Timed Secret Sharing

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***Abstract*—This paper introduces the notion of *timed secret sharing* (TSS), which establishes *lower* and *upper* time bounds for secret reconstruction in a threshold secret sharing scheme. Such time bounds are particularly useful in scenarios where an early or late reconstruction of a secret matters. We propose several new constructions that offer different security properties and show how they can be instantiated efficiently using novel techniques. We highlight how our ideas can be used to break a *public goods game*, which is an issue inherent to threshold secret sharing-based systems, without relying on incentive systems. We achieve this through an upper time bound that can be implemented either via *short-lived proofs*, or the *gradual release of additional shares*, establishing a trade-off between time and fault tolerance. The latter independently provides *robustness* in the event of dropout by some portion of shareholders.**

I. INTRODUCTION

Threshold secret sharing [53] is a widely used primitive in cryptography and distributed computing. A (t, n)-threshold secret sharing scheme lets a dealer distribute a secret s among n shareholders such that any subset of at least $t + 1$ shares can recover s, whereas no subset of at most t shares reveal any information about s. This primitive is useful in a wide range of applications from password-protection [8], [35] and federated learning [39], to verifiable management of on-chain secrets [37] and many more.

Protocols using secret sharing usually specify conditions under which shareholders release their shares to reconstruct the secret [20], [28]. In many cases, these conditions depend on the notion of time in one way or another. In practice, however, shareholders may violate these time-dependent conditions intentionally or unintentionally by releasing their shares too early or late. These issues may arise due to the use of unsynchronized clocks by the shareholders [7], [12], [32] or due to a (temporary) dishonest majority [22], [23]. The latter could occur when incentives are misaligned so that shareholders collude and reconstruct secrets earlier than what specified [34], [43].

Motivations. The practical applications of threshold secret sharing motivate this work. We elaborate on two concrete scenarios as follows.

Maximal Extractable Value. In cryptocurrency platforms consensus nodes, such as proof-of-stake validators may engage in *maximal extractable value* (MEV) processes to gain some benefit from users by learning their transactions and affect their ordering in the block. A principal MEV countermeasure deploys threshold secret sharing to protect the privacy of transactions up to a time where their inclusion/ordering in a block is ensured [34], [43]. [1]

However, it largely overlooks the fact that consensus nodes have significant incentives to prematurely reconstruct the secrets to capitalize on MEV rewards. This type of collusion (*i.e.,* dishonest majority) does not violate the protocol's liveness (*i.e.,* reconstruction) as the success of MEV depends on the completion of the secret reconstruction, and thus colluding parties are incentivized to make progress. In many cases, such behavior is particularly problematic since corrupt shareholders can carry out the process without leaving any public traces and thus collusion is *unobservable* [50].[2]

Public goods game. An independent issue with threshold secret sharing-based schemes is that they essentially constitute a *public goods game* [4], [11]. This is essentially because only a subset of the shareholders needs to release their shares to reconstruct the secret. Consequently, the shareholders may choose to remain inactive, hoping that others will step forward and contribute. As a mitigation mechanism an incentive system is usually assumed [6], [37] which may, however, not be available or feasible to implement under all circumstances.

Our schemes with lower and upper time bounds T_1 and T_2, respectively, address the aforementioned issues: T_1 prevents shareholders from reconstructing the secret early, and T_2 prevents the dilemma of a public goods game without having to rely on financial incentives, providing an alternative solution. We stress that the motivations for lower and upper time bounds are different and independent. In the case of the former, we must *ensure* that the reconstruction does not occur before T_1. In the case of the latter, our goal is to *encourage* (rational) shareholders to appear early and initiate the reconstruction. For the sake of better consistency, we present the schemes with both time bounds rather than treating them separately.

A. Technical Overview

Our constructions enjoy novel techniques and build upon time-based primitives with efficient instantiation in a modular way. In particular, we use time-lock puzzles (TLPs) [1], [42], [49], *verifiable timed commitments* (VTCs) [56], and *verifiable delay functions* (VDFs) [47], [59]. In the remainder of this section, we give an overview of our proposed constructions.

[1] This is done by encrypting the transaction using a random key and then sharing the key towards validators.

[2] Using time-lock puzzles (TLPs) [49] are not sufficient to address the issue as protected transactions may actually not make it into the block and then lose confidentiality after the TLP has been opened.

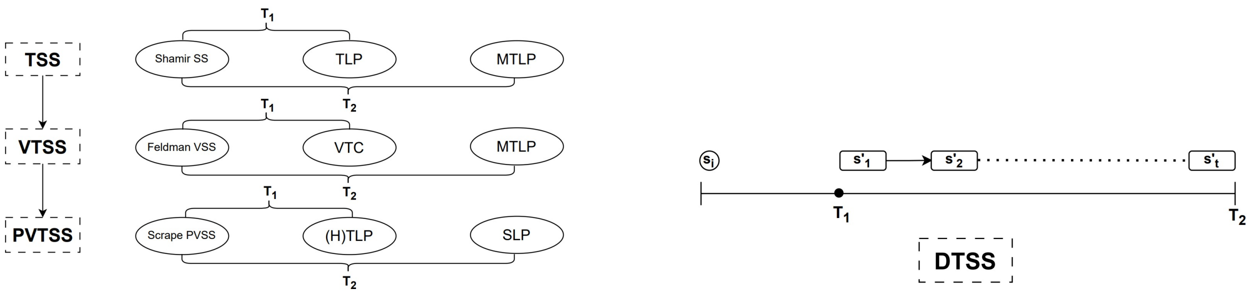

Fig. 1: A visual representation of our protocols

Timed Secret Sharing (TSS). This is our basic construction, where the dealer encapsulates the shares into TLPs [42], [49] to realize a lower time bound T_1. Consequently, no computationally bounded shareholder can learn their share before T_1. Consequently, no computationally bounded adversary can learn the secret before T_1, *even if it corrupts all the shareholders* [23]. Moreover, TLPs that resist parallel computation provide a consistent relative measure of time (*i.e.,* computational timing), eliminating the need for a shared global clock.

Verifiable Timed Secret Sharing (VTSS). We enhance TSS with verification mechanisms, to deal with *malicious* dealers and shareholders. First, we ensure that a malicious dealer cannot distribute *malformed puzzles*, *i.e.,* puzzles that either are not extractable or contain invalid shares. Checking the validity of the puzzle must be performed before shareholder invests time and computational effort to retrieve the share. Second, we ensure that malicious shareholders cannot send invalid shares during the reconstruction phase. To realize VTSS, we make use of verifiable secret sharing (VSS) [29] and verifiable timed commitment (VTC) [56].

Publicly Verifiable Timed Secret Sharing (PVTSS). We also devise a protocol, called PVTSS, that supports *public* verifiability. This is achieved by using publicly verifiable secret sharing (PVSS) scheme [18], [52], an efficient non-interactive zero-knowledge (NIZK) protocol, and the cut-and-choose technique [38] to ensure the validity of the embedded encrypted shares and the extractability of puzzles. Notably, to bind the attestation of the distributed shares to time and impose an upper bound T_2, we utilize short-lived proofs (SLPs) [5]. This approach will guarantee the correct reconstruction of the secret only before T_2. Given that SLP ties the correctness of the protocol to time, it could break a public goods game.

Decrementing-threshold Timed Secret Sharing (DTSS). Another approach to implement an upper time bound T_2 is having shareholders learn *additional* shares *gradually* over time which we realize in the DTSS variant. Such gradual release of shares is equivalent to the gradual reduction of the fault tolerance (*i.e.,* maximum number of corrupted parties tolerated), forcing shareholders to perform reconstruction before the scheme's resilience decays enough so that the adversary can reconstruct the secret on its own.[3] This approach essentially binds the security of the protocol to time and therefore could break a public goods game.[4]

Also, this technique could be of independent interest in scenarios where a sufficient number of (honest) shareholders is not available for reconstruction and thus the additional shares allow the remaining parties to reconstruct the secret, providing *robustness* to the system. As an application, this could help with dropout resilience in secure aggregation protocols for federated learning [41]. Our building block to realize DTSS is a multi-instance TLP [1]. Using this primitive, the shares are gradually released without the need for the dealer to remain online. It does the sharing in one go and the shareholders will learn the shares gradually at regular points in time. Figure 1 provides a high-level representation of our protocols.

B. *Our Contributions*

- We present and formally define (t, n)-*timed secret sharing* (TSS) which enables a dealer to share a secret to a set of n shareholders, allowing a timely reconstruction by a threshold $t+1$ of them within the time interval $[T_1, T_2]$.
- We enhance TSS with verifiability by presenting and formally defining *verifiable timed secret sharing* (VTSS), which protects against a malicious dealer during share distribution and against malicious shareholders during secret reconstruction.
- We further extend VTSS with public verifiability by presenting and formally defining *publicly verifiable timed secret sharing* (PVTSS).
- We present and formally define *decrementing-threshold timed secret sharing* (DTSS) scheme, which illustrates a trade-off between time and fault tolerance in a threshold secret sharing scheme.
- We introduce two novel ideas to break a public goods game in threshold secret sharing systems. One is based on using *short-lived proofs* [5] and the other is based on *gradual release of additional shares*. The latter is also useful to provide *robustness* to the protocol against dropouts.

[3]Note that the actual number of corrupted parties by the adversary f is bounded by threshold t.

[4]This technique could be used both in TSS and VTSS for placing an upper time bound. This is due to the fact that after T_2 everybody can reconstruct the secret on their own.

- We propose efficient constructions for all of the aforementioned schemes.

II. Related Work

There is a large body of literature on the combination of computational timing and cryptographic primitives such as commitment [3], [14], [27], [44], [57], encryption [17], [24], [40], signature [9], [26], [31], [56], and more. The essence of almost all of these works is to enable the receiver(s) to forcefully open the locked object after a predefined period by working through some computational operation.

The work of [56] proposed efficient constructions for encapsulating a signature into a TLP, ensuring the receiver can extract the valid signature after carrying out sequential computation. Roughly speaking, the sender secret shares the signature and embeds each share in a linearly homomorphic TLP [42]. Then, the sender and receiver run a cut-and-choose protocol for verifying the correctness of the puzzles. Moreover, to enable the receiver to compact all the pieces of time-locked signatures and solve one single puzzle, a range proof is used to guarantee that no overflow occurs. With a focus on reducing the interaction in MPC protocols with limited-time secrecy, the authors in [3] developed a gage time capsule (GaTC), allowing a sender to commit to a value that others can obtain after putting a total computational cost which is parallelizable to let solvers claim a monetary reward in exchange for their work. The security guarantee of GaTC is similar to DTSS in the sense that over time it gradually decays, as the adversary can invest more and more computational resources. Doweck and Eyal [27] constructed a multi-party timed commitment that enables a group of parties to jointly commit to a secret to be opened by an aggregator later on via brute-force computation.

The authors in [10] explore multi-party computation with output-independent abort, having each participant in an MPC protocol lock their output until some time in the future. This is to force the adversary to decide whether to cause an abort before learning the output. As performing sequential computations might be beyond the capacity of some users, Thyagarajan et al. [58] developed a system to allow users to outsource their tasks to some servers in a privacy-preserving manner. Srinivasan et al. [55] constructed a TLP that supports unbounded batch-solving while enjoying a transparent setup and a puzzle size independent of the batch size. Although their construction is only of theoretical interest and does not have practical efficiency due to the reliance on indistinguishability obfuscation, it enables a party to solve many puzzle instances simultaneously at the cost of solving one puzzle. One of the motivating reasons for batch-solving is to enable a party to solve the puzzles of others in case a large number of parties abort. We refer the reader to [46] for a more detailed overview of relevant works.

III. Preliminaries

A. Threat Model and Assumptions

We consider a standard synchronous network where each pair of parties in a set $\mathcal{P} = \{P_1, \ldots, P_n\}$ is connected via an authenticated channel, and each message is delivered at most by a known delay. There is also a dealer D that takes the role of distributing the secret among participating parties.

As common in the literature for verifiable secret sharing, we assume the existence of broadcast channels. For a publicly verifiable scheme, we assume the existence of an authenticated public bulletin board. In this work, we consider a static adversary that may corrupt up to t out of n parties before the start of protocol execution. D may also be corrupted. We consider both semi-honest and malicious types of adversaries. In the former, the corrupted parties are assumed to follow the protocol but may try to learn some information by observing the protocol execution. In the latter, however, the corrupted parties are allowed to do any adversarial action of their choice. The adversary's computational power is bounded with respect to a security parameter λ that gives it a negligible advantage in breaking the security of underlying primitives. Such algorithms are often known as probabilistic polynomial time (PPT). Finally, we denote by $[n]$ the set $\{1, \ldots, n\}$ an by $\mathbf{v}$ a vector of elements $\{v_i\}_{i \in [n]}$.

B. Secret Sharing

A (threshold) secret sharing scheme is a cryptographic protocol that enables a dealer D to distribute a secret s among n parties. The scheme typically consists of two main phases; *distribution* and *reconstruction*. In the former, D sends each party their corresponding share, and in the latter, any proper subset of parties reconstruct the secret by pooling their shares.

A (t, n)-threshold secret sharing offers two main properties: (1) correctness: the secret is reconstructed by any subset of at least $t+1$ shares, and (2) t-security: no information is revealed about the secret by gathering t or fewer shares. In this work, we develop our protocols based on the popular Shamir secret sharing [53]. We note that our proposed definitions can capture any (linear) secret sharing.

a) Verifiable Secret Sharing (VSS): The basic (t, n)-threshold secret sharing scheme (e.g., [53]) only provides security against a *semi-honest* adversary. When dealing with malicious adversaries, it is essential for (1) the dealer to prove the validity of the shares it produces in the distribution phase, and (2) the shareholders to prove the validity of the shares they provide in the reconstruction phase. To satisfy these properties, various VSS schemes have been proposed, following the celebrated work by Feldman [29].

b) Publicly Verifiable Secret Sharing (PVSS): To extend the scope of verifiability to the public and not only participating parties, PVSS schemes [18], [19], [52] deploy cryptographic primitives such as encryption and NIZK proofs. PVSS enables anyone to verify the distribution and reconstruction phases. Cascudo and David [18] proposed an efficient scheme called Scrape PVSS, which is an improvement over [52] and has been deployed extensively in many recent cryptographic protocols. The Scrape protocol works as follows. The dealer D chooses a random value $s \xleftarrow{\$} \mathbb{Z}_q$, sets the secret as a group element of form $S = h^s$, splits s into shares $\{s_i\}_{i \in [n]}$, and computes the encrypted shares $\{\hat{s}_i\}_{i\in[n]}$ using corresponding parties' public keys $\{pk_i\}_{i\in[n]}$.

D publishes a set of commitments to shares $\{v_i\}_{i\in[n]}$ together with a proof π_D, enabling anyone to check the consistency of the shares (*i.e.,* shares are evaluations of the same polynomial of proper degree) and validity of the ciphertexts (*i.e.,* encrypted shares correspond to the committed shares). Upon receiving a threshold number of valid shares (*i.e.,* shares with correct decryptions), anyone can use Lagrange interpolation [2] in the exponent to reconstruct the secret S. The authors proposed two versions, one in the random oracle model under the Decisional Diffie-Hellman (DDH) assumption and the other in the plain model under the Decisional Bilinear Squaring (DBS) assumption. We use the non-pairing variant which offers *knowledge soundness*. This is vital to ensure the secret chosen by the adversary is independent of those of honest parties. Also, we require the knowledge soundness property for deploying short-live proofs [5].

C. Time-Lock Puzzles (TLPs)

The idea of TLPs was introduced by Rivest et al. [49]. TLP locks a secret such that it can only be retrieved after a predefined amount of sequential computation. It consists of two algorithms: TLP.Gen, which takes as input a time parameter T and a secret s, and returns a puzzle Z, and TLP.Solve, that takes as input a puzzle Z and returns a secret s. A TLP must satisfy *correctness* and *security*. The correctness ensures that the solution is indeed obtained if the protocol gets executed as specified. The security ensures that no PPT adversary running in parallel obtains the solution within the time bound T, except with negligible probability.

a) Homomorphic Time-lock Puzzles (HTLP): Malavolta and Thyagarajan [42] proposed homomorphic TLP, enabling one to homomorphically combine many instances of TLPs into a single TLP. An HTLP consists of a tuple of algorithms (HTLP.Setup, HTLP.Gen, HTLP.Solve, HTLP.Eval). In particular, HTLP.Setup generates public parameters *pp* on input a security parameter, and HTLP.Eval performs a homomorphic operation on input a set of puzzles to output a single puzzle.

b) Multi-instance Time-lock Puzzle (MTLP): Abadi and Kiayias [1] proposed a primitive called multi-instance TLP. This variant of TLP is suitable for the case where the solver is given multiple puzzles at the same time but must discover each solution at different points in time. It allows solving the instances sequentially one after the other without needing to run parallel computations on them. An MTLP consists of a tuple of algorithms (MTLP.Setup, MTLP.Gen, MTLP.Solve, Prove, Verify), where the last two algorithms are used to check the correctness of a solver's claimed solution.

D. Timed Commitment

An inherent limitation of the well-known time-lock puzzles such as [42], [49] is the lack of verifiability, meaning that the receiver cannot check the validity of the received puzzle unless after putting time and effort into solving it. To fill this gap, a timed commitment scheme [14] enables the receiver to make sure about the well-formedness (*i.e.,* extractability) of the puzzle before performing a sequential computation. In an attempt to make the timed commitment of [14] efficiently verifiable, the recent work of Thyagarajan et al. [56] proposed verifiable timed commitment (VTC), enabling the sender to verifiably[5] commit to signing keys of form $pk = g^{sk}$, $sk \in \{0,1\}^\lambda$. The VTC primitive consists of a tuple of algorithms (VTC.Setup, VTC.Commit, VTC.Verify, VTC.Solve). Note that we deploy VTC to design construction for our verifiable time secret sharing (VTSS) scheme.

E. Sigma Protocols

A zero-knowledge protocol enables proving the validity of a claimed statement by the prover P to the verifier V without revealing any information further. While zero-knowledge protocols involve various settings and notions, we particularly consider the well-known Sigma protocols which are useful building blocks in many cryptographic constructions. Let v denote an instance that is known to both parties and w denote a witness that is only known to the P. Let $R = \{(v; w)\} \in \mathcal{V} \times \mathcal{W}$ denote a relation containing the pairs of instances and corresponding witnesses. A Sigma protocol Σ on $(v; w) \in R$ is an interactive protocol with three movements between P and V. Using Fiat-Shamir heuristic [30] in the random oracle model, one can make the protocol non-interactive with public verifiability. A Sigma protocol satisfies two security properties: (1) *soundness*, ensuring the verifier about the validity of the statement v, and (2) *zero-knowledge*, ensuring the prover about the secrecy of the witness w.

a) Zero Knowledge proof of equality of discrete logarithm: One of the well-used Sigma protocols is discrete logarithm equality (DLEQ) proof. It considers a tuple of publicly known values (g_1, x, g_2, y), where g_1, g_2 are random generators and x, y are two elements of the cyclic group $\mathbb{G}$ of order q. DLEQ proof enables a prover P to prove to the verifier V that it knows a witness α such that $x = g_1^\alpha$ and $y = g_2^\alpha$. A DLEQ proof is an AND-composition of two Sigma protocols for relation $R = \{(v_i; w) : v_i = g_i^w\}$ with the *same* witness and challenge. The following protocol is a Sigma protocol for generating a DLEQ proof due to Chaum-Pedersen [21].

1) P chooses a random element $u \xleftarrow{\$} \mathbb{Z}_q$, computes $a_1 = g_1^u$ and $a_2 = g_2^u$, and sends them to the V.
2) V sends back a randomly chosen challenge $c \xleftarrow{\$} \mathbb{Z}_q$.
3) P computes $r = u + c\alpha$ and sends it to V.
4) V checks if both $g_1^r = a_1 x^c$ and $g_2^r = a_2 y^c$ hold.

Throughout the paper we use the non-interactive version of this protocol which produces a single message DLEQ.P(α, g_1, x, g_2, y) as proof π verified via DLEQ.V(π, g_1, x, g_2, y). The challenge is computed by the prover as $c = H(x, y, a_1, a_2)$, where H is a cryptographic hash function modeled as a random oracle.

[5]Ensuring the extractability together with validity of the committed message that is the discrete logarithm of a public key.

F. Short-lived Proofs

Arun et al. [5] recently introduced the notion of *short-lived proofs* (SLPs) which can be roughly defined as types of proofs with expiration, such that their soundness will disappear after certain time. They are only sound if being observed before a determined time, afterwards, they may be forgery indistinguishable from the valid proofs. At a high level, an SLP is proof of an OR-composition $R \vee R_{VDF}$, where R is an arbitrary relation and R_{VDF} is a VDF evaluation relation. Interestingly, this proof is only convincing to the verifier for a determined time T as forging the proof is possible for anybody after evaluating the VDF. Due to the nature of VDF, short-lived proofs offer efficient public variability. One notable point is that the primitive makes use of a *randomness beacon* [25] which outputs unpredictable values b periodically.

An SLP scheme consists of four algorithms (SLP.Setup, SLP.Gen, SLP.Forge, SLP.Verify) with the following descriptions. SLP.Setup generates public parameters pp on input the security parameter and time parameter T. SLP.Eval takes pp, an input x, a random beacon value b, and generates a proof π. SLP.Forge takes pp, x, b, and produces a proof π. Lastly, SLP.Verify validates the proof π on input pp, x, π, and b. A short-lived proof must satisfy four security properties including *forgeability*, enabling anyone running in time $(1+\epsilon)T$ to generate a valid proof, *soundness*, preventing a malicious prover P^* running with parallel processors to generate a convincing proof in time less than T, *zero knowledge*, preserving the privacy of the witness w, and *indistinguishability*, making the real and forged proofs indistinguishable.

IV. Timed Secret Sharing (TSS)

With timed secret sharing (TSS), we make a secret sharing scheme dependent on time, having the reconstruction phase occur within a determined time interval, $[T_1, T_2]$, where T_1 is the lower time bound and T_2 is the upper time bound. These time bounds might be required by the dealer or as part of the system requirements, or even a combination of these two.

An important consideration, however, is that the dealer's *availability* should not be affected by making the scheme time-based, meaning that the dealer's role should finish after the distribution phase similar to the original setting.

A. TSS Definition

In this section, we present a formal definition of TSS. This definition builds upon the original definition of threshold secret sharing.

Definition 1 (Timed Secret Sharing)**.** A timed secret sharing (TSS) scheme involves the following algorithms.

1) **Initialization:**
   - Setup: $\mathsf{TSS.Setup}(1^\lambda, T_1, T_2) \to pp$, *on input security parameter* λ, *lower time bound* T_1, *and upper time bound* T_2, *outputs public parameters pp.*
2) **Distribution:**
   - Sharing: $\mathsf{TSS.Sharing}(pp, s) \to \{C_i\}_{i\in[n]}$, *on input pp and secret* $s \in S_\lambda$, *outputs a locked share* C_i *with time parameter* T_1 *for each party* P_i *in the set* $\mathcal{P}$.
3) **Reconstruction:**
   - Recovering: $\mathsf{TSS.Recover}(pp, C_i) \to s_i$, *on input pp and* C_i, *recovers the share* s_i. *The algorithm is run by each party* P_i *in* $\mathcal{P}$.
   - Pooling: $\mathsf{TSS.Pool}(pp, \mathcal{S}, T_2) \to s$, *on input pp and a set* $\mathcal{S}$ *of shares (where* $|\mathcal{S}| > t$ *and* $t \in pp$*), outputs the secret* s *if* T_2 *has not elapsed. Otherwise, it outputs* $\perp$.

A correct TSS scheme must satisfy *privacy*, ensuring no share is obtained before T_1 and *security*, ensuring any set of shares less than a threshold $t+1$ reveals no information about the secret before T_2.

Definition 1.1 (Privacy)**.** TSS satisfies privacy if for all parallel algorithms $\mathcal{A}$ whose running time is at most less than T_1 there exists a simulator Sim and a negligible function μ such that for all secret $s \in S_\lambda$, all $\lambda \in \mathbb{N}$, and all $i \in [n]$ it holds

$$\left| \Pr\left[\mathcal{A}(pp, s, C_i) = 1 \;:\; \begin{array}{l} \mathsf{TSS.Setup}(1^\lambda, T_1, T_2) \to pp, \\ \mathcal{A}(pp, 1^\lambda) \to s, \\ \mathsf{TSS.Sharing}(pp, s) \to \{C_i\}_{i\in[n]} \end{array} \right] - \Pr\left[\mathcal{A}(pp, s', C_j) = 1 \;:\; \begin{array}{l} \mathsf{TSS.Setup}(1^\lambda, T_1, T_2) \to pp, \\ \mathcal{A}(pp, 1^\lambda) \to s', \\ \mathsf{Sim}(pp) \to \{C_j\}_{j\in[n]} \end{array} \right] \right| \leq \mu(\lambda)$$

Definition 1.2 (Security)**.** TSS satisfies security if an adversary $\mathcal{A}$ controlling a set $\mathcal{S}'$ of parties, where $|\mathcal{S}'| \leq t$ and $s \in S_\lambda$, learns no information about s. Thus, it must hold

$$\Pr\left[\mathcal{A}(pp, \mathcal{S}', T_2) \to s \;:\; \begin{array}{l} \mathsf{TSS.Setup}(1^\lambda, T_1, T_2) \to pp, \\ \mathsf{TSS.Sharing}(pp, s) \to \{C_i\}_{i\in[n]}, \\ \mathsf{TSS.Recover}(pp, C_i) \to s_i \end{array} \right] \leq \mu(\lambda)$$

B. TSS Construction

We present an instantiation of TSS in Figure 2. To enforce a lower time bound T_1, the dealer uses TLPs [42], [49] to lock the shares into puzzles, enforcing a computational delay for each party to recover their corresponding share. Note that we treat T_2 mostly as a matter of formalization and rely on the underlying assumption of having common knowledge of time for participating parties to realize. We later in Section VII-B show how to relax this assumption using computational timing.

Theorem 1. *If the time-lock puzzle* TLP *and Shamir secret sharing are secure, then timed secret sharing protocol* Π_{TSS} *presented in Figure 2 satisfies privacy and security, w.r.t. definitions 1.1 and 1.2 respectively.*

The proof of the theorem can be found in Appendix H.

V. Verifiable Timed Secret Sharing (VTSS)

So far we assumed all participating parties, including the dealer, follow the protocol faithfully, providing semi-honest security. In this section, we present verifiable timed secret sharing (VTSS), an enhanced TSS which considers malicious adversaries. It protects against a malicious dealer who may send incorrect shares during the distribution phase. and against

Π_{TSS}

1) **Initialization:**
   - Setup: $\mathsf{TSS.Setup}(1^\lambda, T_1, T_2) \rightarrow pp$, the protocol works over $\mathbb{Z}_q$, where $q > n$. The dealer D runs $\mathsf{TLP.Setup}(1^\lambda, T_1)$ and publishes public parameters pp.
2) **Distribution:**
   - Sharing: $\mathsf{TSS.Sharing}(pp, s) \rightarrow \{Z_i\}_{i\in[n]}$, the dealer D picks a secret $s \in Z_p$ to be shared among n parties. It samples a degree-t Shamir polynomial $f(\cdot)$ such that $f(0) = s$ and $f(i) = s_i$ for $i \in [n]$. It runs $\mathsf{TLP.Gen}(1^\lambda, T_1, s_i)$ to create puzzle Z_i with time parameter T_1, locking the share s_i for all $i \in [n]$. Finally, D privately sends each party P_i their corresponding puzzle Z_i.
3) **Reconstruction:**
   - Recovering: $\mathsf{TSS.Recover}(pp, Z_i) \rightarrow s_i$, upon receiving the puzzle Z_i, party P_i starts solving it by running $\mathsf{TLP.Solve}(T_1, Z_i)$ to recover the share s_i.
   - Pooling: $\mathsf{TSS.Pool}(pp, \mathcal{S}, T_2) \rightarrow s$, upon having sufficient number of shares ($\geq t+1$) received before T_2, the reconstructor (a party in $\mathcal{P}$) reconstructs the secret s using Lagrange interpolation at $f(0)$; otherwise, it returns $\perp$.

Fig. 2: Timed Secret Sharing (TSS) protocol

a malicious shareholder who may send an incorrect share during the reconstruction phase.

A. VTSS Definition

In this section, we present a formal definition of VTSS. Our definition extends the original verifiable secret sharing (VSS) of Feldman [29], incorporating the notion of time.

Definition 2 (Verifiable Timed Secret Sharing)**.** A verifiable timed secret sharing (VTSS) scheme involves the following algorithms.

1) **Initialization:**
   - Setup: $\mathsf{VTSS.Setup}(1^\lambda, T_1, T_2) \rightarrow pp$, *on input security parameter λ, lower time bound T_1 and upper time bound T_2, outputs public parameters pp.*
2) **Distribution:**
   - Sharing: $\mathsf{VTSS.Sharing}(pp, s) \rightarrow \{C_i, \pi_i\}_{i\in[n]}$, *on input pp and a secret s, outputs locked share C_i with time parameter T_1 and a proof of validity π_i for each party $P_i \in \mathcal{P}$.*
   - Share verification: $\mathsf{VTSS.Verify_1}(pp, C_i, \pi_i) \rightarrow 1/0$, *on input pp, C_i, and π_i, checks the validity of share to ensure the locked share C_i is well-formed and contains a valid share of secret s. The algorithm returns 1 if both checks pass. Otherwise, it returns 0.*
3) **Reconstruction:**
   - Recovering: $\mathsf{VTSS.Recover}(pp, C_i) \rightarrow s_i$, *on input pp and C_i, forcibly outputs a share s_i. The algorithm is run by each party P_i.*
   - Recovery verification: $\mathsf{VTSS.Verify_2}(pp, s_i, \pi_i) \rightarrow 1/0$, *on input pp, s_i, and π_i, checks the validity of submitted share. The algorithm is run by a verifier $V \in \mathcal{P}$.*
   - Pooling: $\mathsf{VTSS.Pool}(pp, \mathcal{S}, T_2) \rightarrow s$, *on input pp and a set $\mathcal{S}$ of shares (where $|\mathcal{S}| > t$ and $t \in pp$), outputs the secret s if T_2 has not elapsed and $\perp$ otherwise.*

A correct VTSS scheme must satisfy *soundness*, ensuring extractability and verifiability of the shares, *privacy*, and *security*.

Definition 2.1 (Soundness)**.** A VTSS scheme is sound if there exists a negligible function μ such that for all PPT adversaries $\mathcal{A}$ and all $\lambda \in \mathbb{N}$ it holds

$$\Pr\left[b_1 = 1 \vee b_2 = 1 \;:\; \begin{array}{l} \mathsf{VTSS.Setup}(1^\lambda, T_1, T_2) \rightarrow pp, \\ \mathcal{A}(pp) \rightarrow (\{C_i, \pi_i\}_{i\in[n]}, \{s_i, \pi'_i\}), \\ b_1 := \mathsf{VTSS.Verify_1}(pp, C_i, \pi_i) \wedge \nexists s \text{ s.t.} \\ \mathsf{VTSS.Sharing}(pp, s) \rightarrow (\{C_i\}_{i\in[n]}, \cdot), \\ b_2 := \mathsf{VTSS.Verify_2}(pp, s_i, \pi'_i) \wedge \nexists C_i \text{ s.t.} \\ \mathsf{VTSS.Recover}(pp, C_i) \rightarrow s_i \end{array} \right] \leq \mu(\lambda)$$

Definition 2.2 (Privacy)**.** A VTSS satisfies privacy if for all parallel algorithms $\mathcal{A}$ whose running time is at most T_1 there exists a simulator Sim and a negligible function μ such that for all secret $s \in S_\lambda$ and all $\lambda \in \mathbb{N}$, it holds

$$\left| \Pr\left[\mathcal{A}(pp, s, \{C_i, \pi_i\}) = 1 \;:\; \begin{array}{l} \mathsf{VTSS.Setup}(1^\lambda, T_1, T_2) \rightarrow pp, \\ \mathcal{A}(1^\lambda, pp) \rightarrow s \\ \mathsf{VTSS.Sharing}(pp, s) \rightarrow \{C_i, \pi_i\}_{i\in[n]} \end{array} \right] - \Pr\left[\mathcal{A}(pp, s', \{C_j, \pi_j\}) = 1 \;:\; \begin{array}{l} \mathsf{VTSS.Setup}(1^\lambda, T_1, T_2) \rightarrow pp, \\ \mathcal{A}(1^\lambda, pp) \rightarrow s' \\ \mathsf{Sim}(pp) \rightarrow \{C_j, \pi_j\}_{j\in[n]} \end{array} \right] \right| \leq \mu(\lambda)$$

Definition 2.3 (Security)**.** A VTSS satisfies security if there exists a negligible function μ such that for an adversary controlling a subset $\mathcal{S}'$ of parties, where $|\mathcal{S}'| \leq t$ and $s \in S_\lambda$ it holds

$$\Pr\left[\mathcal{A}(pp, \mathcal{S}', T_2) \rightarrow s \;:\; \begin{array}{l} \mathsf{VTSS.Setup}(1^\lambda, T_1, T_2) \rightarrow pp, \\ \mathsf{VTSS.Sharing}(pp, s) \rightarrow \{C_i, \pi_i\}_{i\in[n]}, \\ \mathsf{VTSS.Recover}(pp, C_i) \rightarrow s_i \end{array} \right] \leq \mu(\lambda)$$

B. VTSS Construction

We present a protocol for VTSS in Figure 3. Following Feldman VSS [29], we make a crucial change in the protocol to adapt it for VTSS. Notably, in VTSS we have the dealer commit to the *shares* rather than the *coefficients* of the Shamir polynomial. This modification has two consequences.

Π_{VTSS}

1) **Initialization:**
   - Setup: $\mathsf{VTSS.Setup}(1^\lambda, T_1, T_2) \rightarrow pp$, let g be a generator of a group $\mathbb{G}$ of order q. The dealer D runs $\mathsf{VTC.Setup}(1^\lambda, T_1)$ and publishes a set of public parameters pp.

2) **Distribution:**
   - Sharing: $\mathsf{VTSS.Sharing}(pp, s) \rightarrow \{C_i, \pi_i\}_{i\in[n]}$, D picks a secret $s \xleftarrow{\$} \mathbb{Z}_q$ to be shared among n parties. It samples a degree-t random polynomial $f(\cdot)$ such that $f(0) = s$ and $f(i) = s_i$ for $i \in [n]$. It then commits to f by computing $v_i = g^{s_i}$ and broadcasting $\mathbf{v} = \{v_i\}_{i\in[n]}$. Then, D runs $\mathsf{VTC.Commit}(pp, s_i)$ to create a locked share C_i and a corresponding proof of validity π'_i with respect to v_i, locking the share s_i to be opened forcibly at T_1, $\forall i \in [n]$. Let $\pi_i = \{\pi'_i, \mathbf{v}\}$. D privately sends each party P_i their sharing $\{C_i, \pi'_i\}$.
   - Share verification: $\mathsf{VTSS.Verify_1}(pp, C_i, \pi_i) \rightarrow 1/0$, party P_i runs $\mathsf{VTC.Verify}(pp, v_i, C_i, \pi'_i)$ to check the locked share C_i is well-formed and embeds the share s_i corresponding to v_i. They then validate the consistency of the shares by sampling a code word $\mathbf{y}^\perp \in \mathcal{C}^\perp$, where $\mathbf{y}^\perp = \{y_1^\perp, \ldots, y_n^\perp\}$, and checking if $\prod_{j=1}^n v_j{}^{y_j^\perp} = 1$.
   - Complaint round: If a set of parties of size $\geq t+1$ complain about sharing, then D is disqualified. Otherwise, D reveals the corresponding locked shares with proofs by broadcasting $\{C_i, \pi'_i\}$. If the verification fails (or D does not broadcast), the dealer is disqualified.

3) **Reconstruction:**
   - Recovering: $\mathsf{VTSS.Recover}(pp, C_i) \rightarrow s_i$, each P_i wishing to participate in reconstruction runs $\mathsf{VTC.Solve}(pp, C_i)$ to obtain a share s_i.
   - Recovery verification: $\mathsf{VTSS.Verify_2}(pp, s_i, \pi_i) \rightarrow 1/0$, for each received share s_i from P_i, the reconstructor checks its validity by computing g^{s_i} and comparing it with v_i.
   - Pooling: $\mathsf{VTSS.Pool}(pp, \mathcal{S}, T_2) \rightarrow s$, upon having sufficient number of valid shares (*i.e.,* $\geq t+1$) received before T_2, the reconstrctor (a party in $\mathcal{P}$) reconstructs the secret s using Lagrange interpolation at $f(0)$ or aborts otherwise.

Fig. 3: Verifiable Timed Secret Sharing (VTSS) protocol

First, it allows shareholders to check the consistency of the shares (*i.e.,* all lie on a polynomial of degree t) using properties of error-correcting code, particularly the Reed-Solomon code [48]. This is due to the equivalency of the Shamir secret sharing with Reed-Solomon encoding observed by [45].[6] We restate the basic fact of linear error correcting code in Lemma 2. We remark that in Feldman VSS the checking of each share is done against the commitment to the whole polynomial, but here it is done with respect to an individual commitment to each share. So, it relies on the following lemma [18] to ensure the sharing phase has been performed correctly.

Lemma 2. Let $\mathcal{C}^\perp$ be the dual code of $\mathcal{C}$ that is a linear error correcting code over $\mathbb{Z}_q$ of length n. If $\mathbf{x} \in \mathbb{Z}_q^n \backslash \mathcal{C}$, and $\mathbf{y}^\perp$ is chosen uniformly at random from $\mathcal{C}^\perp$, the probability that the inner product of the vectors $\langle \mathbf{x}, \mathbf{y}^\perp \rangle = 0$ is exactly $1/q$.

Second, it enables us to make a black box use of VTC primitive [56] to *non-interactively* ensure each party P_i that it indeed obtains their correct share s_i at T_1. As mentioned, VTC allows committing to a signing key sk where its corresponding public key $pk = g^{sk}$ is publicly known. Our main insight is that we can think of $v_i = g^{s_i}$ published by the dealer as a public key for each share s_i committed by VTC. So, each party P_i can check the verifiability of their locked share C_i while ensuring the consistency of the shares $\{s_i\}_{i\in[n]}$.

[6]We refer the reader to [18] for a detailed description of the verification procedure.

Theorem 3. *If the verifiable timed commitments* VTC *and Feldman verifiable secret sharing [29] are secure, then verifiable timed secret sharing protocol* Π_{VTSS} *presented in Figure 3 satisfies soundness, privacy, and security, w.r.t. definitions 2.1, 2.2, and 2.3 respectively.*

The proof of the theorem can be found in Appendix I.

VI. Publicly Verifiable Timed Secret Sharing (PVTSS)

In this section, we make our timed secret sharing scheme publicly verifiable, meaning that anyone, not only a participating party, is able to verify different phases of the scheme. The public verification feature eliminates the need for a potential complaint round, as everyone can validate the correctness of the sharing performed by the dealer during the distribution phase. To achieve this, we use a publicly verifiable secret sharing (PVSS) scheme as the main building block that compels parties to behave correctly by non-interactively proving the validity of the messages sent during the distribution and reconstruction phases.

A. PVTSS Definition

In this section, we present a formal definition of PVTSS, based on the existing definition of PVSS, like the ones provided in [18], [19], [52].

Definition 3 (Publicly Verifiable Timed Secret Sharing)**.** A PVTSS scheme involves the following algorithms.

1) **Initialization:**
   - Setup: $\mathsf{PVTSS.Setup}(1^\lambda, T_1, T_2) \rightarrow pp$, *on input security parameter* λ*, lower time bound* T_1*, and upper time bound* T_2*, outputs public parameters* pp*. Each party* P_i *announces a registered public key* pk_i *which the corresponding secret key* sk_i *is only known to them.*
2) **Distribution:**
   - Sharing: $\mathsf{PVTSS.Sharing}(pp, S, \{pk_i\}_{i\in[n]}) \rightarrow \{\{C_i\}_{i\in[n]}, \pi_D\}$*, on input* pp*,* $\{pk_i\}_{i\in[n]}$*, and a secret* S*, generates locked encrypted share* C_i *with time parameter* T_1 *for each party* $P_i \in \mathcal{P}$*. It also generates a proof* π_D *for the correctness of shares.*
   - Share verification: $\mathsf{PVTSS.Verify_1}(pp, \{pk_i, C_i\}_{i\in[n]}, \pi_D) \rightarrow 1/0$*, on input* pp*,* $\{pk_i, C_i\}_{i\in[n]}$*, and* π_D*, checks the validity of the shares. This includes verifying the published locked encrypted shares are well-formed and contain correct shares of secret* S*. The algorithm is run by any verifier* V*.*
3) **Reconstruction:**
   - Recovering: $\mathsf{PVTSS.Recover}(pp, C_i, pk_i, sk_i) \rightarrow \{\tilde{s}_i, \pi_i\}$*, on input* pp*,* C_i*,* pk_i*, and* sk_i*, outputs a decrypted share* $\tilde{s}_i$ *together with proof* π_i *of valid decryption. The algorithm is run by each party* $P_i \in \mathcal{P}$*.*
   - Recovery verification: $\mathsf{PVTSS.Verify_2}(pp, C_i, \tilde{s}_i, \pi_i) \rightarrow \{0, 1\}$*, on input* pp*,* C_i*,* $\tilde{s}_i$*, and* π_i*, checks the validity of the decryption. The algorithm is run by any verifier* V*.*
   - Pooling: $\mathsf{PVTSS.Pool}(pp, \mathcal{S}, T_2) \rightarrow S$*, on input* pp *and a set* $\mathcal{S}$ *of decrypted shares* $\tilde{s}_i$ *(where* $|\mathcal{S}| > t$ *and* $t \in pp$*), outputs the secret* S *if* T_2 *has not elapsed.*

A PVTSS scheme must satisfy the following properties.

Definition 3.1 (Correctness)**.** PVTSS satisfies correctness if for all secret $s \in S_\lambda$ and all $i \in [n]$ it holds that

$$\Pr\left[\begin{array}{l}\mathsf{PVTSS.Verify_1}(pp, \{C_i\}_{i\in[n]},\\ \pi_D, \{pk_i\}_{i\in[n]}) = 1\\ \mathsf{PVTSS.Verify_2}(pp, C_i, \tilde{s}_i, \pi_i) = 1\\ \mathsf{PVTSS.Pool}(pp, \mathcal{S}, T_2) \rightarrow S\end{array} : \begin{array}{l}\mathsf{PVTSS.Setup}(1^\lambda, T_1, T_2) \rightarrow pp,\\ \mathsf{PVTSS.Sharing}(pp, S, \{pk_i\}_{i\in[n]})\\ \rightarrow \{\{C_i\}_{i\in[n]}, \pi_D\},\\ \mathsf{PVTSS.Recover}(pp, C_i, pk_i, sk_i)\\ \rightarrow \{\tilde{s}_i, \pi_i\}\end{array}\right] = 1$$

Definition 3.2 (Soundness)**.** PVTSS scheme is sound if there exists a negligible function μ such that for all PPT adversaries $\mathcal{A}$ and all $\lambda \in \mathbb{N}$ it holds that

$$\Pr\left[b_1 = 1 \vee b_2 = 1 : \begin{array}{l}\mathsf{PVTSS.Setup}(1^\lambda, T_1, T_2) \rightarrow pp,\\ \mathcal{A}(pp) \rightarrow (\{pk_i, C_i\}_{i\in[n]}, \pi_D, \tilde{s}, \pi),\\ b_1 := \mathsf{PVTSS.Verify_1}(pp, \{pk_i, C_i\}_{i\in[n]}, \pi_D)\\ \wedge \nexists s \text{ s.t.}\\ \mathsf{PVTSS.Sharing}(pp, S, \{pk_i\}_{i\in[n]})\\ \rightarrow \{\{C_i\}_{i\in[n]}, \cdot\},\\ b_2 := \mathsf{PVTSS.Verify_2}(pp, C, \tilde{s}, \pi), \wedge \nexists sk \text{ s.t.}\\ \mathsf{PVTSS.Recover}(pp, C, pk, sk) \rightarrow \{\tilde{s}, \cdot\},\end{array}\right] \leq \mu(\lambda)$$

Definition 3.3 (t-Privacy)**.** PVTSS satisfies t-privacy if for all parallel algorithms $\mathcal{A}$ whose running time is at most T_1, and set $I \subset [n]$ with $|I| = t + 1$, there exists a simulator Sim and a negligible function μ such that for all secret $s \in S_\lambda$ and $\lambda \in \mathbb{N}$ it holds that

$$\left|\Pr\left[\mathcal{A}(pp, s, \{C_i\}_{i\in[I]}, \pi_D) = 1 : \begin{array}{l}\mathsf{PVTSS.Setup}(1^\lambda, T_1, T_2) \rightarrow pp,\\ \mathcal{A}(1^\lambda, pp) \rightarrow s,\\ \mathsf{PVTSS.Sharing}(pp, s, \{pk_i\}_{i\in[I]})\\ \rightarrow \{\{C_i\}_{i\in[I]}, \pi_D\}\end{array}\right] - \Pr\left[\mathcal{A}(pp, s', \{C_j\}_{j\in[I]}, \pi_D) = 1 : \begin{array}{l}\mathsf{PVTSS.Setup}(1^\lambda, T_1, T_2) \rightarrow pp,\\ \mathcal{A}(1^\lambda, pp) \rightarrow s',\\ \mathsf{Sim}(pp) \rightarrow (\{C_j\}_{j\in[I]}, \pi_D)\end{array}\right]\right| \leq \mu(\lambda)$$

Definition 3.4 (Security)**.** A PVTSS satisfies security if there exists a negligible function μ such that for an adversary controlling a set $\mathcal{S}'$ of parties/shares, where $|\mathcal{S}'| \leq t$ and $s \in S_\lambda$, together with the public information, denoted by PI, it holds that[7]

$$\Pr\left[\mathcal{A}(pp, \mathcal{S}', \mathsf{PI}, T_2) \rightarrow S : \begin{array}{l}\mathsf{PVTSS.Setup}(1^\lambda, T_1, T_2) \rightarrow pp,\\ \mathsf{PVTSS.Sharing}(pp, s, \{pk_i\}_{i\in[n]})\\ \rightarrow \{\{C_i\}_{i\in[n]}, \pi_D\},\\ \mathsf{PVTSS.Recover}(pp, C_i, pk_i, sk_i)\\ \rightarrow \{\tilde{s}_i, \pi_i\}\end{array}\right] \leq \mu(\lambda)$$

An indistinguishability game given in [33], [51] and adopted by [18] formalizes this. We refer to Appendix K for more details.

B. PVTSS Construction

We present a detailed description of the PVTSS protocol in Figure 4. In what follows, we elaborate on several techniques used in our construction. In particular, it turns out that the public verifiability requirement of the scheme demands taking different approaches toward realizing the lower and upper time bounds.

Dealing with a malicious dealer. What makes the protection mechanism challenging for PVTSS is that *anyone*, before performing sequential computation, should be able to check the correctness of shares including consistency, validity, and extractability of the shares having a set of *encrypted* shares locked by the dealer. That is to say, a solution should *simultaneously* ensure (1) all shares lie on the same polynomial, (2) locked encrypted shares contain the committed shares, and (3) shares are obtainable in time T_1, all concerning some public information. We first discuss how to guarantee consistency and verifiability followed by our approach regarding extractability.

a) Blinded DLEQ: Our solution to meet the first two aforementioned requirements is based on having the dealer blind each encrypted shares $\tilde{s}_i$ using some randomness β_i, put the randomness into a puzzle Z_i, and publish all the puzzles together with locked encrypted shares and commitments for $i \in [n]$. The dealer needs to show that the locked encrypted shares contain the same shares as the commitments, while the consistency of the shares can be checked using the commitments (as discussed in Section V-B). To do so, we slightly modify the DLEQ proof (Section III-E) and make it blinded. It allows proving simultaneous knowledge of two witnesses, one of which is common in two statements. The following is a protocol Π_{BDLEQ} for the language

$$L_{\mathsf{BDLEQ}} = \{(g_1, x, g_2, g_3, y) \mid \exists(\alpha, \beta) : x = g_1^\alpha \wedge y = g_2^\alpha g_3^\beta\}$$

[7]This property is presented as IND1-Secrecy in [33], [51].

1) P chooses two random elements $u_1, u_2 \xleftarrow{\$} \mathbb{Z}_q$, computes $a_1 = g_1^{u_1}$ and $a_2 = g_2^{u_1} g_3^{u_2}$, and sends them to V.
2) V sends back a randomly chosen challenge $c \xleftarrow{\$} \mathbb{Z}_q$.
3) P computes $r_1 = u_1 + c\alpha$ and $r_2 = u_2 + c\beta$ and sends them to V.
4) V checks if both $g_1^{r_1} = a_1 x^c$ and $g_2^{r_1} g_3^{r_2} = a_2 y^c$ hold.

Theorem 4. *Protocol* Π_{BDLEQ} *is a public-coin honest-verifier zero-knowledge argument of knowledge corresponding to the language* L_{BDLEQ}.

The proof of the theorem can be found in Appendix J.

b) Cut-and-choose Technique: The dealer needs to convince the parties they can obtain their shares in time T_1. This is equivalent to saying that Z_i has indeed the value β_i embedded. A typical way to show the correctness of puzzle generation is by utilizing the cut-and-choose technique explored in previous works [9], [55]. At a high level, this technique forces a sender to behave correctly by randomly opening a (fixed) subset of puzzles it has already sent to the receiver based on the receiver's choice.

We remark that it is possible to deploy the cut-and-choose technique in our construction without sacrificing security. Since opening just reveals a (random) set of size t of encrypted shares, we are still guaranteed that the secret remains hidden up to time T_1. Each party is supposed to open their corresponding locked encrypted share, which is not among the opened ones by the dealer.

Realizing an Upper Time Bound. Due to the public verifiability, PVTSS protocol is executed over a public bulletin board. As a result, the secret may be reconstructed/used by any external party. This leads us to deploy *short-lived proofs* (SLPs) [5] for realizing an upper time bound in the construction. Observe that the use of SLPs allows tying the *correctness* of the system to time, meaning that the secret is only guaranteed to be correct if it is reconstructed before the upper time bound. Correctness intuitively states if the distribution phase succeeds, then the reconstruction phase will output the *same* secret initially shared by the dealer. Let us now briefly explain how we make use of SLPs in our construction.

Upper Time Bound with SLPs. Our approach is to take advantage of the *forgeability* property of SLPs in our PVTSS construction. We piggyback on the *proof of decryptions* π_i generated by each party P_i as part of the reconstruction phase, turning them into short-lived proofs where their expiration time matches the upper time bound T_2. Therefore, given the properties of short-lived proofs and also relying on that the secret has *uniformly random* distribution in Scrape PVSS,[8] the correctness of a share submitted by a party P_i is only guaranteed if being observed before T_2, otherwise it could be an invalid share accompanied with a valid proof. As shown in [5], a short-lived proof for any arbitrary relation R for which there exists a Sigma protocol can be efficiently constructed. For completeness, we present the short-lived proof for a relation R using pre-computed VDFs in Figure 6.

In our protocol, we make a black box use of short-lived DLEQ proof generation denoted by $\mathsf{DLEQ.SLP}$ and verification denoted by $\mathsf{DLEQ.SLV}$. It is required that the beacon value b used to compute π_i is not known until the time T_1, with $T = T_2 - T_1$ being the time parameter for the underlying VDF. Therefore, anyone verifying the proof before T_2 knows that it could have not been computed through forgery. We highlight that, to deploy short-lived proofs we need to use the DDH-based version of Scrape PVSS which its DLEQ proof comes with *knowledge soundness* property.

Remark 1. Recently, there have been several works focusing on the notion of forgeability over time, particularly for developing short-lived signature or forward-forgeable signature [5], [54]. To the best of our knowledge, Arun et al. [5] is the only one exploring the time-based forgeability in proof systems. This in turn enables us to deploy their primitive to provide the upper time bound for PVTSS, binding the correctness of the secret reconstruction to time.

Remark 2. We do not assume the availability of an *online* verifier who observes the protocol over time. In fact, due to the characteristic of SLPs, their use is meaningful when the verifier does not necessarily remain online during the reconstruction period $[T_1, T_2]$; otherwise, it can always reject the proofs sent afterward, negating the forgeability property. Moreover, as pointed out in [5], convincingly timestamping the messages published on the bulletin board is also opposed to the usability of SLPs.

In our PVTSS construction, we explicitly feed the upper time bound T_2 and a beacon value b in two algorithms, $\mathsf{PVTSS.Recover}$ and $\mathsf{PVTSS.Verify_2}$. This is essentially due to the necessity of the knowledge of time parameters $T = T_2 - T_1$ and b for short-lived proof generation and verification. Moreover, as discussed in [5], T does not need to be hardcoded when $\mathsf{PVTSS.Setup}$ is run. This allows the use of VDFs with any time parameter $T' > T$, while still generating short-lived proofs with respect to time T. That is, even if different parties use different time parameters with $T' > T$ for their VDF evaluations, only those proofs observed before time T are convincing.

Theorem 5. *If the time-lock puzzle* TLP*, short-lived proofs* SLP*, and Scrape PVSS are secure, then publicly verifiable timed secret sharing protocol* Π_{PVTSS} *(presented in Figure 4) satisfies soundness, t-privacy, and security, w.r.t. definitions 3.2, 3.3, and 3.4 respectively.*

For a proof of theorem see Appendix L.

VII. Decrementing-threshold Timed Secret Sharing (DTSS)

A. Secret Sharing with Additional Shares

As mentioned, a threshold secret sharing scheme guarantees *t-security*. There is also $t+1$-*robustness* assumption, ensuring

[8]This essentially implies any set of shares is indistinguishable from a set of random strings. Note that in normal Shamir secret sharing this is limited to a set of size at most t shares as the secret is not uniformly distributed [13].

Π_{PVTSS}

1) **Initialization:**

– Setup: $\mathsf{PVTSS.Setup}(1^\lambda, T_1) \rightarrow pp$, the public parameters pp include independently chosen generators g_1, g_2, g_3 in a DDH-hard group $\mathbb{G}$, a field $\mathbb{Z}_q$, a hash function $H : \{0,1\}^* \rightarrow I \subset [n]$ with $|I| = t$, and a public bulletin board. Each party P_i announces a registered public key $pk_i = g_1^{sk_i}$ which its secret key sk_i is only known to them.

2) **Distribution:**

– Sharing: $\mathsf{PVTSS.Sharing}(pp, S, \{pk_i\}_{i\in[n]}) \rightarrow \{\{C_i\}_{i\in[n]}, \pi_D\}$, the dealer D randomly chooses $s \xleftarrow{\$} \mathbb{Z}_q$ and defines the secret $S = g_1^s$ to be shared among n parties with public keys $\{pk_i\}_{i\in[n]}$. D computes Shamir shares $f(i) = s_i$, commitments $v_i = g_2^{s_i}$, and encrypted shares $\hat{s}_i = pk_i^{s_i}$ for all $i \in [n]$ using a degree-t Shamir polynomial $f(\cdot)$, where $f(0) = s$. It blinds the encrypted shares $\{\hat{s}_i\}_{i\in[n]}$ using some independent randomness β_i, resulting in $\{c_i\}_{i\in[n]}$, where $c_i = \hat{s}_i g_3^{\beta_i}$. The dealer then locks every randomness β_i in a TLP by running $\mathsf{TLP.Gen}(1^\lambda, T_1, \beta_i)$. Let denote $C_i = \{c_i, Z_i\}$. To show the consistency and validity of the locked encrypted shares, D runs Π_{BDLEQ}, resulting in proof $\pi =: (v_i, e, r_{1,i}, r_{2,i})$ for $i \in [n]$. Finally, D publishes the locked encrypted shares $\{C_i\}_{i\in[n]}$ and proof π_D on a public bulletin board. Moreover, D computes $H(\{C_i\}_{i\in[n]}, \pi) \rightarrow I$ as a random challenge (for cut and choose) and outputs $\pi_D = \{I, \pi, \beta_i, \hat{s}_i\}_{i\in[I]}$.

– Share verification: $\mathsf{PVTSS.Verify_1}(pp, \{C_i\}_{i\in[n]}, \pi_D, \{pk_i\}_{i\in[n]}) \rightarrow 1/0$, the verifier V first validates the consistency of the shares by sampling a code word $\mathbf{y}^\perp \in C^\perp$, where $\mathbf{y}^\perp = \{y_1^\perp, \ldots, y_n^\perp\}$, and checking if $\prod_{j=1}^n v_j{}^{y_j^\perp} = 1$. V then checks the proof π_D is valid. After re-computing I, the verifier checks the puzzles are correctly constructed by invoking $\mathsf{TLP.Gen}$ algorithm and comparing the encrypted share sent by the dealer with the one being unlocked using β_i.

3) **Reconstruction:**

– Recovering: $\mathsf{PVTSS.Recover}(pp, C_i, pk_i, sk_i, b, T_2) \rightarrow \{\tilde{s}_i, \pi_i\}$, after checking the validity of sharing phase, any party P_i wishing to obtain their share at T_1, unlocks the blinding factor β_i by running $\mathsf{TLP.Solve}(pp, Z_i)$, and obtains their share $\tilde{s}_i$ after decrypting $\hat{s}_i$ as $\tilde{s}_i = \hat{s}_j{}^{1/sk_i}$. Then, the party P_i reveals the share $\tilde{s}_i$ together with a short-lived proof $\pi_i =: \{\mathsf{DLEQ.SLP}(sk_i, g_1, pk_i, \tilde{s}_i, \hat{s}_i), \beta_i\}$ of valid decryption. Note that $\mathsf{DLEQ.SLP}$ involves calling $\mathsf{SLP.Gen}$ for the relation $R_{DLEQ} = \{(g_1, pk_i, \tilde{s}_i, \hat{s}_i; sk_i)\}$ given a beacon value b publicly known no sooner than T_1.

– Recovery verification: $\mathsf{PVTSS.Verify_2}(pp, C_i, \tilde{s}_i, \pi_i, b, T_2) \rightarrow 1/0$, any (external) verifier V can check the validity of published share $\tilde{s}_i$ via $\mathsf{DLEQ.SLV}(\pi_i, g_1, pk_i, \tilde{s}_i, \hat{s}_i)$. Note that having C_i, the verifier first obtains $\hat{s}_i$ with β_i.

– Pooling: $\mathsf{PVTSS.Pool}(pp, \mathcal{S}, T_2) \rightarrow S$, upon having sufficient number of shares ($\geq t+1$) received before time T_2, denoted by $\mathcal{S}$, anyone can reconstruct the secret $S = g_1^s$ using Lagrange interpolation in the exponent.

Fig. 4: Publicly Verifiable Timed Secret Sharing (PVTSS) protocol

the availability of a sufficient number of valid shares during the reconstruction phase.

However, it is natural to challenge such a liveness assumption and consider a scenario in which a *large* fraction of honest parties goes offline, particularly when having a determined period for reconstruction, putting the system under threat of failure (*i.e.,* lack of liveness). To be concrete, a possible scenario that may lead to having less than a threshold of honest parties available is explored in [56] known as *denial of spending* (DoSp) attack. Here, the adversary can carry out an attack against honest parties to control, say 51% of parties, while the reconstruction threshold to spend a multi-signature-based transaction is, say 52%. Consequently, the set of available honest parties cannot reach the threshold and their investment will remain locked. We can extend this scenario to a secret sharing, where honest parties may not reach the threshold, at least for some time (*e.g.,* prior to the upper time bound). In a federated learning setting [41], real-world factors such as hardware failure or poor network coverage can indeed cause this issue, leading to shareholders' dropouts.

Our goal is to mitigate the *robustness* assumption using the capabilities of time-based cryptography. We observe this is feasible by having the dealer provide parties with *additional time-locked shares*. By additional, we mean some shares other than the individual one each party already receives during the distribution phase of the protocol. Thus, even if there is less than a threshold of parties (even a single one) available at the reconstruction period (*i.e.,* $[T_1, T_2]$), they will be able to open the additional time-locked shares after carrying out some computation and retrieve the secret.

We remark that a large body of literature on threshold secret sharing assumes all the parties, not only those interacting in the reconstruction phase, learn the secret [18], [36]. Given this, we argue that the availability of a (threshold) number of additional time-lock shares at the proper time (*i.e.,* T_2) does not violate the security of the system since it enables all the parties to eventually learn the secret at the same point via sequential computation if they have not already learned it.

To focus on the core problem, which is preserving the robustness of the system in case of unavailability of (a threshold of) honest parties, we assume the additional time-locked shares are honestly generated. Should a malicious dealer attempt to misbehave, this assumption can be lifted by using mechanisms similar to the ones used in the previous sections.

B. Decrementing-threshold Timed Secret Sharing (DTSS)

It is possible to derive an interesting *trade-off* between time and fault tolerance by having some additional time-locked shares to be realized periodically at *different* points in time. The consequence of this *gradual* release is twofold. Firstly, if necessary, it enables an honest party requiring some more shares (not necessarily t) to reconstruct the secret without going through the sequential computation for the whole period, *i.e.,* $[T_1, T_2]$. They can stop working up to a point where a sufficient number of additional shares is gained. Secondly, as time goes by and the reconstruction is not initiated, the adversary may get more additional shares by investing computational effort, leading to a gradual lessening of the fault tolerance of the system. Looking ahead, this feature happens to be useful to break a public goods game as it ties the security of the system to time; the later parties initiate the reconstruction, the more chances the adversary learns the secret.

C. DTSS Definition

Now, we present a formal definition for our new scheme called decrementing-threshold timed secret sharing (DTSS).

Definition 4 (Decrementing-threshold Timed Secret Sharing)**.** A (t, n) DTSS scheme consists of a tuple of algorithms ($\mathsf{DTSS.Setup}$, $\mathsf{DTSS.Sharing}$, $\mathsf{DTSS.ShaRecover}$, $\mathsf{DTSS.Verify}$, $\mathsf{DTSS.AddRecover}$, $\mathsf{DTSS.Pool}$) as follows.

1) **Initialization:**
   - Setup: $\mathsf{DTSS.Setup}(1^\lambda, T_1, T_2, t) \to \{pp, pk, sk\}$, *on input security parameter λ, lower time bound T_1, and a value t, outputs public parameters pp and key pair (pk, sk) to be used for generating additional locked shares by the dealer D.*
2) **Distribution:**
   - Sharing: $\mathsf{DTSS.Sharing}(pp, s, pk, sk) \to \{\{C_i\}_{i\in[n]}, \mathbf{v}, \{O_j\}_{j\in[t]}\}$, *on input pp, a secret s, and a key pair (pk, sk), outputs locked share C_i with time parameter T_1 together with commitment to shares $\mathbf{v}$ for each party $P_i \in \mathcal{P}$. Moreover, it outputs t additional locked shares $\{O_j\}_{j\in[t]}$, with O_j being locked with time parameter $(j+1)T_1$.*
3) **Reconstruction:**
   - Share recovery: $\mathsf{DTSS.ShaRecover}(pp, C_i) \to s_i$, *on input pp and C_i, outputs a share s_i. The algorithm is run by each party P_i.*
   - Recovery verification: $\mathsf{DTSS.Verify}(pp, s_i, \mathbf{v}) \to 1/0$, *on input pp, s_i, and $\mathbf{v}$, checks the validity of the received share.*
   - Additional share recovery: $\mathsf{DTSS.AddRecover}(pp, pk, \{O_j\}_{j\in[t]}) \to \{s'_j\}$, *on input pp, pk, and $\{O_j\}_{j\in[t]}$, forcibly outputs the additional share s'_j at time $(j+1)T_1$. The algorithm is run by anyone in $\mathcal{P}$ wishing to obtain additional shares.*
   - Pooling: $\mathsf{DTSS.Pool}(pp, \mathcal{S}, T_2) \to s$, *on input pp and a set $\mathcal{S}$ of shares (where $|\mathcal{S}| > t$ and $t \in pp$), outputs the secret s if T_2 has not elapsed.*

We could also include a verification algorithm $\mathsf{VTSS.Verify_2}$ for a verifier to check the validity of the presented additional share by a participating party. We refrain from formalizing this algorithm since we implicitly assume all the parties involved in reconstruction retrieve the additional time-locked shares, negating the verification. However, such a verification algorithm can introduce efficiency as it allows reconstruction of the secret while having *only one* party solve the puzzles containing additional time-locked shares and prove their correctness to others.

Definition 4.1 (Verifiability)**.** A DTSS scheme is verifiable if there exists a negligible function μ such that for all PPT adversaries $\mathcal{A}$, all $\lambda \in \mathbb{N}$, and $i \in [n]$, it holds that

$$\Pr\left[b = 1 \;:\; \begin{array}{l} \mathsf{DTSS.Setup}(1^\lambda, T_1, t) \to \{pp, pk, sk\}, \\ \mathcal{A}(pp) \to (s_i, \mathbf{v}, \cdot), \\ b := \mathsf{DTSS.Verify}(pp, s_i, \mathbf{v}), \wedge \nexists s \text{ s.t.} \\ \mathsf{DTSS.Sharing}(pp, s, pk, sk) \to \{\mathbf{v}, \cdot\} \end{array} \right] \le \mu(\lambda)$$

Definition 4.2 (Privacy)**.** A DTSS satisfies privacy if for all algorithms $\mathcal{A}$ running in time $T < jT_1$, where $1 \le j \le t$, with at most T_1 parallel processors, there exists a simulator Sim and a negligible function μ such that for all secret $s \in S_\lambda$ and $\lambda \in \mathbb{N}$ it holds that

$$\left| \Pr\left[\begin{array}{l} \mathcal{A}(pp, pk, s, \\ C_i, \mathbf{v}, \{O_j\}_{j\in[t]}) = 1 \end{array} : \begin{array}{l} \mathsf{DTSS.Setup}(1^\lambda, T_1) \to \{pp, pk, sk\}, \\ \mathcal{A}(1^\lambda, pp) \to s \\ \mathsf{DTSS.Sharing}(pp, s) \\ \to \{\{C_i\}_{i\in[n]}, \mathbf{v}, \{O_j\}_{j\in[t]}\} \end{array} \right] - \right.$$

$$\left. \Pr\left[\begin{array}{l} \mathcal{A}(pp, pk, s', \\ C_i, \mathbf{v}, \{O_j\}_{j\in[t]}) = 1 \end{array} : \begin{array}{l} \mathsf{DTSS.Setup}(1^\lambda, T_1) \to \{pp, pk, sk\}, \\ \mathcal{A}(1^\lambda, pp) \to s' \\ \mathsf{Sim}(pp) \to \{\{C_i\}_{i\in[n]}, \mathbf{v}, \{O_j\}_{j\in[t]}\} \end{array} \right] \right| \le \mu(\lambda)$$

Definition 4.3 (Security)**.** Let $2T_1, \ldots, (t+1)T_1$ be times at which each additional time-locked share is forcibly obtained. A DTSS is secure if prior to $(j+1)T_1$, where $1 \le j \le t$, the adversary controlling $\le t-(j-1)$ parties learns no information about $s \in S_\lambda$ in a computational sense. Thus, it holds:

$$\Pr\left[\mathcal{A}(pp, pk, \mathcal{S}', T_2) \to s \;:\; \begin{array}{l} \mathsf{DTSS.Setup}(1^\lambda, T_1, t) \to \{pp, pk, sk\} \\ \mathsf{DTSS.Sharing}(pp, s) \\ \to \{\{C_i\}_{i\in[n]}, \mathbf{v}, \{O_j\}_{j\in[t]}\}, \\ \mathsf{DTSS.ShaRecover}(pp, C_i) \to s_i, \\ \mathsf{DTSS.AddRecover}(pp, pk, \{O_j\}_{j\in[t]}) \\ \to \{s'_j\}, 1 \le j \le t. \end{array} \right] \le \mu(\lambda)$$

Definition 4.4 (Robustness)**.** A DTSS is robust if each party in $\mathcal{P}$ can *eventually* reconstruct the secret s, either after receiving a sufficient number of other parties' shares and/or obtaining the additional time-locked shares.

$$\Pr\left[\mathsf{DTSS.Pool}(pp,\mathcal{S},T_2)\rightarrow s \;:\; \begin{array}{l}\mathsf{DTSS.Setup}(1^\lambda,T_1,t)\rightarrow\{pp,pk,sk\}\\ \mathsf{DTSS.Sharing}(pp,s)\\ \rightarrow\{\{C_i\}_{i\in[n]},\mathbf{v},\{O_j\}_{j\in[t]}\},\\ \mathsf{DTSS.ShaRecover}(pp,C_i)\rightarrow s_i,\\ \mathsf{DTSS.AddRecover}(pp,pk,\{O_j\}_{j\in[t]})\\ \rightarrow\{s'_j\}_{j\in[t]}\end{array}\right]=1$$

D. DTSS Construction

We present a construction for DTSS in Figure 5. As mentioned, we would like a protocol in which anyone can obtain each additional share s'_j at time $(j+1)T_1$ given that the dealer's role must end with the distribution phase.[9] In a naive way, the dealer should create t puzzles each embedding one additional share to be opened at t different points in time. However, this inefficient solution comes with a high computation cost as anyone wishing to access the shares needs to solve each puzzle separately in parallel, demanding up to $T_1\sum_{j=1}^{t} j$ squaring. To get away with this issue, we use multi-instance time-lock puzzle (MTLP) [1], a primitive allowing sequential (chained) release of solutions where the overall computation cost of solving t puzzles is equal to that of solving only the last one.

Theorem 6. *If the multi-instance time-lock puzzle* MTLP *and verifiable timed secret sharing* VTSS *are secure, then our DTSS protocol* Π_{DTSS} *presented in Figure 5 satisfies the properties described in Section VII-C.*

For a proof of theorem see Appendix M.

Remark 3. By looking closely, we see the trade-off between time and fault tolerance can be added as a property to any threshold secret sharing scheme. More precisely, the gradual release of fault tolerance over time is implied by the release of the shares and not the type of underlying secret sharing scheme. In addition, the use of additional time-locked shares implicitly provides an upper time bound. This is because by the time T_2 – which the last puzzle is supposed to open – the secret is revealed to all parties in $\mathcal{P}$.

VIII. Discussion

In the following, we explore and discuss several aspects of our constructions.

a) TSS as a generalization of secret sharing: A timed secret sharing scheme can be considered as a time-based generalization of a normal secret sharing scheme. That is, if we set $T_1=0$ and $T_2=\infty$, then the resulting scheme is a normal secret sharing. Also, the *independency* of the methods used to realize the lower and upper time bounds makes it possible to consider them separately depending on the applications.

b) On the setup phase: In all of our schemes, Setup algorithm is responsible for generating a set of public parameters pp, encapsulating the parameters for the underlying secret sharing and time-based cryptographic primitive. In particular, our VTSS construction in Figure 3 requires a trusted setup to generate the parameters for the underlying VTC primitive. This is due to the linearly homomorphic TLP of [42] deployed in VTC construction. The functionality of the primitive depends on such an assumption; otherwise, either the puzzle is not solvable or one can efficiently solve it upon receipt. Using class groups of imaginary quadratic fields [16] as a family of groups of unknown order instead of the well-known RSA group is an option to reduce the trust, but comes with higher (offline) computational investment for the puzzle generator to compute the parameters through sequential computation [42]. Deploying the class groups solely does not eliminate the need for a trusted setup as it is still feasible that a malicious sender fools a receiver into accepting locked shares that will never be opened. Moreover, the VDF used in SLPs can be instantiated efficiently via class groups [59] without making any trusted setup assumption.

c) On the use of SLPs: As previously mentioned, by deploying SLPs in PVTSS protocol we capture the notion of upper time bound as the system is *safely* usable until T_2. As we show next, after T_2 any reconstruction fails with overwhelming probability. This necessitates the availability of a reconstructor during the protocol execution for a correct reconstruction. Moreover, we deploy short-lived proofs using precomputed VDFs [5] which do not offer reusable forgeability, *i.e.,* forging a proof for any statement v without computing a new VDF. However, this essentially fits a secret sharing setting (in particular, PVSS) which is inherently single-use.

d) Failure probability: We here briefly analyze the probability of a reconstruction failure after T_2 when deploying SLPs. Let t be the number of adversarial shares and n be the total number of shares publicly available. Given that the incorporation of even *one* invalid share results in an invalid reconstruction and the fact that shares are uniformly distributed, the success probability can be computed as $p=\frac{p_1}{p_2}$, where $p_1=\binom{n-t}{t+1}$ and $p_2=\binom{n}{t+1}$. We can easily show that by a proper choice of the parameters n,t the reconstruction fails with overwhelming probability. Setting $t=\lceil\frac{n}{2}\rceil-1$, we have $p\leq n2^{-(\lceil\frac{n}{2}\rceil+1)}$ which is a negligible value in λ for a choice of $n=\lambda$.

e) Breaking public goods game: A common method to break the public goods game is to reward those parties who publish their shares sooner via harnessing the financial capabilities of the blockchain systems [6], [11], [37]. That is, the shareholder receives some *reward* if their submitted share is among the first $t+1$ shares published on the chain. This in turn creates a race and motivates the shareholder to show up sooner. Our two ideas, namely using short-lived proofs and gradual release of additional shares, can be considered as *orthogonal* methods that are *off-chain*. More precisely, using SLPs forces shareholders to publish their shares before some time, otherwise, they may not be able to recover the correct secret. Moreover, using a gradual release of additional shares can also play the same role; however, by causing the threat of security reduction over time. As a result, shareholders are pushed to act as soon as possible to avoid any pitfalls.

[9] Without loss of generality we assume $T_2=(t+1)T_1$, accommodating the periodic release of additional shares.

Π_{DTSS}

1) **Initialization:**
– Setup: $\mathsf{DTSS.Setup}(1^\lambda, T_1, t) \rightarrow \{pp, pk, sk\}$, the dealer D invokes two algorithms of $\mathsf{VTSS.Setup}(1^\lambda, T_1, T_2)$ and $\mathsf{MTLP.Setup}(1^\lambda, T_1, t+1)$, and publishes the set of public parameters pp, pk.

2) **Distribution:**
– Sharing: $\mathsf{DTSS.Sharing}(pp, s, pk, sk) \rightarrow \{\{C_i\}_{i\in[n]}, \mathbf{v}, \{O_j\}_{j\in[t]}\}$, the dealer D first picks a secret $s \xleftarrow{\$} \mathbb{Z}_q$ and invokes $\mathsf{VTSS.Sharing}(pp, s)$ to generate n locked shares $\{C_i\}_{i\in[n]}$ and $\mathbf{v}$. Moreover, it computes t additional shares $f(a_j) = s'_j$ for $j \in [t]$, where $f(0) = s$ and $\{a_1, \ldots, a_t\}$ are some known distinct points. Finally, it invokes $\mathsf{MTLP.Gen}(\mathbf{m}, pk, sk)$, where $\mathbf{m} = \{\perp, s'_1, \ldots, s'_t\}$ to generate an MTLP containing $\{s'_j\}_{j\in[t]}$.

3) **Reconstruction:**
– Share recovery: $\mathsf{DTSS.ShaRecover}(pp, C_i) \rightarrow s_i$, each party P_i runs $\mathsf{VTSS.Recover}(pp, C_i)$ to recover their share s_i.
– Recovery verification: $\mathsf{DTSS.Verify}(pp, s_i, \mathbf{v}) \rightarrow \{0, 1\}$, any reconstructor $V \in \mathcal{P}$ runs $\mathsf{VTSS.Verify_2}(pp, s_i, \mathbf{v})$ to check the validity of the received share s_i.
– Additional share recovery: $\mathsf{DTSS.AddRecover}(pp, pk, \{O_j\}_{j\in[t]}) \rightarrow \{s'_j\}_{j\in[t]}$, anyone wishing to obtain additional time-locked shares $\{s'_j\}_{j\in[t]}$ runs $\mathsf{MTLP.Solve}(pp, \{O_j\}_{j\in[t]})$.
– Pooling: $\mathsf{DTSS.Pool}(pp, \mathcal{S}, T_2) \rightarrow s$, upon having sufficient number of valid shares (*i.e.,* $\geq t+1$), the reconstrctor $V \in \mathcal{P}$ reconstructs the secret s using Lagrange interpolation at $f(0)$.

Fig. 5: Decrementing-threshold Timed Secret Sharing (DTSS) protocol

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APPENDIX

A. Time-lock Puzzles (TLP)

Definition 5 (Time-lock Puzzle)**.** A time-lock puzzle (TLP) consists of the following two algorithms:

1) $\mathsf{TLP.Gen}(1^\lambda, T, s) \rightarrow Z$, *a probabilistic algorithm that takes time parameter* T *and a secret* s*, and generates a puzzle* Z*.*
2) $\mathsf{TLP.Solve}(T, Z) \rightarrow s$, *a deterministic algorithm that solves the puzzle* Z *and retrieves the secret* s*.*

We recall the correctness and security definition of standard time-lock puzzles:

Correctness [42]**.** A TLP scheme is correct if for all $\lambda \in \mathbb{N}$, all polynomials $T(\cdot)$ in λ, and all $s \in S_\lambda$, it holds that

$$\Pr\left[\mathsf{TLP.Solve}(T(\lambda), Z) \rightarrow s : \mathsf{TLP.Gen}(1^\lambda, T(\lambda), s) \rightarrow Z\right] = 1$$

Security [42]**.** A TLP scheme is secure with gap $\epsilon < 1$ if there exists a polynomial $\tilde{T}(\cdot)$ such that for all polynomials $T(\cdot) \geq \tilde{T}(\cdot)$ and every polynomial-size adversary

$\mathcal{A} = \{\mathcal{A}_\lambda\}_{\lambda \in N}$ of depth $\leq T^\epsilon(\lambda)$, there exists a negligible function $\mu(\cdot)$, such that for all $\lambda \in \mathbb{N}$ and $s_0, s_1 \in \{0,1\}^\lambda$ it holds that Pr $[\mathcal{A}(Z) \to b : \mathsf{TLP.Gen}(1^\lambda, T(\lambda), s_b) \to Z,$ $b \xleftarrow{\$} \{0,1\}] \leq \frac{1}{2} + \mu(\lambda)$.

In particular, the seminal work of [49] introduced the notion of *encrypting to the future* using an RSA-based TLP. Loosely speaking, the sender encrypts a message m under a key k derived from the solution s to a puzzle Z. So, anyone can obtain m after running $\mathsf{TLP.Solve}(T, Z)$, and learning the key.

B. Homomorphic Time-Lock Puzzles (HTLP)

Definition 6 (Homomorphic Time-Lock Puzzles [42])**.** Let $\mathcal{C} = \{\mathcal{C}_\lambda\}_{\lambda \in \mathbb{N}}$ be a class of circuits and S_λ be a finite domain. A homomorphic time-lock puzzle (HTLP) with respect to $\mathcal{C}$ and with solution space S_λ is a tuple of algorithms (HTLP.Setup, HTLP.Gen, HTLP.Solve, HTLP.Eval) as follows.

1) $\mathsf{HTLP.Setup}(1^\lambda, T) \to pp$, *a probabilistic algorithm that takes a security parameter* 1^λ *and time parameter* T*, and generates public parameters* pp.
2) $\mathsf{HTLP.Gen}(pp, s) \to Z$, *a probabilistic algorithm that takes public parameters* pp *and a solution* $s \in S_\lambda$*, and generates a puzzle* Z.
3) $\mathsf{HTLP.Solve}(pp, Z) \to s$, *a deterministic algorithm that takes public parameters* pp *and puzzle* Z*, and retrieves a secret* s.
4) $\mathsf{HTLP.Eval}(C, pp, Z_1, \ldots, Z_n) \to Z'$, *a probabilistic algorithm that takes a circuit* $C \in \mathcal{C}_\lambda$ *and a set of* n *puzzles* $(Z_1, \ldots, Z_n)$*, and outputs a puzzle* Z'.

Security [42]**.** An HTLP scheme (HTLP.Setup, HTLP.Gen, HTLP.Solve, HTLP.Eval) is secure with gap $\epsilon < 1$ if there exists a polynomial $\tilde{T}(\cdot)$ such that for all polynomials $T(\cdot) \geq \tilde{T}(\cdot)$ and every polynomial-size adversary $(\mathcal{A}_1, \mathcal{A}_2) = \{(\mathcal{A}_1, \mathcal{A}_2)_\lambda\}_{\lambda \in N}$ where the depth of $\mathcal{A}_2$ is bounded from above by $T^\epsilon(\lambda)$, there exists a negligible function $\mu(\cdot)$, such that for all $\lambda \in \mathbb{N}$ it holds that

$$\Pr\left[\mathcal{A}_2(pp, Z, \tau) \to b \;:\; \begin{array}{l} \mathcal{A}_1(1^\lambda) \to (\tau, s_0, s_1) \\ \mathsf{HTLP.Setup}(1^\lambda, T(\lambda)) \to pp \\ b \xleftarrow{\$} \{0,1\} \\ \mathsf{HTLP.Gen}(pp, s_b) \to Z \end{array}\right] \leq \frac{1}{2} + \mu(\lambda)$$

The puzzle is defined over a group of unknown order and is of the form $Z = (u, v)$, where $u = g^r$ and $v = h^{r.N}(1+N)^s$. One notable point regarding the construction is that a trusted setup assumption is needed to generate the public parameters $pp = (T, N, g, h)$, where N is a safe modulus [10] and $h = g^{2^T}$. Such a setup phase is responsible for generating the parameters as specified and keeping the random coins secret; otherwise, either the puzzle is not solvable or one can efficiently solve it in time $t \ll T$. Having said that, the authors in [42] point out that this assumption can be removed if construction gets instantiated over class groups instead of an RSA group of unknown order. However, this comes at the cost of a higher computational overhead by the puzzle generator.

[10] A safe modulus is a product of two safe primes $P = 2p'+1, Q = 2q'+1$, where p' and q' are prime numbers.

C. Multi-instance Time-lock Puzzle (MTLP)

Definition 7 (Multi-instance Time-lock Puzzle [1])**.** A Multi-instance Time-lock Puzzle (MTLP) consists of the following five algorithms.

1) $\mathsf{MTLP.Setup}(1^\lambda, T, z) \to \{pk, sk, \vec{\mathbf{d}}\}$, *a probabilistic algorithm that takes a security parameter* λ*, a time parameter* T*, and the number of puzzle instances* z*, and outputs a key pair* (pk, sk) *and a secret witness vector* $\vec{\mathbf{d}}$.
2) $\mathsf{MTLP.Gen}(\vec{\mathbf{m}}, pk, sk, \vec{\mathbf{d}}) \to \{\vec{\mathbf{o}}, \vec{\mathbf{h}}\}$, *a probabilistic algorithm that takes a message vector* $\vec{\mathbf{m}}$*, the public-private key* (pk, sk)*, secret witness vector* $\vec{\mathbf{d}}$*, and outputs a puzzle vector* $\vec{\mathbf{o}}$ *and a commitment vector* $\vec{\mathbf{h}}$.
3) $\mathsf{MTLP.Solve}(pk, \vec{\mathbf{o}}) \to \vec{\mathbf{s}}$, *a deterministic algorithm that takes the public key* pk *and the puzzle vector* $\vec{\mathbf{o}}$*, and outputs a solution vector* $\vec{\mathbf{s}}$*, where* s_j *is of form* $m_j \,||\, d_j$.
4) $\mathsf{Prove}(pk, s_j) \to \pi_j$, *a deterministic algorithm that takes the public key* pk *and a solution* s_j*, and outputs a proof* π_j.
5) $\mathsf{Verify}(pk, \pi_j, h_j) \to \{0,1\}$, *a deterministic algorithm that takes the public key* pk*, proof* π_j*, and commitment* h_j*. If verification succeeds, it outputs 1, otherwise 0.*

Security [1]**.** A multi-instance time-lock puzzle is secure if for all λ and T, any number of puzzle: $z \geq 1$, any j (where $1 \leq j \leq z$), any pair of randomised algorithm $\mathcal{A} : (\mathcal{A}_1, \mathcal{A}_2)$, where $\mathcal{A}_1$ runs in time $O(poly(jT, \lambda))$ and $\mathcal{A}_2$ runs in time $\delta(jT) < jT$ using at most $\pi(T)$ parallel processors, there exists a negligible function $\mu(.)$ such that

$$\Pr\left[\begin{array}{l} \mathcal{A}_2(pk, \ddot{o}, \tau) \to \ddot{a} \\ \text{s.t.} \\ \ddot{a} : (b_i, i) \\ m_{b_i, i} = m_{b_j, j} \end{array} : \begin{array}{l} \mathsf{MTLP.Setup}(1^\lambda, \Delta, z) \to (pk, sk, \vec{\mathbf{d}}) \\ \mathcal{A}_1(1^\lambda, pk, z) \to (\tau, \vec{\mathbf{m}}) \\ \forall j', 1 \leq j' \leq z : b_{j'} \xleftarrow{\$} \{0,1\} \\ \mathsf{MTLP.Gen}(\vec{\mathbf{m}}', pk, sk, \vec{\mathbf{d}}) \to \ddot{o} \end{array}\right] \leq \frac{1}{2} + \mu(\lambda)$$

D. Verifiable Delay Function

Definition 8 (Verifiable Delay Function)**.** A verifiable delay function (VDF) consists of the following three algorithms:

1) $\mathsf{VDF.Setup}(1^\lambda, T) \to pp$, *a probabilistic algorithm that takes security parameter* λ *and time parameter* T*, and generates system parameters* pp.
2) $\mathsf{VDF.Eval}(pp, x) \to \{y, \pi\}$, *a deterministic algorithm that given system parameters* pp *and a randomly chosen input* x*, computes a unique output* y *and a proof* π.
3) $\mathsf{VDF.Verify}(pp, x, y, \pi) \to \{0,1\}$, *a deterministic algorithm that verifies* y *indeed is a correct evaluation of the* x*. If verification succeeds, the algorithm outputs 1, and otherwise 0.*

Intuitively, there are three security properties that a valid VDF should satisfy. There must be a run time constraint of $(1 + \epsilon)T$ for a positive constant ϵ to limit the evaluation algorithm, called *ϵ-evaluation*. The VDF should have *sequentially*, meaning no adversary using parallel processors can successfully compute the output without executing proper sequential computation. Lastly, the VDF evaluation should be a function with *uniqueness* property. That is, the verification algorithm must accept only one output per input.

a) VDF constructions: Among a variety of constructions, VDFs based on repeated squaring have gained more attention as they offer a simple evaluation function that is more compatible with the hardware and provides better accuracy in terms of the time needed to perform the computation. The two concurrent works of [47], [59] suggest evaluating the function $y = x^{2^T}$ over a hidden-order group. Despite similarities in construction, they present two independent ways of proof generation. Particularly, the one proposed by Wesolowski [59] enjoys the luxury of having a constant size proof and verification cost. In addition, Wesolowski's construction can be instantiated over class groups of imaginary quadratic fields [16] which do not require a trusted setup assumption.

E. Verifiable Timed Commitment

Definition 9 (Verifiable Timed Commitment [56])**.** A verifiable timed commitment consists of the following algorithms:

1) $\mathsf{VTC.Setup}(1^\lambda, T) \to pp$, *a probabilistic algorithm that takes a security parameter* 1^λ *and time parameter* T*, and generates public parameters* pp*.*
2) $\mathsf{VTC.Commit}(pp, s) \to \{C, \pi\}$, *a probabilistic algorithm that takes public parameters* pp *and a secret* s*, and generates a commitment* C *and proof* π*.*
3) $\mathsf{VTC.Verify}(pp, pk, C, \pi) \to \{0, 1\}$, *a deterministic algorithm that takes public parameters* pp*, a public key* pk*, the commitment* C*, and proof* π*, and checks if the commitment contains a valid* s *with respect to* pk*.*
4) $\mathsf{VTC.Solve}(pp, C) \to s$, *a deterministic algorithm that takes commitment* C*, and outputs a secret* s*.*

Intuitively, a correct VTC should satisfy *soundness*, ensuring the commitment C indeed embeds a valid secret s with respect to pk, and *privacy*, ensuring that no parallel adversary with a running time of less than T succeeds in extracting s, except with negligible probability.

F. Sigma Protocols

Let $R = \{(v; w)\} \in \mathcal{V} \times \mathcal{W}$ denote a relation containing the pairs of instances and corresponding witnesses. A Sigma protocol Σ on the $(v; w) \in R$ is an interactive protocol with three movements between P and V as follows.

1) $\Sigma.\mathsf{Ann}(v, w) \to a$, runs by P and outputs a message a to V.
2) $\Sigma.\mathsf{Cha}(v) \to c$, runs by V and outputs a message c to P.
3) $\Sigma.\mathsf{Res}(v, w, c) \to r$, runs by P and outputs a message r to V.
4) $\Sigma.\mathsf{Ver}(v, a, c, r) \to \{0, 1\}$, runs by V and outputs 1 if statement holds.

A Sigma protocol has three main properties including *completeness*, *knowledge soundness*, and *zero-knowledge*. Completeness guarantees the verifier gets convinced if parties follow the protocol. Special soundness states that a malicious prover P^* cannot convince the verifier of a statement without knowing its corresponding witness except with a negligible probability. This is formalized by considering an efficient algorithm called *extractor* to extract the witness given a pair of valid protocol transcripts with different challenges showing the computational infeasibility of having such pairs and therefore guaranteeing the knowledge of the witness by P. The notion of zero-knowledge ensures that no information is leaked to the verifier regarding the witness. This is formalized by considering an efficient algorithm called *simulator* which given the instance v, and also the challenge c, outputs a simulated transcript that is indistinguishable from the transcript of the actual protocol execution. Note that this property only needs to hold against an *honest verifier* which seems to be a limitation of the description, but allows for having much more efficient constructions compared to generic models. The interactive protocol described above can be easily turned into a non-interactive variant using the Fiat-Shamir heuristic [30] in the random oracle model, making it publicly verifiable with no honest verifier assumption.

G. Short-lived Proofs

Definition 10 (Short-lived Proofs [5])**.** A short-lived proof scheme includes a tuple of the following algorithms:

1) $\mathsf{SLP.Setup}(1^\lambda, T) \to pp$, *a probabilistic algorithm that takes security parameter* λ *and time parameter* T*, and generates public parameters* pp*.*
2) $\mathsf{SLP.Gen}(pp, v, w, b) \to \pi$, *a probabilistic algorithm that takes a* $(v; w) \in R$ *and a random value* b*, and generates a proof* π*.*
3) $\mathsf{SLP.Forge}(pp, v, b) \to \pi$, *a probabilistic algorithm that takes any instance* v *and a random value* b*, and generates a proof* π*.*
4) $\mathsf{SLP.Verify}(pp, v, \pi, b) \to 1/0$, *a probabilistic algorithm verifying that* π *indeed is a valid short-lived proof of the instance* v*. If verification succeeds, the algorithm outputs 1, and otherwise 0.*

Note that the definition assumes there exists a *randomness beacon* which outputs an unpredictable value b periodically at certain times. There are various ways to implement such beacons including using a public blockchain [15], financial market [25], and more. Such an assumption is necessary to eliminate the need for having a shared global clock (*i.e.,* timestamping). As parties agree on the initial point in time (implied by b), the proof π tied to b must have been observed before time T to be convincing, otherwise might be a forgery.

a) SLP using Sigma protocols.: Short-lived proofs can be instantiated both using generic (non-interactive) zero-knowledge proofs and efficient Sigma protocols. However, as shown in [5], making a Sigma protocol short-lived is rather tricky as it needs some modification in the protocol for OR-composition to be secure according to SLP properties. The modification is done in such a way to let the honest prover create an SLP in a short time without needing to wait for time T to compute the VDF but forces the malicious prover to do the sequential computation, preventing her from computing a forgery before time T. More accurately, in an Or-composition the prover can convince the verifier even if it only knows the witness to one of the relations. To do so, the verifier lets the prover somehow cheat by using the simulator for the relation that it does not know the witness for. Thus, having one degree of freedom the prover chooses two sub-challenges c_1 and c_2 under the constraint that $c_1 + c_2 = c$. Note that the prover is free to fix one of them and compute the other one under the constraints. The observation made in [5] to let the honest prover quickly generate the short-lived proof is to involve the beacon b in the generation of the challenge. Therefore, an honest prover just needs to pre-compute the VDF on a random value b^* allowing her to use it when computing the forgery by freely setting one of the sub-challenges, say c_2, to $b^* \oplus b$ and letting $c_1 = c \oplus c_2$. A malicious prover, however, should compute the VDF on demand as it does not know a witness w for the relation R and c_1 gets fixed by the simulator, taking away the possibility of setting c_2 as specified.

As an optimization, some alternative ways for generating a VDF solution by the honest prover instead of pre-computing a VDF from scratch have been proposed by Arun et al. that we refer the reader to [5] for more details.

H. Proof of Theorem 1

Proof. Correctness is straightforward. The privacy property follows directly from that of the underlying TLP which implies the indistinguishability of a puzzle produced by algorithm $\mathsf{TSS.Sharing}$ and the one produced by Sim. Since all the puzzles are communicated through private channels, no party can learn the other's share after T_1. Finally, the security stems from the underlying threshold secret sharing, where a subset of shares $\mathcal{S}'$ whose size is less than t reveals no information about the secret s. □

I. Proof of Theorem 3

Proof. Correctness is straightforward. The soundness property of the protocol follows directly from that of the underlying Π_{VTC} primitive for every single share s_i committed with respect to the v_i in $\mathbf{v}$. A maliciously generated $\mathbf{v}$ can pass the verification check $\mathsf{VTSS.Verify}_1$ only with probability $1/q$. A maliciously submitted s_i by P_i cannot pass the verification check $\mathsf{VTSS.Verify}_2$, except with negligible probability. The privacy property also follows directly from that of the underlying Π_{VTC} which implies the indistinguishability of a puzzle produced by $\mathsf{VTC.Sharing}$ and the one produced by Sim. Note that the commitment to shares $\mathbf{v}$ does not reveal any information about the secret s under the DL assumption. It is important to note that for the assumption to hold the secret s should have a random distribution. Observe that before T_1 the privacy property essentially implies the security; afterward, the security follows directly from that of Feldman VSS due to the security of the commitment $\mathbf{v}$. □

J. Blinded DLEQ Proof

Proof. We show that the Π_{BDLEQ} satisfies the properties of a Sigma protocol. Completeness holds, as

$$g_1^{r_1} = g_1^{u_1 + c\alpha} = g_1^{u_1} g_1^{c\alpha} = a_1 x^c$$

$$g_2^{r_1} g_3^{r_2} = g_2^{u_1 + c\alpha} g_3^{u_2 + c\beta} = g_2^{u_1} g_3^{u_2} (g_2^{u_1} g_3^{u_2})^c = a_2 y^c$$

For knowledge soundness, given two accepting transcripts $(a_1, a_2; c; r_1, r_2)$ and $(a_1, a_2; c'; r_1', r_2')$ the witness (α, β) can be found as follows

$$g_1^{r_1} = a_1 x^c,\ g_2^{r_1} g_3^{r_2} = a_2 y^c \ \ ; \ \ g_1^{r_1'} = a_1 x^{c'},\ g_2^{r_1'} g_3^{r_2'} = a_2 y^{c'}$$

$$g_1^{r_1 - r_1'} = x^{c - c'} \Leftrightarrow x = g_1^{\frac{r_1 - r_1'}{c - c'}}$$

$$g_2^{r_1 - r'_1} g_3^{r_2 - r_2'} = y^{c - c'} \Leftrightarrow y = g_2^{\alpha} g_3^{\frac{r_2 - r_2'}{c - c'}}$$

Hence, the witness β can be found as $\beta = (r_2 - r_2')/(c - c')$ given the witness $\alpha = (r_1 - r_1')/(c - c')$.
Let c be a given challenge. Zero-knowledge property is implied by the fact that the following two distributions, namely real protocol distribution and simulated distribution, are identically distributed.

$$\mathtt{Real} : \{(a_1, a_2; c; r_1, r_2) : u_1, u_2 \xleftarrow{\$} \mathbb{Z}_q, a_1 = g_1^{u_1}, a_2 = g_2^{u_1} g_3^{u_2}; r_1 = u_1 + c\alpha, r_2 = u_2 + c\beta\}$$

$$\mathtt{Sim} : \{(a_1, a_2; c; r_1, r_2) : r_1, r_2 \xleftarrow{\$} \mathbb{Z}_q; a_1 = g_1^{r_1} x^{-c}, a_2 = g_2^{r_1} g_3^{r_2} y^{-c}\}$$

Note that the probability of occurring for each distribution is the same and equals $1/q^2$. □

K. Security Game for PVSS

Definition 11 (Indistinguishably of Secrets [18])**.** A PVSS is said to be secure if any polynomial time adversary $\mathcal{A}$ corrupting at most t parties has a negligible probability in the following game played against a challenger.

1) *Playing the role of a dealer, the challenger runs the Setup step of the PVTSS and sends all the public information to $\mathcal{A}$. Moreover, it creates the key pairs for the honest parties and sends the corresponding public keys to $\mathcal{A}$.*
2) *$\mathcal{A}$ creates and sends the public keys of the corrupted parties to the challenger.*
3) *The challenger randomly picks the values s and s' in the space of the secret. It then chooses $b \leftarrow \{0, 1\}$ uniformly at random and runs the Sharing step of the protocol with s as secret. It sends $\mathcal{A}$ all public information generated in that phase together with s_b.*
4) *$\mathcal{A}$ outputs a guess b'.*
   The advantage of $\mathcal{A}$ is defined as $|\Pr[b' = b] - 1/2|$.

L. Proof of Theorem 5

Proof. As our protocol follows closely the one in [18], we analyze the security properties with respect to the new techniques we apply.

Before T_2, the correctness is straightforward. Afterward, the correctness may fail with overwhelming probability due to the forgeability and indistinguishability properties of the underlying SLPs together with the uniform distribution of the

Π_{SLP}

1) Initialization: On input a random value b^*, compute $\mathsf{VDF.Eval}(pp, b^*) \to \{y^*, \pi^*_{VDF}\}$
2) Proof generation: $\mathsf{SLP.Gen}(pp, v, w, b) \to \pi$,
 – Compute $\Sigma.\mathsf{Announce}(v, w) \to a$
 – Compute $c = H(v \parallel b \parallel a)$
 – Set sub-challenge $c_2 = b^* \oplus b$
 – Compute sub-challenge $c_1 = c \oplus c_2$
 – Compute $\Sigma.\mathsf{Response}(v, w, a, c_1) \to r$
 – Output $\pi =: \{a, c_1, r, c_2, y^*, \pi^*_{VDF}\}$
3) Forgery: $\mathsf{SLP.Forge}(pp, v, b) \to \tilde{\pi}$,
 – Compute $\Sigma.\mathsf{Simulator}(v) \to (\tilde{a}, \tilde{c_1}, \tilde{r})$
 – Compute $c = H(v \parallel b \parallel \tilde{a})$
 – Set sub-challenge $c_2 = c \oplus \tilde{c_1}$
 – Compute $\mathsf{VDF.Eval}(pp, b \oplus c_2) \to \{y, \pi_{VDF}\}$
 – Output $\tilde{\pi} =: \{\tilde{a}, \tilde{c_1}, \tilde{r}, c_2, y, \pi_{VDF}\}$
4) Proof verification: $\mathsf{SLP.Verify}(pp, v, \pi/\tilde{\pi}, b) \to \{0, 1\}$
 – Compute $c = H(v \parallel b \parallel a)$
 – Accept if:
  • $c = c_1 \oplus c_2$
  • $\Sigma.\mathsf{Verify}(v, a, c_1, r) = 1$
  • $\mathsf{VDF.Verify}(pp, b \oplus c_2, y, \pi_{VDF}) = 1$

Fig. 6: Short-lived proof for a relation $R = \{(v; w)\}$ using pre-computed VDFs [5]

secret s (and thus shares s_i). Anyone observing the public bulletin board after T_2 cannot distinguish an erroneous decryption share $\tilde{s_i}$ from a valid one as both pass the verification check $\mathsf{PVTSS.Verify}_2$. The soundness of the protocol follows from the underlying cut-and-choose argument and BDLEQ's soundness property. Note that by choosing parameters properly the soundness error for the cut-and-choose technique can be negligible in n. The property of t-privacy stems from the fact that given a random set of t opened locked encrypted shares produced by $\mathsf{VTC.Sharing}$, the simulator Sim can produce a locked encrypted share indistinguishable from any locked encrypted share that remained unopened due to the privacy properties of the underlying TLP. Security of the protocol follows directly from the underlying PVSS protocol. Note that blinded encrypted shares c_i distributed by the dealer provide semantic security due to the independent randomness β_i, while the original encryption method used in [18] to generate $\hat{s_i}$ is not IND-CPA-secure. □

M. Proof of Theorem 6

Proof. Correctness is straightforward. Verifiability is implied by the underlying Π_{VTSS} protocol. Privacy follows from that of Π_{VTSS} together with the underlying Π_{MTLP} protocol for additional time-locked shares. Moreover, the commitments to shares $\mathbf{v}$ do not reveal any information about the secret s under the DL assumption. Security is satisfied concerning the gradual release of additional time-locked shares s'_j over time. That is, the adversary can forcibly learn s'_j by $(j+1)T_1$, reducing fault tolerance to $t - j$. The protocol is robust as each party P_i can eventually learn the secret by the time T_2 due to the t additional time-locked shares. □