Key-and-Signature Compact Multi-Signatures for Blockchain: A Compiler with Realizations

Shaoquan Jiang, Dima Alhadidi and Hamid Fazli Khojir

Abstract—Multi-signature is a protocol where a set of signatures jointly sign a message so that the final signature is significantly shorter than concatenating individual signatures together. Recently, it finds applications in blockchain, where several users want to jointly authorize a payment through a multi-signature. However, in this setting, there is no centralized authority and it could suffer from a rogue key attack where the attacker can generate his own public keys. Further, to minimize the storage on blockchain, it is desired that the aggregated public-key and the aggregated signature are both as short as possible. In this paper, we find a compiler that converts a kind of identification (ID) scheme (which we call a linear ID) to a multi-signature so that both the aggregated public-key and the aggregated signature have a size independent of the number of signers. Our compiler is provably secure. The advantage of our result is that we reduce a multi-party problem to a weakly secure two-party problem. We realize our compiler with two ID schemes. The first is Schnorr ID. The second is a new lattice-based ID scheme, which via our compiler gives the first regular lattice-based multi-signature scheme with a key-and-signature size independent of the number of signers without a restart during the signing process.

Index Terms—Blockchain, Multi-Signature, Identification, Lattice, Random Oracle

1 INTRODUCTION

A multi-signature scheme allows a group of signers to jointly generate a signature while no subset of them can represent all the members to generate it. It was first introduced by Itakura and Nakamura [26]. A trivial method is to ask each signer to generate a signature on the message and concatenate their signatures together. However, this is not efficient: (1) the signature size is linear in the number of signers $n$; (2) we need to provide $n$ signer public-keys to verifier; (3) the verification needs to verify $n$ signatures. This indicates that the complexities of the communication, verification and receiver storage are all linear in $n$. In the blockchain setting, this is not desired as the signature will be transmitted, verified and stored in all the complete nodes on the blockchain network. Thus, it is desired to construct a multi-signature scheme that has a signature with these measures independent of $n$.

Early multi-signature schemes [30, 44] assumed all keys including attacker keys are generated honestly. In Bitcoin [41], every user can choose his own public-key. However, this might raise a very serious issue. For example, if a user wants to generate a multi-signature with users of 3 public-keys $g^{x_1}, g^{x_2}, g^{x_3}$, he could choose $s$ randomly and compute his public-key as $pk = g^{s(g^{x_1+x_2+x_3})^{-1}}$. If the aggregated public-key (which is the only public-key provided to the verifier) is the multiplication of the four public-keys, then attacker knows its secret and hence can forge a multi-signature. This is called a rogue key attack. How to construct a key-and-signature compact multi-signature scheme secure against any possible rogue key attack is an important question.

1.1 Related Works

A multi-signature scheme [26] is a special case of aggregate signature [12] where each signer of the latter can sign a possibly different message. In this work, we only discuss a multi-signature scheme with a motivation of blockchain application where the public-key is arbitrary and the target is to minimize the aggregated public-key and signature size. Micali et al. [39] requires an interactive key generation among signers and hence is not suitable. Boldyreva [11] and Lu et al. [32] require signers to add proof of possession (PoP) to their public-keys, which is typically a signature of the user’s public-key. The main disadvantage of this assumption is the increase of the public-key size. In the signing process, it also requires a signer to verify the PoP of all the other signers. In addition, this assumption is not compatible with an ordinary signature where PoP is not required.

Bellare and Neven [8] converted the Schnorr signature [48] into a multi-signature by linearly adding the signature together. Their protocol is of 3-round but without the key aggregation. Bagherzandi et al. [3], Ma et al. [36], Syta et al. [51] and Maxwell et al. [38] attempted to construct a 2-round multi-signature scheme which essentially tries to remove the preliminary committing message which is a hash of the first message in an ID scheme (see [8] for example). However, Drijvers et al. [17] pointed out that all these schemes have proof flaws. They then proved that a slightly modified scheme of Bagherzandi et al. [3] is secure under the PoP assumption. Other 2-round proposals that support the key-and-signature aggregation are due to Alper and Burdges [2] and Nick et al. [42], [43], where Nick et al. [43] employed a generic NIZK proof while the other two proposals [2], [42] are efficient in terms of the size of aggregated key and signature as well as the cost of signature verification (similar to the original Schnorr signature). Boneh et al. [13] proved the security of a modified version of Maxwell et al. [38] via
an added preliminary committing message and hence it is a 3-round scheme. Bellare and Dai [4] proposed a 2-round multi-signature scheme with a tight reduction without the key aggregation.

The above constructions are all based on variants of the discrete logarithm assumption. It is important to find out quantum-resistant schemes while this is not easy. For instance, lattice-based scheme [28] is insecure [31]. Also, the proof for a ring-SIS based scheme [27] is invalid. They reduced to find a short W for ring-SIS problem AW = 0 with public parameter A. However, their obtained W is trivially zero which does not contradict the ring-SIS assumption. Some schemes [14, 21, 22, 37] need an exponential number of restarts during the signing process, due to a noticeable probability of an abort event. Before our work, there is no solution for this (unless a predefined bound on the number of signings is given). The hardness of resolving this restart issue is discussed in [25]. Some schemes [19, 37] are provably secure only when all the keys (including attacker’s keys) are generated honestly which is not suitable for blockchain. Damgård et al. [16] and Fleischhacker et al. [20] do not support key aggregations while the latter can only allow a signer to sign a predefined (polynomial) number of signatures. Thus, currently no multi-signature scheme can support a key-and-signature aggregation without a restart and allow an unlimited number of signing. Since we consider the polynomial time adversary, this “unlimited number” should be understood as any polynomial (that is not predetermined in the system).

1.2 Contribution

In this paper, we consider the key-and-signature compact multi-signature. That is, both key and signature support aggregation and have a size independent of the number of signers. Toward this, we formulate the linear identification scheme (ID) and propose a compiler that transforms a linear ID to a key-and-signature compact multi-signature scheme, where the signature size and the aggregated public-key are independent of the number of signers. The advantage of our compiler is that we reduce the multi-party signature problem to a weakly secure two-party identification problem. This allows researchers to deal with a much simpler problem and potentially to propose more efficient multi-signature schemes. We formulate the linearity of ID via the $\mathcal{R}$-module from algebra. Our compiler is provably secure.

We realize our compiler with two ID schemes. The first is Schnorr ID scheme. The second one is a new ID scheme over ring that is secure under ring-LWE and ring-SIS assumption. Our ID scheme via the compiler gives the first key-and-signature compact multi-signature without a restart during the signing process (see Fig. 1 for a comparison with other schemes), where a signer can do any polynomial number of signing (unlike [20], which can only do a predetermined number of signings). The security of ID schemes is formulated in terms of unforgeability against an aggregated key of multi-users with at least one of them honest. Our ID schemes are proven secure through a new forking lemma (called nested forking lemma). Our forking algorithm has a nested rewinding and is more effective than the previous algorithms which fork at two or more spots sequentially.

2 Preliminaries

Notations. We will use the following notations.

- $x \leftarrow S$ samples $x$ uniformly random from a set $S$.
- For a randomized algorithm $A$, $u = A(x;r)$ denotes the output of $A$ with input $x$ and randomness $r$, while $u \leftarrow A(x)$ denotes the random output (with unspecified randomness).
- We use $\Pr(r)$ to denote the probability $\Pr(R = r)$; for Boolean variable $G$, $\Pr(G)$ means $\Pr(G = 1)$.
- PPT stands for probabilistic polynomial time.
- Min-entropy $H_{\infty}(X) = -\log(\max_x \log P_X(x))$.
- $A|B$ stands for $A$ concatenating with $B$.
- Non-negative function $\text{negl}(\lambda)$ is negligible if $\text{negl}(\lambda) < \lambda^{-k}$ for any constant $k \in \mathbb{N}$ and when $\lambda$ is large enough.
- $\nu$ denotes set $\{1, \ldots, \nu\}$.

2.1 Ring and Module

In this section, we review a math concept: module (for details, see [29]). We start with the concept of ring. A ring $A$ is a set, associated with multiplication and addition operators, respectively written as a product and a sum, satisfying the following conditions:

- R-1. $A$ is a commutative group under addition operator $+$ with the identity element denoted by $0$.
- R-2. $A$ is associative under multiplication operator: for $a, b, c \in A$, $(ab)c = a(bc)$. Also, it has a unit element 1: $1a = a$.
- R-3. It satisfies the distributive law: for $a, b, c \in A$, $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$.

In this paper, we only consider a commutative ring: if $a, b \in A$, then $ab = ba$. That is, when we say ring, it always means a commutative ring. Note that a non-zero element in a ring does not necessarily have a (multiplicative) inverse, where $b$ is an inverse of $a$ if $ab = 1$. For instance, in $\mathbb{Z}_{10}$, 3 is an inverse of 7 while 5 does not have an inverse. If $A$ is a commutative ring with $0 \neq 1$ and every non-zero element in $A$ has an inverse, then $A$ is a field.

Now we introduce the concept module.
Definition 1. Let $R$ be a ring. An Abelian group $M$ (with group operator $\oplus$) is a $R$-module, if (1) it has defined a multiplication operator $\cdot$ between $R$ and $M$: for any $r \in R$, $m \in M$, $r \cdot m \in M$; (2) the following conditions are satisfied: for any $r, s \in R$ and $x, y \in M$,

1. $r \cdot (x \oplus y) = (r \cdot x) \oplus (r \cdot y)$;
2. $(r+s) \cdot x = (r \cdot x) \oplus (s \cdot x)$;
3. $(rs) \cdot x = r \cdot (s \cdot x)$;
4. $1_R \cdot x = x$ with $1_R$ the multiplicative identity of $R$.

We remark that the group operator $\oplus$ for $M$ is not necessarily the regular number addition (e.g., it can be the integer multiplication). One can easily verify that when $R$ is a field, a $R$-module is a vector space. For instance, a vector space $V \subseteq \mathbb{R}^n$ is a $R$-module. In the following, we give some other $R$-modules, where $R$ is not a field.

Example 1. Let $q$ be a prime and $M$ is a group of order $q$ with generator $g$ (i.e., $M = \langle g \rangle$). Examples of $M$ are a subgroup of $\mathbb{Z}_p^*$ or a prime group on an elliptic curve. For $x, y \in M$, $xy$ denotes its group operation in $M$. Then, $M$ is a $\mathbb{Z}_q$-module with $\cdot$ defined as $r \cdot m = r^m$, for $r \in \mathbb{Z}_q$ and $m \in M$. It is well-defined: since $m^q = 1$, any representative $r$ in $\mathbb{Z}_q$ such as $r, r \cdot q$ gives the same result $r \cdot m$. For $r, s \in \mathbb{Z}_q$, $r \cdot s \cdot x$, $y \in M$, we check the module conditions: (1) $s \cdot (xy) = (sx)^y = (s \cdot x) \cdot (s \cdot y)$; (2) $(r+s) \cdot m = m^{r+s} = m^r \cdot m^s = (r \cdot m)(s \cdot m)$; (3) $(rs) \cdot x = x^{rs} = (x^r)(x^s)$; (4) $1 \cdot x = x^1 = x$.

Example 2. For any integer $n > 0$, $M = \mathbb{Z}_n$ (as an additive group) is a $\mathbb{Z}_n$-module, where $\cdot$ is simply the modular multiplication. The verification of module properties is straightforward.

Example 3. Let $n$ be a positive integer. Then, the polynomial ring $M = \mathbb{Z}_n[x]$ (as an additive group) is a $\mathbb{Z}_n$-module with $\cdot$ being the modular $n$-multiplication: for $s \in \mathbb{Z}_n$, $m = \sum_{i=0}^{t} u_i x^i$, $s \cdot m = \sum_{i=0}^{t} u_i s x^i$, where $u_i s$ is the multiplication over $\mathbb{Z}_n$. All the module properties can be straightforwardly verified.

3 NESTED FORKING LEMMA

The original forking lemma was formulated by Pointcheval and Stern [46] to analyze Schnorr signature [48]. It basically shows that if the attacker can forge a Schnorr signature in the random oracle model [7] with a non-negligible probability, then it can generate two forgeries when reminding to the place where the random oracle value was revised. Bellare and Neven [8] generalized the forking lemma to a general algorithm $A$, without resorting to a signature scheme. This was further generalized by Bagherzandi et al. [3] so that $A$ is rewound to many places. However, the algorithm needs $O(n^2 q/\epsilon)$ rewinding, where $q$ is the number of random values in one run of $A$ (which is the number of random oracle queries in typical cryptographic applications) and $\epsilon$ is the successful probability of $A$ when $n$ is the number of rewinding spots. However, this is not efficient and can even be (sub)exponential for a non-negligible $\epsilon$. The main issue comes from the fact the rewinding for each spot is repeated independently until a new success is achieved. But it does not relate different rewindings. In this section, we give a new forking lemma for two rewinding spots (say at index $i, j$ with $i < j$) while it can be generalized to $n$ rewinding spots. The new feature here is that the rewinding is nested. To see this, suppose that the first run of $A$ uses the list of random values: $h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_j, \ldots, h_q$ and the rewinding spots are chosen at index $i$ and $j$. Then, we execute $A$ for another $3$ runs with rewinding that respectively use the following lists of random values:

$$h_1, \ldots, h_{i-1}, h_i, \ldots, h_{j-1}, h_j, \ldots, h_q; \quad (1)$$
$$h_1, \ldots, h_{i-1}, h_i, \ldots, h_{j-1}, h_j, \ldots, h_q; \quad (2)$$
$$h_1, \ldots, h_{i-1}, h_i, \ldots, h_{j-1}, h_j, \ldots, h_q; \quad (3)$$

That is, execution $(1)$ rewinds the initial execution to index $j$; execution $(2)$ rewinds the initial execution to index $i$ while execution $(3)$ rewinds the (rewound) execution $(2)$ to index $j$. With these related executions, we are able to claim that all the rewinding run successfully with probability at least $\Omega(\epsilon^3)$, which is still non-negligible. The advantage of this nested forking is that it can be directly used to extract a secret hidden in recursive random oracle evaluations.

To taste the usefulness of this nested forking, consider the secret extraction task from an attacker’s “forgery” $z = (ax + y) c + r$ (over $\mathbb{F}_q$ for a prime $q$). Assume that $a, r, c$ are computed in this order, where $a, c$ is known but produced by random oracle while $r$ is unknown but produced by attacker. Suppose that $x, y$ are invariant with $x$ being the secret to be extracted. Let $(a, c) = (h_i, h_j)$ in the initial execution. Let $A$ be the algorithm that uses this forger (who outputs $z$) as a subroutine and outputs $i/j[(a, c, z)$. Using the initial and rewinding executions at Eqs. (1)-(3) of $A$, we get the following outputs:

$$i, j, h_i, h_j, z = (h_i x + y) h_j + r \quad (4)$$
$$i, j, h_i, h'_j, z' = (h_i x + y) h'_j + r \quad (5)$$
$$i, j, h_i, h'_j, z = (h_i x + y) h_j + r \quad (6)$$
$$i, j, h_i, h'_j, z = (h_i x + y) h'_j + r \quad (7)$$

We remark that Eq. $(4)(5)$ use the same $r$ as $r$ is determined before computing $h_j$ while these two executions are identical prior to $h_j$ (as we rewind to the spot $h_j$). The rewinding $h$-values are distinct with high probability. So from Eqs (7)(6), we can compute $h_i x + y$; from Eqs (4)(5), we can compute $h_i x + y$. Solving the linear equation gives $x$. We remark that this secret extraction is useful only if all rewinding return the same $j, i$ with $j > i \geq 1$. Our forking lemma claims that this happens with probability $\Omega(\epsilon^3)$.

Our algorithm will use the following notations.

$$h[1, \ldots, q] \triangleq h_1, \ldots, h_q \quad \text{a sequence of elements;}$$

$$h[1, \ldots, i, \ldots, q] \triangleq h_1, \ldots, h_{i-1}, h_i, \ldots, h_q;$$

$$h[1, \ldots, i, \ldots, j+1, \ldots, q] \triangleq h_1, \ldots, h_{i-1}, h_i, \ldots, h_{j+1}, \ldots, h_q.$$ Other variants such as $h[1, \ldots, i, \ldots, j+1, \ldots, q]$ can be defined similarly. Our forking algorithm is in Fig. 2.

Before our forking lemma, we give two facts.

Fact 1. For any random variable $I, R$ and any function $F()$ on $I, R$, we have

$$\Pr(I = i \land F(I, R) = f) = \Pr(I = i \land F(i, R) = f).$$
Lemma 1. Let \( q \geq 2 \) be a fixed integer and \( H \) be a set of size \( N \geq 2 \). Let \( A \) be a randomized algorithm that on input \( x, h_1, \ldots, h_q \) returns a triple, the first two elements of which are integers from \( \{0, 1, \ldots, q\} \) and the last element of which is a side output. Let \( IG \) be a randomized algorithm (called input generator). The accepting probability of \( A \), denoted by \( acc \), is defined as the probability that \( I, J \geq 1 \) in the experiment

\[
x \leftarrow IG; \ h_1, \ldots, h_q \leftarrow H; \\
(I', J, \sigma) \leftarrow A(x, h_1, \ldots, q).
\]

The forking algorithm \( F_A \) associated with \( A \) is a randomized algorithm with input \( x \) that proceeds as in Fig. 2. Let \( frk \) be \( Pr[F_A(x) \neq \text{Fail} : x \leftarrow IG] \). Then,

\[
frk \geq \frac{8 \cdot acc^4}{q^3(q-1)^3} - \frac{3}{N}.
\]

Proof. With respect to \( Flag_1 \), we define \( Flag_1^* \) as event

\[
(I_0 = \cdots = I_3 \geq 1) \land (J_0 = \cdots = J_3 \geq 1) \land (J_0 > I_0).
\]

Then, it is easy to check that \( F_A(x) \neq \text{Fail} \) is equivalent to \( Flag_1^* \land Flag_2 = 1 \). Since \( h_{I_0} = h_{I_0} \) (resp. \( h_{J_0} = h_{J_0} \) or, \( h_{I_0} = h_{J_0} \)) in \( Flag_2 \) holds with probability \( 1/N \), we have

\[
frk = Pr(Flag_1^* \land Flag_2 = 1) \\
\geq Pr(Flag_1^*) - 3/N.
\]

Notice that

\[
Pr(Flag_1^*) = 1
\]

\[
= \sum_{i=1}^{q} \sum_{j=i+1}^{q} Pr(\wedge_{k=0}^{3} \{I_b = i \land J_b = j\}).
\]

Let \( A_1 \) (resp. \( A_2, A_{12} \)) be three variants of algorithm \( A \) with the only difference in the output which is the first element (resp. the second element, the first two elements) of \( A \)'s output. For instance, keeping symbols in \( F_A \).

\[
J_1 = A_2(x, h_1, \ldots, j_0 - 1, \ldots, q_0); \rho;
\]
\[
I_2 = A_1(x, h_1, \ldots, i_0 - 1, \ldots, q_0); \rho.
\]

Assigning \( I_0 = i \) and \( J_0 = j \), we denote

\[
J_1' = A_2(x, h_1, \ldots, j_0 - 1, \ldots, q) \rho;
\]
\[
I_2' = A_1(x, h_1, \ldots, i_0 - 1, \ldots, q) \rho.
\]

We can similarly define \( I_1', I_2', I_3', I_1'' \). So \( I_0, J_0 \) for \( b \geq 1 \) is a function of \( A \)'s inputs and randomness and when assigning \( I_0 = i \) and \( J_0 = j \), they become \( I_0', J_0' \). Hence, we can apply fact 1 to evaluate Eq. (10). This gives

\[
Pr(Flag_1^*) = 1
\]

\[
= \sum_{i=1}^{q} \sum_{j=i+r}^{q} Pr(\wedge_{k=0}^{3} \{I_b = i \land J_b = j\}).
\]

where \( I_0, J_0 \) is rewritten as \( I_0', J_0' \) (note: \( I_0', J_0' \) is undefined above) for brevity (so the term \( \{I_0 = i \land J_0 = j\} \) becomes \( \{I_0' = i \land J_0' = j\} \)). Notice \( \wedge_{b=0}^{3} \{I_b = i \land J_b = j\} \) is a random variable, with randomness \( (R, B) \), where \( R := (x, \rho, h_1, \ldots, h_{i-1}) \) and \( B := (h_i, \ldots, h_q, h_j, \ldots, h_q) \). So we can define

\[
\wedge_{b=0}^{3} (I_b' = i \land J_b' = j) = G(R, B)
\]

for some boolean function \( G \).

Besides, by checking the inputs of \( I_0', J_0' \), we can see that

\[
\wedge_{b=2}^{3} (I_b' = i \land J_b' = j) = G(R, B')
\]
with $B = (h_1, \ldots, h_q, b_1, \ldots, b_q)$.

Hence, applying Fact 2 to Eq. (16), we have

$$
\Pr(\text{Flag}^*_i = 1) = \sum_{1 \leq i < j \leq q} P_R(r) \Pr^2(I_{br} = i \land J_{br} = j) \\
= \sum_{1 \leq i < j \leq q} P_R(r) \Pr^2(I_{br} = i \land J_{br} = j) = \sum_{1 \leq i < j \leq q} P_R(r) \Pr^2(I_{br}, J_{br} = (i, j)). 
$$

(19)

where $I_{br}^*$ (resp. $J_{br}^*$) is $I_i^*$ (resp. $J_i^*$) with $R = r$.

Notice that $(I_{br}^*, J_{br}^*) = (i, j)$ is a boolean random variable (i.e., the result is true only if the equality holds), determined by $h_i, \ldots, h_q$. We can define

$$
G'(S, C) \overset{\text{def}}{=} \{(I_{br}^*, J_{br}^*) = (i, j)\}
$$

(20)

for some function $G'$, where $S = h_i, \ldots, h_{j-1}$ and $C = h_j, \ldots, h_q$.

Since the input of $(I_{br}^*, J_{br}^*)$ is $S$ and $(h_j, \ldots, h_q)$,

$$
\{(I_{br}^*, J_{br}^*) = (i, j)\} = G'(S, C')
$$

(21)

with $C' = h_j, \ldots, h_q$.

Thus, Eq. (19) is

$$
\Pr(\text{Flag}^*_i = 1) = \sum_r P_R(r) \Pr^2(G'(S, C) \land G'(S, C')).
$$

(22)

Hence, we can apply Fact 2 to Eq. (22) and obtain

$$
\Pr(\text{Flag}^*_i = 1) = \sum_{1 \leq i < j \leq q} P_R(r) \sum_s P_S(s) \Pr^2((I_{br}^*, J_{br}^*) = (i, j))^2 \\
\geq \sum_{1 \leq i < j \leq q} \sum_{r,s} P_R(r) P_S(s) \Pr^2((I_{br}^*, J_{br}^*) = (i, j))^2 \\
\geq \sum_{1 \leq i < j \leq q} \sum_{r,s} \Pr(((I_0^*, J_0^*) = (i, j))^4 \\
\geq \sum_{1 \leq i < j \leq q} \Pr(((I_0^*, J_0^*) = (i, j))^4 / (q^3(q - 1)^3/2^3)
$$

where $(I_{br}^*, J_{br}^*) = (I_{br}, J_{br})$ with $S = s$, the first two inequalities follow from Cauchy-Schwarz inequality\(^1\) (the first one is over distribution $P_R(\cdot)$ and the second one is over distribution $P_R(\cdot)P_S(\cdot)$); the last inequality is to apply Cauchy-Schwarz inequality $\sum_{i=1}^n x_i^2 \geq (\sum_{i=1}^n x_i)^2/n$ twice by noticing that $y_j^2 = (y_i^2)^2$ so that the first time we use $x_i = y_i^2$ for Cauchy-Schwarz inequality. Finally, notice that $I_0^* = I_0$ and $J_0^* = J_0$ by definition. Also, $\sum_{1 \leq i < j \leq q} \Pr(((I_0^*, J_0^*) = (i, j))$ is exactly acc by definition. It follows that $\Pr(\text{Flag}^*_i = 1) \geq \frac{\text{acc}^4}{q^3(q - 1)^3}/2^3$. From Eq. (9), we have $\frac{\text{acc}}{q} \geq \frac{8\text{acc}}{q^3(q - 1)^3}/3/\sqrt{N}$. □

1. $\sum_i p_i x_i^2 \geq (\sum_i p_i x_i)^2$, if $p_i \geq 0$ and $\sum_i p_i = 1$

4 Model of Multi-Signature

In this section, we introduce the model of multi-signature. It consists of the multi-signature definition and the security formalization.

4.1 Syntax

Multi-signature is a signature with a group of signers, where each of them has a public-key and a private key. They jointly generate a signature. The interaction between them proceeds in rounds. Signers are pair-wise connected but the channel is not secure. The signing protocol is to generate a signature so that the successful verification would indicate that all signers have agreed to sign the message. The target is to generate a compact signature that is shorter than concatenating all signers’ individual signatures together.

**Definition 2.** A multi-signature is a tuple of algorithms (Setup, KeyGen, Sign, Verify), described as follows.

**Setup.** Given security parameter $\lambda$, it generates a system parameter $\text{param}$ that serves as part of the input for KeyGen, Sign, Verify (but for brevity, we omit it).

**KeyGen.** It takes $\text{param}$ as input and outputs for a user a private key $sk$ and a public-key $pk$.

**Sign.** Assume $n$ users with public-keys $(pk_1, \ldots, pk_n)$ want to jointly sign a message $M \in \{0, 1\}^*$. Then, each user $i$ takes its private key $sk_i$ as input and interacts with other signers. Finally, each of them outputs a signature $\sigma$ (note: this is for simplicity only; in literature, usually a designated leader outputs $\sigma$). Besides, there is a function $F$ that aggregates $(pk_1, \ldots, pk_n)$ into a compact public-key $pk = F(pk_1, \ldots, pk_n)$.

**Verify.** Upon $(\sigma, M)$ with the aggregated public-key $pk = F(pk_1, \ldots, pk_n)$, verifier takes $\sigma, M$ and $pk$ as input, outputs 1 (for accept) or 0 (for reject).

**Remark.** The verify algorithm only uses the aggregated key $pk$ to verify the signature. This is important for blockchain, where the recipient only uses $pk$ as the public-key. Also, the redeem signature only uses the multi-signature $\sigma$. It is desired that both $pk$ and $\sigma$ are independent of $n$ while no attacker can forge a valid signature w.r.t. this short $pk$. Even though, our definition generally does not make any restriction on $pk$ and it especially can be $(pk_1, \ldots, pk_n)$.

4.2 Security Model

In this section, we introduce the security model [13] of a multi-signature. It formalizes the existential unforgeability. Essentially, it says that no attacker can forge a valid signature on a new message as long as the signing group contains an honest member. Toward this, the attacker can access to a signing oracle and create fake public-keys at will. The security is defined through a game between a challenger CHAL and an attacker $A$.

Initially, CHAL runs Setup($1^\lambda$) to generate system parameter $\text{param}$ and executes KeyGen to generate a public-key $pk^*$ and a private key $sk^*$. It then provides $pk^*$ [param to $A$ who interacts with CHAL through signing oracle below.
Sign $O_s(PK, M)$. Here $PK$ is a set of distinct public-keys with $pk^* \in PK$ and $M \in \{0,1\}^*$ is any message. Upon this query, CHAL represents $pk^*$ and $A$ represents $PK\setminus\{pk^*\}$ to run the signing protocol on message $M$. Finally, $O_s$ outputs the multi-signature $\sigma$ (if it succeeds) or $\perp$ (if it fails).

Forgery. Finally, $A$ outputs a signature $\sigma^*$ for a message $M^* \in \{0,1\}^*$, w.r.t. a set of distinct public-keys $(pk^*_1, \ldots, pk^*_N)$ s.t. $pk^*_i = pk^*$ for some $i$. A succeeds if two conditions are met: (a) Verify$(pk^*, \sigma^*, M^*) = 1$ (where $pk^* = F(pk^*_1, \ldots, pk^*_N)$); (b) no query $(pk^*_1, \ldots, pk^*_N, M^*)$ was issued to $O_s$. Denote a success forgery event by succe.

We remark that since $M^*$ are proposed by attacker, it might be arbitrarily correlated to $pk^*_1, \ldots, pk^*_N$. Now we can define the security of a multi-signature.

Definition 3. A multi-signature scheme $(\text{Setup}, \text{KeyGen}, \text{Sign}, \text{Verify})$ is existentially unforgeable against chosen message attack (or EU-CMA for short), if satisfies the correctness and existential unforgeability below.

- Correctness. For $(sk_1, pk_1), \ldots, (sk_n, pk_n)$ generated by KeyGen, the signature generated by signing algorithm on a message $M$ will pass the verification, except for a negligible probability.
- Existential Unforgeability. For any PPT adversary $A$, $\Pr(\text{succe}(A))$ is negligible.

The multi-signature scheme is said $t$-EU-CMA, if it is EU-CMA w.r.t. adversary $A$ who always restricts the number of signers, in each signing query and also in the final forgery, to be at most $t$.

5 Model of Canonical Linear Identification

In this section, we introduce a variant model of canonical identification (ID) scheme and extend it with linearity. We label the ID scheme with a parameter $\tau$. This is needed later to cover our lattice-based ID scheme as an example ID for realizing our multi-signature method.

Definition 4. A canonical identification scheme with parameter $\tau \in \mathbb{N}$ is a tuple of algorithms $ID = (\text{Setup}, \text{KeyGen}, P, V, \Theta)$, where $\text{Setup}$ takes security parameter $\lambda$ as input and generates a system parameter $\text{param}$; $\text{KeyGen}$ is a key generation algorithm that takes $\text{param}$ as input and outputs a public key $pk$ and a private key $sk$; $P$ is an algorithm, executed by prover; $V$ is a verification algorithm parameterized by $\tau$, executed by Verifier; $\Theta$ is a set. $ID$ scheme is a three-round protocol depicted in Fig. 3, where Prover first generates a committing message $CMT$ with $H_{\infty}(CMT) = \omega(\log \lambda)$, and then Verifier replies with a challenge $CH \leftarrow \Theta$ and finally Prover finishes with a response $Rsp$ which will be either rejected or accepted by $V_r$.

Denote the domain of $sk, pk, CMT, Rsp$ respectively by $SK, PK, CMT, RSP$. In the following, we define linearity and simulability for an ID scheme. Simulatability follows from [1]. The linearity property is new.

Remark. Many authors (e.g., [1]) formalized the ID scheme so that $CMT$ depends on the prover’s key pair $(pk, sk)$.

In our formulation, it only depends on system parameter $\text{param}$. This is consistent with some existing popular ID schemes such as Schnorr ID [48]. This allows us to generate $CMT$ without having $pk$ fixed first. Especially, we can generate $CMT$ that is compatible with both a dummy public-key 0 and a real public-key $pk$. We will use this flexibility to avoid the abort event in a lattice-based ID (while an abortion event has been a serious issue in the literature [1], [25]).

Linearity. A canonical ID scheme $ID = (\text{Setup}, \text{KeyGen}, P, V, \Theta)$ is linear if it satisfies the following conditions.

i. $SK, PK, CMT, RSP$ are $\mathcal{R}$-modules for some ring $\mathcal{R}$ with $\Theta \subseteq \mathcal{R}$ (as a set);
ii. For any $\lambda_1, \ldots, \lambda_t \in \Theta$ and public/private pairs $(sk_i, pk_i) (i = 1, \ldots, t)$, we have that $sk = \sum_{i=1}^t \lambda_i \cdot sk_i$ is a private key of $pk = \sum_{i=1}^t \lambda_i \cdot pk_i$.

Note: Operator $\cdot$ between $\mathcal{R}$ and $SK$ (resp. $PK, CMT, RSP$) might be different. But we use the same symbol $\cdot$, as long as it is clear from the context.

iii. Let $\lambda_i \leftarrow \Theta$ and $(pk_i, sk_i) \leftarrow \text{KeyGen}(1^n)$, for $i = 1, \ldots, t$. If $CMT|CH|Rsp$ is a faithfully generated transcript of the ID scheme w.r.t. $pk_i$, then with probability $1 - \text{negl}(\lambda)$,

$$V_r(pk, CMT|CH|Rsp) = 1,$$

where $pk = \sum_{i=1}^t \lambda_i \cdot pk_i, CMT = \sum_{i=1}^t \lambda_i \cdot CMT_i$ and $Rsp = \sum_{i=1}^t \lambda_i \cdot Rsp_i$.

Note: we require Eq. (23) to hold only if the keys and transcripts are faithfully generated. If some are contributed by attacker, this equality might fail. This property will only be used to guarantee the correctness of our multi-signature framework. That is, if all signers are honest, they will generate the multi-signature that passes the verification. If it includes a dishonest player, then there is no guarantee for the signature validity. This is fine as an attacker can have many ways to make the output invalid.

Simulatability. $ID$ is simulatable if there exists a PPT algorithm $\text{SIM}$ s.t. for $(sk, pk) \leftarrow \text{KeyGen}(1^n)$, $CH \leftarrow \Theta$ and $(CMT, Rsp) \leftarrow \text{SIM}(CH, pk, \text{param})$, it holds that $CMT|CH|Rsp$ is indistinguishable from a real transcript, even if the distinguisher is given $pk|\text{param}$ and has access to oracle $O_{id}(sk, pk)$, where $O_{id}(sk, pk)$ is as follows: $(st, CMT) \leftarrow P(\text{param})$, $CH \leftarrow \Theta$, $Rsp \leftarrow P(st|sk|pk, CH)$, output $CMT|CH|Rsp$.

Now we define the security for an ID scheme. Essentially, it is desired that an attacker is unable to impersonate a prover w.r.t. an aggregated public-key, where at least one of the participating public-keys is generated honestly. Later we will convert an ID scheme with this security into a EU-CMA secure multi-signature. In our definition, the prover does not access to additional information. He is not given extra capability, either. Thus, our definition is rather weak.

Definition 5. A canonical identification scheme $ID = (\text{Setup}, \text{KeyGen}, P, V, \Theta)$ with linearity and $\tau \in \mathbb{N}$ is secure if it satisfies correctness and security below.

Correctness. When no attack presents, Prover will convince Verifier, except for a negligible probability.
Security. For any PPT adversary $A$, $Pr(\text{Exp}_{ID,A} = 1)$ is negligible, where $\text{Exp}_{ID,A}$ is defined below with $pk_i \in PK$ for $i \in [t]$ and $\overline{pk} = \sum_{i=1}^{t} \lambda_i \cdot pk_i$.

**Experiment** $\text{Exp}_{ID,A}(\lambda)$

- $\text{param} \leftarrow \text{Setup}(1^\lambda)$;
- $(pk_1, sk_1) \leftarrow \text{KeyGen}(\text{param})$;
- $(st_0, pk_2, \ldots, pk_t) \leftarrow A(\text{param}, pk_1)$

Let $\lambda_1, \ldots, \lambda_t \equiv \Theta$

$st_1$,$CMT \leftarrow A(st_0, \lambda_1, \ldots, \lambda_t)$;

$CH \leftarrow \Theta$ $Rsp \leftarrow P(st|sk|pk, CH)$

Output. Upon $Rsp_j$, $j = 1, \ldots, t$, user $i$ computes $Rsp_i = \sum_{j=1}^{t} \lambda_j \cdot Rsp_j$, and outputs the aggregated public-key $\overline{pk}_i$ and multi-signature $CMT_i|CH|Rsp$.

**Verify.** Upon signature $(CMT, Rsp)$ on message $M$ with the aggregated public key $\overline{pk}_i$, it outputs $V_j(\overline{pk}_i, CM|CH|Rsp)$, where $CH = H_1(\overline{pk}_i|CM|M)$.

Correctness. This states that when $\{(sk_i, pk_i)\}_{i=1}^{t}$ are honestly generated and signers faithfully execute the signing protocol, then the resulting multi-signature $(CMT, Rsp)$ will pass the verification. This directly follows from linearity (iii).

Remark. (1) Since $\overline{pk}_i$ is the aggregated public-key, we assume that it will be correctly computed and available to verifier, which is true for the Bitcoin application. (2) The most damaging attack to a multi-signature is the rogue key attack, where an attacker chooses his public-key after seeing other signers’ public-keys. By doing this, the attacker could manage to reach an aggregated key for which he knows the private key. In our construction, attacker can not achieve this. To see this, let us assume that $PK = \{pk_1, \ldots, pk_n\}$ is the set of public-keys for the multi-signature with all but $pk_n$ are generated by attacker. Hence, $\overline{pk} = H_0(pk_n, PK) \cdot pk_n + \sum_{i=1}^{n-1} H_0(pk_i, PK) \cdot pk_i$. The hash-value weights can be computed only after $PK$ has been determined. Since $pk_n$ is the honest user’s key, it is quite random. So, $H_0(pk_n, PK)$ (hence $H_0(pk_n, PK) \cdot pk_n$ and also $pk_n$) will be random, given other variables in $\overline{pk}$. So it is unlikely that attacker can predetermine $\overline{pk}$ and so the rogue key attack can not succeed.

### 6 From Canonical Linear ID Scheme to Key-and-Signature Compact Multi-Signature

In this section, we show how to convert a linear ID scheme into a multi-signature so that the aggregated public-key and signature are both compact. The idea is to linearly add the member signatures (resp. public-keys) with individual weights where weights depend on all public-keys.

#### 6.1 Construction

Let $ID = (\text{Setup}_id, \text{KeyGen}_id, P, V_r, \Theta)$ be a canonical linear ID with parameter $\tau \in \mathbb{N}$. $H_0, H_1$ are two random oracles from $\{0,1\}^*$ to $\Theta$ with $\Theta \subseteq R$, where $R$ is the ring defined for the linearity property of $ID$. Our multi-signature scheme $\Pi = (\text{Setup, KeyGen, Sign, Verify})$ is as follows.

**Setup.** Sample and output $\text{param} \leftarrow \text{Setup}_id(1^\lambda)$. Note: $\text{param}$ should be part of the input to the algorithms below. But for brevity, we omit it from now.

**KeyGen.** Sample $(pk, sk) \leftarrow \text{KeyGen}_id(\text{param})$; output a public-key $pk$ and private key $sk$.

**Sign.** Suppose that users with public-keys $pk_i, i = 1, \ldots, t$ want to jointly sign a message $M$. Let $\lambda_i = H_0(pk_i, PK)$ and $\overline{pk} = \sum_{i=1}^{t} \lambda_i \cdot pk_i$, where $PK = \{pk_1, \ldots, pk_t\}$. They run the following procedure.

- R-1. User $i$ takes $(st_i, CMT_i) \leftarrow P(\text{param})$ and sends $r_i := H_0(CMT_i|pk_i)$ to other users.
- R-2. Upon $r_j$, for all $j$ (we do not restrict $j \neq i$ for simplicity), user $i$ sends $CMT_i$ to other users.
- R-3. Upon $CMT_j, j = 1, \ldots, t$, user $i$ verifies if $r_j = H_0(CMT_j|pk_j)$. If no, it aborts; otherwise, it computes $\overline{CMT} = \sum_{j=1}^{t} \lambda_j \cdot CMT_j$ and also $CH = H_1(\overline{pk} \cdot CMT_j|M)$. Finally, it computes $Rsp_i = P(st_i|sk_i|pk_i, CH)$ and sends it to other signers.
- R-4. Upon $Rsp_j$, $j = 1, \ldots, t$, user $i$ computes $Rsp_i = \sum_{j=1}^{t} \lambda_j \cdot Rsp_j$ and outputs the aggregated public-key $\overline{pk}_i$ and multi-signature $CMT_i|CH|Rsp$.

**Verify.** Upon signature $(CMT, Rsp)$ on message $M$ with the aggregated public key $\overline{pk}_i$, it outputs $V_j(\overline{pk}_i, CM|CH|Rsp)$, where $CH = H_1(\overline{pk}_i|CM|M)$.

#### 6.2 Security Theorem

In this section, we prove the security of our scheme. The idea is as follows. We notice that the multi-signature is $(CMT, Rsp)$ that satisfies $V_j(\overline{pk}_i, CM|CH|Rsp) = 1$, where $CH = H_0(\overline{pk} \cdot CM|M)$. Assume $PK = \{pk_1, \ldots, pk_t\}$, where $pk_i$ is an honest user’s key and other keys are created by attacker. We want to reduce the multi-signature security to the security of ID scheme. In this case, $\overline{pk}$ will be the aggregated key with weights $\lambda_i = H_0(pk_i, PK)$. If an attacker can forge a multi-signature with respect to $\overline{pk}$, we want to convert it into an impersonation attack to the ID scheme w.r.t. $\overline{pk}$. There are two difficulties for this task. First, we need to simulate the signing oracle without $sk_1$, where we have to compute the response $Rsp$ for user of $pk_1$ without $sk_1$. Our idea is to use the simulability of the ID scheme to help: take a random CH and simulate an ID
transcript $\text{CMT}'[\text{CH}]\text{Rsp}'$. Then, we send $\text{CMT}_{1} = \text{CMT}'$ as the committing message. The simulation will be well done if we can manage to define $\text{CH}$ as $H_{1}(pk)[\text{CMT}]M$. This will be fine if $pk[\text{CMT}]M$ was never queried to $H_{1}$ oracle. Fortunately, this is true with high probability: due to the initial registration message at round $R-1$, attacker can not know $\text{CMT}_{1}$ before registering $\text{CMT}_{1}$ using $r_{1}j$ (hence $\text{CMT}_{j}$ is known to us through oracle $H_{0}$). Hence, $\text{CH}$ will have a min-entropy of $H_{\infty}(\text{CMT}_{1})$, which is super-logarithmic and hence can not be guessed. That is, $pk[\text{CMT}]M$ was unlikely to be queried to $H_{1}$ before. Hence, the signing oracle will be simulated without difficulty. The second difficulty is how to convert the forgery into an impersonating attack. In the ID attack, $\text{CH}$ is provided by challenger while in the forgery, $\text{CH}$ is the hash value from $H_{1}$. The attacker could make a query $pk[\text{CMT}]M$ to $H_{1}$ oracle. However, the problem is that we do not know if this query is used by attacker for his final forgery output or not. Hence, we do not know which $\text{CMT}$ should be sent to our ID challenger and consequently we do not know which $H_{1}$-query should be answered with our challenger’s $\text{CH}$. Fortunately, this is not a big issue as we can guess which $H_{1}$-query from attacker will be used for his forgery. There are a polynomial number of such queries. Our random guess only degrades the success probability by a polynomial fraction. This completes our idea. Now we give full details below.

**Theorem 1.** Let $H_{0}, H_{1} : \{0, 1\}^{*} \rightarrow \Theta$ be random oracles.

Assume that $h \leftarrow \Theta$ is invertible in $\mathcal{R}$ with probability $1 - \text{neg}(\lambda)$. Let $\mathcal{T}D = (\text{Setup}_{id}, \text{KeyGen}_{id}, P_{\nu}, \Theta)$ be a secure ID with linearity and simulability. Then, our multi-signature scheme is $\text{EU-CMA}$ secure.

**Proof.** We show that if the multi-signature is broken by $\mathcal{D}$ with non-negligible probability $\epsilon$, then we can construct an attacker $\mathcal{B}$ to break $\mathcal{T}D$ scheme with a non-negligible probability $\epsilon'$. Given the challenge public-key $pk_{1}^{*}$, $\mathcal{B}$ needs to come up with some other public-keys $pk_{2}^{*}, \ldots, pk_{d}^{*}$ for some $\nu$ of his choice and receives a list of random numbers $\lambda_{1}^{*}, \ldots, \lambda_{\nu}^{*}$ from $\mathcal{D}$ to convince the verifer (his challenger). Toward this, $\mathcal{B}$ will simulate an environment for $\mathcal{D}$ and use the responses from $\mathcal{D}$ to help complete his attack activity. The details follow.

Upon receiving the challenge public-key $pk_{1}^{*}$ and system parameter $\text{param}$, $\mathcal{B}$ samples $t_{\nu} \leftarrow \{0, 1\}^{\nu}$, where $q_{H_{0}}^{*}$ is the upper bound on the number of new queries (i.e., not queried before) of format $(pk, PK)$ to random oracle $H_{0}$ s.t. $pk, pk_{1}^{*} \in PK$ (call it a *Type-I irregular query*). In addition, a new query of format $\text{CMT}[pk^{*}]M$ to oracle $H_{1}$ after the $t_{\nu}$th Type-I irregular query will be called a *Type-II irregular query*, where $\text{CMT} \in \text{CMT}_{0}, pk^{*} = \sum_{i=1}^{\nu} H_{0}(pk_{i}^{*}, PK^{*}) \bullet pk_{i}^{*}$ and $PK^{*} = \{pk_{1}^{*}, \ldots, pk_{\nu}^{*}\}$ is the public-key set for the $t_{\nu}$th Type-I irregular query. Let $q_{\nu}^{*}$ be the upper bound on the number of the Type-II irregular queries. It then samples $t_{\nu} \leftarrow \{0, 1\}^{\nu} \cdot q_{\nu}^{*}, \mathcal{B}$ invokes $\mathcal{D}$ with $pk_{1}^{*}$ and $\text{param}$ and answers his random oracle queries and signing queries as follows.

**Random Oracle $H_{1}(\cdot)$**. For simplicity, we maintain one random oracle $H$ with $H_{0}(x) = H(0, x)$ and $H_{1}(x) = H(1, x)$. The query $x$ to $H_{0}$ is automatically interpreted as query $b|x$ to $H$. With this in mind, it maintains a hash list $H_{L}$ (initially empty), consisting of records of form $(u, y)$, where $y = H(u)$. Upon a query $b|x$, it first checks if there was a record $(b, y, x)$ in $H_{L}$ for some $y$. If yes, it returns $y$; otherwise, there are three cases (all irregular queries will be in these cases as they were not queried by definition).

- **x is not a (Type-I or Type-II) irregular query to $H_{0}$**. In this case, it takes $y \leftarrow \Theta$ and adds $(b, y, x)$ into $H_{L}$.
- **x is a Type-I irregular query to $H_{0}$ (thus $b = 0$)**. In this setting, there are two cases.

- **x is not the $t_{\nu}$th irregular query**. In this case, for each $pk' \in PK$, it takes $h \leftarrow \Theta$ and adds $(0, (pk', PK), h)$ into $H_{L}$. Note for convenience, we treat each new record in $H_{L}$ as created due to a hash query (from either simulator $\mathcal{B}$ or $\mathcal{D}$). For the technical reason, for given $PK$ with $pk_{1}^{*} \in PK$, we treat $(0, (pk_{1}^{*}, PK), \ast)$ as the last record created in $H_{L}$ among all records of $(0, (pk', PK), \ast)$ with $pk' \in PK$.

  Our treatment is well-defined and perfectly consistent with random oracle, as our treatment on Type-I irregular query has a convention: all records of $(pk', PK)$ with $pk', pk_{1}^{*} \in PK$ will be recorded in $H_{L}$ simultaneously whenever it receives a Type-I irregular query (which is $0|x$ in our case).

- **x is the $t_{\nu}$th irregular query**. In this case, let $0|x = 0x, (pk, PK^{*})$ with $PK^{*} = \{pk_{1}^{*}, \ldots, pk_{\nu}^{*}\}$ for some $\nu \geq 2$. $\mathcal{B}$ sends $(0, (pk_{1}^{*}, PK^{*}), \lambda_{1}^{*})$ into $H_{L}$ for $i = 1, \ldots, \nu$: This treatment is perfectly consistent with random oracles: a Type-I irregular query by definition is an *unrecorded* query (i.e., not queried before) and $(0, (pk', PK^{*}))$ for each $pk' \in PK^{*}$ will be recorded in $H_{L}$ within one hash query (thus none of them was queried before).

- **x is a Type-II irregular query to $H_{0}$ (thus $b = 1$)**. In this setting, there are two cases.

  - **x is not the $t_{\nu}$th Type-II irregular query**. In this case, it takes $y \leftarrow \Theta$ and adds $(1, x, y)$ into $H_{L}$.

  - **x is the $t_{\nu}$th Type-II irregular query**. In this case, it parses $x = pk\text{CMT}M$ with $CMT^{*} \in CMT$. Then, it sends $CMT^{*}$ to its challenger and receive $CH^{*}$. Then, it adds $(1, x, CH^{*})$ to $H_{L}$.

After our treatment above, $x$ now has been recorded in $H_{L}$. Then, the oracle returns $y$ for $(b, x, y)$ in $H_{L}$.

**Sign $\text{O}_{\nu}(pk_{1}, \cdots, pk_{\nu}, M)$**. By our security model, we can assume that $pk_{1}^{*} = pk_{i}$ for some $t$. Then, $\mathcal{B}$ plays the role of user $pk_{i}$ while $\mathcal{D}$ plays users of $pk_{j}$ for $j \neq t$ in the signing algorithm. The action of $\mathcal{B}$ is as follows.

- **R-1.** $\mathcal{B}$ generates $r_{1} \leftarrow \Theta$ and sends to other signers (played by $\mathcal{D}$).

- **R-2.** Upon $(r_{j})_{j \neq t}$ from $\mathcal{D}$, $\mathcal{B}$ first issues hash queries $(pk_{i}, PK)$ for each $pk_{i} \in PK$ to compute $\lambda_{i} = H_{0}(pk_{i}, PK)$, where $PK = \{pk_{1}, \ldots, pk_{\nu}\}$. Then, it computes $pk\text{CMT}^{*}$, takes $h \leftarrow \Theta$ and runs $\text{SIM}(h, pk_{i}, \text{param})$ to simulate an ID transcript $(\text{CMT}', h, \text{Rsp}')$. Then, he defines $\text{CMT}_{i} = \text{CMT}'$. He
as well as adds \( (0|\text{CMT}_t|pk_t,r_t) \) into \( L_H \) if \( (0|\text{CMT}_t|\ast) \) is not recorded in \( L_H \); and otherwise, it aborts with \( \bot \) (denoted by event \( \text{Bad}_0 \)). Next, for each \( j \neq t \), it searches a record \( (0|\text{CMT}_j|pk_j,r_j) \) in \( L_H \) for some \( \text{CMT}_j \), which results in two cases.

(i) If \( (0|\text{CMT}_j|pk_j,r_j) \) for all \( j \neq t \) are found in \( L_H \), it computes \( \text{CMT} = \sum_{i=1}^n \lambda_i \cdot \text{CMT}_i \) and checks whether \( (1|pk|\text{CMT}^*|M,y) \in L_H \) for some \( y \). If this does not exist, it records \( (1|pk|\text{CMT}^*|M,h) \) into \( L_H \) and defines \( CH = h \) and sends \( \text{CMT}_j \) to \( D \); otherwise (denote this event by \( \text{Bad}_1 \)), \( B \) aborts with \( \bot \).

(ii) If \( (0|\text{CMT}_j|pk_j,r_j) \) does not exist in \( L_H \) for some \( j^* \), it sends \( \text{CMT}_i \) to \( D \) (normally). However, we remark that \( \text{CMT}_j \), later in Step R-3 (from \( j^* \)) satisfies \( H_0(\text{CMT}_j,pk_j,r_j) = r_j \) (which will be checked there) only negligibly (so this case will not raise a simulation difficulty), as the hash value is even undefined yet and hence equals \( r_j \) with probability \( 1/|\Theta| \) only (which will be ignored from now).

- **R-3.** Upon \( \{\text{CMT}_j\}_{j \neq t} \), \( B \) checks if \( H_0(\text{CMT}_j|pk_j) = r_j \) for each \( j \). If it does not hold for some \( j \), \( B \) outputs \( \bot \) (normally); otherwise, it sends \( \text{Rsp}_j := \text{Rsp}^* \) to \( D \). We clarify two events: (1) some \( \text{CMT}_j \) found in R-2(i) is different from that received in the current step. In this case, the check in the current step is consistent with a negligible probability only as \( H \) for two different inputs are independent. (2) R-2(ii) occurs to some \( j^* \) (so \( \text{CMT}_j^* \) is not found there) while \( \text{CMT}_j \) received in the current step is consistent with \( r_j \). As seen above, this holds with probability \( 1/|\Theta| \) only. Ignoring these events, \( CH \) and \( \{\text{CMT}_j\}_{j \neq t} \) have already been determined in R-2(ii) and such \( \{\text{CMT}_j\}_{j \neq t} \) are consistent with those received in the current step.

- **Output.** Upon \( \text{Rsp}_j \) for \( j \neq t \), it computes \( \text{Rsp} = \sum_{j=1}^{n} \lambda_j \cdot \text{Rsp}_j \). The final signature is \( (\text{CMT}, \text{Rsp}) \) with the aggregated key \( pk^t \).

Finally, \( D \) outputs a forgery \( (\alpha, \beta) \) for message \( M^t \) and public keys \( PK^t \). If \( |\alpha|_{PK^t} M^t \neq \text{CMT}^*|PK^t|M^* \) (from the Type-II irregular query) or \( \alpha = \beta \) is an invalid signature w.r.t. \( (M^t, PK^t) \) (when verified using \( V_\mathcal{C}(\cdot) \) ), \( B \) aborts with \( \bot \); otherwise, he defines \( \text{Rsp}^* = \beta \) and sends it back to his ID challenger. This completes the description of \( B \).

We now analyze the success probability of \( B \). First, the view of \( D \) is identical to the real game, except for the following events.

a. In step R-2 of oracle \( O_t \), \( (\text{CMT}_t^*, h, \text{Rsp}^*) \) is simulated by SIM (instead of being computed using \( sk_t \)). However, by hybrid reduction to simulability of \( \mathcal{I} \), the view of \( D \) is indistinguishable from his view when this transcript is generated using \( sk_t^\ast \).

b. In step R-2 of oracle \( O_t \), when \( (0|\text{CMT}_t|pk_t,y) \in L_H \), \( \text{Bad}_0 \) occurs for some \( y \) (hence the view of \( D \) is inconsistent if \( y \neq r_t \)). However, since \( \text{CMT}_t^* \) (i.e., \( \text{CMT}_t \)) is just simulated in this oracle query and \( H_\infty(\text{CMT}_t^*) = \omega(\log \lambda) \), \( \text{CMT}_t^* \) is independent of current records in \( L_H \). Hence, \( \text{Bad}_0 \) occurs with probability at most \( Q/2H_\infty(\text{CMT}_t^*) \) (negligible), where \( Q \) is the number of records in \( L_H \). We ignore this negligible probability from now on.

c. In step R-2 (ii), if \( (1|pk|\text{CMT}^*|M,y) \in L_H \) for some \( y \), then event \( \text{Bad}_1 \) occurs. In this case, \( A \) can not define \( CH = h \) and the simulation can not continue. However, since \( \text{CMT} = \lambda \cdot \text{CMT}^* + \sum_{j \neq t} \lambda_j \cdot \text{CMT}_j \) and \( \text{CMT}^* \) is simulated in the current oracle and hence independent of the rest variables in this equation. Hence, as long as \( \lambda_i \) is invertible (which is violated only negligibly), \( \text{CMT} \) has a min-entropy at least \( H_\infty(\text{CMT}) = \omega(\log \lambda) \). Thus, similar to \( \text{Bad}_0 \) event, \( \text{Bad}_1 \) occurs negligibly only.

d. Finally, when \( D \) outputs \( (\alpha, \beta) \) for message \( M^t \) and public-key set \( PK^t \), \( B \) will abort if \( PK^t_{\alpha|M^t} \neq PK^t_{\alpha|\text{CMT}^*|M^*} \). Since \( (\alpha, \beta) \) has been verified, a Type-I irregular query \( (pk, PK^t) \) and a Type-II irregular query \( \alpha|pk^t|M \) must have been issued: the first query \( (pk, PK^t) \) for some \( pk \in PK^t \) is the Type-I irregular query while the first query of \( \alpha|pk^t|M \) is the Type-II query; the existence of such queries are guaranteed as the verification of \( (\alpha, \beta) \) by \( B \) will certainly issue these queries.

Since \( \ell_H^t \) and \( \ell_{ch}^t \) are chosen uniformly random, the Type-I irregular query \( \ell_H^t \) and \( \ell_{ch}^t \) Type-II irregular query happen to equal the foregoing queries w.r.t. \( (\alpha, \beta) \) with probability \( 1/q_{\ell H^t}, 1/q_{\ell_{ch}^t} \geq 1/q_0 \), where \( q_0 \) (resp. \( q_1 \)) is the upper bound on \( \ell_H^t \) queries to \( H_0 \) (resp. \( H_1 \)).

From the analysis of (a)(b)(c), their occurrence changes the adversary view negligibly. Ignoring this, from item d, when \( \ell_H^t \) and \( \ell_{ch}^t \) is chosen correctly, the view of \( D \) is indistinguishable from its view in the real game. On the other hand, it is easy to verify that conditional on this correct choice, a valid forgery indicates a successful attack by \( B \). Hence, \( B \) can break the ID security with probability at least \( \epsilon/q_0/q_1 \), non-negligible. This contradicts the security of our ID scheme.

\[ \square \]

If the adversary always restricts the number of signers in the signing query and the forgery to be at most \( T \), then Theorem 1 immediately implies the following corollary.

**Corollary 1.** Let \( T \geq 2 \) and \( H_0, H_1 \) be two random oracles. Assume that \( h \leftarrow \Theta \) is invertible in \( \mathcal{R} \) with probability \( 1 - \text{negl}(\lambda) \). Let \( \mathcal{I} = (\text{Setup}_{id}, \text{KeyGen}_{id}, P, V_\mathcal{C}, \Theta) \) be a \( T \)-secure ID scheme with linearity and simulability. Then, our multi-signature scheme is \( T \)-EU-CMA secure.

## 7 Realizations

In this section, we will realize our compiler with ID schemes: Schnorr ID scheme and a lattice-based ID scheme. The first scheme is similar to Boneh et al. [13]. But we keep it as it is very simple and efficient and can demonstrate the usage of our compiler. The second one is new and breaks a barrier that the previous schemes can not overcome.

### 7.1 Realization I: Schnorr Identification

In this section, we apply our compiler to the well-known Schnorr ID scheme [48]. Toward this, we only need to show...
that it is linear with simulability and security. For clarity, we first review this scheme.

Let \(q\) be a large prime. Consider a prime group of order \(q\) with a random generator \(g\) (e.g., the group on elliptic curve secp256k1 of \(g^2 = x^3 + 7\) for Bitcoin). The Schnorr identification is depicted in Fig. 4. This scheme can be regarded as a realization of the parameterized ID scheme but the parameter \(\tau\) is never used. In the following, we show that Schnorr ID scheme satisfies the three properties.

**Linearity.** Notice that \(SK = RSP = R = \Theta = Zq\), \(CMT = PK = \langle g \rangle\). We now verify the linearity property.

i. As seen in Section 2.1, \(Zq\) and \(\langle g \rangle\) are both \(Zq\)-modules, where the multiplication \(\ast\) between \(R = \mathbb{Z}_q\) and \(\mathbb{Z}_q\) is the multiplication of \(Zq\), while \(\ast\) between \(R = \mathbb{Z}_q\) and \(\mathbb{Z}_q\) is exponentiation: \(\ast m = m^a\). Hence, \(SK, PK, CMT, RSP\) are \(R\)-modules.

ii. Let \(pk = g^a\) with \(sk = s_i\), \(i = 1, \ldots, n\). Let \(\lambda_1, \ldots, \lambda_n \in R\). Then, \(sk = \sum_{i=1}^n \lambda_i \cdot sk_i = \sum_{i=1}^n \lambda_i s_i\), where the addition is the group operation for \(SK\) (i.e., addition in \(\mathbb{Z}_q\)). Note the group operation for \(PK\) is the multiplication in \(\langle g \rangle\). Hence, \(pk = \prod_{i=1}^n \lambda_i \ast pk_i = \prod_{i=1}^n \lambda_i s_i = g^{\sum_{i=1}^n \lambda_i s_i}\). Thus, \(sk \in SK\) is the private key of \(pk \in PK\).

iii. Let \(X_i \parallel z_i\) be a transcript of \(ID\) w.r.t. \(pk = g^a\) and \(sk = s_i, i = 1, \ldots, n\). For \(\lambda_i \in R\), \(X = \prod_{i=1}^n \lambda_i \ast X_i = \prod_{i=1}^n \lambda_i z_i = \sum_{i=1}^n \lambda_i z_i\). If \(g^s = pk^s X_i\), then \(g^s = \prod_{i=1}^n g^s z_i = \prod_{i=1}^n (pk^s X_i)^{\lambda_i}\). Hence, \(g^s = pk^s X\), desired!

**Simulability.** Let \(pk = g^a\) be the public-key and \(sk = s\) be the private key. For \(c \leftarrow \mathbb{Z}_q\), we define \(SIM\) by taking \(z \leftarrow \mathbb{Z}_q\) and \(X = g^c pk^c\). The simulated ID transcript is \(X \parallel z\). Obviously, this transcript (i.e., it passes the verification). Now we show that for any (even unbounded) distinguisher \(D\) that has oracle access to \(O_{id}\) can not distinguish the output of \(SIM\) from the real ID transcript. Notice for both simulated and real transcripts \(X_i \parallel z\), it satisfies \(g^s = pk^s X\). Hence, \(X = g^s\) for some \(x \in \mathbb{Z}_q\) and \(z = cs + x\). In the real transcript, \(x \leftarrow \mathbb{Z}_q\) while the simulated transcript \(z \leftarrow \mathbb{Z}_q\). Hence, given \(c, (x, z)\) (hence \(X, z\)) in both transcripts has the same distribution. Since \(c\) is uniformly random in \(\mathbb{Z}_q\) in the simulation, the simulated and real transcripts have the same distribution (independent of adversary view before the challenge which includes the responses from \(O_{id}\)). Thus, the adversary view, given oracle access to \(O_{id}\), in both cases has the same distribution. The simulability follows.

**Security.** We now prove the security of Schnorr ID scheme under Definition 5.

**Lemma 2.** Under discrete logarithm assumption, Schnorr ID scheme is secure w.r.t. Definition 5.

**Proof.** The correctness is obvious. We now consider the security property. If there exists an adversary \(D\) that breaks the Schnorr ID scheme with non-negligible probability \(\epsilon\), then we construct an adversary \(A\) that breaks discrete logarithm in \(\langle g \rangle\) with a non-negligible probability \(\epsilon'\). The idea is to make use of \(D\) to construct an algorithm \(A\) for the nested forking lemma and then use the output of the forking algorithm to derive the discrete logarithm for the challenge. Upon a challenge \(A_1 = g^x\) and parameters \(q, g, A\) constructs \(A((A_1, g, q), \lambda_1, c, \rho)\) (as follows (so \(h_1 | h_2 = \lambda_1 \mid c\) with \(q = 2\) in the forking algorithm), where \(\overline{A} = \sum_{i=1}^f A_i \lambda_i\).

**Algorithm** \(A((A_1, g, q), \lambda_1, c, \rho)\)

Parse \(\rho\) as two parts: \(\rho = \rho_0, \rho_1\) \((st_0, A_2, \ldots, A_t) \leftarrow D(q, g, A_1; \rho_0)\)

\(\lambda_2, \ldots, \lambda_t \leftarrow \mathbb{Z}_q\) using randomness \(\rho_1\)

\(st_1 | X \leftarrow D(st_0, \lambda_1, \ldots, \lambda_t);\)

\(z \leftarrow D(st_1, \rho_2);\)

If \(g^x = \overline{A} \cdot X;\) then \(b = 1;\)

else \(b = 0;\)

**output** \((b, 2b, \{A_1 | \lambda_1\} | X | \rho | c | q | g)\).

From the description of \(A\) and the forking algorithm \(F_A\) (for the forking lemma), the rewinding in the forking algorithm \(F_A\) only changes \(\lambda_1\) and/or \(c\) as well as those affected by \((\lambda_1, c)\). In terms of forking lemma terminology, we have \((h_1, h_2) = (\lambda_1, c)\) and \(I_0 = 1, J_0 = 2\) (for a successful execution; otherwise, \(A\) will abort when \(I_0 \leq J_0\)). Let us now analyze algorithm forking algorithm \(F_A\). When four executions are executed successfully (i.e., \(b = 1\) for all cases), then the output for each execution will be described as follows. Let \(A_i = g^{a_i} \ast i, i = 1, \ldots, t.\)

- **Execution 0.** It outputs \((1, 2, \{A_i | \lambda_i\} | X | c | q | g)\). As the verification passes,

\[
z = \left(\sum_{i=1}^t \lambda_i a_i\right) c + x, \tag{24}\]

where \(X = g^x\).

- **Execution 1.** Compared with execution 0, the input only changes \(c\) to \(c\). From the code of \(A\), the output is \((1, 2, \{A_i | \lambda_i\} | X | \hat{c} | q | g)\). As the verification passes,

\[
\hat{z} = \left(\sum_{i=1}^t \lambda_i a_i\right) \hat{c} + x. \tag{25}\]

- **Execution 2.** Compared with execution 0, the input changes \(\lambda_1\) to \(\hat{\lambda}_1\) and \(c\) to \(\hat{c}\). From the code of \(A\),
the output is $(1, 2, \{A_i|\lambda_i\}|_i^t, A_i|\lambda_i|X_i'|\varepsilon_i|g(q)$. As the verification passes,

$$\Bar{\varepsilon} = (\Bar{\lambda}_i a_1 + \sum_{i=2}^{t} \lambda_i a_i) \Bar{c} + x',$$

(26)

where $X' = g^{x'}$.

- Execution 3. Compared with execution 0, the input changes $\lambda_1$ to $\Bar{\lambda}_1$ and $c$ to $\Bar{c}$. From the code of A, the output is $(1, 2, \{A_i|\lambda_i\}|_i^t, A_i|\lambda_i|X_i'|\varepsilon_i|g(q)$. As the verification passes,

$$\Bar{\varepsilon} = (\Bar{\lambda}_i a_1 + \sum_{i=2}^{t} \lambda_i a_i) \Bar{c} + x'.$$

(27)

From Eqs. (27)/(26), A can derive $\lambda_1 a_1 + \sum_{i=2}^{t} \lambda_i a_i$, as long as $c \neq c'$. From Eqs. (25)/(24), A can derive $\lambda_1 a_1 + \sum_{i=2}^{t} \lambda_i a_i$, as long as $c \neq c$. This can further give $a_1$, as long as $\lambda_1 \neq \lambda_1$ in $Z_q$. Finally, if the forking algorithm does not fail, then the four executions succeeds and $(c \neq c') \land (\Bar{c} \neq c') \land (\lambda_1 \neq \lambda_1) = \text{True}$. By forking lemma, it does not fail with probability at least \(8\epsilon^4/(2 \cdot 1)^3 - 3/|\Theta| = \epsilon^4 - 3/3q\). Hence, A can obtain $a_1$ with probability at least $\epsilon^4 - 3/3q$, non-negligible. This contradicts to the discrete logarithm assumption.

Key-and-Signature Compact Multi-Signature from Schnorr ID Scheme. Since Schnorr ID scheme satisfies the linearity, simulability and security, the multi-signature from this scheme using our compiler is obtained. For clarity, we give the complete signature in the following. Let $pk_i = g^{x_i}$ be the public-key with private key $sk_i = s_i$ for $i = 1, \ldots, n$. When users $PK = \{pk_1, \ldots, pk_n\}$ want to jointly sign a message $M$, they act as follows.

- **R-1.** User $i$ generates $X_i = g^{x_i}$ for $x_i \leftarrow Z_q$ and sends $H_0(X_i|pk_i)$ to other users.

- **R-2.** Upon $\{r_j\}_{j=1}^n$, user $i$ sends $X_i$ to other users.

- **R-3.** Upon $\{X_j\}_{j=1}^n$, user $i$ checks $r_j = H_0(X_j|pk_j)$ for all $j$. If not, he rejects; otherwise, he computes

$$\Bar{pk} = \prod_{i=1}^{n} pk_i^{H_0(pk_i, PK)}$$

(28)

$$\Bar{X} = \prod_{i=1}^{n} X_i^{H_0(pk_i, PK)}.$$  

(29)

Then, he computes

$$c = H_1(\Bar{pk})|\Bar{X}|M), \quad z_i = s_ic + x_i$$

(30)

and sends $z_i$ to leader.

- **Output.** Receiving all $z_i$’s, user $i$ computes

$$\Bar{z} = \sum_{j=1}^{n} H_0(pk_j, PK)z_j.$$  

Finally, it outputs $(\Bar{X}, \Bar{z})$ as the multi-signature of $M$ with the aggregated public-key $\Bar{pk}$ (note: the compiler protocol includes $n$ in the aggregated key; we omit it here as it is not used in the verification).

- **Verification.** To verify signature $(\Bar{X}, \Bar{z})$ for $M$ with the aggregated public-key $\Bar{pk}$, it computes $c = H_1(\Bar{pk})|\Bar{X}|M)$. It accepts only if $g^c = \Bar{pk} \cdot \Bar{X}$.

We denote this signature scheme by Schnorr-MultiSig. Notice that $c \leftarrow Z_q$ is invertible in $\mathbb{R}$ with probability $1 - 1/q$. As it satisfies linearity, simulability and security, by Theorem 1, we have the following.

**Corollary 2.** Let $H_0, H_1$ be two random oracles. If Discrete logarithm assumption in $\langle g \rangle$ holds, then Schnorr-MultiSig is EU-CMA.

**Remark.** Boneh et al. [13] proposed a method that transforms Schnorr ID to a key-and-signature compact multi-signature. Their protocol is an improvement of Maxwell et al. [38] to overcome a simulation flaw. Their protocol is also 3-round. Their scheme is computationally more efficient in the signing process than ours. However, our sizes of aggregated public-key and signature as well as the verification cost are all the same as theirs (which are also identical to that of the original Schnorr signature with a single signer). Aggregated public-key and signature have impacts on the storage at a large number of blockchain nodes and the verification cost has the impact on the power consumption on these nodes. The signing cost is relatively not so important as it only has impact on the signers. Boneh et al. [13] uses $\lambda_i s_i$ as a secret for public-key $pk_i^\lambda$ to generate a member signature $X_i|z_i$, and the final multi-signature $\Bar{X} = \prod_i X_i$ and $\Bar{z} = \sum_i z_i$. Their main saving (over us) is to avoid $n$ exponentiations in computing our $\Bar{X}$. One might be motivated to modify our general compiler so that it uses $\lambda_i \bullet pk_i$ (whose private key is $\lambda_i \cdot sk_i$) to generate a member signature $CMT_i | Rsp_i$, so that the final multi-signature is $CMT | Rsp$ with $CMT = \sum_i CMT_i$ and $Rsp = \sum_i Rsp_i$. However, this looking secure scheme has a simulation issue in general when we prove Theorem 1: it is required that $\{\text{SIM}(\text{CH}, \lambda \bullet pk)\}_{\lambda}$ is indistinguishable from the list of real transcripts for a fixed but random $pk$ while it is not clear how this can be proven generally.

**Implementation.** We have provided a solidity implementation for our Schnorr-MultiSig. The prime group uses the standard elliptic curve secp256k1. Our initial motivation is to run the multi-signature computation over blockchain. However, we find the gas usage is high. In our test for three signers, the gas usage for all signers in total is 33629066. However, there is certainly no reason to execute the multi-signature itself on the blockchain as there is no trust issue here. We really only need to verify the final aggregated multi-signature on the chain. In this case, the gas usage drops to a reasonable value 2398282 that is 7% of the above usage. The implementation is not optimized. The source code of this implementation is available at [github](https://github.com/JSQ2023/Schnorr-Multi-Signature).

**7.2 Realization II: a new lattice-based ID scheme**

In this section, we propose a new ID scheme from lattice and then apply our compiler to obtain a lattice-based multi-signature scheme. This is the first lattice-based multi-signature that has both a compact public-key and a compact signature without a restart during the signing process.

**Notations.** The following notations are specific for this section (in addition to the list in Section 2).
As a convention for lattice over ring, this section uses security parameter $n$ (a power of 2), instead of $\lambda$;

- $q$ is a prime with $q \equiv 3 \mod 8$;
- $R = \mathbb{Z}[x]/(x^n + 1)$; $R_q = \mathbb{Z}_q[x]/(x^n + 1)$; $R_q^*$ is the set of invertible elements in $R_q$;
- for a vector $w$, we implicitly assume it is a column vector and the $i$th component is $u_i$ or $w[i]$;
- for a matrix or vector $X, X^T$ is its transpose;
- $1$ denotes the all-$1$ vector $(1, \cdots, 1)^T$ of dimension $n$ that is clear from the context;
- for $u = \sum_{i=0}^{n-1} u_i x^i$ with $u_i \in \mathbb{Z}_q$, $||u||_\infty = \max_i |u_i|$;
- $\alpha \in \mathbb{Z}_q$ always uses the default representative with $\{q, 2q + 1\}$ to make $\alpha \in [0, q]$ for any $\alpha$.

### 7.2.1 Ring-LWE and Ring-SIS

In this section, we introduce the ring-LWE and ring-SIS assumptions (see [33], [35], [45] for details). For $\sigma > 0$, distribution $D_{R, \sigma}$ assigns the probability proportional to

$$e^{-\pi \sigma^2 |y|^2/n}$$

for any $y \in \mathbb{Z}_q^n$ and 0 for other cases. As in [1], $y \leftarrow D_{R, \sigma}$ samples $y = \sum_{i=0}^{n-1} y_i x^i$ from $R$ with $y_i \leftarrow D_{R, \sigma}$.

The Ring Learning With Error (Ring-LWE$_{q, \sigma, m}$) problem over $R$ with standard deviation $\sigma$ is defined as follows. Initially, it takes $s \leftarrow D_{R, \sigma}$ as secret. It then takes $a \leftarrow R_q$, $c \leftarrow D_{R, \sigma}$ and outputs $(a, s + c)$. The problem is to distinguish $(a, s + c)$ from a tuple $(a, b)$ for $a, b \leftarrow R_q$. The Ring-LWE$_{q, \sigma, m}$ assumption is to say that no PPT algorithm can solve Ring-LWE$_{q, \sigma, m}$ problem with a non-negligible advantage. According to [18], ring-LWE assumption with $\sigma = \Omega(n^{3/4})$ is provably hard and so it is safe to assume $\sigma = \Omega(n)$.

The Small Integer Solution problem with parameters $q, m, \beta$ over ring $R$ (ring-SIS$_{q, m, \beta}$) is as follows: given $m$ uniformly random elements $a_1, \cdots, a_m$ over $R_q$, find $(t_1, \cdots, t_m)$ so that $||t_i||_\infty \leq \beta$ and $a_1 t_1 + \cdots + a_m t_m = 0$ (note: here we use $|| \cdot ||_\infty$ norm while the literature regularly uses square-root norm $|| \cdot ||$). However, the gap is only a factor $n$ on $\beta$ and does not affect the validity of the assumption according to the current research status for ring-SIS). We consider the case $m = 3$. Recall that prime $q = 3 \mod 8$.

- By [9, Theorem 1], we can factor $x^n + 1 = \Phi_1(x)\Phi_2(x)$ for some irreducible polynomials $\Phi_1(x), \Phi_2(x)$ of degree $n/2$.

For instance, $1024 + 1 \mod 1187$ by maple is factored as $(512 + 504x + 256 - 1) * (512 - 504x + 256 - 1)$.

So by Chinese remainder theorem, $a_1$ is invertible, except for probability $2q^{-n/2}$. Hence, ring-SIS is equivalent to the case of invertible $a_2$ which is further equivalent to problem $a_1 t_1 + t_2 + a_3 t_3 = 0$, as we can multiply it by $a_2^{-1}$. By [15], [33], the best quantum polynomial algorithm for ring-SIS problem with $q, m$ can only solve $\beta = 2^{\tilde{\Omega}(\sqrt{n})}$ case. Thus, it is safe to assume Ring-SIS$_{q, m, \beta}$ for any polynomial $\beta$ or even $\beta = 2^{\Omega(n)}$.

### 7.2.2 Construction

We now describe our new ID scheme from ring $R$. Initially, take $s_1, s_2 \leftarrow D_{R, \sigma}, a \leftarrow R_q^*$ and compute $u = as_1 + s_2$. The system parameter is $a$; the public key is $u$ and the private key is $(s_1, s_2)$. Our ID scheme is as follows; also see Fig. 5.

1. Prover generates $y_1, y_2 \leftarrow Y^\mu$ and computes $v = ay_1 + y_2$ and sends $v$ to Verifier, where $\mu \geq \log^2 n$.
2. Receiver samples $c \leftarrow C$ and sends it to Prover.
3. Upon $c$, Prover does the following:
   a. Compute $z_1 = s_1 c + 1 + y_1$, $z_2 = s_2 c + 1 + y_2$;
   b. Let $A = \{j \mid z_{1j}, z_{2j} \in \mathbb{Z}\}$ (recall that for any vector $u$, $u_j$ is its $j$th component). If $A = \emptyset$, abort; otherwise, take $j^* \leftarrow A$ and compute
      $$z_1 = z_{1j^*} + \sum_{j \neq j^*} y_{1j}, z_2 = z_{2j^*} + \sum_{j \neq j^*} y_{2j}.$$
4. Upon $z_1, z_2$, Verifier checks
   $$\sum_{i=1}^{\mu} v_i q_i z_i + 2u - uc = 0$$

where $\eta_1 = 5\pi n^2 \sqrt{\mu} \log^3 n$ and $t$ is a positive integer (see the remark below) and recall that (as a convention) $v_i$ is the $i$th component of $v$. If all are valid, it accepts; otherwise, it rejects.

**Remark.** We give two clarifications.

1. $\eta_1$ is defined to depend on $t$. However, the correctness does not need this dependency. Actually, $\eta_1 = 3\pi n^{1.5} \sqrt{\mu} \log^2 n$ suffices for this. However, the dependency of $\eta_1$ on $t$ is necessary for the linearity and later for the multi-signature. Especially, $\eta_1$ is used to support linearity with $t$ transcripts; for the multi-signature case, $t$ stands for the number of signers.

2. It should be pointed out that the choice of $j^*$ (it exists) does not affect $z_1, z_2$ at all as $z_i = s_i c + 1 + \sum_{j=1}^{\mu} y_{ij}$ for $i = 1, 2$. In addition, the probability that $j^*$ does not exist is exponentially small in $n$ and so defining $j^*$ is unnecessary. However, we keep it for ease of analysis later.

**Correctness.** We now prove the correctness with $\eta_1$ replaced by a smaller value $\eta_1 = 3\pi n^{1.5} \sqrt{\mu} \log^2 n$. When all signers are honest, the protocol is easily seen to be correct if we can show $A = \emptyset$ or $||z_2||_\infty > \eta_1$ has a negligible probability. The former is shown in Lemma 5 below. For the latter, notice that $z_1 = s_1 c + y_1 + \cdots + y_{1\mu}$. If we use $w \in R$ to denote the coefficient vector of the polynomial $w$, then

$$y_1 + \cdots + y_{1\mu} = y_1 + \cdots + y_{1\mu}. \quad (31)$$

Notice each component of $y_1$ is uniformly random in $\{-\sigma^11.5 \log^3 n, \cdots, \sigma^11.5 \log^3 n\}$. By Hoeffding inequality (https://en.wikipedia.org/wiki/Hoeffding%27s_inequality) on each of the vector component in Eq. (31), $||y_i||_\infty > 2\sigma^11.5 \sqrt{\mu} \log^2 n$ only has a probability at most $2ne^{-\log^2 n}$. By Lemma 3 below, $||sc||_\infty > \sigma^{1/2} \log^2 n$ with probability at most $e^{-\Omega(\log^2 n)}$. Hence, correctness holds for bound $\eta_1$, except for probability at most $e^{-\Omega(\log^2 n)}$ (note: for brevity, this quantity should be understood as there exists constant $C$ so that the exception probability is at most $e^{-C \log^2 n}$; we will later keep this convention without a mention).
In this section, we analyze our ID scheme. We start with some preparations. The following lemma is adapted from [1, Lemma 4], where our restriction that the element $c$ of $C$ has a degree at most $n/2$, does not affect the proof.

**Lemma 3.** [1] If $s \leftarrow D_{R,σ}$ and $c \leftarrow C$, then

$$\Pr(\|sc\|_\infty \leq σn^{1/2}\log^2 n) \geq 1 - e^{-Ω(\log^2 n)},$$

where $e = 2.71828 \ldots$ is the Euler’s number.

The lemma below was in the proof of [1, Lemma 3].

**Lemma 4.** [1] Fix $γ \in R$ with $\|γ\|_\infty \leq σn^{1/2} \log^3 n$. Then, for $y \leftarrow Y$, we have

$$\Pr(γ + y = 0) \geq 1 - \frac{1}{en}.$$

**Proof.** Notice $z_{bj} = sc + y_{bj}$ for $b = 1, 2$. By Lemma 3, $\|sc\|_\infty \leq σn^{1/2} \log^3 n$ with probability $1 - e^{-Ω(\log^2 n)}$. Fixing $sc$ (that satisfies this condition), $z_{bj}$ for $b = 1, 2, j = 1, \ldots, μ$ are independent and thus by Lemma 4, $A = 0$ with probability at most $(1 - \frac{1}{e})^μ < (1 - 1/e^2)^μ$, exponentially small. Together with the probability for $\|sc\|_\infty \leq σn^{1/2} \log^3 n$, we conclude the lemma.

**Lemma 5.** Let $A$ be the index set in our ID scheme. Then,

$$\Pr(A = \emptyset) < e^{-Ω(\log^2 n)} < 1 - e^{-Ω(\log^2 n)}.$$

**Proof.** Notice $z_{bj} = sc + y_{bj}$ for $b = 1, 2$. By Lemma 3, $\|sc\|_\infty \leq σn^{1/2} \log^3 n$ with probability $1 - e^{-Ω(\log^2 n)}$. Fixing $sc$ (that satisfies this condition), $z_{bj}$ for $b = 1, 2, j = 1, \ldots, μ$ are independent and thus by Lemma 4, $A = 0$ with probability at most $(1 - \frac{1}{e})^μ < (1 - 1/e^2)^μ$, exponentially small. Together with the probability for $\|sc\|_\infty \leq σn^{1/2} \log^3 n$, we conclude the lemma.

**Lemma 6.** If $u \leftarrow C$, then $u$ is invertible in $R_q$ with probability $1 - (1 + 2 \log n)^{-n/2}$.

**Proof.** Recall that $q = 3 \mod 8$ in this section. By Blake et al. [9, Theorem 1], $u^n + 1 = \Phi_1(x)\Phi_2(x) \mod q$, where $\Phi_1(x), \Phi_2(x)$ have degree $n/2$ and are irreducible over $\mathbb{Z}_q$. By Chinese remainder theorem, $u$ is invertible in $R_q$ if and only if it is non-zero mod $\Phi_0(x)$ for both $b = 1, 2$. Since $u$ has a degree at most $n/2$, $u$ remains unchanged after mod $\Phi_0(x)$. Hence, it is invertible in $R_q$ if and only if $u$ is non-zero. This has a probability $1 - (1 + 2 \log n)^{-n/2}$.

**Discussion.** On the other hand, $\Phi_1(x), \Phi_2(x)$ are exactly according to the real distribution. Thus, our simulation of $z_{1j}, z_{2j}, v_j$ is statistically close to that in the real transcript. The closeness holds (even given adversary view, which includes the responses from $O_{id}$). Hence, the simulability follows.

We now show the simulability of our ID scheme. Given the public-key $u$ and $c \leftarrow C$, we define the simulator $SIM$ as follows.

- Sample $j^* \leftarrow [μ]$ and $z_{1j^*}, z_{2j^*} \leftarrow Z$; compute $v_{j^*} = a z_{1j^*} + z_{2j^*} - uc$.
- For $j \in [μ] - \{j^*\}$, sample $y_{1j}, y_{2j} \leftarrow Y$ and compute $v_j = ay_{1j} + y_{2j}$.
- Compute $z_b = z_{bj} + \sum_{j \neq j^*} y_{bj}, b = 1, 2$.
- Output $v = (v_1, \ldots, v_μ)$ and $z_1, z_2$.

This simulation is valid by the following lemma.

**Lemma 7.** The output of $SIM$ is statistically close to the real transcript, even if the distinguisher has oracle access to $O((s_1, s_2), u)$, where $(s_1, s_2) \leftarrow D_{R,σ}$ is the private key and $u = as_1 + s_2$ is the public-key.

**Proof.** First, we can assume $A \neq \emptyset$ for the real transcript as by Lemma 5 this is violated negligibly only. Then, by symmetry, $j^*$ for the real transcript is uniformly random over $\{1, \ldots, μ\}$. By the definition of $j^*$, we know that $z_{1j^*}, z_{2j^*}$ both belong to $Z$. In this case, by Lemma 4, $sc + y_{1j^*}, sc + y_{2j^*}$ for the real transcript with given $sc$ satisfying $\|sc\|_\infty < σn^{1/2} \log^3 n$ are independent and uniformly random over $Z$. By Lemma 3, we conclude that $z_{1j^*}$ and $z_{2j^*}$ are statistically close to uniform over $Z$ if they belong to $Z$. On the other hand, when $z_{1j^*}$, and $z_{2j^*}$ are fixed, $v_{j^*}$ is given, $v_j$ is exactly according to the real distribution. Thus, our simulation is statistically close to the real transcript. This holds even given adversary view, which includes the responses from $O_{id}$. Hence, the simulability follows.

Fig. 5. Our Lattice-based ID Scheme (Note: Membership checks $c \in C$ at Prover is important but omitted in the figure; $1$ is the vector of all $1$ of length $μ$.)
probability. In the multi-signature setting, this restart event explodes exponentially in the number of signers or the security parameter. Our construction avoids this issue by providing a vector $v$ of committing messages. For each $v_i$, we do not guarantee that the response $z_1, z_2$ lies in a good set (in which the abortion event can be avoided). But we have a high probability that one of index $i$ will have this property. By symmetry, this $i$ is uniformly random in $[\mu]$. That allows to achieve the simulability without an abort.

**Security.** Now we prove the security of our ID scheme, where the attacker needs to generate $z_1, z_2$ (given challenge $c$) to pass the verification w.r.t. an aggregated public-key $\pi$. We show that this is unlikely by the ring-SIS assumption.

**Lemma 8.** Under ring-LWE$_{q,\sigma,2n}$ and ring-SIS$_{\sigma,\beta}$ assumptions, our scheme is $t^\ast$-secure (with respect to Definition 5), where $\beta = 16\eta_1, \sqrt{n} \log^2 n$ and $\sigma = \Omega(n)$.

**Proof.** If there exists an adversary $A$ that breaks our ring-based ID scheme with non-negligible probability $\epsilon$, then we construct an adversary $\tilde{A}$ that breaks ring-SIS assumption with a non-negligible probability $\epsilon'$. The idea is to make use of $A$ to construct an algorithm $\tilde{A}$ for the nested forking lemma and then uses the output of the forking algorithm to obtain a solution for ring-SIS problem. Upon a challenge $u_1$ and $u_2$ (both uniformly over $R_q$), $\tilde{A}$ needs to find short $\alpha_1, \alpha_2, \alpha_3 \in R$ so that $a\alpha_1 + a\alpha_2 + u_1\alpha_3 = 0$. Toward this, $\tilde{A}$ constructs an algorithm $A((u_1, a), \lambda_1, c; \rho)$ as follows (so $q = 2$ in the forking algorithm), where $\lambda_1, c \leftarrow C$ and $\pi = \sum_{i=1}^{\lambda_1} \lambda_i; u_i \in R_q$ (in the description of $A$) and $t \leq t^\ast$.

**Algorithm $A((u_1, a), \lambda_1, c; \rho)$**

- **Parse $\rho$ as two parts:** $\rho = \rho_0|\rho_1$
  - $(st_0, u_2, \cdots, u_t) \leftarrow D(u_1, a; \rho_0)$
  - $\lambda_2, \cdots, \lambda_t \leftarrow C$ using randomness $\rho_1$
  - $st_1, v_1 \leftarrow D(st_0, \lambda_1, \cdots, \lambda_t)$
  - $(z_1, z_2) \leftarrow D(st_1, c)$

- If $\|z_0\|_\infty < \eta_1$ and $\sum_{j=1}^{\mu} v_j = az_1 + z_2 - \pi c$, then $b = 1$;
  - else $b = 0$;

**Output** $(b, 2b, \{u_i|\lambda_1\}|v|z_1|z_2|c|a)$.

From the description of $A$ and the forking algorithm $F_A$ (for the forking lemma), the rewinding in $F_A$ only updates $\lambda_1$ and/or $c$ as well as variables affected by $(\lambda_1, c)$. In terms of forking lemma terminology, we have $(h_1, h_2) = (\lambda_1, c)$ and $I_0 = 1, J_0 = 2$ (for a successful execution; otherwise, $A$ will abort when $I_0 \leq J_0$). Let us now analyze algorithm forking algorithm $F_A$. When four executions are executed successfully (i.e., $b = 1$ for all cases), then the output for each execution will be described as follows.

- **Execution 0.** It outputs $(1, 2, \{u_i|\lambda_1\}|v|z_1|z_2|c|a)$.
  - Since it succeeds, $\|z_0\|_\infty \leq \eta_1 (b = 1, 2)$ and
  $$\sum_{i=1}^{\mu} v_i = az_1 + z_2 - \pi c.$$  

- **Execution 1.** Compared with execution 0, the input only changes $c$ to $\hat{c}$. From the code of $A$, the output is $(1, 2, \{u_i|\lambda_1\}|v|\hat{z}_1|\hat{z}_2|c|a)$. Since it succeeds, $\|\hat{z}_0\|_\infty \leq \eta_1 (b = 1, 2)$ and
  $$\sum_{i=1}^{\mu} v'_i = a\hat{z}_1 + \hat{z}_2 - \pi \hat{c}.$$  

- **Execution 2.** Compared with execution 0, the input changes $\lambda_1$ to $\lambda_1$ and changes $c$ to $\hat{c}$. From the code of $A$, the output is $(1, 2, \{u_i|\lambda_1\}|v|\hat{z}_1|\hat{z}_2|c|a)$. Since it succeeds, $\|\hat{z}_0\|_\infty \leq \eta_1 (b = 1, 2)$ and
  $$\sum_{i=1}^{\mu} v''_i = a\hat{z}_1 + \hat{z}_2 - \overline{\pi} \hat{c},$$

where $\overline{\pi'} = \lambda_1 u_1 + \sum_{i=2}^{\lambda_1} \lambda_i u_i$.

- **Execution 3.** Compared with execution 0, the input changes $\lambda_1$ to $\lambda_1$ and changes $c$ to $\hat{c}$. From the code of $A$, the output is $(1, 2, \{u_i|\lambda_1\}|v|\hat{z}_1|\hat{z}_2|c|a)$. Since it succeeds, $\|\hat{z}_0\|_\infty \leq \eta_1 (b = 1, 2)$ and
  $$\sum_{i=1}^{\mu} v'''_i = a\hat{z}_1 + \hat{z}_2 - \overline{\pi'} \hat{c}.$$  

From Eqs. (35)(34), $A$ can derive
$$a(\hat{z}_1 - z_1) + (\hat{z}_2 - z_2) - \overline{\pi'}(\hat{c} - \tau) = 0.$$  

From Eqs. (33)(32),
$$a(\hat{z}_1 - z_1) + (\hat{z}_2 - z_2) - \overline{\pi}(\hat{c} - \tau) = 0.$$  

Notice that Eq. (36)$\times (\hat{c} - \tau)$-Eq. (37)$\times (\hat{c} - \tau)$ gives
$$a\alpha_1 + a\alpha_2 - u_1\alpha_3 = 0,$$  

where
$$\alpha_1 = (\hat{z}_1 - z_1)(\hat{c} - \tau) - (\hat{z}_1 - z_1)(\hat{c} - \tau) = 0,$$
$$\alpha_2 = (\hat{z}_2 - z_2)(\hat{c} - \tau) - (\hat{z}_2 - z_2)(\hat{c} - \tau) = 0,$$
$$\alpha_3 = (\lambda_1 - \lambda_1)(\hat{c} - \tau) = 0.$$  

Hence, $(\alpha_1, \alpha_2, -\alpha_3)$ forms a solution to ring-SIS problem with parameter $(a, 1, u)$. It suffices to verify that each $\alpha_i$ is short and also at least one of them is non-zero. For the second condition, it suffices to make sure that the probability for $\alpha_3 = 0$ is small. Notice that by Chinese remainder theorem, $\alpha_3 = 0$ implies $\lambda_1 = \lambda_1$ mod $\Phi_1(x)$ or $\tau = \tau$ mod $\Phi_1(x)$ or $\hat{c} = c$ mod $\Phi_1(x)$. Similarly, this must also hold for modular $\Phi_2(x)$ but it suffices to consider $\Phi_1(x)$ only. Since $\lambda_1, \lambda_1, \hat{c}$ is uniformly random over $C$, each of the equality holds with probability $(1 + 2 log n)^{-n/2}$ only and hence $Pr(\alpha_3 = 0) \leq 3(1 + 2 log n)^{-n/2}$, negligible! For the first condition, we first show that $\alpha_3$ is short. Notice that $|\hat{c} - \hat{\tau}|_\infty \leq 2 log n$ and $|\hat{z}_1 - z_1|_\infty \leq 2\eta_1$. Further, the constant term of $(\hat{c} - \tau)(\hat{z}_1 - z_1)$ is
$$((\hat{c} - \tau)|0\rangle \cdot (\hat{z}_1 - z_1)|0\rangle - \sum_{k=1}^{2-1} (\hat{c} - \tau)|k\rangle \cdot (\hat{z}_1 - z_1)|n - k\rangle$$

which, by Hoeffding inequality on the randomness of $\hat{c} - \tau$, has an absolute value at most $\sqrt{n/2} log n \cdot 8\eta_1 log n \leq 8\eta_1\sqrt{n} \log^2 n$, with probability at least $1 - e^{-\Omega(log^2 n)}$. The constant term of $(\hat{z}_1 - z_1)(\hat{c} - \tau)$ is similar. Hence, $|\alpha_3|_\infty \leq 16\eta_1\sqrt{n} \log^2 n$, with probability at least $1 - e^{-\Omega(log^2 n)}$. The general case of $|\alpha_1|_\infty$ is similar. Hence, $|\alpha_1|_\infty \leq \eta_1$. From this we conclude that $\hat{c}$ is short with probability at least $1 - e^{-\Omega(log^2 n)}$.
16ηn/√n log^2 n with probability 1 − e^{−Ω(log^2 n)}. Similarly, \|α_2\|_∞ has the same property. We can use the above proof technique to show that \|\hat{f}(c\cdot e\cdot \vec{c})\|_{∞} ≤ 8 \log n \cdot \sqrt{n} \log^2 n with probability 1 − e^{−Ω(log^2 n)}. Since λ_1, λ_1 is uniformly random over C, using the same technique, we have \|α_3\|_∞ ≤ \sqrt{n} \log n \cdot 32 \sqrt{n} \log^4 n = 32n \log^2 n, with probability 1 − e^{−Ω(log^2 n)}. Thus, we find a ring-SIS solution \((α_1, α_2, −α_3)\) of length at most 16ηn/√n log^2 n. Assume that the probability that D succeeds in one execution is \(\hat{c}\). Then, by forking lemma, it succeeds in four executions with probability \(\hat{c}^4 - 3(1 + 2 \log n) \cdot n/2\). This implies that \(A\) breaks the ring-SIS assumption with at least \(\hat{c}^4 - 3(1 + 2 \log n) \cdot n/2 = e^{−Ω(log^2 n)}\).

Finally, notice that the input \(u_1\) is uniformly random over \(R_q\) while in our ID scheme \(u_1 = a s_1 + s_2\) for \(s_1, s_2 \leftarrow D_{R,q}\). However, under ring-LWE assumption, it is immediate that \(\hat{c} ≥ e - \text{negl}(n)\). Hence, \(A\) can succeed with probability at least \(\hat{c}^4 - \text{negl}(n)\), this contradicts the assumption of ring-SIS. □

**Linearity.** Let \(SK = RSP = (R_q, R_q), CM = R^u, PK = R_q, R_q\). We now verify the linearity.

i. Obviously, \(SK\) is a \(R\)-module under the operation \(\ast\) for \((s_1, s_2) \in SK\) and \(c \in R\), \((s_1, s_2) = (cs_1, cs_2)\), where \(cs_1\) and \(cs_2\) are multiplications in \(R\). Other cases are similar.

ii. If \((s_1, s_2) \in SK\) and \(λ_i \in C\) for \(i = 1, \cdots, t\), then \(\sum_{i=1}^{t}(λ_i s_1, λ_i s_2) = (\sum_{i=1}^{t} λ_i s_1, \sum_{i=1}^{t} λ_i s_2)\) is obviously the private key of \(\sum_{i=1}^{t} λ_i (a s_1 + s_2) = a(\sum_{i=1}^{t} λ_i s_1) + (\sum_{i=1}^{t} λ_i s_2)\). However, we emphasize that this key is not necessarily short. But for randomly generated \((pk_i, sk_i, λ_i)\)’s, Lemma 9 implicitly implies that the aggregated private key has length at most \(2 \sqrt{\log n}\) (except for probability \(e^{−Ω(log^2 n)}\)); see max \(|S_v|\) with \(|S_v|\) given in the proof of Lemma 9).

iii. If \(\{(v_i, c, z_{1i}, z_{2i})\}_{i=1}^{\mu}\) are honestly generated accepting transcripts w.r.t the honestly generated public/private key pairs \(\{(u_i, (s_{1i}, s_{2i}))\}_{i=1}^{\mu}\), then

\[
\sum_{j=1}^{\mu} v_{ij} = a z_{1i} + z_{2i} - u_i c. \tag{42}
\]

Together with Lemma 9 below, for \(h_1, \cdots, h_t \leftarrow C\), \((\sum_{i=1}^{t} h_i v_i, c, \sum_{i=1}^{t} h_i z_{1i}, \sum_{i=1}^{t} h_i z_{2i})\) passes the verification at the verifier. That is, it satisfies (except for probability \(e^{−Ω(log^2 n)}\))

\[
||\sum_{i=1}^{t} h_i z_{1i}||_{∞} \leq \eta_t, \quad ||\sum_{i=1}^{t} h_i z_{2i}||_{∞} \leq \eta_t,
\]

\[
\sum_{j=1}^{\mu} (t_{ij} v_{ij}) = a(\sum_{i=1}^{t} h_i z_{1i}) + (\sum_{i=1}^{t} h_i z_{2i}) - (\sum_{i=1}^{t} h_i u_i) c,
\]

where \(\eta_t = 5σn^2 \sqrt{t\mu} \log^6 n\). The linearity follows.

**Lemma 9.** Fix integer \(t \geq 2\) and \(σ \geq \omega(\log n)\). Assume \(s_i \leftarrow D_{R,q}, h_i \leftarrow C, y_{ij} \leftarrow \mathcal{Y}\) for \(i \in [t], j \in [\mu], c \leftarrow C\). Let

\[
Z = \sum_{j=1}^{\mu} y_{ij}.
\]

Then, \(||Z||_∞ \leq \eta_t\) with probability \(1 - e^{−Ω(log^2 n)}\).

**Proof.** Notice

\[
Z[0] = \sum_{j=1}^{n-1} S_v \cdot c[v] - \sum_{i=1}^{t} \sum_{k=0}^{n-1} h_i[n-k] \cdot Y_{ik},
\]

where \(Y_{ik} = \sum_{j=1}^{\mu} y_{ij}[k], h_i[n] \overset{def}{=} -h_i[0]\) and

\[
S_v = \sum_{i=1}^{t} \sum_{k=0}^{n-1} h_i[n-k] s_i[k-v].
\]

By [24, Lemma 4.2], \(|S_v|, n \leq \sigma \log n\), except for probability \(e^{−Ω(log^2 n)}\). When this is satisfied, terms \(h_i[n-k] s_i[k-v] \in S_v\) are independent random variables in the range \([-\sigma \log^2 n, \sigma \log^2 n]\). By Hoeffding inequality, \(|S_v| \leq 2\sqrt{\sigma n \log^3 n}\), except for probability \(e^{−Ω(log^2 n)}\). Since \(y_{ij}[k]\) is uniformly random over \([-\sigma n^{1.5} \log^3 n, \sigma n^{1.5} \log^3 n]\), by Hoeffding inequality, \(|Y_{ik}| \leq 2\sqrt{\log n}\) except for a probability \(e^{−Ω(log^2 n)}\). Assuming these inequalities for \(S_v\) and \(Y_{ik}\), we know that from Hoeffding inequality again,

\[
\left|\sum_{i=1}^{n-1} S_v \cdot c[v]\right| \leq 4\sigma n \sqrt{\log^5 n}
\]

\[
\left|\sum_{i,k} h_i[n-k] \cdot Y_{ik}\right| \leq \sqrt{\log n} \cdot 4\sigma \sqrt{\log n} \cdot \sqrt{\log n}
\]

except for probability \(e^{−Ω(log^2 n)}\). Hence, we conclude that \(|Z[0]| \leq 5\sigma n^2 \sqrt{t\mu} \log^6 n\), except for \(e^{−Ω(log^2 n)}\). We can similarly bound \(Z[i]\) for \(i \geq 1\) and so \(|Z||_∞ \leq 5\sigma n^2 \sqrt{t\mu} \log^6 n\), except for probability \(e^{−Ω(log^2 n)}\). □

7.2.4 Key-and-Signature Compact Multi-signature Scheme from our ID scheme.

With the simulability, linearity and security for our ID, we can use our compiler to convert it into a secure multi-signature. We now describe this scheme as follows.

Let \((s_{1i}, s_{2i})\) be the private key of public-key \(u_i = a s_{1i} + s_{2i}\) for \(i = 1, \cdots, t\). If the users of \(u_1, \cdots, u_t\) want to jointly sign \(M\), they compute the aggregated public-key \(πt\) and execute the protocol as follows, where \(H, H_1 : \{0, 1\}^* \rightarrow C\) and we define \(\overrightarrow{v} = \sum_{i=1}^{t} H_0(u_i, U) v_i\) for any list of variables \(w_1, \cdots, w_t\) in the description below and \(U = (u_1, \cdots, u_t)\) (e.g., \(\overrightarrow{v} = \sum_{i=1}^{t} H_0(u_i, U) \cdot v_i\)).

- **R-1.** User generates \(y_{1i}, y_{2i} \leftarrow \mathcal{Y}^\mu\), computes \(v_i = ay_{1i} + y_{2i}\), and sends \(H_0(v_i, |u_i|)\) to other users.

- **R-2.** Upon receiving all \(r_{ij} = 1, \cdots, t, \) user sends \(v_i\) to other users.

- **R-3.** Upon all \(v_j\), user checks if \(r_j = H_0(v_j, |u_j|)\). If verification fails, it rejects; otherwise, it computes \(v = H_1(π|\overrightarrow{v}|, M)\) as well as the response \((z_{1i}, z_{2i})\) for challenge \(c\) in the ID scheme with committing message \(v_i\).

- **output.** After receiving \((z_{1j}, z_{2j})\) for \(j \in [t]\), user \(i\) computes multi-signature \((z_{1i}, z_{2i}, \overrightarrow{v})\). The aggregated public-key is \(πt\).

- **Verify.** Upon \((z_{1i}, z_{2i}, \overrightarrow{v})\), it verifies the following with \(πt\) and accepts only if it is valid:
\[ \|z_1\|_\infty \leq \eta_t, \quad \|z_2\|_\infty \leq \eta_t, \quad (44) \]
\[ \sum_{j=1}^{\mu} \bar{v}_j = a\bar{z}_1 + \bar{z}_2 - \bar{c}, \quad (45) \]

where \( \eta_t = 5\sigma n^2 \sqrt{\log^6 n} \). Denote this multi-signature scheme by RLWE-MultSig. From our compiler and the properties of our ID scheme, we obtain the following.

**Corollary 3.** Let \( \eta_t^* = 5\sigma n^2 \sqrt{\log^6 n} \), \( \sigma = \Omega(n) \) and \( \beta_t = 16\eta_t^* \sqrt{n} \log^2 n \). Let \( H_0, H_1 \) be two random oracles. Then, under Ring-LWE_{q,2n} and Ring-SIS_{q,3,\beta_t} assumptions, RLWE-MultSig is \( t^*\)EU-CMA secure. Especially, if these harness assumptions hold for \( t^* = 2^{\sqrt{n}} \), then RLWE-MultSig is EU-CMA secure.

**Remark.** As the best algorithm [15], [33] can only solve Ring-SIS_{q,3,\beta} with \( \beta = 2^{O(\sqrt{n})} \), it is safe to assume Ring-SIS_{q,3,\beta} with any polynomial \( \beta \). If the assumption is sound for \( \beta = 2^{\sqrt{n}} \), our multi-signature scheme is EU-CMA secure, as for a PPT adversary, the number of signers in a signing query or forgery is polynomially bounded.

**Implementation.** Our analysis is conducted in the asymptotic notation. The parameters are not optimized. But still, we have provided a proof-of-concept implementation on Ubuntu 20.04 VM using Python for the protocol with 3 signers. For \( n = 1024 \) and \( q = 2^{91} + 11259 \), the protocol has a total runtime of about 30 seconds. We found that the main cost comes from polynomial multiplications for computing \( \bar{v}_i \) and \( \bar{v} \). It is not surprising as we do not use the fast multiplication algorithm. When \( n = 1024 \), \( \bar{v} \) requires to do 100 multiplications of polynomial of degree 1023 over \( \mathbb{F}_q \). This should be greatly improved if a fast Fourier transform (FFT) is applied. Our implementation can not be directly used on a blockchain for the transaction as the existing blockchains are not based on lattice and the gas consumption will be high also. However, once a multi-signature is generated, we use the following contract to achieve the payment.

```
contract FlexPay {
  mapping (address=>uint256) CTR;
  mapping (address=>uint) Balance;

  function Counter (address addr) public view {
    return CTR[addr];
  }

  function Pay (bytes calldata barPK,\|barX\|\|barz, address to, uint val) public {
    from=address_of_barPK;
    M=to||val||CTR[from];
    require(Ver(barX, barz, barPK, M)=true);
    require (Balance[from]>=val);
    decrease Balance[from] by val;
    payable(to).transfer(val);
    increment CTR[from];
  }

  function RecvPay (address addr) public payable {
    increase Balance[addr] by msg.value;
  }
}
```

In this code, contract FlexPay can be regarded as a bank of all addresses including the address (say, \( addr_0 \)) of \( PK \).

Although \( FTK \) is not an Ethereum public-key, \( addr_0 \) can still receive Ether as any uint256 value is an address. Anyone can pay to \( addr_0 \) through \( RecvPay \) function. In this case, the balance \( Balance[addr_0] \) of \( addr_0 \) in FlexPay is updated. If the members of \( FTK \) want to pay money \( val \) to \( addr_0 \), they can first generate a multi-signature (using our python code) and then use this signature to run Pay function. The result is that the account Balance[addr_0] is decreased by \( val \) while the contract transfers \( val \) money to address \( to \). To avoid the double spending using the same multi-signature, a counter for each address is maintained and it is increased after a multi-signature payment on the current counter is consumed. The multi-signature is generated using the message \( M = to||val||CTR[addr_0] \), where a counter is used which can be retrieved using \( Counter \) function before jointly generating the multi-signature. The signers can communicate through a public server as a channel. Since the signature model does not require a secure channel, this server can simply be any TCP server (especially, no secure connection such as TLS is required).

Our python multi-signature source code is available at https://github.com/JSQ2023/Ring-LWE-Multi-Signature. The contract’s Web3-based connection with a python code based multi-signature execution (as well as its parameter optimization) does not seem to be an easy task. We take it as our continued work and will post the detailed implementation on the same site in the near future.

### 8 Conclusion

In this paper, we proposed a compiler that converts a type of identification scheme to a key-and-signature compact multi-signature. This special type of ID owns a linear property. The aggregated public-key and multi-signature are of size both independent of the number of signers. We formulated this compiler through linear ID via the language of \( R \) module and proved the security through a new forking lemma called nested forking lemma. Under our compiler, the compact multi-signature problem has been reduced from a multi-party problem to a two-party problem. We realized our compiler with Schnorr ID scheme and a new lattice-based scheme. Our lattice multi-signature is the first of its kind that is key-and-signature compact without a restart in the signing process.

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### References


Shaoquan Jiang received the B.S. and M.S. degrees in mathematics from the University of Science and Technology of China, Hefei, China, in 1996 and 1999, respectively. He received the Ph.D degree in Electrical and Computer Engineering from the University of Waterloo, Waterloo, ON, Canada, in 2005.

From 1999 to 2000, he was a research assistant at the Institute of Software, Chinese Academy of Sciences, Beijing; from 2005 to 2013, he was a faculty member at the University of Electronic Science and Technology of China, Chengdu, China; from 2013 to 2020, he was a faculty member at Mianyang Normal University, Mianyang, China. Since May 2020, he is a faculty at University of Windsor. He was a postdoc at the University of Calgary from 2006 to 2008 and a visiting research fellow at Nanyang Technological University from 2008 to 2009. His interests are security protocols, network security, blockchain, (post-)quantum cryptography and information theoretical security.

Dima Alhadidi is an assistant professor in the School of Computer Science at the University of Windsor. She received her PhD degree in Computer Science and Software Engineering from Concordia University. Before joining the University of Windsor, she was an assistant professor at the University of New Brunswick and Zayed University, a researcher at the Canadian Institute for Cybersecurity, and a research associate at Concordia University. She has been selected by an independent panel of judges to be honored as one of Canada’s 2021 Top Women in Cybersecurity. Her research addresses data privacy and security issues in emerging technologies such as cloud computing and healthcare.

Hamid Fazli Khojir obtained the B.Sc in Computer Engineering from the University of Tehran in 2020 and the M.Sc in Computer Science from the University of Windsor in 2023. Hamid won the Vector Institute for Artificial Intelligence during his graduate study. He completed an internship at Linux Foundation by working on a privacy-preserving federated learning project based on the Hyperledger ecosystem. Hamid is Software Designer in CamCloud, which provides cloud-based solutions for integrating security cameras. His research interests are privacy and machine learning.