# Tight Security of TNT 

Reinforcing Khairallah's Birthday-bound Attack

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#### Abstract

In a recent paper, Khairallah demonstrated a birthday-bound attack on TNT, thereby invalidating its (beyond-the-birthday-bound) CCA security claims. In this short note, we reestablish a birthday-bound CCA security bound for TNT. Furthermore, using a minor variant of Khairallah's attack, we show that our security bound is tight. We provide a rigorous and complete attack advantage calculations to further enhance the confidence in Khairallah's proposed attack strategy.


Keywords: TNT, LRW1, tight security, birthday-bound attack

## 1 Introduction

The TWEAK-AND-TWEAK or TNT construction is a block cipher mode of operation introduced [1] by Bao et al. at Eurocrypt 2020. Given $E:\{0,1\}^{\kappa} \times\{0,1\}^{n} \rightarrow$ $\{0,1\}^{n}$ - a block cipher family indexed by $\kappa$-bit secret key - the TNT construction gives a family of tweakable block cipher (TBC) TNT $[E]:\{0,1\}^{3 \kappa} \times$ $\{0,1\}^{n} \times\{0,1\}^{n} \rightarrow\{0,1\}^{n}$, indexed by a $3 \kappa$-bit secret key and an $n$-bit public tweak.


Fig. 1: The Tweak-aNd-Tweak Construction.

Formally, for any key triple $K:=\left(K_{1}, K_{2}, K_{3}\right) \in\{0,1\}^{3 \kappa}$, the cipher is defined (also see Figure 1) by the mapping

$$
(T, M) \stackrel{\text { TNT }[E]_{K}}{\longmapsto} E_{K_{3}}\left(T \oplus E_{K_{2}}\left(T \oplus E_{K_{1}}(M)\right)\right) .
$$

The construction is highly appreciated in the community for its simple design and high provable security guarantee. It can be viewed as a cascaded extension
of LRW1, one of the seminal TBC designs [6] by Liskov et al. However, unlike LRW1, which achieves security up to roughly $2^{n / 2}$ queries and that too just against chosen-plaintext attacks (CPA), TNT has been proved 11 to be secure up to roughly $2^{2 n / 3}$ chosen-plaintext and chosen-ciphertext attack (CCA) queries. In fact, Guo et al. have further shown [3] that the security is guaranteed up to roughly $2^{3 n / 4} \mathrm{CPA}$ queries. In 9, Zhang et al. studied a more general problem. They employed the coupling technique to show that an arbitrary $r \geq 2$-round cascading of LRW1, defined by the mapping

$$
(T, M) \stackrel{r-\mathrm{LRW} 1_{K}}{\longrightarrow} E_{K_{r}}\left(T \oplus \ldots E_{K_{2}}\left(T \oplus E_{K_{1}}(M)\right) \ldots\right),
$$

is secure up to roughly $2^{\frac{r-1}{r+1} n}$ CCA queries, where $K=\left(K_{1}, \ldots, K_{r}\right) \in\{0,1\}^{r \kappa}$ is the $r \kappa$-bit key of the construction. Note that, TNT is equivalent to 3 -LRW1, and thus, guaranteed by the Zhang et al. bound to be CCA secure up to roughly $2^{n / 2}$ queries - a significant degradation as compared to the original analysis.

### 1.1 Khairallah's Birthday-Bound CCA Attack on TNT

Recently, Khairallah proposed [5] an elegant CCA attack on TNT that requires roughly $2^{n / 2}$ queries. This completely invalidates the existing CCA security analysis [1] on TNT. At a high level, the attack works in three main steps:

1. For some fixed choice of $\Delta \neq 0^{n} \in\{0,1\}^{n}$, choose a set of carefully crafted tweak values

$$
\left\{T_{1}, \ldots, T_{q}, T_{1} \oplus \Delta, \ldots, T_{q} \oplus \Delta\right\}
$$

such that $T_{i} \neq T_{j}$ and $T_{i} \neq T_{j} \oplus \Delta$ for all $1 \leq i \neq j \leq q$.
2. For some fixed choice of $M \in\{0,1\}^{n}$, for all $1 \leq i \leq q$ :
(a) Make encryption query $\left(T_{i}, M\right)$ and suppose the response is $C_{i}$.
(b) Make decryption query $\left(T_{i} \oplus \Delta, C_{i}\right)$ and suppose the response is $M_{i}^{\prime}$.
3. Count the number of colliding $\left(M_{i}^{\prime}=M_{j}^{\prime}\right)$ pairs $(i, j)$.
4. If the count is more than the expected number of collisions for a with replacement sampling of size $q$, then return 1 .

The tweak values are chosen in such a way that ensures tweak-distinctness at each query. This ensures that for any uniform tweakable random permutation of $\{0,1\}^{n}$, each of the output is chosen uniformly at random from $\{0,1\}^{n}$ and independent of all other outputs.

In the real world, however, since the output of an encryption query is directly fed to a decryption query, the effect of the third block cipher $E_{k_{3}}$ is effectively canceled. This helps in compressing the construction and identifying two different sources of collisions, each roughly following the collision distribution of a with replacement sampling, leading to nearly twice as many collisions as expected in the ideal world.

### 1.2 Our Contributions

Our contributions are twofold.
First, in section 3, we provide a simple proof of birthday-bound CCA security for TNT.

Note that, Khairallah specifically mentions [5] that the CCA security bound 9] of $r$-LRW1 implies birthday-bound security for TNT $=3$-LRW1. While this seems to be the case in an asymptotic sense, the general coupling-based analysis introduces some unnecessary constant factors in the concrete advantage. On the other hand, our H-coefficients based proof is much more compact and gives cleaner bound. Besides, given the recent invalidation of TNT's security claims, we think that multiple security proofs using different techniques will lead to a greater confidence in the revised security claim.

We remark that the attack analysis in 5] is somewhat incomplete. So, as a second contribution, in section 4, we provide a detailed attack algorithm (based on Khairallah's attack) coupled with a careful analysis of the attack advantage. In the course of this rigorous exercise, we also identify a special condition ${ }^{1}$ on the choice of tweak values that is sufficient (and probably necessary) for the successful accomplishment of the attack.

## 2 Preliminaries

Notations: Throughout, we fix a positive integer $n$, and $N:=2^{n}$. For any $0<r \leq m,(m)_{r}:=m(m-1) \ldots(m-r+1)$ denotes the $r$ falling factorial of $m$. For any positive integer $q, x^{q}$ denotes a $q$-tuple $\left(x_{1}, \ldots, x_{q}\right)$. For any finite set $\mathcal{X}, X \leftarrow \Phi \mathcal{X}$ denote the uniform at random sampling of $X$ from $\mathcal{X}$. We write $X_{1}, \ldots, X_{q} \stackrel{\text { wor }}{\leftrightarrows} \mathcal{S}$ to denote WOR (without replacement sampling). More precisely, $\left(X_{1}, \ldots, X_{q}\right) \leftarrow \$\left\{x^{q} \in \mathcal{S}^{q}: x_{i}\right.$ 's are distinct $\}$.

Distinguisher and Its Advantage: Let $\mathbf{F}$ and $\mathbf{G}$ be two oracles and $\mathcal{A}$ be a distinguisher aiming to distinguish the oracles $\mathbf{F}$ and $\mathbf{G}$. The distinguisher works in two steps.

1. The algorithm $\mathcal{A}^{\mathcal{O}}$ obtains a transcript $\tau:=\tau\left(\mathcal{A}^{\mathcal{O}}\right)=\left(x^{q}, y^{q}\right)$ after interacting with its oracle $\mathcal{O}$.
2. After all interaction is over, it returns a bit based on the transcript $\tau$.

We now define the distinguishing advantage as

$$
\Delta_{\mathcal{A}}(\mathbf{F} ; \mathbf{G}):=\left|\operatorname{Pr}\left(\mathcal{A}^{\mathbf{F}} \rightarrow 1\right)-\operatorname{Pr}\left(\mathcal{A}^{\mathbf{G}} \rightarrow 1\right)\right|
$$

An extended oracle provides some additional values to the distinguisher after all interaction is over. Let $S$ and $S^{\prime}$ be the additional values provided by $\mathbf{F}$ and $\mathbf{G}$ respectively. If S is sampled from a set $\mathcal{S}$ then $(\mathbf{F}, \mathrm{S})$ is called an $\mathcal{S}$-extended oracle. The final response of a distinguisher $\mathcal{A}$ interacting with an extended

[^0]oracle is based on the extended transcript. Note that, the extended transcripts are defined as
$$
\bar{\tau}\left(\mathcal{A}^{\mathbf{F}}\right)=\left(\tau\left(\mathcal{A}^{\mathbf{F}}\right), \mathrm{S}\right), \quad \bar{\tau}\left(\mathcal{A}^{\mathbf{G}}\right)=\left(\tau\left(\mathcal{A}^{\mathbf{G}}\right), \mathrm{S}^{\prime}\right)
$$

### 2.1 Tweakable Block Cipher and Its Security

A tweakable random permutation $\widetilde{E}:\{0,1\}^{\kappa} \times\{0,1\}^{n} \times\{0,1\}^{n} \rightarrow\{0,1\}^{n}$ is keyed family of permutations of $\{0,1\}^{n}$ indexed by a $\kappa$-bit key and a $n$-bit tweak, i.e., for all $(K, T) \in\{0,1\}^{\kappa} \times\{0,1\}^{n}, \widetilde{E}_{K}(T, \cdot):=\widetilde{E}(K, T, \cdot)$ is a permutation of $\{0,1\}^{n}$. We write $\widetilde{E}_{K}^{-1}(T, \cdot)$ to denote the inverse of $\widetilde{E}_{K}(T, \cdot)$.

Definition 1. Let $\widetilde{\boldsymbol{\pi}}$ denote a random tweakable permutation of $\{0,1\}^{n}$ with $n$ bit tweak values. If the distinguisher has access to both forward and backward queries, we write the system as $\widetilde{\boldsymbol{\pi}}^{ \pm}$(similar notation for any system).

$$
\operatorname{Adv}_{\widetilde{E}}^{\mathrm{tprp}}(\mathcal{A})=\Delta_{\mathcal{A}}\left(\widetilde{E}_{K} ; \widetilde{\boldsymbol{\pi}}\right), \quad \operatorname{Adv}_{\widetilde{E}}^{\mathrm{tsprp}}(\mathcal{A})=\Delta_{\mathcal{A}}\left(\widetilde{E}_{K}^{ \pm} ; \widetilde{\boldsymbol{\pi}}^{ \pm}\right)
$$

where $K$ is chosen uniformly at random.
When tweak space has size 1 , the random tweakable permutation is essentially the same as the random permutation.

### 2.2 H -Technique

We describe the extended version (see [4) of the H-technique. The basic or standard version, also called the H -coefficients technique [8], is a simple instantiation of the extended version (viewing the adjoined random variable as a degenerated or fixed constant).

Lemma 1 (Extended H-technique). Suppose $\overline{\mathbf{F}}:=(\mathbf{F}, \mathrm{S})$ and $\overline{\mathbf{G}}:=\left(\mathbf{G}, \mathrm{S}^{\prime}\right)$ are two $\mathcal{S}$-extended oracles. Suppose there is a set of extended transcripts $\mathcal{V}_{\text {bad }}$ such that for all $\left(x^{q}, y^{q}, s\right) \notin \mathcal{V}_{\text {bad }}, \operatorname{Pr}\left(\mathbf{G}\left(x^{q}\right)=y^{q}, \mathrm{~S}^{\prime}=s\right)>0$, and

$$
\frac{\operatorname{Pr}\left(\mathbf{F}\left(x^{q}\right)=y^{q}, \mathrm{~S}=s\right)}{\operatorname{Pr}\left(\mathbf{G}\left(x^{q}\right)=y^{q}, \mathrm{~S}^{\prime}=s\right)} \geq(1-\epsilon)
$$

for some $\epsilon \geq 0$. Then, for any distinguisher $\mathcal{A}$,

$$
\begin{equation*}
\Delta_{\mathcal{A}}(\mathbf{F} ; \mathbf{G}) \leq \operatorname{Pr}\left(\left(\tau\left(\mathcal{A}^{\mathbf{G}}\right), \mathrm{S}^{\prime}\right) \in \mathcal{V}_{\text {bad }}\right)+\epsilon \tag{1}
\end{equation*}
$$

A proof of the (extended) H-technique is available in multiple sources, including 817124.

## 3 TNT and Its Security

Hereafter, we only consider the TNT construction in information-theoretic setting. A computational equivalent of all the subsequent results can be easily obtained by the boilerplate hybrid argument.

Accordingly, we instantiate TNT based on three independent random permutations $\boldsymbol{\pi}_{1}, \boldsymbol{\pi}_{2}$, and $\boldsymbol{\pi}_{3}$ of $\{0,1\}^{n}$. Recall that, the TNT construction is defined by the mapping

$$
(T, M) \stackrel{\mathrm{TNT}}{\longmapsto} \boldsymbol{\pi}_{3}\left(T \oplus \boldsymbol{\pi}_{2}\left(T \oplus \boldsymbol{\pi}_{1}(M)\right)\right),
$$

and additionally, for $r \geq 2$, the $r$-LRW1 construction is defined by the mapping

$$
(T, M) \xrightarrow{r-\mathrm{LRW} 1} \boldsymbol{\pi}_{r}\left(T \oplus \ldots \boldsymbol{\pi}_{2}\left(T \oplus \boldsymbol{\pi}_{1}(M)\right) \ldots\right),
$$

where $\boldsymbol{\pi}_{1}, \ldots, \boldsymbol{\pi}_{r}$ are mutually independent uniform random permutations of $\{0,1\}^{n}$. Clearly, TNT is equivalent to 3 -LRW1.

### 3.1 Birthday-bound Security of TNT

In [5], Khairallah relies on the TSPRP bound by Zhang et al. to demonstrate the tightness of his attack. However, we observe that the generic bound in 9 introduces some constant factors, and in general, an independent security proof, using a different proof technique, will instill greater confidence in the revised security claims of TNT.

In light of the above discussion, it is clear that the security of TNT is in a limbo. Here, we salvage a birthday-bound security for TNT based on three independent random permutations $\boldsymbol{\pi}_{1}, \boldsymbol{\pi}_{2}$, and $\boldsymbol{\pi}_{3}$ of $\{0,1\}^{n}$. For an oracle $\mathcal{O}$, we abuse the $\mathcal{O}\left(x^{q}\right)=y^{q}$ to denote $\mathcal{O}\left(x_{1}\right)=y_{1}, \ldots, \mathcal{O}\left(x_{q}\right)=y_{q}$.

Theorem 1. For all $q \geq 1$ we have

$$
\operatorname{Adv}_{T N T}^{\text {tsprp }}(q) \leq \frac{q^{2}}{2^{n}} .
$$

Proof. The statement is vacuously true for $q \geq 2^{n / 2}$. We will use the extended H-technique to prove the statement for $1 \leq q<2^{n / 2}$.
Extended Systems. Let $\mathbf{F}$ be the response system corresponding to TNT and $\widetilde{\boldsymbol{\pi}}$ be the system corresponding to a tweakable random permutation. If $\left(\mathrm{T}_{i}, \mathrm{M}_{i}\right)$ is the encryption query with a tweak $\mathrm{T}_{i}$ we write the response as $\mathrm{C}_{i}$. Similarly, if $\left(\mathrm{T}_{i}, \mathrm{C}_{i}\right)$ is the decryption query with a tweak $\mathrm{T}_{i}$ we write the response as $M_{i}$. After all queries have been made, the transcript $\tau$ is defined as the tuple $\left(\mathrm{T}^{q}, \mathrm{M}^{q}, \mathrm{C}^{q}\right)$.

We now define $\left(\{0,1\}^{n}\right)^{2 q}$-extended response systems by adjoining two internal values $X^{q}$ and $\mathbf{Y}^{q}$. In the case of $\mathbf{F}, \mathrm{X}^{q}$ and $\mathrm{Y}^{q}$ correspond to the output of $\boldsymbol{\pi}_{1}$ and input of $\boldsymbol{\pi}_{2}$, respectively, and thus they are well defined from the definition of TNT. In the ideal system $\widetilde{\boldsymbol{\pi}}$, we sample $\mathbf{X}^{q}, \mathrm{Y}^{q}$ as follows for all $i \in[q]$ :

1. $\mathrm{X}_{i}=\mathrm{X}_{j}$ whenever $\mathrm{M}_{j}=\mathrm{M}_{i}$ for $j<i$. Otherwise (for all $j<i, \mathrm{M}_{j} \neq \mathrm{M}_{i}$ ), we sample

$$
\mathrm{X}_{i} \leftarrow \$\{0,1\}^{n} \backslash\left\{x \in\{0,1\}^{n}: \exists j<i, \mathrm{X}_{j}=x\right\}
$$

2. $\mathrm{Y}_{i}=\mathrm{Y}_{j}$ whenever $\mathrm{C}_{j}=\mathrm{C}_{i}$ for $j<i$. Otherwise (for all $j<i, \mathrm{C}_{j} \neq \mathrm{C}_{i}$ ), we sample

$$
\mathrm{Y}_{i} \leftarrow \Phi\{0,1\}^{n} \backslash\left\{y \in\{0,1\}^{n}: \exists j<i, \mathrm{Y}_{j}=y\right\} ;
$$

We denote the extended transcript as $\bar{\tau}(\widetilde{\boldsymbol{\pi}})=\left(\mathrm{T}^{q}, \mathrm{M}^{q}, \mathrm{C}^{q}, \mathrm{X}^{q}, \mathrm{Y}^{q}\right)$ if follow the ideal world sampling. Similarly, $\bar{\tau}(\mathbf{F})$ denotes the extended transcript for the real-world interaction.

Bad Transcript and Its Analysis. An extended transcript $\left(t^{q}, m^{q}, c^{q}, x^{q}, y^{q}\right)$ is called bad if and only if (i) there is a collision among $u^{q}$ values where $u_{i}=x_{i}+t_{i}$ or (ii) there is a collision among $v^{q}$ values where $v_{i}=y_{i}+t_{i}$. Let $\mathcal{V}_{\text {bad }}$ denote the set of all bad extended transcripts. Now, $\bar{\tau}(\widetilde{\boldsymbol{\pi}}) \in \mathcal{V}_{\text {bad }}$ if either for some $i<j$, $\mathrm{X}_{i}+\mathrm{T}_{i}=\mathrm{X}_{j}+\mathrm{T}_{j}$ or $\mathrm{Y}_{i}+\mathrm{T}_{i}=\mathrm{Y}_{j}+\mathrm{T}_{j}$. Now, it is easy to see that for any fixed $i<j, \operatorname{Pr}\left(\mathrm{X}_{i}+\mathrm{T}_{i}=\mathrm{X}_{j}+\mathrm{T}_{j}\right) \leq\left(2^{n}-1\right)^{-1}$ and similarly for the other case. So, by using the union bound,

$$
\operatorname{Pr}\left(\bar{\tau}(\widetilde{\boldsymbol{\pi}}) \in \mathcal{V}_{\mathrm{bad}}\right) \leq \frac{q(q-1)}{2^{n}-1} \leq \frac{q^{2}}{2^{n}}
$$

Analysis of Good Transcripts. For a good transcript $\bar{\tau}=\left(t^{q}, m^{q}, c^{q}, x^{q}, y^{q}\right)$, we know that $\left(m^{q}, x^{q}\right),\left(y^{q}, c^{q}\right)$, and $\left(u^{q}, v^{q}\right)$ are permutation consistent and hence for the real world we have

$$
\begin{aligned}
\operatorname{Pr}(\bar{\tau}(\mathbf{F})=\bar{\tau}) & =\operatorname{Pr}\left(\boldsymbol{\pi}_{1}\left(m^{q}\right)=x^{q}\right) \times \operatorname{Pr}\left(\boldsymbol{\pi}_{2}\left(u^{q}\right)=v^{q}\right) \times \operatorname{Pr}\left(\boldsymbol{\pi}_{3}\left(y^{q}\right)=c^{q}\right) \\
& =\frac{1}{\left(2^{n}\right)_{m}} \times \frac{1}{\left(2^{n}\right)_{q}} \times \frac{1}{\left(2^{n}\right)_{c}}
\end{aligned}
$$

where $m$ and $c$ denote the the number of distinct values present in $m^{q}$ and $c^{q}$ respectively. In the ideal world, we have,

$$
\begin{aligned}
\operatorname{Pr}(\bar{\tau}(\widetilde{\boldsymbol{\pi}})=\bar{\tau}) & =\operatorname{Pr}\left(\widetilde{\boldsymbol{\pi}}\left(t^{q}, m^{q}\right)=c^{q}\right) \times \frac{1}{\left(2^{n}\right)_{m}} \times \frac{1}{\left(2^{n}\right)_{c}} \\
& \leq \frac{1}{\left(2^{n}\right)_{q}} \times \frac{1}{\left(2^{n}\right)_{m}} \times \frac{1}{\left(2^{n}\right)_{c}}
\end{aligned}
$$

where the final inequality follows from the fact that $\operatorname{Pr}\left(\widetilde{\boldsymbol{\pi}}\left(t^{q}, m^{q}\right)=c^{q}\right)$ maximizes when $t_{i}=t_{j}$ for all $1 \leq i<j \leq q$. The result follows from the extended H-technique Lemma 1 .

## 4 Cryptanalysis of TNT

### 4.1 Attack Algorithm $\mathcal{A}^{*}$

Fix a message $M \in\{0,1\}^{n}$, a subspace $\mathcal{T} \subseteq\{0,1\}^{n}$ of size $q$ (assuming $q$ is a power of 2 ), and a $\Delta \notin \mathcal{T}$. We write $\mathcal{T}=\left\{T_{1}, \ldots, T_{q}\right\}$. Let $\boldsymbol{\pi}_{1}(M)=\hat{M}$ (unknown secret). For all $T_{i} \in \mathcal{T}$ :

1. Make encryption query $\left(T_{i}, M\right)$ and suppose the response is $C_{i}$.
2. Make decryption query $\left(T_{i} \oplus \Delta, C_{i}\right)$ and suppose the response is $X_{i}$.
3. Return 1, if for some $j<i, X_{i}=X_{j}$.

### 4.2 Advantage Calculation



Fig. 2: Cancellation of $\boldsymbol{\pi}_{3}$.


Fig. 3: The $\pi_{\Delta}$ permutation.

Ideal world collision probability. The ideal world probability of obtaining a collision can be derived as follows

$$
\begin{aligned}
\underset{\mathrm{Id}}{\operatorname{Pr}}\left[\exists i, j \in[q]: X_{i}=X_{j}\right] & =1-\operatorname{Pr}\left(\forall i, j \in[q]: X_{i} \neq X_{j}\right) \\
& =1-\frac{(N)_{q}}{N^{q}}
\end{aligned}
$$

We denote this ideal probability as $\mathbf{c p}(q):=1-\frac{\left(N_{q}\right)}{N^{q}}$ for future use.

Real world collision probability. Note that since the same message $M$ is used in every query by the attacker we have $U_{i} \oplus U_{j}=T_{i} \oplus T_{j}$ for all $i, j \in[q]$. If two responses collide, i.e., $X_{i}=X_{j}$ then we must have $V_{i} \oplus V_{j}=T_{i} \oplus T_{j}$. Therefore there will be a collision in the $i$-th and $j$-th responses if and only if $U_{i} \oplus V_{i}=U_{j} \oplus V_{j}$.
From Fig. 3 we can observe that $U_{i} \oplus V_{i}=U_{j} \oplus V_{j}$, or equivalently $U_{i} \oplus V_{i}^{\prime}=$ $U_{j} \oplus V_{j}^{\prime}$, holds if and only if:

- Either $\widehat{U}_{i} \oplus \widehat{U}_{j}=\Delta$
- $\operatorname{Or}\left(\widehat{U}_{i} \oplus \widehat{U}_{j} \neq \Delta\right) \wedge\left(\boldsymbol{\pi}_{2}^{-1}\left(\widehat{U}_{i} \oplus \Delta\right) \oplus U_{i}=\boldsymbol{\pi}_{2}^{-1}\left(\widehat{U}_{j} \oplus \Delta\right) \oplus U_{j}\right)$

Let us define the following three events

$$
\begin{aligned}
& \mathrm{E}_{0}:=\left(\exists i, j \in[q]: U_{i} \oplus V_{i}=U_{j} \oplus V_{j}\right) \\
& \mathrm{E}_{1}:=\left(\exists i, j \in[q]: \widehat{U}_{i} \oplus \widehat{U}_{j}=\Delta\right) \\
& \mathrm{E}_{2}:=\left(\exists i, j \in[q]: \pi_{2}^{-1}\left(\widehat{U}_{i} \oplus \Delta\right) \oplus U_{i}=\pi_{2}^{-1}\left(\widehat{U}_{j} \oplus \Delta\right) \oplus U_{j}\right)
\end{aligned}
$$

The above observation says that $\mathrm{E}_{0} \Leftrightarrow \mathrm{E}_{1} \cup \mathrm{E}_{2}$. Then we can write

$$
\begin{equation*}
\operatorname{Pr}\left(\mathrm{E}_{0}\right)=\operatorname{Pr}\left(\mathrm{E}_{1}\right)+\operatorname{Pr}\left(\mathrm{E}_{1}^{c} \wedge \mathrm{E}_{2}\right) \tag{2}
\end{equation*}
$$

Calculating $\operatorname{Pr}\left(\mathrm{E}_{1}^{c}\right)$. Assuming the underlying permutation $\boldsymbol{\pi}_{2}$ as a random permutation, $\operatorname{Pr}\left(\mathrm{E}_{1}^{c}\right)$ is same as the probability that $\widehat{U}_{i} \oplus \widehat{U}_{j} \neq \Delta, \forall i, j \in[q]$, where $\widehat{U}_{1}, \ldots, \widehat{U}_{q} \stackrel{\text { wor }}{\leftrightarrows}\{0,1\}^{n}$. Suppose $\widehat{U}_{1}, \ldots, \widehat{U}_{t}$ is chosen such that

$$
\begin{equation*}
\left(\widehat{U}_{i} \oplus \widehat{U}_{j} \neq \Delta\right) \wedge\left(\widehat{U}_{i} \neq \widehat{U}_{j}\right), \forall i, j \in[t] \tag{3}
\end{equation*}
$$

Then the possible choices for $\widehat{U}_{t+1}$ are exactly $\{0,1\}^{n} \backslash \mathcal{S}_{t}$, where

$$
\mathcal{S}_{t}:=\left\{\widehat{U}_{1}, \ldots, \widehat{U}_{t}\right\} \cup\left\{\widehat{U}_{1} \oplus \Delta, \ldots, \widehat{U}_{t} \oplus \Delta\right\}
$$

Since $\widehat{U}_{1}, \ldots \widehat{U}_{t}$ satisfies condition (3) we have $\left|\mathcal{S}_{t}\right|=2 t$. Thus total number of ways of selecting $\widehat{U}_{1}, \ldots, \widehat{U}_{q}$ is $N(N-2) \cdots(N-2 q)$. Hence we have,

$$
\begin{align*}
\operatorname{Pr}\left(\mathrm{E}_{1}^{c}\right) & =\frac{N(N-2) \cdots(N-2 q+2)}{(N)_{q}} \\
& \leq \frac{(N)_{q}}{N^{q}}=(1-\mathbf{c p}(q)) \tag{4}
\end{align*}
$$

Hence, $\operatorname{Pr}\left(\mathrm{E}_{1}\right) \geq \mathbf{c p}(q)$. So, we get that the probability of collision of responses in the real world is bounded as follows

$$
\begin{aligned}
\operatorname{Pr}_{\operatorname{Re}}\left(\exists i, j \in[q]: X_{i}=X_{j}\right) & =\operatorname{Pr}\left(\mathrm{E}_{0}\right) \\
& =\operatorname{Pr}\left(\mathrm{E}_{1}\right)+\operatorname{Pr}\left(\mathrm{E}_{1}^{c} \wedge \mathrm{E}_{2}\right) \\
& \geq \mathbf{c p}(q)+\operatorname{Pr}\left(\mathrm{E}_{1}^{c} \wedge \mathrm{E}_{2}\right) \\
& =\operatorname{Pr}_{\text {ld }}\left(\exists i, j \in[q]: X_{i}=X_{j}\right)+\operatorname{Pr}\left(\mathrm{E}_{1}^{c} \wedge \mathrm{E}_{2}\right)
\end{aligned}
$$

Hence the advantage of our distinguisher $\mathcal{A}^{*}$ will be

$$
\begin{equation*}
\operatorname{Adv}_{\mathrm{TNT}}^{\text {tsprp }}\left(\mathcal{A}^{*}\right) \geq \operatorname{Pr}\left(\mathrm{E}_{1}^{c} \wedge \mathrm{E}_{2}\right) \tag{5}
\end{equation*}
$$

So, it is sufficient to provide a lower bound for $\operatorname{Pr}\left(\mathrm{E}_{1}^{c} \wedge \mathrm{E}_{2}\right)$ which is the same as $\operatorname{Pr}\left(\mathrm{E}_{2} \mid \mathrm{E}_{1}^{c}\right) \times \operatorname{Pr}\left(\mathrm{E}_{1}^{c}\right)$.

$$
\begin{align*}
\operatorname{Pr}\left(\mathrm{E}_{1}^{c}\right) & =\frac{N(N-2) \cdots(N-2 q+2)}{(N)_{q}} \\
& =(1-\mathbf{c p}(q)) \prod_{i=1}^{q-1}\left(1-\frac{i^{2}}{(N-i)^{2}}\right) \\
& \geq(1-\mathbf{c p}(q)) \prod_{i=1}^{q-1}\left(1-\frac{i^{2}}{(N-q)^{2}}\right) \\
& \geq(1-\mathbf{c p}(q))\left(1-\frac{q(q-1)(2 q-1)}{6(N-q)^{2}}\right) \\
& \geq(1-\mathbf{c p}(q))\left(1-\frac{2 q^{3}}{N^{2}}\right) \tag{6}
\end{align*}
$$

In the last inequality, we assume that $q \leq N / 2$.

Calculating $\operatorname{Pr}\left(\mathrm{E}_{2} \mid \mathrm{E}_{1}^{c}\right)$. Given the condition that $\widehat{U}_{i} \oplus \Delta \neq \widehat{U}_{j}, \forall i, j \in[q]$, we have that $\boldsymbol{\pi}_{2}^{-1}\left(\widehat{U}_{i} \oplus \Delta\right) \notin \mathcal{U}:=\left\{U_{1}, \ldots, U_{q}\right\}$. Note that the set $\mathcal{U}=\mathcal{T} \oplus \boldsymbol{\pi}_{1}(M)$ is the affine space obtained from the subspace $\mathcal{T}$ by translating it by $\boldsymbol{\pi}_{1}(M)$. Now, declaring the variables $\widehat{V}_{i}:=\pi_{2}^{-1}\left(\widehat{U}_{i} \oplus \Delta\right)$ and noting that $U_{i} \oplus U_{j}=T_{i} \oplus T_{j}$, we have that $\left(\mathrm{E}_{2} \mid \mathrm{E}_{1}^{c}\right)$ is same as the event,

$$
\bigvee_{i \neq j \in[q]}\left(\widehat{V}_{i} \oplus \widehat{V}_{j}=T_{i} \oplus T_{j}\right), \text { where } \widehat{V}_{1}, \ldots, \widehat{V}_{q} \stackrel{\text { wor }}{\longleftarrow} \mathcal{U}^{c}:=\{0,1\}^{n} \backslash \mathcal{U}
$$

For every $i \neq j \in[q]$, we define the events $\mathrm{E}_{\{i, j\}}=\left(\widehat{V}_{i} \oplus \widehat{V}_{j}=T_{i} \oplus T_{j}\right)$ where $\widehat{V}_{1}, \ldots, \widehat{V}_{q} \stackrel{\text { wor }}{\leftrightarrows} \mathcal{U}^{c}$. Note that for any distinct $i, j$,

$$
\widehat{V}_{i}, \widehat{V}_{j} \stackrel{\text { wor }}{\longleftarrow} \mathcal{U}^{c} .
$$

In general, any subset follows WOR distribution. Using this observation we have $\operatorname{Pr}\left(\mathrm{E}_{\{i, j\}}\right)=(N-q-1)^{-1}$. This is true because any choice of $\widehat{V}_{i}$ from the set $\mathcal{U}^{c}$, we have $\widehat{V}_{i} \oplus T_{i} \oplus T_{j} \notin \mathcal{U}$. By using a similar argument, one can show that

$$
\operatorname{Pr}\left(\mathrm{E}_{\{i, j\}} \wedge \mathrm{E}_{\{k, \ell\}}\right) \leq \frac{1}{(N-q-1)(N-q-3)}
$$

Hence, by using Bonferroni's inequality and denoting $\alpha=\frac{\binom{q}{2}}{N-q-1}$ we have

$$
\begin{align*}
\operatorname{Pr}\left(\bigvee_{i \neq j \in[q]} \mathrm{E}_{\{i, j\}}\right) & \geq \frac{\binom{q}{2}}{N-q-1}-\frac{\binom{q}{2}^{2}}{2(N-q-1)(N-q-3)} \\
& =\alpha\left(1-\frac{\binom{q}{2}}{2(N-q-3)}\right) \\
& \geq \alpha\left(1-\frac{\alpha}{2}\left(1+\frac{2}{(N-q-3)}\right)\right) \tag{7}
\end{align*}
$$

Note that $\mathbf{c p}(q) \leq \alpha$ (by union bound). Thus, using Equations (5)- (7), we have

$$
\operatorname{Adv}_{\mathrm{TNT}}^{\mathrm{tsprp}}\left(\mathcal{A}^{*}\right) \geq \alpha(1-\alpha)\left(1-\frac{\alpha}{2}-\frac{\alpha}{N-q-3}\right)\left(1-\frac{2 q^{3}}{N^{2}}\right)
$$

Let $q_{0}$ be the number for which $\alpha=1 / 2$. Clearly, $q_{0}=O\left(2^{n / 2}\right)$. So, if the attacker chooses $q=\left\lceil q_{0}\right\rceil, \mathbf{A d v}_{\text {TNT }}^{\text {tsprp }}\left(\mathcal{A}^{*}\right) \geq \frac{1}{8}-\lambda(n)$, where $\lambda(n)=O\left(\frac{q_{0}^{3}}{N^{2}}\right)=O\left(2^{-n / 2}\right)$ is negligible for large $n$.

## 5 Conclusion

In this short note, we established the tight birthday bound security for TNT. Our attack algorithm is a minor variant of Khairallah's attack [5], albeit with a more rigorous treatment and detailed advantage calculations.

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[^0]:    ${ }^{1}$ This condition is missing in Khairallah's analysis.

