instances of \( I_{2,i}^{(2,i)} \). Moreover, the outputs will be common for all the parties in \( \mathcal{H} \). Furthermore, the parties in \( \mathcal{H} \) will have a subset \( CORE \), which corresponds to all the \( I_{2,i}^{(2,i)} \) instances, which produces output 1.

Note that \( CD \setminus CORE \in \mathcal{Z}_s \) holds, since we have shown that the \( I_{2,i}^{(2,i)} \) instances, corresponding to the parties \( P_j \in B \) will produce output 1, implying \( B \subseteq CORE \). And \( CD \setminus B \in \mathcal{Z}_s \) holds. Now since \( \mathcal{P} \setminus CD \in \mathcal{Z}_s \) and \( CD \setminus CORE \in \mathcal{Z}_s \), it follows that \((H \cap CORE) \neq \emptyset\), since the \( Q_{2,1}(\mathcal{P}, \mathcal{Z}_s, Z_a) \) condition is satisfied.

We next proceed to prove our claim about the existence of \( B_{qa} \). Since the instance \( I_{2,1}^{(1,q)} \) has produced output 1, it follows that at least one party from \( \mathcal{H} \), say \( P_k \), must have participated with input 1 in the instance \( I_{2,1}^{(1,q)} \). This is because if all the parties in \( \mathcal{H} \) participates with input 0 in the instance \( I_{2,1}^{(1,q)} \), then from the \( Z_a \)-validity of \( BA \) in the asynchronous network (Theorem 3.1), all the parties in \( \mathcal{H} \) would have obtained the output 0 from the instance \( I_{2,1}^{(1,q)} \), which is a contradiction. Now since \( P_k \) has participated with input 1 in the instance \( I_{2,1}^{(1,q)} \), it follows there exists a subset of parties \( A_{qa,k} \), where \( CD \setminus A_{qa,k} \in \mathcal{Z}_s \), such that \((P_k, S_{qa})\) is present in the set \( C_\ell \), corresponding to every \( P_k \in A_{qa,k} \). We show that the set \( A_{qa,k} \) constitutes the candidate \( B_{qa} \). For this, note that for any \( P_k \in A_{qa,k} \), party \( P_k \) includes \((P_k, S_{qa})\) to \( C_i \), only after receiving a message \((CanCS, P_k, S_{qa}, C_i, W_{qa,q}^{(1)} q=1,...,|Z_q|, BS_{qa}^{(1)}, \{s_q^{(2)}\}_{q \in BS_{qa}^{(1)}}\) from the broadcast of \( P_k \) and verifying it. It then follows from Lemma E.8 that eventually, every party in \( \mathcal{H} \) would have received
The communication complexity of the protocol is dominated by the instances of Proof. The number of Protocol Lemma E.9. Hence each party $P_t \in \mathcal{H}$ would eventually include $(P_t, S_{q_{\text{core}}})$ to the set $C_i$, corresponding to every $P_t \in A_{q_{\text{core}},k}$.

We next claim that corresponding to every $P_t \in$ CORE, every $P_t \in \mathcal{H}$ eventually receives a message $(\text{CanCS}, P_t, S_{q_{\text{core}}}, \{W^{(t)}_{q_{\text{core}},q} \}_{q=1, \ldots, |Z_s|}, BS^{(t)}_{q_{\text{core}}}, \{s^{(t)}_q \}_{q \in BS^{(t)}_{q_{\text{core}}}} )$ from the broadcast of $P_t$. The proof for this is very similar to the proof of the previous claim and relies on the properties of $\Pi_{\text{BA}}$ in the asynchronous network. So consider an arbitrary $P_t \in$ CORE. This implies that the instance $\Pi_{\text{BA}}^{(2,t)}$ have produced output 1, which further implies that at least one party from $\mathcal{H}$, say $P_m$, have participated with input 1 during the instance $\Pi_{\text{BA}}^{(2,t)}$. If not, then from the $Z_s$-validity of $\Pi_{\text{BA}}$ in the asynchronous network (Theorem 3.1), the instance $\Pi_{\text{BA}}^{(2,t)}$ would have produced output 0, which is a contradiction. Now since $P_m$ participates with input 1 in the instance $\Pi_{\text{BA}}^{(2,t)}$, it follows that $P_m$ must have received a message $(\text{CanCS}, P_t, S_{q_{\text{core}}}, \{W^{(t)}_{q_{\text{core}},q} \}_{q=1, \ldots, |Z_s|}, BS^{(t)}_{q_{\text{core}}}, \{s^{(t)}_q \}_{q \in BS^{(t)}_{q_{\text{core}}}} )$ from the broadcast of $P_t$ and included $(P_t, S_{q_{\text{core}}})$ to $C_m$. It then follows from Lemma E.8 that eventually, every party $P_i$ in $\mathcal{H}$ would receive $(\text{CanCS}, P_t, S_{q_{\text{core}}}, \{W^{(t)}_{q_{\text{core}},q} \}_{q=1, \ldots, |Z_s|}, BS^{(t)}_{q_{\text{core}}}, \{s^{(t)}_q \}_{q \in BS^{(t)}_{q_{\text{core}}}} )$ from $P_t$ and includes $(P_t, S_{q_{\text{core}}})$ to $C_i$.

Till now we have shown that almost-surely, all the $|Z_s|$ instances $\Pi_{\text{BA}}^{(1,*)}$ of $\Pi_{\text{BA}}$ and then all the $|CD|$ instances $\Pi_{\text{BA}}^{(2,*)}$ of $\Pi_{\text{BA}}$ will eventually produce some output, for all the parties in $\mathcal{H}$. Moreover, all the parties in $\mathcal{H}$ will eventually have a common $q_{\text{core}} \in \{1, \ldots, |Z_s|\}$ and a common set CORE $\subseteq \mathcal{P}$, where CORE has at least one honest party. Furthermore, corresponding to every $P_t \in$ CORE, each $P_t \in \mathcal{H}$ would eventually received a message $(\text{CanCS}, P_t, S_{q_{\text{core}}}, \{W^{(t)}_{q_{\text{core}},q} \}_{q=1, \ldots, |Z_s|}, BS^{(t)}_{q_{\text{core}}}, \{s^{(t)}_q \}_{q \in BS^{(t)}_{q_{\text{core}}}} )$ from the broadcast of $P_t$. The rest of the proof will be now same as Lemma E.4, except that we now rely on Lemma E.6 and Lemma E.7. To avoid repetition, we do not produce the formal details here.

We finally derive the communication complexity of the protocol.

**Lemma E.9.** Protocol $\Pi_{\text{MDVSS}}$ incurs a communication of $O(|Z_s|^2 \cdot n^9 \cdot \log |F| \cdot |\sigma|)$ bits. In addition, $O(|Z_s| + n)$ instances of $\Pi_{\text{BA}}$ are invoked.

**Proof.** The number of $\Pi_{\text{BA}}$ instances follows easily from the protocol inspection. The communication complexity of the protocol is dominated by the instances of $\Pi_{\text{SVM}}$ and $\Pi_{\text{Auth}}$ invoked in the protocol. There are $O(|Z_s| \cdot n)$ instances of $\Pi_{\text{SVM}}$ and $O(|Z_s| \cdot n^3)$ instances of $\Pi_{\text{Auth}}$ invoked. The communication complexity now follows from the communication complexity of $\Pi_{\text{SVM}}$ (Lemma 6.3) and $\Pi_{\text{Auth}}$ (Theorem 4.1).

Theorem 7.1 now follows from Lemma E.4, Lemma E.8 and Lemma E.9.
F Properties of the Triple-Generation Protocol

In this section, we prove the properties of the triple-generation protocol and related sub-protocols.

F.1 Properties of the Protocol $\Pi_{LSh}$

We first prove the properties of the protocol $\Pi_{LSh}$ (see Fig 10 for the formal description).

**Lemma 8.1.** Let $r$ be a random value which is linearly secret-shared with IC-signatures with $GW_1, \ldots, GW_{|Zs|}$ being the underlying core-sets. Then protocol $\Pi_{LSh}$ achieves the following where $D$ participates with the input $s$.

- If $D$ is honest, then the following hold, where $T_{LSh} = T_{Rec} + T_{BC}$.
  - $Z_s$-**Correctness:** If the network is synchronous, then except with probability $O(n^3 \cdot \epsilon_{ICP})$, the honest parties output $[s]$ at the time $T_{LSh}$, with $GW_1, \ldots, GW_{|Zs|}$ being the underlying core-sets.
  - $Z_s$-**Correctness:** If the network is asynchronous, then except with probability $O(n^3 \cdot \epsilon_{ICP})$, the honest parties eventually output $[s]$, with $GW_1, \ldots, GW_{|Zs|}$ being the underlying core-sets.
  - Privacy: Irrespective of the network type, the view of the adversary remains independent of $s$.
  - If $D$ is corrupt then either no honest party computes any output or there exists some value, say $s^\star$, such that the following hold.
    - $Z_s$-**Commitment:** If the network is synchronous, then except with probability $O(n^3 \cdot \epsilon_{ICP})$, the honest parties eventually output $[s^\star]$, with $GW_1, \ldots, GW_{|Zs|}$ being the underlying core-sets. Moreover, if any honest party computes its output at the time $T$, then all honest parties will have their respective output by the time $T + \Delta$.
    - $Z_s$-**Commitment:** If the network is asynchronous, then except with probability $O(n^3 \cdot \epsilon_{ICP})$, the honest parties eventually output $[s^\star]$, with $GW_1, \ldots, GW_{|Zs|}$ being the underlying core-sets.
    - Communication Complexity: $O(|Zs| \cdot n^3 \cdot \log |F| + n^4 \cdot \log |F| \cdot |\sigma|)$ bits are communicated by the honest parties.

Proof. Let us first consider an honest dealer. Moreover, we consider a synchronous network. Form the $Z_s$-correctness of $\Pi_{Rec}$ in the synchronous network (Lemma 6.2), after time $T_{Rec}$, the dealer $D$ will reconstruct $r$, except with probability $O(n^3 \cdot \epsilon_{ICP})$. Moreover, from the privacy property of $\Pi_{Rec}$, $r$ will be random from the point of the view of the adversary. Since $D$ is honest, from the $Z_s$-validity of $\Pi_{BC}$ in the synchronous network (Theorem 3.2), all honest parties will receive $s$ from the broadcast of $D$ at the time $T_{Rec} + T_{BC}$, where $s = s + r$. The parties then take the default linear secret-sharing $[s]$ of $s$ with the IC-signatures, with $GW_1, \ldots, GW_{|Zs|}$ being the underlying core-sets. Since $r$ is also linearly secret-shared with IC-signatures, with $GW_1, \ldots, GW_{|Zs|}$ being the underlying core-sets, from the linearity property of the secret sharing, it follows that $[s - r]$ will be the
same as a linear secret-sharing of \( s \) with IC-signatures, with \( GW_1, \ldots, GW_{|Z_s|} \) being the underlying core-sets. This proves the \( Z_s \)-correctness. The privacy of \( s \) follows since \( r \) remains random for the adversary and hence \( s \) does not reveal any information about \( s \) to the adversary. The \( Z_a \)-correctness and privacy for an honest dealer in an asynchronous network follows using similar arguments as above, except that we now rely on the \( Z_a \)-correctness of \( \Pi_{\text{Rec}} \) in the asynchronous network (Lemma 6.2) and the \( Z_a \)-validity of \( \Pi_{\text{BC}} \) in the asynchronous network (Theorem 3.2).

We next consider a corrupt dealer and a synchronous network. Let \( P_h \) be the first honest party which computes some output in the protocol, at the time \( T \). This implies that \( P_h \) has received some value \( s \) from the broadcast of \( D \). From the \( Z_s \)-consistency and \( Z_s \)-fallback consistency of \( \Pi_{\text{BC}} \) in the synchronous network (Theorem 3.2), it follows that all the honest parties will receive \( s \) from the broadcast of \( D \), latest by the time \( T + \Delta \). We define

\[
s^* \overset{\text{def}}{=} s - r.
\]

Since the (honest) parties take the default linear secret-sharing \([s]\) of \( s \) with the IC-signatures, with \( GW_1, \ldots, GW_{|Z_s|} \) being the underlying core-sets and since \( r \) is also linearly secret-shared with IC-signatures, with \( GW_1, \ldots, GW_{|Z_s|} \) being the underlying core-sets, from the linearity property of the secret sharing, it follows that \([s - r]\) will be the same as a linear secret-sharing of \( s^* \) with IC-signatures, with \( GW_1, \ldots, GW_{|Z_s|} \) being the underlying core-sets. This completes the proof of the \( Z_s \)-commitment in the synchronous network. The proof of the \( Z_a \)-commitment in the asynchronous network is similar as above, except that we now rely on the \( Z_a \)-weak consistency and \( Z_a \)-fallback consistency of \( \Pi_{\text{BC}} \) in the asynchronous network (Theorem 3.2) and \( Z_a \)-correctness of \( \Pi_{\text{Rec}} \) in the asynchronous network (Lemma 6.2).

In the protocol, one instance of \( \Pi_{\text{Rec}} \) with \(|R| = 1\) and one instance of \( \Pi_{\text{BC}} \) with \( \ell = \log |F| \) bits are invoked. The communication complexity now follows from the communication complexity of \( \Pi_{\text{Rec}} \) (Lemma 6.2) and communication complexity of \( \Pi_{\text{BC}} \) (Theorem 3.2).

### F.2 Properties of the Protocol \( \Pi_{\text{BasicMult}} \)

In this section, we prove the properties of the protocol \( \Pi_{\text{BasicMult}} \) (see Fig 11 for the formal details). While proving these properties, we will assume that no honest party is present in the set \( GD \); looking ahead, this will be ensured in the protocol \( \Pi_{\text{RandMultCI}} \) (presented in Section 8.3), where the set \( GD \) is maintained.

We begin by showing that no summand-sharing party during the first two phases are from the discarded set of parties.

**Lemma 8.2.** During any instance \( \Pi_{\text{BasicMult}}(Z_s, Z_a, S_{Z_s}, [a], [b], GW_1, \ldots, GW_{|Z_s|}, GD, \text{iter}) \) of \( \Pi_{\text{BasicMult}} \), if \( P_j \in \text{Selected}_{\text{iter}} \) then \( P_j \notin GD \), irrespective of the network type.
Proof. Let $P_j$ be an arbitrary party belonging to the set $\text{Selected}_{\text{iter}}$. This implies that $P_j$ is included to $\text{Selected}_{\text{iter}}$, either during Phase I or Phase II. If $P_j \in \mathcal{GD}$, then no honest party will participate with input 1 in the instance $\Pi_{\text{BA}}^{(\text{phi}, \text{hop}, j)}$ during Phase I for any value of hop and instance $\Pi_{\text{BA}}^{(\text{phi}, j)}$ during Phase II. Consequently, from the $Z_a$-validity and $Z_a$-validity of $\Pi_{\text{BA}}$ (Theorem 3.1), party $P_j$ will not be added to $\text{Selected}_{\text{iter}}$, which is a contradiction.

We next show that the first phase will get over for the honest parties after a fixed time in a synchronous network and eventually in an asynchronous network. Towards this we show that if the honest parties start any hop during the first phase, then they will complete it after a fixed time in a synchronous network and eventually in an asynchronous network.

**Lemma 8.3.** Suppose that no honest party is present in $\mathcal{GD}$. If the honest parties start participating during hop number hop of Phase I of $\Pi_{\text{BasicMult}}$ with iteration number iter, then except with probability $O(n^3 \cdot \epsilon_{\text{ICP}})$, the hop takes $T_{\text{LSH}} + 2T_{\text{BA}}$ time to complete in a synchronous network, or almost-surely completes eventually in an asynchronous network.

*Proof.* Let $Z^*$ be the set of corrupt parties and let $\mathcal{H} = \mathcal{P} \setminus Z^*$ be the set of honest parties. We note that since $Z_a \subset Z_s$, irrespective of the network type, there exists some set in $S_{Z_a}$, say $S_h$, such that $S_h \in S_{Z_s}$. Since the honest parties participate in hop number hop, it implies that there exists no $S_q \in S_{Z_s}$, such that $\text{SIS}_{\text{iter}}^{(S_q)} = \emptyset$. Particularly, this implies that $\text{SIS}_{\text{iter}}^{(S_h)} \neq \emptyset$. Hence there exists some $P_j \in \mathcal{H}$, such that $P_j \notin \text{Selected}_{\text{iter}}$. This is because if $\mathcal{H} \subseteq \text{Selected}_{\text{iter}}$, then clearly $\text{SIS}_{\text{iter}}^{(S_h)} = \emptyset$ and hence the parties in $\mathcal{H}$ will not participate in hop number hop.

Let us first consider a synchronous network. During the hop number hop, every $P_j \in \mathcal{H}$ such that $P_j \notin \text{Selected}_{\text{iter}}$ will invoke an instance of $\Pi_{\text{LSH}}$ with the input $c_{\text{iter}}^{(j)}$. Then from the $Z_a$-correctness of $\Pi_{\text{LSH}}$ in the synchronous network (Lemma 8.1), after time $T_{\text{LSH}}$, the parties in $\mathcal{H}$ output $[c_{\text{iter}}^{(j)}]$, except with probability $O(n^3 \cdot \epsilon_{\text{ICP}})$. Since $(\mathcal{H} \cap \mathcal{GD}) = \emptyset$, all the parties in $\mathcal{H}$ will participate in the instance $\Pi_{\text{BA}}^{(\text{phi}, \text{hop}, j)}$ with input 1. Hence from the $Z_a$-validity of $\Pi_{\text{BA}}$ in the synchronous network (Theorem 3.1), all the parties in $\mathcal{H}$ output 1 during the instance $\Pi_{\text{BA}}^{(\text{phi}, \text{hop}, j)}$, after time $T_{\text{LSH}} + T_{\text{BA}}$. Consequently, the parties in $\mathcal{H}$ start participating with input 0 in the remaining $\Pi_{\text{BA}}$ instances $\Pi_{\text{BA}}^{(\text{phi}, \text{hop}, k)}$ (if any), for which no input is provided yet. Hence from the $Z_a$-security of $\Pi_{\text{BA}}$ in the synchronous network, at the time $T_{\text{LSH}} + 2T_{\text{BA}}$, the parties in $\mathcal{H}$ compute some output during the $\Pi_{\text{BA}}$ instances invoked during hop number hop and hence complete the hop number hop.

Next, consider an asynchronous network. From the $Z_a$-correctness of $\Pi_{\text{LSH}}$ in the asynchronous network (Lemma 8.1), corresponding to every $P_j \in \mathcal{H}$ such that $P_j \notin \text{Selected}_{\text{iter}}$, the parties in $\mathcal{H}$ eventually output $[c_{\text{iter}}^{(j)}]$, except with probability $O(n^3 \cdot \epsilon_{\text{ICP}})$, during the instance $\Pi_{\text{LSH}}^{(\text{phi}, \text{hop}, j)}$. Now there are now two cases.
- **Case 1** - There exists some $P_j \in \mathcal{H}$ where $P_j \notin \text{Selected}_{\text{iter}}$ and some party $P_i \in \mathcal{H}$, such that $P_i$ has started participating with input 0 in the instance $\Pi_{BA}^{(\text{phi, hop}, j)}$. This implies that $P_i$ has computed the output 1 in some $\Pi_{BA}$ instance during the Phase I, say $\Pi_{BA}^{(\text{phi, hop}, m)}$. And hence $P_i$ starts participating in all the remaining $\Pi_{BA}$ instances of hop number hop of Phase I (if any) with the input 0. From the $Z_a$-consistency of $\Pi_{BA}$ in the asynchronous network, all the parties in $\mathcal{H}$ will also eventually compute the output 1 during the instance $\Pi_{BA}^{(\text{phi, hop}, m)}$ and will start participating in all the remaining $\Pi_{BA}$ instances of hop number hop of Phase I (if any) with the input 0. Consequently, from the $Z_a$-security of $\Pi_{BA}$ in the asynchronous network (Theorem 3.1), almost-surely, the parties in $\mathcal{H}$ eventually compute some output during all the $\Pi_{BA}$ instances of hop number hop and hence complete the hop number hop.

- **Case 2** - No honest party has yet started participating with input 0 in any of the BA instances of Phase I corresponding to the honest parties: In this case, the honest parties will eventually start participating with input 1 in the $\Pi_{BA}$ instances $\Pi_{BA}^{(\text{phi, hop}, j)}$, corresponding to the parties $P_j \in \mathcal{H}$ where $P_j \notin \text{Selected}_{\text{iter}}$. Hence from the $Z_a$-validity of $\Pi_{BA}$ in the asynchronous network (Theorem 3.1), it follows that eventually, there will be some $P_j \in \mathcal{H}$ where $P_j \notin \text{Selected}_{\text{iter}}$, such that all the parties in $\mathcal{H}$ compute the output 1 during the instance $\Pi_{BA}^{(\text{phi, hop}, j)}$. The rest of the proof is similar to what is argued in the previous case.

We next show that in protocol $\Pi_{BasicMult}$, the honest parties compute some output, after a fixed time in a synchronous network and eventually in an asynchronous network.

**Lemma 8.4.** If no honest party is present in $GD$, then in protocol $\Pi_{BasicMult}$, except with probability $O(n^3 \cdot \epsilon_{ICP})$, all honest parties compute some output by the time $T_{BasicMult} = (2n + 1) \cdot T_{BA} + (n + 1) \cdot T_{LSh} + T_{Rec}$ in a synchronous network, or almost-surely, eventually in an asynchronous network.

**Proof.** In the protocol $\Pi_{BasicMult}$, to compute an output, the (honest) parties need to complete the three phases. We first show that irrespective of the network type, the honest parties will complete Phase II, provided they complete Phase I. This is because the honest parties will participate in all the $\Pi_{BA}$ instances $\Pi_{BA}^{(\text{phi, hop}, j)}$ of phase II with some input, after waiting exactly for $T_{LSh}$. Consequently, once phase I is completed, from the $Z_a$-security of $\Pi_{BA}$ in the synchronous network (Theorem 3.1), it takes $T_{LSh} + T_{BA}$ time for the honest parties to compute outputs in the $\Pi_{BA}$ instances during phase II in a synchronous network. On the other hand, the $Z_a$-security of $\Pi_{BA}$ in the asynchronous network (Theorem 3.1) guarantees that almost-surely, all honest parties eventually compute some output during the $\Pi_{BA}$ instances during phase II in an asynchronous network. Now once Phase I and Phase II are completed, it takes $T_{Rec}$ time for the parties to complete Phase III in a synchronous network (follows from Lemma 6.2), while in an asynchronous network, it gets completed eventually (see Lemma 6.2).
We now show that Phase I always gets completed for the honest parties. Lemma 8.3 guarantees that if the honest parties start any hop during Phase I, then it gets completed for all the honest parties after time $T_{LS} + 2T_{BA}$ in a synchronous network or eventually in an asynchronous network. From the protocol steps, it follows that there can be at most $n$ hops during Phase I. This is because once the set of honest parties is included in the set of summand-sharing parties $\text{Selected}_{iter}$ during Phase I, then the parties will exit Phase I.

We next show that the adversary does not learn anything additional about $a$ and $b$ during the protocol.

**Lemma 8.5.** If no honest party is present in $GD$, then the view of the adversary remains independent of $a$ and $b$ throughout the protocol, irrespective of the network type.

**Proof.** Let $Z^*$ be the set of corrupt parties and let $H = \mathcal{P} \setminus Z^*$ be the set of honest parties. The view of the adversary remains independent of $a$ and $b$ during Phase I and Phase II. This is because the privacy of $\Pi_{LS}$ (Lemma 8.1) ensures that the view of the adversary remains independent of the summand-sums shared by the honest summand-sharing parties during Phase I and Phase II. Let $S_h \subseteq \mathbb{Z}_n$ be the group consisting of only honest parties; i.e. $(S_h \cap Z^*) = \emptyset.$ To prove that the view of the adversary remains independent of $a$ and $b$ during Phase III, we show that irrespective of the network type, any summand of the form $(h, q)$ or $(p, h)$ will not be present in $SIS_{iter}$ during this phase corresponding to any $p, q \in \{1, \ldots, |\mathbb{Z}_n|\}$, implying that the shares $[a]_h$ and $[b]_h$ does not get publicly reconstructed. Note that $(h, q) \notin SIS^{(j)}_{iter}$ for any $P_j \in Z^*$, as otherwise it would imply that a corrupt $P_j \in (S_h \cap Z^*)$, which is a contradiction. Similarly, $(p, h) \notin SIS^{(j)}_{iter}$ for any $P_j \in Z^*$.

Let us first consider a synchronous network and consider an arbitrary ordered pair $(h, q)$. Consider the case when $(h, q)$ has not been removed from $SIS_{iter}$ during Phase I in any of the hops. Then, during Phase II, the pair $(h, q)$ will get statically re-assigned to the $SIS^{(j)}_{iter}$ set of some $P_j \in H$, such that $P_j \notin \text{Selected}_{iter}$. From the protocol steps, $P_j$ will include the summand $[a]_h \cdot [b]_q$ while computing $c^{(j)}_{iter}$ and share $c^{(j)}_{iter}$ through an instance of $\Pi_{LS}$. From the $Z^*_s$-correctness of $\Pi_{LS}$ in the synchronous network, the parties in $H$ will output $[c^{(j)}_{iter}]$ during the instance of $\Pi_{LS}$ invoked by $P_j$. Since $P_j \notin (\text{Selected}_{iter} \cup GD)$, it follows that all the parties in $H$ will participate in the instance $\Pi^{(\phi_{BA})}_{iter}$ with input 1 and compute the output 1. Consequently, $(h, q)$ will be removed from the updated $SIS_{iter}$ during Phase II, if not removed during Phase I. By the same logic, any ordered pair of the form $(p, h)$ will also be removed from $SIS_{iter}$, by the end of Phase II.

Next, consider an asynchronous network and an arbitrary $(h, q)$. In this case, we show that $(h, q)$ will eventually be removed from $SIS_{iter}$ during Phase I itself. Let the parties in $H$ complete Phase I. This implies that there exists some $S_t \subseteq \mathbb{Z}_n$ such that $SIS^{(S_t)}_{iter} = \emptyset$ when Phase I gets over. We show that $(h, q)$ was present in $SIS^{(S_t)}_{iter}$ at the beginning of Phase I, when the parties initialize $SIS^{(S_t)}_{iter}$.
That is, at the time of initialization, there was some \( P_j \in (S_\ell \cap H \cap S_q) \) such that \((h, q) \in \text{SIS}_{\text{iter}}^{(j)}\). For this, it is enough to show that \((S_\ell \cap H \cap S_q) \neq \emptyset\), which follows from the fact that \(Z_s\) and \(\overline{Z_s}\) satisfy the condition \(Q^{(2\cdot1)}(P, Z_s, Z_a)\) and \((S_\ell \cap H \cap S_q) = P \setminus (Z_\ell \cup Z^* \cup Z_q)\), where \(Z^* \in Z_a\) (because we are considering an asynchronous network) and \(Z_\ell, Z_q \in Z_s\) (follows from the construction of \(\overline{S_z}\))

Let \( P_j \in (S_\ell \cap H \cap S_q) \). Hence at the time of the initialization, \((h, q) \in \text{SIS}_{\text{iter}}^{(j)}\), which further implies that \((h, q) \in \text{SIS}_{\text{iter}}^{(j)} \subseteq \text{SIS}_{\text{iter}}^{(S_j)}\) at the time of initialization. And hence \((h, q)\) must have been removed from \(\text{SIS}_{\text{iter}}\) by the end of Phase I, since \( P_j \) would have been selected as the summand-sharing party in one of the hops during Phase I. The same logic also applies to any arbitrary \((p, h)\), implying that \((p, h)\) would have been removed from \(\text{SIS}_{\text{iter}}\) by the end of Phase I itself.

We next show that if all the parties behave honestly in the protocol, the parties output a linear-secret sharing of \(a \cdot b\) with IC-signatures.

**Lemma 8.6.** If no honest party is present in \(\mathcal{GD}\) and if all parties in \(\mathcal{P} \setminus \mathcal{GD}\) behave honestly, then in protocol \(\Pi_{\text{BasicMult}}\), the honest parties output a linear secret-sharing of \(a \cdot b\) with IC-signatures, with \(\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z|}\) being the underlying core-sets, irrespective of the network type.

**Proof.** Note that \((\text{Selected}_{\text{iter}} \cap \mathcal{GD}) = \emptyset\), which follows from Lemma 8.2. Now consider an arbitrary \((p, q) \in \text{SIS}_{\text{iter}}\). We claim that irrespective of the network type, if all the parties behave honestly, then the summand \([a]_p \cdot [b]_q\) is considered on behalf of exactly one party while secret-sharing the summand-sums. That is there is exactly one \( P_j \) across the three phases, such that the following hold:

\[
 c^{(j)}_{\text{iter}} = \ldots + [a]_p [b]_q + \ldots
\]

Assuming that the claim is true, the proof then follows from the linearity property of secret-sharing and the fact that \(c_{\text{iter}} = c^{(1)}_{\text{iter}} + \ldots + c^{(n)}_{\text{iter}}\) holds. That is, corresponding to each \( P_j \in \text{Selected}_{\text{iter}}\), the value \(c^{(j)}_{\text{iter}}\) is secret-shared through an instance of \(\Pi_{\text{LSH}}\), with \(\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z|}\) being the underlying core-sets, which guarantees that \(c^{(j)}_{\text{iter}}\) is linearly secret-shared with IC-signatures, with \(\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z|}\) being the underlying core-sets. Moreover, by the end of Phase II, the parties will be publicly knowing the set \(\text{Selected}_{\text{iter}}\), which follows from the security properties of \(\Pi_{\text{BA}}\). Furthermore, after Phase II, the parties will be publicly reconstructing the shares \([a]_p\) and \([b]_q\) corresponding to every \((p, q)\), which are still present in \(\text{SIS}_{\text{iter}}\). Since corresponding to each \( P_j \in \mathcal{P} \setminus \text{Selected}_{\text{iter}}\), the parties take the default linear secret-sharing of \(c^{(j)}_{\text{iter}}\) with IC-signatures and \(\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z|}\) being the underlying core-sets, it follows that \(c\) will be linearly secret-shared with IC-signatures and \(\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z|}\) being the underlying core-sets. Moreover, \(c = a \cdot b\) holds, since each summand \([a]_p \cdot [b]_q\) is considered exactly once, as per our claim. We now proceed to prove our claim.

We first show that there exists at least one \( P_j \) in one of the three phases, such that the following holds:

\[
 c^{(j)}_{\text{iter}} = \ldots + [a]_p [b]_q + \ldots
\]
For this, consider the following cases.

- **Case 1** — During Phase I, there is some \( P_j \in \text{Selected}_{\text{iter}} \), such that \((p, q)\) was present in the set \( \text{SIS}_{\text{iter}}^{(j)} \) when \( P_j \) was added to \( \text{Selected}_{\text{iter}} \). In this case, there is nothing to show.

- **Case 2** — At the end of Phase I, \((p, q)\) is still present in \( \text{SIS}_{\text{iter}} \).

This implies that at the end of Phase I, \(( \text{Selected}_{\text{iter}} \cap S_p \cap S_q ) = \emptyset \). Since \( Z_s \) and \( Z_a \) satisfy the condition \( Q^{(2,1)}(P, Z_s, Z_a) \), it follows that \((S_p \cap S_q) \neq \emptyset \). Let \( P_j \triangleq \min(S_p, S_q) \). Note that \( P_j \not\in \text{Selected}_{\text{iter}} \) at the end of Phase I, otherwise \((p, q)\) would have been removed from \( \text{SIS}_{\text{iter}} \), which is a contradiction. Now if \( P_j \not\in GD \), then clearly the summand \([a]_p \cdot [b]_q\) will be present in \( c_{\text{iter}}^{(j)} \), shared by \( P_j \) during the Phase II. Else, during Phase III, the parties will publicly reconstruct \([a]_p\) and \([b]_q\) and consequently \([a]_p \cdot [b]_q\) will be present in the default secret-sharing of \( c_{\text{iter}}^{(j)} \), taken on the behalf of \( P_j \).

To complete the proof of our claim, we next show that \((p, q)\) cannot be present in the summand-sum of more than one party across the three phases. On contrary, let \( P_j \) and \( P_k \) be two distinct parties, such that the following holds across the three phases:

\[
c_{\text{iter}}^{(j)} = \ldots + [a]_p [b]_q + \ldots \quad \land \quad c_{\text{iter}}^{(k)} = \ldots + [a]_p [b]_q + \ldots
\]

Now there are three following cases.

- **Case 1** — \( P_j, P_k \in \text{Selected}_{\text{iter}} \) at the end of Phase I: From the security properties of \( \Pi_{\text{BA}} \), the parties will agree on which party to add to \( \text{Selected}_{\text{iter}} \) during every hop during Phase I. Moreover, from the protocol steps, exactly one party is selected as a summand-sharing party and added to \( \text{Selected}_{\text{iter}} \) in each hop. Suppose that \( P_j \) was added to \( \text{Selected}_{\text{iter}} \) during hop \((j)\), and that \( P_k \) was added to \( \text{Selected}_{\text{iter}} \) during hop \((k)\). Moreover, without loss of generality, let \( \text{hop}^{(j)} < \text{hop}^{(k)} \). From the protocol steps, it follows that \( P_j, P_k \not\in GD \); as otherwise no honest party would have voted for \( P_j \) and \( P_k \) as a candidate summand-sharing party and consequently, \( P_j, P_k \not\in \text{Selected}_{\text{iter}} \). Now as per the lemma conditions, all the parties (including \( P_k \)) behave honestly. Hence, the ordered pair \((p, q)\) would be removed from \( \text{SIS}_{\text{iter}}^{(k)} \) at the end of \( \text{hop}^{(j)} \). Consequently, \( P_k \) will not include the summand \([a]_p \cdot [b]_q\) while computing \( c_{\text{iter}}^{(k)} \) during hop number \( \text{hop}^{(k)} \), which is a contradiction.

- **Case 2** — \( P_j \in \text{Selected}_{\text{iter}} \) at the end of Phase I and \( P_k \not\in \text{Selected}_{\text{iter}} \) at the end of Phase I: In this case, \((p, q)\) will be removed from \( \text{SIS}_{\text{iter}} \) at the end of Phase I. Hence, it cannot get re-assigned to any other party \( P_k \) after Phase I and hence cannot belong to \( \text{SIS}_{\text{iter}}^{(k)} \). Consequently, the summand \([a]_p \cdot [b]_q\) will not be considered while computing \( c_{\text{iter}}^{(k)} \), which is a contradiction.

- **Case 3** — \( P_j \not\in \text{Selected}_{\text{iter}} \) and \( P_k \not\in \text{Selected}_{\text{iter}} \) at the end of Phase I: In this case, the summand \([a]_p \cdot [b]_p\) is deterministically and statically reassigned to the least-indexed party from the set \((S_p \cap S_q)\), as per the min function. Hence \((p, q)\) will be re-assigned to either \( P_j \) or \( P_k \), but not both.
Consequently, the summand \[a_p \cdot b_q\] will be considered while computing either \[c_{\text{iter}}^{(j)}\] or \[c_{\text{iter}}^{(k)}\], but not both, which is a contradiction.

**Lemma 8.7.** Protocol \(\Pi_{\text{BasicMult}}\) incurs a communication of \(O(|Z_s| \cdot n^5 \cdot \log |F| + n^6 \cdot \log |F| \cdot |\sigma|)\) bits and makes \(O(n^2)\) calls to \(\Pi_{\text{BA}}\).

**Proof.** During Phase I, there can be \(O(n)\) hops, where during each hop, a party from \(P \setminus \mathcal{G}D\) secret-shares a field element through an instance of \(\Pi_{\text{LSh}}\). Moreover, \(n\) instances of \(\Pi_{\text{BA}}\) are invoked to agree upon the summand-sharing party of the hop. Hence, \(O(n^2)\) instances of \(\Pi_{\text{LSh}}\) and \(O(n^2)\) instances of \(\Pi_{\text{BA}}\) are required during Phase I. During Phase II, \(O(n)\) instances of \(\Pi_{\text{LSh}}\) and \(O(n)\) instances of \(\Pi_{\text{BA}}\) are required. Finally during Phase III, up to \(O(|Z_s|)\) shares need to be publicly reconstructed. The communication complexity now follows from the communication complexity of \(\Pi_{\text{LSh}}\) (Lemma 8.1) and communication complexity of \(\Pi_{\text{RecShare}}\) with \(|R| = n\) (Lemma 6.1).

As a corollary of Lemma 8.7, we derive the following corollary, which determines the maximum number of instances of \(\Pi_{\text{LSh}}\) which are invoked during an instance of \(\Pi_{\text{BasicMult}}\). Looking ahead, this will be useful to later calculate the maximum number of instances of \(\Pi_{\text{LSh}}\) which need to be invoked as part of our final multiplication protocol. This will be further useful to determine the number of linearly secret-shared values with IC-signatures and core-sets \(\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z_s|}\), which need to be generated through the protocol \(\Pi_{\text{Rand}}\) beforehand.

**Corollary 8.1.** During any instance of \(\Pi_{\text{BasicMult}}\), there can be at most \(n^2 + n\) instances of \(\Pi_{\text{LSh}}\) invoked.

**Proof.** The proof follows from the fact that during Phase I, there can be up to \(n^2\) instances of \(\Pi_{\text{LSh}}\), if \(\mathcal{G}D = \emptyset\) and during Phase II, there can be up to \(n - 1\) instances of \(\Pi_{\text{LSh}}\), if only one party is added to \(\text{Selected}_{\text{iter}}\) during Phase I.

### F.3 Properties of the Protocol \(\Pi_{\text{RandMultCI}}\)

In this section, we prove the properties of the protocol \(\Pi_{\text{RandMultCI}}\) (see Fig 12 for the formal description). We begin by showing that irrespective of the network type, the (honest) parties compute linearly secret-shared \(a_{\text{iter}}, b_{\text{iter}}, b'_{\text{iter}}\) and \(r_{\text{iter}}\) with IC-signatures, which are random from the point of view of the adversary.

**Lemma 8.8.** In protocol \(\Pi_{\text{RandMultCI}}\), the following hold.

- **Synchronous Network:** Except with probability \(O(n^3 \cdot \epsilon_{\text{CP}})\), honest parties will have linearly secret-shared \(a_{\text{iter}}, b_{\text{iter}}, b'_{\text{iter}}\) and \(r_{\text{iter}}\) with IC-signatures, with \(\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z_s|}\) being the underlying core-sets, by the time \(T_{\text{LSh}} + 2T_{\text{BA}}\). Moreover, adversary’s view is independent of \(a_{\text{iter}}, b_{\text{iter}}, b'_{\text{iter}}\) and \(r_{\text{iter}}\).

- **Asynchronous Network:** Except with probability \(O(n^3 \cdot \epsilon_{\text{CP}})\), almost-surely, honest parties will eventually have linearly secret-shared \(a_{\text{iter}}, b_{\text{iter}}, b'_{\text{iter}}\) and \(r_{\text{iter}}\) with IC-signatures, with \(\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z_s|}\) being the underlying core-sets. Moreover, adversary’s view is independent of \(a_{\text{iter}}, b_{\text{iter}}, b'_{\text{iter}}\) and \(r_{\text{iter}}\).
Proof. We first consider a synchronous network. Let \( Z^* \in \mathcal{Z}_s \) be the set of corrupt parties and let \( \mathcal{H} = \mathcal{P} \setminus Z^* \) be the set of honest parties. Corresponding to each \( P_j \in \mathcal{H} \), the honest parties compute the output \([a_{\text{iter}}^{(j)}, b_{\text{iter}}^{(j)}, c_{\text{iter}}^{(j)}] \) and \([r_{\text{iter}}^{(j)}] \) during the instances of \( \Pi_{LSH} \) invoked by \( P_j \) at the time \( T_{LSH} \), except with probability \( \mathcal{O}(n^3 \cdot \epsilon_{CP}) \). This follows from the \( Z_s \)-correctness of \( \Pi_{LSH} \) in the synchronous network (Lemma 8.1). Consequently, at the time \( T_{LSH} \), all the parties in \( \mathcal{H} \) will be present in the set \( \mathcal{C}_i \) of every \( P_i \in \mathcal{H} \). Hence corresponding to each \( P_j \in \mathcal{H} \), each \( P_i \in \mathcal{H} \) starts participating with input 1 in the instance \( \Pi_{BA}^{(j)} \) at the time \( T_{LSH} \). Hence from the \( Z_s \)-validity and \( Z_s \)-guaranteed liveness properties of \( \Pi_{BA} \) in the synchronous network (Theorem 3.1), it follows that at the time \( T_{LSH} + T_{BA} \), all the parties in \( \mathcal{H} \) compute the output 1 during the instance \( \Pi_{BA}^{(j)} \), corresponding to every \( P_j \in \mathcal{H} \). Consequently, at the time \( T_{LSH} + T_{BA} \), all the parties in \( \mathcal{H} \) will start participating in the remaining \( \Pi_{BA} \) instances for which no input has been provided yet (if there are any). And from the \( Z_s \)-guaranteed liveness and \( Z_s \)-consistency properties of \( \Pi_{BA} \) in the synchronous network (Theorem 3.1), these \( \Pi_{BA} \) instances will produce common outputs for every honest party by the time \( T_{LSH} + 2T_{BA} \). Hence, at the time \( T_{LSH} + 2T_{BA} \), the honest parties will have a common \( \mathcal{C}S \), where \( \mathcal{P} \setminus \mathcal{C}S \in \mathcal{Z}_s \) and where \( \mathcal{H} \subseteq \mathcal{C}S \).

We next wish to show that corresponding to every \( P_j \in \mathcal{C}S \), there exists some quadruplet of values, which are linearly secret-shared with IC-signatures, with \( \mathcal{GW}_1, \ldots, \mathcal{GW}_{|\mathcal{Z}_s|} \) being the underlying core-sets.

Consider an arbitrary party \( P_j \in \mathcal{C}S \). If \( P_j \in \mathcal{H} \), then whatever we wish to show is correct, as shown above. Next, consider a corrupt \( P_j \in \mathcal{C}S \). Since \( P_j \in \mathcal{C}S \), it follows that the instance \( \Pi_{BA}^{(j)} \) produces the output 1 for all honest parties. This further implies that at least one honest \( P_i \) must have computed some output during the instances of \( \Pi_{LSH} \) invoked by \( P_j \), by the time \( T_{LSH} + T_{BA} \) (implying that \( P_j \in \mathcal{C}_i \)) and participated with input 1 in the instance \( \Pi_{BA}^{(j)} \). Otherwise, all honest parties would participate with input 0 in the instance \( \Pi_{BA}^{(j)} \) at the time \( T_{LSH} + T_{BA} \) and then from the \( Z_s \)-validity of \( \Pi_{BA} \) in the synchronous network, every honest party would have computed the output 0 in the instance \( \Pi_{BA}^{(j)} \) and hence \( P_j \) will not be present in \( \mathcal{C}S \), which is a contradiction. Now if \( P_j \) has computed some output during the instances of \( \Pi_{LSH} \) invoked by \( P_j \) at the time \( T_{LSH} + T_{BA} \), then from the \( Z_s \)-commitment of \( \Pi_{LSH} \) in the synchronous network (Lemma 8.1), it follows that except with probability \( \mathcal{O}(n^3 \cdot \epsilon_{CP}) \), there exist values \((a_{\text{iter}}^{(j)}, b_{\text{iter}}^{(j)}, c_{\text{iter}}^{(j)}, r_{\text{iter}}^{(j)})\), which will be linearly secret-shared with IC-signature, with \( \mathcal{GW}_1, \ldots, \mathcal{GW}_{|\mathcal{Z}_s|} \) being the underlying core-sets, by the time \( T_{LSH} + T_{BA} + \Delta \).

Since \( \Delta < T_{BA} \), it follows that by the time \( T_{LSH} + 2T_{BA} \), the honest parties will have \([a_{\text{iter}}^{(j)}, b_{\text{iter}}^{(j)}, c_{\text{iter}}^{(j)}, r_{\text{iter}}^{(j)}] \). From the linearity property of secret-sharing, it then follows that by the time \( T_{LSH} + T_{BA} + \Delta \), the values \( a_{\text{iter}}, b_{\text{iter}}, c_{\text{iter}}, r_{\text{iter}} \) will be linearly secret-shared with IC-signatures, with \( \mathcal{GW}_1, \ldots, \mathcal{GW}_{|\mathcal{Z}_s|} \) being the underlying core-sets.

From the privacy of \( \Pi_{LSH} \) (Lemma 8.1), the view of the adversary will be independent of the values \((a_{\text{iter}}^{(j)}, b_{\text{iter}}^{(j)}, c_{\text{iter}}^{(j)}, r_{\text{iter}}^{(j)})\), corresponding to the parties \( P_j \in \mathcal{H} \). As \( \mathcal{H} \) is non-empty, it follows that \( a_{\text{iter}}, b_{\text{iter}}, c_{\text{iter}}, r_{\text{iter}} \) will be indeed random.
from the point of view of the adversary. This completes the proof for the case of synchronous network.

We next consider an asynchronous network. Let $Z^* \in Z_a$ be the set of corrupt parties and let $\mathcal{H} = \mathcal{P} \setminus Z^*$ be the set of honest parties. Notice that $\mathcal{P} \setminus \mathcal{H} \in Z_s$, since $Z_a \subset Z_s$. Now irrespective of the way messages are scheduled, there will be eventually a subset of parties $\mathcal{P} \setminus Z$ for some $Z \in Z_s$, such that all the parties in $\mathcal{H}$ participate with input 1 in the instances of $\Pi_{BA}$, corresponding to the parties in $\mathcal{P} \setminus Z$. This is because, corresponding to every $P_j \in \mathcal{H}$, all the parties in $\mathcal{H}$ eventually compute some output during the instances of $\Pi_{LS}$, invoked by $P_j$ except with probability $O(n^3 \cdot \epsilon_{ICP})$, which follows from the $Z_a$-correctness of $\Pi_{LS}$ in the asynchronous network (Lemma 8.1). So even if the corrupt parties $P_j$ do not invoke their respective instances of $\Pi_{LS}$, there will be a set of $\Pi_{BA}$ instances corresponding to the parties in $\mathcal{P} \setminus Z$ for some $Z \in Z_s$, in which all the parties in $\mathcal{H}$ eventually participate with input 1. Consequently, from the $Z_a$-almost-surely liveness and $Z_a$-consistency properties of $\Pi_{BA}$ in the asynchronous network (Theorem 3.1), these $\Pi_{BA}$ instances eventually produce the output 1 for all the parties in $\mathcal{H}$. Hence, all the parties in $\mathcal{H}$ eventually participate with some input in the remaining $\Pi_{BA}$ instances, which almost-surely produce some output for every honest party eventually. From the properties of $\Pi_{BA}$ in the asynchronous network, it then follows that all the honest parties output the same $\mathcal{CS}$.

Now consider an arbitrary $P_j \in \mathcal{CS}$. It implies that the honest parties computed the output 1 during the instance $\Pi_{BA}^{(j)}$, which further implies that at least one honest $P_i$ participated with input 1 in $\Pi_{BA}^{(j)}$ after computing its output in the instances of $\Pi_{LS}$ invoked by $P_j$. If $P_j$ is honest, then the $Z_a$-correctness of $\Pi_{LS}$ in the asynchronous network guarantees that except with probability $O(n^3 \cdot \epsilon_{ICP})$, the values $(a_i^{(j)}, b_i^{(j)}, b_i^{(j)}', r_i^{(j)})$ will be eventually linearly secret-shared with IC-signatures, with $\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z_s|}$ being the underlying core-sets. On the other hand, even if $P_j$ is corrupt, the $Z_a$-commitment of $\Pi_{LS}$ in the asynchronous network (Lemma 8.1) guarantees that there exist values $(a_i^{(j)}, b_i^{(j)}, b_i^{(j)}', r_i^{(j)})$ which are eventually linearly secret-shared with IC-signatures, with $\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z_s|}$ being the underlying core-sets, except with probability $O(n^3 \cdot \epsilon_{ICP})$. From the linearity property of secret-sharing, it then follows that eventually, the values $a_{iter}, b_{iter}, b_i^{iter}$ and $r_{iter}$ will be linearly secret-shared with IC-signatures, with $\mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z_s|}$ being the underlying core-sets. The privacy of $a_{iter}, b_{iter}, b_i^{iter}$ and $r_{iter}$ is similar as for the synchronous communication network and the fact that $(\mathcal{H} \cap \mathcal{CS}) \neq \emptyset$ still holds in the asynchronous network.

We next claim that all honest parties will eventually agree on whether the instances of $\Pi_{Basic\text{Mult}}$ in $\Pi_{Rand\text{MultCl}}$ have succeeded or failed.

**Lemma 8.9.** Consider an arbitrary $iter$, such that all honest parties participate in the instance $\Pi_{Rand\text{MultCl}}(\mathcal{P}, Z_a, Z_s, \mathcal{G}_s, \mathcal{GW}_1, \ldots, \mathcal{GW}_{|Z_s|}, \mathcal{GD}, iter)$, where no honest party is present in $\mathcal{GD}$. Then except with probability $O(n^3 \cdot \epsilon_{ICP})$, all honest parties reconstruct a (common) value $d_{iter}$ and set $\text{flag}_{iter}$ to a common
Boolean value, at the time $T_{LSh} + 2T_{BA} + T_{BasicMult} + 3T_{Rec}$ in a synchronous network, or eventually in an asynchronous network.

Proof. From Lemma 8.8, the honest parties have $[a_{iter}], [b_{iter}], [b'_{iter}]$, and $[r_{iter}]$ at the time $T_{LSh} + 2T_{BA}$ in a synchronous network, or eventually in an asynchronous network, except with probability $O(n^3 \cdot \epsilon_{ICP})$, with $GW_1, \ldots, GW_{|Z|}$ being the underlying core-sets. From Lemma 8.4, it follows that the honest parties have $\{[c^{(1)}_{iter}], \ldots, [c^{(n)}_{iter}], [c_{iter}]\}$ and $\{[c'^{(1)}_{iter}], \ldots, [c'^{(n)}_{iter}], [c'_{iter}]\}$ from the corresponding instances of $\Pi_{BasicMult}$, either after time $T_{BasicMult}$ or eventually, based on the network type, where $GW_1, \ldots, GW_{|Z|}$ are the underlying core-sets. From Lemma 6.2, the honest parties reconstruct $e_{iter}$ from the corresponding instance of $\Pi_{Rec}$ after time $T_{Rec}$ in a synchronous network or eventually in an asynchronous network, except with probability $O(n^3 \cdot \epsilon_{ICP})$. From the linearity property of secret-sharing, it then follows that the honest parties compute $e_{iter}$ and hence reconstruct $e_{iter}$ from the corresponding instance of $\Pi_{Rec}$ after time $T_{Rec}$ in a synchronous network or eventually in an asynchronous network, except with probability $O(n^3 \cdot \epsilon_{ICP})$. Moreover, $GW_1, \ldots, GW_{|Z|}$ will be the underlying core-sets for $e_{iter}$. Again, from the linearity property of secret-sharing, it follows that the honest parties compute $[d_{iter}]$ with $GW_1, \ldots, GW_{|Z|}$ being the underlying core-sets, followed by reconstructing $d_{iter}$ from the corresponding instance of $\Pi_{Rec}$, which takes $T_{Rec}$ time in a synchronous network or happens eventually in an asynchronous network. Thus, the honest parties will have $d_{iter}$ either at the time $T_{LSh} + 2T_{BA} + T_{BasicMult} + 3T_{Rec}$ in a synchronous network, or eventually in an asynchronous network. Now depending upon the value of $d_{iter}$, the honest parties set $flag_{iter}$ to either 0 or 1.

We next claim that if no cheating occurs, then the honest parties output a multiplication-triple, which is linearly secret-shared with IC-signatures.

Lemma 8.10. Consider an arbitrary $iter$, such that all honest parties participate in the instance $\Pi_{RandMult}((\mathcal{P}, Z, S_{Z}, GW_1, \ldots, GW_{|Z|}, GD, iter)$, where no honest party is present in $GD$. If no party in $\mathcal{P} \setminus GD$ behaves maliciously, then $d_{iter} = 0$ and the honest parties output $([a_{iter}], [b_{iter}], [c_{iter}])$ at the time $T_{LSh} + 2T_{BA} + T_{BasicMult} + 3T_{Rec}$ in a synchronous network or eventually in an asynchronous network, where $c_{iter} = a_{iter} \cdot b_{iter}$ and where $GW_1, \ldots, GW_{|Z|}$ are the underlying core-sets.

Proof. If no party in $\mathcal{P} \setminus GD$ behaves maliciously, then from Lemma 8.6, the honest parties compute $[c_{iter}]$ and $[c'_{iter}]$ from the respective instances of $\Pi_{BasicMult}$, such that $c_{iter} = a_{iter} \cdot b_{iter}$ and $c'_{iter} = a_{iter} \cdot b'_{iter}$ holds and where $GW_1, \ldots, GW_{|Z|}$ are the underlying core-sets. Moreover, from Lemma 8.9, the honest parties will compute $d_{iter}$ at the time $T_{LSh} + 2T_{BA} + T_{BasicMult} + 3T_{Rec}$ in a synchronous network or eventually in an asynchronous network. Furthermore, if $c_{iter} = a_{iter} \cdot b_{iter}$ and $c'_{iter} = a_{iter} \cdot b'_{iter}$ holds, the value $d_{iter}$ will be 0 and consequently, the honest parties will output $([a_{iter}], [b_{iter}], [c_{iter}])$. Furthermore, it is easy to see that $GW_1, \ldots, GW_{|Z|}$ will be the underlying core-sets. This is because all the secret-shared values in the protocol are linearly secret-shared with IC-signatures, with $GW_1, \ldots, GW_{|Z|}$ being the underlying core-sets.
We next show that if \(d_{\text{iter}} \neq 0\), then the honest parties include at least one new maliciously-corrupt party in the set \(GD\).

**Lemma 8.11.** Consider an arbitrary \(\text{iter}\), such that all honest parties participate in the instance \(\Pi_{\text{RandMultCl}}(P, Z, Z_s, S_{Z_s}, G|\{\mathcal{W}_1, \ldots, \mathcal{W}_{\ell+1}\}, GD, \text{iter})\), where no honest party is present in \(GD\). If \(d_{\text{iter}} \neq 0\), then except with probability \(O(n^2 \cdot \epsilon_{\text{CP}})\), the honest parties update \(GD\) by adding a new maliciously-corrupt party in \(GD\), either at the time \(T_{\text{RandMultCl}} = T_{\text{LSh}} + 2T_{\text{BA}} + T_{\text{BasicMult}} + 4T_{\text{Rec}}\) in a synchronous network or eventually in an asynchronous network.

**Proof.** Let \(d_{\text{iter}} \neq 0\) and let \(\text{Selected}_{\text{iter}}\) be the set of summand-sharing parties across the two instances of \(\Pi_{\text{RandMultCl}}\) executed in \(\Pi_{\text{RandMultCl}}\). That is:

\[
\text{Selected}_{\text{iter}} \overset{\text{def}}{=} \text{Selected}_{\text{iter}, c} \cup \text{Selected}_{\text{iter}, c'}.
\]

Note that there exists no \(P_j \in \text{Selected}_{\text{iter}}\) such that \(P_j \in GD\), which follows from Lemma 8.2. We claim that there exists at least one party \(P_j \in \text{Selected}_{\text{iter}}\), such that corresponding to \(c^{(j)}_{\text{iter}}\) and \(c^{(j)}_{\text{iter}}\), the following holds:

\[
r_{\text{iter}} \cdot c^{(j)}_{\text{iter}} + c^{(j)}_{\text{iter}} \neq r_{\text{iter}} \cdot \sum_{(p,q) \in \text{SIS}(j)} [a_{\text{iter}}]_p [b_{\text{iter}}]_q + \sum_{(p,q) \in \text{SIS}(j)} [a_{\text{iter}}]_p [b'_{\text{iter}}]_q.
\]

Assuming the above holds, the proof now follows from the fact that once the parties reconstruct \(d_{\text{iter}} \neq 0\), they proceed to reconstruct the shares \(\{[a_{\text{iter}}]_p [b_{\text{iter}}]_q, [b'_{\text{iter}}]_q\}_{s \in S_{Z_s}}\) through appropriate instances of \(\Pi_{\text{ReCShare}}\) and the values \(c^{(1)}_{\text{iter}}, \ldots, c^{(n)}_{\text{iter}}, c^{(1)}_{\text{iter}}, \ldots, c^{(n)}_{\text{iter}}\) through appropriate instances of \(\Pi_{\text{ReC}}\). From Lemma 8.9, Lemma 6.1 and Lemma 6.2, this happens by the time \(T_{\text{LSh}} + 2T_{\text{BA}} + T_{\text{BasicMult}} + 4T_{\text{Rec}}\) in a synchronous network or eventually in an asynchronous network, except with probability \(O(n^3 \cdot \epsilon_{\text{CP}})\). Upon reconstructing these values, party \(P_j\) will be included in the set \(GD\). Moreover, it is easy to see that \(P_j\) is a maliciously-corrupt party since, for every honest \(P_j \in \text{Selected}_{\text{iter}}\), the following conditions hold:

\[
(c^{(j)}_{\text{iter}}) = \sum_{(p,q) \in \text{SIS}(j)} [a_{\text{iter}}]_p [b_{\text{iter}}]_q \quad \text{and} \quad (c^{(j)}_{\text{iter}}) = \sum_{(p,q) \in \text{SIS}(j)} [a_{\text{iter}}]_p [b'_{\text{iter}}]_q.
\]

We prove the above claim through a contradiction. Let the following condition hold for each \(P_j \in \text{Selected}_{\text{iter}}:\n\)

\[
r_{\text{iter}} \cdot c^{(j)}_{\text{iter}} + c^{(j)}_{\text{iter}} = r_{\text{iter}} \cdot \sum_{(p,q) \in \text{SIS}(j)} [a_{\text{iter}}]_p [b_{\text{iter}}]_q + \sum_{(p,q) \in \text{SIS}(j)} [a_{\text{iter}}]_p [b'_{\text{iter}}]_q.
\]

Next, summing the above equation over all \(P_j \in \text{Selected}_{\text{iter}}\), we get that the following holds:

\[
\sum_{P_j \in \text{Selected}_{\text{iter}}} r_{\text{iter}} c^{(j)}_{\text{iter}} + c^{(j)}_{\text{iter}} = \sum_{P_j \in \text{Selected}_{\text{iter}}} r_{\text{iter}} \cdot \sum_{(p,q) \in \text{SIS}(j)} [a_{\text{iter}}]_p [b_{\text{iter}}]_q + \sum_{(p,q) \in \text{SIS}(j)} [a_{\text{iter}}]_p [b'_{\text{iter}}]_q.
\]
This implies that the following holds:

\[ r_{\text{iter}} \cdot \sum_{p_j \in \text{Selected}_{\text{iter}}} c^{(j)}_{\text{iter}} + c'_{\text{iter}} = r_{\text{iter}} \cdot \sum_{p_j \in \text{Selected}_{\text{iter}}} \sum_{(p,q) \in \text{SIS}_{\text{iter},c}^{(j)}} [d_{\text{iter}}]_p [b_{\text{iter}}]_q + \sum_{(p,q) \in \text{SIS}_{\text{iter},c'}} [d'_{\text{iter}}]_p [b'_{\text{iter}}]_q. \]

Now based on the way \( a_{\text{iter}}, b_{\text{iter}}, b'_{\text{iter}}, c_{\text{iter}} \) and \( c'_{\text{iter}} \) are defined, the above implies that the following holds:

\[ r_{\text{iter}} \cdot c_{\text{iter}} + c'_{\text{iter}} = r \cdot a_{\text{iter}} \cdot b_{\text{iter}} + a_{\text{iter}} \cdot b'_{\text{iter}}. \]

This further implies that

\[ r_{\text{iter}} \cdot c_{\text{iter}} + c'_{\text{iter}} = (r_{\text{iter}} \cdot b_{\text{iter}} + b'_{\text{iter}}) \cdot a_{\text{iter}}. \]

Since in the protocol \( e_{\text{iter}} \triangleq r_{\text{iter}} \cdot b_{\text{iter}} + b'_{\text{iter}} \), the above implies that

\[ r_{\text{iter}} \cdot c_{\text{iter}} + c'_{\text{iter}} = e_{\text{iter}} \cdot a_{\text{iter}} \Rightarrow e_{\text{iter}} \cdot a_{\text{iter}} - r_{\text{iter}} \cdot c_{\text{iter}} - c'_{\text{iter}} = 0 \Rightarrow d_{\text{iter}} = 0, \]

where the last equality follows from the fact that in the protocol, \( d_{\text{iter}} \triangleq e_{\text{iter}} \cdot a_{\text{iter}} - r_{\text{iter}} \cdot c_{\text{iter}} - c'_{\text{iter}} \). However, \( d_{\text{iter}} = 0 \) is a contradiction since, according to the hypothesis of the lemma, we are given that \( d_{\text{iter}} \neq 0 \).

We next show that if the honest parties output a secret-shared triple in the protocol, then except with probability \( \frac{1}{|\mathbb{F}|} \), the triple is a multiplication-triple. Moreover, the triple will be random for the adversary.

**Lemma 8.12.** Consider an arbitrary \( \text{iter} \), such that all honest parties participate in the instance \( II_{\text{RandMultCl}}(\mathcal{P}, \mathbb{Z}_2, S_\mathbb{Z}_2, \mathcal{GW}_1, \ldots, \mathcal{GW}_1, \mathcal{GD}, \text{iter}) \), where no honest party is present in \( \mathcal{GD} \). If \( d_{\text{iter}} = 0 \), then the honest parties output linearly secret-shared \( (a_{\text{iter}}, [b_{\text{iter}}], [c_{\text{iter}}]) \) with IC-signatures with \( \mathcal{GW}_1, \ldots, \mathcal{GW}_1 \) being the underlying core-sets, at the time \( T_{\text{LSh}} + 2T_{\text{BA}} + T_{\text{BasicMult}} + 3T_{\text{Rec}} \) in a synchronous network or eventually in an asynchronous network where, except with probability \( \frac{1}{|\mathbb{F}|} \), the condition \( c_{\text{iter}} = a_{\text{iter}} \cdot b_{\text{iter}} \) holds. Moreover, the view of \( \text{Adv} \) will be independent of \( (a_{\text{iter}}, b_{\text{iter}}, c_{\text{iter}}) \).

**Proof.** Let \( d_{\text{iter}} = 0 \). From Lemma 8.4, all honest parties will agree that \( d_{\text{iter}} = 0 \), either at the time \( T_{\text{LSh}} + 2T_{\text{BA}} + T_{\text{BasicMult}} + 3T_{\text{Rec}} \) in a synchronous network or eventually in an asynchronous network. Then, from the protocol steps, the honest parties output \( ([a_{\text{iter}}], [b_{\text{iter}}], [c_{\text{iter}}]) \), with \( \mathcal{GW}_1, \ldots, \mathcal{GW}_1 \) being the underlying core-sets. In the protocol \( d_{\text{iter}} \overset{\text{def}}{=} e_{\text{iter}} \cdot a_{\text{iter}} - r_{\text{iter}} \cdot c_{\text{iter}} - c'_{\text{iter}} \), where \( e_{\text{iter}} \overset{\text{def}}{=} r_{\text{iter}} \cdot b_{\text{iter}} + b'_{\text{iter}} \). Since \( d_{\text{iter}} = 0 \) holds, it implies that the honest parties have verified that the following holds:

\[ r_{\text{iter}}(a_{\text{iter}} \cdot b_{\text{iter}} - c_{\text{iter}}) = (c'_{\text{iter}} - a_{\text{iter}} \cdot b'_{\text{iter}}). \]

We note that \( r_{\text{iter}} \) will be a random element from \( \mathbb{F} \) and will be unknown to \( \text{Adv} \) till it is publicly reconstructed, which follows from Lemma 8.8. We also note
that $r_{iter}$ will be unknown to Adv, till the outputs for the underlying instances of $Π_{BasicMult}$ are computed, and the honest parties have $[c_{iter}]$ and $[c'_{iter}]$. This is because, in the protocol, the honest parties start participating in the instance of $Π_{Rec}$ to reconstruct $r_{iter}$, only after they compute $[c_{iter}]$ and $[c'_{iter}]$. Now we have the following cases with respect to whether any party from $POP \setminus GD$ behaved maliciously during the underlying instances of $Π_{BasicMult}$.

- **Case I:** $c_{iter} = a_{iter} \cdot b_{iter}$ and $c'_{iter} = a_{iter} \cdot b'_{iter}$ — In this case, $(a_{iter}, b_{iter}, c_{iter})$ is a multiplication-triple.
- **Case II:** $c_{iter} = a_{iter} \cdot b_{iter}$, but $c'_{iter} \neq a_{iter} \cdot b'_{iter}$ — This case is never possible, as this will lead to the contradiction that $r_{iter}(a_{iter} \cdot b_{iter} - c_{iter}) \neq (c'_{iter} - a_{iter} \cdot b'_{iter})$ holds.
- **Case III:** $c_{iter} \neq a_{iter} \cdot b_{iter}$, but $c'_{iter} = a_{iter} \cdot b'_{iter}$ — This case is possible only if $r_{iter} = 0$, as otherwise this will lead to the contradiction that $r_{iter}(a_{iter} \cdot b_{iter} - c_{iter}) \neq (c'_{iter} - a_{iter} \cdot b'_{iter})$ holds. However, since $r_{iter}$ is a random element from $F$, it implies that this case can occur only with probability at most $\frac{1}{|F|}$.
- **Case IV:** $c_{iter} \neq a_{iter} \cdot b_{iter}$ as well as $c'_{iter} \neq a_{iter} \cdot b'_{iter}$ — This case is possible only if $r_{iter} = (c'_{iter} - a_{iter} \cdot b'_{iter}) \cdot (a_{iter} \cdot b_{iter} - c_{iter})^{-1}$, as otherwise this will lead to the contradiction that $r_{iter}(a_{iter} \cdot b_{iter} - c_{iter}) \neq (c'_{iter} - a_{iter} \cdot b'_{iter})$ holds. However, since $r_{iter}$ is a random element from $F$, it implies that this case can occur only with probability at most $\frac{1}{|F|}$.

Hence, we have shown that except with probability at most $\frac{1}{|F|}$, the triple $(a_{iter}, b_{iter}, c_{iter})$ is a multiplication-triple. To complete the proof, we need to argue that the view of Adv in the protocol will be independent of the triple $(a_{iter}, b_{iter}, c_{iter})$. For this, we first note that $a_{iter}, b_{iter}$ and $b'_{iter}$ will be random for the adversary at the time of their generation, which follows from Lemma 8.8. From Lemma 8.5, Adv learns nothing additional about $a_{iter}, b_{iter}$ and $b'_{iter}$ during the two instances of $Π_{BasicMult}$. While Adv learns the value of $c_{iter}$, since $b'_{iter}$ is a uniformly distributed for Adv, for every candidate value of $b'_{iter}$ from the view-point of Adv, there is a corresponding value of $b_{iter}$ consistent with the $c_{iter}$ learnt by Adv. Hence, learning $c_{iter}$ does not add any new information about $(a_{iter}, b_{iter}, c_{iter})$ to the view of Adv. Moreover, Adv will be knowing beforehand that $d_{iter}$ will be 0 and hence, learning this value does not change the view of Adv regarding $(a_{iter}, b_{iter}, c_{iter})$.

We next derive the communication complexity of the protocol $Π_{RandMultCI}$.

**Lemma 8.13.** Protocol $Π_{RandMultCI}$ incurs a communication of $O(|Z| \cdot n^5 \cdot \log |F| + n^6 \cdot \log |F| \cdot |σ|)$ bits and makes $O(n^2)$ calls to $Π_{BA}$.

**Proof.** To generate $[a_{iter}], [b_{iter}], [b'_{iter}]$ and $[r_{iter}]$, $O(n)$ instances of $Π_{LSh}$ and $Π_{BA}$ are invoked. To compute $[c_{iter}]$ and $[c'_{iter}]$, two instances of $Π_{BasicMult}$ are invoked. To publicly reconstruct $c_{iter}$ and $d_{iter}$, two instances of $Π_{Rec}$ are invoked with $|R| = n$. Finally, if $d_{iter} \neq 0$, then $3 \cdot |S_{Z_{Z}}|$ instances of $Π_{RecShare}$ and $2n$ instances of $Π_{Rec}$ are invoked, with $|R| = n$. The communication complexity now follows from the communication complexity of $Π_{LSh}$ (Lemma 8.1), communication complexity
of $\Pi_{\text{BasicMult}}$ (Lemma 8.7), communication complexity of $\Pi_{\text{RecShare}}$ (Lemma 6.1) and communication complexity of $\Pi_{\text{Rec}}$ (Lemma 6.2).

F.4 Properties of the Protocol $\Pi_{\text{TripGen}}$

In this section, we prove the properties of the protocol $\Pi_{\text{TripGen}}$ (see Fig 13 for the formal details.) We begin by showing that each party computes an output in the protocol.

**Lemma 8.14.** Let $t$ be the size of the largest set in $Z_s$. Then except with probability $O(n^3 \cdot \epsilon_{\text{ICP}})$, the honest parties compute an output during $\Pi_{\text{TripGen}}$, by the time $T_{\text{TripGen}} = (t + 1) \cdot T_{\text{RandMultCI}}$ in a synchronous network, or almost-surely, eventually in an asynchronous network, where $T_{\text{RandMultCI}} = T_{\text{LSh}} + 2T_{\text{BA}} + T_{\text{BasicMult}} + 4T_{\text{Rec}}$.

**Proof.** From Lemma 8.9, except with probability $O(n^3 \cdot \epsilon_{\text{ICP}})$, the honest parties will know the outcome of each iteration $\text{iter}$, since all honest parties set the Boolean variable $\text{flag}_{\text{iter}}$ to a common value. For every iteration $\text{iter}$ where $\text{flag}_{\text{iter}}$ is set to 1, from Lemma 8.11, a new corrupt party is added to $\mathcal{GD}$. Thus, after at most $t$ iterations, all the corrupt parties will be included in $\mathcal{GD}$ and the parties will set $\text{flag}_{\text{iter}}$ to 0 in the next iteration. Moreover, they will output $(G_{W_1}, \ldots, G_{W_{|Z_s|}}, [a_{\text{iter}}], [b_{\text{iter}}], [c_{\text{iter}}])$, computed during the corresponding instance of $\Pi_{\text{RandMultCI}}$.

We next claim that the output computed by the honest parties is indeed a multiplication-triple.

**Lemma 8.15.** If the honest parties output $(G_{W_1}, \ldots, G_{W_{|Z_s|}}, [a_{\text{iter}}], [b_{\text{iter}}], [c_{\text{iter}}])$ during the protocol $\Pi_{\text{TripGen}}$, then $a_{\text{iter}}, b_{\text{iter}}$ and $c_{\text{iter}}$ are linearly secret-shared with IC-signatures, with $G_{W_1}, \ldots, G_{W_{|Z_s|}}$ being the underlying core-sets. Moreover, $c_{\text{iter}} = a_{\text{iter}} b_{\text{iter}}$ holds, except with probability $\frac{1}{|F|}$. Furthermore, the view of the adversary remains independent of $a_{\text{iter}}, b_{\text{iter}}$ and $c_{\text{iter}}$.

**Proof.** Follows from Lemma 8.12.

We finally derive the communication complexity of the protocol $\Pi_{\text{TripGen}}$.

**Lemma 8.16.** Protocol $\Pi_{\text{TripGen}}$ incurs a communication of $O(|Z_s| \cdot n^6 \cdot \log |F| + n^7 \cdot \log |F| \cdot |\sigma|)$ bits and makes $O(n^3)$ calls to $\Pi_{\text{BA}}$.

**Proof.** The proof follows from the communication complexity of $\Pi_{\text{RandMultCI}}$ (Lemma 8.13) and the fact that $O(n)$ instances of $\Pi_{\text{RandMultCI}}$ are invoked in the protocol.

G Properties of the Circuit Evaluation Protocol

In this section, we prove the properties of the protocol $\Pi_{\text{cktEval}}$ (see Fig 14 for the formal description). We begin by showing that the honest parties compute some output during the pre-processing phase.
Lemma G.1. Protocol $\Pi_{\text{defEval}}$ achieves the following during the pre-processing phase.

- **Synchronous Network**: Except with probability $O(n^3 \cdot \epsilon_{\text{ICP}})$, at the time $T_{\text{Rand}}$, the honest parties have $\{r^{(i)}\}_{i=1,...,L}$, which are linearly secret-shared with IC-signatures, with $GW_1, \ldots, GW_{|Z_s|}$ being the underlying core-sets, where $L \overset{df}{=} n^3 \cdot c_M + 4n^2 \cdot c_M + n^2 + n$. Moreover, the view of the adversary remains independent of the values $\{r^{(i)}\}_{i=1,...,L}$. At the time $T_{\text{Rand}} + T_{\text{TripGen}}$, the honest parties have triples $\{(a^{(i)}, b^{(i)}, c^{(i)})\}_{i=1,...,c_M}$, which are linearly secret-shared with IC-signatures, with $GW_1, \ldots, GW_{|Z_s|}$ being the underlying core-sets, where $c^{(i)} = a^{(i)} \cdot b^{(i)}$, except with probability $\frac{1}{|Z|}$. The view of the adversary will be independent of the multiplication-triples.

- **Asynchronous Network**: Except with probability $O(n^3 \cdot \epsilon_{\text{ICP}})$, almost-surely, the honest parties eventually have $\{r^{(i)}\}_{i=1,...,L}$, which are linearly secret-shared with IC-signatures, with $GW_1, \ldots, GW_{|Z_s|}$ being the underlying core-sets, where $L \overset{df}{=} n^3 \cdot c_M + 4n^2 \cdot c_M + n^2 + n$. Moreover, the view of the adversary remains independent of the values $\{r^{(i)}\}_{i=1,...,L}$. Furthermore, the honest parties eventually have triples $\{(a^{(i)}, b^{(i)}, c^{(i)})\}_{i=1,...,c_M}$, which are linearly secret-shared with IC-signatures, with $GW_1, \ldots, GW_{|Z_s|}$ being the underlying core-sets, where $c^{(i)} = a^{(i)} \cdot b^{(i)}$, except with probability $\frac{1}{|Z|}$. The view of the adversary will be independent of the multiplication-triples.

**Proof.** The proof follows from the $Z_s$-correctness, $Z_a$-correctness and privacy of the protocol $\Pi_{\text{Rand}}$ (Theorem 7.2) and from the properties of $\Pi_{\text{TripGen}}$ in the asynchronous and synchronous network (Lemma 8.14 and Lemma 8.15). We also note that the multiplication-triples will be linearly secret-shared with IC-signatures, with $GW_1, \ldots, GW_{|Z_s|}$ being the underlying core-sets. This is because there will be at most $n^3 \cdot c_M + 4n^2 \cdot c_M + n^2 + n$ instances of $\Pi_{\text{LSH}}$ invoked as part of $\Pi_{\text{TripGen}}$ for generating $c_M$ multiplication-triples. And prior to invoking the instance of $\Pi_{\text{TripGen}}$, the honest parties would have already generated $n^3 \cdot c_M + 4n^2 \cdot c_M + n^2 + n$ number of linearly secret-shared random pads with IC-signatures with $GW_1, \ldots, GW_{|Z_s|}$ being the underlying core-sets through the instance of $\Pi_{\text{Rand}}$, which can serve $n^3 \cdot c_M + 4n^2 \cdot c_M + n^2 + n$ instances of $\Pi_{\text{LSH}}$.

We next show that during the input phase, the inputs of all honest parties will be linearly secret-shared with IC-signatures in a synchronous network and in an asynchronous network, the inputs of a subset of the parties will be linearly secret-shared with IC-signatures.

Lemma G.2. Protocol $\Pi_{\text{defEval}}$ achieves the following during the input phase.

- **Synchronous Network**: Except with probability $O(n^3 \cdot \epsilon_{\text{ICP}})$, at the time $T_{\text{Rand}} + T_{\text{TripGen}} + T_{\text{LSH}} + 2T_{\text{BA}}$, the honest parties will have a common subset $\mathcal{CS}$ where $\mathcal{P} \setminus \mathcal{CS} \subset Z_s$, such that all honest parties will be present in $\mathcal{CS}$. Moreover, corresponding to every $P_j \in \mathcal{CS}$, there will be some value, say $x^{(j)}$, which is the same as $x^{(i)}$, which will be linearly secret-shared with IC-signatures, with $GW_1, \ldots, GW_{|Z_s|}$ being the underlying core-sets. Fur-
Furthermore, the view of the adversary will be independent of the $x^{(j)}$ values, corresponding to the honest parties $P_j \in \mathcal{CS}$.

- **Asynchronous Network:** Except with probability $O(n^3 \cdot \epsilon_{ICP})$, almost-surely, the honest parties will eventually have a common subset $\mathcal{CS}$ where $\mathcal{P} \setminus \mathcal{CS} \in \mathbb{Z}_s$. Moreover, corresponding to every $P_j \in \mathcal{CS}$, there will be some value, say $x^{*^{(j)}}$, which is the same as $x^{(j)}$, which will be eventually linearly secret-shared with IC-signatures, with $GW_1, \ldots, GW_{|Z_s|}$ being the underlying core-sets. Furthermore, the view of the adversary will be independent of the $x^{(j)}$ values, corresponding to the honest parties $P_j \in \mathcal{CS}$.

**Proof.** We first consider a synchronous network. From the protocol steps, the honest parties start participating in the input phase, only after computing output during the instance of $\Pi_{TripGen}$, which from Lemma G.1 happens at the time $T_{Rand} + T_{TripGen}$. We also note that there can be at most $n^3 \cdot c_M + 4n^2 \cdot c_M + n^2$ instances of $\Pi_{LSh}$, invoked as part of the instance of $\Pi_{TripGen}$, which will utilize the secret-shared values $\{r_{\mathcal{P}^{(j)}}\}_{j=1, \ldots, n^3 \cdot c_M + 4n^2 \cdot c_M + n^2}$ as pads. Consequently, the remaining linearly secret-shared values $\{r_{\mathcal{P}^{(j)}}\}_{j=n^3 \cdot c_M + 4n^2 \cdot c_M + n^2 + 1, \ldots, n^3 \cdot c_M + 4n^2 \cdot c_M + n^2 + n}$ will be still available to the honest parties for being used as pads in up to $n$ instances of $\Pi_{LSh}$, since these pads will be random from the adversary’s point of view.

Let $Z^* \in \mathbb{Z}_s$ be the set of corrupt parties and let $\mathcal{H} = \mathcal{P} \setminus Z^*$ be the set of honest parties. We claim that by the time $T_{Rand} + T_{TripGen} + T_{LSh} + 2T_{BA}$, except with probability $O(n^3 \cdot \epsilon_{ICP})$, all the parties in $\mathcal{H}$ will have a common subset $\mathcal{CS}$, where $\mathcal{P} \setminus \mathcal{CS} \in \mathbb{Z}_s$ and where $\mathcal{H} \subseteq \mathcal{CS}$. The proof for this is exactly the same as that of Lemma 8.8. Namely, at the time $T_{Rand} + T_{TripGen} + T_{LSh}$, all the parties in $\mathcal{H}$ would start participating with input 1 in the BA instances $\Pi_{BA}^{(j)}$, corresponding to the parties $P_j \in \mathcal{H}$, since by this time, the instances of $\Pi_{LSh}$ invoked by the parties in $\mathcal{H}$ will produce output for all the parties in $\mathcal{H}$. Consequently, these BA instances will produce output 1 at the time $T_{Rand} + T_{TripGen} + T_{LSh} + T_{BA}$, after which all the parties in $\mathcal{H}$ will start participating with input 0 in the remaining BA instances (if any). Consequently, by the time $T_{Rand} + T_{TripGen} + T_{LSh} + 2T_{BA}$, all the $n$ instances of $\Pi_{BA}$ will produce some output and the parties in $\mathcal{H}$ will have a common $\mathcal{CS}$. Next, it can be shown that corresponding to every $P_j \in \mathcal{CS}$, there exists some value, say $x^{*^{(j)}}$, which is the same as $x^{(j)}$, such that the parties in $\mathcal{H}$ have a linear secret-sharing with IC-signatures of $x^{*^{(j)}}$, with $GW_1, \ldots, GW_{|Z_s|}$ being the underlying core-sets. The proof for this follows similar lines as that of Lemma 8.8. It is easy to see that $x^{*^{(j)}}$ will be linearly secret-shared with IC-signatures with $GW_1, \ldots, GW_{|Z_s|}$ being the underlying core-sets. This is because the instance of $\Pi_{LSh}$ invoked by $P_j$ utilizes $r_{\mathcal{P}^{(j)}}^{(n^3 \cdot c_M + 4n^2 \cdot c_M + n^2 + j)}$ as the pad, which is linearly secret-shared with IC-signatures with $GW_1, \ldots, GW_{|Z_s|}$ being the underlying core-sets.

The proof of the lemma for an asynchronous network is almost the same as above and follows using similar arguments as used to prove Lemma 8.8 for the case of asynchronous network.
Finally, the privacy of the inputs \( x^{(j)} \) of the parties \( P_j \in (\mathcal{H} \cap \mathcal{S}) \) follows from the privacy of \( \Pi_{\text{LSR}} \) and the fact that the underlying pads \( r^{(n^2-c_{\text{LS}}+4n^2-c_{\text{LS}}+n^2+j)} \) used in the corresponding instances of \( \Pi_{\text{LSR}} \) are still random for the adversary, after the instance of \( \Pi_{\text{TripGen}} \).

We next show that the honest parties compute an output during the circuit-evaluation phase.

**Lemma G.3.** Protocol \( \Pi_{\text{cktEval}} \) achieves the following during the circuit-evaluation phase, where \( D_M \) denotes the multiplicative depth of \( \text{ckt} \).

- **Synchronous Network:** Except with probability \( O(n^3 \cdot \epsilon_{\text{CP}}) \), at the time \( T_{\text{Rand}} + T_{\text{TripGen}} + T_{\text{LSH}} + 2T_{\text{BA}} + (D_M + 1) \cdot T_{\text{Rec}} \), the honest parties will have \( y \), where \( y = f(x^{(1)}, \ldots, x^{(n)}) \), such that \( x^{(j)} = x^{(j)} \) for every honest party \( P_j \in \mathcal{S} \) and where \( x^{(j)} = 0 \) for every \( P_j \notin \mathcal{S} \). Moreover, all honest parties will be present in \( \mathcal{S} \). Furthermore, the view of the adversary will be independent of the \( x^{(j)} \) values, corresponding to the honest parties \( P_j \in \mathcal{S} \).

- **Asynchronous Network:** Except with probability \( O(n^3 \cdot \epsilon_{\text{CP}}) \), almost-surely, the honest parties will eventually have \( y \), where \( y = f(x^{(1)}, \ldots, x^{(n)}) \), where \( x^{(j)} = x^{(j)} \) for every honest party \( P_j \in \mathcal{S} \) and where \( x^{(j)} = 0 \) for every \( P_j \notin \mathcal{S} \). Furthermore, the view of the adversary will be independent of the \( x^{(j)} \) values, corresponding to the honest parties \( P_j \in \mathcal{S} \).

**Proof.** Let us first consider a synchronous network. Let \( Z^* \in Z_s \) be the set of corrupt parties and let \( \mathcal{H} = \mathcal{P} \setminus Z^* \) be the set of honest parties. From Lemma G.1, at the time \( T_{\text{Rand}} + T_{\text{TripGen}} \), the parties in \( \mathcal{H} \) will have the triples \( \{(a^{(\ell)}, b^{(\ell)}, c^{(\ell)})\}_{\ell=1,\ldots,c_M} \), which are linearly secret-shared with IC-signatures, with \( GW_1, \ldots, GW_{|Z_s|} \) being the underlying core-sets and where \( c^{(\ell)} = a^{(\ell)} \cdot b^{(\ell)} \), except with probability \( 1 \). Moreover, from Lemma G.2, at the time \( T_{\text{Rand}} + T_{\text{TripGen}} + T_{\text{LSH}} + 2T_{\text{BA}} \), the honest parties will have a common subset \( \mathcal{S} \) where \( \mathcal{P} \setminus \mathcal{S} \in Z_s \), such that all honest parties will be present in \( \mathcal{S} \). Furthermore, corresponding to every \( P_j \in \mathcal{S} \), there will be some value, say \( x^{(j)} \), which is the same as \( x^{(j)} \) for an honest \( P_j \), which will be linearly secret-shared with IC-signatures, with \( GW_1, \ldots, GW_{|Z_s|} \) being the underlying core-sets. At the end of the input phase, the parties take 0 as the input on behalf of the parties \( P_j \notin \mathcal{S} \) and take the default linear secret-sharing of 0 with IC-signatures, with \( GW_1, \ldots, GW_{|Z_s|} \) being the underlying core-sets. To prove the lemma, we show that all the gates in \( \text{ckt} \) are correctly evaluated. Namely, for every gate in \( \text{ckt} \), given the gate-inputs in a linearly secret-shared fashion with IC-signatures with \( GW_1, \ldots, GW_{|Z_s|} \) being the underlying core-sets, the parties compute the gate-output in a linearly secret-shared fashion with IC-signatures with \( GW_1, \ldots, GW_{|Z_s|} \) being the underlying core-sets. While this is true for the linear gates in \( \text{ckt} \), which follows from the linearity of the secret-sharing, the same is true even for the multiplication gates, except with probability \( O(n^3 \cdot \epsilon_{\text{CP}}) \). This is because, for every multiplication gate, the parties deploy a linearly secret-shared multiplication-triple from the pre-processing phase and apply Beaver’s method. And the masked gate-inputs are correctly reconstructed.
through instances of $\Pi_{Rec}$, except with probability $\mathcal{O}(n^3 \cdot \epsilon_{ICP})$. Since all the independent multiplication gates at the same multiplicative depth can be evaluated in parallel, to evaluate the multiplication gates, it will take a total $D_M \cdot T_{Rec}$ time. Finally, once the circuit-output is ready in a secret-shared fashion, it is publicly reconstructed through an instance of $\Pi_{Rec}$, which takes $T_{Rec}$ time and produces the correct output, except with probability $\mathcal{O}(n^3 \cdot \epsilon_{ICP})$. The privacy of the inputs of the honest parties in $CS$ follows from the privacy of $\Pi_{LSh}$ (Lemma 8.1) and the fact no additional information is revealed during the evaluation of multiplication gates. This is because the underlying multiplication-triples which are deployed while applying Beaver’s method are random for the adversary.

The proof for the case of asynchronous network follows similar arguments as above and depends upon the properties of the pre-processing phase and input phase in the asynchronous network (Lemma G.1 and Lemma G.2).

We finally show that the honest parties terminate the protocol.

**Lemma G.4.** If the network is synchronous, then except with probability $\mathcal{O}(n^3 \cdot \epsilon_{ICP})$, the honest parties terminate the protocol at the time $T_{Rand} + T_{TripGen} + 2T_{Lsh} + 2T_{BA} + (D_M + 1) \cdot T_{Rec} + \Delta$. If the network is asynchronous, then except with probability $\mathcal{O}(n^3 \cdot \epsilon_{ICP})$, almost-surely, the honest parties eventually terminate the protocol.

**Proof.** Let us first consider a synchronous network. Let $Z^* \in Z_s$ be the set of corrupt parties and let $H = P \setminus Z^*$ be the set of honest parties. From Lemma G.3, except with probability $\mathcal{O}(n^3 \cdot \epsilon_{ICP})$, at the time $T_{Rand} + T_{TripGen} + T_{Lsh} + 2T_{BA} + (D_M + 1) \cdot T_{Rec}$, all the parties in $H$ will have $y$. Hence every party in $H$ will send a ready message for $y$ to all the parties, which gets delivered within $\Delta$ time, while the parties in $Z^*$ may send a ready message for some $y'$ where $y' \neq y$. Now since $Z_s$ does not satisfy the $\mathbb{Q}(1)(Z^*, Z_s)$ condition, it follows that no party in $H$ will ever send a ready message for any $y'$ where $y' \neq y$. As $P \setminus H \in Z_s$, it follows that at the time $T_{Rand} + T_{TripGen} + T_{Lsh} + 2T_{BA} + (D_M + 1) \cdot T_{Rec} + \Delta$, all the parties in $H$ will have sufficient number of ready messages for $y$ and hence they terminate with output $y$.

Next, consider an asynchronous network. Let $Z^* \in Z_a$ be the set of corrupt parties and let $H = P \setminus Z^*$ be the set of honest parties. Note that $P \setminus Z^* \subseteq Z_s$, since $Z_s \subseteq Z_a$. From Lemma G.3, except with probability $\mathcal{O}(n^3 \cdot \epsilon_{ICP})$, almost-surely, all the parties in $H$ will eventually compute $y$. Hence every party in $H$ will eventually send some ready message. We claim that no party in $H$ will ever send a ready message for any $y' \neq y$. On the contrary, let $P_i \in H$ be the first party, which sends a ready message for $y' \neq y$. From the protocol steps, it follows that $P_i$ sends the ready message for $y'$ after computing $y'$ during the circuit-evaluation phase. Otherwise, there should exist a subset of parties $A$ where $Z_s$ satisfies $\mathbb{Q}(1)(A, Z_s)$ condition (implying that $A$ has at least one party from $H$), who should have sent the ready message for $y'$ to $P_i$, which is not possible, since we are assuming $P_i$ to be first party from $H$ to send a ready message for $y'$. From Lemma G.3, $P_i$ will not compute $y'$ and hence will not send a ready message for $y'$. Now since every party in $H$ eventually computes $y$ in the circuit-evaluation
phase, it eventually sends a ready message for $y$. And since $P \setminus Z^* \in Z_s$ and $P \setminus Z^* \notin Z_s$, it follows that irrespective of the behaviour of the corrupt parties, the parties in $H$ will eventually receive a sufficient number of ready messages for $y$, to terminate with output $y$.

Let $P_h$ be the first party from $H$, who terminates with output $y$. This implies that there exists a subset of parties $W$ with $P \setminus W \in Z_s$, who sends a ready message for $y$ to $P_h$. Now consider the set $(H \cap W)$. The set satisfies the $Q(1)(H \cap W, Z_s)$ condition, due to the $Q(2,1)(P, Z_s, Z_a)$ condition. The ready messages of these parties (for $y$) get eventually delivered to every party in $H$. Consequently, every party in $H$ (including $P_h$) who has not yet sent any ready message will eventually send the ready message for $y$, which gets eventually delivered to all the parties. And as a result, every party in $H$ will eventually have a sufficient number of ready messages for $y$, to terminate with the output $y$.

We next derive the communication complexity of the protocol.

**Lemma G.5.** Protocol $\Pi_{\text{cktEval}}$ incurs a communication of $O(|Z_s|^2 \cdot n^{12} \cdot \log |F| \cdot |\sigma|)$ bits and makes $O(n^3)$ calls to $\Pi_{\text{BA}}$.

**Proof.** The communication complexity is dominated by the instance of $\Pi_{\text{Rand}}$ to generate $L \overset{\text{def}}{=} n^3 \cdot c_M + 4n^2 \cdot c_M + n^2 + n$ random secret-shared values and the instance of $\Pi_{\text{TripGen}}$ to generate $L = c_M$ number of secret-shared multiplication-triples. The proof now follows from the communication complexity of $\Pi_{\text{Rand}}$ (Theorem 7.2) and the communication complexity of the (generalized) $\Pi_{\text{TripGen}}$ protocol (Lemma 8.16).

Theorem 9.1 now easily follows from Lemma G.1-G.5.