Sprints: Intermittent Blockchain PoW Mining

Michael Mirkin*, Lulu Zhou†, Ittay Eyal*, Fan Zhang†
*Technion
†Yale University

Abstract—**Cryptocurrencies and decentralized platforms are rapidly gaining traction since Nakamoto's discovery of Bitcoin's blockchain protocol. These systems use Proof of Work (PoW) to achieve unprecedented security for digital assets. However, the significant power consumption and ecological impact of PoW are leading policymakers to consider stark measures against them and prominent systems to explore alternatives. But these alternatives imply stepping away from key security aspects of PoW.**

We present Sprints, a blockchain protocol that achieves almost the same security guarantees as PoW blockchains, but with an order-of-magnitude lower ecological impact, taking into account both power and hardware. To achieve this, Sprints forces miners to mine intermittently. It interleaves Proof of Delay (PoD, e.g., using a Verifiable Delay Function) and PoW, where only the latter bares a significant resource expenditure. We prove that in Sprints the attacker's success probability is the same as that of legacy PoW. To evaluate practical performance, we analyze the effect of shortened PoW duration, showing a minor reduction in resilience (49% instead of 50%). We confirm the results with a full implementation using 100 patched Bitcoin clients in an emulated network.

1. Introduction

Proof of Work (PoW) cryptocurrencies offer a decentralized form of money, with monetary policy dictated by code. Participants can acquire coins and perform transactions without needing permission from other parties or centralized exchanges. PoW cryptocurrencies, starting with Nakamoto's Bitcoin [1], have gained significant success with market capitalization in the hundreds of billions [2] and attract the attention of major financial institutions [3, 4]. But PoW cryptocurrencies consume significant resources, with Bitcoin's power consumption surpassing Argentina's in 2022 [5]. Environmental concerns are leading to policy changes, including bans [6, 7]. Nevertheless, the stable valuation of the PoW Bitcoin, despite the proliferation of non-PoW alterantives, demonstrates the demand for PoW guarantees. It implies the need for a protocol that provides such guaratees with lower environmental impact.

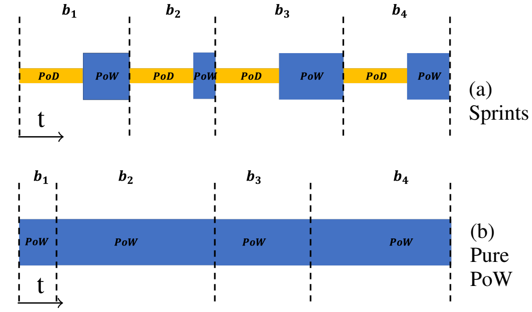
Indeed, both theoretical work and operational systems address this issue. Previous work (§2), showed [8] that resource expenditure cannot be reduced by tuning the protocol parameters. Proof of Storage [9] requires participants to dedicate storage resources instead of computation but is about 50% cheaper to attack [10]. Proof of Stake (PoS) protocols (e.g., [11–16]) take a different approach: Instead of physical resource expenditure, PoS uses on-chain deposits, thus the likelihood of a miner being able to create a new block and add it to the blockchain is determined by the amount of cryptocurrency they have *staked*, i.e., held in their account. With this approach, no physical resource is expended. But PoS protocols require stronger assumptions, such as long-term connectivity and availability [17–19].

***Sprints*.** We introduce *Sprints* (§4), a blockchain protocol that maintains the advantages of PoW but with significantly lower resource consumption. The key idea is to force miners to perform PoW intermittently, i.e., to insert periods of time during which miners pause mining. Like Bitcoin, *Sprints* miners collect transactions (e.g., payment orders) from users and batch them into *blocks* that they broadcast. Each block contains a hash reference to its predecessor, so the blocks together form a tree with an agreed-upon root. The system state is thus obtained by processing the transactions in the longest path in the tree, called the *longest chain*.

In addition to PoW — statistical proofs that the miner expended computational effort, *Sprints* also requires PoD, proving the miner waited for a certain time before the PoW computation. Unlike PoW, PoD computation does not require significant computational resources. Thus, a miner alternates between producing PoD with nominal power expenditure and PoW with high power expenditure – Figure 1 illustrates this process.

Like PoW systems where participation is profitable only for miners with efficient hardware [20–22], *Sprints* is profitable only for miners with efficient PoD hardware. We thus assume all miners have similar, efficient, PoD hardware;

Figure 1: Pure PoW and *Sprints* over time.



others with weaker hardware would not participate.

Security. At first glance, it might seem that an attacker has an advantage compared to a *pure PoW* system like Bitcoin: She does not have to stop calculating PoW while performing the PoD calculation, whereas honest miners will. We prove (§5) that such behavior specifically is futile, and that in general there is no sacrifice of security compared to a pure PoW system. Due to the distinct characteristics of *Sprints*, previous proof techniques [10, 23, 24] are not applicable. Specifically, the PoD puzzle delay eliminates the memorylessness of pure PoW systems and prevents the use of the common Markov chain analysis. Consequently, we are compelled to devise a novel proof approach.

The key step is showing that in a race where the attacker competes with the honest miners, the attacker has to solve PoW and PoD puzzles sequentially, thus, parallel mining would not benefit her. Intuitively, the only race between the honest chain and that of the attacker is the PoW puzzles, as both spend the same amount of time solving PoD puzzles for the same number of blocks. The implication is that the attacker's probability of winning the race is the same as in a pure PoW blockchain. We use this result to show that if we consider a race where the honest miners and the adversary are building two chains that extend the same block and the adversary controls less than half of the PoW computational power, then her probability of building a chain that is at least r blocks deep and is longer than the honest chain is bounded by $2^{-\Omega(r)}$. Finally, we utilize this result to show that if the propagation delay is negligible, *Sprints* provides the same guarantees as pure PoW.

Implementation and evaluation. If we compare Sprints to a pure PoW system with the same block interval, since part of the block generation is computing PoD, PoW duration in Sprints is shorter. Our theoretical analysis shows that with shorter PoW average duration it is more likely for honest miners to generate blocks with the same parent, forming *forks*. The implication is that the longest chain extension is slower and the threshold attacker size is smaller.

To confirm this analysis, we implemented *Sprints* by patching the standard Bitcoin client [25] and measured the frequency of forks as a function of both the network propagation delay and the PoW duration. With system parameters similar to those of the operational Bitcoin system [26], when using only 5% of block interval for PoW, the threshold attacker size is 49%, compared to Bitcoin's 50%.

Implementing *Sprints* involves addressing several practical challenges (§6). To address spam prevention while maintaining efficient propagation, we lazily validate PoDs, i.e. nodes do not wait for the PoD validation to complete before propagating the block, allowing for the same propagation delay as a pure PoW system without sacrificing spam prevention [27]. Additionally, we adjust the difficulty of both PoW and PoD to maintain constant block intervals as hardware for PoW and PoD improves. We do this by estimating parameters using the second moments of the interval probability distribution.

Ecological benefits. The analysis *Sprints*'s ecological footprint (§7) is more involved than comparing the ratio of PoW per block. We show that *Sprints* miners use their budget to purchase more mining equipment compared to pure PoW, for the short duration where they PoW-mine. In other words, *Sprints* shifts a portion of the *operating expenses* (*OPEX*) to *capital expenditure* (*CAPEX*), which we show to be more environmentally friendly by comparing the emissions from the hardware lifecycle to the emissions from electricity consumption. We use the CO2e (carbon dioxide equivalent) metric [28] to compare systems, which allows us to quantify the emissions resulting from power consumption and the hardware lifecycle.

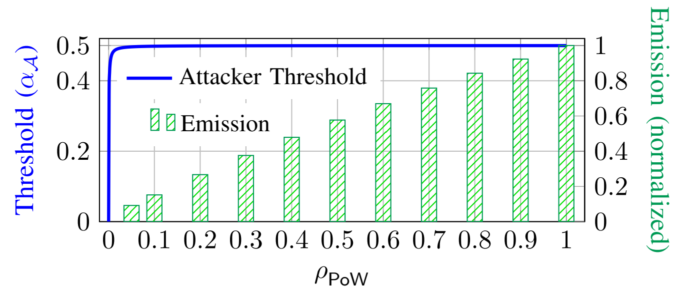


Figure 2: Attack threshold and normalized emission under as a function of portion of PoW time out of block interval

Combining the security and emission analyses, Figure 2 shows how the attack threshold and emission reduction ratio change with the PoW time ratio when using Bitcoin-like parameters (100ms network delay, 600s block interval [26]). At 5% of the block interval the CO2e is 9.2% that of Bitcoin.

In summary, our main contributions are:

- *Sprints*, a novel blockchain protocol that introduces intermittent mining, alternating between PoD and PoW;
- proof that the security threshold is the same as in PoW;
- tuning of both PoD and PoW based on block interval;
- evaluation with full implementation over an emulated network demonstrating effective difficulty adjustment and close-to-optimal (49%) attack threshold values; and
- ecological analysis showing over $10x$ CO2e reduction.

2. Related Work

We review energy-efficient alternatives to PoW and their trade-offs.

The most well-established energy-efficient alternative to PoW is Proof of Stake (PoS) [11–16]. Instead of solving hash puzzles, PoS miners participate in a mining lottery with a winning probability proportional to their token holding in the system. However, these protocols require stronger assumptions than PoW protocols regarding network connectivity, validator behavior, or the availability of a randomness oracle [17–19]. Additionally, in PoS protocols new nodes need cooperation from existing nodes to join the network [29], in contrast to *Sprints*, where new nodes can join the network by mining blocks using external computations.

Several protocols take advantage of Verifiable Delay Functions (VDFs). Verifiable Delay Functions (VDFs) [30–34] are a class of functions that cannot be accelerated by a parallel computation and can be verified efficiently. Long and Wei [35] propose a PoS protocol that incorporates a variation of Verifiable Delay Functions (VDFs) that has a random delay. However, the threshold for a successful private attack is less than 27% compared to almost 50% in *Sprints*. PoSAT [36] has a similar construction. PoSAT is vulnerable to nothing-at-stake attacks, where an attacker can mine on multiple forks for no additional cost. The authors, therefore, divide the rounds into epochs which prevents new players from joining the network during an epoch. The system has a threshold of 50% for a private attack only when the epochs are infinitely long, which turns it into a permissioned system. Thus, PoSAT has a tradeoff between decentralization and security.

HEB [37] also utilizes on-chain resources uses mechanism design to reduce electricity consumption by 50%, but reduces resilience to malicious attacks by 2, while *Sprints* has almost the same threshold as pure PoW with an 8x reduction in power consumption.

Several approaches reduce expenditure with particular types of PoW. REM [38] and PoET [39] use trusted hardware to reduce resource expenditure. As the protocols are based on trusted hardware, they rely on a trusted party to guarantee the hardware's integrity. *Sprints* makes no such assumptions.

Chia [9] shifts the costs of miners from electricity to hardware by combining Proof-of-Space (PoS) and Proof-of-Time (PoT). It reduces electricity consumption by replacing much of the mining costs with storage costs. However, it is resilient to attackers with under 30% of the network storage resources [10], which is significantly lower than the almost 50% threshold of *Sprints*.

Another approach to deal with this challenge is to use permissioned protocols [40–42], which are a class of protocols that require pre-authorization of nodes to participate in the protocol. However, permissioned protocols are not decentralized and require a trusted third party to authorize new nodes.

As for the analysis of PoW protocols, Dembo et al. [10] analyze the security of blockchain systems by showing that a so-called *private attack* is the worst-case attack. They define the notion of Nakamoto blocks and use their existence to prove that the system is secure. We use a different approach, where we avoid the notion of Nakamoto blocks and instead describe a single race between the attacker and the honest miners. We use this race to prove directly that the security requirements hold.

3. Model

The system consists of a set of participants called *miners*. Each miner maintains a tree data structure whose vertices are called *blocks*. Each block contains *transactions*, commands issued by system users, which are the *payload* of the block. All miners start with the same block, called *genesis*, that serves as the root of the tree. In addition to the payload, each block contains metadata. The metadata of all blocks (except the genesis block) includes a hash that points to its *parent* block; we say that a block *extends* the pointed-to block. A path with the most blocks in the tree is called a *longest chain*. There may be several longest chains and one of them, chosen by a deterministic arbitrary algorithm, is called the *main chain*. The *height*[1] of a block b is the number of blocks in the path from the genesis to b.

Time progresses in discrete steps $t = 0, 1, ...$ (as in, e.g., [43]). In each step, miners can *work* on two types of *puzzles*. The first type is *Proof-of-Delay (PoD)*, a function that maps an arbitrarily sized input and difficulty parameter to a small output. Given a random input, a PoD requires a deterministic number of steps, called *delay period* and denoted by Δ_{PoD}. In each step, a miner can choose to mine on multiple PoD puzzles. After working on a particular PoD puzzle in Δ_{PoD} distinct steps, a miner has its solution, which cannot be guessed except with negligible probability. The formal specification of a PoD is the same as that of a VDF [30] with the added requirement that all the players have the same delay period. This assumption is reasonable since miners who cannot solve the PoD puzzle in the given delay period will not be able to compete in the protocol, and thus will be forced to leave the system.

The second puzzle type is *Proof-of-Work*, a probabilistic puzzle that has an independent probability of being solved in any given step. In each step, a miner can choose to mine on a PoW puzzle. Miner p has a probability $P_w(p)$ of solving it in each step independent of previous attempts. The number of steps until success thus has a Geometric distribution with parameter $P_w(p)$.

Combining metadata M, payload D and puzzle solutions Z, a block is the tuple (M,D,Z).

The miners communicate over a Δ-synchronous broadcast network [10, 23]: If a miner sends a message in step t, then all miners receive it by step $t+\Delta$. We assume that $\Delta < \Delta_{PoD}$.

Thus, at the beginning of every step, a miner receives messages sent in previous steps. The miner can then perform local computations, i.e., work on one PoW and multiple[2] puzzles. Then the environment notifies the miners if they found solutions. Finally, the miner can broadcast blocks.

There are a total of $n+1$ miners in the system, where n are *honest*, i.e., miners that follow a predefined protocol, and a single miner is controlled by an *adversary* $\mathcal{A}$ and acts arbitrarily.

Note that the adversary can only work on a single PoW puzzle in a single step in our model. However, working on multiple PoW puzzles in parallel can be approximated by frequent puzzle changes.

The adversary controls the message delay, constrained by the bound Δ.

Next, we define a predicate that validates the correctness of a block and its puzzles. A *validity function* $V(b)$ returns *true* if some predefined conditions on the block contents are met and *false* otherwise. A block b is *valid* if $V(b) = true$.

1. Called depth in graph-theory literature.

2. The number of puzzles is polynomial in a system security parameter; we omit these details to simplify the presentation.

Invalid blocks with either invalid payloads or proofs are simply ignored.

An *execution* is a series of states of the system that develops based on the miners' algorithms and the environment's coin flips, i.e., the outcome of PoW puzzle-solving attempts. Each execution has a certain probability of occurring. Denote by Σ the set of all executions. Given step t_0, denote by π the t_0*–prefix* that is the collection of all executions in Σ that agree on the state of the system at step t_0. Given an execution σ, denote by $\mathcal{T}^\sigma(t)$ the tree that corresponds to the execution in step t, called a *mother tree* [10]. The mother tree consists of all valid blocks in the system (published or not) until step t. Note that $\mathcal{T}^\sigma(t)$ represents the state of the tree at the beginning of the step. Denote the depth of a valid block b by $d(b)$. Given a execution σ, denote the depth of the longest chain(s) in $\mathcal{T}^\sigma(t)$ by $d^\sigma(t)$. Each player p has a local copy of the mother tree, which is updated according to the blocks she receives, it is denoted by $\mathcal{T}_p^\sigma(t)$.

Given an execution σ, denote by σ_p the execution view of miner p. It includes all the information she received and the results of her local computations.

In each step, each honest miner performs an *action* that comprises the mining target for PoW and PoD and the blocks a miner publishes. The action is defined by the *mining function* $q(\sigma_p)$, given a view σ_p of miner p. The vector $Q_H^\sigma(t)$ includes the honest miners' actions in step t defined by the mining function. We denote by $Q_\mathcal{A}^\sigma(t)$ the action of the adversary at step t and execution σ. The adversarial action is decided based on a predefined *strategy* A, which is a map from step t and a state of execution at step $t-1$ to a vector that represents the mining targets for PoW and PoD, the blocks being published and the delay the adversary imposes on messages.

A longest chain protocol is thus defined by a pair – a validity function and a mining function, $(V(\cdot), q(\cdot))$. For each node, the sequence of payloads along the main chain, excluding the payloads of the last r blocks (for some r), is called a *ledger*. We use the notions of *persistence* and *progress*[3], similar to the definitions in previous work [10, 23, 24]. Given a block b, we denote by $d_b^\sigma(t,p)$ the depth of the longest chain in the view of miner p in step t that contains b and by $d_{\neg b}^\sigma(t,p)$ the depth of the longest chain in the view of miner p in step t that does not contain b.

Definition 1. *A protocol* $(V(\cdot),q(\cdot))$ *implements a ledger if it satisfies the following two conditions:*

Persistence *Given* $\varepsilon>0$*, step* t_f*, there exists* $r\in\mathbb{N}$ *s.t. for all adversarial strategy* A *given a randomly drawn execution* $\sigma\leftarrow\Sigma_A$*, for every step* $t\le t_f$ *and every block* $b\in\sigma$*, if* b *is at depth* i *of the main chain and* $d_b^\sigma(t-\Delta,p)>d_{\neg b}^\sigma(t-\Delta,p)+r$ *for an honest miner* p*, then for every* $t'>t$ *it holds that* b *is in depth* i *in the main chain of the view of all other honest miners with probability at least* $1-\varepsilon$*.*

Progress[3] *Given* $\varepsilon>0$ *and* t_f *there exists* $\delta\in\mathbb{N}$*, such that for all steps* $t_0\le t_f$ *and adversarial strategy* A *the probability that a random execution* $\sigma\in\Sigma_A$ *does not include a block that was mined in* $[t_0,t_0+\delta]$ *in the main chain of all honest miners for some step* $t'>t_0+\delta$ *is smaller than* ε*.*

3. Our Progress is called Liveness in some prior work [10, 23, 24] although it is a safety property [44].

4. *Sprints*

We now present the *Sprints* protocol, which implements a ledger.

We only need to define a mining function $q(\sigma_p)$ for a player p and a validity function $V(b)$.

Sprints requires each block to contain two puzzle solutions, the first is a proof of delay b^{PoD} that enforces serial computation, and the second is a proof of work b^{PoW}. The mining function chooses the deepest block, partial or full, in the view of the miner and returns the puzzle that is required to be solved: If the deepest block is a full block, then the mining function first works on the PoD of the next block, as follows. Given a metadata M that contains the previous block's hash, the mining function returns the action to take a step in the PoD puzzle of M. If the deepest block is partial, i.e., the PoD puzzle of the new block was already solved, then the mining function works on the PoW puzzle, as follows. Given the PoD puzzle solution of the metadata, b^{PoD}, and the payload of the new block, D, the mining function returns the action to take a step in the PoW puzzle of $b^{PoD}||D$. If the PoW step is successful and the miner obtains a solution b^{PoW}, then the miner publishes the new block with the metadata, the payload, and the puzzle solutions: $b=(M,D,(b^{PoD},b^{PoW}))$.

The validity function $V(b)$ validates a block b by checking that the PoD and PoW puzzles are valid, using the validity functions $V_{PoD}(\cdot)$ and $V_{PoW}(\cdot)$, respectively:

$$V(b):=V_{PoD}(b^{PoD},M)\wedge V_{PoW}(b^{PoW},b^{PoD}||D).$$

Note that the PoD does not require the payload of the block, while the PoW requires the payload of the block.

If a node learns of a chain longer than the block it is currently working on, it discards its work and begins generating a block extending the new chain.

5. Security

We prove that *Sprints* achieves persistence and progress. To prove progress, we must show that, given a sufficiently long duration, at least one honestly-mined block within that time will remain in the main chain forever with high probability. We decompose this problem into two components: (1) Establish that given $r>0$, with high probability, honest miners will discover r blocks within a long enough period. And (2) show that the probability that the adversary forms a tree deeper than the tree observed by honest miners tree by at least r blocks is bounded by $2^{-\Omega(r)}$.

Both proofs employ a reduction to a *simple game*. In this game the adversary and honest miners initially mine their respective block trees, stemming from a shared genesis block. The adversary's goal is to reach or surpass the length of the honest miners' tree once both trees have reached a length of at least r blocks. Subsequently, we examine an infinite set of games where the adversary achieves victory when both trees have lengths of at least r,$r+1$,$r+2$,..., and so forth. We prove that the sum of the probabilities of the adversary winning each game is bounded by $2^{-\Omega(r)}$. To do this we find the attack that is most likely to succeed and bound the probability of its success; we say this attacker is the *worst attacker*. In particular, we should find which PoW and which PoD the worst attacker should calculate at each step. For PoD we assume that the adversary mines all possible PoD blocks in parallel This assumption can only make the adversary stronger since any other adversary can be emulated by the one described above. Subsequently, we prove that the optimal strategy, defined as the one that maximizes the length of the adversary's subtree while minimizing the honest subtree for any given step, is the strategy that PoW mines on the deepest block in the adversary's subtree as long as no PoW solution exists of the same depth, i.e., a complete PoD puzzle does not yet extend this PoW puzzle. In the latter scenario, the adversary halts PoW until a PoD puzzle is found.

Now, after proving it is sufficient to consider this simple worst attack, we note that when the adversary's subtree and the honest subtree have the same length, both subtrees contain an equal number of PoD blocks, so the competition only depends on the PoW Computations. This simplifies the analysis and allows us to analyze the game as a pure PoW game, and conclude the proof.

Before detailing the proof, we present some terminology and notation (§5.1). We then discuss optimal strategies for the adversary (§5.2) and the honest miners (§5.3) subtrees. Next, we use these strategies to analyze the simple game (§5.4). Finally, we prove persistence and progress and the security of *Sprints* (§5.5).

5.1. Terminology and Notation

We define a race between the honest miners and an adversary with strategy A. We assume that an adversary always chooses to PoD-mine on all possible blocks in parallel, as every strategy A_1 that PoD-mines only some blocks creates the same tree as the one created by a strategy A_2 that is identical to A_1 but PoD-mines everywhere. Therefore, for succinctness, we define a mining action at step t as the block that the adversary chooses to PoW mine on; if there is no such block, we say that the adversary's action is $\perp$, i.e., empty action. The adversary chooses a delay for each block b and honest miner p. To simplify the analysis, we assume that the attacker can delay the block even for the miner who found it. This assumption only strengthens the adversary, allowing us to bound the adversary's success probability.

Given a depth i, execution σ and honest block b at depth i, we define two *sub-trees* (portrayed in Figure 3): for all steps t, $\mathcal{T}^{\sigma}_{H}(t)$ is the sub-tree of $\mathcal{T}^{\sigma}(t)$ that includes b, its descendants that are known by all honest miners and its ancestors; tree $\mathcal{T}^{\sigma}_{\mathcal{A}}(t)$ is the sub-tree of $\mathcal{T}^{\sigma}(t)$ that includes b and its private adversarial descendants and ancestors. Note that $\mathcal{T}^{\sigma}(t)$ can include adversarial blocks.

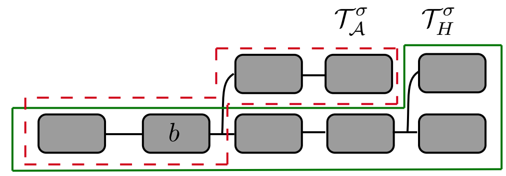


Figure 3: Scheme of $\mathcal{T}^{\sigma}_{\mathcal{A}}$ and $\mathcal{T}^{\sigma}_{H}$ depending on b.

A block data structure containing only the PoD puzzle, $(M, D, (b^{PoW}, \perp))$, is called a *partial block* and a valid block with all proofs is called a *full block*. Denote the depth of a partial block b by $\tilde{d}(b)$. Given an execution σ, we define the *PoD depth* of a tree as the deepest block in the mother tree that has a completed PoD puzzle on the subtree, it is denoted by $\tilde{d}^{\sigma}_{b}(t)$ for $\mathcal{T}^{\sigma}_{\mathcal{A}}(t)$ and by $\tilde{d}^{\sigma}_{\neg b}(t)$ for $\mathcal{T}^{\sigma}_{H}(t)$. For some π, denote by $\Sigma^{\pi_i}_{A}$ the subset of Σ^{π}, where the adversary follows a strategy A.

Next, we define the optimality of an adversarial strategy. We look separately at $\mathcal{T}^{\sigma}_{H}(t)$ and $\mathcal{T}^{\sigma}_{\mathcal{A}}\ (t)$ and find a single strategy that minimizes the depth of the former and maximizes the depth of the latter for all steps.

5.2. Adversary tree

We focus on the optimal strategy of the adversary on her private tree $\mathcal{T}^{\sigma}_{\mathcal{A}}(t)$.

Given a prefix π of length t_0, and given an execution $\sigma \in \pi$ we start by considering only the actions of an adversary that target the subtree $\mathcal{T}^{\sigma}_{\mathcal{A}}(t)$. At first, we assume that given some step $t_f > t_0$, the adversary aims to maximize the depth of $\mathcal{T}^{\sigma}_{\mathcal{A}}(t_f)$. Later we generalize this to a strategy that aims to maximize the depth of $\mathcal{T}^{\sigma}_{\mathcal{A}}(t)$ for all $t > t_0$.

We define the notion of an *optimal strategy*. As a first step, we define a (t_f,ℓ)-optimal strategy that maximizes the probability that at step t_f the depth of $\mathcal{T}^{\sigma}_{\mathcal{A}}$ is at least ℓ. For a set of executions Σ^{π}_{A}, denote by $\Pr_{\Sigma^{\pi}_{A}}[\cdot]$ the conditional probability $\Pr[\cdot|\sigma \in \Sigma^{\pi}_{A}]$.

Definition 2. *An attacker strategy A is* (t_f,ℓ)-optimal *if, for all prefixes π of length $t_0 < t_f$ and strategies A', it holds that* $\Pr_{\Sigma^{\pi}_{A}}[d^{\sigma}_{\neg b}(t_f) \geq \ell] \geq \Pr_{\Sigma^{\pi}_{A'}}[d^{\sigma}_{\neg b}(t_f) \geq \ell]$. *Strategy A is* optimal *if it is (t_f,ℓ)-optimal for all t_f and ℓ.*

For execution σ, denote by $B_{max}(\mathcal{T}^{\sigma}_{\mathcal{A}}(t))$ the *tips of the chain* the set of deepest partial blocks in step t whose PoD puzzle is completed locally for player p. Denote their depth by $\tilde{d}^{\sigma}_{\neg b}(t)$.

Definition 3 (longest-chain mining)**.** *Given an execution σ, an attacker strategy that chooses a block in $B_{max}(\mathcal{T}^{\sigma}_{\mathcal{A}}(t))$ for $t \geq t_0$ is* longest-chain mining *(*LCM*).*

We show that any longest-chain mining strategy is (t_f,ℓ)-optimal.

Lemma 1. *For all prefixes π of length t_0, ℓ and steps t_f, an LCM strategy A played from step t_0 is (t_f,ℓ)-optimal.*

Before proving Lemma 1, we introduce another lemma, which will also be useful later. We show that given two execution sets where the first has higher partial depth at step t_0 and Δ_{PoD} steps later, it has a higher probability to be deeper at step $t_f > t_0$ if the attacker follows a (t_f,ℓ)-optimal strategy.

Lemma 2. *Let there be two prefixes π_1 and π_2 of length t_0, given t_f and ℓ, and a (t_f,ℓ)-optimal strategy A. Consider two sets of executions, Σ_1 and Σ_2, such that $\Sigma_i = \Sigma_A^{\pi_i}$. We assume that for π_1 and π_2, in step t_0 no PoD puzzle is being calculated so that at some point in the future, an execution from Σ_2 will be deeper than any execution from Σ_2 due to this puzzle. Formally:*

$$\forall \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2, t \in [t_0, t_0+\Delta_{PoD}] : \tilde{d}^{\sigma_1}_{\neg b}(t) \geq \tilde{d}^{\sigma_2}_{\neg b}(t). \quad (1)$$

Then it holds that

$$\Pr_{\Sigma_1}[d^{\sigma}_{\neg b}(t_f) \geq \ell] \geq \Pr_{\Sigma_2}[d^{\sigma}_{\neg b}(t_f) \geq \ell].$$

The full proof of Lemma 1 and Lemma 2 is given in §B.2. The main idea is that we prove both lemmas by double induction starting from t_f. We run the induction for decreasing step period and prove both lemmas together at each step of the induction. We first show that the base of the induction holds for $t_0 \in [q_0, t_f - 1]$ for both lemmas Next, we assume that both lemmas hold for $t_0 \in [q_0 - n, t_f - 1]$ for some n. We then prove the induction step for Lemma 2 for $t_0 = t_f - n - 1$ using the assumption. Finally, we prove the induction step for Lemma 1 for $t_0 = t_f - n - 1$ using the result we just proved for Lemma 2 for $t_0 = t_f - n - 1$. This concludes the proof of both lemmas.

After showing that LCM is a (t_f,ℓ)-optimal strategy, we show that there is no benefit for the adversary from PoW mining while she calculates the deepest block's PoD puzzle. For this purpose, we first define useless actions we call *backward mining*, where the adversary mines a PoW puzzle that would not extend the depth of the chain:

Definition 4. *Given an execution σ, an action $Q^{\sigma}_{\mathcal{A}}(t) = b_{PoD}$ at step t is* backward mining *if the adversary already computed a partial block b'_{PoD} that extends a block in $\mathcal{T}^{\sigma}_H$, such that $\tilde{d}(b_{PoD}) < \tilde{d}(b'_{PoD})$ or if there is a full block b such that $\tilde{d}(b_{PoD}) = d(b)$.*

Next, we define an upgrade of the LCM strategy without useless backward mining actions.

Definition 5. *A strategy is (t_f,ℓ)-*intermittent LCM *if it is (t_f,ℓ)-optimal and does not perform backward mining actions.*

It remains to show that every LCM can be transformed to (t_f,ℓ)-intermittent LCM without affecting its optimality.

Lemma 3. *For all t_f, ℓ and an LCM strategy A, the intermittent LCM strategy A' that is identical to A, except that every backward mining action is replaced with $\perp$, is (t_f,ℓ)-optimal.*

Proof. Given a prefix π of length t_0, We look at two sets of executions. (1) Σ^{π}_A for a (t_f,ℓ)-optimal strategy A that for some $\sigma \in \Sigma^{\pi}_A$ chooses some backward mining action $Q^{\sigma}_{\mathcal{A}}(t_0) = b$. and (2) $\Sigma^{\pi}_{A'}$, where the strategy A' chooses $\perp$ at step t_0 but for $t > t_0$ it is identical to A. As before, the probability to find a block exactly at t_0 is $P_w(\mathcal{A})$. Note that this probability is not relevant for $\Sigma^{\pi}_{A'}$. Denoted the subset of Σ^{π}_A where no block is found at step t_0, by Σ_1. For all $\sigma_1 \in \Sigma_1$, there exists a unique $\sigma_2 \in \Sigma^{\pi}_{A'}$ such that $\mathcal{T}^{\sigma_1}_b \equiv \mathcal{T}^{\sigma_2}_b$ and vice versa. σ_1 and σ_2 agree on all random coins from t_0+1. If a block is found at step t_0, the tree depth of both executions is still equal, therefore:

$$\Pr_{\Sigma_1}[d^{\sigma}_{\neg b}(t_f) \geq \ell]) = \Pr_{\Sigma^{\pi}_{A'}}[d^{\sigma}_{\neg b}(t_f) \geq \ell]. \quad (2)$$

The block that is found in step t_0 for all $\sigma_1 \in \Sigma^{\pi}_A \setminus \Sigma_1$ does not extends the depth of $\mathcal{T}^{\sigma_1}_b$. Therefore, for all $\sigma_2 \in \Sigma^{\pi}_{A'}$, it holds that $\tilde{d}^{\sigma_1}_b(t) \leq \tilde{d}^{\sigma_2}_b(t)$ for $t \in [t_0+1, t_0+\Delta_{PoD}+1]$. The conditions of Lemma 2 thus hold, so:

$$\Pr_{\Sigma^{\pi}_A \setminus \Sigma_1}[d^{\sigma}_{\neg b}(t_f) \geq \ell] \leq \Pr_{\Sigma^{\pi}_{A'}}[d^{\sigma}_{\neg b}(t_f) \geq \ell]. \quad (3)$$

Using complete probability:

$$\begin{aligned}
&\Pr_{\Sigma^{\pi}_A}[d^{\sigma}_{\neg b}(t_f) \geq \ell] = \\
&\quad P_w(\mathcal{A}) \cdot \Pr_{\Sigma^{\pi}_A \setminus \Sigma_1}[d^{\sigma}_{\neg b}(t_f) \geq \ell] + \\
&\quad (1 - P_w(\mathcal{A})) \cdot \Pr_{\Sigma_1}[d^{\sigma}_{\neg b}(t_f) \geq \ell]) \\
&\overset{\text{Equation (2) and Equation (3)}}{\leq} \\
&\quad P_w(\mathcal{A}) \cdot \Pr_{\Sigma^{\pi}_{A'}}[d^{\sigma}_{\neg b}(t_f) \geq \ell] + \\
&\quad (1 - P_w(\mathcal{A})) \cdot \Pr_{\Sigma^{\pi}_{A'}}[d^{\sigma}_{\neg b}(t_f) \geq \ell] = \\
&\quad \Pr_{\Sigma^{\pi}_{A'}}[d^{\sigma}_{\neg b}(t_f) \geq \ell].
\end{aligned}$$

Note that although $P_w(\mathcal{A})$ has no meaning in the context of $\Sigma^{\pi}_{A'}$, we used a simple algebraic trick to disassemble $\Pr_{\Sigma^{\pi}_{A'}}[d^{\sigma}_{\neg b}(t_f) \geq \ell]$ to two parts.

We apply the described process recursively, each time eliminating a single backward mining action. We end with a new (t_f,ℓ)-optimal intermittent LCM strategy as required by the lemma. ☐

We can now conclude that any intermittent LCM strategy is an optimal strategy.

Corollary 1. *Intermittent LCM is an optimal strategy.*

Proof. By Lemma 3, for all ℓ and t_f any intermittent LCM strategy is (t_f,ℓ)-optimal, so it is an optimal strategy. ☐

5.3. Honest tree

We now focus on the subtree $\mathcal{T}^{\sigma}_H(t)$. We look for a strategy that minimizes the depth of the tree.

Definition 6 (Maliciously optimal strategy)**.** *A strategy A is (t_f,ℓ)-*maliciously optimal *if, for all prefixes π of length $t_0 < t_f$ with block b, and for all strategies A', it holds that*

$$\Pr_{\Sigma^{\pi}_A}[d^{\sigma}_b(t_f) \geq \ell] \leq \Pr_{\Sigma^{\pi}_{A'}}[d^{\sigma}_b(t_f) \geq \ell].$$

We call A a maliciously optimal *if it is (t_f,ℓ)-maliciously optimal for all $t_f > t_0$ and for all ℓ.*

We consider a strategy where the attacker does not mine new blocks on $\mathcal{T}_H^\sigma(t)$ and delays any honest block by Δ, which is the maximal delay she can impose.

Definition 7. *Given a prefix π, a strategy A is a* Maximum Delay and No Mining strategy (MDNM) *if the adversary chooses to mine no blocks and to maximize the delay for the arrival of all honest blocks to be Δ for all miners.*

Denote the hash rate of the honest miners that received the block that is a tip of the chain in execution σ at step t by $\lambda^\sigma(t)$. Similarly, denote by $\lambda_m^\sigma(t)$ the total hash rate of miners that heard of blocks of depth at least m.

We show that an MDNM strategy maintains some advantages.

Lemma 4. *Given two prefixes π_1 and π_2 of length t_0, ℓ, and a MDNM strategy A, such that for all $\sigma_1 \in \Sigma_A^{\pi_1}$, $\sigma_2 \in \Sigma_A^{\pi_2}$ it holds that*

$$\forall t \in [t_0, t_0+\Delta_{PoD}] : d_b^{\sigma_1}(t) \geq d_b^{\sigma_2}(t) \tag{4}$$

and

$$\forall t \in [t_0, t_0+\Delta+\Delta_{PoD}] : \lambda^{\sigma_1}_{d_b^{\sigma_2}(t)}(t) \geq \lambda^{\sigma_2}(t), \tag{5}$$

it holds that for all $\ell > 0$ and step t_f:

$$\Pr_{\Sigma_A^{\pi_1}}[d_b^\sigma(t_f) > \ell] \geq \Pr_{\Sigma_A^{\pi_2}}[d_b^\sigma(t_f) > \ell].$$

Proof. We fix t_f and ℓ and prove by induction.

Basis. For $t_0 \in [t_f - \Delta_{PoD}, t_f]$ the lemma holds by Equation (4) and Equation (5).

Assumption. Denote by $q_n = t_f - \Delta_{PoD} - n$. We assume that the lemma holds for all $t_0 \in [q_n, t_f]$.

Step. We prove the lemma holds for $t_0 = q_{n+1}$. For some prefix π, the probability that any honest miner would find a PoW block of depth $d_b^{\pi_1}(t_0)+1$, i.e., deeper by 1 than $\mathcal{T}_H^{\sigma_1}$, is denoted by $P_{suc}(\pi)$. Denote the correspondent event by $e_{suc}^\sigma(t)$. From Equation (5) it holds that:

$$P_{suc}(\pi_1) > P_{suc}(\pi_2). \tag{6}$$

We partition $\Sigma_A^{\pi_1}$ into two subsets, Σ_1 and Σ_2, where $e_{suc}^\sigma(t_0)$ and $\neg e_{suc}^\sigma(t_0)$ hold, respectively. To use the induction assumption, we show that the requirement in Equation (5) holds for the period starting in step t_0+1. For all $t \in [t_0+1, t_0+\Delta+\Delta_{PoD}+1]$, $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$, it holds that $\lambda^{\sigma_1}_{d_b^{\sigma_2}(t)}(t) \geq \lambda^{\sigma_2}(t)$ because for σ_2, no new block is found in t_0 thus every miner in σ_1 knows of any block that a miner in σ_2 knows in the same step. Additionally, for σ_1 miners' new mining target would necessarily be deeper than $d_b^{\sigma_2}(t)$, thus, for all $t \in [t_0+1, t_0+\Delta_{PoD}+1]$ it holds $d_b^{\sigma_1}(t) \geq d_b^{\sigma_2}(t)$. From the induction assumption:

$$\Pr_{\Sigma_1}[d_b^\sigma(t_f) \geq \ell] \geq \Pr_{\Sigma_2}[d_b^\sigma(t_f) \geq \ell]. \tag{7}$$

Next we compare $\Sigma_A^{\pi_1}$ and $\Sigma_A^{\pi_2}$ separately for subsets that contain executions where a block deeper than $d_b^{\pi_1}(t_0)+1$ was found in the end of step t_0 and to subsets where the block was not found.

Next, our goal is to show that the requirement in Equation (5) holds. We look at $t = t_0+\Delta_{PoD}+1$, $\sigma_1 \in \Sigma_A^{\pi_1}$ and $\sigma_2 \in \Sigma_A^{\pi_2}$. If a block of depth $d_b^{\pi_1}(t_0)+1$ was found in the end of step t_0, because the depth of σ_2 cannot increase more than $d_b^{\sigma_1}(t_0)+1$ it holds $d_b^{\sigma_1}(t) \geq d_b^{\sigma_2}(t)$. If a block of depth $d_b^{\pi_1}(t_0)+1$ was not found in the end of step t_0, it holds that $d_b^{\sigma_1}(t_0)+1 > d_b^{\sigma_2}(t)$, therefore, $d_b^{\sigma_1}(t) \geq d_b^{\sigma_1}(t_0) \geq d_b^{\sigma_2}(t)$.

For $t = t_0+\Delta+\Delta_{PoD}+1$: If a block of depth $d_b^{\pi_1}(t_0)+1$ was found at the end of step t_0, the attacker delays blocks with maximum delay Δ for all honest miners, thus, they all hear about the block in step $t_0+\Delta+1$. The PoD puzzle is completed by the beginning of step t so that all miners in σ_1 and σ_2 are now mining on a block with depth $d_b^{\sigma_1}(t_0)+1$. Because $d_b^{\sigma_1}(t_0)+1 \geq d_b^{\sigma_2}(t)$, it holds $\lambda^{\sigma_1}_{d_b^{\sigma_2}(t)}(t) \geq \lambda^{\sigma_2}(t)$.

If a block of depth $d_b^{\pi_1}(t_0)+1$ was not found at the end of step t_0, by step t all miners in σ_1 heard of a block with depth $d_b^{\sigma_1}(t_0)$, as more than Δ step has passed since such block was found. Therefore, it must hold $\lambda^{\sigma_1}_{d_b^{\sigma_2}(t)}(t) \geq \lambda^{\sigma_2}(t)$.

We assume that the assumption in Equation (4) that is true for $[t_0, t_0+\Delta_{PoD}]$, and Equation (5) that is true for $[t_0, t_0+\Delta+\Delta_{PoD}]$. Using what we showed above, it holds that: $\forall t \in [t_0+1, t_0+\Delta_{PoD}+1] : d_b^{\sigma_1}(t) \geq d_b^{\sigma_2}(t)$ and $\forall t \in [t_0+1, t_0+\Delta+\Delta_{PoD}+1] : \lambda^{\sigma_1}_{d_b^{\sigma_2}(t)}(t) \geq \lambda^{\sigma_2}(t)$.

Based on the induction assumption we get:

$$\Pr_{\Sigma_A^{\pi_1}}[d_b^\sigma(t_f) \geq \ell | e_{suc}^\sigma(t_0)] \geq \Pr_{\Sigma_A^{\pi_2}}[d_b^\sigma \geq \ell | e_{suc}^\sigma(t_0)] \tag{8}$$

and

$$\Pr_{\Sigma_A^{\pi_1}}[d_b^\sigma(t_f) \geq \ell | \neg e_{suc}^\sigma(t_0)] \geq \Pr_{\Sigma_A^{\pi_2}}[d_b^\sigma \geq \ell | \neg e_{suc}^\sigma(t_0)]. \tag{9}$$

We can now conclude (see §B.1 for the full details):

$$\Pr_{\Sigma_A^{\pi_1}}[d_b^\sigma(t_f) \geq \ell] = \Pr_{\Sigma_A^{\pi_2}}[d_b^\sigma(t_f) \geq \ell]. \tag{10}$$

This concludes the step of the induction and thus the proof of the lemma. □

Next, we use use Lemma 4 to prove:

Lemma 5. *An MDNM strategy is maliciously optimal.*

Proof. To simplify the proof, we conservatively assume that the adversary can add an unbounded number of blocks to the tree in all steps, obtaining the required PoD and PoW instantly. If MDNM is optimal for such a powerful adversary it is also optimal for the weaker adversary defined in the model.

We prove that the MDNM is (t_f, ℓ)-maliciously optimal for all t_f and ℓ. Given a prefix π of length t_0, and some t_f and ℓ, we prove using induction. Denote the depth of the deepest block that the adversary adds at step t by $d_*^\sigma(t)$. If the adversary does not add a block in step t we take $d_*^\sigma(t) = 0$.

Basis. First, we prove for $t_0 \in [t_f - \Delta_{PoD}, t_f]$. By step t_f the chain can grow by at most one block. Any strategy where the block that the adversary adds extends the total depth would deterministically increase the depth at t_f at least as $d_*^\sigma(t) = 0$ would in the worst case. Any strategy where the new block does not change the depth would have an identical result, as miners extend the first block they hear of a specific depth. Therefore, not adding a block is not worse than any strategy and therefore is optimal.

Similarly, not delaying a block that extends the chain would be an inferior strategy, and delaying a block that does not extend the chain would be identical to allowing it to be published as it does not change other miners' behavior.

Therefore, MDNM is maliciously optimal for $t_0 \in [t_f - \Delta_{PoD}, t_f]$ as required.

Assumption. Denote $q_n = t_f - \Delta_{PoD} - n$. We assume that the lemma holds for $t_0 \in [q_n, t_f]$.

Step. We prove it holds for $t_0 = q_{n+1}$. From the induction assumption, during $[q_n,\ t_f]$ the maliciously optimal strategy is to not introduce new blocks and delay every honest block by Δ. Let A be the MDNM strategy and A' be a strategy where the action at t_0 is arbitrary and is MDNM for $t \in [t_0+1, t_f]$. Denote the sets of executions where the adversary uses A and A' by Σ_A and $\Sigma_{A'}$, respectively. Denote the probability that the honest miners find a block that extends the depth at step t_0 by P_{FB} and the correspondent event by $e^{\sigma}_{FB}(t_0)$. Note that this probability is identical for both Σ_A and $\Sigma_{A'}$, as all the honest miners have the same mining target at the beginning of step t_0. We are interested in using Lemma 4 for both subsets.

Given $\sigma_2 \in \Sigma_A$ and $\sigma_1 \in \Sigma_{A'}$, if the honest miners find a block on the tip in step t_0 for both σ_1 and σ_2, for some step $t \in [t_0+1, t_0+\Delta_{PoD}+1]$ it holds $d^{\sigma_1}_b(t) \geq d^{\sigma_2}_b$. This is because for σ_2 the depth increase exactly by 1 for t_0+1 and does not change until at least $t_0 + \Delta_{PoD} + 1$, while for σ_1 the depth increase at least by 1 by t_0+1. It also holds that $\lambda^{\sigma_1}_{d^{\sigma_2}_b(t)}(t) \geq \lambda^{\sigma_1}(t) \geq \lambda^{\sigma_2}(t)$, from the definition of $\lambda^{\sigma_1}_{d^{\sigma_2}_b(t)}(t)$ and given a miner p, p receives the block in σ_2 not later than p in σ_1, thus, the target of every such miner p in σ_1 cannot be deeper than in σ_2.

Using similar considerations for the case where the miner does not find a block on the tip, we conclude that $\lambda^{\sigma_1}_{d^{\sigma_2}_b(t)}(t) \geq \lambda^{\sigma_1}(t) \geq \lambda^{\sigma_2}(t)$ and $d^{\sigma_1}_b(t) \geq d^{\sigma_2}_b$ for $t \in [t_0+1, t_0+\Delta+\Delta_{PoD}+1]$ in both cases. The condition in Lemma 4 holds. Therefore:

$$
\begin{aligned}
&\Pr_{\Sigma^{\pi}_{A}}[d^{\sigma}_b(t_f) \geq \ell] = \\
&P_{FB} \cdot \Pr_{\Sigma^{\pi}_{A}}[d^{\sigma}_b \geq \ell | e^{\sigma}_{FB}(t_0)] + (1-P_{FB}) \cdot \Pr_{\Sigma^{\pi}_{A}}[d^{\sigma}_b \geq \ell | \neg e^{\sigma}_{FB}(t_0)] \leq \\
&P_{FB} \cdot \Pr_{\Sigma^{\pi}_{A'}}[d^{\sigma}_b \geq \ell | e^{\sigma}_{FB}(t_0)] + (1-P_{FB}) \cdot \Pr_{\Sigma^{\pi}_{A'}}[d^{\sigma}_b \geq \ell | \neg e^{\sigma}_{FB}(t_0)] \leq \\
&\Pr_{\Sigma^{\pi}_{A'}}[d^{\sigma}_b(t_f) \geq \ell].
\end{aligned}
$$

This concludes the induction, showing MDNM is a maliciously optimal strategy. □

5.4. Basic race

We now introduce a two-epoch race where the adversary tries to fork a chain of blocks. We then find an upper bound on the probability that the adversary succeeds in the attack. From now on, we assume that $\Delta = 0$.

Given an adversarial strategy A and execution σ, we define a block $b^{\sigma}_{\mathsf{s}}(q)$ as the first published depth-q block in σ.

We consider a race where the adversary mines a secret tree denoted by $\mathcal{T}^{\sigma}_{\mathcal{A}}(t,q)$, whose root is $b^{\sigma}_{\mathsf{s}}(q)$ and it is a function of step t and depth q of $b^{\sigma}_{\mathsf{s}}(q)$. We assume that the adversary did not mine any blocks before a block at depth q is published. The adversary never sends blocks from $\mathcal{T}^{\sigma}_{\mathcal{A}}(t,q)$ to the honest miners until the end of the race and they stay secret. We also consider a public subtree $\mathcal{T}^{\sigma}_{H}(t,q)$ whose blocks are public and has $b^{\sigma}_{\mathsf{s}}(q)$ as its root. All blocks that have $b^{\sigma}_{\mathsf{s}}(q)$ as an ancestor and are not in $\mathcal{T}^{\sigma}_{\mathcal{A}}(t,q)$ are in $\mathcal{T}^{\sigma}_{H}(t,q)$. The attacker can mine on $\mathcal{T}^{\sigma}_{H}(t,q)$ and publish the blocks at any time. Given an integer $r > 0$, we say that the adversary has won the race if, at some step t, it holds that $d(\mathcal{T}^{\sigma}_{\mathcal{A}}(t,q)) \geq q+r$ and $d(\mathcal{T}^{\sigma}_{\mathcal{A}}(t,q)) \geq d(\mathcal{T}^{\sigma}_{H}(t,q))$. Note that in this race the adversary cannot mine before a block of depth q is published by miners. Therefore, we conclude that the probability that the adversary wins the race does not depend on q.

Given a random execution $\sigma \in \Sigma_A$, we denote the probability that the adversary wins the race by $\chi(A,r)$ and the respective event by $e_{\chi}(A,r,q)$.

With foresight, we define the sum

$$S(r) = \max_{A} \sum_{i=0}^{\infty} \chi(A, r+i) \tag{11}$$

and show that $S(r) = 2^{-\Omega(r)}$. We later use this result to prove persistence and progress.

Lemma 6. *For a minority attacker ($\alpha_{\mathcal{A}} < \alpha_H$) and $\Delta = 0$ it holds that $S(r) = 2^{-\Omega(r)}$.*

Proof. First we bound $\chi(A,m)$.

For a block $b^{\sigma}_{\mathsf{s}}(q)$ of depth q, we break the race into two epochs. The first epoch ends when the honest miners find m blocks and the second epoch ends when the adversary wins the race. Denote by $d_{\mathcal{A}}$ the number of blocks that the adversary has at the beginning of the first epoch and by $\Pr[d_{\mathcal{A}} = k]$ the probability of a given k.

Next, to simplify the calculations we observe that different values of Δ_{PoD} do not affect the probability that the adversary wins the race. We prove that the attacker has the same probability to win for $\Delta_{PoD} > 0$ and for $\Delta_{PoD} = 0$. Intuitively, the introduction of PoD affects both the honest and the adversary miners equally, so that we can cancel it out. This is only true for strategies that mine blocks sequentially, such as LCM mining with interruption. In this case, the number of PoD blocks is the same for both $\mathcal{T}^{\sigma}_{\mathcal{A}}(t,q)$ and $\mathcal{T}^{\sigma}_{H}(t,q)$.

We observe that given an intermittent LCM strategy A there is a strategy A' where the adversary stops mining after she finds m blocks earlier than the honest miners find their first m blocks. Note that in this scenario whether the adversary stops mining or not, she already won the race. Strategy A' has the same probability to win the race as A, as they are different only when the adversary has found m blocks, and won the race. For strategy A', the race always ends when both $\mathcal{T}^{\sigma}_{H}$ and $\mathcal{T}^{\sigma}_{\mathcal{A}}$ have identical depth. Therefore, because both the honest and the adversary trees have an identical number of PoD blocks at the end of the race, we can cancel out the effect of PoD blocks on both trees. In other words, for the strategy A' every execution where the adversary wins for $\Delta_{PoD} > 0$ has a parallel execution for $\Delta_{PoD} = 0$ with the same probability to happen. Therefore, the probability of winning for $\Delta_{PoD} > 0$ is the same as for $\Delta_{PoD} = 0$.

Denote by $P_{\mathsf{w}}(k)$ the probability that the attacker wins the race in the second epoch for a given k. The probability of the attacker winning is:

$$\chi(A,m) \le \max_{A} \chi(A,m) = \sum_{k=0}^{m-1} P_{\mathsf{w}}(k) \cdot \Pr[d_{\mathcal{A}}=k] + \sum_{k=m}^{\infty} \Pr[d_{\mathcal{A}}=k]. \quad (12)$$

We use the stars and bars [45] problem in combinatorics to calculate $\Pr[d_{\mathcal{A}}=k] = \alpha_{\mathcal{A}}^{k} \cdot \alpha_{H}^{m} \binom{k+m-1}{m-1}$ and gambler's ruin [46] for $P_{\mathsf{w}}(k) = (\frac{\alpha_{\mathcal{A}}}{\alpha_H})^{m-k}$.

Plugging into Equation (12) we get:

$$\chi(A,m) \le \alpha_{\mathcal{A}}^{m} \cdot \sum_{k=0}^{m-1} \alpha_{H}^{k} \cdot \binom{k+m-1}{m-1} + \alpha_{H}^{m} \cdot \underbrace{\sum_{k=m}^{\infty} \alpha_{\mathcal{A}}^{k} \cdot \binom{k+m-1}{m-1}}_{a_k} \le$$
$$\alpha_{\mathcal{A}}^{m} \cdot \alpha_{H}^{m-1} \cdot m \cdot \binom{2m-2}{m-1} + \alpha_{\mathcal{A}}^{m} \cdot \alpha_{H}^{m} \cdot \binom{2m-1}{m} \cdot C \le$$
$$\frac{\alpha_{\mathcal{A}} \cdot m}{\sqrt{\pi(m-1)}} \cdot (4\alpha_{\mathcal{A}}\alpha_H)^{m-1} + \frac{2 \cdot C}{\sqrt{\pi m}} \cdot (4\alpha_{\mathcal{A}}\alpha_H)^{m} = 2^{-\Omega(m)}, \quad (13)$$

where C is a constant that does not depend on n. We used the fact that $4\alpha_{\mathcal{A}}\alpha_H \le 1$, the known inequality $\binom{2r}{r} \le \frac{4^r}{\sqrt{\pi r}}$, the fact that a_H is monotonic for $k<r$ and that $\frac{b_{k+1}}{a_k} \le 2\alpha_{\mathcal{A}}$ for $k \ge r$.

As the tail of a geometric sum decreases exponentially, we conclude that $\sum_{k=r}^{\infty} \chi(A,k) = 2^{-\Omega(r)}$, thus due to Equation (11) and Equation (13) we conclude that $S(r) \le \sum_{i=0}^{\infty} \max_{A_i} \chi(A_i, r+i) = 2^{-\Omega(r)}$. □

5.5. Persistence and Progress

We are now ready to prove our main results. We first prove that persistence holds.

Lemma 7. *For $\Delta=0$, and a minority attacker, persistence holds.*

Proof. Given $\sigma \in \Sigma_A$, step t_f and $r > 0$, denote by $\neg\mathit{Pers}(\sigma,t_0,r)$ the event that $r-$persistence is violated for some $t < t_f$ in σ. We say that a block violates persistence if it holds for this block at some step $t \le t_f$ that the chain that includes the block is longer by at least r blocks than any chain that does not contain the block, but for some $t' > t$ the block is not in the main chain. We observe that $r-$persistence does not hold if for at least one block persistence is violated. Next, we consider a scenario where the attacker tries to violate persistence for any block at a specific depth i. Denote by $\neg\mathit{Pers}_i(\sigma,r)$ the event where persistence was violated for a block at depth i.

Finally, we break the race into a collection of sub-races where the adversary tries to violate persistence for a specific depth i by mining only on a specific private tree that starts at a specific honest block at depth $j<i$, denoted by $\mathcal{T}_{\neg j}^{\sigma}(t',i,r)$, we denote the event where the attacker was able to do so by $\neg\mathit{Pers}_i^j(\sigma,r)$. As before we denote by $\mathcal{T}_j^{\sigma}(t',i,r)$ the tree that starts at depth j and includes all blocks that are not in the adversary's private tree. Observe that persistence is violated if two conditions hold: (1) at some step t the chain that includes b at depth i in $\mathcal{T}_j^{\sigma}(t',i,r)$ is deeper by r than $\mathcal{T}_{\neg j}^{\sigma}$, and (2) at some step $t' > t$, $\mathcal{T}_{\neg j}^{\sigma}(t',i,r)$ is deeper than $\mathcal{T}_j^{\sigma}(t',i,r)$. The adversary can maximize the probability of the first condition to be 1, if she does not publish her blocks and keeps her chain private. Note to maximize the probability of the second condition the attacker should choose a strategy that maximizes $\mathcal{T}_{\neg j}^{\sigma}(t',i,r)$ and minimizes $\mathcal{T}_j^{\sigma}(t',i,r)$ for all t', as we saw in Corollary 1 and Lemma 5 LCM mining with interruptions on $\mathcal{T}_{\neg j}^{\sigma}(t',i,r)$ and MDNM mining on $\mathcal{T}_j^{\sigma}(t',i,r)$ achieve these goals respectively. Moreover, these two strategies can be used at the same time and therefore an optimal strategy to maximize the second condition is their combination. Therefore, we can think of the game where the attacker tries to maximize $\Pr[\neg\mathit{Pers}_i^j(\sigma,r)|\sigma \in \Sigma_A]$ as a race described in §5.4. Thus, it holds $\chi(A,r+i-j) = \Pr[\neg\mathit{Pers}_i^j(\sigma,r)|\sigma \in \Sigma_A]$. There are at most i possible honest blocks on the main chain before the i-th block, thus,

$$\Pr[\neg\mathit{Pers}_i(\sigma,r)|\sigma \in \Sigma_{A_i}] \le \Pr[\bigcup_{j=0}^{i} \neg\mathit{Pers}_i^j(\sigma,r)|\sigma \in \Sigma_{A_i}]$$

Denote by t_i^{σ} the time when the first block at depth i was mined. We bound the probability that persistence does not hold for all A', q and r:

$$\Pr[\neg\mathit{Pers}(\sigma,t_0,r)|\sigma \in \Sigma_{A'}] \le \sum_{i=1}^{t_f} S(r) = t_f \cdot S(r) = 2^{-\Omega(r)} \quad (14)$$

The full details of Equation (14) are in §B.3. We used union bound and Lemma 6.

We choose r such that: $\Pr[\neg\mathit{Pers}(A',t_f,r)] \le \varepsilon$ for all A' and thus we conclude that for every ε, there exists r such that persistence holds with a probability at least $1-\varepsilon$.

Note that because the connection between r and t_f is polynomial, as they are both polynomial in a security parameter [23], for every ε there exists large enough t_f so there is r that is significantly smaller than t_f so that the probability of persistence is at least $1-\varepsilon$. Thus, persistence holds non-vacuously. □

Next, we prove that progress holds.

Lemma 8. *For $\Delta = 0$, and a minority attacker, progress holds.*

Proof. Given some $\varepsilon > 0$, we show that there exists δ so that progress holds with a probability larger than $1-\varepsilon$ for all steps $t_0 \le t_f$. To find such δ, we first denote by r a parameter, such that δ is a function of r and ε. To bound the probability that progress does not hold, we look at the first r honest blocks generated after step t_0; denote this set of blocks by $B^{\sigma}(t_0,r)$. Our goal is to find a δ such that all the blocks in $B^\sigma(t_0,r)$ were mined in the period $[t_0,t_0+\delta]$ with high probability. For the adversary to exclude the blocks from $B^\sigma(t_0,r)$ from the main chain she has to build a chain that does not include any of them, i.e, this chain has to win the race against the tree that includes blocks from $B^\sigma(t_0,r)$. We look at the chain of all the ancestors of blocks in $B^\sigma(t_0,r)$ without the blocks themselves. Given this chain, we look at the subtrees that start at honest blocks and include only adversarial blocks. Similarly to what we did with persistence, we separate into sub-races where every race is between an adversarial subtree and the honest chain.

Given an adversarial strategy A and $\sigma\in\Sigma_A$ we denote by $\Pr[\neg \mathit{Prog}_q^j(\sigma,t_0,r)|\sigma\in\Sigma_A]$, the probability that the attacker was able to exclude all the first r honest blocks after the first block of depth q, denote by b_q, by building a chain that has a common honest ancestor with $B^\sigma(t_0,r)$ at depth j. We denote this common ancestor by b_j and the step when it was mined by t_j.

We denote by $\mathcal{T}_H^\sigma(t,t_0,r)$ the tree that includes all blocks from $B^\sigma(t_0,r)$, their ancestors, and their descendants. We denote by $\mathcal{T}_\mathcal{A}^\sigma(t,t_0,r)$ the trees that includes all the ancestors of honest block b_j and all the descendants of b_j that are not in $B^\sigma(t_0,r)$ and are not decedents of blocks in $B^\sigma(t_0,r)$.

From Corollary 1 intermittent LCM is an optimal strategy on $\mathcal{T}_\mathcal{A}^\sigma(t,t_0,r)$. From Lemma 5, MDNM is a maliciously optimal strategy on $\mathcal{T}_H^\sigma(t,t_0,r)$. As intermittent LCM and MDNM can be executed in parallel, and as they are both optimal and maliciously optimal respectively on their respective trees, it is guaranteed that if the adversary uses both of them on $\mathcal{T}_\mathcal{A}^\sigma(t,t_0,r)$ and $\mathcal{T}_H^\sigma(t,t_0,r)$, she will maximize the probability that none of the blocks in $B^\sigma(t_0,r)$ will be in the main chain forever.

Given an adversarial strategy A, denote the probability that non of the first r honest blocks after step t_0 are in the main chain forever, by $P_1(A,t_0,r)$ and the corresponding event by $e_1(\sigma,t_0,r)$. Denote by $\neg \mathit{Prog}_q(\sigma,t_0,r)$ the event that the first r honest blocks that were mined after b_q was published do not stay in the main chain forever. As we showed, $\Pr[\neg \mathit{Prog}_q^j(\sigma,r)|\sigma\in\Sigma_A]$ is maximal when the adversary's strategy A is to mine using intermittent LCM on the adversarial chain and MDNM on the honest miners' chain. There are at most q honest blocks that are ancestors of the block from $B^\sigma(t_0,r)$. Thus,

$$\Pr[\neg \mathit{Prog}_q(\sigma,t_0,r)|\sigma\in\Sigma_A]\le \Pr[\bigcup_{j=0}^{q}\neg \mathit{Prog}_q^j(\sigma,t_0,r)|\sigma\in\Sigma_A]\le \max_{A'}\Pr[\bigcup_{j=0}^{q}e_\chi(\sigma,r+j)|\sigma\in\Sigma_{A'}]\le S(r).$$

Denote by $e_\mathsf{d}(\sigma,t_0,q)$ the event where the deepest public block before step t_0 is of depth q.

Due to complete probability and the Poisson tail bounds [47], it holds that:

$$P_1(A',t_0,r)=\sum_{q=0}^{\infty}\Pr[\neg \mathit{Prog}_q(\sigma,t_0,r)\wedge e_\mathsf{d}(\sigma,t_0,q)|\sigma\in\Sigma_{A'}]\le \max_A\sum_{q=0}^{n-1}\Pr[\neg \mathit{Prog}_q(\sigma,t_0,r)|\sigma\in\Sigma_A]+2^{-\Omega(n-\lambda_T t)}\le \sum_{q=0}^{n-1}S(r)+2^{-\Omega(n-\lambda_T t)}.$$

Therefore, we can choose n and r_1 such that for $\frac{\varepsilon}{2}$ the probability $P_1(A,t,r_1)$ that one of the next r_1 blocks will not stay in the main chain forever is smaller than $\frac{\varepsilon}{2}$.

Next, for all r, we define $\delta(r)\triangleq r^2+r\cdot\Delta_{PoD}$, where we add Δ_{PoD} so that the period is long enough to account for the times of proof of delays puzzles. We calculate the probability $P_2(A,\delta,r)$ (with the corresponding event $e_2(A,\delta,r)$) that there are fewer than r honest blocks within a period δ using Erlang distribution. For simplicity we look at $\Delta_{PoD}=0$, as for $\Delta_{PoD}>0$ we can increase the number of steps by $r\cdot\Delta_{PoD}$ to account for the times of proof of delay:

$$P_2(A,\delta,r)=1-\mathrm{Erlang}(r^2;r,\lambda_H)=\sum_{n=0}^{r-1}\underbrace{\frac{1}{n!}e^{-\lambda_H r^2}(\lambda_H r^2)^n}_{a_n}\overset{(1)}{\le} e^{-\lambda_H r^2}\cdot{\lambda_H}^r\cdot e^{2r\log r}=2^{-\Omega(r)}.$$

For (1) we use the fact that $\frac{a_n}{a_{n-1}}\le\lambda_H r^2$ and the formula for a sum of geometric series. Thus, there is r_2 such that the probability P_2 is smaller than $\frac{\varepsilon}{2}$. We choose $r_3=\max(r_1,r_2)$. Using the union bound:

$$\Pr[e_1(A,t,r_3)\cup e_2(A,\delta,r_3)]\le P_1(A,t,r_3)+P_2(A,\delta,r_3)\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

We conclude that the probability for a block in period $\delta(r_3)$ to stay in the main chain forever is at least $1=1-\varepsilon$. Therefore, at least one block that was mined in the period $[t,t+\delta(r_3)]$ will be included in the main chain forever with high probability and thus progress holds. □

We now combine both lemmas to prove our main theorem:

Theorem 1. Sprints *implements a ledger (Definition 1).*

Proof. Using Lemma 7 and Lemma 8 we can conclude that the *Sprints* fulfills the requirements in Definition 1. □

6. Implementation and Practical Considerations

We turn our attention to practical aspects of *Sprints*. We implement *Sprints* using VDF for PoD, apply optimization to reduce the network delay, and dynamically adjust PoW and PoD difficulty (§6.1). Our attack threshold analysis (§6.2) and experiments (§6.3) with practical fork rates show that the security of our protocol is close (98%) to that of Bitcoin.

6.1. Implementation

We implemented a prototype of *Sprints* by modifying Bitcoin Core [48]. We made two major changes. First, we modified the data structure of block headers to include proofs of delay and adapted the mining process as well as the block validation process accordingly. Second, we modified the difficulty adjustment algorithm to adjust both the PoD and the PoW parameters.

To generate a block, a miner first calculates a VDF using the previous block hash as input. Then, the PoD is included in the block header and a standard proof of work with the new block header as input is calculated.

PoD implementation. In Pure PoW systems, a miner can change the content of the block while mining, up until she finds a solution. Replacing the block header (i.e., the PoW puzzle) does not increase the time to solution; there is no sunk cost. But PoD is not memoryless; therefore, to allow miners this flexibility in *Sprints*, the PoD does not depend on the block contents. A miner thus solves the PoD once and can use the same result independent of the block content.

We implement PoD as a Pietrzak VDF [31]. To overcome domination by a single party with an algorithmic breakthrough on a single VDF, a more robust way to implement PoD is requiring the miners to calculate multiple distinct VDFs in parallel. The difficulty adjustment method of parallel VDFs is outside our scope and we defer it to future work.

Reducing propagation latency. In Bitcoin, blocks are verified before propagation to prevent spam. While PoW verification is quick, PoD verification is considerably slower. In our implementation, PoD verification typically takes 100ms to 500ms. Worse yet, since blocks are verified at every hop, repeated PoD verification can add significant propagation latency [49].

We observe that we can postpone the VDF verification when propagating since PoW alone already creates a significant barrier for denial of service (rapid publication of invalid blocks). Specifically, each node that receives a block verifies the PoW, then concurrently broadcast the block and validates the PoD. A node processes a block only if its PoD is verified. Since both proofs are verified before a block is processed, this network-level optimization does not affect the logic of the consensus mechanism.

Difficulty adjustment. We assume only principals with the fastest hardware for VDF computation participate in the protocol, but the best hardware is expected to get more efficient over time [50]. Meanwhile, the total hash power may fluctuate [51] due to hardware improvement or miners joining and leaving the system. Therefore, Sprints adjusts the difficulty parameters for PoD and PoW to keep both the block interval and the mean PoW ratio constant.

Let Δ_{PoD} denote the PoD time and Δ_{PoW} the average PoW time. We have $\Delta_{block} = \Delta_{PoW} + \Delta_{PoD}$. The estimation of Δ_{PoD} and Δ_{PoW} is based on a recent history we call the *adjustment period*. Since we only have samples of block interval Δ_{block}, we use both the mean $\bar{x}$ and the variance s^2 of block intervals in the adjustment period to estimate

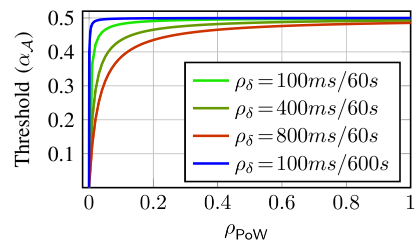


Figure 4: Attacker threshold against PoW time ratio ρ_{PoW} with different delay ratio ρ_δ.

Δ_{PoD} and Δ_{PoW}. Since the variance of the distribution is only due to its PoW element, which is exponentially distributed, the mean PoW time estimate is the root of the variance, $E(\Delta_{PoW}) = \sqrt{s^2}$. The PoD time estimate is thus its complement, $E(\Delta_{PoD}) = \bar{x} - \sqrt{s^2}$. We can thus adjust the difficulty to match our target PoD time and PoW mean time.

We numerically compare the estimation accuracy of PoW mean time between Bitcoin (first moment) and *Sprints* (second moment, see above). As in Bitcoin, we use a 2016 block adjustment period with 10 minute PoW for both processes. Based on 30k instances, the average error is only slightly higher in Sprints (0.09%) than in Bitcoin (0.02%) and so is the standard deviation (3.14% vs. 2.22%), showing that difficulty adjustment is practical for both puzzle types.

6.2. Attacker threshold analysis

So far we analyzed *Sprints*' security without network delays, We now consider the effect of network delay on *Sprints* miners. The probability of forks is a function of the ratio between proof-of-work time and proof-of-delay time. Intuitively, the less proof-of-work time, the more likely for nodes to finish proof-of-delay around the same time which leads to forks.

Forked blocks are blocks not on the main chain. We refer to the ratio between forked blocks and all blocks as *fork rate*, denoted by ϕ. Conservatively assuming the adversary does not incur any forks (this means the adversary has strong control over the network which makes the result stronger), the honest mining power is reduced by $(1-\phi)$, thus the adversary threshold becomes $\alpha_{\mathcal{A}} < (1-\alpha_{\mathcal{A}})(1-\phi)$ [52], i.e., $\alpha_{\mathcal{A}} < \frac{1-\phi}{2-\phi}$.

To derive ϕ, we first calculate the probability that when a miner i mines a block at height h, another miner j also mines a block at height h before learning of i's block, creating a fork. Denote the propagation delay from miner i to miner j by T_{ij}. Let $\beta_j \in [0,1)$ denote node j's share of mining power. Assuming difficulty is well adjusted, miner j can mine a block in time Δ_{PoW} with probability β_j. Therefore, when $\Delta_{PoW} \neq 0$, node j can mine a block in time T_{ij} with probability $\frac{\beta_j T_{ij}}{\Delta_{PoW}}$. Once miner i generates a block, the mean number of forks generated by other miners is, therefore, $\sum_{j \neq i} \frac{\beta_j T_{ij}}{\Delta_{PoW}}$. Calculating the total probability over all miners i, we obtain the average number of forks n_f as follows:

$$n_f = \sum_{i=1}^{N} \beta_i \left(\sum_{j=1, j \neq i}^{N} \frac{\beta_j T_{ij}}{\Delta_{PoW}} \right). \quad (15)$$

Data propagation in a decentralized blockchain system is performed with gossip over a p2p unstructured overlay network. The propagation delay is due to the transmission of blocks over multiple hops, and we express it based on the average one-hop delay between a pair of nodes, denoted δ. For each pair of miners denote by d_{ij} the number of hops between them, so the overall propagation delay is $T_{ij} = d_{ij} \times \delta$.

Assuming all nodes have same mining power, i.e. $\beta_i = \frac{1}{N}$ $(i = 1, 2, \cdots, N)$, Equation (15) could be simplified as $n_f = \frac{N-1}{N} \cdot \frac{d_{avg}\delta}{\Delta_{PoW}}$ where d_{avg} is the average number of hops between any two nodes in the network. d_{avg} is determined by the network topology. Note that we have a coefficient $\frac{N-1}{N}$ because the average distance does not take the distance to the node itself into account. The fork rate is then

$$\phi = \frac{n_f}{1+n_f} = \frac{N-1}{N \frac{\Delta_{PoW}}{d_{avg}\delta} + (N-1)}. \quad (16)$$

According to the relationship between fork rate and attack threshold, the attack threshold is

$$\alpha_{\mathcal{A}} = \frac{1}{2 + \frac{N-1}{N} \cdot \frac{d_{avg}\delta}{\Delta_{PoW}}}.$$

The fork rate and attacker threshold are thus a function of the *delay ratio* between the one-hop propagation delay and the block interval, denoted $\rho_\delta = \frac{\delta}{\Delta_{block}}$ and the *PoW ratio* between PoW mean duration and the block interval, denoted as $\rho_{\mathsf{PoW}} = \frac{\Delta_{PoW}}{\Delta_{block}}$.

Figure 4 visualizes the attacker threshold for various ρ_δ and ρ_{PoW} values. With the increase of PoW ratio ρ_{PoW} and decrease of delay ratio ρ_δ, the attacker threshold increases and approaches 0.5. For example, suppose the one-hop network latency is 100ms (approximately the average latency in Bitcoin [26]) and the block interval is 600s (i.e., $\rho_\delta = \frac{100ms}{600s}$), even if *Sprints* performs PoW only for 5% of the time ($\rho_{\mathsf{PoW}} = 0.05$), the attacker threshold is still 49% (compared to 50% in Bitcoin.)

6.3. Evaluation

We have derived an analytical relationship between attacker threshold, fork rates, and network parameters. Now we empirically validate the analysis by running *Sprints* with real-world parameters. We describe our setup (6.3.1), validate our theoretical results (6.3.2), and evaluate *Sprints* under practical parameters (6.3.3).

6.3.1. Setup. We deploy a network of 100 nodes running *Sprints* on our testbed with two 64-core AMD processors (256 hardware threads in total). Each node is given two hardware threads, so they have roughly the same mining power. Like previous work [53], we create a random topology by connecting each node to four random neighbors and we fix the topology throughout the experiments. We added network latency to outbound traffic using the Linux `tc` tool. We did not explicitly limit the bandwidth since messages sent in our experiments are small; this is representative of the nominal block size in practice (e.g., Bitcoin's Compact Blocks [54] and Prism's Proposal blocks [53]).

From propagation traces, we identified that the average number of hops of block propagation in our network is $d_{avg} = 4.5$ (see §A.2 for details).

6.3.2. Theory Validation. We now use our experimental setup to validate the analysis of §6.2. In all experiments, we run *Sprints* until 100 blocks are generated and calculate the fork rate from the log.

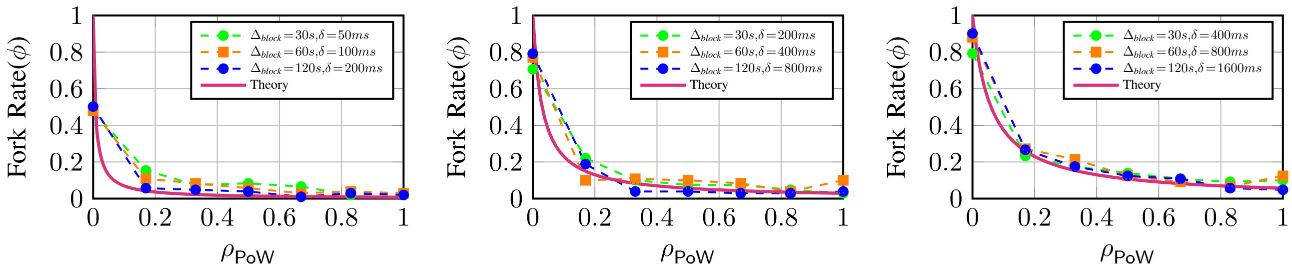
Parameter choice. According to Equation (16), when N and d_{avg} are fixed, fork rates are determined by ρ_δ (network latency normalized by block interval) and ρ_{PoW} (PoW ratio). We choose $\rho_\delta \in \{\frac{100ms}{60s}, \frac{400ms}{60s}, \frac{800ms}{60s}\}$ to cover a wide range of latencies. For each ρ_δ, we run experiments with different block intervals $\Delta_{block} \in (30s, 60s, 120s)$ and thus different one-hop network latency $\delta = \rho_\delta \times \Delta_{block}$.

Results. Figure 5 plots the results. The three subgraphs correspond to the values of ρ_δ. In all graphs, the y axis is the fork rate and the x-axis is the PoW ratio ρ_{PoW} ranging from zero to one. In each graph, there are four lines. The solid line shows fork rate by Equation (16). Markers, connected by dashed lines, show the experimental results for each block interval Δ_{block} (and thus network latency $\delta = \Delta_{block} \times \rho_\delta$). Each dot represents an experiment with 100 blocks.

We observe that the experimental results are closer to the theory with a larger one-hop delay δ. This is because the analysis only takes the network delay into account, while in reality there are other sources of delay, e.g., disk I/O, block validation, etc. These additional delays are more significant when δ is small. Nonetheless, the experimental results of each graph are close to each other, confirming that the fork rate is affected mostly by the ratio ρ_δ.

Finally, note that the measured fork rate without PoW is much smaller than the theory. From Equation (16), when taking $\Delta_{PoW} \to 0$, all the miners produce a block at about the same time. The fork rate is then $\phi = \frac{N-1}{N}$ (about one). However, in practice, variance in network delays and computation times reduce the synchrony of block generation even with $\Delta_{PoW} = 0$, resulting in a smaller fork rate.

6.3.3. Real-world parameters. We conclude by studying *Sprints*'s behavior with practical, Bitcoin-like parameters: We measure the fork rate under $\Delta_{block} = 600s$ and the one-hop network delay $\delta = 100ms$ [26]. Figure 6 shows the results. The x axis is the ratio of PoW time ρ_{PoW}, while the left y axis shows the fork rate, and the right y axis shows the attacker threshold. The red line plots the fork rate in theory and the blue dots are the experiment results. Each dot represents 3 experiments, each with 100 blocks. The green line shows the attacker threshold derived from fork rates, and the orange dots represent the attacker threshold derived from experimental fork rates. According to Figure 6, *Sprints* achieves a good attacker threshold even



(a) $\rho_\delta = 100ms/60s$ (b) $\rho_\delta = 400ms/60s$ (c) $\rho_\delta = 800ms/60s$

Figure 5: Fork rate under different ρ_δ, ρ_{PoW} and Δ_{block}. Each graph corresponds to a different delay ratio ρ_δ. Solid lines plot the theoretical analysis in Equation (16). Dotted lines plot the experiment results under different Δ_{block}.

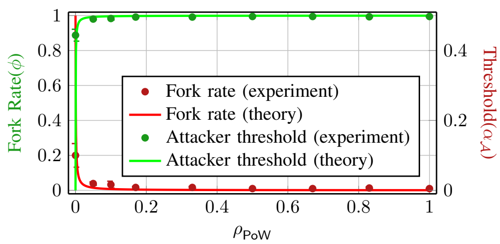


Figure 6: Fork rates and attack threshold under Bitcoin-like parameters (100ms network delay and 600s block interval.)

with small ρ_{PoW}. For example, when $\rho_{\mathsf{PoW}} = 0.05$, *Sprints* can withstand an attacker with $\alpha_{\mathcal{A}} = 49\%$ mining power, close to the ideal attacker threshold 50%.

7. Ecological Benefit Quantification

The primary objective of *Sprints* is to reduce environmental impact while preserving the security standards that are characteristic of conventional pure Proof-of-Work (PoW) protocols. To facilitate a fair comparison between *Sprints* and pure PoW, we compare them assuming the same revenue per second and profit margins for both systems and equal electricity costs so that the aggregate expenses, encompassing hardware acquisition and electricity-related costs, are equivalent in both systems. We compare the environmental impact of the electricity and hardware that is utilized in both systems using CO2e (Carbon dioxide equivalent [28]). CO2e is a measure expressing the total impact of greenhouse gas emissions including power, hardware manufacturing, transportation and disposal in terms of the amount of CO2 that would have the same environmental impact.

Although *Sprints* reduces each mining rig's electricity consumption by limiting its activity to a ρ_{PoW} portion of the time, this does not lead to a proportional reduction in the overall electricity consumption. This occurs due to the variation in the number of mining rigs between the two systems, i.e., a portion of the expenses are reallocated from electricity consumption to hardware acquisition and miners in *Sprints* purchase more mining hardware. However, The added electricity consumption due to the increased number of rigs in the system is less significant than the reduction in consumption achieved by shortening each rig's active time. For instance, when $\rho_{\mathsf{PoW}} = 0.05$, the total electricity consumption in *Sprints* is 15.7 times lower than in pure PoW.

While reducing electricity consumption is important, our primary objective is to decrease the overall environmental impact. To assure this, we take into account the environmental consequences of hardware production and disposal, using data from previous studies [55–57]. Our results (Figure 2) show that despite an increase in total hardware in *Sprints*, the system's overall environmental impact decreases while taking into account the environmental impact of hardware production and disposal. For example, when ρ_{PoW} is set to 5%, the total environmental impact of *Sprints* is reduced by 90.8% compared to pure PoW.

In our analysis, we do not consider the power consumption and environmental effects of PoD computation, as they are insignificant compared to PoW mining, owing to their non-parallelizable nature (similar to VDFs). The acquisition of additional hardware would not expedite the puzzle-solving process, and a single PoD device per miner is sufficient. We anticipate that PoD ASICs will be energy-efficient, similar to VDF ASICs, where the Ethereum Foundation's intention to develop cost-effective, optimized VDF ASICs in the form of USB sticks [58].

The full details of our analysis are presented in §C.

8. Conclusion

We present *Sprints*, a hybrid PoD-PoW protocol that shifts costs from OPEX to CAPEX, decreasing the ecological footprint with the same security threshold as pure PoW. Moreover, we show that even when keeping the same block interval as Bitcoin and reducing the PoW portion to 5%, the security threshold is only reduced from 50% to 49%, achieving a reduction of 10.9x in resource expenditure.

The *Sprints* design of a hybrid PoW-PoD system can pave the way to an eco-friendly decentralized PoW blockchain protocol.

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Appendix A.: Details of implementation and experiments

A.1. Block Time estimation in difficulty adjustment

In *Sprints*, the block interval follows a “lifted” exponential distribution: $\Delta_{block} \sim Exp(\lambda) + \Delta_{PoD}$. Unlike Bitcoin, we cannot estimate Δ_{PoW} directly from the mean (the first moment) of the block interval. Instead, we use both the first moment ($\bar{x}$) and the second moment (s^2). Since Δ_{PoD} is a constant, the variance of Δ_{block} is equal to the variance of Δ_{PoW}. Using the method of moments [59], we get the following estimations: $E(\Delta_{PoW}) = \sqrt{s^2}$, $E(\Delta_{PoD}) = \bar{x} - \sqrt{s^2}$.

A.2. Modeling the delay of the experiment network

Several factors contribute to d_{avg}. First, the average distance of our network is ~ 2.5, so nodes are on average 2-3 hops away. Moreover, the compact block feature transmits blocks in multiple rounds to reduce bandwidth consumption [54], which adds extra hops. We assume the block header propagates through first (taking 2.5 hops on average), then the recipient requests the block content from its peer following the compact block protocol, which adds two more hops in most cases. Thus the total number of hops is $d_{avg} = 2.5 + 2 = 4.5$.

Appendix B.: Proofs Details

B.1. Equation 10

We provide the development of Equation (10).

$$
\begin{aligned}
\Pr_{\Sigma_A^{\pi_1}}[d_b^\sigma(t_f) \geq \ell] &= P_{suc}(\pi_1) \cdot \Pr_{\Sigma_A^{\pi_1}}[d_b^\sigma(t_f) \geq \ell | e_{suc}^\sigma(t_o)] \\
&+ (1 - P_{suc}(\pi_1)) \cdot \Pr_{\Sigma_A^{\pi_1}}[d_b^\sigma(t_f) \geq \ell | \neg e_{suc}^\sigma(t_o)]) = \\
&P_{suc}(\pi_1) \cdot (\Pr_{\Sigma_A^{\pi_1}}[d_b^\sigma(t_f) \geq \ell | e_{suc}^\sigma(t_o)] \\
&- \Pr_{\Sigma_A^{\pi_1}}[d_b^\sigma(t_f) \geq \ell | \neg e_{suc}^\sigma(t_o)]) \\
&+ \Pr_{\Sigma_A^{\pi_1}}[d_b^\sigma(t_f) \geq \ell | \neg e_{suc}^\sigma(t_o)]) \\
&\overset{\text{Equation (6), value in parentheses positive from Equation (7)}}{\geq} \\
&P_{suc}(\pi_2) \cdot (\Pr_{\Sigma_A^{\pi_1}}[d_b^\sigma(t_f) \geq \ell | e_{suc}^\sigma(t_o)] \\
&- \Pr_{\Sigma_A^{\pi_1}}[d_b^\sigma(t_f) \geq \ell | \neg e_{suc}^\sigma(t_o)]) \\
&+ \Pr_{\Sigma_A^{\pi_1}}[d_b^\sigma(t_f) \geq \ell | \neg e_{suc}^\sigma(t_o)]) \\
&\overset{Equation\ (8), Equation\ (9)}{\geq} \\
&P_{suc}(\pi_2) \cdot \Pr_{\Sigma_A^{\pi_2}}[d_b^\sigma(t_f) \geq \ell | e_{suc}^\sigma(t_o)] \\
&+ (1 - P_{suc}(\pi_2)) \cdot \Pr_{\Sigma_A^{\pi_2}}[d_b^\sigma(t_f) \geq \ell | \neg e_{suc}^\sigma(t_o)] = \\
&\Pr_{\Sigma_A^{\pi_2}}[d_b^\sigma(t_f) \geq \ell].
\end{aligned}
$$

B.2. Proofs of Lemmas 1 and 2

We prove both lemma Lemma 1 and Lemma 2 together:

Proof. We prove by backward induction on $q_n \triangleq t_f - \Delta_{PoD} - n$.

As the basis of induction, we prove that both lemmas hold for all $t_o \in [q_0, t_f - 1]$.

Basis (Lemma 1). Note a PoW puzzle for a block can only be solved after the block's PoD puzzle. Therefore, a new block with both PoW and PoD puzzles, cannot be created in the period $[q_0, t_f - 1]$. In contrast, a PoD puzzle that has started before q_0 and was completed during $[q_0, t_f - 1]$, can be extended by a PoW puzzle.

Given a prefix π of length t_o, we define a set $\Sigma_{A_1}^\pi$ all executions starting with π, with all subsequent actions are chosen according to an LCM strategy A_1. Additionally, we define the set $\Sigma_{A_2}^\pi$ which is the set of all executions where the actions are chosen according to some strategy A_2. Denote the probability that the depth in step t_f has grown by 1 compared to the depth at t_o for strategy i by $g_i \triangleq \Pr_{\Sigma_{A_i}^\pi}[d_{\neg b}^\sigma(t_f) = d_{\neg b}^\sigma(t_o) + 1]$. It holds that $g_1 \geq g_2$, because in both cases the probability to find the PoW puzzle is identical and the LCM strategy chooses to mine on the deepest partial block would increase the depth by 1. Denote by $\delta(q)$ the Kronecker delta that returns 1 if the predicate q is true and 0 otherwise. The probability that the depth at t_f is greater than ℓ for strategy $i \in [1,2]$ is

$$
\begin{aligned}
&\Pr_{\Sigma_{A_i}^\pi}[d_{\neg b}^\sigma(t_f) \geq \ell] = \\
&\quad g_i \cdot \delta(d_{\neg b}^\pi(t_o) + 1 \geq \ell) + (1 - g_i) \cdot \delta(d_{\neg b}^\pi(t_o) \geq \ell) \qquad (17)
\end{aligned}
$$

Since $g_1 \geq g_2 \geq 0$ and $\delta(d_{\neg b}^\pi(t_o) + 1 \geq \ell) \geq \delta(d_{\neg b}^\pi(t_o) \geq \ell)$, it follows from Equation (17) that

$$
\Pr_{\Sigma_{A_1}^\pi}[d_{\neg b}^\sigma(t_f) \geq \ell] \geq \Pr_{\Sigma_{A_2}^\pi}[d_{\neg b}^\sigma(t_f) \geq \ell]
$$

as required by Lemma 1.

Basis (Lemma 2). The lemma holds for $t_o \in [q_0, t_f - 1]$ because of the requirement in Equation (1).

Assumption. We assume that both lemmas hold for $t_o \in [q_n, t_f]$ and prove they hold for $t_o = q_{n+1}$.

Step (Lemma 1). Consider a prefix π of length t_0, a strategy A_1 that is LCM and a strategy A_2 that is LCM after step q_n. We look at two sets of executions $\Sigma^{\pi}_{A_1}$ and $\Sigma^{\pi}_{A_2}$. The probability of finding a PoW puzzle in q_{n+1} is $P_w(\mathcal{A})$ in both $\Sigma^{\pi}_{A_1}$ and $\Sigma^{\pi}_{A_2}$. Denote the successful mining event at step t of execution σ, by e^{σ}_t. Denote by Σ_1 and Σ_2 the subsets of $\Sigma^{\pi}_{A_1}$ and $\Sigma^{\pi}_{A_2}$ where a PoW puzzle is not found in step q_{n+1}: $\Sigma_i \triangleq \{\sigma \in \Sigma^{\pi}_{A_i} | \neg e^{\sigma}_{q_{n+1}}\}$. Because both subsets have the same prefix until step q_{n+1} and optimal by assumption strategies are used after, it holds that:

$$\Pr_{\Sigma_1}[d^{\sigma}_{\neg b}(t_f) \geq \ell] = \Pr_{\Sigma_2}[d^{\sigma}_{\neg b}(t_f) \geq \ell]. \quad (18)$$

For $\Sigma^{\pi}_{A_1} \setminus \Sigma_1$ and $\Sigma^{\pi}_{A_2} \setminus \Sigma_2$, where the attacker solves a PoW puzzle in step q_{n+1}, relevant PoD can start only at step q_n. Therefore, the Lemma 2 assumption (Equation (1)) holds, and we can use Lemma 2 for q_n, obtaining

$$\Pr_{\Sigma^{\pi}_{A_1} \setminus \Sigma_1}[d^{\sigma}_{\neg b}(t_f) \geq \ell] \geq \Pr_{\Sigma^{\pi}_{A_2} \setminus \Sigma_2}[d^{\sigma}_{\neg b}(t_f) \geq \ell]. \quad (19)$$

Using complete probability:

$$\begin{aligned}
&\Pr_{\Sigma^{\pi}_{A_1}}[d^{\sigma}_{\neg b}(t_f) \geq \ell] = \\
&P_w(\mathcal{A}) \cdot \Pr_{\Sigma^{\pi}_{A_1} \setminus \Sigma_1}[d^{\sigma}_{\neg b}(t_f) \geq \ell] + (1 - P_w(\mathcal{A})) \cdot \Pr_{\Sigma_1}[d^{\sigma}_{\neg b}(t_f) \geq \ell] \\
&\overset{\textit{Equation (18) and Equation (19)}}{\geq} \\
&P_w(\mathcal{A}) \cdot \Pr_{\Sigma^{\pi}_{A_2} \setminus \Sigma_2}[d^{\sigma}_{\neg b}(t_f) \geq \ell] + (1 - P_w(\mathcal{A})) \cdot \Pr_{\Sigma_2}[d^{\sigma}_{\neg b}(t_f) \geq \ell] \\
&= \Pr_{\Sigma^{\pi}_{A_2}}[d^{\sigma}_{\neg b}(t_f) \geq \ell].
\end{aligned}$$

This concludes the step proof for Lemma 1.

Step (Lemma 2). Taking $t_0 = q_{n+1}$, according to the lemma assumption, for all $\sigma_1 \in \Sigma_1$, $\sigma_2 \in \Sigma_2$ and $t \in [q_{n+1}, q_{n+1} + \Delta_{PoD}]$, it holds that $\tilde{d}^{\sigma_1}_b(t) \geq \tilde{d}^{\sigma_2}_b(t)$. If the adversary solves a PoW puzzle in q_{n+1}, she will complete the PoD puzzle in $q_{n+1} + \Delta_{PoD}$. Given any $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$, we showed in the step for Lemma 1 that a (t_f, ℓ)-optimal LCM strategy A chooses a block $b_1 \in B_{max}(\mathcal{T}^{\sigma_1}_{\mathcal{A}}(q_{n+1}))$ and $b_2 \in B_{max}(\mathcal{T}^{\sigma_2}_{\mathcal{A}}(q_{n+1}))$.

We again separate for $e^{\sigma}_{q_{n+1}}$ and $\neg e^{\sigma}_{q_{n+1}}$ for both sets. Note that whether a PoW puzzle is found or not in both σ_1 and σ_2 at step q_{n+1}, it holds for all $t \in [q_n, q_n + \Delta_{PoD}]$ that $\tilde{d}^{\sigma_1}_b(t) \geq \tilde{d}^{\sigma_2}_b(t)$: For $t \in [q_n, q_n + \Delta_{PoD} - 1]$ it is true from the assumption for Equation (1). As for $q_n + \Delta_{PoD}$, $\tilde{d}^{\sigma_2}_b(q_n + \Delta_{PoD})$ cannot outgrow $\tilde{d}^{\sigma_1}_b(q_n + \Delta_{PoD})$ because the PoD puzzles that can finish at this step all started in step q_n where the depth of σ_1 is greater or equal of that of σ_2. The probability that a PoW block is found for q_n is $P_w(\mathcal{A})$ in both sets of executions Σ_1 and Σ_2. Therefore, we can use Lemma 2 twice for $e^{\sigma}_{q_{n+1}}$ and for $\neg e^{\sigma}_{q_{n+1}}$ where the conditions of the lemma hold also for $q_n + \Delta_{PoD}$. Therefore,

$$\Pr_{\Sigma_1}[d^{\sigma}_{\neg b}(t_f) \geq \ell | e^{\sigma}_{q_{n+1}}] \geq \Pr_{\Sigma_2}[d^{\sigma}_{\neg b}(t_f) \geq \ell | e^{\sigma}_{q_{n+1}}], \quad (20)$$

and

$$\Pr_{\Sigma_1}[d^{\sigma}_{\neg b}(t_f) \geq \ell | \neg e^{\sigma}_{q_{n+1}}] \geq \Pr_{\Sigma_2}[d^{\sigma}_{\neg b}(t_f) \geq \ell | \neg e^{\sigma}_{q_{n+1}}]. \quad (21)$$

Using complete probability:

$$\begin{aligned}
&\Pr_{\Sigma_1}[d^{\sigma}_{\neg b}(t_f) \geq \ell] = \\
&\quad P_w(\mathcal{A}) \cdot \Pr_{\Sigma_1}[d^{\sigma}_{\neg b}(t_f) \geq \ell | e^{\sigma}_{q_{n+1}}] + \\
&(1 - P_w(\mathcal{A})) \cdot \Pr_{\Sigma_1}[d^{\sigma}_{\neg b}(t_f) \geq \ell | \neg e^{\sigma}_{q_{n+1}}]) \\
&\overset{\textit{Equation (20) and Equation (21)}}{\geq} \\
&P_w(\mathcal{A}) \cdot \Pr_{\Sigma_2}[d^{\sigma}_{\neg b}(t_f) \geq \ell | e^{\sigma}_{q_{n+1}}] + \\
&(1 - P_w(\mathcal{A})) \cdot \Pr_{\Sigma_2}[d^{\sigma}_{\neg b}(t_f) \geq \ell | \neg e^{\sigma}_{q_{n+1}}]) \\
&= \Pr_{\Sigma_2}[d^{\sigma}_{\neg b}(t_f) \geq \ell].
\end{aligned}$$

This concludes the step of Lemma 2 and thus the proof of Lemma 1 and Lemma 2 for all t_0. □

B.3. Equation (14)

We provide the full details of the inequality in Equation (14).

$$\begin{aligned}
&\Pr[\neg \textit{Pers}(\sigma, t_0, r) | \sigma \in \Sigma_{A'}] \leq \\
&\max_A \Pr[\neg \textit{Pers}(\sigma, t_0, r) | \sigma \in \Sigma_A] = \\
&\max_A \Pr[\bigcup_{i=1}^{t_f} \neg \textit{Pers}_i(\sigma, r) \wedge t^{\sigma}_i \leq t_f | \sigma \in \Sigma_A] \leq \\
&\max_A \sum_{i=1}^{t_f} \Pr[\neg \textit{Pers}_i(\sigma, r) \wedge t^{\sigma}_i \leq t_f | \sigma \in \Sigma_A] \leq \\
&\sum_{i=1}^{t_f} \max_A \Pr[\neg \textit{Pers}_i(\sigma, r) \wedge t^{\sigma}_i \leq t_f | \sigma \in \Sigma_A] \leq \\
&\sum_{i=1}^{t_f} \max_{A_i} \Pr[\neg \textit{Pers}_i(\sigma, r) | \sigma \in \Sigma_{A_i}] \leq \\
&\sum_{i=1}^{t_f} \max_{A_i} \Pr[\bigcup_{j=0}^{i} \neg \textit{Pers}^j_i(\sigma, r) | \sigma \in \Sigma_{A_i}] = \\
&\sum_{i=1}^{t_f} \max_{A_i} \Pr[\bigcup_{j=0}^{i} e_{\chi}(\sigma, r + j) | \sigma \in \Sigma_{A_i}] \leq \\
&\sum_{i=1}^{t_f} S(r) = t_f \cdot S(r) = 2^{-\Omega(r)}
\end{aligned}$$

We used union bound and Lemma 6.

Appendix C.: Ecological Impact Analysis

We analyze the ecological impact of Sprints compared to pure PoW by presenting a model for comparison (§C.1), quantifying the improvement, parameterized by the PoW time ratio ρ_{PoW} (§C.2) and instantiate it with a practical value (§C.3).

C.1. Modeling mining expenditure

Following the model of Tsabary et al. [8], we assume miners' expenses consist of capital expenditure (CAPEX, the cost of acquiring mining hardware) and operating expenses (OPEX, the cost of electricity). Denote the CAPEX and OPEX costs *per device per second* by C and O, respectively. Specifically, C is the cost of each device divided by its lifetime and O is the average cost of electricity used by one device in each second.

We compare two scenarios, scenario p representing Nakamoto's pure Proof-of-Work (PoW) consensus and scenario s, *Sprints*, with $\rho^s_{\mathsf{PoW}} < 1$. Both scenarios generate a constant revenue of R_0 per second, allowing for a fair and balanced evaluation between the two cases. We explain how the ρ^s_{PoW} influences C^s and O^s, which are the CAPEX and OPEX costs respectively in scenario s.

CAPEX. Mining rigs' life expectancy is limited by both technological advances and wear [20, 60]. Frequent on-off switching accelerates wear through thermal cycling, leading to mechanical failure [60, 61]. We introduce a factor μ to account for this difference in life expectancy. The CAPEX cost per machine per second in both scenarios is related by the factor μ: $C^s = \mu C^p$. Determining the precise value of μ is challenging; however, as indicated by a miners' technical guide [60], it is greater than 1. We conservatively assume $\mu = 1$, as any value of $\mu > 1$ would result in a higher CAPEX expense for *Sprints*, thereby further shifting the costs from electricity to hardware procurement. By assuming this, we set a lower bound for total savings, suggesting that actual environmental benefits could surpass our conservative estimates.

OPEX. We assume that the electricity prices are homogenous for both Bitcoin and *Sprints*. Since the rigs in *Sprints* only run ρ_{PoW} of the time, the average OPEX cost per machine per second reduces, and $O^s = \rho^s_{\mathsf{PoW}} O^p$, as a mining rig is active only for ρ^s_{PoW} of the time. The assumption is conservative because industrial consumers often encounter peak demand-based charges [62], which lead to higher OPEX costs per hash for *Sprints* compared to pure PoW when a greater number of rigs operate simultaneously which would further increase the total cost of electricity and make *Sprints* more energy and environmentally efficient than what we account for in our model.

Total revenue. We use the factor θ to account for the profit margin of the miners, i.e.,

$$R_0 = \theta \cdot N^s (C^s + O^s) = \theta \cdot N^p (C^p + O^p). \tag{22}$$

C.2. Reduction of ecological footprint

First, we find the relation between the number of mining rigs in the two systems. Denote the numbers of mining devices in the pure-PoW system and in *Sprints* by N^p and N^s, respectively. Since the rewards are the same, the overall expenditure per second is the same in both systems. It follows that:

$$N^p (O^p + C^p) = N^s (\rho^s_{\mathsf{PoW}} O^p + \mu C^p)$$
$$\Rightarrow \frac{N^p}{N^s} = \frac{\rho_{\mathsf{PoW}} O^p + \mu C^p}{O^p + C^p} = \frac{\rho_{\mathsf{PoW}} + \frac{\mu C^p}{O^p}}{1 + \frac{C^p}{O^p}}.$$

We first estimate the ratio $\frac{C^p}{O^p}$, based on real-world statistics from the Bitcoin system. As of March 29, 2022, in Bitcoin, considering the block reward and inter-block time, the base revenue amounts to $R_0 = 499 \frac{\$}{\text{sec}}$. Take the miners' electricity price to be $0.04 \frac{\$}{\text{kWh}}$ [63]. The total amount of electricity consumed by Bitcoin annually is 89 TWh [64]. We assume a competitive market, as in [8, 63, 65, 66], the per-second total mining cost is close to the per-second revenue ($\theta \approx 1$), as we assume small profit margins. Therefore, using Equation (22) it holds that $N^p \cdot O^p = 113 \frac{\$}{\text{sec}}$. Using the constants values from Bitcoin and O^p and assuming that $\theta \approx 1$ we derive $N^p \cdot C^p \approx 386 \frac{\$}{\text{sec}}$. Therefore, the ratio of CAPEX and OPEX in Bitcoin is $\frac{C^p}{O^p} = 3.42$. If we assume $\theta = 1.05$ the ratio will increase by 6.5%.

Electricity consumption reduction. Denote the rate between the electricity cost per second in *Sprints* and pure PoW by $Q_H(\rho^s_{\mathsf{PoW}}) \triangleq \frac{N^p O^p}{N^s O^s} = \frac{N^p}{N^s \rho^s_{\mathsf{PoW}}} = \frac{\rho^s_{\mathsf{PoW}} + \frac{\mu C^p}{O^p}}{(1 + \frac{C^p}{O^p}) \rho^s_{\mathsf{PoW}}}$. This equation shows that when $\mu = 1$ and the PoW time ratio is 5%, the ratio of electricity consumed in Bitcoin and *Sprints* is 15.7:1.

Emission reduction. To justify *Sprints*'s environmental impact, we need to show that the increase in the number of mining devices used has a smaller environmental impact than the reduction in electricity consumption.

We employ the concept of carbon dioxide equivalent [28] (denoted as CO_2e) to gauge the environmental impact of a mining device. This measure takes into account the emissions of all greenhouse gases, converting them to CO_2 based on their respective environmental impacts. It provides a unified metric to assess the overall contribution of different gases to climate change. We define the total amount of CO_2e emission during the production of each mining device as E_C and the total emission of mining during the lifetime of a device as E_O. While the estimation of non-energetic components carries uncertainty [56], our analysis shows that they are considerably smaller than electricity-related waste, suggesting that any variations in these estimates would not alter the overall conclusions. Additionally, we use a rough estimate based on data regarding PC production carbon footprint [57]. Using PC production data as a proxy for mining rig production is a practical approach, given their manufacturing similarities. We estimate each rig implies an emission of 200 kg CO_2 in production and delivery ($E_C = 200$). Based on de Vries [55] we estimate a mining rig emits 8400 kg CO_2 during its lifetime ($E_O = 8400$), considering the popular Antminer S9 mining rig.

WWe define the emission reduction ratio Q_e as the ratio between Bitcoin and *Sprints* emissions. Considering the device's total lifetime emission and that *Sprints* allows mining in ρ_{PoW} of the time, the emission ratio for each device in Bitcoin and *Sprints* is $\frac{E_C + E_O}{E_C + \rho^s_{\mathsf{PoW}} E_O}$. Therefore, the emission reduction ratio as a function of the load ρ^s_{PoW} is

$$Q_e(\rho^s_{\mathsf{PoW}}) = \frac{N^p (E_C + E_O)}{N^s (E_C + \rho^s_{\mathsf{PoW}} E_O)}$$
$$= \frac{\rho^s_{\mathsf{PoW}} + \frac{\mu C^p}{O^p}}{1 + \frac{C^p}{O^p}} \frac{(E_C + E_O)}{(E_C + \rho^s_{\mathsf{PoW}} E_O)}.$$

Figure 2 shows the relationship of ρ_{PoW} and Q_e.

C.3. Finding the optimal ρ_{PoW}

Lowering ρ_{PoW} reduces the ecological footprint but also the attack threshold (§6), forming a tradeoff.

Our analysis shows that the attacker threshold decreases very slowly as we gain more reduction in emission (i.e,. as Q_e increases). For example, for $\rho_{\mathsf{PoW}} = 0.05$ the reduction in emission is 10.9x and *Sprints* achieves a threshold of $\alpha_{\mathcal{A}} = 49\%$ mining power based on experiment fork rates.