Revisiting the Efficiency of Asynchronous Multi Party Computation Against General Adversaries

Ananya Appan\(^*\) \quad Anirudh Chandramouli\(^†\) \quad Ashish Choudhury\(^‡\)

Abstract

In this paper, we design secure multi-party computation (MPC) protocols in the asynchronous communication setting with optimal resilience. Our protocols are secure against a computationally-unbounded malicious adversary, characterized by an adversary structure \( \mathcal{Z} \), which enumerates all possible subsets of potentially corrupt parties. Our protocols incur a communication of \( O(|\mathcal{Z}|^2) \) and \( O(|\mathcal{Z}|) \) bits per multiplication for perfect and statistical security respectively. These are the first protocols with this communication complexity, as such protocols were known only in the synchronous communication setting (Hirt and Tschudi, ASIACRYPT 2013).

1 Introduction

Secure multi-party computation (MPC) \([34, 19, 6, 33]\) is a fundamental problem in secure distributed computing. Consider a set of \( n \) mutually-distrusting parties \( P = \{P_1, \ldots, P_n\} \), where a subset of parties can be corrupted by a computationally-unbounded malicious (Byzantine) adversary \( \text{Adv} \). Informally, an MPC protocol allows the parties to securely compute any function \( f \) of their private inputs, by keeping their respective inputs private. The most popular way of characterizing \( \text{Adv} \) is through a threshold, by assuming that it can corrupt any subset of up to \( t \) parties. In this setting, MPC with perfect security (where no error is allowed in the outcome) is achievable iff \( t < n/3 \) \([6]\), while statistical security (where a negligible error is allowed) is achievable iff \( t < n/2 \) \([33]\). Hirt and Maurer \([22]\) generalized the threshold model by introducing the general-adversary model (also known as the non-threshold setting). In this setting, \( \text{Adv} \) is characterized by an adversary structure \( \mathcal{Z} = \{Z_1, \ldots, Z_n\} \subset 2^P \), which enumerates all possible subsets of potentially corrupt parties, where \( \text{Adv} \) can select any subset of parties \( Z^* \in \mathcal{Z} \) for corruption. Modelling the distrust in the system through \( \mathcal{Z} \) allows for more flexibility (compared to the threshold model), especially when \( P \) is not too large. In the general-adversary model, MPC with perfect and statistical security is achievable iff \( \mathcal{Z} \) satisfies the \( Q^{(5)}(P, \mathcal{Z}) \) and \( Q^{(2)}(P, \mathcal{Z}) \) conditions respectively.

In terms of efficiency, MPC protocols against general adversaries are less efficient than those against threshold adversaries, by several orders of magnitude. Protocols against threshold adversaries typically incur a communication of \( n^{O(1)} \) bits per multiplication, compared to \( |\mathcal{Z}|^{O(1)} \) bits per multiplication required against general adversaries\(^2\). Since \( |\mathcal{Z}| \) could be exponentially large in \( n \), the exact exponent is very important. For instance, as noted in \([23]\), if \( n = 25 \), then \( |\mathcal{Z}| \) is expected to be around one million, and a protocol with a communication complexity of \( O(|\mathcal{Z}|^2 \cdot \text{Poly}(n)) \) bits is preferred over a protocol with a communication complexity of \( O(|\mathcal{Z}|^3 \cdot \text{Poly}(n)) \) bits.

Our Motivation and Results: All the above results hold in the synchronous communication setting, where the parties are assumed to be globally synchronized, with strict upper bounds on the message delay. Such strict time-outs are, however, extremely difficult to maintain in real-world networks like the Internet, which are better modelled by the asynchronous communication setting \([8]\). Here, no timing assumptions are made and messages can be arbitrarily, but finitely delayed, with every message sent being delivered eventually. Asynchronous protocols are more complex and less efficient when compared to their synchronous counter-parts, since a slow (but

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\(^*\)International Institute of Information Technology, Bangalore India. Email: ananya.appan@iiitb.ac.in.

\(^†\)International Institute of Information Technology, Bangalore India. Email: anirudh.c@iiitb.ac.in.

\(^‡\)International Institute of Information Technology, Bangalore India. Email: ashish.choudhury@iiitb.ac.in.

\(^1\)We say that \( \mathcal{Z} \) satisfies the \( Q^{(k)}(P, \mathcal{Z}) \) condition \([22]\), if the union of no \( k \) sets from \( \mathcal{Z} \) covers \( P \).

\(^2\)The cost of any generic MPC protocol is typically dominated by the overhead associated with the multiplication operations in \( f \).
honest) sender party cannot be distinguished from a corrupt sender party, who does not send any message. To avoid an endless wait, the parties cannot afford to wait to receive messages from all the parties, which results in disregarding messages from a subset of potentially honest parties. Against threshold adversaries, perfectly-secure and statistically-secure asynchronous MPC (AMPC) is achievable, iff $t < n/4$ \cite{5} and $t < n/3$ \cite{7,1} respectively. By using the player-partitioning argument \cite{22}, these results can be generalized to show that against general adversaries, perfect and statistical security require $Z$ to satisfy the $Q^{(1)}(P, Z)$ and $Q^{(3)}(P, Z)$ conditions respectively.

Compared to synchronous MPC protocols, AMPC protocols are not very well-studied \cite{5,7,4,31,13,12}, especially against general adversaries. While perfectly-secure AMPC against general adversaries has been studied in \cite{25,12}, to the best of our knowledge, there exists no statistically-secure AMPC protocol against general adversaries. We design communication efficient AMPC protocols against general adversaries, both with perfect and statistical security, whose efficiency is comparable with the most efficient MPC protocols in the synchronous communication setting. Our results put in the context of relevant existing results are presented in Table 1, where $F$ denotes a finite field over which all computations are performed, and $\kappa$ denotes the statistical-security parameter.

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<th>Synchronous</th>
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Table 1: Communication complexity of different MPC protocols against general adversaries in terms of $|Z|$

Our protocols are in the pre-processing model, where the parties generate random secret-shared multiplication-triples. The parties then evaluate ckt in a secret-shared fashion, where Beaver’s method \cite{3} is used to evaluate the multiplication gates using the generated multiplication-triples. Our protocols for the pre-processing phase closely follow \cite{23}. However, there are several non-trivial challenges while adapting these protocols to the asynchronous world. Since our protocols are slightly technical, we refer to Section 3 for the technical overview of our protocols.

## 2 Preliminaries, Definitions and Existing Asynchronous Primitives

We assume that the parties in $P = \{P_1, \ldots, P_n\}$ are connected by pair-wise secure channels. The adversary $\text{Adv}$ is assumed to be malicious and static, and decides the set of corrupt parties at the beginning of the protocol execution. Parties not under the control of $\text{Adv}$ are called honest. Given $P' \subseteq P$, we say that $Z$ satisfies the $Q^{(k)}(P', Z)$ condition, if for every $Z_{i_1}, \ldots, Z_{i_k} \in Z$, the condition $P' \not\subseteq Z_{i_1} \cup \ldots \cup Z_{i_k}$ holds.

We assume that the parties want to compute a function $f$, represented by a publicly known arithmetic circuit ckt over a finite field $\mathbb{F}$ consisting of linear and non-linear gates, with $M$ being number of multiplication gates. Without loss of generality, we assume that each $P_i \in P$ has an input $x^{(i)}$ for $f$, and that all the parties want to learn the single output $y = f(x^{(1)}, \ldots, x^{(n)})$. We follow the asynchronous communication model of \cite{5,8}. Unlike the previous unconditionally-secure AMPC protocols \cite{7,4,31,13,12}, we prove the security of our protocols using the UC framework \cite{9,13,10}, whose details are presented in Appendix A.

In our protocols, we use a secret-sharing based on the one from \cite{29}, defined with respect to a sharing specification $\mathbb{S}$, which is a tuple of subsets of $P$. A sharing specification $\mathbb{S}$ is said to be $Z$-private, if for every $Z \in Z$, there is an $S \in \mathbb{S}$, such that $Z \cap S = \emptyset$. A sharing specification $\mathbb{S}$ satisfies the $Q^{(k)}(\mathbb{S}, Z)$ condition if for every $Z_{i_1}, \ldots, Z_{i_k} \in Z$ and every $S \in \mathbb{S}$, the condition $S \not\subseteq Z_{i_1} \cup \ldots \cup Z_{i_k}$ holds. In our protocols, we use the sharing specification $\mathbb{S} = (S_1, \ldots, S_h) \triangleq \{P \setminus Z | Z \in Z\}$, which guarantees that $\mathbb{S}$ is $Z$-private. This $\mathbb{S}$ satisfies the $Q^{(3)}(\mathbb{S}, Z)$ and $Q^{(2)}(\mathbb{S}, Z)$ conditions, if $Z$ satisfies the $Q^{(4)}(P, Z)$ and $Q^{(3)}(P, Z)$ conditions respectively.

**Definition 2.1** \cite{29,23}. A value $s \in \mathbb{F}$ is said to be secret-shared with respect to $\mathbb{S} = (S_1, \ldots, S_h)$, if there exist shares $s_1, \ldots, s_h$, such that $s = s_1 + \ldots + s_h$ and for $q = 1, \ldots, h$, share $s_q$ is known to every (honest) party in $S_q$.

A sharing of $s$ is denoted by $[s]$, where $[s]_q$ denotes the $q^{th}$ share. Note that $P_i$ will hold the shares $\{[s]_q\}_{P_i \in S_q}$. The above secret-sharing is linear, as $[c_1 s_1 + c_2 s_2] = c_1 [s_1] + c_2 [s_2]$ for any publicly-known $c_1, c_2 \in \mathbb{F}$.
Asynchronous Reliable Broadcast (Acast): Acast allows a designated sender $P_S \in \mathcal{P}$ to identically send a message $m \in \{0, 1\}^\ell$ to all the parties. If $P_S$ is honest, then all honest parties eventually output $m$. If $P_S$ is corrupt and some honest party outputs $m^*$, then every other honest party eventually outputs $m^*$. The above requirements are formalized by an ideal functionality $F_{\text{Acast}}$, presented in Appendix A. In [27], a perfectly-secure Acast protocol is presented with a communication complexity of $O(n^2\ell)$ bits, provided $Z$ satisfies the $Q^{(3)}(\mathcal{P}, Z)$ condition. The security of the protocol in [27] is not proven in the UC framework. For completeness, we do this in Appendix A.

Asynchronous Byzantine Agreement (ABA): In an ABA protocol [12] [28] [2], every party has a private bit and the (honest) parties eventually obtain a common output bit almost-surely with probability 1, where the output bit is the input bit of an honest party, if all honest parties have the same input. The above requirements are formalized through the functionality $F_{\text{ABA}}$, presented in Appendix A. We assume the existence of a perfectly-secure ABA protocol for $F_{\text{ABA}}$ with UC-security (see [26] [27] for such protocols if $Z$ satisfies the $Q^{(3)}(\mathcal{P}, Z)$ condition). The number of ABA instances will be independent of the size of ckt and so, we do not focus on the exact details.

Verifiable Secret-Sharing (VSS): A VSS protocol allows a designated dealer $P_D \in \mathcal{P}$ to verifiably secret-share its input $s \in \mathbb{F}$. If $P_D$ is honest, then the honest parties eventually complete the protocol with $[s]$. The verifiability guarantees that if $P_D$ is corrupt and some honest party completes the protocol, then all honest parties eventually complete the protocol with a secret-sharing of some value. These requirements are formalized through the functionality $F_{\text{VSS}}$ (Fig 1). The functionality, upon receiving a vector of shares from $P_D$, distributes the appropriate shares to the respective parties. The dealer’s input is defined implicitly as the sum of provided shares. We will use $F_{\text{VSS}}$ in our protocols as follows: $P_D$ on having the input $s$, sends a random vector of shares $(s_1, \ldots, s_h)$ to $F_{\text{VSS}}$ where $s_1 + \ldots + s_h = s$. If $P_D$ is honest, then the view of $\text{Adv}$ will be independent of $s$, if $S$ is $Z$-private. Hence, the probability distribution of shares learnt by $\text{Adv}$ will be independent of the dealer’s input.

**Functionality $F_{\text{VSS}}$**

$F_{\text{VSS}}$ proceeds as follows for each party $P_i \in \mathcal{P}$ and an adversary $S$, and is parametrized by a sharing specification $S = (S_1, \ldots, S_h)$, adversary structure $Z$ and a dealer $P_D$. Let $Z^*$ be the set of corrupt parties.

- On receiving (dealer, sid, $P_D$, $(S_1, \ldots, S_h)$) from $P_D$ (or from $S$ if $P_D \not\in Z^*$), set $s = \sum_{q=1}^h s_q$ and for $q = 1, \ldots, h$, set $[s]_q = s_q$. Generate a request-based delayed output (share, sid, $P_D$, $\{[s]_q\}_{P_i \in S_q}$) for each $P_i \not\in Z^*$.

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*If $P_D$ is corrupt, then $S$ may not send any input to $F_{\text{VSS}}$, in which case the functionality will not generate any output. See Appendix A for the meaning of request-based delayed output in asynchronous ideal world.*

**Figure 1:** The ideal functionality for VSS for session id sid.

In [12], a perfectly-secure VSS protocol $\Pi_{\text{PVSS}}$ is presented, provided $S$ satisfies the $Q^{(3)}(S, Z)$ condition (which holds for our $S$). The protocol (after a minor modification) incurs a communication of $O(|Z| \cdot n^2 \log |\mathbb{F}| + n^4 \log n)$ bits. In [12], the UC-security of $\Pi_{\text{PVSS}}$ was not shown and for completeness, we do so in Appendix B.

Default Secret-Sharing: The perfectly-secure protocol $\Pi_{\text{PerDefSh}}$ takes a public input $s \in \mathbb{F}$ and $S = (S_1, \ldots, S_h)$ to non-interactively generate $[s]$, where the parties collectively set $[s]_1 = s$ and $[s]_2 = \ldots = [s]_h = 0$.

Reconstruction Protocols: Let the parties hold $[s]$ with respect to some $S = (S_1, \ldots, S_h)$ which satisfies the $Q^{(2)}(S, Z)$ condition. Then, [12] presents a perfectly-secure protocol $\Pi_{\text{PerRecShare}}$ to reconstruct $[s]_q$ for any given $q \in \{1, \ldots, h\}$ and a perfectly-secure protocol $\Pi_{\text{PerRecShare}}$ to reconstruct $s$. The protocols incur a communication of $O(n^2 \log |\mathbb{F}|)$ and $O(|Z| \cdot n^2 \log |\mathbb{F}|)$ bits respectively (see Appendix B for the details).

3 Perfectly-Secure Pre-Processing Phase Protocol with $Q^{(4)}(\mathcal{P}, Z)$ Condition

Throughout this section, we assume that $Z$ satisfies the $Q^{(4)}(\mathcal{P}, Z)$ condition. We present a perfectly-secure protocol which generates a secret-sharing of $M$ random multiplication-triples, unknown to the adversary. The
protocol realizes the ideal functionality $F_{\text{Triples}}$ (Fig. 2) which allows the ideal-world adversary to specify the shares for each of the output triples on the behalf of corrupt parties. The functionality then “completes” the sharings of all the triples randomly, while keeping them “consistent” with the shares specified by the adversary.

**Functionality $F_{\text{Triples}}$**

$F_{\text{Triples}}$ proceeds as follows, running with the parties $P$ and an adversary $S$, and is parametrized by an adversary-structure $Z$ and $Z$-private sharing specification $S = (S_1, \ldots, S_h) = (P \setminus Z | Z \in Z)$. Let $Z^*$ denote the set of corrupt parties.

- If there exists a set of parties $A$ such that $P \setminus A \in Z$ and every $P_i \in A$ has sent the message (triples, sid, $P_i$), then send (triples, sid, $A$) to $S$ and prepare the output as follows.
  - Generate secret-sharing of $M$ random multiplication-triples. To generate one such sharing, randomly select $a, b \in F$, compute $c = ab$ and execute the steps labelled Single Sharing Generation for $a, b$ and $c$.
  - Let $\{(a(\ell)), [b(\ell)], [c(\ell)]\}_{\ell \in \{1, \ldots, M\}}$ be the resultant secret-sharing of the multiplication-triples. Send a request-based delayed output (triples, sid, $\{(a(\ell))_q, [b(\ell)]_q, [c(\ell)]_q\}_{\ell \in \{1, \ldots, M\}, p \in S_q}$) to each $P_i \in P \setminus Z^*$ (no need to send the respective shares to the parties in $Z^*$, as $S$ already has the shares of all the corrupt parties).

**Single Sharing Generation**: Do the following to generate a secret-sharing of a given value $s$.

- Upon receiving (shares, sid, $\{s_q\}_{S_q \cap Z^* \neq \emptyset}$) from $S$, randomly select $s_q \in F$ corresponding to each $S_q \in S$ for which $S_q \cap Z^* = \emptyset$, such that $\sum_{S_q \cap Z^* \neq \emptyset} s_q + \sum_{S_q \cap Z^* = \emptyset} s_q = s$ holds.

Figure 2: The ideal functionality for the asynchronous pre-processing phase with session id sid.

We now present a protocol for securely realizing $F_{\text{Triples}}$. To design the protocol, we need a multiplication protocol which takes as input $\{(a(\ell)), [b(\ell)]\}_{\ell = 1, \ldots, M}$ and securely generates $\{c(\ell)\}_{\ell = 1, \ldots, M}$, where $c(\ell) = a(\ell)b(\ell)$, without revealing any additional information about $a(\ell)$ and $b(\ell)$. For simplicity, we first explain and present the protocol assuming $M = 1$, where the inputs are $[a]$ and $[b]$, and the goal is to securely generate a random sharing $[ab]$.

Our starting point is the synchronous multiplication protocol of [23, 29]. Note that $ab = \sum_{(p,q) \in S \times S} [a]_p [b]_q$. The main idea of $S_p \cap S_q \neq \emptyset$, a publicly-known party from $S_p \cap S_q$ can be designated to act as a dealer and generate a random sharing of the summand $[a]_p [b]_q$. For efficiency, every designated “summand-sharing party” can sum up all the summands assigned to it and generate a random sharing of the sum instead. If no summand-sharing party behaves maliciously, then the sum of all secret-shared sums leads to a secret-sharing of $ab$. To deal with maliciously-correct summand-sharing parties, [23] first designed an optimistic multiplication protocol $\Pi_{\text{OptMult}}$, which takes an additional parameter $Z \in \mathcal{Z}$ and generates a secret-sharing of $ab$, provided $\text{Adv}$ corrupts a set of parties $Z^* \subseteq Z$. The idea used in $\Pi_{\text{OptMult}}$ is the same as above, except that the summand-sharing parties are now restricted to the subset $P \setminus Z$. Since the parties will not be knowing the identity of correct parties in $Z^*$, they run $\Pi_{\text{OptMult}}$ once for each $Z \in \mathcal{Z}$. This guarantees that at least one of these instances generates a secret-sharing of $ab$. By comparing the output sharings generated in all the instances of $\Pi_{\text{OptMult}}$, the parties can detect whether any cheating has occurred. If no cheating is detected, then any of the output sharings can serve as the sharing of $ab$. Else, the parties consider a pair of conflicting $\Pi_{\text{OptMult}}$ instances (whose resultant output sharings are different) and proceed to a cheater-identification phase. In this phase, based on the values shared by the summand-sharing parties in the conflicting $\Pi_{\text{OptMult}}$ instances, the parties identify at least one corrupt summand-sharing party. This phase necessarily requires the participation of all the summand-sharing parties from the conflicting $\Pi_{\text{OptMult}}$ instances. Once a corrupt summand-sharing party is identified, the parties disregard all output sharings of $\Pi_{\text{OptMult}}$ instances involving that party. This process of comparing the output sharings of $\Pi_{\text{OptMult}}$ instances and identifying corrupt parties continues, until all the remaining output sharings are for the same value.

**Challenges in the Asynchronous Setting**: There are two main non-trivial challenges while applying the above ideas in an asynchronous setting. First, in $\Pi_{\text{OptMult}}$, a potentially corrupt party may never share the sum of the

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*This provision is made because in our pre-processing phase protocol, the real-world adversary will have full control over the shares of the corrupt parties corresponding to the random multiplication-triples generated in the protocol.*
instances are executed to ensure that any summand \([a]_p[b]_q\) is considered for sharing by at least one honest party. However, care has to be taken to ensure that any summand \([a]_p[b]_q\) is not shared multiple times (more on this later).

The second challenge is that once the parties identify a pair of conflicting \(\Pi_{\text{OptMult}}\) instances, the potentially corrupt summand-sharing parties from these instances may not participate in the cheater-identification phase, thus causing the parties to wait indefinitely. To get around this problem, the multiplication protocol proceeds in iterations, where in each iteration, the parties run an instance of the asynchronous \(\Pi_{\text{OptMult}}\) (outlined above) for each \(Z \in \mathcal{Z}\), compare the outputs from each instance, and then proceed to the respective cheater-identification phase if the outputs are not the same. However, the summand-sharing parties from previous iterations are not allowed to participate in future iterations until they participate in the cheater-identification phase of all the previous iterations. This prevents the corrupt summand-sharing parties in previous iterations from acting as summand-sharing parties in future iterations until they clear their “pending tasks”, in which case they are caught and discarded for ever. We stress that the honest parties are eventually “released” to act as summand-sharing parties in future iterations. Thus, even if the corrupt summand-sharing parties from previous iterations are “stuck” forever, the parties eventually progress to the next iteration in case the current iteration “fails”. Once the parties reach an iteration where the outputs of all the \(\Pi_{\text{OptMult}}\) instances are the same, the protocol stops. We show that there will be at most \(t(tn+1)+1\) iterations, where \(t\) is the cardinality of the maximum-sized subset in \(\mathcal{Z}\).

Based on the above discussion, we next present protocols \(\Pi_{\text{OptMult}}, \Pi_{\text{MultCI}}\) and \(\Pi_{\text{Mult}}\). Protocol \(\Pi_{\text{MultCI}}\) represents an iteration where the parties run an instance of \(\Pi_{\text{OptMult}}\) for each \(Z \in \mathcal{Z}\) and execute a cheater-identification phase if the iteration fails. Protocol \(\Pi_{\text{Mult}}\) calls the protocol \(\Pi_{\text{MultCI}}\) multiple times, till it reaches a “successful” instance of \(\Pi_{\text{MultCI}}\) (where the outputs of all the instances of \(\Pi_{\text{OptMult}}\) are the same). In these protocols, the parties maintain the following dynamic sets: (a) \(W_{(i)}^{(t)}\): Denotes the wait-listed parties maintained by \(P_i\), corresponding to instance number \(t\) of \(\Pi_{\text{MultCI}}\) in \(\Pi_{\text{Mult}}\); (b) \(L_D^{(i)}\): Denotes the set of parties locally discarded by \(P_i\) during the cheater-identification phase of instance number \(t\) of \(\Pi_{\text{MultCI}}\) in \(\Pi_{\text{Mult}}\); and (c) \(G_D\): Denotes the set of parties, globally discarded by all (honest) parties across various instances of \(\Pi_{\text{MultCI}}\) in \(\Pi_{\text{Mult}}\).

These sets will be maintained such that no honest party is ever included in the sets \(G_D\) and \(L_D^{(i)}\) of any honest \(P_i\). Moreover, any honest party which is included in \(W_{(i)}^{(t)}\) set of any honest \(P_i\) is eventually removed from \(W_{(i)}^{(t)}\).

### 3.1 Optimistic Multiplication Protocol

Protocol \(\Pi_{\text{OptMult}}\) is executed with respect to a given \(Z \in \mathcal{Z}\) and iteration number \(t\). Each party in \(\mathcal{P} \setminus Z\) tries to act as a summand-sharing party and shares the sum of all the summands it is “capable” of. To avoid “repetition” of summands, the parties select distinct summand-sharing parties in hops and “mark” the summands whose sum is shared by the selected summand-sharing party in a hop, ensuring that they are not considered in future hops. To agree on the summand-sharing party of each hop, the parties execute an instance of the agreement on common subset (ACS) primitive \([5]\), where one instance of ABA is invoked on behalf of each candidate summand-sharing party. While voting for a candidate party in \(\mathcal{P} \setminus Z\) during a hop, the parties ensure that the candidate has indeed secret-shared some sum, and that it was not 1) selected in an earlier hop; 2) in the waiting list or the list of locally-discarded parties of any previous iteration; 3) in the list of globally-discarded parties.

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Protocol \(\Pi_{\text{OptMult}}(\mathcal{P}, Z, S, [a], [b], Z, t)\)

- Initialization
  - Initialize the set of ordered pair of indices of all summands: \(\text{Summands}_{(Z, t)} = \{(p, q)\}_{p,q=1,...,|S|}\).
  - Initialize the summand indices corresponding to \(P_j \in \mathcal{P} \setminus Z\): \(\text{Summands}_{(Z, t)}^{(j)} = \{(p, q)\}_{p \in S_p \cap S_q}\).
  - Initialize the set of summands-sharing parties: \(\text{Selected}_{(Z, t)} = \emptyset\). Initialize the hop number hop = 1.
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\[5\] The reason for two different discarded sets is that the various instances of cheater-identification corresponding to the failed \(\Pi_{\text{MultCI}}\) instances are executed asynchronously, thus resulting in a corrupt party to be identified by different honest parties during different iterations.
Do the following till **Summands**$_{(Z,\text{iter})} \neq \emptyset$:

- **Sharing Summands:**
  
  1. If $P_i \notin Z$ and $P_i \notin \text{Selected}(Z,\text{iter})$, then compute $c^{(i)}_{(Z,\text{iter})} = \sum_{(p,q) \in \text{Summands}^{(i)}_{(Z,\text{iter})}} [a]_p [b]_q$. Randomly select the shares $c^{(i)}_{(Z,\text{iter})}, \ldots, c^{(i)}_{(Z,\text{iter})}$, such that $c^{(i)}_{(Z,\text{iter})} + \ldots + c^{(i)}_{(Z,\text{iter})} = c^{(i)}_{(Z,\text{iter})}$. Call $\mathcal{F}_{\text{VSS}}$ with message $(\text{dealer}, \text{sid}_{\text{hop}, i, \text{iter}}, Z, \{c^{(i)}_{(Z,\text{iter})}, \ldots, c^{(i)}_{(Z,\text{iter})}\})$, where $\text{sid}_{\text{hop}, i, \text{iter}, Z} = \text{hop} || \text{sid} || i || \text{iter} || Z$.
  
  2. Keep requesting for an output from $\mathcal{F}_{\text{VSS}}$ with $\text{sid}_{\text{hop}, j, \text{iter}, Z}$ for $j = 1, \ldots, n$, till an output is received.

- **Selecting Summand-Sharing Party Through ACS:**

  1. For $j = 1, \ldots, n$, send $(\text{vote}, \text{sid}_{\text{hop}, j, \text{iter}}, Z, 1)$ to $\mathcal{F}_{\text{ABA}}$, if *all* the following conditions hold:
     - $P_j \notin \mathcal{G}D$, $P_j \notin Z$ and $P_j \notin \text{Selected}(Z,\text{iter})$. Moreover, $\forall \text{iter}’ < \text{iter}$, $P_j \notin \mathcal{V}_{\text{iter}’}^{(i)}$ and $P_j \notin \mathcal{L}_{\text{iter}’}^{(i)}$.
     - An output $(\text{share}, \text{sid}_{\text{hop}, j, \text{iter}}, Z, P_j, \{c^{(j)}_{(Z,\text{iter})} q\}_{\text{iter}}) \in S_q$ is received from $\mathcal{F}_{\text{VSS}}$ with $\text{sid}_{\text{hop}, j, \text{iter}, Z}$.
  
  2. For $j = 1, \ldots, n$, request for an output from $\mathcal{F}_{\text{ABA}}$ with $\text{sid}_{\text{hop}, j, \text{iter}, Z}$, until an output is received.

  3. If $\exists P_j \in \mathcal{P}$ such that $(\text{decide}, \text{sid}_{\text{hop}, j, \text{iter}, Z}, 1)$ is received from $\mathcal{F}_{\text{ABA}}$ with $\text{sid}_{\text{hop}, j, \text{iter}, Z}$, then for each $P_k \in \mathcal{P}$ for which no vote message has been sent yet, send $(\text{vote}, \text{sid}_{\text{hop}, k, \text{iter}, Z}, 0)$ to $\mathcal{F}_{\text{ABA}}$ with $\text{sid}_{\text{hop}, k, \text{iter}, Z}$.

  4. Once an output $(\text{decide}, \text{sid}_{\text{hop}, j, \text{iter}, Z}, v_j)$ is received from $\mathcal{F}_{\text{ABA}}$ with $\text{sid}_{\text{hop}, j, \text{iter}, Z}$ for all $j \in \{1, \ldots, n\}$, select the least indexed $P_j$ such that $v_j = 1$. Then set $\text{hop} = \text{hop} + 1$ and update the following:
     - $\text{Selected}(Z,\text{iter}) = \text{Selected}(Z,\text{iter}) \cup \{P_j\}$.
     - $\text{Summands}(Z,\text{iter}) = \text{Summands}(Z,\text{iter}) \setminus \text{Summands}^{(i)}_{(Z,\text{iter})}$.
     - $\forall P_k \in \mathcal{P} \setminus \{Z \cup \text{Selected}(Z,\text{iter})\}: \text{Summands}^{(k)}_{(Z,\text{iter})} = \text{Summands}^{(k)}_{(Z,\text{iter})} \setminus \text{Summands}^{(i)}_{(Z,\text{iter})}$.

- $\forall P_j \in \mathcal{P} \setminus \text{Selected}(Z,\text{iter})$, participate in an instance of $\Pi_{\text{PerDefSh}}$ with public input $c^{(j)}_{(Z,\text{iter})} = 0$.

- **Output:** Let $c_{(Z,\text{iter})} = c^{(1)}_{(Z,\text{iter})} + \ldots + c^{(n)}_{(Z,\text{iter})}$, Output $(\{c^{(1)}_{(Z,\text{iter})} q, \ldots, c^{(n)}_{(Z,\text{iter})} q\}_{\text{iter}})_{P_i \in S_q}$.

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The notation $\text{sid}_{\text{hop}, i, \text{iter}, Z}$ is used to distinguish among the different calls to $\mathcal{F}_{\text{VSS}}$ and $\mathcal{F}_{\text{ABA}}$ within each hop.

Figure 3: Optimistic multiplication in $(\mathcal{F}_{\text{VSS}}, \mathcal{F}_{\text{ABA}})$-hybrid for iteration $\text{iter}$ and session id $\text{sid}$, assuming $Z$ to be corrupt.

The above code is executed by each $P_i$, who implicitly uses the dynamic sets $\mathcal{G}D$, $\mathcal{V}_{\text{iter}’}^{(i)}$ and $\mathcal{L}_{\text{iter}’}^{(i)}$ for $\text{iter}’ < \text{iter}$.

Lemma 3.1 is proven in Appendix B. To handle $M$ pairs of inputs in $\Pi_{\text{OptMult}}$, in each hop, every $P_i$ calls $\mathcal{F}_{\text{VSS}}$ $M$ times to share $M$ summands. While voting for a candidate summand-sharing party in a hop, the parties check whether it has shared $M$ values. Hence, there will be $O(n^2M)$ calls to $\mathcal{F}_{\text{VSS}}$, but only $O(n^2)$ calls to $\mathcal{F}_{\text{ABA}}$.

**Lemma 3.1.** Let $Z$ satisfy the $\mathcal{Q}^{(i)}(P, Z)$ condition and $S = \{P \setminus Z | Z \in \mathcal{Z}\}$. Consider an arbitrary $Z \in \mathcal{Z}$ and $\text{iter}$, such that all honest parties participate in the instance $\Pi_{\text{OptMult}}(P, \mathcal{Z}, S, [a], [b], Z, \text{iter})$. Then all honest parties eventually compute $[c_{(Z,\text{iter})}], [c^{(1)}_{(Z,\text{iter})}], \ldots, [c^{(n)}_{(Z,\text{iter})}]$ where $c_{(Z,\text{iter})} = c^{(1)}_{(Z,\text{iter})} + \ldots + c^{(n)}_{(Z,\text{iter})}$ provided no honest party is included in the $\mathcal{G}D$ and $\mathcal{L}_{\text{iter}’}^{(i)}$ sets and each honest party in the $\mathcal{W}_{\text{iter}’}$ sets of every honest $P_i$ is eventually removed, for all $\text{iter}’ < \text{iter}$. If no party in $\mathcal{P} \setminus Z$ acts maliciously, then $c_{(Z,\text{iter})} = ab$. In the protocol, $\text{Adv}$ does not learn anything additional about $a$ and $b$. The protocol makes $O(n^2)$ calls to $\mathcal{F}_{\text{VSS}}$ and $\mathcal{F}_{\text{ABA}}$.

### 3.2 Multiplication Protocol with Cheater Identification

Protocol $\Pi_{\text{MultCI}}$ with cheater identification (Fig 4) takes as inputs an iteration number $\text{iter}$ and $([a], [b])$. If no party behaves maliciously, then the protocol securely outputs $[ab]$. In the protocol, parties execute an instance of $\Pi_{\text{OptMult}}$ for each $Z \in \mathcal{Z}$ and compare the outputs. Since at least one of the $\Pi_{\text{OptMult}}$ instances is guaranteed to output $[ab]$, if all the outputs are same, then no cheating has occurred. Otherwise, the parties identify a pair of conflicting $\Pi_{\text{OptMult}}$ instances with different outputs, executed with respect to $Z$ and $Z’$. Let $\text{Selected}_{(Z,\text{iter})}$ and $\text{Selected}_{(Z’,\text{iter})}$ be the summand-sharing parties in the conflicting $\Pi_{\text{OptMult}}$ instances. The parties next proceed to a cheater-identification phase to identify at least one corrupt party in $\text{Selected}_{(Z,\text{iter})} \cup \text{Selected}_{(Z’,\text{iter})}$.

Each $P_i \in \text{Selected}_{(Z,\text{iter})}$ is made to share the sum of the summands from its summand-list overlapping with the summand-list of each $P_k \in \text{Selected}_{(Z’,\text{iter})}$ and vice-versa. Next, these “partitions” are compared, based on which at least one corrupt party in $\text{Selected}_{(Z,\text{iter})} \cup \text{Selected}_{(Z’,\text{iter})}$ is guaranteed to be identified provided *all* the
parties in $\text{Selected}_{(Z, \text{iter})} \cup \text{Selected}_{(Z', \text{iter})}$ secret-share the required partitions. The cheater-identification phase will be “stuck” if the corrupt parties in $\text{Selected}_{(Z, \text{iter})} \cup \text{Selected}_{(Z', \text{iter})}$ do not participate. To prevent such corrupt parties from causing future instances of $\Pi_{\text{MultCI}}$ to fail, the parties wait-list all the parties in $\text{Selected}_{(Z, \text{iter})} \cup \text{Selected}_{(Z', \text{iter})}$. A party is then “released” only after it has shared all the required values as part of the cheater-identification phase. Every honest party is eventually released from the waiting-list. This wait-listing guarantees that corrupt parties will be barred from acting as summand-sharing parties as part of the $\Pi_{\text{OptMult}}$ instances of future invocations of $\Pi_{\text{MultCI}}$ until they participate in the cheater-identification phase of previous failed instances of $\Pi_{\text{MultCI}}$. Since the cheater-identification phase is executed asynchronously, each party maintains its own set of locally-discarded parties, where corrupt parties are included as and when they are identified.

**Protocol $\Pi_{\text{MultCI}}(\mathcal{P}, Z, S, [a], [b], \text{iter})$**

- **Initialization:** Initialize $\mathcal{W}_{\text{iter}}^{(i)} = \mathcal{L}D_{\text{iter}}^{(i)} = \emptyset$ and $\text{flag}_{\text{iter}}^{(i)} = \perp$. Fix some (publicly-known) $Z' \in Z$.
- **Running Optimistic Multiplication and Checking Pair-wise Differences:**
  - For each $Z \in Z$, participate in the instance $\Pi_{\text{OptMult}}(\mathcal{P}, Z, S, [a], [b], Z, \text{iter})$ with session id $\text{sid}$.
  - Let $\{[c_{(Z, \text{iter})}]q_i, \ldots, [c_{(Z, \text{iter})}]q_i \}_{q_i \in q_i}$ be the output obtained. Moreover, let $\text{Selected}_{(Z, \text{iter})}$ be set of summand-sharing parties and for each $P_j \in \text{Selected}_{(Z, \text{iter})}$, let $\text{Summands}_{(Z, \text{iter})}$ be set of ordered pairs of indices corresponding to the summands whose sum has been shared by $P_j$, during this instance of $\Pi_{\text{OptMult}}$.
  - Corresponding to every $Z \in Z$, participate in an instance of $\Pi_{\text{PerRec}}$ to reconstruct $c_{(Z, \text{iter})} - c_{(Z', \text{iter})}$.
- **Output in Case of Success:** If $\forall Z \in Z, c_{(Z, \text{iter})} - c_{(Z', \text{iter})} = 0$, then set $\text{flag}_{\text{iter}}^{(i)} = 0$ and output $\{c_{(Z', \text{iter})}q_i\}_{q_i \in q_i}$.
- **Waiting-List and Cheater Identification in Case of Failure:** If $\exists Z \in Z : c_{(Z, \text{iter})} - c_{(Z', \text{iter})} \neq 0$, then let $Z$ be the first set such that $c_{(Z, \text{iter})} - c_{(Z', \text{iter})} \neq 0$. Set the conflicting-sets to be $Z, Z'$, $\text{flag}_{\text{iter}}^{(i)} = 1$ and proceed as follows.
  - **Wait-listing Parties:** Set $\mathcal{W}_{\text{iter}}^{(i)} = \text{Selected}_{(Z, \text{iter})} \cup \text{Selected}_{(Z', \text{iter})}$.
  - **Sharing Partition of the Summand-Sums:**
    1. If $P_i \in \text{Selected}_{(Z, \text{iter})}$, compute $d_{(Z, \text{iter})}^{(i)} = \sum_{(p, q) \in \text{Summands}_{(Z, \text{iter})} \cap \text{Summands}_{(Z', \text{iter})}} [a]p [b]q$, for every $P_j \in \text{Selected}_{(Z, \text{iter})}$.
    2. If $P_i \in \text{Selected}_{(Z', \text{iter})}$, compute $e_{(Z', \text{iter})}^{(i)} = \sum_{(p, q) \in \text{Summands}_{(Z', \text{iter})} \cap \text{Summands}_{(Z', \text{iter})}} [a]p [b]q$, for all $P_j \in \text{Selected}_{(Z', \text{iter})}$.
  - **Removing Parties from Wait List:** Set $\mathcal{W}_{\text{iter}}^{(i)} = \mathcal{W}_{\text{iter}}^{(i)} \setminus \{P_j\}$, if all the following criteria pertaining to $P_j$ hold:
    1. $P_j \in \text{Selected}_{(Z, \text{iter})}$: if an output (share, $\text{sid}_{j, k, \text{iter}, Z}, P_j, \{[d_{(Z, \text{iter})}]q_i\}_{q_i \in q_i}$) is received from $\mathcal{F}_{\text{VSS}}$ with session id $\text{sid}_{j, k, \text{iter}, Z}$, corresponding to each $P_k \in \text{Selected}_{(Z', \text{iter})}$.
    2. $P_j \in \text{Selected}_{(Z', \text{iter})}$: if an output (share, $\text{sid}_{j, k, \text{iter}, Z'}, P_j, \{[d_{(Z', \text{iter})}]q_i\}_{q_i \in q_i}$) is received from $\mathcal{F}_{\text{VSS}}$ with session id $\text{sid}_{j, k, \text{iter}, Z'}$, corresponding to every $P_k \in \text{Selected}_{(Z, \text{iter})}$.
  - **Verifying the Summand-Sum Partitions and Locally Identifying Corrupt Parties:**
    1. For every $P_j \in \text{Selected}_{(Z, \text{iter})}$, participate in an instance of $\Pi_{\text{PerRec}}$ to reconstruct the difference value $c_{(Z, \text{iter})}^{(j)} - \sum_{P_k \in \text{Selected}_{(Z', \text{iter})}} d_{(Z', \text{iter})}^{(j)}$. If the difference is not 0, then set $\mathcal{L}D_{\text{iter}}^{(i)} = \mathcal{L}D_{\text{iter}}^{(i)} \cup \{P_j\}$.
    2. For every $P_j \in \text{Selected}_{(Z', \text{iter})}$ participate in an instance of $\Pi_{\text{PerRec}}$ to reconstruct the difference value $c_{(Z', \text{iter})}^{(j)} - \sum_{P_k \in \text{Selected}_{(Z, \text{iter})}} e_{(Z', \text{iter})}^{(j)}$. If the difference is not 0, then set $\mathcal{L}D_{\text{iter}}^{(i)} = \mathcal{L}D_{\text{iter}}^{(i)} \cup \{P_j\}$.
    3. For each ordered pair $(P_j, P_k)$ where $P_j \in \text{Selected}_{(Z, \text{iter})}$ and $P_k \in \text{Selected}_{(Z', \text{iter})}$, participate in an
instance of $\Pi_{\text{PerfRec}}$ to reconstruct $d^{(jk)}(Z,\text{iter}) - e^{(kj)}(Z',\text{iter})$. If the value is not 0, then do the following:

i. Participate in instances of $\Pi_{\text{PerfRec}}$ to reconstruct $d^{(jk)}(Z,\text{iter})$ and $e^{(kj)}(Z',\text{iter})$. Participate in instances of $\Pi_{\text{PerfRecShare}}$ to reconstruct $[a]_p$ and $[b]_q$, such that $(p, q) \in \text{Summands}^{(j)}(Z,\text{iter}) \cap \text{Summands}^{(k)}(Z',\text{iter})$.

ii. Compare $\sum_{(p, q) \in \text{Summands}^{(j)}(Z,\text{iter}) \cap \text{Summands}^{(k)}(Z',\text{iter})} [a]_p [b]_q$ with $d^{(jk)}(Z,\text{iter})$ and $e^{(kj)}(Z',\text{iter})$ and identify the corrupt party $P_c \in \{P_j, P_k\}$. Set $LD^{(i)}_{\text{iter}} = LD^{(i)}_{\text{iter}} \cup \{P_c\}$.

Figure 4: Code for $P_i$ for multiplication with cheater identification for iteration $\text{iter}$ and session id $\text{sid}$, in the $F_{\text{VSS}}$-hybrid setting.

Lemma 3.2 is proved in Appendix B. In the Lemma, we say that an instance of $\Pi_{\text{MultCI}}$ is successful, if $c(Z,\text{curr}) - c(Z',\text{curr}) = 0$ for all $Z \in Z$ with respect to the publicly-known $Z' \in Z$ fixed in the protocol, else the instance fails.

The modifications to $\Pi_{\text{MultCI}}$ for handling $M$ pairs of inputs are simple (see Appendix B), requiring $O(M \cdot |Z| \cdot n^2)$ calls to $F_{\text{VSS}}$, $O(|Z| \cdot n^2)$ calls to $F_{\text{ABA}}$ and a communication of $O((M \cdot |Z|^2 \cdot n^2 + |Z| \cdot n^4) \log |F|)$ bits.

**Lemma 3.2.** Let $Z$ satisfy the $Q(\epsilon)\left(\mathcal{P}, Z\right)$ condition and let all honest parties participate in $\Pi_{\text{MultCI}}(\mathcal{P}, Z, \mathcal{S}, [a], [b], \text{iter})$. Then, $\text{Adv}$ does not learn any additional information about $a$ and $b$. Moreover, the following hold:

- The instance will eventually be deemed to succeed or fail by the honest parties, where for a successful instance, the parties output a sharing of $ab$.
- If the instance is not successful, then the honest parties will agree on a pair $Z, Z' \in Z$ such that $c(Z,\text{iter}) - c(Z',\text{iter}) \neq 0$. Moreover, all honest parties present in the $W^{(i)}_{\text{iter}}$ set of any honest party $P_i$ will eventually be removed and no honest party is ever included in the $LD^{(i)}_{\text{iter}}$ set of any honest $P_i$. Furthermore, there will be a pair of parties $P_j, P_k$ from $\text{Selected}^{(j)}(Z,\text{iter}) \cup \text{Selected}^{(k)}(Z',\text{iter})$, with at least one of them being maliciously-corrupt, such that if both $P_j$ and $P_k$ are removed from the set $W^{(h)}_{\text{iter}}$ of any honest party $P_h$, then eventually the corrupt party(ies) among $P_j, P_k$ will be included in the set $LD^{(i)}_{\text{iter}}$ of every honest $P_i$.
- The protocol needs $O(|Z| \cdot n^2)$ calls to $F_{\text{VSS}}$ and $F_{\text{ABA}}$ and communicates $O((|Z|^2 \cdot n^2 + |Z| \cdot n^4) \log |F|)$ bits.

### 3.3 Multiplication Protocol

Protocol $\Pi_{\text{Mult}}$ (Fig 5) takes $([a], [b])$ and securely generates $[ab]$. The protocol proceeds in iterations, where in each iteration, an instance of $\Pi_{\text{MultCI}}$ is invoked. If the iteration is successful, then the parties take the output of the corresponding $\Pi_{\text{MultCI}}$ instance. Else, they proceed to the next iteration, with the cheater-identification phase of failed $\Pi_{\text{MultCI}}$ instances running in the background. Let $t$ be the cardinality of maximum-sized subset from $Z$. To upper bound the number of failed iterations, the parties run ACS after every $tn + 1$ failed iterations to “globally” include a new corrupt party in $GD$. This is done through calls to $F_{\text{ABA}}$, where the parties vote for a candidate corrupt party, based on the $LD$ sets of all failed iterations. The idea is that during these $tn + 1$ failed iterations, there will be at least one corrupt party who is eventually included in the $LD$ set of every honest party. This is because there can be at most $tn$ distinct pairs of “conflicting-parties” across the $tn + 1$ failed iterations (follows from Lemma 3.2). At least one conflicting pair, say $(P_j, P_k)$, is guaranteed to repeat among the $tn + 1$ failed instances, with both $P_j$ and $P_k$ being removed from the previous waiting-lists. Thus, the corrupt party(ies) among $P_j, P_k$ is eventually included in the $LD$ sets. There can be at most $t(nn + 1)$ failed iterations after which all the corrupt parties will be discarded and the next iteration is guaranteed to be successful, with only honest parties acting as the candidate summand-sharing parties in the underlying instances of $\Pi_{\text{OptMult}}$.

**Protocol** $\Pi_{\text{Mult}}(\mathcal{P}, Z, \mathcal{S}, [a], [b])$

- **Initialization**: Set $t \leftarrow \max\{|Z| : Z \in Z\}$, initialize $GD = \emptyset$ and $\text{iter} = 1$.
- **Multiplication with Cheater Identification**: Participate in the instance $\Pi_{\text{MultCI}}(\mathcal{P}, Z, \mathcal{S}, [a], [b], \text{iter})$ with sid.
  - **Positive Output**: If $\text{flag}^{(i)}_{\text{iter}}$ is set to 0, then output the shares obtained during the $\Pi_{\text{MultCI}}$ instance.
  - **Negative Output**: If $\text{flag}^{(i)}_{\text{iter}}$ is set to 1 during the $\Pi_{\text{MultCI}}$ instance, then proceed as follows.
    - **Identifying a Cheater Party Through ACS**: If $\text{iter} = k \cdot [tn + 1]$ for some $k \geq 1$, then do the following.
Let $L_D^{(1)}$ be the set of locally-discarded parties for the instance $\Pi_{MultCI}(P, Z, S, [a], [b], r)$, for $r = 1, \ldots, \text{iter}$. For $j = 1, \ldots, n$, send $(\text{vote}, \text{sid}_{j, \text{iter}, k})$ to $F_{ABA}$ where $\text{sid}_{j, \text{iter}, k} \leftarrow \text{sid}_{\text{iter}, k} \oplus \{j\} \oplus k$, if for any $r \in \{1, \ldots, \text{iter}\}$, party $P_j$ is present in $L_D^{(1)}$ and $P_j \not\in GD$.

2. For $j = 1, \ldots, n$, keep requesting for an output from $F_{ABA}$ with $\text{sid}_{j, \text{iter}, k}$, until an output is received.

3. If $\exists P_j \in \mathcal{P}$ such that $(\text{decide}, \text{sid}_{j, \text{iter}, k}, 1)$ is received from $F_{ABA}$ with $\text{sid}_{j, \text{iter}, k}$, then for each $P_i \in \mathcal{P}$, for which no vote message has been sent yet, send $(\text{vote}, \text{sid}_{i, \text{iter}, k}, 0)$ to $F_{ABA}$ with $\text{sid}_{i, \text{iter}, k}$.

4. Once an output $(\text{decide}, \text{sid}_{i, \text{iter}, k}, \ell)$ is received from $F_{ABA}$ with $\text{sid}_{i, \text{iter}, k}$ for every $\ell \in \{1, \ldots, n\}$, select the minimum indexed party $P_j$ from $\mathcal{P}$, such that $\ell = j$. Then set $GD = GD \cup \{P_j\}$, set iter = iter + 1 and go to the step labelled Multiplication with Cheater Identification.

- Else set iter = iter + 1 and go to the step Multiplication with Cheater Identification.

Lemma 3.3 is proved in Appendix B. To handle $M$ pairs of inputs, the instances of $\Pi_{MultCI}$ are now executed with $M$ pairs of inputs in each iteration. This requires $O(M \cdot |Z| \cdot n^5)$ calls to $F_{VSS}$, $O(|Z| \cdot n^5)$ calls to $F_{ABA}$ and a communication of $O((M \cdot |Z|^2 \cdot n^5 + |Z| \cdot n^7) \log |P|)$ bits.

**Lemma 3.3.** Let $Z$ satisfy the $Q^{(4)}(\mathcal{P}, Z)$ condition and let $S = (S_1, \ldots, S_n) = \mathcal{P} \setminus Z \subseteq Z$. Then $\Pi_{Mult}$ takes at most $(t n + 1)$ iterations and all honest parties eventually output a secret-sharing of $[ab]$, where $t = \max\{|Z| : Z \subseteq Z\}$. In the protocol, Adv does not learn anything additional about $a$ and $b$. The protocol makes $O(|Z| \cdot n^5)$ calls to $F_{VSS}$ and $F_{ABA}$ and additionally incurs a communication of $O(|Z|^2 \cdot n^5 \log |P| + |Z| \cdot n^7 \log |P|)$ bits.

### 3.4 The Pre-Processing Phase Protocol

The perfectly-secure pre-processing phase protocol $\Pi_{PerTriples}$ is standard. The parties first jointly generate secret-sharing of $M$ random pairs of values, followed by running an instance of $\Pi_{Mult}$ to securely compute the product of these pairs. Protocol $\Pi_{Mult}$ and the proof of Theorem 3.4 is provided in Appendix B.

**Theorem 3.4.** If $Z$ satisfies the $Q^{(4)}(\mathcal{P}, Z)$ condition, then $\Pi_{PerTriples}$ is a perfectly-secure protocol for securely realizing $F_{Triples}$ with UC-security in the $(F_{VSS}, F_{ABA})$-hybrid model. The protocol makes $O(M \cdot |Z| \cdot n^5)$ calls to $F_{VSS}$, $O(|Z| \cdot n^5)$ calls to $F_{ABA}$ and incurs a communication of $O(M \cdot |Z|^2 \cdot n^5 \log |P| + |Z| \cdot n^7 \log |P|)$ bits.

### 4 Statistically-Secure Pre-Processing Phase Protocol with $Q^{(3)}(\mathcal{P}, Z)$ Condition

We first present an asynchronous information-checking protocol (AICP) with $Q^{(3)}(\mathcal{P}, Z)$ condition.

#### 4.1 Asynchronous Information Checking Protocol (AICP)

An ICP [33, 16] is used for authenticating data in the presence of a computationally-unbounded adversary. An AICP [30] extends ICP for the asynchronous setting. In an AICP, there are four entities, a signer $S \in \mathcal{P}$, an intermediary $I \in \mathcal{P}$, a receiver $R \in \mathcal{P}$ and all the parties in $\mathcal{P}$ acting as verifiers (note that $S, I$ and $R$ also act as verifiers). An AICP has two sub-protocols, one for the authentication phase and one for the revelation phase.

In the authentication phase, $S$ has some private input $s \in \mathbb{F}$, which it distributes to $I$ along with some authentication information. Each verifier is provided with some verification information, followed by the parties verifying whether $S$ has distributed consistent information. The data held by $I$ at the end of this phase is called $S$’s IC-Signature on $s$ for intermediary $I$ and receiver $R$, denoted by $\text{ICSig}(S, I, R, s)$. Later, during the revelation phase, $I$ reveals $\text{ICSig}(S, I, R, s)$ to $R$, who “verifies” it with respect to the verification information provided by the verifiers and decides whether to accept or reject $s$. We require the same security guarantees from AICP as expected from digital signatures, namely correctness, unforgeability and non-repudiation. Additionally, we will need the privacy property guaranteeing that if $S, I$ and $R$ are all honest, then Adv does not learn $s$.

Our AICP is a generalization of the AICP of [30], which was designed against threshold adversaries. During the authentication phase, $S$ embeds $s$ in a random $t$-degree signing-polynomial $F(x)$, where $t$ is the cardinality of maximum-sized subset in $Z$, and gives $F(x)$ to $I$. In addition, each verifier $P_i$ is given a random verification-point.
mials and points. The linear combiner is randomly selected by any inconsistency among the data distributed by a SV the verification-points to sufficiently many verifiers in a set S the authentication phase, the parties interact in a “zero-knowledge” fashion to verify the consistency of the distributed information. For this, S additionally distributes a random t-degree masking-polynomial M(x) to I, while each verifier P_i is given a point on M(x) at a distinct α_i. The parties then publicly check the consistency of the F(x), M(x) polynomials and the distributed points, with respect to a random linear combination of these polynomials and points. The linear combiner is randomly selected by I, only when it is confirmed that S has distributed the verification-points to sufficiently many verifiers in a set SV, which we call supporting verifiers. This ensures that S has no knowledge beforehand about the random combiner while distributing the points to SV and hence, any inconsistency among the data distributed by a corrupt S will be detected with a high probability.

**Protocol AICP**

Protocol \( \Pi_{Auth}(P, Z, S, I, R, s) \)

- **Distributing the Polynomials and the Verification Points:** Only S executes the following steps.
  - Randomly select t-degree signing-polynomial \( F(x) \) and masking-polynomial \( M(x) \), such that \( F(0) = s \), where \( t = \max \{ |Z| : Z \in Z \} \). For \( j = 1, \ldots, n \), randomly select \( \alpha_j \in \mathbb{F} \setminus \{0\} \), compute \( v_j = F(\alpha_j), m_j = M(\alpha_j) \).
  - Send \((\text{authPoly}, \text{id}, F(x), M(x))\) to I. For \( j = 1, \ldots, n \), send \((\text{authPoint}, \text{id}, (\alpha_j, v_j, m_j))\) to party \( P_j \).
- **Confirming Receipt of Verification Points:** Each party \( P_i \) (including S, I and R) upon receiving \((\text{authPoint}, \text{id}, (\alpha_i, v_i, m_i))\) from S, sends \((\text{Received}, \text{id}, i)\) to I.
- **Announcing Masked Polynomial and Set of Supporting Verifiers:**
  - I, upon receiving \((\text{Received}, \text{id}, j)\) from a set of parties \( SV \) where \( P \setminus SV \in Z \), randomly picks \( d \in \mathbb{F} \setminus \{0\} \)
    and sends \((\text{sender}, \text{Acast}, \text{id}, (d, B(x), SV))\) to \( F_{\text{Acast}} \), where \( \text{id} = \text{id}||l \) and \( B(x) = df(x) + M(x) \).
  - Every party \( P_i \in P \) keeps requesting for output from \( F_{\text{Acast}} \) with \( \text{id} \) until an output is received.
- **Announcing Validity of Masked Polynomial:**
  - S, upon receiving an output \((l, \text{Acast}, \text{id}, (d, B(x), SV))\) from \( F_{\text{Acast}} \) with \( \text{id} \), checks if \( B(x) \) is a t-degree polynomial, \( \mathbb{P} \setminus SV \in Z \) and \( dv_j + m_j = B(\alpha_j) \) holds for all \( P_j \in SV \). If yes, then it sends \((\text{sender}, \text{Acast}, \text{id}, \text{OK})\) to \( F_{\text{Acast}} \), where \( \text{id} = \text{id}||S \). Else, it sends \((\text{sender}, \text{Acast}, \text{id}, \text{NOK}, s)\) to \( F_{\text{Acast}} \).
  - Every party \( P_i \in P \) keeps requesting for output from \( F_{\text{Acast}} \) with \( \text{id} \) until an output is received.
- **Deciding Whether Authentication is Successful:**
  - Every party \( P_i \) (including S, I and R) upon receiving \((l, \text{Acast}, \text{id}, (d, B(x), SV))\) from \( F_{\text{Acast}} \) with \( \text{id} \), sets the variable authCompleted\(_{S,1,R}^{\text{id},i} \) to 1 or either of the following holds.
    - \((\text{sender}, \text{Acast}, \text{id}, \text{NOK}, s)\) if \( B(x) \) is a t-degree polynomial, \( \mathbb{P} \setminus SV \in Z \) and \( dv_j + m_j = B(\alpha_j) \) holds for all \( P_j \in SV \). If yes, then it sends \((\text{sender}, \text{Acast}, \text{id}, \text{OK})\) to \( F_{\text{Acast}} \) with \( \text{id} \). Else, it sends \((\text{sender}, \text{Acast}, \text{id}, \text{NOK}, s)\) to \( F_{\text{Acast}} \).
  - In this case, \( P_i \) also sets ICSig(S, I, R, s) = s.
- **Revealing Signing Polynomial and Verification Points:**
  - Each party \( P_i \) (including S, I and R) does the following, if authCompleted\(_{S,1,R}^{\text{id},i} \) is set to 1 and ICSig(S, I, R, s) has not been publicly set during \( \Pi_{Auth} \).
    - If \( P_i = 1 \) then send \((\text{revealPoint}, \text{id}, F(x))\) to R, where ICSig(S, I, R, s) has been set to \( F(x) \) during \( \Pi_{Auth} \).
    - If \( P_i \in SV \), then send \((\text{revealPoint}, \text{id}, (\alpha_i, v_i, m_i))\) to R.
- **Accepting or Rejecting the IC-Sig:**
  - The following steps are executed only by R, if authCompleted\(_{S,1,R}^{\text{id},i} \) is set to 1 by R during the protocol \( \Pi_{Auth}(P, Z, S, I, R, s) \), where \( R = P_i \).
    - If R has set ICSig(S, I, R, s) = s during \( \Pi_{Auth} \), then output s. Else, wait till \((\text{revealPoint}, \text{id}, F(x))\) is received from I, where \( F(x) \) is a t-degree polynomial. Then proceed as follows.
      1. If \((\text{revealPoint}, \text{id}, (\alpha_j, v_j, m_j))\) is received from \( P_j \in SV \), then accept \((\alpha_j, v_j, m_j)\) if either \( v_j = F(\alpha_j) \) or \( B(\alpha_j) \neq dv_j + m_j \), where \( B(x) \) is received from \( F_{\text{Acast}} \) with \( \text{id} \), during \( \Pi_{Auth} \).
      2. Wait till a subset of parties SV' \subseteq SV is found, such that SV' \setminus SV' = Z and for every \( P_j \in SV' \), the corresponding revealed point \((\alpha_j, v_j, m_j)\) is accepted. Then output s = F(0).

\*If S broadcasts s along with NOK, then ICSig will be set publicly to s, while if S broadcasts OK then only I sets ICSig to F(x).\*

Figure 6: The asynchronous information-checking protocol against general adversaries for session id in the \( F_{\text{Acast}} \)-hybrid
Lemma 4.1. Let $Z$ satisfy the $Q^{(3)}(\mathcal{P}, Z)$ condition. Then the pair of protocols $(\Pi_{Auth}, \Pi_{Reveal})$ satisfy the following properties, except with probability at most $\epsilon_{AICP} \overset{def}{=} \frac{nt}{|F|-1}$, where $t = \max\{|Z| : Z \in Z\}$.

- **Correctness:** If $S, I$ and $R$ are honest, then each honest $P_i$ eventually sets authCompleted$_{(sid, i)}$ to $1$ during $\Pi_{Auth}$. Moreover, $R$ eventually outputs $s$ during $\Pi_{Reveal}$.

- **Privacy:** If $S, I$ and $R$ are honest, then the view of adversary remains independent of $s$.

- **Unforgeability:** If $S, R$ are honest, $I$ is corrupt and if $R$ outputs $s' \in F$ during $\Pi_{Reveal}$, then $s' = s$ holds.

- **Non-repudiation:** If $S$ is corrupt and $I, R$ are honest and if $I$ has set ICSig(S, I, R, s) during $\Pi_{Auth}$, then $R$ eventually outputs $s$ during $\Pi_{Reveal}$.

Protocol $\Pi_{Auth}$ requires a communication of $O(n \cdot \log |F|)$ bits and $O(1)$ calls to $FAcast$ with $O(n \cdot \log |F|)$-bit messages. Protocol $\Pi_{Reveal}$ requires a communication of $O(n \cdot \log |F|)$ bits.

Lemma 4.1 is proven in Appendix C.1. We use the following notations for AICP in our statistical VSS protocol.

**Notation 4.2 (Notation for Using AICP).** While using $(\Pi_{Auth}, \Pi_{Reveal})$, we will say that:

- “$P_i$ gives ICSig(sid, $P_i$, $P_j$, $P_k$, s) to $P_j$” to mean that $P_i$ acts as $S$ and invokes an instance of the protocol $\Pi_{Auth}$ with session id sid, where $P_j$ and $P_k$ plays the role of $I$ and $R$ respectively.

- “$P_j$ receives ICSig(sid, $P_i$, $P_j$, $P_k$, s) from $P_i$” to mean that $P_j$, as $I$, has set authCompleted$_{(sid, j)}$ to $1$ during protocol $\Pi_{Auth}$ with session id sid, where $P_i$ and $P_k$ plays the role of $S$ and $R$ respectively.

- “$P_j$ reveals ICSig(sid, $P_i$, $P_j$, $P_k$, s) to $P_k$” to mean $P_j$, as $I$, invokes an instance of $\Pi_{Reveal}$ with session id sid, with $P_i$ and $P_k$ playing the role of $S$ and $R$ respectively.

- “$P_k$ accepts ICSig(sid, $P_i$, $P_j$, $P_k$, s)” to mean that $P_k$, as $R$, outputs $s$ during the instance of $\Pi_{Reveal}$ with session id sid, invoked by $P_j$ as $I$, with $P_i$ playing the role of $S$.

### 4.2 Statistically-Secure VSS Protocol with $Q^{(3)}(\mathcal{P}, Z)$ Condition

The high level idea behind our statistically-secure protocol $\Pi_{SVSS}$ (Figure 7) is similar to that of the perfectly-secure VSS protocol $\Pi_{PVSS}$ (see Fig 12 in Appendix B.1). In $\Pi_{PVSS}$, dealer $P_D$, on having the shares $(s_1, \ldots, s_h)$, sends $s_q$ to the parties in $S_q \in S$, followed by the parties in $S_q$ performing pairwise consistency tests of their supposedly common shares and publicly announcing the results. Based on these results, the parties identify a core set $C_q \subseteq S_q$ where $S_q \setminus C_q \in Z$, such that all the (honest) parties in $C_q$ have received the same share $s_q$ from $P_D$. Once such a $C_q$ is identified, then the honest parties in $C_q$, forming a “majority”, can “help” the (honest) parties in $S_q \setminus C_q$ get this common $s_q$. However, since $Z$ now satisfies the $Q^{(3)}(\mathcal{P}, Z)$ condition, $C_q$ may have only one honest party. Consequently, the “majority-based filtering” used by the parties in $S_q \setminus C_q$ to get $s_q$ will fail.

To deal with the above problem, the parties in $S_q$ issue IC-Signatures during the pair-wise tests of their supposedly common shares. The parties then check whether the common share $s_q$ held by the (honest) parties in $C_q$ is “$(P_i, P_j, P_k)$-authenticated” for every $P_i, P_j \in C_q$ and every $P_k \in S_q$; i.e. $P_i$ holds ICSig($P_j, P_k, s_q$). Now, to help the parties $P_k \in S_q \setminus C_q$ obtain the common share $s_q$, every $P_j \in C_q$ reveals IC-signed $s_q$ to $P_k$, signed by every $P_i \in C_q$. Since $C_q$ is bound to contain at least one honest party, a corrupt $P_j$ will fail to forge an honest $P_i$’s IC-signature on an incorrect $s_q$. On the other hand, an honest $P_j$ will be able to eventually reveal the IC-signature of all the parties in $C_q$ on the share $s_q$, which is accepted by $P_k$.

**Protocol $\Pi_{SVSS}$**

- **Distribution of Shares:** $P_D$, on having input $(s_1, \ldots, s_h)$, sends (dist, sid, q, $s_q$) to all $P_i \in S_q$, for $q = 1, \ldots, h$.

- **Pairwise Consistency Tests on IC-Signed Values:** For each $S_q \in S$, each $P_i \in S_q$ does the following.
  - Upon receiving (dist, sid, q, $s_q$) from $D$, give ICSig(sid$_{(i,j,k)}$, $P_i$, $P_j$, $P_k$, $s_q$) to every $P_j \in S_q$, corresponding to every $P_k \in S_q$, where sid$_{(i,j,k)}$ = sid$[P_D||q||i||j]$.  
    - Upon receiving ICSig(sid$_{(i,j,k)}$, $P_j$, $P_k$, $s_q$) from $P_j \in S_q$ corresponding to every party $P_k \in S_q$, if $s_{q_i}$ = $s_q$ holds, then send (sender, Acast, sid$_{(P_i,q)}$, OK$_q(i, j)$) to $FAcast$, where sid$_{(i,j)}$ = sid$[P_D||q||i|j]$.
  - **Constructing Consistency Graph:** For each $S_q \in S$, each $P_i \in \mathcal{P}$ executes the following steps.
    - Initialize a set $C_q$ to $\emptyset$. Construct an undirected consistency graph $G_q^{(3)}$ with $S_q$ as the vertex set.
For every $P_j, P_k \in S_q$, keep requesting an output from $\cal F_{Acast}$ with $\text{sid}_{j,k}^{(P_j, q)}$, until an output is received.

Add the edge $(P_j, P_k)$ to $G_q^{(i)}$ if $(P_j, \text{Acast}, \text{sid}_{j,k}^{(P_j, q)}, \text{OK}_q(j, k))$ and $(P_k, \text{Acast}, \text{sid}_{k,j}^{(P_k, q)}, \text{OK}_q(k, j))$ is received from $\cal F_{Acast}$ with $\text{sid}_{j,k}^{(P_j, q)}$ and $\text{sid}_{k,j}^{(P_k, q)}$ respectively.

- **Identification of Core Sets and Public Announcements:** $P_0$ executes the following steps to compute the core sets.
  - For each $S_q \in \cal S$, check if there exists a subset of parties $W_q \subseteq S_q$, such that $S_q \setminus W_q \in \cal Z$, and the parties in $W_q$ form a clique in the consistency graph $G_q^{(i)}$. If such a $W_q$ exists, then assign $C_q := W_q$.
  - Once $C_1, \ldots, C_h$ are computed, send ($\text{sender}, \text{Acast}, \text{sid}_{P_0}, \{C_q\}_{q \in \cal S}$), where $\text{sid}_{P_0} = \text{sid}\|P_0$.

- **Share computation:** Each $P_i \in \cal P$ executes the following steps.
  - Keep requesting for output from $\cal F_{Acast}$ with $\text{sid}_i$ until an output is received.
  - Upon receiving an output ($\text{sender}, \text{Acast}, \text{sid}_{P_0}, \{C_q\}_{q \in \cal S}$) from $\cal F_{Acast}$ with $\text{sid}_{P_0}$, wait until the parties in $C_q$ form a clique in $G_q^{(i)}$, corresponding to each $S_q \in \cal S$.
    - For $q = 1, \ldots, h$, verify if $S_q \setminus C_q \in \cal Z$.
    - If the verification is successful, then proceed to compute the shares corresponding to each $S_q$ such that $P_i \in S_q$ as follows.
      1. If $P_i \in C_q$ then set $[s]_q = s_{qi}$ and corresponding to every signer $P_j \in C_q$, reveal $\text{ICSig}(\text{sid}_{j,k,i}^{(P_j, q)}, P_j, P_i, s_{qi})$ to every receiver party $P_k \in S_q \setminus C_q$.
      2. If $P_i \notin C_q$, then wait till $P_i$ finds some $P_j \in C_q$ such that $P_i$ has accepted $\text{ICSig}(\text{sid}_{j,k,i}^{(P_j, q)}, P_k, P_j, s_{qj})$ revealed by the intermediary $P_j$, corresponding to every signer $P_k \in C_q$. Then set $[s]_q = s_{qj}$.
  - Upon computing $\{[s]_q\}_{P_i \in S_q}$, output (share, sid, $P_0$, $\{[s]_q\}_{P_i \in S_q}$).

Figure 7: The statistically-secure VSS protocol for session id sid for realizing $\cal F_{VSS}$ in the $\cal F_{Acast}$-hybrid model

The properties of the protocol $\Pi_{VSS}$ stated in Theorem 4.3 are proven in Appendix C.2

**Theorem 4.3.** Let $\cal Z$ satisfy the $\Omega(3)(\cal P, \cal Z)$ condition. Then $\Pi_{VSS}$ UC-securely realizes $\cal F_{VSS}$ in the $\cal F_{Acast}$-hybrid model, except with error probability $|\cal Z| n^3 \epsilon_{\text{AICP}}$, where $\epsilon_{\text{AICP}} \approx \frac{n^2}{|\cal P|}$. The protocol makes $\cal O(|\cal Z| \cdot n^3)$ calls to $\cal F_{Acast}$ with $\cal O(n \cdot \log |\cal P|)$ bit messages and additionally incurs a communication of $\cal O(|\cal Z| \cdot n^4 \log |\cal P|)$ bits. By replacing the calls to $\cal F_{Acast}$ with protocol $\Pi_{Acast}$, the protocol incurs a total communication of $\cal O(|\cal Z| \cdot n^6 \log |\cal P|)$ bits.

### 4.2.1 Statistically-Secure VSS for Superpolynomial $|\cal Z|$

The error probability of $\Pi_{VSS}$ depends linearly on $|\cal Z|$ (Theorem 4.3), which is problematic for a large sized $\cal Z$. We now discuss modifications to the protocols $\Pi_{Auth}/\Pi_{Reveal}$, followed by the modifications in the way they are used in $\Pi_{VSS}$, so as to ensure that the error probability of $\Pi_{VSS}$ is only $n^2 \cdot \epsilon_{\text{AICP}}$, irrespective of the number of invocations of $\Pi_{VSS}$. The idea is to use local “dispute control” as used in [23], where the parties locally discard corrupt parties as and when they are identified to be cheating during any instance of $\Pi_{Auth}/\Pi_{Reveal}$. Once a party $P_j$ is locally discarded by some $P_i$, then $P_i$ “behaves” as if $P_j$ has certainly behaved maliciously in all “future” instances of $\Pi_{Auth}/\Pi_{Reveal}$, irrespective of whether this is not the case or not.

**Modifications in $\Pi_{Auth}$ and $\Pi_{Reveal}$:** Each $P_i$ maintains a list of locally-discarded parties $\cal LD_{(i)}$, which it keeps on populating across all the invoked instances of $\Pi_{Auth}$ and $\Pi_{Reveal}$. In any instance of $\Pi_{Auth}$, if $P_i \in \cal SV$ receives an OK message from $\cal S$ even though $B(\alpha_i) \neq \text{dev}_i + m_i$ holds, then $P_i$ adds $S$ to $\cal LD_{(i)}$. Once $P_i$ adds $S$ to $\cal LD_{(i)}$, then in any future instance of $\Pi_{Reveal}$ involving the signer $S$, party $P_i$, if present in the corresponding $\cal SV$ set, sends a special “dummy” point to the corresponding receiver $R$, instead of the verification-point received from $S$, and this dummy point is always accepted by $R$. This ensures that once the verifier $P_i$ catches a corrupt $S$ trying to break the non-repudiation property by distributing inconsistent verification-point to $P_i$, then in any future instance of $\Pi_{Auth}$, if $P_i$ is added to the corresponding $\cal SV$ set, its verification-point will always be accepted.

Similarly, if in any instance of $\Pi_{Reveal}$ where $P_i$ is the receiver, $P_i$ is sure that it has not accepted the verification-point of some honest verifier belonging to $\cal SV$, then $P_i$ includes the corresponding intermediary $l$ to $\cal LD_{(i)}$. To check this, in $\Pi_{Reveal}$, $P_i$ now additionally checks if there exists a set of verifiers $\cal SV'' \subseteq \cal SV$, where $\cal SV \setminus \cal SV'' \in \cal Z$, such that the verification-points received from all the parties in $\cal SV''$ are not accepted. Once $P_i$ adds $l$ to $\cal LD_{(i)}$, in any future instance of $\Pi_{Reveal}$ involving $l$ as intermediary and $P_i$ as the receiver, $P_i$ rejects the IC-signature revealed by $l$. This ensures that once $P_i$, as a receiver catches $l$ trying to break the unforgeability property, then from then onwards, $P_i$ cannot do so in any other instance of $\Pi_{Reveal}$ involving $P_i$ as the receiver.
Modifications in $\Pi_{SVSS}$: Party $P_i$ now broadcasts a single OK$(i, j)$ message for $P_j$, only after receiving the corresponding signature from all the instances of $\Pi_{Auth}$ involving $P_j$ as the signer and $P_i$ as the intermediary, followed by pair-wise consistency tests. Consequently, $P_D$ now finds a common core set $C$ across all the sets $S_1, \ldots, S_i$, where $S_q \subset C \in Z$ for each $S_q$, and where the parties in $C$ constitute a clique. Moreover, each verifier now waits for all instances of $\Pi_{Auth}$ between a signer $S$ and an intermediary $I$ in $C$ to complete (by checking if the corresponding authCompleted variables are all set to 1), before participating in any instance of $\Pi_{Reveal}$.

The above modification ensures that if a corrupt signer in $C$ gives any verifier an inconsistent verification-point during any instance of $\Pi_{Auth}$, it will be caught and locally discarded, except with probability $\epsilon_{AICP}$. By considering all possibilities for a corrupt signer and an honest verifier, it follows that except with probability at most $n^2 \cdot \epsilon_{AICP}$, the verification-points of all honest verifiers will be accepted by every honest receiver during all the instances of $\Pi_{Reveal}$ in any instance of $\Pi_{SVSS}$. On the other hand, if any corrupt intermediary in $C$ tries to forge a signature on the behalf of an honest party in $C$, then except with probability $\epsilon_{AICP}$, it will be discarded by an honest receiver $R$. From then on, $R$ will always reject any signature revealed by the same intermediary. Hence, by considering all possibilities for a corrupt intermediary and an honest receiver, except with probability $n^2 \cdot \epsilon_{AICP}$, no corrupt intermediary will be able to forge a signature to any honest receiver in any instance of $\Pi_{SVSS}$.

Based on the above discussion, we state the following lemma.

Lemma 4.4. The modified $\Pi_{SVSS}$ has error probability of $n^2 \cdot \epsilon_{AICP}$, independent of the number of invocations.

4.3 Statistically-Secure Protocol for $\mathcal{F}_{Triples}$ in the $(\mathcal{F}_{VSS}, \mathcal{F}_{ABA})$-Hybrid

Our statistically-secure protocol $\Pi_{StatTriples}$ for realizing $\mathcal{F}_{Triples}$ with $\mathbb{Q}^{(3)}(\mathcal{P}, \mathbb{Z})$ condition mostly follows [23]. Here, we discuss the high level ideas and refer to Appendix C for formal details and proofs. To explain the idea at a high-level, we consider the case when $M = 1$ multiplication-triple is generated through $\Pi_{StatTriples}$. The modifications to generate $M$ multiplication-triples are straight forward. Protocol $\Pi_{StatTriples}$ is almost the same as $\Pi_{PerTriples}$, except that we now use a statistically-secure multiplication protocol.

Basic Multiplication Protocol: Our starting point is the basic multiplication protocol of [23] in the synchronous setting. The protocol takes $[a], [b]$, along with a set of globally-discarded parties $\mathcal{G}D$ which are guaranteed to be corrupt, and outputs $[c]$. In the protocol, each summand $[a]_{p} [b]_{q}$ is assigned to a publicly-known designated party from $\mathcal{P} \setminus \mathcal{G}D$. Every designated summand-sharing party then secret-shares the sum of all the assigned summands, based on which the parties compute $[c]$. If no summand-sharing party behaves maliciously, then $[c] = [ab]$ holds.

Similar to $\Pi_{OptMult}$, the main challenge while running the above protocol in the asynchronous setting is that a corrupt summand-sharing party may never share the sum of the assigned summands. To deal with this issue, similar to what was done for $\Pi_{OptMult}$, we ask each party in $\mathcal{P} \setminus \mathcal{G}D$ to share the sum of all possible summands it is capable of, while ensuring that no summand is shared twice. The idea here is that since $\mathbb{Z}$ satisfies the $\mathbb{Q}^{(3)}(\mathcal{P}, \mathbb{Z})$ condition, for every summand $[a]_{p} [b]_{q}$, the set $(S_p \cap S_q) \setminus \mathcal{G}D$ is guaranteed to contain at least one honest party who will share $[a]_{p} [b]_{q}$. Based on this above idea, we design a protocol $\Pi_{BasicMult}$ which is executed with respect to a set $\mathcal{G}D$, and an iteration number iter. Looking ahead, it will be guaranteed that no honest party is ever included in $\mathcal{G}D$. The protocol is similar to $\Pi_{OptMult}$, except that it does not take any subset $Z \in \mathbb{Z}$ as input.

Detectable Random-Triple Generation: Based on $\Pi_{BasicMult}$, we design a protocol $\Pi_{RandMultClt}$, which takes as input an iteration number iter and an existing set of corrupt parties $\mathcal{G}D$. If no party in $\mathcal{P} \setminus \mathcal{G}D$ behaves maliciously, then the protocol outputs a random secret-shared multiplication-triple $[a_{iter}], [b_{iter}], [c_{iter}]$. Else, with probability $\frac{1}{|\mathcal{F}|}$, the parties update $\mathcal{G}D$ by identifying at least one new corrupt party among $\mathcal{P} \setminus \mathcal{G}D$. In the protocol, the parties first generate secret-sharing of random values $a_{iter}, b_{iter}, a'_{iter}$ and $r_{iter}$. Two instances of $\Pi_{BasicMult}$ with inputs $[a_{iter}], [b_{iter}]$ and $[a_{iter}], [b'_{iter}]$ are run to obtain $[c_{iter}]$ and $[c'_{iter}]$ respectively. The parties then reconstruct the “challenge” $r_{iter}$ and publicly check if $[a_{iter}] (r_{iter} [b_{iter}] + [b'_{iter}]) = (r_{iter} [c_{iter}] + [c'_{iter}])$ holds, which should be the case if no cheating has occurred during the instances of $\Pi_{BasicMult}$. If the condition holds, then the parties output $[a_{iter}], [b_{iter}], [c_{iter}]$, which is guaranteed to be a multiplication-triple, except with probability $\frac{1}{|\mathcal{F}|}$. Otherwise, the parties proceed to identify at least one new corrupt party by reconstructing $[a_{iter}], [b_{iter}], [b'_{iter}], [c_{iter}], [c'_{iter}]$ and the sum of the summands shared by the various summand-sharing parties during the instances of $\Pi_{BasicMult}$.
The Statistically-Secure Pre-Processing Phase Protocol: Protocol $\Pi_{\text{StatTriples}}$ proceeds in iterations, where in each iteration an instance of $\Pi_{\text{RandMultCI}}$ is invoked, which either succeeds or fails. In case of success, the parties output the returned multiplication-triple, else, they continue to the next iteration. As a new corrupt party is discarded in each failed iteration, the protocol eventually outputs a multiplication-triple.

**Theorem 4.5.** Let $Z$ satisfy the $Q^{(3)}(P, Z)$ condition. Then $\Pi_{\text{StatTriples}}$ UC-securely realizes $F_{\text{Triples}}$ in the $(F_{\text{VSS}}, F_{\text{ABA}})$-hybrid model, except with error probability of at most $\frac{n^2}{|Z|}$. The protocol makes $O(n^3 \cdot M)$ calls to $F_{\text{VSS}}$ and $O(n^3)$ calls to $F_{\text{ABA}}$, and additionally communicates $O((M \cdot |Z| \cdot n^3 + |Z| \cdot n^4) \log |F|)$ bits.

By replacing the calls to $F_{\text{VSS}}$ with protocol $\Pi_{\text{VSS}}$ (along with the modifications discussed in Section 4.2.1), protocol $\Pi_{\text{StatTriples}}$ UC-securely realizes $F_{\text{Triples}}$ in the $F_{\text{ABA}}$-hybrid model, except with error probability $n^2 \cdot \epsilon_{\text{AICP}}$. The protocol makes $O(n^3)$ calls to $F_{\text{ABA}}$ and incurs a communication of $O(M \cdot |Z| \cdot n^3 \log |F|)$ bits.

5 MPC Protocols in the Pre-Processing Model

The MPC protocol $\Pi_{\text{AMPC}}$ in the pre-processing model is standard. The parties first generate secret-shared random multiplication-triples through $F_{\text{Triples}}$. Each party then randomly secret-shares its input for $\text{ckt}$ through $F_{\text{VSS}}$. To avoid an indefinite wait, the parties agree on a common subset of parties, whose inputs are eventually secret-shared, through ACS. The parties then jointly evaluate each gate in $\text{ckt}$ in a secret-shared fashion by generating a secret-sharing of the gate-output from a secret-sharing of the gate-input(s). Linear gates are evaluated non-interactively due to the linearity of secret-sharing. To evaluate multiplication gates, the parties deploy Beaver’s method [3], using the secret-shared multiplication-triples generated by $F_{\text{Triples}}$. Finally, the parties publicly reconstruct the secret-shared function output.

**Theorem 5.1.** Protocol $\Pi_{\text{AMPC}}$ UC-securely realizes the functionality $F_{\text{AMPC}}$ for securely computing $f$ (see Fig 8 in Appendix A) with perfect security in the $(F_{\text{Triples}}, F_{\text{VSS}}, F_{\text{ABA}})$-hybrid model, in the presence of a static malicious adversary characterized by an adversary-structure $Z$ satisfying the $Q^{(3)}(P, Z)$ condition. The protocol makes one call to $F_{\text{Triples}}$ and $O(n)$ calls to $F_{\text{VSS}}$ and $F_{\text{ABA}}$ and additionally incurs a communication of $O(M \cdot |Z| \cdot n^2 \log |F|)$ bits, where $M$ is the number of multiplication gates in the circuit $\text{ckt}$ representing $f$.

If we replace the calls to $F_{\text{Triples}}$ and $F_{\text{VSS}}$ with perfectly-secure protocol $\Pi_{\text{PerTriples}}$ and $\Pi_{\text{PVSS}}$ respectively, then protocol $\Pi_{\text{AMPC}}$ achieves perfect security in the $F_{\text{ABA}}$-hybrid. On the other hand, replacing the calls to $F_{\text{Triples}}$ and $F_{\text{VSS}}$ in $\Pi_{\text{AMPC}}$ with $\Pi_{\text{StatTriples}}$ and $\Pi_{\text{SVSS}}$ respectively leads to statistical-security. To bound the error probability of the statistically-secure protocol by $2^{-\kappa}$, we select a finite field $F$ such that $|F| > n^{42\kappa}$. Based on the above discussion, we get the following corollaries of Theorem 5.1.

**Corollary 5.2.** If $Z$ satisfies the $Q^{(1)}(P, Z)$ condition, then $\Pi_{\text{AMPC}}$ UC-securely realizes $F_{\text{AMPC}}$ in the $F_{\text{ABA}}$-hybrid model with perfect security. The protocol makes $O(|Z| \cdot n^3)$ calls to $F_{\text{ABA}}$ and incurs a communication of $O(M \cdot (|Z|^2 \cdot n^7 \log |F| + |Z| \cdot n^9 \log n))$ bits, where $M$ is the number of multiplication gates in $\text{ckt}$.

**Corollary 5.3.** If $Z$ satisfies the $Q^{(3)}(P, Z)$ condition, then $\Pi_{\text{AMPC}}$ UC-securely realizes $F_{\text{AMPC}}$ in the $F_{\text{ABA}}$-hybrid model with statistical security. If $|F| > n^{42\kappa}$ for a given statistical-security parameter $\kappa$, then the error probability of the protocol is at most $2^{-\kappa}$. The protocol makes $O(n^3)$ calls to $F_{\text{ABA}}$ and incurs a communication of $O(M \cdot |Z| \cdot n^9 \log |F|)$ bits, where $M$ is the number of multiplication gates in $\text{ckt}$.

References


In this section, we discuss the asynchronous UC framework followed in this paper. The discussion is based on the description of the framework against threshold adversaries as provided in [14] (which is further based on [24, 15]). We adapt the framework for the case of general adversaries. Informally, the security of a protocol is argued by
“comparing” the capabilities of the adversary in two separate worlds. In the real-world, the parties exchange messages among themselves, computed as per a given protocol. In the ideal-world, the parties do not interact with each other, but with a trusted third-party (an ideal functionality), which enables the parties to obtain the result of the computation based on the inputs provided by the parties. Informally, a protocol is considered to be secure if whatever an adversary can do in the real protocol can be also done in the ideal-world.

The Asynchronous Real-World: An execution of a protocol $\Pi$ in the real-world consists of $n$ interactive Turing machines (ITMs) representing the parties in $\mathcal{P}$. Additionally, there is an ITM for representing the adversary $\text{Adv}$. Each ITM is initialized with its random coins and possible inputs. Additionally, $\text{Adv}$ may have some auxiliary input $z$. Following the convention of [8], the protocol operates asynchronously by a sequence of activations, where at each point, a single ITM is active. Once activated, a party can perform some local computation, write on its output tape, or send messages to other parties. On the other hand, if the adversary is activated, it can send messages on the behalf of corrupt parties. The protocol execution is complete once all honest parties obtain their respective outputs. We let $\text{REAL}_{\Pi,\text{Adv}(z),Z^*}(\vec{x})$ denote the random variable consisting of the output of the honest parties and the view of the adversary $\text{Adv}$ during the execution of a protocol $\Pi$. Here, $\text{Adv}$ controls parties in $Z^*$ during the execution of protocol $\Pi$ with inputs $\vec{x} = (x^{(1)}, \ldots, x^{(n)})$ for the parties (where party $P_i$ has input $x^{(i)}$), and auxiliary input $z$ for $\text{Adv}$.

The Asynchronous Ideal-World: A protocol in the ideal-world consists of $n$ dummy parties $P_1, \ldots, P_n$, an ideal-world adversary $\mathcal{S}$ (also called simulator) and an ideal functionality $\mathcal{F}_{\text{AMPC}}$. We consider static corruptions such that the set of corrupt parties $Z^*$ is fixed at the beginning of the computation and is known to $\mathcal{S}$. The functionality $\mathcal{F}_{\text{AMPC}}$ receives the inputs from the respective dummy parties, performs the desired computation $f$ on the received inputs, and sends the outputs to the respective parties. The ideal-world adversary does not see and cannot delay the communication between the parties and $\mathcal{F}_{\text{AMPC}}$. However, it can communicate with $\mathcal{F}_{\text{AMPC}}$ on the behalf of corrupt parties.

Since $\mathcal{F}_{\text{AMPC}}$ models the desired behaviour of a real-world protocol which is asynchronous, ideal functionalities must consider some inherent limitations to model the asynchronous communication model with eventual delivery. For example, in a real-world protocol, the adversary can decide when each honest party learns the output since it has full control over message scheduling. To model the notion of time in the ideal-world, [24] uses the concept of number of activations. Namely, once $\mathcal{F}_{\text{AMPC}}$ has computed the output for some party, it does not ask “permission” from $\mathcal{S}$ to deliver it to the respective party. Instead, the corresponding party must “request” $\mathcal{F}_{\text{AMPC}}$ for the output, which can be done only when the concerned party is active. Moreover, the adversary can “instruct” $\mathcal{F}_{\text{AMPC}}$ to delay the output for each party by ignoring the corresponding requests, but only for a polynomial number of activations. If a party is activated sufficiently many times, the party will eventually receive the output from $\mathcal{F}_{\text{AMPC}}$ and hence, ideal computation eventually completes. That is, each honest party eventually obtains its desired output. As in [14], we use the term “$\mathcal{F}_{\text{AMPC}}$ sends a request-based delayed output to $P_i$”, to describe the above interaction between the $\mathcal{F}_{\text{AMPC}}, \mathcal{S}$ and $P_i$.

Another limitation is that in a real-world AMPC protocol, the (honest) parties cannot afford for all the parties to provide their input for the computation to avoid an endless wait, as the corrupt parties may decide not to provide their inputs. Hence, every AMPC protocol suffers from input deprivation, where inputs of a subset of potentially honest parties (which is decided by the choice of adversarial message scheduling) may get ignored during computation. Consequently, once a “core set” of parties $\mathcal{CS}$ provide their inputs for the computation, where $\mathcal{P} \setminus \mathcal{CS} \in Z$, the parties have to start computing the function by assuming some default input for the left-over parties. To model this in the ideal-world, $\mathcal{S}$ is given the provision to decide the set $\mathcal{CS}$ of parties whose inputs should be taken into consideration by $\mathcal{F}_{\text{AMPC}}$. We stress that $\mathcal{S}$ cannot delay sending $\mathcal{CS}$ to $\mathcal{F}_{\text{AMPC}}$ indefinitely. This is because in the real-world protocol, $\text{Adv}$ cannot prevent the honest parties from providing their inputs indefinitely. The formal description of $\mathcal{F}_{\text{AMPC}}$ is available in Fig 8.
Functionality \( \mathcal{F}_{\text{AMPC}} \)

\( \mathcal{F}_{\text{AMPC}} \) proceeds as follows, running with the parties \( \mathcal{P} = \{P_1, \ldots, P_n\} \) and an adversary \( \mathcal{S} \), and is parametrized by an \( n \)-party function \( f : \mathbb{F}^n \rightarrow \mathbb{F} \) and an adversary structure \( \mathcal{Z} \subseteq 2^\mathcal{P} \).

1. For each party \( P_i \in \mathcal{P} \), initialize an input value \( x^{(i)} = \perp \).
2. Upon receiving a message \((\text{inp}, \text{sid}, \nu)\) from some \( P_i \in \mathcal{P} \) (or from \( \mathcal{S} \) if \( P_i \) is corrupt), do the following:
   - Ignore the message if output has already been computed;
   - Else, set \( x^{(i)} = \nu \) and send \((\text{inp}, \text{sid}, P_i)\) to \( \mathcal{S} \).
3. Upon receiving a message \((\text{coreset}, \text{sid}, \mathcal{C}\mathcal{S})\) from \( \mathcal{S} \), do the following:
   - Ignore the message if \((\mathcal{P} \setminus \mathcal{C}\mathcal{S}) \not\subseteq \mathcal{Z} \) or if output has already been computed;
   - Else, record \( \mathcal{C}\mathcal{S} \) and set \( x^{(i)} = 0 \) for every \( P_j \notin \mathcal{C\mathcal{S}} \).
4. If \( \mathcal{C}\mathcal{S} \) has been recorded and the value \( x^{(i)} \) has been set to a value different from \( \perp \) for every \( P_i \in \mathcal{E}\mathcal{S} \), then compute
   \[
   y \overset{\text{def}}{=} f(x^{(1)}, \ldots, x^{(n)})
   \]
   and generate a request-based delayed output \((\text{out}, \text{sid}, (\mathcal{C}\mathcal{S}, y))\) for every \( P_i \in \mathcal{P} \).

---

Figure 8: The ideal functionality for asynchronous secure multi-party computation for session id \( \text{sid} \).

Similar to the real-world, we let \( \text{IDEAL}_{\mathcal{F}_{\text{AMPC}}, \mathcal{S}(z), Z^*(\bar{x})} \) denote the random variable consisting of the output of the honest parties and the view of the adversary \( \mathcal{S} \), controlling the parties in \( Z^* \), with the parties having inputs \( \bar{x} = (x^{(1)}, \ldots, x^{(n)}) \) (where party \( P_i \) has input \( x_i \)), and auxiliary input \( z \) for \( \mathcal{S} \).

We say that a real-world asynchronous protocol \( \Pi \) securely realizes \( \mathcal{F}_{\text{AMPC}} \) with perfectly-security if and only if for every real-world adversary \( \text{Adv} \), there exists an ideal-world adversary \( \mathcal{S} \) whose running time is polynomial in the running time of \( \text{Adv} \), such that for every possible \( Z^* \), every possible \( \bar{x} \in \mathbb{F}^n \) and every possible \( z \in \{0, 1\}^* \), it holds that the random variables
\[
\left\{ \text{REAL}_{\Pi, \text{Adv}(z), Z^*(\bar{x})} \right\} \quad \text{and} \quad \left\{ \text{IDEAL}_{\mathcal{F}_{\text{AMPC}}, \mathcal{S}(z), Z^*(\bar{x})} \right\}
\]
are identically distributed. That is, the random variables are perfectly-indistinguishable.

For statistically-secure AMPC, the parties and adversaries are parameterized with a statistical-security parameter \( \kappa \), and the above random variables (which are viewed as ensembles, parameterized by \( \kappa \)) are required to be statistically-indistinguishable. That is, their statistical-distance should be a negligible function in \( \kappa \).

The Universal-Composability (UC) Framework: While the real-world / ideal-world based security paradigm is used to define the security of a protocol in the “stand-alone” setting, the more powerful UC framework \([9, 10]\) is used to define the security of a protocol when multiple instances of the protocol might be running in parallel, possibly along with other protocols. Informally, the security in the UC-framework is still argued by comparing the real-world and the ideal-world. However, now, in both worlds, the computation takes place in the presence of an additional interactive process (modeled as an ITM) called the environment and denoted by \( \text{Env} \). Roughly speaking, \( \text{Env} \) models the “external environment” in which protocol execution takes place. The interaction between \( \text{Env} \) and the various entities takes place as follows in the two worlds.

In the real-world, the environment gives inputs to the honest parties, receives their outputs, and can communicate with the adversary at any point during the execution. During the protocol execution, the environment gets activated first. Once activated, the environment can either activate one of the parties by providing some input, or activate \( \text{Adv} \) by sending it a message. Once a party completes its operations upon getting activated, the control is returned to the environment. Once \( \text{Adv} \) gets activated, it can communicate with the environment (apart from
sending the messages to the honest parties). The environment also fully controls the corrupt parties that send all the messages they receive to \( Env \), and follow the orders of \( Env \). The protocol execution is completed once \( Env \) stops activating other parties, and outputs a single bit.

In the ideal-model, the environment \( Env \) gives inputs to the (dummy) honest parties, receives their outputs, and can communicate with \( S \) at any point during the execution. The dummy parties act as channels between \( Env \) and the functionality. That is, they send the inputs received from \( Env \) to functionality and transfer the output they receive from the functionality to \( Env \). The activation sequence in this world is similar to the one in the real-world. The protocol execution is completed once \( Env \) stops activating other parties and outputs a single bit.

A protocol is said to be UC-secure with perfect-security, if for every real-world adversary \( Adv \) there exists a simulator \( S \), such that for any environment \( Env \), the environment cannot distinguish the real-world from the ideal-world. On the other hand, the protocol is said to be UC-secure with statistical-security, if the environment cannot distinguish the real-world from the ideal-world, except with a probability which is a negligible function in the statistical-security parameter \( \kappa \).

The Hybrid Model: In a \( \mathcal{G} \)-hybrid model, a protocol execution proceeds as in the real-world. However, the parties have access to an ideal functionality \( \mathcal{G} \) for some specific task. During the protocol execution, the parties communicate with \( \mathcal{G} \) as in the ideal-world. The UC framework guarantees that an ideal functionality in a hybrid model can be replaced with a protocol that UC-securely realizes \( \mathcal{G} \). This is specifically due to the following composition theorem from [9,10].

**Theorem A.1** ([9,10]). Let \( \Pi \) be a protocol that UC-securely realizes some functionality \( \mathcal{F} \) in the \( \mathcal{G} \)-hybrid model and let \( \rho \) be a protocol that UC-securely realizes \( \mathcal{G} \). Moreover, let \( \Pi^\rho \) denote the protocol that is obtained from \( \Pi \) by replacing every ideal call to \( \mathcal{G} \) with the protocol \( \rho \). Then \( \Pi^\rho \) UC-securely realizes \( \mathcal{F} \) in the model where the parties do not have access to the functionality \( \mathcal{G} \).

### A.1 The Asynchronous Reliable Broadcast (Acast) Functionality and the Protocol

The ideal functionality \( \mathcal{F}_{\text{Acast}} \) capturing the requirements for asynchronous reliable broadcast is presented in Fig 9. The functionality, upon receiving \( m \) from the sender \( P_S \), performs a request-based delayed delivery of \( m \) to all the parties. Notice that if \( P_S \) is corrupt, then the functionality may not receive any message for delivery, in which case parties obtain no output. This models the fact that in any real-world Acast protocol, a potentially corrupt \( P_S \) may not invoke the protocol.

**Functionality \( \mathcal{F}_{\text{Acast}} \)**

\( \mathcal{F}_{\text{Acast}} \) proceeds as follows, running with the parties \( \mathcal{P} = \{ P_1, \ldots, P_n \} \) and an adversary \( S \), and is parametrized by an adversary structure \( \mathcal{Z} \subseteq 2^\mathcal{P} \). Let \( \mathcal{Z}^* \) denote the set of corrupt parties, where \( \mathcal{Z}^* \subseteq \mathcal{Z} \).

- Upon receiving (sender, Acast, sid, \( m \)) from \( P_S \in \mathcal{P} \) (or from \( S \) if \( P_S \in \mathcal{Z}^* \)), do the following:
  - Send \( (P_S, \text{Acast}, \text{sid}, m) \) to \( S \).
  - Send a request-based delayed output \( (P_S, \text{Acast}, \text{sid}, m) \) to each \( P_i \in \mathcal{P} \setminus \mathcal{Z}^* \) (no need to send \( m \) to the parties in \( \mathcal{Z}^* \), as \( S \) gets \( m \) on their behalf).

*If \( P_S \in \mathcal{Z}^* \), then no need to send \((P_S, \text{Acast}, \text{sid}, m)\) to \( S \), as in this case \( m \) is received from \( S \) itself.

Figure 9: The ideal functionality for asynchronous reliable broadcast for session id sid.

We next recall the Acast protocol of [27] and present it in Fig[10].

**Protocol \( \Pi_{\text{Acast}}(P_S, m) \)**

- Code for the Sender \( P_S \) (with input \( m \in \{0, 1\}^\ell \)):
  - Send the message (inp, sid, m) to all the parties in \( \mathcal{P} \).
- Code for each party \( P_i \in \mathcal{P} \) (including \( P_S \)):
Theorem A.2. If \( \mathcal{Z} \) satisfies the \( \mathcal{Q}^{(3)}(\mathcal{P}, \mathcal{Z}) \) condition, then protocol \( \Pi_{\text{Acast}} \) UC-securely realizes \( \mathcal{F}_{\text{Acast}} \) with perfect security. The protocol incurs a communication of \( O(n^2 \ell) \) bits, where \( P_S \) has an input of size \( \ell \) bits.

Proof. The communication complexity simply follows from the fact that in the protocol, each party needs to send \( m \) to every other party. For security, consider an arbitrary adversary \( \text{Adv} \) attacking \( \Pi_{\text{Acast}} \) by corrupting a set of parties \( Z^* \in \mathcal{Z} \), and let \( \text{Env} \) be an arbitrary environment. We present a simulator \( S_{\text{Acast}} \) such that for any set of corrupt parties \( Z^* \in \mathcal{Z} \), the output of the honest parties and the view of the adversary in an execution of \( \Pi_{\text{Acast}} \) with \( \text{Adv} \) is distributed identically to the output of the honest parties and the view of the adversary in an execution with \( S_{\text{Acast}} \) involving \( \mathcal{F}_{\text{Acast}} \) in the ideal world. This further implies that \( \text{Env} \) cannot distinguish between the two executions. The simulator constructs virtual real-world honest parties and invokes \( \text{Adv} \). The simulator simulates the environment and the honest parties towards \( \text{Adv} \) as follows: in order to simulate \( \text{Env} \), the simulator \( S_{\text{Acast}} \) forwards every message it receives from \( \text{Env} \) to \( \text{Adv} \) and vice-versa. To simulate the execution of honest parties, we consider two cases, depending upon whether \( P_S \) is corrupt or not.

Case I: \( P_S \) is honest. In this case, \( S_{\text{Acast}} \) first interacts with \( \mathcal{F}_{\text{Acast}} \) and receives the output \( m \) from the functionality. The simulator then plays the role of \( P_S \) with input \( m \), as well as the role of the honest parties, and interacts with \( \text{Adv} \) as per the steps of \( \Pi_{\text{Acast}} \).

It is easy to see that that view of \( \text{Adv} \) is identical in the real execution and simulated execution. This is because only \( P_S \) has input in the protocol and in the simulated execution, \( S_{\text{Acast}} \) plays the role of \( P_S \) as per \( \Pi_{\text{Acast}} \) after learning the input of \( P_S \) from \( \mathcal{F}_{\text{Acast}} \). Next, conditioned on the view of \( \text{Adv} \), we show that the outputs of the honest parties are identical in both the executions. So consider an arbitrary View of \( \text{Adv} \). Conditioned on View, all honest parties eventually obtain a request-based delayed output \( m \) in the simulated execution, where \( m \) is the input of \( P_S \) as per View. We show that even in the real execution, all honest parties eventually output \( m \). This is because all honest parties eventually complete steps 1 - 4 in the protocol, even if the corrupt parties do not send their message, as the messages of the honest parties \( \mathcal{P} \setminus Z^* \) are eventually selected for delivery and \( \mathcal{P} \setminus Z^* \not\in \mathcal{Z} \); the latter holds, as otherwise \( Z \) does not satisfy the \( \mathcal{Q}^{(2)}(\mathcal{P}, \mathcal{Z}) \) condition. \( \text{Adv} \) may send echo and ready messages for \( m' \), where \( m' \neq m \), on the behalf of corrupt parties. But since \( Z^* \in \mathcal{Z} \) and since \( Z \) satisfies the \( \mathcal{Q}^{(2)}(\mathcal{P}, \mathcal{Z}) \) condition, it follows that no honest party ever generates a ready message for \( m' \), neither in step 2, nor in step 3. Thus the output of the honest parties is identically distributed in both the worlds. Consequently, in this case, we conclude that \( \{ \text{REAL}_{\Pi_{\text{Acast}}, \text{Adv}}(\cdot, \cdot, \cdot) \}_{m \in \{0, 1\}^\ell, z \in \{0, 1\}^\ast} \equiv \{ \text{REAL}_{\mathcal{F}_{\text{Acast}}, S_{\text{Acast}}(\cdot, \cdot, \cdot)} \}_{m \in \{0, 1\}^\ell, z \in \{0, 1\}^\ast} \) holds, where \( \equiv \) denotes perfect indistinguishability.

Case II: \( P_S \) is corrupt. In this case, \( S_{\text{Acast}} \) first plays the role of the honest parties and interacts with \( \text{Adv} \), as per \( \Pi_{\text{Acast}} \). If in this execution, \( S_{\text{Acast}} \) finds that some honest party, say \( P_h \), outputs \( m^* \), then \( S_{\text{Acast}} \) interacts with \( \mathcal{F}_{\text{Acast}} \) by sending \( m^* \) as the input to \( \mathcal{F}_{\text{Acast}} \), on the behalf of \( P_S \). Else, \( S_{\text{Acast}} \) does not provide any input to \( \mathcal{F}_{\text{Acast}} \) on the behalf of \( P_S \).

It is easy to see that the view of \( \text{Adv} \) is identically distributed in the real and the simulated execution. This is because only \( P_S \), which is under the control of \( \text{Adv} \), has an input in the protocol, and \( S_{\text{Acast}} \) plays the role of the honest parties exactly as per the protocol \( \Pi_{\text{Acast}} \). We next show that conditioned on the view of \( \text{Adv} \), the outputs of the honest parties are identically distributed in both the executions.
Consider an arbitrary view View of Adv, during an execution of $\Pi_{Acast}$. If according to View, no honest party obtains an output during the execution of $\Pi_{Acast}$, then the honest parties do not obtain any output in the simulated execution as well. This is because in this case, $S_{Acast}$ does not provide any input on the behalf of $P_S$ to $F_{Acast}$. On the other hand, consider the case when according to View, some honest party $P_h$ outputs $m^*$. In this case, in the simulated execution, all honest parties eventually obtain an output $m^*$, since $S_{Acast}$ provides $m^*$ as the input to $F_{Acast}$ on the behalf of $P_S$. We next show that even in the real execution, all honest parties eventually obtain the output $m^*$.

Since $P_h$ obtained the output $m^*$, it received ready messages for $m^*$ during step 4 of the protocol from a set of parties $P \setminus Z$, for some $Z \in \mathcal{Z}$. Let $\mathcal{H}$ be the set of honest parties whose ready messages are received by $P_h$ during step 4. It is easy to see that $\mathcal{H} \not\in \mathcal{Z}$, as otherwise, $\mathcal{Z}$ does not satisfy the $Q^{(3)}(P, Z)$ condition. The ready messages of the parties in $\mathcal{H}$ are eventually delivered to every honest party and hence, each honest party (including $P_h$) eventually executes step 3 and sends a ready message for $m^*$. It follows that the ready messages of all honest parties $P \setminus Z^*$ are eventually delivered to every honest party (irrespective of whether Adv sends all the required messages), guaranteeing that all honest parties eventually obtain some output. To complete the proof, we show that this output is the same as $m^*$.

For contradiction, let $P_h' \neq P_h$ be an honest party who outputs $m^{**} \neq m^*$. This implies that $P_h'$ received ready message for $m^{**}$ from at least one honest party. From the protocol steps, it follows that an honest party generates a ready message for some potential $m$, only if it either receives echo messages for $m$ during step 2 from a set of parties $P \setminus Z$ for some $Z \in \mathcal{Z}$, or ready messages for $m$ from a set of parties $C \not\in \mathcal{Z}$ during step 3. So, in order that a subset of parties $P \setminus Z$ for some $Z \in \mathcal{Z}$ eventually generates ready messages for some potential $m$ during step 4, it must be the case that some honest party has received echo messages for $m$ during step 1 from a set of parties $P \setminus Z'$ for some $Z' \in \mathcal{Z}$ and has generated a ready message for $m$.

Since $P_h$ received the ready message for $m^*$ from at least one honest party, it must be the case that some honest party has received echo messages for $m^*$ from a set of parties $P \setminus Z_1$ for some $Z_1 \in \mathcal{Z}$. Similarly, since $P_h'$ received the ready message for $m^{**}$ from at least one honest party, it must be the case that some honest party has received echo messages for $m^{**}$ from a set of parties $P \setminus Z_2$ for some $Z_2 \in \mathcal{Z}$. Let $\mathcal{T} = (P \setminus Z_1) \cap (P \setminus Z_2)$. Since $\mathcal{Z}$ satisfies the $Q^{(3)}(P, Z)$ condition, it follows that $\mathcal{Z}$ satisfies the $Q^{(1)}(T, Z)$ condition and hence $T$ is guaranteed to have at least one honest party. This further implies that there exists some honest party who generated an echo message for $m^*$ as well as $m^{**}$ during step 1, which is impossible. This is because an honest party executes step 1 at most once and hence, generates an echo message at most once. Consequently, $\{\text{REAL}_{\Pi_{Acast}, Adv(z), Env(m)}\}_{m \in \{0,1\}^*} \equiv \{\text{IDEAL}_{F_{Acast}, S_{Acast}(z), Env(m)}\}_{m \in \{0,1\}^*}$ holds.

### A.2 Asynchronous Byzantine Agreement (ABA)

In a synchronous BA protocol, each party participates with an input bit to obtain an output bit. The protocol guarantees the following three properties.

- **Agreement**: The output bit of all honest parties is the same.
- **Validity**: If all honest parties have the same input bit, then this will be the common output bit.
- **Termination**: All honest parties eventually complete the protocol.

In an ABA protocol, the above requirements are slightly weakened, since all (honest) parties may not be able to provide their inputs to the protocol, as waiting for all the inputs may turn out to be an endless wait. Hence the decision is taken based on the inputs of a subset of parties $CS$, where $P \setminus CS \in \mathcal{Z}$. Moreover, since the adversary can control the schedule of message delivery, it has full control in deciding the set $CS$.

The formal specification of an ideal ABA functionality is presented in Fig. 11, which is obtained by generalizing the corresponding ideal functionality against threshold adversaries, as presented in [15]. Intuitively, it can be considered as a special case of the ideal AMPC functionality (see Fig. 9), which looks at the set of inputs provided by the set of parties in $CS$, where $CS$ is decided by the ideal-world adversary. If the input bits provided by all the honest parties in $CS$ are the same, then it is set as the output bit. Else, the output bit is set to be the input bit provided by some corrupt party in $CS$ (for example, the first corrupt party in $CS$ according to lexicographic
ordering). In the functionality, the inputs bits provided by various parties are considered to be the “votes” of the respective parties.

<table>
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<th>Functionality $F_{ABA}$</th>
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| $F_{ABA}$ proceeds as follows, running with the parties $P = \{P_1, \ldots, P_n\}$ and an adversary $S$, and is parametrized by an adversary-structure $Z \subseteq 2^P$. Let $Z^*$ denote the set of corrupt parties, where $Z^* \in Z$ and let $H = P \setminus Z^*$. For each party $P_i$, initialize an input value $x^{(i)} = \bot$.
| 1. Upon receiving a message $(vote, sid, b)$ from some $P_i \in P$ (or from $S$ if $P_i$ is corrupt) where $b \in \{0, 1\}$, do the following:
|   - Ignore the message if output has been already computed;
|   - Else, set $x^{(i)} = b$ and send $(vote, sid, P_i, b)$ to $S$.
| 2. Upon receiving a message $(coreset, sid, CS)$ from $S$, do the following:
|   - Ignore the message if $(P \setminus CS) \notin Z$ or if output has been already computed;
|   - Else, record $CS$.
| 3. If the set $CS$ has been recorded and the value $x^{(i)}$ has been set to a value different from $\bot$ for every $P_i \in CS$, then compute the output $y$ as follows and generate a request-based delayed output $(decide, sid, (CS, y))$ for every $P_i \in P$.
|   - If $x^{(i)} = b$ holds for all $P_i \in (H \cap CS)$, then set $y = b$.
|   - Else, set $y = x^{(i)}$, where $P_i$ is the party with the smallest index in $CS \cap Z^*$.

"If $P_i \in Z^*$, then no need to send $(vote, sid, P_i, b)$ to $S$ as the input has been provided by $S$ only.

As in the case of the AMPC functionality $F_{AMPC}$, $S$ cannot delay sending $CS$ indefinitely.

Figure 11: The ideal functionality for asynchronous Byzantine agreement for session id sid.

B Properties of the Perfectly-Secure PreProcessing Phase

In this section, we prove the security properties of all the perfectly-secure subprotocols, followed by the perfectly-secure preprocessing phase. We first start with the perfectly-secure VSS.

B.1 Asynchronous VSS Protocol

In this section, we recall the perfectly-secure VSS protocol $\Pi_{PVSS}$ from [12]. The protocol is designed with respect to an adversary structure $Z$ and a sharing specification $S = (S_1, \ldots, S_h) \overset{def}{=} \{P \setminus Z|Z \in Z\}$, such that $Z$ satisfies the $Q^{(4)}(P, Z)$ condition (this automatically implies that $S$ satisfies the $Q^{(3)}(S, Z)$ condition). The input for the dealer $P_D$ in the protocol is a vector of shares $(s_1, \ldots, s_h)$, the goal being to ensure that the parties output a secret-sharing of $s \overset{def}{=} s_1 + \ldots + s_h$, such that $[s]_q = s_q$, for each $S_q \in S$. The protocol guarantees that even if $P_D$ is corrupt, if some honest party completes the protocol, then every honest party eventually completes the protocol such that there exists some value which has been secret-shared by $P_D$.

The high level idea of the protocol is as follows: the dealer gives the share $s_q$ to all the parties in the set $S_q \in S$. To verify whether the dealer has distributed the same share to all the parties in $S_q$, the parties in $S_q$ perform pairwise consistency tests of the supposedly common share and publicly announce the result. Next, the parties check if there exists a subset of “core” parties $C_q$, where $S_q \setminus C_q \in Z$, who have positively confirmed the pairwise consistency of their supposedly common share. Such a subset $C_q$ is guaranteed for an honest dealer, as the set of honest parties in $S_q$ always constitutes a candidate set for $C_q$. To ensure that all honest parties have the same version of the core sets $C_1, \ldots, C_h$, the dealer is assigned the task of identifying these sets based on the results of the pairwise consistency tests, and making them public. Once the core sets are identified and verified, it is guaranteed that the dealer has distributed some common share to all honest parties within $C_q$. The next goal is to ensure that even the honest parties in $S_q \setminus C_q$ get this common share, which is required as per the semantics of our secret-sharing. For this,
the (honest) parties in \( S_q \setminus \mathcal{C}_q \) “filter” out the supposedly common shares received during the pairwise consistency tests and ensure that they obtain the common share held by the honest parties in \( \mathcal{C}_q \). Protocol \( \Pi_{PVSS} \) is formally presented in Fig. 12.

\[ \boxed{\text{Protocol } \Pi_{PVSS}} \]

- **Distribution of Shares by }\ P_D: If }\ P_i \text{ is the dealer } }\ P_D, \text{ then execute the following steps.}
  1. On having the shares \( s_1, \ldots, s_h \in \mathbb{F} \), send (dist, sid, }\ P_D, q, }\ [s]_q) \text{ to all the parties } \ P_i \in S_q, \text{ corresponding to each } S_q \in S, \text{ where } s \overset{\text{def}}{=} s_1 + \ldots + s_h \text{ and } }\ [s]_q = s_q.

- **Pairwise Consistency Tests and Public Announcement of Results**: For each }\ S_q \in S, \text{ if } \ P_i \in S_q, \text{ then execute the following steps.}
  1. Upon receiving (dist, sid, }\ P_D, q, }\ s_{q_i}) \text{ from } D, \text{ send (test, sid, }\ P_D, q, }\ s_{q_i}) \text{ to every party } \ P_j \in S_q.
  2. Upon receiving (test, sid, }\ P_D, q, }\ s_{q_j}) \text{ from } \ P_j \in S_q, \text{ send (sender, Acast, }\ \text{sid}(P_	ext{b}, q), \text{OK}(q, i, j)) \text{ to } F_{Acast} \text{ if } s_{q_i} = s_{q_j}, \text{ where } \text{sid}(P_	ext{b}, q) = \text{sid}||P_b||q||i||j||}

- **Constructing Consistency Graph**: For each }\ S_q \in S, \text{ execute the following steps.
  1. Initialize }\ C_q \text{ to } 0. \text{ Construct an undirected consistency graph } G_q^{(i)} \text{ with } S_q \text{ as the vertex set.}
  2. For every ordered pair of parties }\ (P_j, P_k) \text{ where } P_j, P_k \in S_q, \text{ keep requesting for an output from } F_{Acast} \text{ with } \text{sid}(P_	ext{b}, q), \text{ till an output is received.}
  3. Add the edge }\ (P_j, P_k) \text{ to } G_q^{(i)} \text{ if outputs } (P_j, \text{Acast, }\ \text{sid}(P_	ext{b}, q), \text{OK}(q, j, k)) \text{ and } (P_k, \text{Acast, }\ \text{sid}(P_	ext{b}, q), \text{OK}(q, k, j)) \text{ are received from } F_{Acast} \text{ with } \text{sid}(P_	ext{b}, q) \text{ and } F_{Acast} \text{ with } \text{sid}(P_	ext{b}, q) \text{ respectively.}

- **Identification of Core Sets and Public Announcements**: If }\ P_i \text{ is the dealer } }\ P_D, \text{ then execute the following steps.
  1. For each }\ S_q \in S, \text{ check if there exists a subset of parties } W_q \subseteq S_q, \text{ such that } S_q \setminus W_q \subseteq Z \text{ and the parties in } \mathcal{W}_q \text{ form a clique in the consistency graph } G_q^{(i)}. \text{ If such a } W_q \text{ exists, then assign } C_q := \mathcal{W}_q.
  2. Upon computing the sets }\ C_1, \ldots, C_h, \text{ send (sender, Acast, }\ \text{sid}(P_b), \{C_q\}_{S_q \in S}) \text{ to } F_{Acast}, \text{ where } \text{sid}(P_b) = \text{sid}||P_D||.

- **Share Computation**: Execute the following steps.
  1. For each }\ S_q \in S \text{ such that } P_i \in S_q, \text{ initialize } [s]_q \text{ to } \bot. \text{ }
  2. Keep requesting for an output from } F_{Acast} \text{ with } \text{sid}(P_b) \text{ until an output is received.}
  3. Upon receiving an output (sender, Acast, }\ \text{sid}(P_b, \{C_q\}_{S_q \in S}) \text{ from } F_{Acast} \text{ with } \text{sid}(P_b), \text{ wait until the parties in } C_q \text{ form a clique in } G_q^{(i)}, \text{ for } q = 1, \ldots, h. \text{ Then, verify if } S_q \setminus C_q \subseteq Z, \text{ for each } q = 1, \ldots, h. \text{ If the verification is successful, then proceed to compute the output as follows.}
  i. Corresponding to each }\ C_q \text{ such that } P_i \in C_q, \text{ set } [s]_q := s_{q_i}.
  ii. Corresponding to each }\ C_q \text{ such that } P_i \notin C_q, \text{ set } [s]_q := s_q, \text{ where (test, sid, }\ P_D, q, s_q) \text{ is received from a set of parties } C_q \text{ such that } C_q \setminus \mathcal{C}_q \subseteq Z.
  4. Once }\ [s]_q \neq \bot \text{ for each } S_q \in S \text{ such that } P_i \in S_q, \text{ output (share, sid, }\ P_D, \{[s]_q\}_{P_i \in S_q}).

\[\text{The notation } \text{sid}(P_b, q) \text{ is used here to distinguish among the different calls to } F_{Acast} \text{ within the session sid.}\]

**Figure 12**: The perfectly-secure VSS protocol for realizing }\ F_{VSS} \text{ in the } F_{Acast}-hybrid model. The above steps are executed by every }\ P_i \in P.

We next prove the security of the protocol }\ \Pi_{PVSS}.

**Theorem B.1**: Consider a static malicious adversary }\ Adv \text{ characterized by an adversary-structure } Z, \text{ satisfying the } Q^{(i)}(P, Z) \text{ condition and let } S = (S_1, \ldots, S_h) \overset{\text{def}}{=} \{P \setminus Z|Z \in Z\} \text{ be the sharing specification}. Then protocol }\ \Pi_{PVSS} \text{ UC-securely realizes the functionality } F_{VSS} \text{ with perfect security in the } F_{Acast}-hybrid model, in the presence of }\ Adv.

**Proof**: Let }\ Adv \text{ be an arbitrary adversary corrupting a set of parties } Z^* \subseteq Z. \text{ Let } Env \text{ be an arbitrary environment. We show the existence of a simulator } \Delta_{PVSS}, \text{ such that for any } Z^* \subseteq Z, \text{ the outputs of the honest parties and the
view of the adversary in the protocol $\Pi_{PVSS}$ is indistinguishable from the outputs of the honest parties and the view of the adversary in an execution in the ideal world involving $S_{PVSS}$ and $F_{VSS}$. The steps of the simulator will be different depending on whether the dealer is corrupt of honest.

If the dealer is honest, then the simulator interacts with $F_{VSS}$ and receives the shares of the corrupt parties corresponding to the sets $S_q \in S$ which they are part of. With these shares, the simulator then plays the role of the dealer as well as the honest parties, as per the steps of $\Pi_{PVSS}$, and interacts with $Adv$. The simulator also plays the role of $F_{Acast}$. If $Adv$ queries $F_{Acast}$ for the result of any pairwise consistency test involving an honest party, the simulator provides the appropriate result. In addition, the simulator records the result of any test involving corrupt parties which $Adv$ sends to $F_{Acast}$. Based on the results of these pairwise consistency tests, the simulator finds the core sets for each $S_q$ and sends these to $Adv$ upon request.

If the dealer is corrupt, the simulator plays the role of honest parties and interacts with $Adv$, as per the steps of $\Pi_{PVSS}$. This involves recording shares which $Adv$ distributes to any honest party (on the behalf of the dealer), as well as performing pairwise consistency tests on their behalf. If $Adv$ sends core sets for each $S_q \in S$ as input to $F_{Acast}$, then the simulator checks if these are valid, and accordingly, sends the shares held by honest parties in these core sets as the input shares to $F_{VSS}$ on the behalf of the dealer. The simulator is presented in Figure 13.

**Simulator $S_{PVSS}$**

$S_{PVSS}$ constructs virtual real-world honest parties and invokes the real-world adversary $Adv$. The simulator simulates the view of $Adv$, namely its communication with $Env$, the messages sent by the honest parties and the interaction with $F_{Acast}$. In order to simulate $Env$, the simulator $S_{PVSS}$ forwards every message it receives from $Env$ to $Adv$ and vice-versa. The simulator then simulates the various phases of the protocol as follows, depending upon whether the dealer is honest or corrupt.

### Simulation When $P_D$ is Honest

**Interaction with $F_{VSS}$:** The simulator interacts with the functionality $F_{VSS}$ and receives a request based delayed output $(\text{share}, \text{sid}, P_D, \{(s_q)_{q \cap Z^*} = \emptyset\}, \text{on the behalf of the parties in } Z^*)$.

**Distribution of Shares by $P_D$:** On the behalf of the dealer, the simulator sends $(\text{dist}, \text{sid}, P_D, q, \{s_q\})$ to $Adv$, corresponding to every $P_i \in Z^* \cap S_q$.

**Pairwise Consistency Tests:** For each $S_q \in S$ such that $S_q \cap Z^* \neq \emptyset$, corresponding to each $P_i \in S_q \cap Z^*$, the simulator does the following.

- On the behalf of every party $P_j \in S_q \setminus Z^*$, sends $(\text{test}, \text{sid}, P_D, q, s_{qj})$ to $Adv$, where $s_{qj} = \{s_q\}$.

- If $Adv$ sends $(\text{test}, \text{sid}, P_D, q, s_{qj})$ on the behalf of $P_i$ to any $P_j \in S_q$, then record it.

**Announcing Results of Consistency Tests:**

- If for any $S_q \in S$, $Adv$ requests an output from $F_{Acast}$ with sid$_{ij}^{(P,b,q)}$, corresponding to parties $P_i \in S_q \setminus Z^*$ and $P_j \in S_q$, then the simulator provides output on the behalf of $F_{Acast}$ as follows.

  - If $P_j \notin S_q \setminus Z^*$, then send the output $(P_i, Acast, \text{sid}_{ij}^{(P,b,q)}, \text{OK}_q(i,j))$.

  - If $P_j \in (S_q \cap Z^*)$, then send the output $(P_i, Acast, \text{sid}_{ij}^{(P,b,q)}, \text{OK}_q(i,j))$, if the message $(\text{test}, \text{sid}, P_D, q, s_{qj})$ has been recorded on the behalf of $P_j$ for party $P_i$, and $s_{qj} = \{s_q\}$ holds.

- If for any $S_q \in S$ and any $P_i \in S_q \cap Z^*$, $Adv$ sends $(P_i, Acast, \text{sid}_{ij}^{(P,b,q)}, \text{OK}_q(i,j))$ to $F_{Acast}$ with sid$_{ij}^{(P,b,q)}$ on the behalf of $P_i$, for any $P_j \in S_q$, then the simulator records it. Moreover, if $Adv$ requests for an output from $F_{Acast}$ with sid$_{ij}^{(P,b,q)}$, then the simulator sends the output $(P_i, Acast, \text{sid}_{ij}^{(P,b,q)}, \text{OK}_q(i,j))$ on the behalf of $F_{Acast}$.

**Construction of Core Sets and Public Announcement:**

- For each $S_q \in S$, the simulator plays the role of $P_D$ and adds the edge $(P_i, P_j)$ to the graph $G^\emptyset$ over the vertex set $S_q$, if the following hold.

  - $P_i, P_j \in S_q$.

  - One of the following is true.

    1. $P_i, P_j \in S_q \setminus Z^*$.

    2. If $P_i \in S_q \setminus Z^*$ and $P_j \in S_q \setminus Z^*$, then the simulator has recorded $(P_i, Acast, \text{sid}_{ij}^{(P,b,q)}, \text{OK}_q(i,j))$ sent by $Adv$ on the behalf of $P_i$ to $F_{Acast}$ with sid$_{ij}^{(P,b,q)}$, and recorded $(\text{test}, \text{sid}, P_D, q, s_{qj})$ on the behalf of $P_i$ for $P_j$ such that $s_{qj} = \{s_q\}$.

    3. If $P_i, P_j \in S_q \cap Z^*$, then the simulator has recorded $(P_i, Acast, \text{sid}_{ij}^{(q)}, \text{OK}_q(i,j))$ and $(P_j, Acast, \text{sid}_{ji}^{(q)}, \text{OK}_q(j,i))$. 


Claim B.2. If $P_D$ is honest, then the view of $Adv$ in the simulated execution of $\PiPVSS$ with $S_{PVSS}$ is identically distributed to the view of $Adv$ in the real execution of $\PiPVSS$ involving honest parties.

Proof. Let $S^* \overset{def}{=} \{S_q \in S \mid S_q \cap Z^* \neq \emptyset\}$. Then the view of $Adv$ during the various executions consists of the following.

We now prove a series of claims which will help us prove the theorem. We start with an honest $P_D$.

**Simulation When $P_D$ is Corrupt**

In this case, the simulator $S_{PVSS}$ interacts with $Adv$ during the various phases of $\PiPVSS$ as follows.

**Distribution of shares by $P_b$:** For $q = 1, \ldots, h$, if $Adv$ sends (dist, sid, $P_b$, $q$, $v$) on the behalf of $P_b$ to any party $P_i \in S_q \setminus Z^*$ and each $P_j \in S_q \setminus Z^*$, the simulator records it and sets $s_{qi}$ to be $v$.

**Pairwise Consistency Tests:** For each $S_q \in S$ such that $S_q \cap Z^* \neq \emptyset$, corresponding to each party $P_i \in S_q \cap Z^*$ and each $P_j \in S_q \setminus Z^*$, the simulator does the following.

- If $s_{qj}$ has been set to some value, then send (test, sid, $P_D$, $q$, $s_{qj}$) to $Adv$ on the behalf of $P_j$.
- If $Adv$ sends (test, sid, $P_D$, $q$, $s_{qj}$) on the behalf of $P_i$, to $P_j$, then record it.

**Announcing Results of Consistency Tests:**

- If for any $S_q \in S$, $Adv$ requests an output from $F_{Acast}$ with sid$^{(P_b,q)}$ corresponding to parties $P_i \in S_q \setminus Z^*$ and $P_j \in S_q$, then the simulator provides the output on the behalf of $F_{Acast}$ as follows, if $s_{qi}$ has been set to some value.
  - If $P_j \in S_q \setminus Z^*$, then send the output $(P_i, Acast, sid^{(P_b,q)}_i, OK_q(i,j))$, if $s_{qi}$ has been set to some value and $s_{qi} = s_{qj}$ holds.
  - If $P_j \in S_q \cap Z^*$, then send the output $(P_i, Acast, sid^{(P_b,q)}_i, OK_q(i,j))$, if (test, sid, $P_D$, $q$, $s_{qj}$) sent by $Adv$ on the behalf of $P_j$ to $P_i$ has been recorded and $s_{qi} = s_{qj}$ holds.
- If for any $S_q \in S$ and any $P_i \in S_q \cap Z^*$, $Adv$ sends $(P_i, Acast, sid^{(P_b,q)}_i, OK_q(i,j))$ to $F_{Acast}$ with sid$^{(P_b,q)}$ on the behalf of $P_i$ for any $P_j \in S_q$, then the simulator records it. Moreover, if $Adv$ requests for an output from $F_{Acast}$ with sid$^{(P_b,q)}$, then the simulator sends the output $(P_i, Acast, sid^{(P_b,q)}_i, OK_q(i,j))$ on the behalf of $F_{Acast}$.

**Construction of Core Sets:** For each $S_q \in S$, the simulator plays the role of the honest parties $P_i \in S_q \setminus Z^*$ and adds the edge $(P_j, P_k)$ to the graph $G^{(i)}$ over vertex set $S_q$, if the following hold.

- $P_i, P_k \in S_q$
- One of the following is true.
  - If $P_j, P_k \in S_q \setminus Z^*$, then the simulator has set $s_{qj}$ and $s_{qk}$ to some values, such that $s_{qj} = s_{qk}$.
  - If $P_j \in S_q \cap Z^*$ and $P_k \in S_q \setminus Z^*$, then the simulator has recorded $(P_j, Acast, sid^{(P_b,q)}_j, OK_q(j,k))$ sent by $Adv$ on the behalf of $P_j$ to $F_{Acast}$ with sid$^{(P_b,q)}_j$ and recorded (test, sid, $P_D$, $q$, $s_{qj}$) on the behalf of $P_j$ for $P_k$ and has set $s_{qk}$ to a value such that $s_{qj} = s_{qk}$.
  - If $P_j, P_k \in S_q \cap Z^*$, then the simulator has recorded $(P_j, Acast, sid^{(P_b,q)}_j, OK_q(j,k))$ and $(P_k, Acast, sid^{(P_b,q)}_k, OK_q(k,j))$ sent by $Adv$ on behalf of $P_j$ and $P_k$ respectively to $F_{Acast}$ with sid$^{(P_b,q)}_j$ and $F_{Acast}$ with sid$^{(P_b,q)}_k$.

**Verification of Core Sets and Interaction with $F_{PVSS}$:**

- If $Adv$ sends (sender, Acast, sid$^{(P_b,q)}_b$, $\{C_q\}_{q \in S}$) to $F_{Acast}$ with sid$^{(P_b,q)}_b$ on the behalf of $P_b$, then the simulator records it. Moreover, if $Adv$ requests an output from $F_{Acast}$ with sid$^{(P_b,q)}_b$, then on the behalf of $F_{Acast}$, the simulator sends the output $(P_D, Acast, sid^{(P_b,q)}_b, \{C_q\}_{q \in S})$.
- If simulator has recorded the sets $\{C_q\}_{q \in S}$, then it plays the role of the honest parties and verifies if $C_1, \ldots, C_h$ are valid by checking if each $S_q \setminus C_q \in \mathcal{Z}$ and if each $C_q$ constitutes a clique in the graph $G^{(i)}$ of every party $P_i \in \mathcal{P} \setminus Z^*$. If $C_1, \ldots, C_h$ are valid, then the simulator sends (share, sid, $P_D$, $\{s_q\}_{s_q \in S}$) to $F_{PVSS}$, where $s_q$ is set to $s_{qi}$ corresponding to any $P_i \in C_q \setminus Z^*$.

Figure 13: Simulator for the protocol $\PiPVSS$ where $Adv$ corrupts the parties in set $Z^* \in \mathcal{Z}$.
- The shares \( \{[s]_q\}_{S_q \in S} \) distributed by \( P_D \): In the real execution, \( \text{Adv} \) receives \([s]_q\) from \( P_D \) for each \( S_q \in S^* \). In the simulated execution, the simulator provides this to \( \text{Adv} \) on behalf of \( P_D \). Clearly, the distribution of the shares is identical in both the executions.

- Corresponding to every \( S_q \in S^* \), messages (test, sid, \( P_D, q, s_{qj} \)) received from party \( P_j \in S_q \setminus Z^* \), as part of pairwise consistency tests, where \( s_{qj} = [s]_q \): While each \( P_j \) sends this to \( \text{Adv} \) in the real execution, the simulator sends this on the behalf of \( P_j \) in the simulated execution. Clearly, the distribution of the messages is identical in both the executions.

- For every \( S_q \in S \) and every \( P_i, P_j \in S_q \), the outputs \( \text{OK}_q(P_i, \text{Acast}, \text{sid}_{ij}^{(P_D,q)}, \text{OK}_q(i,j)) \) of the pairwise consistency tests, received as output from \( F_{\text{Acast}} \) with \( \text{sid}_{ij}^{(P_D,q)} \). To compare the distribution of these messages in the two executions, we consider the following cases, considering an arbitrary \( S_q \in S \) and arbitrary \( P_i, P_j \in S_q \).

  - \( P_i, P_j \in S_q \setminus Z^* \): In both the executions, \( \text{Adv} \) receives \( \text{OK}_q(P_i, \text{Acast}, \text{sid}_{ij}^{(P_D,q)}, \text{OK}_q(i,j)) \) as the output from \( F_{\text{Acast}} \) with \( \text{sid}_{ij}^{(P_D,q)} \).

  - \( P_i \in S_q \setminus Z^*, P_j \in (S_q \cap Z^*) \): In both the executions, \( \text{Adv} \) receives \( \text{OK}_q(P_i, \text{Acast}, \text{sid}_{ij}^{(P_D,q)}, \text{OK}_q(i,j)) \) as the output from \( F_{\text{Acast}} \) with \( \text{sid}_{ij}^{(P_D,q)} \) if and only if \( \text{Adv} \) sent (test, sid, \( P_D, q, s_{qj} \)) on the behalf of \( P_j \) to \( P_i \) such that \( s_{qj} = [s]_q \) holds.

  - \( P_i \in (S_q \cap Z^*) \): In both the executions, \( \text{Adv} \) receives \( \text{OK}_q(P_i, \text{Acast}, \text{sid}_{ij}^{(Q)}, \text{OK}_q(i,j)) \) if and only if \( \text{Adv} \) on the behalf of \( P_i \) has sent \( (P_i, \text{Acast}, \text{sid}_{ij}^{(P_D,q)}, \text{OK}_q(i,j)) \) to \( F_{\text{Acast}} \) with \( \text{sid}_{ij}^{(P_D,q)} \) for \( P_j \).

Clearly, irrespective of the case, the distribution of the \( \text{OK}_q \) messages is identical in both the executions.

- The core sets \( \{C_q\}_{S_q \in S} \): In both the executions, the sets \( C_q \) are determined based on the \( \text{OK}_q \) messages delivered to \( P_D \). So the distribution of these sets is also identical.

\[\square\]

We next claim that if the dealer is honest, then conditioned on the view of the adversary \( \text{Adv} \) (which is identically distributed in both the executions, as per the previous claim), the outputs of the honest parties are identically distributed in both the executions.

Claim B.3. If \( P_D \) is honest, then conditioned on the view of \( \text{Adv} \), the outputs of the honest parties during the execution of \( \Pi_{\text{PVSS}} \) involving \( \text{Adv} \) has the same distribution as the outputs of the honest parties in the ideal-world involving \( S_{\text{PVSS}} \) and \( F_{\text{VSS}} \).

Proof. Let \( P_D \) be honest and let View be an arbitrary view of \( \text{Adv} \). Moreover, let \( \{s_q\}_{S_q \cap Z^* \neq \emptyset} \) be the shares of the corrupt parties, as per View. Furthermore, let \( \{s_q\}_{S_q \cap Z^* = \emptyset} \) be the shares used by \( P_D \) in the simulated execution, corresponding to the set \( S_q \in S \), such that \( S_q \cap Z^* = \emptyset \). Let \( s \eqdef \sum_{S_q \cap Z^* \neq \emptyset} s_q + \sum_{S_q \cap Z^* = \emptyset} s_q \). Then in the simulated execution, each honest party \( P_i \) obtains the output \( \{[s]_q\}_{P_i \in S_q} \) from \( F_{\text{VSS}} \), where \([s]_q = s_q \). We now show that \( P_i \) eventually obtains the output \( \{[s]_q\}_{P_i \in S_q} \) in the real execution as well, if \( P_D \)'s inputs in the protocol \( \Pi_{\text{PVSS}} \) are \( \{s_q\}_{S_q \in S} \).

Since \( P_D \) is honest, it sends the share \( s_q \) to all the parties in the set \( S_q \), which is eventually delivered. Now consider an arbitrary \( S_q \in S \). During the pairwise consistency tests, each honest \( P_k \in S_q \) will eventually send \( s_{qk} = s_q \) to all the parties in \( S_q \). Consequently, every honest \( P_j \in S_q \) will eventually broadcast the message \( \text{OK}_q(j,k) \), corresponding to every honest \( P_k \in S_q \). This is because \( s_{qj} = s_{qk} = s_q \) will hold. These \( \text{OK}_q(j,k) \) messages are eventually received by every honest party, including \( P_D \). This implies that the parties in \( S_q \setminus Z^* \) will eventually form a clique in the graph \( C_q^{(i)} \) of every honest \( P_i \). This further implies that \( P_D \) will eventually find a set \( C_q \) where \( S_q \setminus C_q \in Z \) and where \( C_q \) constitutes a clique in the consistency graph of every honest party. This is because the set \( S_q \setminus Z^* \) is guaranteed to eventually constitute a clique. Hence \( P_D \) eventually broadcasts the sets \( \{C_q\}_{S_q \in S} \), which are eventually delivered to every honest party. Moreover, the verification of these sets will eventually be successful for every honest party.
Next, consider an arbitrary honest \( P_i \in S_q \). If \( P_i \in C_q \), then it has already received the share \( s_q \) from \( P_D \) and \( s_{qj} = s_q \) holds. Hence, \( P_i \) sets \([s]_q \) to \( s_q \). So consider the case when \( P_i \notin C_q \). In this case, \( P_i \) sets \([s]_q \) based on the supposedly common values \( s_{qj} \) received from the parties \( P_j \in S_q \) as part of pairwise consistency tests. Specifically, \( P_i \) checks for a subset of parties \( C_q' \subseteq C_q \), where \( C_q \setminus C_q' \) \( \in Z \), such that every party \( P_j \in C_q' \) has sent the same \( s_{qj} \) value to \( P_i \) as part of the pairwise consistency test. This is because \( Z \) satisfies the \( \mathcal{Q}^{(4)}(P, Z) \) condition. Also, since the \( \mathcal{Q}^{(4)}(P, Z) \) condition is satisfied, the set of honest parties in \( C_q \), namely the parties in \( C_q \setminus Z^* \), always constitute a candidate \( C_q' \) set. This is because every party \( P_j \in C_q \setminus Z^* \) would have sent \( s_{qj} = s_q \) to every party in \( S_q \) during the pairwise consistency test, and these values are eventually delivered. \( \square \)

We next prove certain claims with respect to a corrupt dealer. The first claim is that the view of \( \text{Adv} \) in this case is also identically distributed in both the real as well as simulated execution. This is simply because in this case, the honest parties have no inputs and the simulator simply plays the role of the honest parties exactly as per the steps of the protocol \( \Pi_{PVSS} \) in the simulated execution.

**Claim B.4.** If \( P_D \) is corrupt, then the view of \( \text{Adv} \) in the simulated execution of \( \Pi_{PVSS} \) with \( S_{PVSS} \) is identically distributed as the view of \( \text{Adv} \) in the real execution of \( \Pi_{PVSS} \) involving honest parties.

**Proof.** The proof follows from the fact that if \( P_D \) is corrupt, then \( S_{PVSS} \) participates in a full execution of the protocol \( \Pi_{PVSS} \), by playing the role of the honest parties as per the steps of \( \Pi_{PVSS} \). Hence, there is a one-to-one correspondence between simulated executions and real executions. \( \square \)

We finally claim that if the dealer is corrupt, then conditioned on the view of the adversary (which is identical in both the executions as per the last claim), the outputs of the honest parties are identically distributed in both the executions.

**Claim B.5.** If \( D \) is corrupt, then conditioned on the view of \( \text{Adv} \), the output of the honest parties during the execution of \( \Pi_{PVSS} \) involving \( \text{Adv} \) has the same distribution as the output of the honest parties in the ideal-world involving \( S_{PVSS} \) and \( \mathcal{F}_{VSS} \).

**Proof.** Let \( P_D \) be corrupt and let \( \text{View} \) be an arbitrary view of \( \text{Adv} \). We note that whether valid core sets \( \{C_q\}_{S_q \in S} \) have been generated during the corresponding execution of \( \Pi_{PVSS} \) or not can be found out from \( \text{View} \). We now consider the following cases.

- **No core sets \( \{C_q\}_{S_q \in S} \) are generated per View:** In this case, the honest parties do not obtain any output in either execution. This is because in the real execution of \( \Pi_{PVSS} \), the honest parties compute their output only when they get valid core sets \( \{C_q\}_{S_q \in S} \) from \( P_D \)'s broadcast. If this is not the case, then in the simulated execution, the simulator \( S_{PVSS} \) does not provide any input to \( \mathcal{F}_{VSS} \) on behalf of \( P_D \); hence, \( \mathcal{F}_{VSS} \) does not produce any output for the honest parties.

- **Core sets \( \{C_q\}_{S_q \in S} \) generated per View are invalid:** Again, in this case, the honest parties do not obtain any output in either execution. This is because in the real execution of \( \Pi_{PVSS} \), even if the sets \( \{C_q\}_{S_q \in S} \) are received from \( P_D \)'s broadcast, the honest parties compute their output only when each set \( C_q \) is found to be valid with respect to the verifications performed by the honest parties in their own consistency graphs. If these verifications fail (implying that the core sets are invalid), then in the simulated execution, the simulator \( S_{PVSS} \) does not provide any input to \( \mathcal{F}_{VSS} \) on behalf of \( P_D \), implying that \( \mathcal{F}_{VSS} \) does not produce any output for the honest parties.

- **Valid core sets \( \{C_q\}_{S_q \in S} \) are generated per View:** We first note that in this case, \( P_D \) has distributed some common share, say \( s_q \), determined by \( \text{View} \), to all the parties in \( C_q \setminus Z^* \) during the real execution of \( \Pi_{PVSS} \). This is because all the parties in \( C_q \setminus Z^* \) are honest, and form a clique in the consistency graph of the honest
parties. Hence, each \( P_j, P_k \in C_q \setminus Z^* \) has broadcasted the messages \( \text{OK}_q(j, k) \) and \( \text{OK}_q(k, j) \) after checking that \( s_{qj} = s_{qk} \) holds, where \( s_{qj} \) and \( s_{qk} \) are the shares received from \( P_j \) by \( P_j \) and \( P_k \) respectively. We next show that in the real execution of \( \Pi_{PVSS} \), every party in \( S_q \setminus Z^* \), eventually sets \( |s_q = s_q. \) While this is true for the parties in \( C_q \setminus Z^* \), we consider an arbitrary party \( P_i \in S_q \setminus (Z^* \cup C_q). \) From the protocol steps, \( P_i \) checks for a subset of parties \( C_q' \subseteq C_q \) where \( C_q' \subseteq Z^* \), such that every party \( P_j \in C_q' \) has sent the same \( s_{qj} \) value to \( P_i \) as part of the pairwise consistency test. If \( P_i \) finds such a set \( C_q' \), then it sets \( |s_q \) to the common \( s_{qj} \). We next argue that \( P_i \) will eventually find such a set \( C_q' \) and if such a set \( C_q' \) is found by \( P_i \), then the common \( s_{qj} \) is the same as \( s_q. \) The proof for this is exactly the same, as for Claim \( \text{B.3}. \) Thus, in the real execution, every honest party \( P_i \) eventually outputs \( \{s_q = s_q\} \in S_q. \) From the steps of \( S_{PVSS} \), the simulator sends the shares \( \{s_q\} \in S_q \) to \( F_{VSS} \) on the behalf of \( P_i \) in the simulated execution. Consequently, in the ideal world, \( F_{VSS} \) will eventually deliver the shares \( \{s_q = s_q\} \in S_q \) to every honest \( P_i \). Hence, the outputs of the honest parties are identical in both the worlds.

\[ \square \]

The proof of the theorem now follows from Claims \( \text{B.2}, \text{B.5}. \)

Reducing the Broadcast Complexity of the Protocol \( \Pi_{PVSS} \): Protocol \( \Pi_{PVSS} \) as presented in [12] has a broadcast complexity, which is proportional to the size of \( Z. \) More specifically, in the protocol, \( P_D \) needs to compute a core set \( C_q \) corresponding to each \( S_q \in S. \) For finding these \( C_q \) sets, every pair of (honest) parties need to broadcast an \( \text{OK}_q \) message for each other by calling \( F_{Acast}. \) This results in the number of bits being broadcasted, proportional to \( |S|, \) where \( |S| = |Z| \) in our case. A small modification to the protocol can make the broadcast complexity, independent of \( |Z|. \) The idea is to let every party broadcast a single \( \text{OK} \) message for every other party, if the pairwise consistency test with that party is successful across all the sets \( S_q \) to which both the parties belong. In a more detail, party \( P_i \) sends an \( \text{OK}(i, j) \) message to \( F_{Acast}, \) only after checking whether \( s_{qi} = s_{qj} \) holds corresponding to every \( S_q \in S, \) such that \( P_j \in S_q \) holds. Consequently, \( P_D \) now checks for the presence of a single core set \( C \) where for \( q = 1, \ldots, |S|, \) the conditions \( C \subseteq S_q \) and \( S_q \setminus C \subseteq Z \) hold. Upon finding such a \( C \) the dealer broadcasts it by sending it to \( F_{Acast}. \) Note that such a \( C \) is eventually obtained for an honest \( P_D. \) This is because the set of parties \( (S_1 \setminus Z^*) \cap \ldots \cap (S_q \setminus Z^*) \) constitutes a candidate \( C \) for an honest \( P_D, \) where \( Z^* \) is the set of corrupt parties. The rest of the protocol steps remain the same. With these modifications, the communication complexity of the protocol \( \Pi_{PVSS} \) is computed as follows: the dealer needs to send the share \( s^{(q)} \) to all the parties in \( S_q, \) and every party in \( S_q \) has to send the received share to every other party in \( S_q \) during pairwise consistency tests. This incurs a communication of \( O(|Z| \cdot n^2 \log |P|) \) bits, since each \( |S_q| = O(n) \) and each share \( s^{(q)} \) can be represented by \( \log |P| \) bits. There will be total \( O(n^2) \) \( \text{OK} \) messages broadcasted, where each message can be represented by \( O(\log n) \) bits, since it represents the index of 2 parties. Moreover, \( P_D \) will broadcast a single core set \( C \) of size \( O(n \log n) \) bits. Based on this discussion, we next state the following theorem for \( \Pi_{PVSS} \).

**Theorem B.6.** Consider a static malicious adversary \( \text{Adv} \) characterized by an adversary-structure \( Z, \) satisfying the \( Q^{(1)}(\mathcal{P}, Z) \) condition and let \( S = (S_1, \ldots, S_h) \) def \( \mathcal{P} \setminus Z|Z \in Z \) be the sharing specification.\footnote{Hence \( S \) satisfies the \( Q^{(1)}(\mathcal{S}, Z) \) condition.} Then protocol \( \Pi_{PVSS} \) UC-securely realizes the functionality \( F_{VSS} \) with perfect security in the \( F_{Acast}^{-}\)-hybrid model, in the presence of \( \text{Adv}. \) The protocol makes \( O(n^2) \) calls to \( F_{Acast} \) with \( O(\log n) \) bit messages, one call to \( F_{Acast} \) with \( O(n \log n) \) bit message and additionally incurs a communication of \( O(|Z| \cdot n^2 \log |P|) \) bits.

By replacing the calls to \( F_{Acast} \) with protocol \( \Pi_{Acast}, \) the protocol incurs a total communication of \( O(|Z| \cdot n^2 \log |P| + n^4 \log n) \) bits.

**B.2 Asynchronous Reconstruction Protocols**

Let \( s \) be a value which is secret-shared with respect to some sharing specification \( S = (S_1, \ldots, S_h), \) such that \( S \) satisfies the \( Q^{(2)}(\mathcal{S}, Z) \) condition. We first present the protocol \( \Pi_{PVSS} \) to reconstruct a single share \( |s_q|, \) corresponding to a designated set \( S_q \in S. \) In the protocol, every party in \( S_q \) sends
the share \([s]_q\) to all the parties outside \(S_q\), who then “filter” out the potentially incorrect versions of \([s]_q\) and output \([s]_q\). Protocol \(\Pi_{\text{PerRecShare}}\) is formally presented in Figure 14.

**Protocol \(\Pi_{\text{PerRecShare}}(q)\)**

- **Sending Share to All Parties:** If \(P_i \in S_q\), then execute the following steps.
  1. On having \([s]_q\), send \((\text{share}, \text{sid}, q, [s]_q)\) to all the parties in \(\mathcal{P} \setminus S_q\).

- **Computing Output:** Based on the following conditions, execute the corresponding steps.
  1. \(P_i \in S_q\): Output \([s]_q\).
  2. \(P_i \notin S_q\): Upon receiving \((\text{share}, \text{sid}, q, v)\) from a set of parties \(S'_q \subseteq S_q\) such that \(S_q \setminus S'_q \in Z\), output \([s]_q = v\).

**Figure 14:** Perfectly-secure reconstruction protocol for session id sid to publicly reconstruct the share \([s]_q\) corresponding to \(S_q \in \mathbb{S}\). The public inputs are \(\mathcal{P}, Z\) and \(\mathbb{S}\). The above steps are executed by every \(P_i \in \mathcal{P}\).

**Lemma B.7.** Let \(Z\) be an adversary structure and let \(\mathbb{S} = (S_1, \ldots, S_h)\) be a sharing specification, such that \(\mathbb{S}\) satisfies the \(Q^{(2)}(\mathbb{S}, Z)\) condition. Moreover, let \(s\) be a value, which is secret-shared as per \(\mathbb{S}\). Then for any \(q \in \{1, \ldots, h\}\) and any adversary \(\text{Adv}\) corrupting a set of parties \(Z^* \in Z\), all honest parties eventually output the share \([s]_q\) in the protocol \(\Pi_{\text{PerRecShare}}\). The protocol incurs a communication of \(\mathcal{O}(n^2 \log |F|)\) bits.

**Proof.** Consider an arbitrary honest party \(P_i \in \mathcal{P}\). We consider two cases.

- \(P_i \in S_q\): In this case, \(P_i\) outputs \([s]_q\).

- \(P_i \notin S_q\): In this case, \(P_i\) waits for a subset of parties \(S'_q \subseteq S_q\) where \(S_q \setminus S'_q \in Z\), such that every party \(P_j \in S'_q\) has sent the same share \(v\) to \(P_i\). If \(P_i\) finds such a set \(S'_q\), then it outputs \(v\). To complete the proof, we need to show that \(P_i\) will eventually find such a set \(S'_q\) and if such a set \(S'_q\) is found by \(P_i\), then the common \(v\) is the same as \([s]_q\).

Assuming that \(P_i\) eventually finds such a common \(S'_q\), the proof that the common \(v\) is the same as \([s]_q\) follows from the fact that \(S'_q\) is guaranteed to contain at least one honest party from \(S_q\), who would have sent the share \([s]_q\) to \(P_i\). This is because the \(Q^{(2)}(\mathbb{S}, Z)\) condition is satisfied. Also, since the \(Q^{(2)}(\mathbb{S}, Z)\) condition is satisfied, the set of honest parties in \(S_q\), namely the parties in \(S_q \setminus Z^*\), always constitute a candidate \(S'_q\) set. This is because every party \(P_j \in S_q \setminus Z^*\) would have sent \([s]_q\) to \(P_i\) and these values are eventually delivered to \(P_i\). The communication complexity follows from the protocol steps.

We now present the protocol \(\Pi_{\text{PerRec}}\) (Fig 15), which allows all parties in \(\mathcal{P}\) to reconstruct a secret shared value \(s\). The idea is to run an instance of \(\Pi_{\text{PerRecShare}}\) for each \(S_q \in \mathbb{S}\), and to sum up the shares obtained as the output from each instantiation.

**Protocol \(\Pi_{\text{PerRec}}\)**

- **Reconstructing Shares:** For each \(S_q \in \mathbb{S}\), participate in an instance \(\Pi_{\text{PerRecShare}}(q)\) with sid to obtain the output \([s]_q\).

- **Output Computation:** Output \(s = \sum_{S_q \in \mathbb{S}} [s]_q\).

**Figure 15:** Perfectly-secure reconstruction protocol for session id sid to reconstruct a shared value \(s\). The public inputs of the protocol are \(\mathcal{P}, \mathbb{S}\) and \(Z\). The above steps are executed by every \(P_i \in \mathcal{P}\).

The properties of the protocol \(\Pi_{\text{PerRec}}\) are stated in Lemma B.8, which follow from the protocol steps and Lemma B.7.

**Lemma B.8.** Let \(Z\) be an adversary structure and let \(\mathbb{S} = (S_1, \ldots, S_h)\) be a sharing specification, such that \(\mathbb{S}\) satisfies the \(Q^{(2)}(\mathbb{S}, Z)\) condition. Moreover, let \(s\) be a value which is secret-shared as per \(\mathbb{S}\). Then for every adversary \(\text{Adv}\) corrupting a set of parties \(Z^* \in Z\), all honest parties eventually output \(s\) in the protocol \(\Pi_{\text{PerRecShare}}\). The protocol incurs a communication of \(\mathcal{O}(|\mathbb{S}| \cdot n^2 \log |F|)\) bits, which is \(\mathcal{O}(|Z| \cdot n^2 \log |F|)\) bits if \(|\mathbb{S}| = |Z|\).
In this section, we formally prove the properties of the protocol \( \Pi_{\text{OptMult}} \) (Fig 3). While proving these properties, we will assume that \( Z \) satisfies the \( \mathcal{Q}^{(4)}(P, Z) \) condition. This further implies that the sharing specification \( S = (S_1, \ldots, S_h) \) satisfies the \( \mathcal{Q}^{(3)}(S, Z) \) condition. Moreover, while proving these properties, we also assume that for every iter, no honest party is ever included in the set \( GD \) and all honest parties are eventually removed from the \( W^{(i)}_{\text{iter}}, LD^{(i)}_{\text{iter}} \) sets of every honest \( P_i \) for every iter’ < iter. Note that these conditions are guaranteed in the protocols \( \Pi_{\text{MultCl}} \) and \( \Pi_{\text{Mult}} \) (where these sets are constructed and managed), where \( \Pi_{\text{OptMult}} \) is used as a subprotocol.

Claim B.9. For every \( Z \in \mathcal{Z} \) and every ordered pair \( (p, q) \in \{1, \ldots, h\} \times \{1, \ldots, h\} \), the set \( (S_p \cap S_q) \setminus Z \) contains at least one honest party.

Proof. From the definition of the sharing specification \( S \), we have \( S_p = P \setminus Z_p \) and \( S_q = P \setminus Z_q \), where \( Z_p, Z_q \in \mathcal{Z} \). Let \( Z^* \in \mathcal{Z} \) be the set of corrupt parties during the protocol \( \Pi_{\text{OptMult}} \). If \( (S_p \cap S_q) \setminus Z \) does not contain any honest party, then it implies that \( ((S_p \cap S_q) \setminus Z) \subseteq Z^* \). This further implies that \( P \subseteq Z_p \cup Z_q \cup Z \cup Z^* \), implying that \( Z \) does not satisfy the \( \mathcal{Q}^{(4)}(P, Z) \) condition, which is a contradiction. \( \square \)

Claim B.10. For every \( Z \in \mathcal{Z} \), if all honest parties participate during the hop number hop in the protocol \( \Pi_{\text{OptMult}} \), then all honest parties eventually obtain a common summand-sharing party, say \( p_j \), for this hop, such that the honest parties will eventually hold \( \lfloor c_{(Z, \text{iter})}^{(j)} \rfloor \). Moreover, party \( p_j \) will be distinct from the summand-sharing party selected for any hop number hop’ < hop.

Proof. Since all honest parties participate in hop number hop, it follows that \( \text{Summands}_{(Z, \text{iter})} \neq \emptyset \) at the beginning of hop number hop. This implies that there exists at least one ordered pair \( (p, q) \in \text{Summands}_{(Z, \text{iter})} \). From Claim B.9, there exists at least one honest party in \( (S_p \cap S_q) \setminus Z \), say \( P_k \), who will have both the shares \( [a]_p \) as well as \( [b]_q \) (and hence the summand \( [a]_p[b]_q \)). We also note that \( P_k \) would not have been selected as the common summand-sharing party in any previous hop’ < hop, as otherwise \( P_k \) would have already included the summand \( [a]_p[b]_k \) in the sum \( c_{(Z, \text{iter})}^{(k)} \) shared by \( P_k \) during hop hop’, implying that \( (p, q) \notin \text{Summands}_{(Z, \text{iter})} \). Now, during the hop number hop, party \( P_k \) will randomly secret-share the sum \( c_{(Z, \text{iter})}^{(k)} \) by making a call to \( \mathcal{F}_{\text{VSS}} \) and every honest \( P_i \) will eventually receive an output \( \{\text{share, sid}_{\text{hop}, k, \text{iter}, Z}, P_k, \{\lfloor c_{(Z, \text{iter})}^{(k)} \rfloor\}_{P_i \in S_q}\} \) from \( \mathcal{F}_{\text{VSS}} \) with \( \text{sid}_{\text{hop}, k, \text{iter}, Z} \). Moreover, \( P_k \) will not be present in the set \( GD \) and if \( P_k \) is present in the sets \( W^{(i)}_{\text{iter}}, LD^{(i)}_{\text{iter}} \) of any honest \( P_i \) for any iter’ < iter, then it is eventually removed from these sets.

We next claim that during the hop number hop, there will at least one instance of \( \mathcal{F}_{\text{ABA}} \) corresponding to which all honest parties eventually receive the output 1. For this, we consider two possible cases:

- At least one honest party participates with input 0 in the \( \mathcal{F}_{\text{ABA}} \) instance corresponding to \( P_k \): Let \( P_i \) be an honest party who sends \( \lfloor \text{vote, sid}_{\text{hop}, k, \text{iter}, Z}, 0 \rfloor \) to \( \mathcal{F}_{\text{ABA}} \) with \( \text{sid}_{\text{hop}, k, \text{iter}, Z} \). Then from the steps of \( \Pi_{\text{OptMult}} \), it follows that there exists some \( P_j \in P \), such that \( P_j \) has received \( \{\text{decide, sid}_{\text{hop}, j, \text{iter}, Z}, 1\} \) as the output from \( \mathcal{F}_{\text{ABA}} \) with \( \text{sid}_{\text{hop}, j, \text{iter}, Z} \). Hence, every honest party will eventually receive the output \( \{\text{decide, sid}_{\text{hop}, j, \text{iter}, Z}, 1\} \) as the output from \( \mathcal{F}_{\text{ABA}} \) with \( \text{sid}_{\text{hop}, j, \text{iter}, Z} \).

- No honest party participates with input 0 in the \( \mathcal{F}_{\text{ABA}} \) instance corresponding to \( P_k \): In this case, every honest party will eventually send \( \lfloor \text{vote, sid}_{\text{hop}, k, \text{iter}, Z}, 1 \rfloor \) to \( \mathcal{F}_{\text{ABA}} \) with \( \text{sid}_{\text{hop}, k, \text{iter}, Z} \) and eventually receives the output \( \{\text{decide, sid}_{\text{hop}, k, \text{iter}, Z}, 1\} \) from \( \mathcal{F}_{\text{ABA}} \).

Now, based on the above claim, we can further claim that all honest parties will eventually participate with some input in all the \( n \) instances of \( \mathcal{F}_{\text{ABA}} \) invoked during the hop number hop and hence, all the \( n \) instances of \( \mathcal{F}_{\text{ABA}} \) during the hop number hop will eventually produce an output. Since the summand-sharing party for hop number hop corresponds to the least indexed \( \mathcal{F}_{\text{ABA}} \) instance in which all the honest parties obtain 1 as the output, it follows that eventually the honest parties will select a summand-sharing party. Moreover, this summand-sharing party will be common, as it is based on the outcome of \( \mathcal{F}_{\text{ABA}} \) instances.
Let $P_j$ be the summand-sharing party for the hop number hop. We next show that the honest parties will eventually hold $[c^{(j)}_{(Z, \text{iter})}]$. For this, we note that since $P_j$ has been selected as the summand-sharing party, at least one honest party, say $P_i$, must have sent $(\text{vote}, \text{sid}_{\text{hop},j,\text{iter},Z}, 1)$ to $\mathcal{F}_{\text{ABA}}$ with $\text{sid}_{\text{hop},j,\text{iter},Z}$. If not, then $\mathcal{F}_{\text{ABA}}$ with $\text{sid}_{\text{hop},j,\text{iter},Z}$ will never produce the output $(\text{decide}, \text{sid}_{\text{hop},j,\text{iter},Z}, 1)$ and hence $P_j$ will not be the summand-sharing party for the hop number hop. Now since $P_i$ sent $(\text{vote}, \text{sid}_{\text{hop},j,\text{iter},Z}, 1)$ to $\mathcal{F}_{\text{ABA}}$, it follows that $P_i$ has received an output $(\text{share}, \text{sid}_{\text{hop},j,\text{iter},Z}, P_j, \{[c^{(j)}_{(Z, \text{iter})}]_q \mid p \in S_q\})$ from $\mathcal{F}_{\text{VSS}}$ with $\text{sid}_{\text{hop},j,\text{iter},Z}$. This implies that $P_j$ must have sent the message $(\text{dealer}, \text{sid}_{\text{hop},j,\text{iter},Z}, (c^{(j)}_{(Z, \text{iter})}, \ldots, c^{(n)}_{(Z, \text{iter})}))$ to $\mathcal{F}_{\text{VSS}}$ with $\text{sid}_{\text{hop},j,\text{iter},Z}$. Consequently, every honest party will eventually receive their respective outputs from $\mathcal{F}_{\text{VSS}}$ with $\text{sid}_{\text{hop},j,\text{iter},Z}$ and hence, the honest parties will eventually hold $[c^{(j)}_{(Z, \text{iter})}]$.

Finally, to complete the proof of the claim, we need to show that party $P_j$ is different from the summand-sharing parties selected during the hops $1, \ldots, \text{hop} - 1$. If $P_j$ has been selected as a summand-sharing party for any hop number hop' < hop, then no honest party ever sends $(\text{vote}, \text{sid}_{\text{hop},j,\text{iter},Z}, 1)$ to $\mathcal{F}_{\text{ABA}}$ with $\text{sid}_{\text{hop},j,\text{iter},Z}$. Consequently, $\mathcal{F}_{\text{ABA}}$ with $\text{sid}_{\text{hop},j,\text{iter},Z}$ will never send the output $(\text{decide}, \text{sid}_{\text{hop},j,\text{iter},Z}, 1)$ to any honest party and hence $P_j$ will not be selected as the summand-sharing party for hop number hop, which is a contradiction. 

Claim B.11. In protocol $\Pi_{\text{OptMult}}$, all honest parties eventually obtain an output. The protocol makes $O(n^2)$ calls to $\mathcal{F}_{\text{VSS}}$ and $\mathcal{F}_{\text{ABA}}$.

Proof. From Claim B.9 and B.10, it follows that the number of hops in the protocol is $O(n)$, as in each hop a new summand-sharing party is selected and if all honest parties are included in the set of summand-sharing parties $\text{Selected}_{(Z, \text{iter})}$, then $\text{Summands}_{(Z, \text{iter})}$ becomes $\emptyset$. The proof now follows from the fact that in each hop, there are $O(n)$ calls to $\mathcal{F}_{\text{VSS}}$ and $\mathcal{F}_{\text{ABA}}$. 

Claim B.12. In protocol $\Pi_{\text{OptMult}}$, if no party in $\mathcal{P} \setminus Z$ behaves maliciously, then for each $P_i \in \text{Selected}_{(Z, \text{iter})}$, the condition $c^{(i)}_{(Z, \text{iter})} = \sum_{(p,q) \in \text{Summands}^{(i)}_{(Z, \text{iter})}} [a]_{p}[b]_{q}$ holds and $c_{(Z, \text{iter})} = ab$.

Proof. From the protocol steps, it follows that $\text{Selected}_{(Z, \text{iter})} \cap Z = \emptyset$, as no honest part ever votes for any party from $Z$ as a candidate summand-sharing party during any hop in the protocol. Now since $\text{Selected}_{(Z, \text{iter})} \subseteq (\mathcal{P} \setminus Z)$, if no party in $\mathcal{P} \setminus Z$ behaves maliciously, then it implies that every party $P_i \in \text{Selected}_{(Z, \text{iter})}$ behaves honestly and secret-shares $c^{(i)}_{(Z, \text{iter})}$ by calling $\mathcal{F}_{\text{VSS}}$, where $c^{(i)}_{(Z, \text{iter})} = \sum_{(p,q) \in \text{Summands}^{(i)}_{(Z, \text{iter})}} [a]_{p}[b]_{q}$. Moreover, from the protocol steps, it follows that for every $P_j, P_k \in \text{Selected}_{(Z, \text{iter})}$:

$$\text{Summands}^{(j)}_{(Z, \text{iter})} \cap \text{Summands}^{(k)}_{(Z, \text{iter})} = \emptyset.$$ 

To prove this, suppose $P_j$ and $P_k$ are included in $\text{Selected}_{(Z, \text{iter})}$ during hop number hop$_j$ and hop$_k$ respectively, where without loss of generality, hop$_j <$ hop$_k$. Then from the protocol steps, during hop$_j$, the parties would set $\text{Summands}^{(k)}_{(Z, \text{iter})} = \text{Summands}^{(k)}_{(Z, \text{iter})} \setminus \text{Summands}^{(j)}_{(Z, \text{iter})}$. This ensures that during hop$_k$, there exists no ordered pair $(p, q) \in \{1, \ldots, |S|\} \times \{1, \ldots, |S|\}$, such that $(p, q) \in \text{Summands}^{(j)}_{(Z, \text{iter})} \cap \text{Summands}^{(k)}_{(Z, \text{iter})}$.

Since all the parties $P_i \in \text{Selected}_{(Z, \text{iter})}$ have behaved honestly, from the protocol steps, it also follows that :

$$\bigcup_{P_i \in \text{Selected}_{(Z, \text{iter})}} \text{Summands}^{(i)}_{(Z, \text{iter})} = \{(p, q)\}_{p,q = 1,\ldots,|S|}.$$ 

Finally, from the protocol steps, it follows that $\forall P_j \in \mathcal{P} \setminus \text{Selected}_{(Z, \text{iter})}$, the condition $c^{(j)}_{(Z, \text{iter})} = 0$ holds. Now since $c_{(Z, \text{iter})} = c^{(1)}_{(Z, \text{iter})} + \ldots + c^{(n)}_{(Z, \text{iter})}$, it follows that if no party in $\mathcal{P} \setminus Z$ behaves maliciously, then $c_{(Z, \text{iter})} = ab$ holds.

Claim B.13. In $\Pi_{\text{OptMult}}$, Adv does not learn any additional information about $a$ and $b$. 

31
Proof. Let $Z^* \in Z$ be the set of corrupt parties. To prove the claim, we argue that in the protocol, Adv does not learn any additional information about the shares $\{[a]_p, [b]_p\}_{S_p \cap Z^* = \emptyset}$. For this, consider an arbitrary summand $[a]_p [b]_q$ where $S_p \cap Z^* = \emptyset$ and where $q \in \{1, \ldots, h\}$. Clearly, the summand $[a]_p [b]_q$ will not be available with any party in $Z^*$. Let $P_j$ be the party from $\text{Selected}_{(Z, \text{iter})}$, such that $(p, q) \in \text{Summands}^{(j)}_{(Z, \text{iter})}$; i.e. the summand $[a]_p [b]_q$ is included by $P_j$ while computing the summand-sum $\epsilon^{(j)}_{(Z, \text{iter})}$. Clearly $P_j$ is honest, since $P_j \notin Z^*$. In the protocol, party $P_j$ randomly secret-shares the summand-sum $\epsilon^{(j)}_{(Z, \text{iter})}$, by supplying a random vector of shares for $\epsilon^{(j)}_{(Z, \text{iter})}$ to the corresponding $F_{\text{VSS}}$. Now, since $S$ is $Z$-private, it follows that the shares $\{[\epsilon^{(j)}_{(Z, \text{iter})}]_r\}_{S_r \cap Z^* = \emptyset}$ learnt by Adv in the protocol will be independent of the summand $[a]_p [b]_q$ and hence, independent of $[a]_p$. Using a similar argument, we can conclude that the shares learnt by Adv in the protocol will be independent of the summands $[a]_p [b]_q$ and hence independent of $[b]_q$, where $S_p \cap Z^* = \emptyset$ and where $q \in \{1, \ldots, h\}$. □

Lemma 3.1. Let $Z$ satisfy the $Q^{(4)}(\mathcal{P}, Z)$ condition and let $S = (S_1, \ldots, S_h) = \{\mathcal{P} \setminus Z | Z \in Z\}$. Consider an arbitrary $Z \in Z$ and iter, such that all honest parties participate in the instance $\Pi_{\text{OptMult}}(\mathcal{P}, Z, S, [a], [b], Z, \text{iter})$. Then all honest parties eventually compute $[c_{(Z, \text{iter})}]$ and $([c_{(Z, \text{iter})}^{(1)}], \ldots, [c_{(Z, \text{iter})}^{(n)}])$ where $c_{(Z, \text{iter})} = c_{(Z, \text{iter})}^{(1)} + \ldots + c_{(Z, \text{iter})}^{(n)}$, provided no honest party is ever included in the $\mathcal{GD}$ and $\mathcal{LD}^{(i)}_{\text{iter}}$, sets and every honest party in the $\mathcal{W}^{(i)}_{\text{iter}}$ sets of every honest $P_i$ is eventually removed, for every iter $' < \text{iter}$. If no party in $\mathcal{P} \setminus Z$ behaves maliciously, then $c_{(Z, \text{iter})} = ab$. In the protocol, Adv does not learn any additional information about $a$ and $b$. The protocol makes $O(n^2)$ calls to $F_{\text{VSS}}$ and $F_{\text{ABA}}$.

Proof. The proof follows from Claims B.9, B.13.

We end this section by claiming an important property about the protocol $\Pi_{\text{OptMult}}$, which will be useful later when we analyze the properties of the protocol $\Pi_{\text{Mult}}$ where $\Pi_{\text{OptMult}}$ is used as a sub-protocol.

Claim B.14. For every $Z \in Z$ and every iter, all the following hold for every $P_j \in \text{Selected}_{(Z, \text{iter})}$ during the instance $\Pi_{\text{OptMult}}(\mathcal{P}, Z, S, [a], [b], Z, \text{iter})$.

- There exists at least one honest party $P_i$, such that $P_j$ will not be present in the $\mathcal{W}^{(i)}_{\text{iter}}$ and $\mathcal{LD}^{(i)}_{\text{iter}}$ sets of $P_i$ for any iter $' < \text{iter}$.
- $P_j$ will not be present in the set $\mathcal{GD}$.

Proof. Consider an arbitrary $P_j \in \text{Selected}_{(Z, \text{iter})}$, such that $P_j$ is included in $\text{Selected}_{(Z, \text{iter})}$ during the hop number hop in the instance $\Pi_{\text{OptMult}}(\mathcal{P}, Z, S, [a], [b], Z, \text{iter})$. We prove the first part of the claim through a contradiction. Let $\mathcal{H}$ be the set of honest parties and for every $P_i \in \mathcal{H}$, let there exist some iter $' < \text{iter}$, such that either $P_j \in \mathcal{W}^{(i)}_{\text{iter}}$ or $P_j \in \mathcal{LD}^{(i)}_{\text{iter}}$. This implies that during hop number hop, no $P_i \in \mathcal{H}$ will send (vote, $\text{sid}_{\text{hop}, j, \text{iter}, Z}, 1$) to $F_{\text{ABA}}$ with $\text{sid}_{\text{hop}, j, \text{iter}, Z}$. Consequently, $F_{\text{ABA}}$ with $\text{sid}_{\text{hop}, j, \text{iter}, Z}$ will never return the output (decide, $\text{sid}_{\text{hop}, j, \text{iter}, Z}, 1$) for any honest party and hence, $P_j$ will not be selected as the summand-sharing party for hop number hop, which is a contradiction.

The second part of the claim also follows using a similar argument as above. Namely, if $P_j$ is present in the set $\mathcal{GD}$, then no $P_i \in \mathcal{H}$ will send (vote, $\text{sid}_{\text{hop}, j, \text{iter}, Z}, 1$) to $F_{\text{ABA}}$ with $\text{sid}_{\text{hop}, j, \text{iter}, Z}$ and consequently, $P_j$ will not be selected as the summand-sharing party for hop number hop, which is a contradiction.

B.4 Properties of the Multiplication Protocol $\Pi_{\text{MultCI}}$ with Cheater Identification

In this section, we formally prove the properties of the protocol $\Pi_{\text{MultCI}}$ (see Fig 4 for the formal description of the protocol). While proving these properties, we will assume that $Z$ satisfies the $Q^{(3)}(\mathcal{P}, Z)$ condition. This further implies that the sharing specification $S = (S_1, \ldots, S_h) \overset{\text{def}}{=} \{\mathcal{P} \setminus Z | Z \in Z\}$ satisfies the $Q^{(3)}(S, Z)$ condition. Moreover, we will also assume that no honest party is ever included in the set $\mathcal{GD}$, which will be guaranteed in the protocol $\Pi_{\text{Mult}}$ where the set $\mathcal{GD}$ is constructed and managed, and where $\Pi_{\text{MultCI}}$ is used as a sub-protocol.

We first give the definition of a successful $\Pi_{\text{MultCI}}$ instance, which will be used throughout this section and the next.
**Definition B.15 (Successful \( \Pi_{\text{MultCI}} \) Instance).** For an instance \( \Pi_{\text{MultCI}}(\mathcal{P}, \mathcal{Z}, \mathcal{S}, [a], [b], \text{iter}) \), we define the following.

- The instance is called *successful* if and only if for every \( Z \in \mathcal{Z} \), the value \( c_{(Z, \text{iter})} - c_{(Z', \text{iter})} = 0 \), where \( Z' \in \mathcal{Z} \) is the fixed set used in the protocol.
- If the instance is not successful, then the sets \( Z, Z' \) are called the *conflicting-sets* for the instance, if \( Z \) is the smallest indexed set from \( \mathcal{Z} \) such that \( c_{(Z, \text{iter})} - c_{(Z', \text{iter})} \neq 0 \).

We first show that any instance of \( \Pi_{\text{MultCI}} \) will be eventually found to be either a success or a failure by the honest parties.

**Claim B.16.** For every iter, any instance \( \Pi_{\text{MultCI}}(\mathcal{P}, \mathcal{Z}, \mathcal{S}, [a], [b], \text{iter}) \) will eventually be deemed to either succeed or fail by the honest parties, provided no honest party is ever included in the \( \mathcal{GD} \) and \( \mathcal{LD}^{(i)}_{\text{iter}} \) sets, and all honest parties are eventually removed from the \( \mathcal{W}^{(i)}_{\text{iter}} \) sets of every honest \( P_i \) for every \( \text{iter}' < \text{iter} \). Moreover, for a successful instance, the parties output a sharing of \( ab \). If the instance is not successful, then the parties identify the conflicting-sets \( Z, Z' \) for the instance.

**Proof.** Let \( Z^* \in \mathcal{Z} \) be the set of corrupt parties. If the lemma conditions hold, then it follows from Lemma 3.1 that corresponding to every \( Z \in \mathcal{Z} \), the instance \( \Pi_{\text{OptMult}}(\mathcal{P}, \mathcal{Z}, \mathcal{S}, [a], [b], \text{iter}) \) eventually completes with honest parties obtaining the outputs \( [c_{(1, \text{iter})}], \ldots, [c_{(n, \text{iter})}], [c_{(Z, \text{iter})}] \), where \( c_{(Z, \text{iter})} = c_{(Z^*, \text{iter})} + \ldots + c_{(Z, \text{iter})} \). Moreover, in the \( \Pi_{\text{OptMult}} \) instance corresponding to \( Z^* \), the output \( c_{(Z^*, \text{iter})} \) will be the same as \( ab \), since all the parties in \( \mathcal{P} \setminus Z^* \) will be honest.

Since \( \mathcal{S} \) satisfies the \( \mathcal{Q}^{(3)}(\mathcal{S}, Z) \) condition, it follows that with respect to the fixed \( Z' \in \mathcal{Z} \), the honest parties will eventually reconstruct the difference \( c_{(Z, \text{iter})} - c_{(Z', \text{iter})} \) corresponding to every \( Z \in \mathcal{Z} \). Now there are two possibilities. If all the differences \( c_{(Z, \text{iter})} - c_{(Z', \text{iter})} \) turn out to be 0, then the \( \Pi_{\text{MultCI}} \) instance will be considered to be successful by the honest parties and the honest parties will output \( [c_{(Z', \text{iter})}] \), which is bound to be the same as \( ab \). This is because \( c_{(Z, \text{iter})} - c_{(Z', \text{iter})} = 0 \) and hence \( c_{(Z', \text{iter})} = c_{(Z^*, \text{iter})} = ab \) holds. The other possibility is that all the differences are *not* zero, in which case the instance \( \Pi_{\text{MultCI}} \) will not be considered successful by the honest parties. Moreover, in this case, the parties will set \( (Z, Z') \) as the conflicting-sets for the instance, where \( Z \) is the smallest indexed set from \( Z \) such that \( c_{(Z, \text{iter})} - c_{(Z', \text{iter})} \neq 0 \).

We next prove a series of claims regarding any \( \Pi_{\text{MultCI}} \) instance which is *not* successful. We begin by showing that if any instance of \( \Pi_{\text{MultCI}} \) is *not* successful, then every honest party in eventually removed from the waiting-list of the honest parties for that instance. Moreover, no honest party will be ever included in the \( \mathcal{LD} \) set of any honest party for that instance.

**Claim B.17.** For every iter, if the instance \( \Pi_{\text{MultCI}}(\mathcal{P}, \mathcal{Z}, \mathcal{S}, [a], [b], \text{iter}) \) is not successful, then every honest party from the set \( \text{Selected}_{(Z, \text{iter})} \cup \text{Selected}_{(Z', \text{iter})} \) is eventually removed from the waiting set \( \mathcal{W}^{(i)}_{\text{iter}} \) of every honest party \( P_i \). Moreover, no honest party is ever included in the \( \mathcal{LD}^{(i)}_{\text{iter}} \) set of any honest party \( P_i \).

**Proof.** Suppose that the instance \( \Pi_{\text{MultCI}}(\mathcal{P}, \mathcal{Z}, \mathcal{S}, [a], [b], \text{iter}) \) is not successful. This implies that the parties identify a pair of conflicting-sets \( (Z, Z') \), such that \( c_{(Z, \text{iter})} - c_{(Z', \text{iter})} \neq 0 \). From the protocol steps, every honest party \( P_i \) initializes \( \mathcal{W}^{(i)}_{\text{iter}} \) to \( \text{Selected}_{(Z, \text{iter})} \cup \text{Selected}_{(Z', \text{iter})} \). Let \( P_j \) be an arbitrary honest party belonging to \( \text{Selected}_{(Z, \text{iter})} \cup \text{Selected}_{(Z', \text{iter})} \). From the protocol steps, party \( P_j \) secret-shares all the required values \( d^{(j, k)}_{(Z, \text{iter})}, c^{(j, k)}_{(Z', \text{iter})} \) by calling appropriate instances of \( \mathcal{F}_{\text{VSS}} \) and eventually these values are secret-shared, with every honest \( P_j \) receiving the appropriate shares from corresponding \( \mathcal{F}_{\text{VSS}} \) instances. Consequently, \( P_j \) will eventually be removed from the set \( \mathcal{W}^{(i)}_{\text{iter}} \). Moreover, since \( P_j \) is an honest party, the \( d^{(j, k)}_{(Z, \text{iter})}, c^{(j, k)}_{(Z', \text{iter})} \) values shared by \( P_j \) will be correct and consequently, the conditions for including \( P_j \) to the \( \mathcal{LD}^{(i)}_{\text{iter}} \) set of any honest party \( P_i \) will fail. That is, if \( P_j \in \text{Selected}_{(Z, \text{iter})} \), then the parties will find that \( c^{(j)}_{(Z, \text{iter})} - \sum_{P_k \in \text{Selected}_{(Z', \text{iter})}} d^{(j, k)}_{(Z, \text{iter})} = 0 \). On the other
hand, if \( P_j \in \text{Selected}_{(Z', \text{iter})} \), then the parties will find that \( c_{(Z', \text{iter})}^{(j)} - \sum_{P_k \in \text{Selected}_{(Z', \text{iter})}} d_{(Z', \text{iter})}^{(jk)} = 0 \). Moreover, if there exists any \( P_k \in \text{Selected}_{(Z, \text{iter})} \cup \text{Selected}_{(Z', \text{iter})} \) such that either \( d_{(Z, \text{iter})}^{(jk)} \neq e_{(Z', \text{iter})}^{(jk)} \) or \( d_{(Z, \text{iter})}^{(jk)} \neq e_{(Z', \text{iter})}^{(jk)} \), then after reconstructing the values shared by \( P_j \) and the shares held by \( P_j \), the parties will find that \( P_j \) has behaved honestly and hence, \( P_j \) will not be included to the \( \mathcal{LD}^{(i)}_{\text{iter}} \) set of any honest \( P_i \).

We next give the definition of a conflicting-pair of parties, which is defined based on the partitions of the summand-sum shared by the summand-sharing parties.

**Definition B.18 (Conflicting-Pair of Parties).** Let \( \Pi_{\text{MultCl}}(\mathcal{P}, Z, S, [a], [b], \text{iter}) \) be an instance of \( \Pi_{\text{MultCl}} \) which is not successful and let \( Z, Z' \) be the corresponding conflicting-sets for the instance. A pair of parties \((P_j, P_k)\) is said to be a conflicting-pair of parties for this \( \Pi_{\text{MultCl}} \) instance if all the following hold:

\[ P_j \in \text{Selected}_{(Z, \text{iter})}, P_k \in \text{Selected}_{(Z', \text{iter})}; \]
\[ d_{(Z, \text{iter})}^{(jk)} \neq e_{(Z', \text{iter})}^{(jk)}; \]

We next show that if an instance of \( \Pi_{\text{MultCl}} \) is not successful, then certain conditions hold with respect to the summand-sums and the respective partitions shared by the summand-sharing parties during the underlying instances of \( \Pi_{\text{OptMult}} \) and the cheater-identification phase of the \( \Pi_{\text{MultCl}} \) instance.

**Claim B.19.** Let \( \Pi_{\text{MultCl}}(\mathcal{P}, Z, S, [a], [b], \text{iter}) \) be an instance of \( \Pi_{\text{MultCl}} \) which is not successful and let \( Z, Z' \) be the corresponding conflicting-sets for the instance. Moreover, let \( Z^* \) be the set of corrupt parties. Then, one of the following must hold true for some \( P_j \in Z^* \).

1. \( P_j \in \text{Selected}_{(Z, \text{iter})} \) and \( c_{(Z, \text{iter})}^{(j)} - \sum_{P_k \in \text{Selected}_{(Z', \text{iter})}} d_{(Z', \text{iter})}^{(jk)} \neq 0 \).
2. \( P_j \in \text{Selected}_{(Z', \text{iter})} \) and \( c_{(Z', \text{iter})}^{(j)} - \sum_{P_k \in \text{Selected}_{(Z, \text{iter})}} e_{(Z, \text{iter})}^{(jk)} \neq 0 \).
3. There is some \( P_k \in \text{Selected}_{(Z, \text{iter})} \cup \text{Selected}_{(Z', \text{iter})} \) such that either \((P_j, P_k)\) or \((P_k, P_j)\) constitutes a conflicting-pair of parties.

**Proof.** Since the instance of \( \Pi_{\text{MultCl}} \) is not successful and \( Z, Z' \) constitute conflicting-sets, it follows that \( c_{(Z', \text{iter})} \neq c_{(Z, \text{iter})} \). Assume for the sake of contradiction that none of the conditions in the claim is true. Then, we can infer the following:

\[
c_{(Z, \text{iter})} = \sum_{P_j \in \text{Selected}_{(Z, \text{iter})}} c_{(Z, \text{iter})}^{(j)} = \sum_{P_j \in \text{Selected}_{(Z, \text{iter})}} \sum_{P_k \in \text{Selected}_{(Z', \text{iter})}} d_{(Z', \text{iter})}^{(jk)} = \sum_{P_j \in \text{Selected}_{(Z, \text{iter})}} \sum_{P_k \in \text{Selected}_{(Z', \text{iter})}} e_{(Z', \text{iter})}^{(jk)} = \sum_{P_k \in \text{Selected}_{(Z', \text{iter})}} \sum_{P_j \in \text{Selected}_{(Z, \text{iter})}} e_{(Z, \text{iter})}^{(jk)} = \sum_{P_k \in \text{Selected}_{(Z', \text{iter})}} c_{(Z', \text{iter})}^{(k)} = c_{(Z', \text{iter})},
\]

where the first equation follows from the definition of \( c_{(Z, \text{iter})} \), the second equation holds because, as per our assumption, \( c_{(Z, \text{iter})}^{(j)} - \sum_{P_k \in \text{Selected}_{(Z', \text{iter})}} d_{(Z', \text{iter})}^{(jk)} = 0 \) for every \( P_j \in \text{Selected}_{(Z, \text{iter})} \), the third equation holds because,
as per our assumption, there is no conflicting-pair of parties, the fifth equation holds because as per our assumption \(c_j(Z',\text{iter}) - \sum_{P_k \in \text{Selected}(Z',\text{iter})} e_{(k)}(Z',\text{iter}) = 0\) for every \(P_k \in \text{Selected}(Z',\text{iter})\) and the last equation follows from the definition of \(c_j(Z',\text{iter})\). However, \(c(Z,\text{iter}) = c(Z',\text{iter})\) is a contradiction. \(\square\)

We next define a characteristic function with respect to the partitions of the summands-sum shared by the summand-sharing parties, to “characterize” instances of \(\Pi_{\text{MultCI}}\) which are not successful. Looking ahead, this will be helpful to upper bound the number of failed \(\Pi_{\text{MultCI}}\) instances in the protocol \(\Pi_{\text{Mult}}\).

**Definition B.20 (Characteristic Function).** Let \(\Pi_{\text{MultCI}}(P, Z, S, [a], [b], \text{iter})\) be an instance of \(\Pi_{\text{MultCI}}\) which is not successful and let \(Z, Z'\) be the corresponding conflicting-sets for the instance. Then the characteristic function \(f_{\text{char}}\) for this instance is defined as follows.

- If there is some \(P_j \in \text{Selected}(Z,\text{iter})\) such that \(c_j(Z,\text{iter}) - \sum_{P_k \in \text{Selected}(Z',\text{iter})} d_{(k)}(Z,\text{iter}) \neq 0\), then \(f_{\text{char}}(\text{iter}) \stackrel{\text{def}}{=} (P_j, P_k)\), where \(P_k\) is the smallest-indexed party from \(P \setminus \{P_j\}\).

- Else, if there is some \(P_j \in \text{Selected}(Z,\text{iter})\) such that \(c_j(Z,\text{iter}) - \sum_{P_k \in \text{Selected}(Z',\text{iter})} e_{(k)}(Z',\text{iter}) \neq 0\), then \(f_{\text{char}}(\text{iter}) \stackrel{\text{def}}{=} (P_k, P_j)\), where \(P_k\) is the smallest-indexed party from \(P \setminus \{P_j\}\).

- Else, \(f_{\text{char}}(\text{iter}) \stackrel{\text{def}}{=} (P_j, P_k)\), where \(P_j, P_k\) is a conflicting-pair of parties, corresponding to the \(\Pi_{\text{MultCI}}\) instance.

Before we proceed, we would like to stress that if \(f_{\text{char}}\) is defined either with respect to the first or the second condition, then party \(P_k\) in the pair \((P_j, P_k)\) serves as a “dummy” party. This is just for notational convenience to ensure uniformity so that \(f_{\text{char}}\) is always a pair of parties irrespective of whether it is defined with respect to the first, second or third condition.

From the definition, it is easy to see that if \(f_{\text{char}}(\text{iter}) = (P_j, P_k)\), then at least one party among \(P_j, P_k\) is maliciously-corrupt. We next claim that the characteristic function is well defined.

**Claim B.21.** Let \(\Pi_{\text{MultCI}}(P, Z, S, [a], [b], \text{iter})\) be an instance of \(\Pi_{\text{MultCI}}\) which is not successful and let \(Z, Z'\) be the corresponding conflicting-sets for the instance. Then \(f_{\text{char}}(\text{iter})\) is well defined.

**Proof.** Proof follows from Claim B.19. \(\square\)

We next prove an important property by showing that if \(f_{\text{char}}(\text{iter}) = (P_j, P_k)\) for some instance of \(\Pi_{\text{MultCI}}\) which is not successful, and if both \(P_j\) and \(P_k\) have been removed from the waiting-list of some honest party for that instance, then the corrupt party(ies) among \(P_j, P_k\) will eventually be discarded by all honest parties.

**Claim B.22.** Let \(\Pi_{\text{MultCI}}(P, Z, S, [a], [b], \text{iter})\) be an instance of \(\Pi_{\text{MultCI}}\) which is not successful and let \(Z, Z'\) be the corresponding conflicting-sets for the instance. Moreover, let \(f_{\text{char}}(\text{iter}) = (P_j, P_k)\). If both \(P_j\) and \(P_k\) are removed from the set \(W_{\text{iter}}^{(h)}\) of any honest party \(P_h\), then the corrupt party(ies) among \(P_j, P_k\) will eventually be included in the set \(\mathcal{L}(\text{iter})\) of every honest \(P_i\).

**Proof.** Let \(f_{\text{char}}(\text{iter}) = (P_j, P_k)\), where without loss of generality, \(P_j \in \text{Selected}(Z,\text{iter})\) and \(P_k \in \text{Selected}(Z',\text{iter})\). From the definition of characteristic function (Def B.20), one of the following holds for \(P_j\) and \(P_k\):

- \((P_j, P_k)\) constitutes a conflicting-pair: In this case, \(d_{(j)}(Z,\text{iter}) \neq e_{(k)}(Z',\text{iter})\). Since the honest \(P_h\) has removed both \(P_j\) and \(P_k\) from \(W_{\text{iter}}^{(h)}\) from the protocol steps, the outputs (share, \(\text{sid}_{j,k,\text{iter},Z}, P_j, \{d_{(j)}(Z,\text{iter})\}_h P_h \in S_q\)) and (share, \(\text{sid}_{j,k,\text{iter},Z'}, P_k, \{e_{(k)}(Z',\text{iter})\}_h P_h \in S_q\)) have been obtained by \(P_h\) from \(\mathcal{F}_{\text{VSS}}\) with \(\text{sid}_{j,k,\text{iter},Z}\) and

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*Footnotes:

1. If there are multiple parties \(P_j\) satisfying this condition, then we consider the \(P_j\) with the smallest index.
2. If there are multiple parties \(P_j\) satisfying this condition, then we consider the \(P_j\) with the smallest index.
3. If there are multiple conflicting-pairs, then we consider the one having parties with the smallest indices.
sid_{k,j,iter,Z'} respectively. Consequently, each honest party will eventually receive its respective share corresponding to \(d_{(Z,iter)}^{(j,k)}\) and \(e_{(Z',iter)}^{(j,k)}\) from the corresponding \(F_{\text{VSS}}\) instances. Hence, each honest party will be able to locally compute its share of \(d_{(Z,iter)}^{(j,k)} - e_{(Z',iter)}^{(j,k)}\) and participate in the instance of \(\Pi_{\text{PerRec}}\) to reconstruct the difference. Since \(S\) satisfies the \(Q^{(3)}(S, Z)\) condition, all honest parties will eventually reconstruct \(d_{(Z,iter)}^{(j,k)} - e_{(Z',iter)}^{(j,k)}\) and find that the difference is not 0. Consequently, the honest parties will participate in appropriate instances of \(\Pi_{\text{PerRec}}\) to reconstruct the values \(d_{(Z,iter)}^{(j,k)}\), \(e_{(Z',iter)}^{(j,k)}\), and instances of \(\Pi_{\text{PerRecShare}}\) to reconstruct the shares \([a]_p\) and \([b]_q\), such that \((p, q) \in \text{Summands}_{(Z,iter)}^{(j)} \cap \text{Summands}_{(Z',iter)}^{(k)}\). Now, either \(d_{(Z,iter)}^{(j,k)}\) or \(e_{(Z',iter)}^{(j,k)}\) will not be equal to \(\sum_{(p,q) \in \text{Summands}_{(Z,iter)}^{(j)} \cap \text{Summands}_{(Z',iter)}^{(k)}} [a]_p [b]_q\), as otherwise \(d_{(Z,iter)}^{(j,k)} = e_{(Z',iter)}^{(j,k)}\) will hold, which is a contradiction. Consequently, every honest party \(P_i\) will eventually add the corrupt party(ies) among \(P_j, P_k\) to \(\mathcal{LD}_{\text{iter}}^{(i)}\).

- **The condition** \(c_{(Z,iter)}^{(j)} - \sum_{P_h \in \text{Selected}_{(Z',iter)}} d_{(Z,iter)}^{(j,k)} \neq 0\) holds: Since the honest \(P_h\) has removed \(P_j\) from \(\mathcal{W}_{\text{iter}}^{(h)}\), then from the protocol steps, corresponding to every \(P_k \in \text{Selected}_{(Z',iter)}\), party \(P_h\) has received the output \((\text{share, sid}_{j,k,iter,Z', P_j, \{d_{(Z,iter)}^{(j,k)}\} P_h \in S_q})\) from \(F_{\text{VSS}}\) with \(\text{sid}_{j,k,iter,Z'}\). Consequently, for every \(P_k \in \text{Selected}_{(Z',iter)}\), all honest parties eventually receive their respective shares corresponding to \(d_{(Z,iter)}^{(j,k)}\) from the respective \(F_{\text{VSS}}\) instances. In the protocol, all honest parties participate in an instance of \(\Pi_{\text{PerRec}}\) with their respective shares corresponding to \(d_{(Z,iter)}^{(jk)}\) from \(F_{\text{VSS}}\) instances. Consequently, each honest party will eventually receive its respective share corresponding to \(d_{(Z,iter)}^{(j,k)}\) and \(e_{(Z',iter)}^{(j,k)}\) from the corresponding \(F_{\text{VSS}}\) instances. Hence, each honest party will be able to locally compute its share of \(d_{(Z,iter)}^{(j,k)} - e_{(Z',iter)}^{(j,k)}\) and participate in the instance of \(\Pi_{\text{PerRec}}\) to reconstruct the difference. Since \(S\) satisfies the \(Q^{(3)}(S, Z)\) condition, all honest parties will eventually reconstruct \(d_{(Z,iter)}^{(j,k)} - e_{(Z',iter)}^{(j,k)}\) and find that the difference is not 0. Consequently, the honest parties will participate in appropriate instances of \(\Pi_{\text{PerRec}}\) to reconstruct the values \(d_{(Z,iter)}^{(j,k)}\), \(e_{(Z',iter)}^{(j,k)}\), and instances of \(\Pi_{\text{PerRecShare}}\) to reconstruct the shares \([a]_p\) and \([b]_q\), such that \((p, q) \in \text{Summands}_{(Z,iter)}^{(j)} \cap \text{Summands}_{(Z',iter)}^{(k)}\). Now, either \(d_{(Z,iter)}^{(j,k)}\) or \(e_{(Z',iter)}^{(j,k)}\) will not be equal to \(\sum_{(p,q) \in \text{Summands}_{(Z,iter)}^{(j)} \cap \text{Summands}_{(Z',iter)}^{(k)}} [a]_p [b]_q\), as otherwise \(d_{(Z,iter)}^{(j,k)} = e_{(Z',iter)}^{(j,k)}\) will hold, which is a contradiction. Consequently, every honest party \(P_i\) will eventually add the corrupt party(ies) among \(P_j, P_k\) to \(\mathcal{LD}_{\text{iter}}^{(i)}\).

- **The condition** \(c_{(Z',iter)}^{(k)} - \sum_{P_j \in \text{Selected}_{(Z,iter)}} e_{(Z',iter)}^{(j,k)} \neq 0\) holds: This case is symmetric to the previous case and using a similar argument as above, we can conclude that each honest \(P_i\) will eventually include the corrupt \(P_k\) to \(\mathcal{LD}_{\text{iter}}^{(i)}\).

We next claim that the adversary does not learn anything additional about \(a\) and \(b\) in the protocol.

**Claim B.23.** In \(\Pi_{\text{MultCl}}\), \(\text{Adv}\) does not learn any additional information about \(a\) and \(b\).

**Proof.** From Claim **B.13** \(\text{Adv}\) does not learn any additional information about \(a\) and \(b\) from the instances of \(\Pi_{\text{OptMult}}\) executed in \(\Pi_{\text{MultCl}}\). Corresponding to every \(Z \in Z\), \(\text{Adv}\) learns the difference \(c_{(Z,iter)} - c_{(Z',iter)}\) which are all 0, unless the adversary cheats. In case of cheating, the reconstructed differences \(c_{(Z,iter)} - c_{(Z',iter)}\) depend completely upon the inputs of the adversary and hence learning these differences does not add anything additional about \(a\) and \(b\) to the adversary’s view. Next, corresponding to every honest \(P_j \in \text{Selected}_{(Z,iter)} \cup \text{Selected}_{(Z',iter)}\), the shares corresponding to \(d_{(Z,iter)}^{(j,k)}\) or \(e_{(Z',iter)}^{(j,k)}\) learnt by \(\text{Adv}\) will be distributed uniformly, since \(S\) is \(Z\)-private and hence, these shares do not add anything additional about \(a\) and \(b\) to the adversary’s view. Moreover, for every honest \(P_j \in \text{Selected}_{(Z,iter)} \cup \text{Selected}_{(Z',iter)}\), \(\text{Adv}\) will know beforehand that the differences \(c_{(Z,iter)}^{(j)} - \sum_{P_h \in \text{Selected}_{(Z',iter)}} d_{(Z,iter)}^{(j,k)}\) as well as \(c_{(Z',iter)}^{(j)} - \sum_{P_h \in \text{Selected}_{(Z,iter)}} e_{(Z',iter)}^{(j,k)}\) will be 0 and hence, learning these differences does not add anything additional about \(a\) and \(b\) to adversary’s view. On the other hand, for every corrupt \(P_j \in \text{Selected}_{(Z,iter)} \cup \text{Selected}_{(Z',iter)}\), the above differences completely depend upon the adversary’s inputs and hence, reveal no additional information. Finally, if for any ordered pair of parties \((P_j, P_k)\), the condition
$d_{(Z,\text{iter})}^{(jk)} \neq c_{(Z',\text{iter})}^{(kj)}$ holds, then at least one among $P_j$ and $P_k$ is corrupt. Consequently, the shares $[a]_p$ and $[b]_q$ where $(p, q) \in \text{Summands}_{(Z,\text{iter})}^{(j)} \cap \text{Summands}_{(Z',\text{iter})}^{(k)}$ reconstructed in this case are already known to the adversary, and do not add anything new to the view of the adversary regarding $a$ and $b$. $\square$

Claim B.24. Protocol $\Pi_{\text{MultCl}}$ needs $\mathcal{O}(|Z| \cdot n^2)$ calls to $\mathcal{F}_{\text{VSS}}$ and $\mathcal{F}_{\text{ABA}}$ and incurs an additional communication of $\mathcal{O}(|Z|^2 \cdot n^2 \log |\mathbb{F}| + |Z| \cdot n^4 \log |\mathbb{F}|)$ bits.

Proof. In the protocol, corresponding to each $Z \in \mathcal{Z}$, an instance of $\Pi_{\text{OptMult}}$ is executed. From Lemma 3.1, this will require $\mathcal{O}(|Z| \cdot n^2)$ calls to $\mathcal{F}_{\text{VSS}}$ and $\mathcal{F}_{\text{ABA}}$. There are $\mathcal{O}(|Z|)$ instances of $\Pi_{\text{PerRec}}$ to reconstruct $\mathcal{O}(|Z|)$ difference values for checking whether the instance is successful or not, incurring a communication of $\mathcal{O}(|Z|^2 \cdot n^2 \log |\mathbb{F}|)$ bits. If the instance is not successful, then there are $\mathcal{O}(n^2)$ calls to $\mathcal{F}_{\text{VSS}}$ to share various summand-sum partitions. To check whether the correct partitions are shared, $\mathcal{O}(n^2)$ values need to be publicly reconstructed through these many instances of $\Pi_{\text{PerRec}}$, which incurs a communication of $\mathcal{O}(|Z| \cdot n^4 \log |\mathbb{F}|)$ bits. $\square$

The proof of Lemma 3.2 now follows from Claims B.16, B.24.

Lemma 3.2. Let $Z$ satisfy the $\mathcal{Q}^{(4)}(\mathcal{P}, \mathcal{Z})$ condition and let $S = (S_1, \ldots, S_h) = \{P \setminus Z | Z \in \mathcal{Z}\}$. Moreover, let all honest parties participate in the instance $\Pi_{\text{MultCl}}(\mathcal{P}, \mathcal{Z}, S, [a], [b], \text{iter})$. Then the following hold.

– The instance will eventually be deemed to succeed or fail by the honest parties, where for a successful instance, the parties output a sharing of $ab$. 

– If the instance is not successful, then the honest parties will agree on a pair $Z, Z' \in \mathcal{Z}$ such that $c_{(Z,\text{iter})} - c_{(Z',\text{iter})} \neq 0$. Moreover, all honest parties present in the $\mathcal{V}_{\text{iter}}^{(i)}$ set of any honest party $P_i$ will eventually be removed and no honest party is ever included in the $\mathcal{LD}_{\text{iter}}^{(i)}$ set of any honest party $P_i$. Furthermore, there will be a pair of parties $P_j, P_k$ from $\text{Selected}_{(Z,\text{iter})} \cup \text{Selected}_{(Z',\text{iter})}$, with at least one of them being maliciously-corrupt, such that if both $P_j$ and $P_k$ are removed from the set $\mathcal{V}_{\text{iter}}^{(h)}$ of any honest party $P_h$, then eventually the corrupt party(ies) among $P_j, P_k$ will be included in the set $\mathcal{LD}_{\text{iter}}^{(i)}$ of every honest party $P_i$.

– In the protocol, Adv does not learn any additional information $a$ and $b$.

– The protocol needs $\mathcal{O}(|Z| \cdot n^2)$ calls to $\mathcal{F}_{\text{VSS}}$ and $\mathcal{F}_{\text{ABA}}$ and incurs an additional communication of $\mathcal{O}(|Z|^2 \cdot n^2 \log |\mathbb{F}| + |Z| \cdot n^4 \log |\mathbb{F}|)$ bits.

$\Pi_{\text{MultCl}}$ for Inputs $\{(a^{(f)}_\ell), (b^{(f)}_\ell)\}_{\ell=1,\ldots,M}$: The modifications to the protocol $\Pi_{\text{MultCl}}$ for handling $M$ pairs of secret-shared inputs is simple. The parties now run instances of $\Pi_{\text{OptMult}}$ handling $M$ pairs of inputs. Corresponding to every pair $(Z, Z')$, the parties reconstruct $M$ differences. If any of these differences is non-zero, the parties focus on the smallest-indexed $([a^{(f)}], [b^{(f)}])$ such that $c^{(f)}_{(Z,\text{curr})} - c^{(f)}_{(Z',\text{curr})} \neq 0$. The parties then proceed to the cheater identification phase with respect to $(Z, Z')$ and $(a^{(f)}), (b^{(f)})$. The protocol will require $\mathcal{O}(M \cdot |Z| \cdot n^2)$ calls to $\mathcal{F}_{\text{VSS}}, \mathcal{O}(|Z| \cdot n^2)$ calls to $\mathcal{F}_{\text{ABA}}$ and additionally communicates $\mathcal{O}(M \cdot |Z|^2 \cdot n^2 \log |\mathbb{F}| + |Z| \cdot n^4 \log |\mathbb{F}|)$ bits.

B.5 Properties of the Multiplication Protocol $\Pi_{\text{Mult}}$

In this section, we formally prove the properties of the protocol $\Pi_{\text{Mult}}$ (see Fig 5 for the formal description of the protocol). While proving these properties, we will assume that $Z$ satisfies the $\mathcal{Q}^{(4)}(\mathcal{P}, \mathcal{Z})$ condition. This further implies that the sharing specification $S = (S_1, \ldots, S_h) \overset{\text{def}}{=} \{P \setminus Z | Z \in \mathcal{Z}\}$ satisfies the $\mathcal{Q}^{(3)}(\mathcal{S}, \mathcal{Z})$ condition.

We begin with the definition of a successful iteration in protocol $\Pi_{\text{Mult}}$.

Definition B.25 (Successful Iteration). In protocol $\Pi_{\text{Mult}}$, an iteration $\text{iter}$ is called successful, if every honest $P_i$ sets $\text{flag}_{\text{iter}}^{(i)} = 0$ during the corresponding instance $\Pi_{\text{MultCl}}(\mathcal{P}, \mathcal{Z}, S, [a], [b], \text{iter})$ of $\Pi_{\text{MultCl}}$. 

37
We next claim that during each iteration of the protocol $\Pi_{\text{Mult}}$, the honest parties will know whether the iteration is successful or not.

**Claim B.26.** For any iter, if all honest parties participate in iteration number $\text{iter}$ of the protocol $\Pi_{\text{Mult}}$ and if no honest party is ever included in the set $\mathcal{GD}$, then all honest parties will eventually agree on whether the iteration is successful or not.

**Proof.** We prove the claim through induction on $\text{iter}$. The statement is obviously true for $\text{iter} = 1$, since during the instance $\Pi_{\text{MultCI}}(P, Z, S, [a], [b], 1)$, all honest parties $P_i$ will eventually set $\text{flag}^{(i)}_1$ to a common value from $\{0, 1\}$ (follows from Lemma 3.2). Assume that the statement is true for $\text{iter} = r$. Now consider $\text{iter} = r + 1$ and let all honest parties participate in iteration number $r + 1$ by invoking the instance $\Pi_{\text{MultCI}}(P, Z, S, [a], [b], r + 1)$. From the protocol steps, since the honest parties participate in iteration number $r + 1$, it implies that none of the previous $r$ iterations were successful. From Lemma 3.2, all honest parties from the sets $\mathcal{W}_r^{(i)}, \ldots, \mathcal{W}_r^{(i)}$ will eventually be removed for every honest $P_i$. Moreover, no honest party will ever be included in the sets $\mathcal{LD}_r^{(i)}$, $\ldots, \mathcal{LD}_r^{(i)}$. Furthermore, as per the lemma condition, no honest party is ever included in the set $\mathcal{GD}$. It now follows from Claim B.16 and Lemma 3.2 that during the instance $\Pi_{\text{MultCI}}(P, Z, S, [a], [b], r + 1)$, all honest parties $P_i$ will eventually set $\text{flag}^{(i)}_{r + 1}$ to a common value from $\{0, 1\}$ and learn whether the iteration is successful or not. □

We next claim that if any iteration of $\Pi_{\text{Mult}}$ is successful, then honest parties output $[ab]$ in that iteration.

**Claim B.27.** If the iteration number $\text{iter}$ in $\Pi_{\text{Mult}}$ is successful, then honest parties output $[ab]$ during iteration number $\text{iter}$.

**Proof.** Let iteration number $\text{iter}$ in $\Pi_{\text{Mult}}$ be successful. This implies that every honest $P_i$ sets $\text{flag}^{(i)}_{\text{iter}} = 0$ during the corresponding instance $\Pi_{\text{MultCI}}(P, Z, S, [a], [b], \text{iter})$ of $\Pi_{\text{MultCI}}$ and hence this instance of $\Pi_{\text{MultCI}}$ is successful. The proof now follows from Lemma 3.2. □

We next prove that after every $tn + 1$ consecutive unsuccessful iterations of $\Pi_{\text{Mult}}$, a new corrupt party is globally discarded.

**Claim B.28.** Let $t = \max\{|Z| : Z \in \mathcal{Z}\}$. In $\Pi_{\text{Mult}}$, for every $k \geq 1$, if none of the iterations $(k - 1)(tn + 1) + 1, \ldots, k(tn + 1)$ is successful, then eventually, a new corrupt party is included in the set $\mathcal{GD}$.

**Proof.** Let $Z^* \in \mathcal{Z}$ be the set of corrupt parties during the execution of $\Pi_{\text{Mult}}$. We prove the claim through strong induction over $k$.

**Base case:** $k = 1$. We first note that from the protocol steps, the condition $\mathcal{GD} = \emptyset$ holds for each of the iterations $1, \ldots, tn + 1$, during the corresponding instance of $\Pi_{\text{MultCI}}$ in these iterations. Consequently, from Claim B.26, for the iterations $1, \ldots, tn + 1$, the honest parties agree on whether the iteration is successful or not. Let none of the iterations $1, \ldots, tn + 1$ be successful. This implies that for $\text{iter} = 1, \ldots, tn + 1$, none of the instances $\Pi_{\text{MultCI}}(P, Z, S, [a], [b], \text{iter})$ of $\Pi_{\text{MultCI}}$ is successful. This further implies that for every $\text{iter} \in \{1, \ldots, tn + 1\}$, there exists a well-defined unordered pair of parties $(P_j, P_k)$, such that $f_{\text{char}}(\text{iter}) = (P_j, P_k)$, with at least one among $P_j, P_k$ being maliciously-corr upt (follows from Claim B.21). Let $C$ denote the set of all pairs of “characteristic parties” for the first $tn + 1$ instances of $\Pi_{\text{MultCI}}$. That is, \[ C \overset{\text{def}}{=} \{(P_j, P_k) : f_{\text{char}}(\text{iter}) = (P_j, P_k) \text{ and } \text{iter} \in \{1, \ldots, tn + 1\}\}. \]

It then follows that $|C| \leq tn$. This is because $|Z^*| \leq t$, implying that can be at most $tn$ distinct (unordered) pairs of parties, where at least one of the parties in the pair is corrupt. Since the cardinality of $C$ is smaller than the number of failed $\Pi_{\text{MultCI}}$ instances, from the pigeonhole principle, we can conclude that there exist at least two iterations $r, r' \in \{1, \ldots, tn + 1\}$ where $r < r'$, such that $f_{\text{char}}(r) = f_{\text{char}}(r') = (P_j, P_k)$. □
Now, let us focus on the failed instances $\Pi_{\text{MultCI}}(P, Z, S, [a], [b], r)$ and $\Pi_{\text{MultCI}}(P, Z, S, [a], [b], r')$, corresponding to iteration number $r$ and $r'$ respectively in $\Pi_{\text{mult}}$. Let $W_r^{(i)}$ and $LD_r^{(i)}$ be the dynamic sets maintained by every party $P_i$ during the instance $\Pi_{\text{MultCI}}(P, Z, S, [a], [b], r)$. Note that at the time of initializing $W_r^{(i)}$, both $P_j$ as well as $P_k$ will be present in $W_r^{(i)}$ (this follows from the protocol steps of $\Pi_{\text{MultCI}}$). Let $Z, Z' \subseteq Z$ be the conflicting sets for the failed instance $\Pi_{\text{MultCI}}(P, Z, S, [a], [b], r')$. From the definition of characteristic function $f_{\text{char}}$, it follows that $P_j, P_k \in \text{Selected}_{(Z, r)} \cup \text{Selected}_{(Z', r')}$. Hence, $P_j$ (resp. $P_k$) is selected as a summand-sharing party and hence $P_j, P_k \notin \text{Selected}_{(Z, r)} \cup \text{Selected}_{(Z', r')}$. This further implies that there exists at least one honest party, say $P_h$, such that both $P_j$ as well as $P_k$ are removed by $P_h$ from the set $W_r^{(h)}$. This is because if both $P_j$ as well as $P_k$ are still present in the $W_r^{(i)}$ set of all honest parties during the instances $\Pi_{\text{OptMult}}(P, Z, S, [a], [b], Z', r')$ and $\Pi_{\text{OptMult}}(P, Z, S, [a], [b], Z', r')$, then neither $P_j$ nor $P_k$ will be selected as a summand-sharing party and hence $P_j, P_k \notin \text{Selected}_{(Z, r') \cup \text{Selected}_{(Z', r')}}$ (follows from Claim B.14), which is a contradiction. Now, if both $P_j$ and $P_k$ are removed from $W_r^{(h)}$, then from Claim B.22, the corrupt party(ies) among $P_j, P_k$ will be eventually included in the $LD_r^{(i)}$ set of every honest $P_i$. For simplicity and without loss of generality, let $P_j$ be the corrupt party among $P_j, P_k$.

In the protocol $\Pi_{\text{Mult}}$, once the parties find that iteration number $tn + 1$ has failed, they run an instance of ACS to identify a cheating party across the first $tn + 1$ failed instances, where the parties vote for candidate cheating parties based on the contents of their local $LD$ sets. To complete the proof for the base case, we need to show that ACS will eventually output a common corrupt party for all the honest parties. The proof for this is similar to that of Claim B.10. Namely, as argued above, the corrupt party $P_j$ from the pair $(P_j, P_k)$ above will be eventually included in the $LD_r^{(i)}$ set of every honest $P_i$. We first show that there will be at least one instance of $F_{\text{ABA}}$, corresponding to which all honest parties eventually receive the output 1. For this, we consider two possible cases:

- **At least one honest party participates with input 0 in the $F_{\text{ABA}}$ instance corresponding to $P_j$:** Let $P_l$ be an honest party, who sends $(\text{vote}, \text{sid}_{l,tn+1,1}, 0)$ to $F_{\text{ABA}}$ with $\text{sid}_{j,tn+1,1}$. Then from the steps of $\Pi_{\text{Mult}}$, it follows that there exists some $P_i \in P$, such that $P_i$ has received $(\text{decide}, \text{sid}_{i,tn+1,1}, 1)$ as the output from $F_{\text{ABA}}$ with $\text{sid}_{i,tn+1,1}$. Hence, every honest party will eventually receive the output $(\text{decide}, \text{sid}_{i,tn+1,1}, 1)$ from $F_{\text{ABA}}$ with $\text{sid}_{i,tn+1,1}$.

- **No honest party participates with input 0 in the $F_{\text{ABA}}$ instance corresponding to $P_j$:** In this case, every honest party will eventually send $(\text{vote}, \text{sid}_{j,tn+1,1}, 1)$ to $F_{\text{ABA}}$ with $\text{sid}_{j,tn+1,1}$. This is because $P_j$ will be eventually included in the $LD_r^{(i)}$ set of every honest $P_i$. Consequently, every honest party eventually receives the output $(\text{decide}, \text{sid}_{j,tn+1,1}, 1)$ from $F_{\text{ABA}}$.

Now based on the above argument, we can further infer that all honest parties will eventually participate with some input in all the $n$ instances of $F_{\text{ABA}}$ invoked after the first $tn + 1$ failed iterations and hence, all the $n$ instances of $F_{\text{ABA}}$ will eventually produce an output. Let $P_m$ be the smallest indexed party such that $F_{\text{ABA}}$ with $\text{sid}_{m,tn+1,1}$ has returned the output $(\text{decide}, \text{sid}_{m,tn+1,1}, 1)$. Hence, all honest parties eventually include $P_m$ to $GD$.

Finally, it is easy to see that $P_m \in Z^*$. This is because if $P_m \notin Z^*$, then $P_m$ is honest. From Claim B.17 it follows that $P_m$ will not be included in the $LD_r^{(i)}$ of any honest $P_i$ for any $i \in \{1, \ldots, tn + 1\}$. Consequently, no honest $P_i$ will ever send $(\text{vote}, \text{sid}_{m,tn+1,1}, 1)$ to $F_{\text{ABA}}$ with $\text{sid}_{m,tn+1,1}$. Hence, $F_{\text{ABA}}$ with $\text{sid}_{m,tn+1,1}$ will never return the output $(\text{decide}, \text{sid}_{m,tn+1,1}, 1)$, which is a contradiction. This completes the proof for the base case.

**Inductive Step:** Let the statement be true for $k = 1, \ldots, k'$. Now consider the case when $k = k' + 1$. Let $GD_1, \ldots, GD_{k'}$ be the set of discarded cheating parties after the iteration number $tn + 1, \ldots, k'(tn + 1)$ respectively. From the inductive hypothesis, $GD_1 \subseteq GD_2 \subseteq \ldots \subseteq GD_{k'}$ and no honest party is present in $GD_{k'}$. Consequently, from the protocol steps and from Claim B.26, for the iterations $k'(tn + 1) + 1, \ldots, (k' + 1)(tn + 1)$, the honest parties agree on whether the iteration is successful or not. Let none of the iterations $k'(tn + 1) + 1, \ldots, (k' + 1)(tn + 1)$
1, ..., (k' + 1)(tn + 1) be successful. This implies that for iter = k'(tn + 1) + 1, ..., (k' + 1)(tn + 1), none of the instances Π_{MultCI}(P, Z, S, [a], [b], iter) of Π_{MultCI} is successful. In the protocol, once the parties find that the iteration number (k' + 1)(tn + 1) is not successful, they proceed to select a common cheating party through ACS.

Let LD{i} iter, W{i} iter be the dynamic sets maintained by each party P_i across the iterations 1, ..., (k' + 1)(tn + 1).

We first note that none of the parties from GD{k'} will be selected as a summand-sharing party in any of the underlying Π_{OptMult}(P, Z, S, [a], [b], Z, iter) instances, for any iter ∈ {k'(tn + 1) + 1, ..., (k' + 1)(tn + 1)} and any Z ∈ {Z} (this follows from Claim B.14). We also note that there will be at least one party from Z, which is not present in GD{k'}; i.e. GD{k'} ⊂ Z. If not, then only honest parties will be selected as summand-sharing parties in all the underlying instances of Π_{OptMult} during the iteration number k'(tn + 1) + 1 and hence, the iteration number k'(tn + 1) + 1 in Π_{Mult} would be successful, which is a contradiction. Since the iteration number k'(tn + 1) + 1, ..., (k' + 1)(tn + 1) constitutes tn + 1 failed iterations, by applying the same pigeonhole-principle based argument as applied for the base case, we can infer that there exists a pair of iterations r, r′ ∈ {k'(tn + 1) + 1, ..., (k' + 1)(tn + 1)} where r < r′, such that

\[ f_{\text{char}}(r) = f_{\text{char}}(r') = (P_j, P_k), \]

with at least one among P_j and P_k being maliciously-corrump. Moreover, the corrupt party(ies) among P_j, P_k will be from the set Z \GD{k'}, since the parties from GD{k'} will not be selected as a summand-sharing party during the iteration number r and r′. Next, following the same argument as used for the base case, we can infer that the corrupt party(ies) among P_j and P_k will be eventually included in the LD_r iter set of every honest P_i. This will further imply all the n instances of F_{ABA} with sid_{1,(k'+1)(tn+1),(k'+1)}, ..., sid_{nk,(k'+1)(tn+1),(k'+1)} will eventually return an output for all the honest parties, such that at least one of the F_{ABA} instances with sid_{1,(k'+1)(tn+1),(k'+1)} corresponding to the party P_i will return an output (decide, sid_{1,(k'+1)(tn+1),(k'+1)}). Let \( P_m \) be the smallest indexed party corresponding to which the F_{ABA} instance with sid_{1,(k'+1)(tn+1),(k'+1)} returns the output (decide, sid_{m,(k'+1)(tn+1),(k'+1)}). Hence the honest parties will update GD to GD_k' \cup \{ P_m \}. To complete the proof, we need to show that P_m \not\in GD{k'} and P_m \in Z. The former follows from the fact that if P_m \not\in GD{k'}, then it implies that then no honest party ever sends (vote, sid_{m,(k'+1)(tn+1),(k'+1)}) to F_{ABA} with sid_{m,(k'+1)(tn+1),(k'+1)} and consequently, F_{ABA} with sid_{m,(k'+1)(tn+1),(k'+1)} will never return the output (decide, sid_{m,(k'+1)(tn+1),(k'+1)}). On the other hand, P_m \in Z follows using a similar argument as used for the base case.

An immediate corollary of Claim B.28 is that there can be at most t(tn + 1) consecutive failed iterations in the protocol Π_{Mult}.

**Corollary B.29.** In protocol Π_{Mult}, there can be at most t(tn + 1) consecutive failed iterations, where \( t \) def max\{|Z| : Z \in Z\}.

We next claim that it will take at most (tn + 1) + 1 iterations in the protocol Π_{Mult} to guarantee that there is at least one successful iteration.

**Claim B.30.** In protocol Π_{Mult}, it will take at most (tn + 1) + 1 iterations to ensure that one of these iterations is successful.

**Proof.** Follows easily from Claim B.26 and Corollary B.29.

We next claim that the adversary does not learn anything additional about a and b in the protocol.

**Claim B.31.** In protocol Π_{Mult}, Adv does not learn anything additional about a and b.

**Proof.** Follows directly from the fact that in every iteration of Π_{Mult}, Adv does not learn anything additional about a and b, which in turn follows from Claim B.23.

Lemma 3.3 now follows from Claim B.30 Claim B.27 and Claim B.31 where the communication complexity follows from the communication complexity of Π_{MultCI} and the fact that there are t(tn + 1) + 1 = \( \mathcal{O}(n^3) \) instances of Π_{MultCI} executed inside the protocol Π_{Mult}.

**Lemma 3.3.** Let Z satisfy the \( \mathcal{Q}^4(P, Z) \) condition and let \( S = (S_1, ..., S_h) = \{ P \setminus Z | Z \in Z \} \). Then
\( \Pi_{\text{Mult}} \) will take at most \( t(n+1) \) iterations and all honest parties eventually output a secret-sharing of \([ab]\), where \( t = \max\{|Z| : Z \in \mathcal{Z}\} \). In the protocol, adversary does not learn anything additional about \( a \) and \( b \). The protocol makes \( O(|Z| \cdot n^5) \) calls to \( \mathcal{F}_{\text{VSS}} \) and \( \mathcal{F}_{\text{ABA}} \) and additionally incurs a communication of \( O(|Z|^2 \cdot n^5 \log |F| + |Z| \cdot n^7 \log |F|) \) bits.

**B.6 Perfectly-Secure Pre-Processing Phase Protocol \( \Pi_{\text{PerTriples}} \) and Its Properties**

Protocol \( \Pi_{\text{PerTriples}} \) for securely realizing \( \mathcal{F}_{\text{Triples}} \) with \( M = 1 \) in the \( (\mathcal{F}_{\text{VSS}}, \mathcal{F}_{\text{ABA}}) \)-hybrid model is presented in Fig. 16.

**Protocol \( \Pi_{\text{PerTriples}}(P, Z, S) \)**

1. **Stage I: Generating a Secret-Sharing of Random Pair of Values.**
   - **Sharing Random Pairs of Values:**
     1. Randomly select \( a^{(i)}, b^{(i)} \in F \) and shares \( (a_1^{(i)}, \ldots, a_h^{(i)}) \) and \( (b_1^{(i)}, \ldots, b_h^{(i)}) \), such that \( a_1^{(i)} + \ldots + a_h^{(i)} = a^{(i)} \) and \( b_1^{(i)} + \ldots + b_h^{(i)} = b^{(i)} \). Call \( \mathcal{F}_{\text{VSS}} \) with \((\text{dealer}, \text{sid}_{1,1}, (a_1^{(i)}, \ldots, a_h^{(i)}))\) and \( \mathcal{F}_{\text{VSS}} \) with \((\text{dealer}, \text{sid}_{1,2}, (b_1^{(i)}, \ldots, b_h^{(i)}))\) for \( \text{sid}_{1,1} \) and \( \text{sid}_{1,2} \), where \( \text{sid}_{1,1} = \text{sid}(i)|1 \) and \( \text{sid}_{1,2} = \text{sid}(i)|2 \).
     2. For \( j = 1, \ldots, n \), keep requesting for an output from \( \mathcal{F}_{\text{VSS}} \) with \( \text{sid}_{j,1} \) and \( \text{sid}_{j,2} \), till an output is received.
   - **Selecting a Common Subset of Parties Through ACS**
     1. If \( (\text{share}, \text{sid}_{j,1}, P_j, \{[a^{(j)}]_q \}_{P_i \in S_q}) \) and \( (\text{share}, \text{sid}_{j,2}, P_j, \{[b^{(j)}]_q \}_{P_i \in S_q}) \) are received from \( \mathcal{F}_{\text{VSS}} \) with \( \text{sid}_{j,1} \) and \( \text{sid}_{j,2} \) respectively, then send \((\text{vote}, \text{sid}_{j,1})\) to \( \mathcal{F}_{\text{ABA}} \), where \( \text{sid}_{j} \overset{\text{def}}{=} \text{sid}(j) \).
     2. For \( j = 1, \ldots, n \), request for output from \( \mathcal{F}_{\text{ABA}} \) with \( \text{sid}_{j} \), till an output is received.
     3. If there exists a subset of parties \( \mathcal{G}_P \) such that \( P \setminus \mathcal{G}_P \in \mathcal{Z} \) and \((\text{decide}, \text{sid}_{j,1})\) is received from \( \mathcal{F}_{\text{ABA}} \) with \( \text{sid}_{j} \) corresponding to every \( P_j \in \mathcal{G}_P \), then send \((\text{vote}, \text{sid}_{j,0})\) to \( \mathcal{F}_{\text{ABA}} \) with \( \text{sid}_{j} \) corresponding to every \( P_j \), for which no message has been sent yet.
     4. Once \((\text{decide}, \text{sid}_{j,1})\) is received from \( \mathcal{F}_{\text{ABA}} \) for \( j = 1, \ldots, n \), set \( CS = \{P_j : v_j = 1\} \).
     5. Let \( a \overset{\text{def}}{=} \sum_{P_j \in CS} a^{(j)} \), \( b \overset{\text{def}}{=} \sum_{P_j \in CS} b^{(j)} \). Locally compute the shares \( \{[a]_q \}_{P_i \in S_q} \) and \( \{[b]_q \}_{P_i \in S_q} \).

   **Stage II: Generating the Product.**
   - Participate in the instance \( \Pi_{\text{Mult}}(P, Z, S, [a], [b]) \) with \( \text{sid} \) and compute \( \{[c]_q \}_{P_i \in S_q} \). Output \( \{[a]_q, [b]_q, [c]_q \}_{P_i \in S_q} \).

Figure 16: A perfectly-secure protocol to securely realize \( \mathcal{F}_{\text{Triples}} \) with \( M = 1 \) in \( (\mathcal{F}_{\text{VSS}}, \mathcal{F}_{\text{ABA}}) \)-hybrid model for session id \( \text{sid} \). The above code is executed by every party \( P_i \).

**Protocol \( \Pi_{\text{PerTriples}} \) for Generating L Multiplication-Triples:** The modifications in \( \Pi_{\text{PerTriples}} \) to generate \( M \) multiplication-triples are straight forward. During the first stage, each party secret-shares \( M \) pairs of values, by calling \( \mathcal{F}_{\text{VSS}} 2M \) number of times. While running ACS, a party votes “positively” for party \( P_j \), only after receiving an output from all the \( 2M \) instances of \( \mathcal{F}_{\text{VSS}} \) corresponding to \( P_j \). During the second stage, the instance of \( \Pi_{\text{Mult}} \) will now take \( M \) pairs of secret-shared inputs.

We now prove the security of the protocol \( \Pi_{\text{PerTriples}} \) in the \( (\mathcal{F}_{\text{VSS}}, \mathcal{F}_{\text{ABA}}) \)-hybrid model. While proving these properties, we will assume that \( Z \) satisfies the \( Q^{(4)}(P, Z) \) condition. This further implies that the sharing specification \( S = (S_1, \ldots, S_h) = \{P \setminus Z | Z \in \mathcal{Z}\} \) satisfies the \( Q^{(3)}(S, Z) \) condition.

**Theorem 3.4** If \( Z \) satisfies the \( Q^{(4)}(P, Z) \) condition, then \( \Pi_{\text{PerTriples}} \) is a perfectly-secure protocol for securely realizing \( \mathcal{F}_{\text{Triples}} \) with UC-security in the \( (\mathcal{F}_{\text{VSS}}, \mathcal{F}_{\text{ABA}}) \)-hybrid model. The protocol makes \( O(M \cdot |Z| \cdot n^5) \) calls to \( \mathcal{F}_{\text{VSS}} \), \( O(|Z| \cdot n^5) \) calls to \( \mathcal{F}_{\text{ABA}} \) and additionally incurs a communication of \( O(M \cdot |Z|^2 \cdot n^5 \log |F| + |Z| \cdot n^7 \log |F|) \) bits.

**Proof.** The communication complexity and the number of calls to \( \mathcal{F}_{\text{VSS}} \) and \( \mathcal{F}_{\text{ABA}} \) follow from the protocol steps and the communication complexity of the protocol \( \Pi_{\text{Mult}} \). So we next prove the security. For ease of explanation,
we consider the case where only one multiplication-triple is generated in $\Pi_{\text{PerTriples}}$; i.e. $M = 1$. The proof can be easily modified for any general $M$.

Let $\text{Adv}$ be an arbitrary adversary, attacking the protocol $\Pi_{\text{PerTriples}}$ by corrupting a set of parties $Z^* \in \mathcal{Z}$, and let $\text{Env}$ be an arbitrary environment. We show the existence of a simulator $\mathcal{S}_{\text{PerTriples}}$ (Fig. 17), such that for any $Z^* \in \mathcal{Z}$, the outputs of the honest parties and the view of the adversary in the protocol $\Pi_{\text{PerTriples}}$ is indistinguishable from the outputs of the honest parties and the view of the adversary in an execution in the ideal world involving $\mathcal{S}_{\text{PerTriples}}$ and $\mathcal{F}_{\text{Triples}}$.

The high level idea of the simulator is as follows. Throughout the simulation, the simulator itself performs the role of the ideal functionality $\mathcal{F}_{\text{VSS}}$ and $\mathcal{F}_{\text{ABA}}$ whenever required. During the first stage, whenever $\text{Adv}$ sends a pair of vector of shares to $\mathcal{F}_{\text{VSS}}$ on the behalf of a corrupt party, the simulator records these vectors. On the other hand, for the honest parties, the simulator picks pairs of random values and random shares for those values, and distributes the appropriate shares to the corrupt parties, as per $\mathcal{F}_{\text{VSS}}$. During ACS, to select the common subset of parties, the simulator itself performs the role of $\mathcal{F}_{\text{ABA}}$ and simulates the honest parties as per the steps of the protocol and $\mathcal{F}_{\text{ABA}}$. This allows the simulator learn the common subset of parties $\mathcal{C}\mathcal{S}$. Notice that the secret-sharing of the pairs of values shared by all the parties in $\mathcal{C}\mathcal{S}$ will be available with the simulator. While the secret-sharing of pairs of the honest parties in $\mathcal{C}\mathcal{S}$ are selected by the simulator itself, for every corrupt party $P_j$ which is added to $\mathcal{C}\mathcal{S}$, at least one honest party $P_i$ should participate with input 1 in the corresponding call to $\mathcal{F}_{\text{ABA}}$. This implies that the honest party $P_i$ must have received some shares from $\mathcal{F}_{\text{VSS}}$ corresponding to the vector of shares which $P_j$ sent to $\mathcal{F}_{\text{VSS}}$. Since in the simulation, the role of $\mathcal{F}_{\text{VSS}}$ is played by the simulator itself, it implies that the vector of shares used by $P_j$ will be known to the simulator.

Once the simulator learns $\mathcal{C}\mathcal{S}$ and the secret-sharing of the pairs of values shared by the parties in $\mathcal{C}\mathcal{S}$, during the second stage, the simulator simulates the rest of the interaction between the honest parties and $\text{Adv}$ as per the protocol steps, by itself playing the role of the honest parties. Moreover, in the underlying instances of $\Pi_{\text{OptMult}}, \Pi_{\text{MultCI}}$ and $\Pi_{\text{Mult}}$, the simulator itself performs the role of $\mathcal{F}_{\text{VSS}}$ and $\mathcal{F}_{\text{ABA}}$ whenever required. Notice that simulator will be knowing the values which should be shared by the respective parties through $\mathcal{F}_{\text{VSS}}$ during the underlying instances of $\Pi_{\text{OptMult}}$ and $\Pi_{\text{MultCI}}$. This is because these values are completely determined by the secret-sharing of the pairs of values shared by the parties in $\mathcal{C}\mathcal{S}$, which will be known to the simulator. Consequently, in the simulated execution, the simulator will be knowing which instances of $\Pi_{\text{MultCI}}$ are successful and which iterations of $\Pi_{\text{Mult}}$ are successful. Once the simulated execution is over, the simulator learns the shares of the corrupt parties corresponding to the output multiplication-triple in the simulated execution. The simulator then communicates these shares on the behalf of the corrupt parties during its interaction with $\mathcal{F}_{\text{Triples}}$. This ensures that the shares of the corrupt parties remain the same in both the worlds.

In the steps of the simulator, to distinguish between the values used by the various parties during the real execution and simulated execution, the variables in the simulated execution will be used with a $\tilde{}$ symbol.

**Simulator $\mathcal{S}_{\text{PerTriples}}$**

$\mathcal{S}_{\text{PerTriples}}$ constructs virtual real-world honest parties and invokes the real-world adversary $\text{Adv}$. The simulator simulates the view of $\text{Adv}$, namely its communication with $\text{Env}$, the messages sent by the honest parties, and the interaction with $\mathcal{F}_{\text{VSS}}$ and $\mathcal{F}_{\text{ABA}}$. In order to simulate $\text{Env}$, the simulator $\mathcal{S}_{\text{PerTriples}}$ forwards every message it receives from $\text{Env}$ to $\text{Adv}$ and vice-versa. The simulator then simulates the various stages of the protocol as follows.

- **Stage I: Generating a Secret-Sharing of a Random Pair of Values.**
  - **Sharing Random Pairs of Values:**
    - The simulator simulates the operations of the honest parties during this phase by picking random random pairs of values and random vector of shares for those values on their behalf. Namely, when $\text{Adv}$ requests for output from $\mathcal{F}_{\text{VSS}}$ with $\text{sid}_{j,1}$ and $\text{sid}_{j,2}$ for any $P_j \notin Z^*$, the simulator picks random values $\tilde{a}^{(j)} \in \mathbb{F}$ and random shares $(\tilde{a}_1^{(j)}, \ldots, \tilde{a}_h^{(j)})$ and $(\tilde{b}_1^{(j)}, \ldots, \tilde{b}_h^{(j)})$, such that $\tilde{a}_1^{(j)} + \ldots + \tilde{a}_h^{(j)} = \tilde{a}^{(j)}$ and $\tilde{b}_1^{(j)} + \ldots + \tilde{b}_h^{(j)} = \tilde{b}^{(j)}$. The simulator then sets $[\tilde{a}^{(j)}]_q = [\tilde{a}_1^{(j)}]_q + [\tilde{a}_2^{(j)}]_q = [\tilde{b}^{(j)}]_q$ for $q = 1, \ldots, h$, and responds to $\text{Adv}$ with the output (share, $\text{sid}_{j,1}, P_j, \{[\tilde{a}^{(j)}]_q\}_{q \in \mathbb{Z}, \neq 0}$) and (share, $\text{sid}_{j,2}, P_j, \{[\tilde{b}^{(j)}]_q\}_{q \in \mathbb{Z}, \neq 0}$) on the behalf of $\mathcal{F}_{\text{VSS}}$ with $\text{sid}_{j,1}$ and $\text{sid}_{j,2}$ respectively.
    - Whenever $\text{Adv}$ sends (dealer, $\text{sid}_{i,1}, (\tilde{a}_1^{(i)}, \ldots, \tilde{a}_h^{(i)})$) and (dealer, $\text{sid}_{i,2}, (\tilde{b}_1^{(i)}, \ldots, \tilde{b}_h^{(i)})$) to $\mathcal{F}_{\text{VSS}}$ with $\text{sid}_{i,1}$
We first show that there always exists some set that for every $P_1 \in Z^*$, the simulator sets $[\tilde{a}(i)]_q = \tilde{a}_i^{(i)}$ and $[\tilde{b}(i)]_q = \tilde{b}_i^{(i)}$ for $q = 1, \ldots , h$, where $\tilde{a}(i) \overset{df}{=} \tilde{a}_1^{(i)} + \ldots + \tilde{a}_h^{(i)}$ and $\tilde{b}(i) \overset{df}{=} \tilde{b}_1^{(i)} + \ldots + \tilde{b}_h^{(i)}$.

- **Selecting a Common Subset of Parties (ACS):** The simulator simulates the interface to $F_{ABA}$ for $\text{Adv}$ by itself performing the role of $F_{ABA}$ and playing the role of the honest parties, as per the steps of the protocol. When the first honest party completes this phase during the simulated execution, $S_{\text{PerTriples}}$ learns the set $CS$. It then sets $\tilde{a} \overset{df}{=} \sum_{P_j \in CS} \tilde{a}_j^{(i)}$, $\tilde{b} \overset{df}{=} \sum_{P_j \in CS} \tilde{b}_j^{(i)}$ and computes $[\tilde{a}] = \sum_{P_j \in CS} [\tilde{a}_j^{(i)}]$, $[\tilde{b}] = \sum_{P_j \in CS} [\tilde{b}_j^{(i)}]$.

- **Stage II: Generating the Product.** The simulator plays the role of the honest parties as per the protocol and interacts with $\text{Adv}$ for the instance $\Pi_{\text{Mult}}(P, Z, [a], [b])$, where during the instance, the simulator uses the shares $\{(\tilde{a}_q, \tilde{b}_q) \}_{P_j \in S_d}$ on the behalf of every $P_i \notin Z^*$. Moreover, during this instance of $\Pi_{\text{Mult}}$, the simulator simulates the interface to $F_{ABA}$ for $\text{Adv}$ during the underlying instances of $\Pi_{\text{OptMult}}$ and during cheater identification, by itself performing the role of $F_{ABA}$, as per the steps of the protocol. Furthermore, during the underlying instances of $\Pi_{\text{OptMult}}$ and $\Pi_{\text{MultCl}}$, whenever required, the simulator itself plays the role $F_{\text{VSS}}$.

- **Interaction with $F_{\text{Triples}}$:** Let $\{(\tilde{c}_q)_{S_q \cap Z^* \neq \emptyset}\}$ be the output shares of the parties in $Z^*$, during the instance $\Pi_{\text{Mult}}(P, Z, S, [a], [b])$. The simulator sends $(\text{shares}, \text{sid}, \{(\tilde{a}_q, \tilde{b}_q, \tilde{c}_q)_{S_q \cap Z^* \neq \emptyset}\})$ to $F_{\text{Triples}}$, on the behalf of the parties in $Z^*$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure17.png}
\caption{Simulator for the protocol $\Pi_{\text{PerTriples}}$ with $M = 1$ where $\text{Adv}$ corrupts the parties in set $Z^* \in Z$}
\end{figure}

We now prove a series of claims, which will help us to finally prove the theorem. We first claim that in any execution of $\Pi_{\text{PerTriples}}$, a set $CS$ is eventually generated.

**Claim B.32.** In any execution of $\Pi_{\text{PerTriples}}$, a common set $CS$ is eventually generated where $P \setminus CS \in Z$, such that for every $P_j \in CS$, there exists a pair of values held by $P_j$, which are eventually secret-shared.

**Proof.** We first show that there always exists some set $Z \in Z$ such that in the $F_{ABA}$ instances corresponding to every party in $P \setminus Z$, all honest parties eventually obtain an output 1. For this, we consider the following two cases.

- **If some honest party $P_i$ has participated with vote input 0 in any instance of $F_{ABA}$ during step 3 of the ACS phase:** this implies that there exists a subset $GP_i$ for $P_i$ where $P \setminus GP_i \in Z$, such that $P_i$ receives the output $(\text{decide}, \text{sid}_{j,1})$ from $F_{ABA}$ with $\text{sid}_j$, corresponding to every $P_j \in GP_i$. Consequently, every honest party will eventually receive the same outputs from these $F_{ABA}$ instances. Since $P \setminus GP_i \in Z$, we get that there exists some set $Z \in Z$ such that the $F_{ABA}$ instances corresponding to every party in $P \setminus Z$ responded with output 1, which is what we wanted to show.

- **No honest party has participated with vote input 0 in any instance of $F_{ABA}$:** In the protocol, each party $P_j \notin Z^*$ sends its vector of shares to $F_{VSS}$ with $\text{sid}_{j,1}$ and $\text{sid}_{j,2}$ and every honest party eventually receives its respective shares from these vectors as the output from the corresponding instances of $F_{VSS}$. Hence, corresponding to every $P_j \notin Z^*$, all honest parties eventually participate with input $(\text{vote}, \text{sid}_{j,1})$ during the instance of $F_{ABA}$ with $\text{sid}_j$, and this $F_{ABA}$ instance will eventually respond with output $(\text{decide}, \text{sid}_{j,1})$. Since $Z^* \in Z$, it then follows that even in this case, there exists some set $Z \in Z$ such that the $F_{ABA}$ instances corresponding to every party in $P \setminus Z$ responded with output 1.

We next show that all honest parties eventually receive an output from all the instances of $F_{ABA}$. Since we have shown there exists some set $Z \in Z$ such that the $F_{ABA}$ instances corresponding to every party in $P \setminus Z$ eventually returns the output 1, it thus follows that all honest parties eventually participate with some vote inputs in the remaining $F_{ABA}$ instances and hence will eventually obtain some output from these $F_{ABA}$ instances as well. Since the set $CS$ corresponds to the $F_{ABA}$ instances in which the honest parties obtain 1 as the output, it thus follows that eventually, the honest parties obtain some $CS$ where $P \setminus CS \in Z$. Moreover, the set $CS$ will be common, as it is based on the outcome of $F_{ABA}$ instances.

Now consider an arbitrary $P_j \in CS$. This implies that the parties obtain 1 as the output from the $j^{th}$ instance of $F_{ABA}$. This further implies that at least one honest party $P_i$ participated in this $F_{ABA}$ instance with vote input 1. This is possible only if $P_i$ received its respective shares from the instances of $F_{VSS}$ with $\text{sid}_{j,1}$ and $\text{sid}_{j,2}$, further implying that $P_i$ has provided some vector of shares $\{a_1^{(i)}, \ldots , a_h^{(i)}\}$ and $\{b_1^{(i)}, \ldots , b_h^{(i)}\}$ as inputs to these $F_{VSS}$ instances. It now follows easily that eventually, all honest parties will have their respective shares corresponding to
the vectors of shares provided by $P_j$, implying that the honest parties will eventually hold $[a(j)]$ and $[b(j)]$, where
\[ a(j) \overset{\text{def}}{=} a_1^{(j)} + \ldots + a_h^{(j)} \quad \text{and} \quad b(j) \overset{\text{def}}{=} b_1^{(j)} + \ldots + b_h^{(j)}. \]

We next show that the view generated by $S_{\text{PerTriples}}$ for Adv is identically distributed to Adv’s view during the real execution of $\Pi_{\text{PerTriples}}$.

**Claim B.33.** The view of Adv in the simulated execution with $S_{\text{PerTriples}}$ is identically distributed as the view of Adv in the real execution of $\Pi_{\text{PerTriples}}$.

**Proof.** We first note that in the real-world (during the real execution of $\Pi_{\text{PerTriples}}$), the view of Adv consists of the following:

1. The vector of shares $(a_1^{(j)}, \ldots, a_h^{(j)})$ and $(b_1^{(j)}, \ldots, b_h^{(j)})$ (if any) for $F_{\text{VSS}}$ with $\text{sid}_{j,1}$ and $\text{sid}_{j,2}$ respectively, corresponding to $P_j \in Z^*$.
2. Shares $\{[a(j)]_q, [b(j)]_q\} \cap Z \neq \emptyset$, corresponding to $P_j \notin Z^*$.
3. Inputs of the various parties during the $F_{\text{ABA}}$ instances as part of ACS and the outputs from the $F_{\text{ABA}}$ instances.
4. The view generated for Adv during the instance of $\Pi_{\text{Mult}}$.

The vectors of shares in (1) are the inputs of Adv and hence they are identically distributed in both the real as well as simulated execution of $\Pi_{\text{PerTriples}}$, so let us fix these vectors. In the real execution, every $P_j \notin Z^*$ picks its pair of values randomly and the vectors of shares for $F_{\text{VSS}}$, corresponding to these values, uniformly at random. On the other hand, in the simulated execution, the simulator picks the pair of values and their shares randomly on the behalf of $P_j$. Now since the sharing specification $S = (S_1, \ldots, S_h) \overset{\text{def}}{=} \{P \setminus Z \mid Z \in Z\}$ is $Z$-private, it follows that the distribution of the shares in (2) is identical in both the real, as well as the simulated execution. Specifically, conditioned on the shares in (2), the underlying pairs of values shared by the parties $P_j \notin Z^*$ are uniformly distributed. Since the partial view of Adv containing (1) and (2) are identically distributed, let us fix them. Now conditioned on (1) and (2), it is easy to see that the partial view of Adv consisting of (3) is identically distributed in both the executions. This is because the outputs of the $F_{\text{ABA}}$ instances are determined deterministically based on the inputs provided by the various parties in these $F_{\text{ABA}}$ instances. Furthermore, the inputs of the parties in these $F_{\text{ABA}}$ instances depend upon the order in which various parties receive outputs from various $F_{\text{VSS}}$ instances, which is completely determined by Adv, since message scheduling is under the control of Adv. Since in the simulated execution, the simulator provides the interface to various instances of $F_{\text{ABA}}$ to Adv in exactly the same way as $F_{\text{ABA}}$ would have been accessed by Adv in the real execution, it follows that the partial view of Adv containing (1), (2) and (3) is identically distributed in both the executions and so let us fix this. This also fixes the set $CS$, which according to Claim B.32, is guaranteed to be generated.

Let $[a]$ and $[b]$ be the secret-sharing held by the honest parties after stage I, conditioned on the view of Adv in (1), (2) and (3). Note that in the simulated execution, the simulator will be knowing the complete sharing $[a]$ and $[b]$. This is because $[a]$ and $[b]$ are computed deterministically based on the secret-sharing of the pairs of the values shared by the parties in $CS$, all of which will be completely known to the simulator in the simulated execution. To complete the proof of the claim, we need to show that the partial view of Adv consisting of (4) is identically distributed in both the executions (conditioned on (1), (2) and (3)). However, this follows from the privacy of $\Pi_{\text{OptMult}}$, $\Pi_{\text{MultCI}}$ and $\Pi_{\text{Mult}}$ (Claims B.13, B.23 and B.31) and the fact that in the simulated execution, simulator plays the role of the honest parties during the instance of $\Pi_{\text{Mult}}$ exactly as per the steps of $\Pi_{\text{Mult}}$, where the simulator will be completely knowing the shares of both $[a]$ and $[b]$ corresponding to both the honest as well as corrupt parties. Consequently, it will be knowing the shares with which different parties have to participate in the underlying instances of $\Pi_{\text{OptMult}}$ and $\Pi_{\text{MultCI}}$. Moreover, in the simulated execution, the simulator honestly plays the role of $F_{\text{VSS}}$ and $F_{\text{ABA}}$ in these $\Pi_{\text{OptMult}}$ and $\Pi_{\text{MultCI}}$ instances. This guarantees that the view of Adv during the real execution of the $\Pi_{\text{Mult}}$ instance is exactly the same as the view of Adv during the simulated execution of $\Pi_{\text{Mult}}$. □
Finally, we show that conditioned on the view of Adv, the outputs of the honest parties are identically distributed in both the worlds.

**Claim B.34.** Conditioned on the view of Adv, the output of the honest parties are identically distributed in the real execution of $\Pi_{\text{PerTriples}}$ involving Adv, as well as in the ideal execution involving $S_{\text{PerTriples}}$ and $F_{\text{Triples}}$.

**Proof.** Let View be an arbitrary view of Adv, and let $\{(a^{(j)}, b^{(j)})\}_j \in CS$ be the secret-sharing of the pairs of values as per View, shared by the parties in $CS$. Note that $CS$ is bound to have at least one honest party. This is because $P \setminus CS \in Z$ and if $CS \subseteq Z^*$, then it implies that $Z$ does not satisfy the $Q^{(2)}(P, Z)$ condition, which is a contradiction. From the proof of Claim B.33, it follows that corresponding to every honest $P_j \in CS$, the pairs $(a^{(j)}, b^{(j)})$ are uniformly distributed conditioned on the shares of these pairs, as determined by View. Let us fix these pairs.

We show that in the real-world, the honest parties eventually output $([a], [b], [c])$, where conditioned on View, the triple $(a, b, c)$ is uniformly random multiplication-triple over $F$. From the protocol steps, the parties set $[a] \defeq \sum_{P_j \in CS} [a^{(j)}]$, $[b] \defeq \sum_{P_j \in CS} [b^{(j)}]$. Since corresponding to every $P_j \in CS$, the honest parties eventually hold $[a^{(j)}]$ and $[b^{(j)}]$ (follows from Claim B.32), it follows that the honest parties eventually hold $[a]$ and $[b]$. Moreover, since $[c]$ is computed as the output of the instance $\Pi_{\text{Mult}}(P, Z, S, [a], [b])$, it follows from Lemma 3.3 that the honest parties will eventually hold $[c]$, where $c = ab$. We next show that conditioned on View, the multiplication-triple $(a, b, c)$ is uniformly distributed over $F$. However, this follows from the fact that there exists a one-to-one correspondence between the random pairs shared by the honest parties in $CS$ and $(a, b)$. Namely, from the view point of Adv, for every candidate pair $(a^{(j)}, b^{(j)})$ shared by the honest parties $P_j \in CS$, there exists a unique $(a, b)$ which is consistent with View. Since the pairs shared by the honest parties $P_j$ are uniformly distributed and independent of View, it follows that $(a, b)$ is also uniformly distributed. Since $c = ab$ holds, it follows that $(a, b, c)$ is uniformly distributed.

To complete the proof, we now show that conditioned on the shares $\{(a_q, b_q, c_q)\}_{q \in Z \setminus \emptyset}$ (which are determined by View), the honest parties output a secret-sharing of some random multiplication-triple in the ideal-world which is consistent with the shares $\{(a_q, b_q, c_q)\}_{q \in Z \setminus \emptyset}$. However, this simply follows from the fact that in the ideal-world, the simulator $S_{\text{PerTriples}}$ sends the shares $\{(a_q, b_q, c_q)\}_{q \in Z \setminus \emptyset}$ to $F_{\text{Triples}}$ on the behalf of the parties in $Z^*$. As an output, $F_{\text{Triples}}$ generates a random secret-sharing of some random multiplication-triple consistent with the shares provided by $S_{\text{PerTriples}}$. $\square$

The proof of Theorem 3.4 now follows from Claim B.33 and Claim B.34. $\square$

C Properties of the Statistically-Secure Pre-Processing Phase

In this section, we prove the security properties of all the statistically-secure subprotocols, followed by the statistically-secure preprocessing phase. We first start with the AICP.

C.1 Properties of Our AICP

In this section, we formally prove the properties of our AICP. While proving these properties, we assume that $Z$ satisfies the $Q^{(3)}(P, Z)$ condition. We first show that when $S$, $I$ and $R$ are honest, then all honest parties set the local bit indicating that the authentication has completed to 1. Furthermore, $R$ will accept the signature revealed by $I$.

**Claim C.1 (Correctness).** If $S$, $I$ and $R$ are honest, then each honest $P_i$ eventually sets authCompleted$_{(\text{sid}, i)}^{(S, I, R)}$ to 1 during $\Pi_{\text{Auth}}$. Moreover, $R$ eventually outputs $s$ during $\Pi_{\text{Reveal}}$.

**Proof.** Let $S$, $I$ and $R$ be honest and let $H$ be the set of honest parties among $P$. Moreover, let $Z^* \in Z$ be the set of corrupt parties. We first show that each honest party $P_i$ eventually sets authCompleted$_{(\text{sid}, i)}^{(S, I, R)}$ to 1 during $\Pi_{\text{Auth}}$.  


During $\Pi_{\text{Auth}}$, S chooses the signing-polynomial $F(x)$ such that $s = F(0)$ holds. S will then send the signing-polynomial $F(x)$ and masking-polynomial $M(x)$ to l, and the corresponding verification-point $(\alpha_i, v_i, m_i)$ to each verifier $P_i$, such that $v_i = F(\alpha_i)$ and $m_i = M(\alpha_i)$ holds. Consequently, each verifier in $H$ will eventually receive its verification-point and indicates this to l. Since $P \setminus H = Z^* \in Z$, it follows that l will eventually find a set $\mathcal{S}V$, such that $\mathcal{P} \setminus \mathcal{S}V \in Z$, and where each verifier in $\mathcal{S}V$ has indicated to l that it has received its verification-point. Consequently, l will compute $B(x) = dF(x) + M(x)$, and broadcast $(d, B(x), \mathcal{S}V)$, which is eventually delivered to every honest party, including S. Moreover, S will find that $B(\alpha_j) = dv_j + m_j$ holds for all the verifiers $P_j \in \mathcal{S}V$. Consequently, S will broadcast an OK message, which is eventually received by every honest party $P_i$, who then sets authCompleted$_{S,1,R}^{(\text{sid},i)}$ to 1. Moreover, l will set $\text{ICSig}(S, l, r, s)$ to $F(x)$.

During $\Pi_{\text{Reveal}}$, l will send $F(x)$ to R, and each verifier $P_i \in H \cap \mathcal{S}V$ will send its verification points $(\alpha_i, v_i, m_i)$ to R. These points and the polynomial $F(x)$ are eventually received by R. Moreover, the condition $v_i = F(\alpha_i)$ will hold true for these points, and consequently these points will be accepted. Since $\mathcal{S}V \setminus (H \cap \mathcal{S}V) \subseteq Z^* \in Z$, it follows that R will eventually find a subset $\mathcal{S}V' \subseteq \mathcal{S}V$ where $\mathcal{S}V \setminus \mathcal{S}V' \in Z$, such that the points corresponding to all the parties in $\mathcal{S}V'$ are accepted. This implies that R will eventually output $s = F(0)$.

We next show that when S, l and R are honest, then the adversary does not learn anything about s during either $\Pi_{\text{Auth}}$ or $\Pi_{\text{Reveal}}$.

**Claim C.2 (Privacy).** If S, l and R are honest, then the view of adversary Adv throughout $\Pi_{\text{Auth}}$ and $\Pi_{\text{Reveal}}$ is independent of s.

**Proof.** Let $t = \max\{|Z| : Z \in Z\}$ and let $Z^* \in Z$ be the set of corrupt parties. For simplicity and without loss of generality, let $|Z^*| = t$. During $\Pi_{\text{Auth}}$, the adversary Adv learns $t$ verification-points $\{\alpha_i, v_i, m_i\}_{P_i \in Z^*}$. However, since $F(x)$ is a random $t$-degree polynomial with $F(0) = s$, the points $\{\alpha_i, v_i\}_{P_i \in Z^*}$ are distributed independently of s. That is, for every candidate $s \in \mathbb{F}$ from the point of view of Adv, there is a corresponding unique $t$-degree polynomial $F(x)$, such that $F(\alpha_i) = v_i$ holds corresponding to every $P_i \in Z^*$.

During $\Pi_{\text{Auth}}$, the adversary Adv also learns $d$ and the blinded-polynomial $B(x) = dF(x) + M(x)$, along with the points $\{\alpha_i, v_i\}_{P_i \in Z^*}$. However, this does not add any new information about s to the view of the adversary. This is because $M(x)$ is a random $t$-degree polynomial. Hence for every candidate $M(x)$ polynomial from the point of view of Adv where $M(\alpha_i) = m_i$ holds for every $P_i \in Z^*$, there is a corresponding unique $t$-degree polynomial $F(x)$, such that $F(\alpha_i) = v_i$ holds corresponding to every $P_i \in Z^*$, and where $dF(x) + M(x) = B(x)$. We also note that in $\Pi_{\text{Auth}}$, the signer S does not broadcast s, which follows from the Claim C.1. Finally, Adv does not learn anything new about s during $\Pi_{\text{Reveal}}$, since the verification-points and the signing-polynomial are sent only to R.

We next prove the unforgeability property.

**Claim C.3 (Unforgeability).** If S, R are honest, l is corrupt and if R outputs s′ during $\Pi_{\text{Reveal}}$, then s′ = s holds, except with probability at most $\frac{nt}{|\mathcal{P}|-1}$.

**Proof.** Let $H$ be the set of honest parties in $\mathcal{P}$ and let $Z^*$ be the set of corrupt parties. Since R outputs s′ during $\Pi_{\text{Reveal}}$, it implies that during $\Pi_{\text{Auth}}$, the variable authCompleted$_{S,1,R}^{(\text{sid},i)}$ is set to 1 by R, if $R = P_i$. This further implies that S has broadcasted either an OK or an NOK message during $\Pi_{\text{Auth}}$, which further implies that l has broadcasted some blinded-polynomial $B(x)$ during $\Pi_{\text{Auth}}$. Now there are now two possible cases.

- **S has broadcasted NOK along with s during $\Pi_{\text{Auth}}**: In this case, every honest party including R would set IC Sig$(S, l, R, s)$ to s during $\Pi_{\text{Auth}}$. Moreover, during $\Pi_{\text{Reveal}}$, the receiver R outputs s. Hence, in this case, $s′ = s$ holds with probability 1.

- **S has broadcasted OK during $\Pi_{\text{Auth}}**: This implies that during $\Pi_{\text{Auth}}$, l had broadcasted a $t$-degree blinded-polynomial $B(x)$, along with the set $\mathcal{S}V$. Furthermore, S has verified that $\mathcal{P} \setminus \mathcal{S}V \in Z$ and $B(\alpha_i) = dv_i + m_i$ holds for every verifier $P_i \in \mathcal{S}V$. Now during $\Pi_{\text{Reveal}}$, if l sends $F(x)$ as IC Sig$(S, l, R, s)$ to R, then again $s′ = s$ holds with probability 1. So consider the case when l sends $F′(x)$ as IC Sig$(S, l, R, s)$ to R, where $F′(x)$ is a $t$-degree polynomial such that $F′(x) \neq F(x)$ and where $F′(0) = s′$. In this case, we show that
except with probability at most \( \frac{n}{|F|-1} \), the verification-point of no honest verifier from \( SV \) will get accepted by \( R \) during \( \Pi_{Reveal} \), with respect to \( F'(x) \). Now assuming that this statement is true, the proof follows from the fact that in order for \( F'(x) \) to be accepted by \( R \), it should accept the verification-point of at least one honest verifier from \( SV \) with respect to \( F'(x) \). This is because \( R \) should find a subset of verifiers \( SV' \subseteq SV \) whose corresponding verification-points are accepted, where \( SV \setminus SV' \in Z \). So clearly, the set of corrupt verifiers in \( SV \) cannot form a candidate \( SV' \). This is because since \( Z \) satisfies the \( Q'(P, Z) \) condition, it satisfies the \( Q'(SV, Z) \) condition as \( P \setminus SV \in Z \). This further implies that \( Z \) satisfies the \( Q'(SV', Z) \) condition as \( SV \setminus SV' \in Z \). Hence, any candidate for \( SV' \) must contain at least one honest party from \( SV \).

Consider an arbitrary verifier \( P_i \in H \cap SV \) from which \( R \) receives the verification-point \((\alpha_i, v_i, m_i)\) during \( \Pi_{Reveal} \). This point can be accepted only if either of the following holds.

- \( v_i = F'(\alpha_i) \): This is possible with probability at most \( \frac{1}{|F|-1} \). This is because \( F'(x) \) and \( F(x) \), being distinct \( t \)-degree polynomials can have at most \( t \) points in common, and since the evaluation-point \( \alpha_i \) corresponding to \( P_i \), being randomly selected from \( F - \{0\} \), will not be known to \( l \).
- \( dv_i + m_i \neq B(\alpha_i) \): This is impossible, as otherwise \( S \) would have broadcasted \( s \) and NOK during \( \Pi_{Auth} \), which is a contradiction.

Now as there could be up to \( n - 1 \) honest verifiers in \( SV \), it follows from the union bound that except with probability at most \( \frac{n}{|F|-1} \), the polynomial \( F'(x) \) will not be accepted.

We next prove the non-repudiation property.

Claim C.4 (Non-Repudiation). If \( S \) is corrupt and \( l, R \) are honest and if \( l \) has set \( ICSig(S, l, R, s) \) during \( \Pi_{Auth} \), then \( R \) eventually outputs \( s \) during \( \Pi_{Reveal} \), except with probability at most \( \frac{n}{|F|-1} \).

Proof. Let \( H \) be the set of honest parties in \( P \) and \( Z^* \in Z \) be the set of corrupt parties. Since \( l \) has set \( ICSig(S, l, R, s) \) during \( \Pi_{Auth} \), it implies that it has that it has set the variable \( authCompleted_{\lambda_i}^{(sid, i)} \) to 1 during \( \Pi_{Auth} \), if \( l = P_i \). This further implies that \( l \) has broadcasted a blinded-polynomial \( B(x) \), the linear combiner \( d \) and the set \( SV \), where \( B(x) = dF(x) + M(x) \) and where \( F(x) \) and \( M(x) \) are the signing and masking polynomials received by \( l \) from \( S \). Moreover, \( S \) has broadcasted either an OK message or a NOK message. Consequently, all honest parties \( P_i \), including \( R \), eventually set \( authCompleted_{\lambda_i}^{(sid, i)} \) to 1. Now there are two possible cases.

- **S has broadcasted NOK, along with \( s \) during \( \Pi_{Auth} \):** In this case, all honest parties, including \( l \) and \( R \), set \( ICSig(S, l, R, s) \) to \( s \) during \( \Pi_{Auth} \). Moreover, from the steps of \( \Pi_{Reveal} \), \( R \) outputs \( s \) during \( \Pi_{Reveal} \). Thus, the claim holds in this case with probability 1.

- **S has broadcasted OK during \( \Pi_{Auth} \):** In this case, \( l \) sets \( ICSig(S, l, R, s) \) to \( F(x) \), where \( F(0) = s \). During \( \Pi_{Reveal} \), \( l \) sends \( F(x) \) to \( R \). Moreover, every verifier \( P_i \in H \cap SV \) eventually sends its verification-point \((\alpha_i, v_i, m_i)\) to \( R \). We next show that except with probability at most \( \frac{n}{|F|-1} \), all these verification-points are accepted by \( R \). Now assuming that this statement is true, the proof follows from the fact that \( H \cap SV = SV \setminus Z^* \). Consequently, \( R \) eventually accepts the verification-points from a subset of parties \( SV' \subseteq SV \) where \( SV \setminus SV' \in Z \) and outputs \( s \).

Consider an arbitrary verifier \( P_i \in H \cap SV \) whose verification-point \((\alpha_i, v_i, m_i)\) is received by \( R \) during \( \Pi_{Reveal} \). Now there are two possible cases, depending upon the relationship that holds between \( F(\alpha_i) \) and \( v_i \) during \( \Pi_{Auth} \).

- \( v_i = F(\alpha_i) \) holds: In this case, according to the protocol steps of \( \Pi_{Reveal} \), the point \((\alpha_i, v_i, m_i)\) is accepted by \( R \).
- \( v_i \neq F(\alpha_i) \) holds: In this case, we claim that except with probability at most \( \frac{1}{|F|-1} \), the condition \( dv_i + m_i \neq B(\alpha_i) \) will hold, implying that the point \((\alpha_i, v_i, m_i)\) is accepted by \( R \). This is because the only way \( dv_i + m_i = B(\alpha_i) \) holds is when \( S \) distributes \((\alpha_i, v_i, m_i)\) to \( P_i \) where \( v_i \neq F(\alpha_i) \) and \( m_i \neq M(\alpha_i) \) holds, and \( l \) selects \( d = (M(\alpha_i) - m_i) \cdot (v_i - F(\alpha_i))^{-1} \). However, \( S \) will not be knowing the random \( d \) from \( F \setminus \{0\} \) which \( l \) picks while distributing \( F(x) \), \( M(x) \) to \( l \), and \( (\alpha_i, v_i, m_i) \) to \( P_i \).

Now, as there can be up to \( n - 1 \) honest verifiers in \( SV \), from the union bound, it follows that except with probability at most \( \frac{n}{|F|-1} \), the verification-point of all honest verifiers in \( SV \) are accepted by \( R \).
We next derive the communication complexity of $\Pi_{Auth}$ and $\Pi_{Reveal}$.

**Claim C.5.** Protocol $\Pi_{Auth}$ incurs a communication of $O(n \cdot \log |F|)$ bits and $O(1)$ calls to $F_{Acast}$ with $O(n \cdot \log |F|)$-bit messages. Protocol $\Pi_{Reveal}$ requires a communication of $O(n \cdot \log |F|)$ bits.

**Proof.** During $\Pi_{Auth}$, signer $S$ sends $t$-degree polynomials $F(x)$ and $M(x)$ to $I$, and verification-points to each verifier. This requires a communication of $O(n \cdot \log |F|)$ bits. Intermediary $I$ needs to broadcast $B(x)$, $d$ and the set $SVS$, which requires one call to $F_{Acast}$ with a message of size $O(n \cdot \log |F|)$ bits. Moreover, $S$ may need to broadcast $s$, which requires one call to $F_{Acast}$ with a message of size $O(\log |F|)$ bits. During $\Pi_{Reveal}$, $I$ may send $F(x)$ to $R$, and each verifier may send its verification-point to $R$. This will require a communication of $O(n \cdot \log |F|)$ bits. □

Lemma 4.1 now follows from Claims C.1-C.5.

**Lemma 4.1.** Let $Z$ satisfy the $Q(3)(P, Z)$ condition. Then the pair of protocols $(\Pi_{Auth}, \Pi_{Reveal})$ satisfies the following properties, except with probability at most $\epsilon_{AICP} = \frac{nt}{|F|-1}$, where $t = \max\{|Z| : Z \in Z\}$.

- **Correctness:** If $S, I$ and $R$ are honest, then each honest $P_i$ eventually sets $authCompleted_{\Pi_{Auth}}(\text{id}, i)$ to $1$ during $\Pi_{Auth}$. Moreover, $R$ eventually outputs $s$ during $\Pi_{Reveal}$.
- **Privacy:** If $S, I$ and $R$ are honest, then the view of adversary remains independent of $s$.
- **Unforgeability:** If $S, R$ are honest, $I$ is corrupt and if $R$ outputs $s' \in F$ during $\Pi_{Reveal}$, then $s' = s$ holds.
- **Non-repudiation:** If $S$ is corrupt and $I, R$ are honest and if $I$ has set $lCSig(S, I, R, s)$ during $\Pi_{Auth}$, then $R$ eventually outputs $s$ during $\Pi_{Reveal}$.

Protocol $\Pi_{Auth}$ incurs a communication of $O(n \cdot \log |F|)$ bits and $O(1)$ calls to $F_{Acast}$ with $O(n \cdot \log |F|)$-bit messages. Protocol $\Pi_{Reveal}$ requires a communication of $O(n \cdot \log |F|)$ bits.

### C.2 Statistical VSS Protocol

In this section, we prove the properties of $\Pi_{SVSS}$ (see Fig 7 for the formal description of the protocol) stated in Theorem 4.3. Throughout the section, we assume that $Z$ satisfies the $Q(3)(P, Z)$ condition, implying that $S = (S_1, \ldots, S_h) \overset{def}{=} \{P \setminus Z | Z \in Z\}$ satisfies the $Q(2)(S, Z)$ condition.

**Theorem 4.3** Let $Z$ satisfy the $Q(3)(P, Z)$ condition and let $S = (S_1, \ldots, S_h) = \{P \setminus Z | Z \in Z\}$. Then protocol $\Pi_{SVSS}$ UC-securely computes $F_{VSS}$ in the $F_{Acast}$-hybrid model, except with an error probability of at most $|Z| \cdot n^3 \cdot \epsilon_{AICP}$, where $\epsilon_{AICP} \approx \frac{n^2}{|F|}$. The protocol makes $O(|Z| \cdot n^3)$ calls to $F_{Acast}$ with $O(n \cdot \log |F|)$-bit messages and additionally incurs a communication of $O(|Z| \cdot n^4 \log |F|)$ bits. By replacing the calls to $F_{Acast}$ with protocol $\Pi_{Acast}$, the protocol incurs a total communication of $O(|Z| \cdot n^6 \log |F|)$ bits.

**Proof.** In the protocol, the dealer needs to send the share $s_q$ to all the parties in $S_q$, and this requires a communication of $O(|Z| \cdot n \log |F|)$ bits. An instance of $\Pi_{Auth}$ and $\Pi_{Reveal}$ is executed with respect to every ordered triplet of parties $P_i, P_j, P_k \in S_q$, leading to $O(|Z| \cdot n^3)$ instances of $\Pi_{Auth}$ and $\Pi_{Reveal}$ being executed. The communication complexity now follows from the communication complexity of $\Pi_{Auth}$ and $\Pi_{Reveal}$ (Claim C.5) and from the communication complexity of the protocol $\Pi_{Acast}$ (Theorem A.2).

We next prove the security of the protocol. Let $Adv$ be an arbitrary adversary, attacking the protocol $\Pi_{SVSS}$ by corrupting a set of parties $Z^* \in Z$, and let $Env$ be an arbitrary environment. We show the existence of a simulator $\Sigma_{SVSS}$, such that for any $Z^* \in Z$, the outputs of the honest parties and the view of the adversary in the protocol $\Pi_{SVSS}$ is indistinguishable from the outputs of the honest parties and the view of the adversary in an execution in the ideal world involving $\Sigma_{SVSS}$ and $F_{VSS}$, except with probability at most $|Z| \cdot n^3 \cdot \epsilon_{AICP}$, where $\epsilon_{AICP} \approx \frac{n^2}{|F|}$ (see Lemma 4.1). The simulator is very similar to the simulator $\Sigma_{PVSS}$ for the protocol $\Pi_{PVSS}$ (see Fig 13 in Appendix B.1), except that the simulator now has to simulate giving and accepting signatures on the behalf of honest parties, as part of pairwise consistency checks. In addition, for each $S_q \in S$, the simulator has to simulate
revealing signatures to the corrupt parties in $S_q \setminus C_q$ on the behalf of the honest parties in $C_q$. The simulator is formally presented in Figure [18]

**Simulator SVSS**

SVSS constructs virtual real-world honest parties and invokes the real-world adversary $Adv$. The simulator simulates the view of $Adv$, namely its communication with $Env$, the messages sent by the honest parties and the interaction with $F_{Acast}$. In order to simulate $Env$, the simulator $Sp_{SVSS}$ forwards every message it receives from $Env$ to $Adv$ and vice-versa. The simulator then simulates the various phases of the protocol as follows, depending upon whether the dealer is honest or corrupt.

**Simulation When $P_D$ is Honest**

Interaction with $F_{SVSS}$: the simulator interacts with the functionality $F_{SVSS}$ and receives a request based delayed output (share, $sid, P_D, \{[s]_q\}_{S_q \cap Z^* \neq \emptyset}$, on the behalf of the parties in $Z^*$).

Distribution of Shares: On the behalf of the dealer, the simulator sends $(dist, sid, P_D, q, [s]_q)$ to $Adv$, corresponding to every $P_i \in Z^* \setminus S_q$.

**Pairwise Consistency Tests on IC-Signed Values:**

- For each $S_q \in \mathbb{S}$ such that $S_q \cap Z^* \neq \emptyset$, corresponding to each $P_i \in S_q \cap Z^*$, the simulator does the following.
  - On the behalf of every party $P_i \in S_q \setminus Z^*$ as a signer and every $P_k \in S_q$ as a receiver, perform the role of the signer and the honest verifiers as per the steps of $\Pi_{auth}$ and interact with $Adv$ on the behalf of the honest parties to give $\text{ICSig}(sid_{i,j,k}^{P_i,q}), P_i, P_k, s_{qj} \rightarrow P_i$, where $s_{qj} = [s]_q$.
  - On the behalf of every $P_i, P_j \in S_q$ as intermediary and receiver respectively, perform the role of the honest parties as per the steps of $\Pi_{auth}$ and interact with $Adv$ on the behalf of the honest parties, if $Adv$ gives the signature $\text{ICSig}(sid_{i,j,k}^{P_i,q}, P_i, P_j, P_k, s_{qi})$ to $P_j$ on the behalf of the signer $P_i$. Upon receiving the signature $\text{ICSig}(sid_{i,j,k}^{P_i,q}, P_i, P_j, P_k, s_{qi})$ from $P_i$, record it.

- For each $S_q \in \mathbb{S}$ and for every $P_i, P_j \in S_q \setminus Z^*$ the simulator simulates $P_i$ giving $\text{ICSig}(sid_{i,j,k}^{P_i,q}, P_i, P_j, P_k, v)$ to $P_j$, corresponding to every $P_k \in S_q$, by playing the role of the honest parties and interacting with $Adv$ on their behalf, as per the steps of $\Pi_{auth}$ in the respective $\Pi_{auth}$ instances. Based on the following conditions, the simulator chooses the value $v$ in these instances as follows.
  - $S_q \cap Z^* \neq \emptyset$: Choose $v$ to be $[s]_q$.
  - $S_q \cap Z^* = \emptyset$: Pick a random element from $\mathbb{F}$ as $v$.

**Announcing Results of Pairwise Consistency Tests:**

- If for any $S_q \in \mathbb{S}$, $Adv$ requests an output from $F_{Acast}$ with $sid_{i,j,k}^{P_i,q}$ corresponding to parties $P_i \in S_q \setminus Z^*$ and $P_j \in S_q$, then the simulator provides the output on the behalf of $F_{Acast}$ as follows.
  - If $P_j \in S_q \setminus Z^*$, then send the output $(P_i, Acast, sid_{i,j,k}^{P_i,q}, OK_q(i,j))$.
  - If $P_i \in (S_q \cap Z^*)$, then send the output $(P_i, Acast, sid_{i,j,k}^{P_i,q}, OK_q(i,j))$, if $\text{ICSig}(sid_{i,j,k}^{P_i,q}, P_j, P_k, s_{qi})$ has been recorded on the behalf of $P_j$ as a signer, corresponding to the intermediary $P_i$ and every $P_k \in S_q$ as a receiver, such that $s_{qi} = [s]_q$ holds.

- If for any $S_q \in \mathbb{S}$ and any $P_i \in S_q \cap Z^*$, $Adv$ sends $(P_i, Acast, sid_{i,j,k}^{P_i,q}, OK_q(i,j))$ to $F_{Acast}$ with $sid_{i,j,k}^{P_i,q}$ on the behalf of $P_i$ for any $P_j \in S_q$, then the simulator records it. Moreover, if $Adv$ requests an output from $F_{Acast}$ with $sid_{i,j,k}^{P_i,q}$, then the simulator sends the output $(P_i, Acast, sid_{i,j,k}^{P_i,q}, OK_q(i,j))$ on the behalf of $F_{Acast}$.

**Construction of Core Sets and Public Announcement:**

- For each $S_q \in \mathbb{S}$, the simulator plays the role of $P_D$ and adds the edge $(P_i, P_j)$ to the graph $G_q^{(D)}$ over the vertex set $S_q$, if any one of the following is true.

  1. $P_i, P_j \in S_q \setminus Z^*$.
  2. If $P_i \in S_q \setminus Z^*$ and $P_j \in S_q \setminus Z^*$, then the simulator has recorded $(P_i, Acast, sid_{i,j,k}^{P_i,q}, OK_q(i,j))$ sent by $Adv$ on the behalf of $P_i$ to $F_{Acast}$ with $sid_{i,j,k}^{P_i,q}$, and has recorded $\text{ICSig}(sid_{i,j,k}^{P_i,q}, P_i, P_j, P_k, s_{qi})$ on the behalf of $P_i$ as a signer and $P_j$ as an intermediary corresponding to every party $P_k \in S_q$ as a receiver, such that $s_{qi} = [s]_q$ holds.
  3. If $P_i, P_j \in S_q \cap Z^*$, then the simulator has recorded $(P_i, Acast, sid_{i,j,k}^{q}, OK_q(i,j))$ and $(P_j, Acast, sid_{i,j,k}^{q}, OK_q(j,i))$ sent by $Adv$ on behalf $P_i$ and $P_j$ respectively, to $F_{Acast}$ with $sid_{i,j,k}^{P_i,q}$ and $F_{Acast}$ with $sid_{i,j,k}^{P_j,q}$.
For each $S_q \in \mathcal{S}$, the simulator finds a set $C_q$ which forms a clique in $G_q^P$, such that $S_q \setminus C_q \in \mathcal{Z}$. When $\text{Adv}$ requests output from $\mathcal{F}_{\text{Acast}}$ with $\text{sid}_P$, the simulator sends the output $(\text{sender}, \text{Acast}, \text{sid}_P, \{C_q\}_{S_q \in \mathcal{S}})$ on the behalf of $\mathcal{F}_{\text{Acast}}$.

**Share Computation:** Once $C_1, \ldots, C_q$ are computed, then for each $S_q \in \mathcal{S}$, simulator does the following for every $P_i \in (S_q \setminus C_q) \cap \mathcal{Z}$ and every $P_j \in C_q \setminus \mathcal{Z}$.

- Simulate the revelation of the signature $\text{ICSig}(\text{sid}_{j,k,i}, P_k, P_j, P_i, s_{qj})$ to $P_i$ on the behalf of the intermediary $P_j$ corresponding to every signer $P_k \in C_q$, where $s_{qj} = [s]_q$, by playing the role of the honest parties as per $\Pi_{\text{Reveal}}$ and interacting with $\text{Adv}$.

**Simulation When $P_d$ is Corrupt**

In this case, the simulator $\mathcal{S}_{\text{SVSS}}$ interacts with $\text{Adv}$ during the various phases of $\Pi_{\text{SVSS}}$ as follows.

**Distribution of Shares:** For $q = 1, \ldots, h$, if $\text{Adv}$ sends $(\text{dist}, \text{sid}_P, q, v)$ on the behalf of $P_d$ to any party $P_i \in S_q \setminus \mathcal{Z}$, then the simulator records it and sets $s_{qi}$ to $v$.

**Pairwise Consistency Tests on IC-Signed Values:**

- For each $S_q \in \mathcal{S}$ such that $S_q \cap \mathcal{Z} \neq \emptyset$, corresponding to each party $P_i \in S_q \cap \mathcal{Z}$ and each $P_j \in S_q \setminus \mathcal{Z}$, the simulator does the following.
  - If $s_{qj}$ has been set to some value, then simulate giving $\text{ICSig}(\text{sid}_{j,k,i}, P_j, P_k, P_i, s_{qj})$ to $\text{Adv}$ on the behalf of $P_j$ as a signer, corresponding to every $P_k \in \mathcal{P}$ as receiver, by playing the role of the honest parties as per the steps of $\Pi_{\text{Auth}}$.
  - Upon receiving $\text{ICSig}(\text{sid}_{i,j,k}, P_i, P_j, P_k, s_{qi})$ from $\text{Adv}$ on the behalf of $P_i$ as a signer, corresponding to $P_j \in S_q$ as an intermediary and $P_k \in S_q$ as a receiver, record $\text{ICSig}(\text{sid}_{i,j,k}, P_i, P_j, P_k, s_{qj})$.
  - For each $S_q \in \mathcal{S}$ such that $S_q \cap \mathcal{Z} = \emptyset$, corresponding to each party $P_i, P_j \in S_q$, the simulator does the following.
    - Upon setting $s_{qi}$ to some value, simulate $P_i$ giving $\text{ICSig}(\text{sid}_{i,j,k}, P_i, P_j, P_k, s_{qi})$ to $P_j$, corresponding to every receiver $P_k \in S_q$, by playing the role of the honest parties and interacting with $\text{Adv}$ as per the steps of $\Pi_{\text{Auth}}$.

**Announcing Results of Pairwise Consistency Tests:**

- If for any $S_q \in \mathcal{S}$, $\text{Adv}$ requests an output from $\mathcal{F}_{\text{Acast}}$ with $\text{sid}_{i,j,k}$ corresponding to parties $P_i \in S_q \setminus \mathcal{Z}$ and $P_j \in S_q$, then the simulator provides the output on the behalf of $\mathcal{F}_{\text{Acast}}$ as follows, if $s_{qj}$ has been set to some value.
  - If $P_j \in S_q \setminus \mathcal{Z}$, then send the output $(P_i, \text{Acast}, \text{sid}_{i,j,k}, \text{OK}_q(i, j))$, if $s_{qj}$ has been set to some value and $s_{qi} = s_{qj}$ holds.
  - If $P_j \in S_q \cap \mathcal{Z}$, then send the output $(P_i, \text{Acast}, \text{sid}_{i,j,k}, \text{OK}_q(i, j))$, if $\text{ICSig}(\text{sid}_{i,j,k}, P_i, P_j, P_k, s_{qj})$ has been recorded on the behalf of $P_j$ as a signer for the intermediary $P_i$, corresponding to every $P_k \in S_q$ as a receiver, such that $s_{qj} = s_{qi}$ holds.
- If for any $S_q \in \mathcal{S}$ and any $P_i \in S_q \cap \mathcal{Z}$, $\text{Adv}$ sends $(P_i, \text{Acast}, \text{sid}_{i,j,k}, \text{OK}_q(i, j))$ to $\mathcal{F}_{\text{Acast}}$ with $\text{sid}_{i,j,k}$ on the behalf of $P_i$ for any $P_j \in S_q$, then the simulator records it. Moreover, if $\text{Adv}$ requests for an output from $\mathcal{F}_{\text{Acast}}$ with $\text{sid}_{i,j,k}$, then the simulator sends the output $(P_i, \text{Acast}, \text{sid}_{i,j,k}, \text{OK}_q(i, j))$ on the behalf of $\mathcal{F}_{\text{Acast}}$.

**Construction of Core Sets:** For each $S_q \in \mathcal{S}$, the simulator plays the role of the honest parties $P_i \in S_q \setminus \mathcal{Z}$ and adds the edge $(P_j, P_k)$ to the graph $G_q(i)$ over vertex set $S_q$, if any one of the following is true.

- If $P_j, P_k \in S_q \setminus \mathcal{Z}$, then the simulator has set $s_{qj}$ and $s_{qk}$ to some values, such that $s_{qj} = s_{qk}$ holds.
- If $P_j \in S_q \setminus \mathcal{Z}$ and $P_k \in S_q \setminus \mathcal{Z}$, then all the following should hold.
  - The simulator has recorded $(P_j, \text{Acast}, \text{sid}_{i,j,k}, \text{OK}_q(j, k))$ sent by $\text{Adv}$ on the behalf of $P_j$ to $\mathcal{F}_{\text{Acast}}$ with $\text{sid}_{i,j,k}$.
  - The simulator has recorded $\text{ICSig}(\text{sid}_{i,j,k}, P_i, P_j, P_k, P_m, s_{qj})$ on the behalf of $P_j$ as a signer and $P_k$ as an intermediary, corresponding to every receiver $P_m \in S_q$.
  - The simulator has set $s_{qk}$ to a value such that $s_{qj} = s_{qk}$ holds.
- If $P_j, P_k \in S_q \setminus \mathcal{Z}$, then the simulator has recorded $(P_j, \text{Acast}, \text{sid}_{i,j,k}, \text{OK}_q(j, k))$ and $(P_k, \text{Acast}, \text{sid}_{i,j,k}, \text{OK}_q(k, j))$ sent by $\text{Adv}$ on behalf of $P_j$ and $P_k$ respectively, to $\mathcal{F}_{\text{Acast}}$ with $\text{sid}_{i,j,k}$ and $\mathcal{F}_{\text{Acast}}$ with $\text{sid}_{i,j,k}$.

**Verification of Core Sets and Interaction with $\mathcal{F}_{\text{SVSS}}$:**

- If $\text{Adv}$ sends $(\text{sender}, \text{Acast}, \text{sid}_P, \{C_q\}_{S_q \in \mathcal{S}})$ to $\mathcal{F}_{\text{Acast}}$ with $\text{sid}_P$, then the simulator records it. Moreover, if $\text{Adv}$ requests for an output from $\mathcal{F}_{\text{Acast}}$ with $\text{sid}_P$, then on the behalf of $\mathcal{F}_{\text{Acast}}$, the simulator sends the output $(P_d, \text{Acast}, \text{sid}_P, \{C_q\}_{S_q \in \mathcal{S}})$.
- If simulator has recorded the sets $\{C_q\}_{S_q \in \mathcal{S}}$, then it plays the role of the honest parties and verifies if for $q = 1, \ldots, h$, etc.
the set $C_q$ is valid with respect to $S_q$, by checking if $S_q \setminus C_q \in \mathcal{Z}$ and if $C_q$ constitutes a clique in the graph $G_q^{(i)}$ of every party $P_i \in \mathcal{P} \setminus \mathcal{Z}^*$. If $C_1, \ldots, C_q$ are valid, then the simulator sends $(\text{share}, \text{sid}, P_D, \{s_q\}_{S_q \in \mathcal{S}})$ to $\mathcal{F}_{\text{PVSS}}$, where $s_q$ is set to $s_{qi}$ corresponding to any $P_i \in C_q \setminus \mathcal{Z}^*$.

Figure 18: Simulator for the protocol $\Pi_{\text{SVSS}}$ where Adv corrupts the parties in set $Z^* \in \mathcal{Z}$

We now prove a series of claims, which helps us to prove the theorem. We start with an honest $P_D$.

**Claim C.6.** If $P_D$ is honest, then the view of Adv in the simulated execution of $\Pi_{\text{SVSS}}$ with $\mathcal{S}_{\text{PVSS}}$ is identically distributed to the view of Adv in the real execution of $\Pi_{\text{SVSS}}$ involving honest parties.

**Proof.** Let $S^* \overset{\text{def}}{=} \{S_q \in \mathcal{S} \mid S_q \cap Z^* \neq \emptyset\}$. Then the view of Adv during the two executions consists of the following.

- **The shares $\{[s_q]\}_{S_q \in S^*}$ distributed by $P_D$:*** In the real execution, Adv receives $[s_q]$ from $P_D$ for each $S_q \in S^*$. In the simulated execution, the simulator provides this to Adv on behalf of $P_D$. Clearly, the distribution of the shares is identical in both the executions.

- **Corresponding to every $S_q \in S^*$ and every triplet of parties $P_i, P_j, P_k$ where $P_j \in S_q \setminus \mathcal{Z}^*$, $P_i \in S_q \cap \mathcal{Z}^*$ and $P_k \in S_q$, the signature $\text{ICSig}([\text{sid}_{i,j,k}^{(P, q)}, P_j, P_i, P_k, s_{ij}])$ received from $P_j$ as part of pairwise consistency tests:** While $P_j$ sends this to Adv in the real execution, the simulator sends this on behalf of $P_j$ in the simulated execution. Clearly, the distribution of the messages learnt by Adv during the corresponding instances of $\Pi_{\text{Auth}}$ is identical in both the executions.

- **Corresponding to every $S_q \in \mathcal{S}$, every pair of parties $P_i, P_j \in S_q \setminus \mathcal{Z}^*$ and every $P_k \in S_q$, the view generated when $P_i$ gives $\text{ICSig}([\text{sid}_{i,j}^{(P, q)}, P_j, P_i, P_k, v])$ to $P_j$:*** We consider the following two cases.
  - $S_q \in S^*$: In both the real and simulated execution, the value of $v$ is $[s_q]$. Since the simulator simulates the interaction of honest parties with Adv during the simulated execution, the distribution of messages is identical in both the executions.
  - $S_q \notin S^*$: In the simulated execution, the simulator chooses $v$ to be a random element from $\mathbb{F}$, while in the real execution, $v$ is $[s_q]$. However, due to the privacy property of $\Pi_{\text{Auth}}$ (Claim C.2), the view of Adv is independent of the value of $v$ in either of the executions. Hence, the distribution of the messages is identical in both the executions.

- **For every $S_q \in \mathcal{S}$ and every $P_i, P_j \in S_q$, the outputs $(P_i, \text{Acast}, \text{sid}_{i,j}^{(P, q)}, \text{OK}_{q}(i, j))$ of the pairwise consistency tests, received from $\mathcal{F}_{\text{Acast}}$ with $\text{sid}_{i,j}^{(P, q)}$:*** To compare the distribution of these messages in the two executions, we consider the following cases, considering an arbitrary $S_q \in \mathcal{S}$ and arbitrary $P_i, P_j \in S_q$.
  - $P_i, P_j \in S_q \setminus \mathcal{Z}^*$: In both the executions, Adv receives $(P_i, \text{Acast}, \text{sid}_{i,j}^{(P, q)}, \text{OK}_{q}(i, j))$ as the output from $\mathcal{F}_{\text{Acast}}$ with $\text{sid}_{i,j}^{(P, q)}$.
  - $P_i \in S_q \setminus \mathcal{Z}^*, P_j \in (S_q \cap \mathcal{Z}^*)$: In both the executions, Adv receives $(P_i, \text{Acast}, \text{sid}_{i,j}^{(P, q)}, \text{OK}_{q}(i, j))$ as the output from $\mathcal{F}_{\text{Acast}}$ with $\text{sid}_{i,j}^{(P, q)}$ if and only if Adv gave $\text{ICSig}([\text{sid}_{i,j,k}^{(P, q)}, P_j, P_i, P_k, s_{ij}])$ on behalf of $P_j$ to $P_i$, corresponding to every $P_k \in S_q$, such that $s_{ij} = [s_q]$ holds.
  - $P_i \in (S_q \cap \mathcal{Z}^*)$: In both the executions, Adv receives $(P_i, \text{Acast}, \text{sid}_{i,j}^{(P, q)}, \text{OK}_{q}(i, j))$ if and only if Adv on the behalf of $P_i$ has sent $(P_i, \text{Acast}, \text{sid}_{i,j}^{(P, q)}, \text{OK}_{q}(i, j))$ to $\mathcal{F}_{\text{Acast}}$ with $\text{sid}_{i,j}^{(P, q)}$ for $P_j$.

Clearly, irrespective of the case, the distribution of the OK messages is identical in both the executions.

- **The core sets $\{C_q\}_{S_q \in \mathcal{S}}$:*** In both the executions, the sets $C_q$ are determined based on the OK messages delivered to $P_D$. So the distribution of these sets is also identical.

- **Corresponding to every $S_q \in S^*$, for every triplet of parties $P_i, P_j, P_k$ where $P_i \in C_q \setminus \mathcal{Z}^*, P_j \in (S_q \setminus \mathcal{C}_q) \cap \mathcal{Z}^*$ and $P_k \in C_q$, the signatures $\text{ICSig}([\text{sid}_{i,j}^{(P, q)}, P_k, P_i, P_j, s_{ij}])$ revealed by party $P_i$ to $P_j$, signed by party $P_k$:*** We note that the distribution of core sets $C_q$ is the same in both the executions. In the real execution, $P_i$, upon receiving $\text{ICSig}([\text{sid}_{i,j,k}^{(P, q)}, P_k, P_i, P_j, s_{ij}])$ during $\Pi_{\text{Auth}}$, checks if $s_{ij} = [s_q]$ holds, before adding the edge $(P_i, P_k)$ in $G_q^{i}$. Since $P_D$ is honest, $s_{qi} = [s_q]$. In the simulated execution as well, the simulator reveals
ICSig\((\text{sid}_{k,j,i}, P_k, P_j, P_i, s_{qk})\) to Adv, where \(s_{qk} = [s]_q\). Hence, the distribution of messages is identical in both executions.

We next claim that if the dealer is honest, then conditioned on the view of the adversary Adv (which is identically distributed in both the executions, as per the previous claim), the outputs of the honest parties are identically distributed in both the executions.

**Claim C.7.** If \(P_D\) is honest, then conditioned on the view of Adv, the output of the honest parties during the execution of \(\Pi_{SVSS}\) involving Adv has the same distribution as the output of the honest parties in the ideal-world involving \(\mathbf{SVSS}\) and \(\mathbf{PVSS}\), except with probability at most \(|Z| \cdot n^3 \cdot \epsilon_{AICP}\), where \(\epsilon_{AICP} \approx \frac{\epsilon}{|Z|}\).

**Proof.** Let \(P_D\) be honest and let View be an arbitrary view of Adv. Moreover, let \(\{s_q\}_{s_q \cap Z^* = \emptyset}\) be the shares of the corrupt parties, as per View. Furthermore, let \(\{s_q\}_{s_q \cap Z^* = \emptyset}\) be the shares used by \(P_D\) in the simulated execution corresponding to the set \(S_q \in \set S\), such that \(S_q \cap Z^* = \emptyset\). Let \(s = \sum s_q\). Then, in the simulated execution, each honest party \(P_i\) obtains the output \([s]_q\) from \(\mathbf{PVSS}\), where \([s]_q = s_q\). We now show that except with probability at most \(|Z| \cdot n^3 \cdot \epsilon_{AICP}\), each honest \(P_i\) eventually obtains the output \([s]_q\) in the real execution as well, if \(P_D\)'s inputs in the protocol \(\Pi_{SVSS}\) are \(\{s_q\}_{s_q \in \set S}\).

Since \(P_D\) is honest, it sends the share \(s_q\) to all the parties in the set \(S_q\), which is eventually delivered. Now consider any \(S_q \in \set S\). During the pairwise consistency tests, each honest \(P_k \in S_q\) will eventually send ICSig\((\text{sid}_{k,j,m}, P_k, P_j, P_m, s_{qk})\) to all the parties \(P_j \in S_q\), with respect to every receiver \(P_m \in \mathcal P\), where \(s_{qk} = s_q\). Consequently, every honest \(P_j \in S_q\) will eventually broadcast the OK\(_q(j, k)\) message, corresponding to every honest \(P_k \in S_q\). This is because, by the correctness of AICP (Claim C.1), \(P_j\) will receive \(s_{qk}\), and \(s_{qj} = s_q = s_{qk}\) will hold. So, every honest party (including \(P_D\)) eventually receives the OK\(_q(j, k)\) messages. This implies that the parties in \(S_q \setminus Z^*\) will eventually form a clique in the graph \(G^{(i)}\) of every honest \(P_i\). This further implies that \(P_D\) will eventually find a set \(C_q\) such that \(S_q \setminus C_q \in Z\) and where \(C_q\) constitutes a clique in the consistency graph of every honest party. This is because the set \(S_q \setminus Z^*\) is guaranteed to eventually constitute a clique. Hence, \(P_D\) eventually broadcasts the sets \(\{C_q\}_{C_q \in \set S}\), which are eventually delivered to every honest party. Moreover, the verification of these sets will eventually be successful for every honest party.

Next consider an arbitrary \(S_q\) and an arbitrary honest \(P_i \in S_q\). If \(P_i \in C_q\), then it has already received the share \(s_{qi}\) from \(P_D\) and \(s_{qi} = s_q\) holds. Hence, \(P_i\) sets \([s]_q\) to \(s_q\). So consider the case when \(P_i \notin C_q\). In this case, \(P_i\) waits to find some \(P_j \in C_q\) such that \(P_i\) accepts the signature ICSig\((\text{sid}_{k,j,i}, P_k, P_j, P_i, s_{qj})\) from intermediary \(P_j\), corresponding to every signer \(P_k \in C_q\) and upon finding such a \(P_j\), party \(P_i\) sets \([s]_q\) to \(s_{qj}\). We show that except with probability at most \(n \cdot \epsilon_{AICP}\), party \(P_i\) will eventually find a candidate \(P_j\) satisfying the above condition. Moreover, if \(P_i\) finds a candidate \(P_j\) satisfying the above condition, then except with probability at most \(n \cdot \epsilon_{AICP}\), the condition \(s_{qj} = s_q\) holds. As \(P_i\) can have up to \(O(n)\) candidates for \(P_j\), it will follow from the union bound that except with probability at most \(n^2 \cdot \epsilon_{AICP}\), party \(P_i\) will eventually compute \([s]_q\). Now assuming these statements are true, the proof follows from the union bound and the fact that \(S_q\) can be any set out of \(Z\) subsets in \(\set S\) and for any \(S_q\), there could be up to \(O(n)\) honest parties \(P_i \in S_q \setminus C_q\). We next proceed to prove the above two statements.

Since \(\set S\) satisfies the \(Q^{(2)}(\set S, Z)\) condition and \(S_q \setminus C_q \in Z\), it follows that \(Z\) satisfies the \(Q^{(1)}(C_q, Z)\) condition and hence \(C_q\) contains at least one honest party, say \(P_h\). Consider any arbitrary \(P_k \in C_q\). From the protocol steps, \(P_h\) has broadcasted the OK\(_q(h, k)\) after receiving ICSig\((\text{sid}_{k,h,i}, P_k, P_h, P_i, s_{qh})\) from \(P_k\) during \(\Pi_{Auth}\) and verifying that \(s_{qk} = s_{qh}\), where \(s_{qh} = s_q\). It then follows from Lemma [4.1] that except with probability at most \(\epsilon_{AICP}\), party \(P_i\) will accept the signature ICSig\((\text{sid}_{k,h,i}, P_k, P_h, P_i, s_{qh})\) revealed by \(P_h\). Hence, except with probability at most \(n \cdot \epsilon_{AICP}\), party \(P_i\) will eventually accept the signature ICSig\((\text{sid}_{k,h,i}, P_k, P_h, P_i, s_{qh})\) corresponding to all \(P_k \in C_q\), revealed by \(P_h\).

Finally, consider an arbitrary \(P_j \in C_q\), such that \(P_j\) has accepted the signature ICSig\((\text{sid}_{k,j,i}, P_k, P_j, P_i, s_{qj})\) corresponding to all \(P_k \in C_q\) and sets \([s]_q\) to \(s_{qj}\). Now one of these signatures corresponds to the signer \(P_k = P_h\). If \(P_j\) is corrupt, then it follows from Lemma [4.1] that except with probability at most \(\epsilon_{AICP}\), the condition \(s_{qj} = s_{qh}\)
holds. As there can be up to $O(n)$ honest parties $P_h$ in $C_q$, it follows that $P_j$ will fail to reveal signature of any honest party from $C_q$ on any $s_{qj} \neq s_q$, except with probability at most $n \cdot \epsilon_{AICP}$. Since there can be up to $O(n)$ corrupt parties $P_j \in C_q$, it then follows from the union bound that except with error probability $n^2 \cdot \epsilon_{AICP}$, no corrupt party from $C_q$ will be able to forge the signature of any honest party from $C_q$ on an incorrect $s_q$. 

We next prove certain claims with respect to a corrupt dealer. The first claim is that the view of Adv in this case is also identically distributed in both the real as well as simulated execution. This is simply because in this case, the honest parties have no inputs and the simulator simply plays the role of the honest parties, exactly as per the steps of the protocol $\Pi_{SVSS}$ in the simulated execution.

**Claim C.8.** If $P_D$ is corrupt, then the view of Adv in the simulated execution of $\Pi_{SVSS}$ with $S_{PVSS}$ is identically distributed to the view of Adv in the real execution of $\Pi_{SVSS}$ involving honest parties.

**Proof.** The proof follows from the fact that if $P_D$ is corrupt, then $S_{PVSS}$ participates in a full execution of the protocol $\Pi_{SVSS}$ by playing the role of the honest parties as per the steps of $\Pi_{SVSS}$. Hence, there is a one-to-one correspondence between simulated executions and real executions. 

We finally claim that if the dealer is corrupt, then conditioned on the view of the adversary (which is identical in both the executions as per the last claim), the outputs of the honest parties are identically distributed in both the executions.

**Claim C.9.** If $D$ is corrupt, then conditioned on the view of Adv, the output of the honest parties during the execution of $\Pi_{SVSS}$ involving Adv has the same distribution as the output of the honest parties in the ideal-world involving $S_{PVSS}$ and $F_{VSS}$, except with probability at most $|Z| \cdot n^3 \cdot \epsilon_{AICP}$, where $\epsilon_{AICP} \approx \frac{n^2}{|P|}$.

**Proof.** Let $P_D$ be corrupt and let $View$ be an arbitrary view of Adv. We note that it can be found out from $View$ whether valid core sets $\{C_q\}_{S_q \in \mathbb{S}}$ have been generated during the corresponding execution of $\Pi_{SVSS}$ or not. We now consider the following cases.

- **No core sets $\{C_q\}_{S_q \in \mathbb{S}}$ are generated as per View:** In this case, the honest parties do not obtain any output in either execution. This is because in the real execution of $\Pi_{SVSS}$, the honest parties compute their output only when they get valid core sets $\{C_q\}_{S_q \in \mathbb{S}}$ from $P_D$’s broadcast. If this is not the case, then in the simulated execution, the simulator $S_{PVSS}$ does not provide any input to $F_{VSS}$ on behalf of $P_D$; hence, $F_{VSS}$ does not produce any output for the honest parties.

- **Core sets $\{C_q\}_{S_q \in \mathbb{S}}$ generated as per View are invalid:** Again, in this case, the honest parties do not obtain any output in either execution. This is because in the real execution of $\Pi_{SVSS}$, even if the sets $\{C_q\}_{S_q \in \mathbb{S}}$ are received from $P_D$’s broadcast, the honest parties compute their output only when each $C_q$ set is found to be valid with respect to the verifications performed by the honest parties in their own consistency graphs. If these verifications fail (implying that the core sets are invalid), then in the simulated execution, the simulator $S_{PVSS}$ does not provide any input to $F_{VSS}$ on behalf of $P_D$, implying that $F_{VSS}$ does not produce any output for the honest parties.

- **Valid core sets $\{C_q\}_{S_q \in \mathbb{S}}$ are generated as per View:** We first note that in this case, $P_D$ has distributed some common share, say $s_q$, as determined by View, to all the parties in $C_q \setminus Z^*$, during the real execution of $\Pi_{SVSS}$. This is because all the parties in $C_q \setminus Z^*$ are honest, and form a clique in the consistency graph of the honest parties. Hence, each $P_j, P_k \in C_q \setminus Z^*$ has broadcasted the messages $OK_q(j,k)$ and $OK_q(k,j)$ after checking that $s_{qj} = s_{qk}$ holds, where $s_{qj}$ and $s_{qk}$ are the values received from $P_D$ by $P_j$ and $P_k$ respectively. We next show that in the real execution of $\Pi_{SVSS}$, except with probability at most $n^3 \cdot \epsilon_{AICP}$, all honest parties in $S_q \setminus Z^*$ eventually set $[s]_q$ to $s_q$. While this is obviously true for the parties in $C_q \setminus Z^*$, the proof when $P_i \in S_q \setminus (Z^* \cup C_q)$ is exactly the same, as in Claim C.7.

Since $|S| = |Z|$, it then follows that in the real execution, except with probability at most $n^3 \cdot \epsilon_{AICP}$, every honest party $P_i$ eventually outputs $\{[s]_q = s_q\}_{P_i \in S_q}$. From the steps of $S_{PVSS}$, the simulator sends the shares $\{s_q\}_{S_q \in \mathbb{S}}$ to $F_{VSS}$ on the behalf of $P_D$ in the simulated execution. Consequently, in the simulated execution,
Non-robust basic multiplication protocol in the C.3 The Basic Multiplication Protocol

\[ \Pi \]

Protocol \( \Pi \) is almost the same as the protocol \( \Pi_{\text{OptMult}} \), except that it does not take any subset \( Z \in \mathcal{Z} \) as input. Consequently, the various dynamic sets and session ids maintained in the protocol will not be notated with \( Z \) (unlike the protocol \( \Pi_{\text{OptMult}} \)).

\[
\text{Initialization: Initialize } \text{Summands}_{\text{iter}} = \{(p,q)\}_{p,q=1,...,|\mathcal{Z}|}, \text{Selected}_{\text{iter}} = \emptyset, \text{hop} = 1 \text{ and corresponding to each } P_j \in \mathcal{P} \setminus \mathcal{G} \mathcal{D}, \text{set } \text{Summands}^{(j)}_{\text{iter}} = \{(p,q)\}_{p \in S_p \cap S_q}.
\]

- Do the following till \( \text{Summands}_{\text{iter}} \neq \emptyset ):
  - **Sharing Summands:** Same as in \( \Pi_{\text{OptMult}} \), except that \( P \) randomly secret-shares \( c^{(i)}_{\text{iter}} = \sum_{(p,q) \in \text{Summands}^{(i)}_{\text{iter}}} [a]_p [b]_q \)
    
    by calling \( F_{\text{VSS}} \) with \( \text{sid}_{\text{hop},i} \overset{\text{def}}{=} \text{sid}[\text{hop}]|i \), if \( P \notin \text{Selected}_{\text{iter}} \).
  - **Selecting Summand-Sharing Party Through ACS:** Same as in \( \Pi_{\text{OptMult}} \), except that \((\text{vote}, \text{sid}_{\text{hop},j}, 1)\) is sent to \( F_{\text{ABA}} \) with \( \text{sid}_{\text{hop},j} \) corresponding to any \( P_j \in \mathcal{P} \), if all the following hold:
    - \( P_j \notin \mathcal{G} \mathcal{D} \), \( P_j \notin \text{Selected}_{\text{iter}} \) and an output \( \text{share, sid}_{\text{hop},j}, \{(c^{(j)}_{\text{iter}})_{p \in S_p}\} \) is received from \( F_{\text{VSS}} \) with \( \text{sid}_{\text{hop},j} \), corresponding to the dealer \( P_j \).
    - If \( P_j \) is selected as common summand-sharing party for this hop, then update the following:
      - \( \text{Selected}_{\text{iter}} = \text{Selected}_{\text{iter}} \cup \{P_j\} \).
      - \( \text{Summands}_{\text{iter}} = \text{Summands}_{\text{iter}} \setminus \text{Summands}^{(j)}_{\text{iter}} \).
      - \( \forall P_k \in \mathcal{P} \setminus \{\mathcal{G} \mathcal{D} \cup \text{Selected}_{\text{iter}}\}, \text{Summands}^{(k)}_{\text{iter}} = \text{Summands}^{(k)}_{\text{iter}} \setminus \text{Summands}^{(j)}_{\text{iter}} \).
      - \( \text{hop} = \text{hop} + 1 \).
    - \( \forall P_j \in \mathcal{P} \setminus \text{Selected}_{\text{iter}}, \) participate in an instance of \( \Pi_{\text{perDefSh}} \), with public input \( c^{(j)}_{\text{iter}} \).
  - **Output:** Set \( c_{\text{iter}} \overset{\text{def}}{=} c^{(1)}_{\text{iter}} + \ldots + c^{(n)}_{\text{iter}} \). Output \( \{[c^{(1)}_{\text{iter}}]_q, \ldots, [c^{(n)}_{\text{iter}}]_q, [c_{\text{iter}}]_q\} \) for all \( P \).

Figure 19: Non-robust basic multiplication protocol in the \( (F_{\text{VSS}}, F_{\text{ABA}}) \)-hybrid model for session id sid. The above code is executed by every party \( P \).

We next formally prove the properties of the protocol \( \Pi_{\text{BasicMult}} \). While proving these properties, we will assume that \( \mathcal{Z} \) satisfies the \( Q_3^{(3)}(\mathcal{P}, \mathcal{Z}) \) condition. This further implies that the sharing specification \( S = (S_1, \ldots, S_h) \overset{\text{def}}{=} \{P \setminus Z | Z \in \mathcal{Z} \} \) satisfies the \( Q_2^{(2)}(S, \mathcal{Z}) \) condition. Moreover, while proving these properties, we assume that no honest party is ever included in the set \( \mathcal{G} \mathcal{D} \). Note that this will be ensured in the protocol \( \Pi_{\text{RandMultCI}} \) where \( \Pi_{\text{BasicMult}} \) is used as a subprotocol. We first show that the intersection of any two sets in \( S \) contains at least one honest party outside \( \mathcal{G} \mathcal{D} \).

**Claim C.10.** For every \( Z \in \mathcal{Z} \) and every ordered pair \( (p,q) \in \{1,\ldots,h\} \times \{1,\ldots,h\} \), the set \( (S_p \cap S_q) \setminus \mathcal{G} \mathcal{D} \) contains at least one honest party.

**Proof.** From the definition of the sharing specification \( S \), we have \( S_p = \mathcal{P} \setminus Z_p \) and \( S_q = \mathcal{P} \setminus Z_q \), where \( Z_p, Z_q \in \mathcal{Z} \).

Let \( Z^* \in \mathcal{Z} \) be the set of corrupt parties during the protocol \( \Pi_{\text{BasicMult}} \). Now, \( S_p \cap S_q = (\mathcal{P} \setminus Z_p) \cap (\mathcal{P} \setminus Z_q) = \mathcal{P} \setminus (Z_p \cup Z_q) \). This means that \( (S_p \cap S_q) \cup Z_p \cup Z_q = \mathcal{P} \). If \( (S_p \cap S_q) \subseteq Z^* \), then \( Z^* \cup Z_p \cup Z_q = \mathcal{P} \). This is a violation of the \( Q_3^{(3)}(\mathcal{P}, \mathcal{Z}) \) condition, and hence, \( S_p \cap S_q \) contains at least one honest party. Since \( \mathcal{G} \mathcal{D} \) contains only corrupt parties, \( (S_p \cap S_q) \setminus \mathcal{G} \mathcal{D} \) contains at least one honest party. 

\[ \square \]
We next claim a series of properties related to protocol $\Pi_{\text{BasicMult}}$ whose proofs are almost identical to the proof of the corresponding properties for protocol $\Pi_{\text{OptMult}}$. Hence, we skip the formal proofs.

**Claim C.11.** For any $\iter$, if all honest parties participate during the hop number hop in the protocol $\Pi_{\text{BasicMult}}(\mathcal{P}, \mathcal{Z}, \mathcal{S}, [a], [b], \iter)$, then all honest parties eventually obtain a common summand-sharing party, say $P_j$, for this hop, such that the honest parties will eventually hold $c_i^{(j)}$. Moreover, party $P_j$ will be distinct from the summand-sharing party selected for any hop number hop$' <$ hop.

**Proof.** The proof is identical to that of Claim B.10, except that we now use Claim C.10 to argue that for every ordered pair $(p, q) \in \text{Summands}_{\iter}$, there exists at least one honest party in $(S_p \cap S_q) \setminus \GD$, say $P_k$, who will have both the shares $[a]_p$ as well as $[b]_q$ (and hence the summand $[a]_p [b]_q$).

**Claim C.12.** In protocol $\Pi_{\text{BasicMult}}$, all honest parties eventually obtain an output. The protocol makes $O(n^2)$ calls to $F_{\text{VSS}}$ and $F_{\text{ABA}}$.

**Proof.** The proof is similar to that of Claim B.11.

**Claim C.13.** During protocol $\Pi_{\text{BasicMult}}$, $\Adv$ learns nothing about $a$ and $b$.

**Proof.** The proof is similar to that of Claim B.13.

**Claim C.14.** In $\Pi_{\text{BasicMult}}$, if no party in $\mathcal{P} \setminus \GD$ behaves maliciously, then for each $P_i \in \text{Selected}_{\iter}$, the condition $c_i = \sum_{(p, q) \in \text{Summands}_{\iter}} [a]_p [b]_q$ holds, which further implies that $c = ab$ holds.

**Proof.** The proof is similar to that of Claim B.12.

Lemma C.15 now follows from Claims C.10-C.14.

**Lemma C.15.** Let $\mathcal{Z}$ satisfy the $Q(3)(\mathcal{P}, \mathcal{Z})$ condition and let $\mathcal{S} = (S_1, \ldots, S_h) = \{\mathcal{P} \setminus \mathcal{Z} | \mathcal{Z} \in \mathcal{Z}\}$. Consider an arbitrary $\iter$, such that all honest parties participate in the instance $\Pi_{\text{BasicMult}}(\mathcal{P}, \mathcal{Z}, \mathcal{S}, [a], [b], \GD, \iter)$. Then all honest parties eventually compute $[c_\iter]$ and $[c_\iter^{(1)}], \ldots, [c_\iter^{(n)}]$ where $c_\iter = c_\iter^{(1)} + \ldots + c_\iter^{(n)}$, provided no honest party is ever included in the $\GD$. If no party in $\mathcal{P} \setminus \GD$ behaves maliciously, then $c_\iter = ab$ holds. In the protocol, $\Adv$ does not learn any additional information about $a$ and $b$. The protocol makes $O(n^2)$ calls to $F_{\text{VSS}}$ and $F_{\text{ABA}}$.

We claim another property of $\Pi_{\text{BasicMult}}$, which will be useful later while analyzing the properties of $\Pi_{\text{RandMultCI}}$, where $\Pi_{\text{BasicMult}}$ is used as a sub-protocol.

**Claim C.16.** For any $\iter$, if $P_j \in \text{Selected}_{\iter}$ during the instance $\Pi_{\text{BasicMult}}(\mathcal{P}, \mathcal{Z}, \mathcal{S}, [a], [b], \GD, \iter)$, then $P_j \not\in \GD$.

**Proof.** The proof is similar to that of Claim B.14.

We finally end this section by discussing the modifications to the protocol $\Pi_{\text{BasicMult}}$ for handling $M$ pairs of inputs.

**Protocol $\Pi_{\text{BasicMult}}$ for $M$ pairs of inputs:** Protocol $\Pi_{\text{BasicMult}}$ can be easily modified if executed with input $\{(a^{(el)}, b^{(el)})\}_{e=1,...,M}$. The modifications will be along similar lines to those done for $\Pi_{\text{OptMult}}$. Consequently, there will be $O(n^2 M)$ calls to $F_{\text{VSS}}$, but only $O(n^2)$ calls to $F_{\text{ABA}}$.

### C.4 Protocol $\Pi_{\text{RandMultCI}}$ for Detectable Random-Triple Generation and Its Properties

Protocol $\Pi_{\text{RandMultCI}}$ for one triple is formally presented in Fig 20.
Protocol $\Pi_{\text{RandMultCI}}(\mathcal{P}, Z, S, \mathcal{GD}, \text{iter})$

- Generating Secret-Sharing of Random Values: The parties jointly generate $[a_{\text{iter}}], [b_{\text{iter}}], [b'_{\text{iter}}]$ and $[r_{\text{iter}}]$, where $a_{\text{iter}}, b_{\text{iter}}, b'_{\text{iter}}$, and $r_{\text{iter}}$ are random from the view of $\text{Adv}$, by using a similar procedure as in $\Pi_{\text{PerTriples}}$. For this, each $P_i \in \mathcal{P}$ acts as a dealer, picks random $a_{i,\text{iter}}, b_{i,\text{iter}}, b'_{i,\text{iter}}, r_{i,\text{iter}}$ from $\mathbb{F}$ and generates random $[c_{i,\text{iter}}], [d_{i,\text{iter}}], [e_{i,\text{iter}}], [f_{i,\text{iter}}]$, and $[r_{i,\text{iter}}]$, by making calls to $\mathcal{F}_{\text{VSS}}$. The parties then agree on a common subset of parties $CS$ through ACS as in $\Pi_{\text{PerTriples}}$, such that $\mathcal{P} \setminus CS \in Z$ and for each $P_j \in CS$, the honest parties eventually hold $[a_{j,\text{iter}}], [b_{j,\text{iter}}], [b'_{j,\text{iter}}]$ and $[r_{j,\text{iter}}]$. The parties then set

$$[a_{\text{iter}}] = \sum_{P_j \in CS} [a_{j,\text{iter}}], \quad [b_{\text{iter}}] = \sum_{P_j \in CS} [b_{j,\text{iter}}], \quad [b'_{\text{iter}}] = \sum_{P_j \in CS} [b'_{j,\text{iter}}], \quad \text{and} \quad [r_{\text{iter}}] = \sum_{P_j \in CS} [r_{j,\text{iter}}].$$

- Running Multiplication Protocol and Reconstructing the Random Challenge:
  - The parties participate in instances $\Pi_{\text{BasicMult}}(\mathcal{P}, Z, S, [a_{\text{iter}}], [b_{\text{iter}}], \mathcal{GD}, \text{iter})$ and $\Pi_{\text{BasicMult}}(\mathcal{P}, Z, S, [a_{\text{iter}}], [b'_{\text{iter}}], \mathcal{GD}, \text{iter})$ to get outputs $\{ [c_{1,\text{iter}}], \ldots, [c_{n,\text{iter}}], [e_{\text{iter}}] \}$ and $\{ [d_{1,\text{iter}}], \ldots, [d_{n,\text{iter}}], [e'_{\text{iter}}] \}$ respectively. Let $\text{Selected}_{\text{iter}, c}$ and $\text{Selected}_{\text{iter}, c'}$ be the summand-sharing parties for the two instances respectively. Moreover, for $P_j \in \text{Selected}_{\text{iter}, c}$, let $\text{Summands}_{\text{iter}, c}$ be the set of ordered pairs of indices corresponding to the summands whose sum has been shared by $P_j$ during the instance $\Pi_{\text{BasicMult}}(\mathcal{P}, Z, S, [a_{\text{iter}}], [b_{\text{iter}}], \mathcal{GD}, \text{iter})$. Similarly, for $P_j \in \text{Selected}_{\text{iter}, c'}$, let $\text{Summands}_{\text{iter}, c'}$ be the set of ordered pairs of indices corresponding to the summands whose sum has been shared by $P_j$ during the instance $\Pi_{\text{BasicMult}}(\mathcal{P}, Z, S, [a_{\text{iter}}], [b'_{\text{iter}}], \mathcal{GD}, \text{iter})$.
  - Once the parties obtain their respective outputs from the instances of $\Pi_{\text{BasicMult}}$, they participate in an instance of $\Pi_{\text{PerRec}}$ with shares corresponding to $[r_{\text{iter}}]$, to reconstruct $r_{\text{iter}}$.

- Detecting Errors in Instances of $\Pi_{\text{BasicMult}}$:
  - The parties locally compute $[e_{\text{iter}}] = r_{\text{iter}}[b_{\text{iter}}] + [b'_{\text{iter}}]$ and then participate in an instance of $\Pi_{\text{PerRec}}$ with shares corresponding to $[e_{\text{iter}}]$, to reconstruct $e_{\text{iter}}$.
  - The parties locally compute $[d_{\text{iter}}] = e_{\text{iter}}[a_{\text{iter}}] - r_{\text{iter}}[c_{\text{iter}}] - [e_{\text{iter}}]$ and then participate in an instance of $\Pi_{\text{PerRec}}$ with shares corresponding to $[d_{\text{iter}}]$, to reconstruct $d_{\text{iter}}$.

- Output Computation in Case of Success: If $d_{\text{iter}} \neq 0$, then every party $P_i \in \mathcal{P}$ sets the Boolean variable $\text{flag}_{\text{iter}} = 1$ and outputs $\{ ([a_{i,\text{iter}}], [b_{i,\text{iter}}], [b'_{i,\text{iter}}]) \}_{P_i \in S_{\text{iter}}}$. Cheater Identification in Case of Failure: If $d_{\text{iter}} \neq 0$, then every party $P_i \in \mathcal{P}$ sets the Boolean variable $\text{flag}_{\text{iter}} = 1$ and proceeds as follows.
  - Participate in appropriate instances of $\Pi_{\text{PerRecShare}}$ to reconstruct the shares $\{ ([a_{i,\text{iter}}], [b_{i,\text{iter}}], [b'_{i,\text{iter}}]) \}_{P_i \in S_{\text{iter}}}$ and appropriate instances of $\Pi_{\text{PerRec}}$ to reconstruct $c_{\text{iter}}, c_{\text{iter}}, c'_{\text{iter}}, \ldots, c_{\text{iter}}$.
  - Set $\mathcal{GD} = \mathcal{GD} \cup \{ P_i \}$, if $P_i \in \text{Selected}_{\text{iter}, c} \cup \text{Selected}_{\text{iter}, c'}$ and the following holds for $P_i$:

$$r_{\text{iter}} \cdot c_{\text{iter}} + r_{\text{iter}}' \neq r_{\text{iter}} \sum_{(p,q) \in \text{Summands}_{\text{iter}, c}} [a_{i,\text{iter}}][b_{i,\text{iter}}]_{p,q} + \sum_{(p,q) \in \text{Summands}_{\text{iter}, c'}} [a_{i,\text{iter}}][b'_{i,\text{iter}}]_{p,q}.$$

Figure 20: Detectable triple generation protocol in the $(\mathcal{F}_{\text{VSS}}, \mathcal{F}_{\text{ABA}})$-hybrid model

We now formally prove the properties of the protocol $\Pi_{\text{RandMultCI}}$. While proving these properties, we will assume that $Z$ satisfies the $Q(3)(\mathcal{P}, Z)$ condition. This further implies that $S = (S_1, \ldots, S_h) \triangleq \{ \mathcal{P} \setminus Z | Z \in Z \}$ satisfies the $Q(2)(S, Z)$ condition.

We first claim that the honest parties eventually compute $[a_{\text{iter}}], [b_{\text{iter}}], [b'_{\text{iter}}]$ and $[r_{\text{iter}}]$.

**Claim C.17.** Consider an arbitrary iter, such that all honest parties participate in the instance $\Pi_{\text{RandMultCI}}(\mathcal{P}, Z, S, \mathcal{GD}, \text{iter})$, where no honest party is present in $\mathcal{GD}$. Then the honest parties eventually compute $[a_{\text{iter}}], [b_{\text{iter}}], [b'_{\text{iter}}]$ and $[r_{\text{iter}}]$.

**Proof.** The proof is similar to the proof of Claim [B.32].

We next claim that all honest parties will eventually agree on whether the instances of $\Pi_{\text{BasicMult}}$ in $\Pi_{\text{RandMultCI}}$ has succeeded or failed.
Claim C.18. Consider an arbitrary $\text{iter}$, such that all honest parties participate in the instance $\Pi_{\text{RandMultCl}}(P, Z, S, GD, \text{iter})$, where no honest party is present in $\mathcal{G}D$. Then all honest parties eventually hold the outputs $d_{\text{iter}}$. Consequently, each honest $P_i$ eventually sets $\text{flag}_{\text{iter}}^{(i)}$ to either 0 or 1.

Proof. From Claim C.17 the honest parties eventually hold $[a_{\text{iter}}], [b_{\text{iter}}], [b'_{\text{iter}}]$ and $[r_{\text{iter}}]$. From Lemma C.15 it follows that the honest parties eventually hold the outputs $\{[c^{(1)}_{\text{iter}}], \ldots, [c^{(n)}_{\text{iter}}], [c_{\text{iter}}]\}$ and $\{[c'^{(1)}_{\text{iter}}], \ldots, [c'^{(n)}_{\text{iter}}], [c'_{\text{iter}}]\}$ from the corresponding instances of $\Pi_{\text{BasicMult}}$. From Lemma B.8 the honest parties eventually reconstruct $r_{\text{iter}}$ from the corresponding instance of $\Pi_{\text{PerRec}}$. From the linearity property of secret-sharing, it then follows that the honest parties eventually hold $c_{\text{iter}}$, and hence eventually reconstruct $c_{\text{iter}}$ from the corresponding instance of $\Pi_{\text{PerRec}}$. Again, from the linearity property of secret-sharing, it follows that the honest parties eventually hold $d_{\text{iter}}$, followed by eventually reconstructing $d_{\text{iter}}$ from the corresponding instance of $\Pi_{\text{PerRec}}$. Now based on whether $d_{\text{iter}}$ is 0 or not, each honest $P_i$ eventually sets $\text{flag}_{\text{iter}}^{(i)}$ to either 0 or 1.

We next claim that if no party in $P \setminus \mathcal{G}D$ behaves maliciously, then the honest parties eventually hold a secretly-shared multiplication-triple.

Claim C.19. Consider an arbitrary $\text{iter}$, such that all honest parties participate in the instance $\Pi_{\text{RandMultCl}}(P, Z, S, GD, \text{iter})$, where no honest party is present in $\mathcal{G}D$. If no party in $P \setminus \mathcal{G}D$ behaves maliciously, then $d_{\text{iter}} = 0$ and the honest parties eventually hold $([a_{\text{iter}}], [b_{\text{iter}}], [c_{\text{iter}}])$, where $c_{\text{iter}} = a_{\text{iter}} \cdot b_{\text{iter}}$ holds.

Proof. If no party in $P \setminus \mathcal{G}D$ behaves maliciously, then from Lemma C.15 the honest parties eventually compute $[c_{\text{iter}}]$ and $[c'_{\text{iter}}]$ from the respective instances of $\Pi_{\text{BasicMult}}$, such that $c_{\text{iter}} = a_{\text{iter}} \cdot b_{\text{iter}}$ and $c'_{\text{iter}} = a_{\text{iter}} \cdot b'_{{\text{iter}}}$ holds. From Claim C.18 the honest parties will eventually reconstruct $d_{\text{iter}}$. Moreover, since $c_{\text{iter}} = a_{\text{iter}} \cdot b_{\text{iter}}$ and $c'_{\text{iter}} = a_{\text{iter}} \cdot b'_{{\text{iter}}}$ holds, the value $d_{\text{iter}}$ will be 0 and consequently, the honest parties will output $([a_{\text{iter}}], [b_{\text{iter}}], [c_{\text{iter}}])$.

We next show that if $d_{\text{iter}} \neq 0$, then the honest parties eventually include at least one new maliciously-corrupt party in the set $\mathcal{G}D$.

Claim C.20. Consider an arbitrary $\text{iter}$, such that all honest parties participate in the instance $\Pi_{\text{RandMultCl}}(P, Z, S, GD, \text{iter})$, where no honest party is present in $\mathcal{G}D$. If $d_{\text{iter}} \neq 0$, then the honest parties eventually update $\mathcal{G}D$ by adding a new maliciously-corrupt party in $\mathcal{G}D$.

Proof. Let $d_{\text{iter}} \neq 0$ and let $\text{Selected}_{\text{iter}}$ be the set of summand-sharing parties across the two instances of $\Pi_{\text{BasicMult}}$ executed in $\Pi_{\text{RandMultCl}}$; i.e. $\text{Selected}_{\text{iter}} \triangleq \text{Selected}_{\text{iter},c} \cup \text{Selected}_{\text{iter},c'}$. Note that there exists no $P_j \in \text{Selected}_{\text{iter}}$ such that $P_j \in \mathcal{G}D$, which follows from Claim C.16. We claim that there exists at least one party $P_j \in \text{Selected}_{\text{iter}}$, such that corresponding to $c_{\text{iter}}^{(j)}$ and $c'_{\text{iter}}^{(j)}$, the following holds:

$$r_{\text{iter}} \cdot c_{\text{iter}}^{(j)} + c'_{\text{iter}}^{(j)} \neq r_{\text{iter}} \cdot \sum_{(p,q) \in \text{Summands}_{\text{iter},c}} [a_{\text{iter}}]_p [b_{\text{iter}}]_q + \sum_{(p,q) \in \text{Summands}_{\text{iter},c'}} [a_{\text{iter}}]_p [b'_{\text{iter}}]_q.$$

Assuming the above holds, the proof now follows from the fact that once the parties reconstruct $d_{\text{iter}} \neq 0$, they proceed to reconstruct the shares $[[a_{\text{iter}}]_q, [b_{\text{iter}}]_q, [b'_{\text{iter}}]_q]_{S_i \in \mathcal{S}}$ through appropriate instances of $\Pi_{\text{PerRecShare}}$ and the values $c_{\text{iter}}^{(1)}, \ldots, c_{\text{iter}}^{(n)}$, $c'_{\text{iter}}^{(1)}, \ldots, c'_{\text{iter}}^{(n)}$ through appropriate instances of $\Pi_{\text{PerRec}}$. Upon reconstructing these values, party $P_j$ will be eventually included in the set $\mathcal{G}D$. Moreover, it is easy to see that $P_j$ is a maliciously-corrupt party, since for every honest $P_j \in \text{Selected}_{\text{iter}}$, the condition $c_{\text{iter}}^{(j)} = \sum_{(p,q) \in \text{Summands}_{\text{iter},c}} [a_{\text{iter}}]_p [b_{\text{iter}}]_q$ and $c'_{\text{iter}}^{(j)} = \sum_{(p,q) \in \text{Summands}_{\text{iter},c'}} [a_{\text{iter}}]_p [b'_{\text{iter}}]_q$ holds.

We prove the above claim through a contradiction. So let the following condition hold for each $P_j \in \text{Selected}_{\text{iter}}$:

$$r_{\text{iter}} \cdot c_{\text{iter}}^{(j)} + c'_{\text{iter}}^{(j)} = r_{\text{iter}} \cdot \sum_{(p,q) \in \text{Summands}_{\text{iter},c}} [a_{\text{iter}}]_p [b_{\text{iter}}]_q + \sum_{(p,q) \in \text{Summands}_{\text{iter},c'}} [a_{\text{iter}}]_p [b'_{\text{iter}}]_q.$$
Next, summing the above equation over all $P_j \in \text{Selected}_{\text{iter}}$, we get that the following holds:

$$
\sum_{P_j \in \text{Selected}_{\text{iter}}} r_{\text{iter}} \cdot c_{\text{iter}}^{(j)} + c_{\text{iter}}^{(j)} = \sum_{P_j \in \text{Selected}_{\text{iter}}} r_{\text{iter}} \cdot \sum_{(p,q) \in \text{Summands}_{\text{iter},c}} [a_{\text{iter}}]_p [b_{\text{iter}}]_q + \sum_{(p,q) \in \text{Summands}_{\text{iter},c'}} [a_{\text{iter}}]_p [b_{\text{iter}}']_q.
$$

This implies that the following holds:

$$
r_{\text{iter}} \cdot \sum_{P_j \in \text{Selected}_{\text{iter}}} c_{\text{iter}}^{(j)} + c_{\text{iter}}^{(j)} = r_{\text{iter}} \cdot \sum_{(p,q) \in \text{Summands}_{\text{iter},c}} [a_{\text{iter}}]_p [b_{\text{iter}}]_q + \sum_{(p,q) \in \text{Summands}_{\text{iter},c'}} [a_{\text{iter}}]_p [b_{\text{iter}}']_q.
$$

Now based on the way $a_{\text{iter}}, b_{\text{iter}}, b_{\text{iter}}', c_{\text{iter}}$ and $c_{\text{iter}}'$ are defined, the above implies that the following holds:

$$
r_{\text{iter}} \cdot c_{\text{iter}} + c_{\text{iter}}' = r \cdot a_{\text{iter}} \cdot b_{\text{iter}} + a_{\text{iter}} \cdot b_{\text{iter}}'
$$

This further implies that

$$
r_{\text{iter}} \cdot c_{\text{iter}} + c_{\text{iter}}' = (r_{\text{iter}} \cdot b_{\text{iter}} + b_{\text{iter}}') \cdot a_{\text{iter}}
$$

Since in the protocol $e_{\text{iter}} \stackrel{def}{=} r_{\text{iter}} \cdot b_{\text{iter}} + b_{\text{iter}}'$, the above implies that

$$
r_{\text{iter}} \cdot c_{\text{iter}} + c_{\text{iter}}' = e_{\text{iter}} \cdot a_{\text{iter}} \Rightarrow e_{\text{iter}} \cdot a_{\text{iter}} - r_{\text{iter}} \cdot c_{\text{iter}} - c_{\text{iter}}' = 0 \Rightarrow d_{\text{iter}} = 0,
$$

where the last equality follows from the fact that in the protocol, $d_{\text{iter}} \stackrel{def}{=} e_{\text{iter}} \cdot a_{\text{iter}} - r_{\text{iter}} \cdot c_{\text{iter}} - c_{\text{iter}}'$. However $d_{\text{iter}} = 0$ is a contradiction, since according to the hypothesis of the claim, we are given that $d_{\text{iter}} \not= 0$. \hfill $\Box$

We next show that if the honest parties output a secret-shared triple in the protocol, then except with probability $\frac{1}{|P|}$, the triple is a multiplication-triple. Moreover, the triple will be random for the adversary.

**Claim C.21.** Consider an arbitrary iter, such that all honest parties participate in the instance $\Pi_{\text{RandMultCl}}(P, Z, S, \mathcal{G}D, \text{iter})$, where no honest party is present in $\mathcal{G}D$. If $d_{\text{iter}} = 0$, then the honest parties eventually output $([a_{\text{iter}}], [b_{\text{iter}}], [c_{\text{iter}}]),$ where except with probability $\frac{1}{|P|}$, the condition $c_{\text{iter}} = a_{\text{iter}} \cdot b_{\text{iter}}$ holds. Moreover, the view of Adv will be independent of $(a_{\text{iter}}, b_{\text{iter}}, c_{\text{iter}})$.

**Proof.** Let $d_{\text{iter}} = 0$. Then from the protocol steps, the honest parties eventually output $([a_{\text{iter}}], [b_{\text{iter}}], [c_{\text{iter}}]).$ In the protocol $d_{\text{iter}} \stackrel{def}{=} e_{\text{iter}} \cdot a_{\text{iter}} - r_{\text{iter}} \cdot c_{\text{iter}} - c_{\text{iter}}'$, where $e_{\text{iter}} \stackrel{def}{=} r_{\text{iter}} \cdot b_{\text{iter}} + b_{\text{iter}}'$. Since $d_{\text{iter}} = 0$ holds, it implies that the honest parties have verified that the following holds:

$$
r_{\text{iter}} (a_{\text{iter}} \cdot b_{\text{iter}} - c_{\text{iter}}) = (c_{\text{iter}}' - a_{\text{iter}} \cdot b_{\text{iter}}').
$$

We also note that $r_{\text{iter}}$ will be a random element from $\mathbb{F}$ and will be unknown to Adv till it is publicly reconstructed. This simply follows from the fact there will be at least one honest party $P_j$ in the set $\mathcal{C}S$, such that the corresponding value $r_{\text{iter}}^{(j)}$ shared by $P_j$ will be random from the view-point of Adv. We also note that $r_{\text{iter}}$ will be unknown to Adv, till the outputs for the underlying instances of $\Pi_{\text{BasicMult}}$ are computed, and the honest parties hold $[c_{\text{iter}}]$ and $[c_{\text{iter}}'].$ This is because in the protocol, the honest parties start participating in the instance of $\Pi_{\text{PerRec}}$ to reconstruct $r_{\text{iter}}$, only after they obtain their respective shares corresponding to $[c_{\text{iter}}]$ and $[c_{\text{iter}}'].$ Now we have the following cases with respect to whether any party from $\mathcal{P} \setminus \mathcal{G}D$ behaved maliciously during the underlying instances of $\Pi_{\text{BasicMult}}$:

- **Case I:** $c_{\text{iter}} = a_{\text{iter}} \cdot b_{\text{iter}}$ and $c_{\text{iter}}' = a_{\text{iter}} \cdot b_{\text{iter}}'$. In this case, $(a_{\text{iter}}, b_{\text{iter}}, c_{\text{iter}})$ is a multiplication-triple.
- **Case II:** $c_{\text{iter}} = a_{\text{iter}} \cdot b_{\text{iter}}$, but $c_{\text{iter}}' \not= a_{\text{iter}} \cdot b_{\text{iter}}'$. This case is never possible, as this will lead to the contradiction that $r_{\text{iter}} (a_{\text{iter}} \cdot b_{\text{iter}} - c_{\text{iter}}) \not= (c_{\text{iter}}' - a_{\text{iter}} \cdot b_{\text{iter}}')$ holds.
- **Case III:** $c_{\text{iter}} \not= a_{\text{iter}} \cdot b_{\text{iter}}$, but $c_{\text{iter}}' = a_{\text{iter}} \cdot b_{\text{iter}}'$. This case is possible only if $r_{\text{iter}} = 0$, as otherwise this will lead to the contradiction that $r_{\text{iter}} (a_{\text{iter}} \cdot b_{\text{iter}} - c_{\text{iter}}) \not= (c_{\text{iter}}' - a_{\text{iter}} \cdot b_{\text{iter}}')$ holds. However, since $r_{\text{iter}}$ is a random element from $\mathbb{F}$, it implies that this case can occur only with probability at most $\frac{1}{|P|}$.
– Case IV: \( c_{\text{iter}} \neq a_{\text{iter}} \cdot b_{\text{iter}} \) as well as \( c'_{\text{iter}} \neq a_{\text{iter}} \cdot b'_{\text{iter}} \) — This case is possible only if \( r_{\text{iter}} = (c'_{\text{iter}} - a_{\text{iter}} \cdot b'_{\text{iter}}) \cdot (a_{\text{iter}} \cdot b_{\text{iter}} - c_{\text{iter}})^{-1} \), as otherwise this will lead to the contradiction that \( r_{\text{iter}}(a_{\text{iter}} \cdot b_{\text{iter}} - c_{\text{iter}}) \neq (c'_{\text{iter}} - a_{\text{iter}} \cdot b'_{\text{iter}}) \) holds. However, since \( r_{\text{iter}} \) is a random element from \( \mathbb{F} \), it implies that this case can occur only with probability at most \( \frac{1}{|\mathbb{F}|} \).

Hence, we have shown that except with probability at most \( \frac{1}{|\mathbb{F}|} \), the triple \((a_{\text{iter}}, b_{\text{iter}}, c_{\text{iter}})\) is a multiplication-triple. To complete the proof, we need to argue that the view of \( \text{Adv} \) in the protocol, will be independent of the triple \((a_{\text{iter}}, b_{\text{iter}}, c_{\text{iter}})\). For this, we first note that \( a_{\text{iter}}, b_{\text{iter}} \) and \( b'_{\text{iter}} \) will be random for the adversary. The proof for this is similar to that of Claim C.34 and follows from the fact that there will be at least one honest party \( P_j \) in \( CS \), such that the corresponding values \( a^{(j)}_{\text{iter}}, b^{(j)}_{\text{iter}} \) and \( b^{(j)}_{\text{iter}} \) shared by \( P_j \) will be randomly distributed for \( \text{Adv} \). From Lemma C.15 \( \text{Adv} \) learns nothing additional about \( a_{\text{iter}}, b_{\text{iter}} \) and \( b'_{\text{iter}} \) during the two instances of \( \Pi_{\text{BasicMult}} \). While \( \text{Adv} \) learns the value of \( e_{\text{iter}} \), since \( b'_{\text{iter}} \) is a uniformly distributed for \( \text{Adv} \), for every candidate value of \( b'_{\text{iter}} \) from the view-point of \( \text{Adv} \), there is a corresponding value of \( b_{\text{iter}} \) consistent with the \( e_{\text{iter}} \) learnt by \( \text{Adv} \). Hence, learning \( e_{\text{iter}} \) does not add any new information about \((a_{\text{iter}}, b_{\text{iter}}, c_{\text{iter}})\) to the view of \( \text{Adv} \). Moreover, \( \text{Adv} \) will be knowing beforehand that \( d_{\text{iter}} \) will be 0 and hence, learning this value does not change the view of \( \text{Adv} \) regarding \((a_{\text{iter}}, b_{\text{iter}}, c_{\text{iter}})\).

We next derive the communication complexity of the protocol \( \Pi_{\text{RandMultCI}} \).

Claim C.22. Protocol \( \Pi_{\text{RandMultCI}} \) requires \( \mathcal{O}(n^2) \) calls to \( F_{\text{VSS}} \) and \( F_{\text{ABA}} \), and incurs a communication of \( \mathcal{O}(|Z| \cdot n^3 \log |\mathbb{F}|) \) bits.

Proof. Follows from the communication complexity of the protocol \( \Pi_{\text{BasicMult}} \) (Claim C.12) and the fact that if \( d_{\text{iter}} \neq 0 \), then the parties proceed to publicly reconstruct \( \mathcal{O}(n) \) values through instances of \( \Pi_{\text{PerRec}} \) and publicly reconstruct \( \mathcal{O}(|S|) \) number of shares through instances of \( \Pi_{\text{PerRecShare}} \), where \( |S| = |Z| \) for our sharing specification \( S \).

The proof of Lemma C.23 now follows from Claims C.17 – C.22.

Lemma C.23. Let \( Z \) satisfy the \( \mathcal{O}^{(3)}(\mathcal{P}, Z) \) condition and let \( S = (S_1, \ldots, S_h) = \{ \mathcal{P} \setminus Z | Z \in \mathcal{Z} \} \). Consider an arbitrary \( \text{iter} \), such that all honest parties participate in the instance \( \Pi_{\text{RandMultCI}}(\mathcal{P}, Z, S, \mathcal{G}_D, \text{iter}) \), where no honest party is present in \( \mathcal{G}_D \). Then each honest \( P_i \) eventually sets \( \text{flag}^{(\ell)}_{\text{iter}} \) to either 0 or 1. In the former case, the honest parties output \( (a_{\text{iter}}, \{b_{\text{iter}}, c_{\text{iter}}\}) \), such that with probability at least of \( 1 - \frac{1}{|\mathbb{F}|} \), the condition \( c_{\text{iter}} = a_{\text{iter}} \cdot b_{\text{iter}} \) holds. Moreover, the view of \( \text{Adv} \) will be independent of the triple \((a_{\text{iter}}, b_{\text{iter}}, c_{\text{iter}})\). In the latter case, the honest parties will eventually include at least one new maliciously-corrupt party \( P_j \) to \( \mathcal{G}_D \). The protocol makes \( \mathcal{O}(n^2) \) calls to \( F_{\text{VSS}} \) and \( F_{\text{ABA}} \), and incurs a communication of \( \mathcal{O}(|Z| \cdot n^3 \log |\mathbb{F}|) \) bits.

Protocol \( \Pi_{\text{RandMultCI}} \) for \( M \) Triples: The extension of the protocol \( \Pi_{\text{RandMultCI}} \) for generating \( M \) triples is straightforward. The parties first generate \( M \) random shared tuples \( \{(a_{\text{iter}}^{(\ell)}, b_{\text{iter}}^{(\ell)}, c_{\text{iter}}^{(\ell)})\}_{\ell=1,\ldots,M} \) and a single random challenge \( r_{\text{iter}} \). The parties then run \( 2M \) instances of \( \Pi_{\text{BasicMult}} \) to compute \( \{(c_{\text{iter}}^{(\ell)}, c_{\text{iter}}^{(\ell)})\}_{\ell=1,\ldots,M} \), followed by probabilistically checking if all the instances of \( \Pi_{\text{BasicMult}} \) are executed correctly, by using the same \( r_{\text{iter}} \) for all the instances. If cheating is detected in any of the instances, then the parties proceed further to identify at least one new maliciously-corrupt party and update \( \mathcal{G}_D \), as done in \( \Pi_{\text{RandMultCI}} \). The protocol makes \( \mathcal{O}(n^2 \cdot M) \) calls to \( F_{\text{VSS}} \) and \( \mathcal{O}(n^2) \) calls to \( F_{\text{ABA}} \), and incurs a communication of \( \mathcal{O}((M \cdot |Z| \cdot n^2 + |Z| \cdot n^3) \log |\mathbb{F}|) \) bits.

C.5 Statistically-Secure Protocol \( \Pi_{\text{StatTriples}} \) and Its Properties

Protocol \( \Pi_{\text{StatTriples}} \) for generating \( M = 1 \) multiplication-triple is presented in Fig C.21.
Protocol $\Pi_{\text{StatTriples}}(P, Z, S)$

- **Initialization:** Parties initialize $GD = \emptyset$ and $iter = 1$.
- **Detectable Triple Generation:** Parties participate in an instance $\Pi_{\text{RandMultCI}}(P, Z, S, GD, iter)$ with session id $sid_{iter} \overset{\text{def}}{=} sid || iter$. Each $P_i \in P$ then proceeds as follows.
  - **Positive Output:** If flag$_{iter}^{(i)}$ is set to 0 during the instance $\Pi_{\text{RandMultCI}}(P, Z, S, GD, iter)$, then output the shares $\{(a_{iter}^{(i)}, b_{iter}^{(i)}, c_{iter}^{(i)}) | P_i \in S_i\}$ obtained during the instance of $\Pi_{\text{RandMultCI}}$.
  - **Negative Output:** If flag$_{iter}^{(i)}$ is set to 1 during the instance $\Pi_{\text{RandMultCI}}(P, Z, S, GD, iter)$, then set $iter = iter + 1$ and go to the step Detectable Triple Generation.

Figure 21: A statistically-secure protocol for $F_{\text{Triples}}$ with $M = 1$ in $(F_{\text{VSS}}, F_{\text{ABA}})$-hybrid for session id $sid$

Protocol $\Pi_{\text{StatTriples}}$ for Generating $M$ Multiplication-Triples: The only modification will be to call $\Pi_{\text{RandMultCI}}$ for generating $M$ random triples.

We next prove the security of the protocol $\Pi_{\text{StatTriples}}$ in the $(F_{\text{VSS}}, F_{\text{ABA}})$-hybrid model. While proving these properties, we will assume that $Z$ satisfies the $Q^{(3)}(P, Z)$ condition. This further implies that the sharing specification $S = (S_1, \ldots, S_h) \overset{\text{def}}{=} \{P \setminus Z | Z \in Z\}$ satisfies the $Q^{(2)}(S, Z)$ condition.

**Theorem 4.5** Let $Z$ satisfy the $Q^{(3)}(P, Z)$ condition and let $S = (S_1, \ldots, S_h) = \{P \setminus Z | Z \in Z\}$. Then $\Pi_{\text{StatTriples}}$ securely realizes $F_{\text{Triples}}$ with UC-security in the $(F_{\text{VSS}}, F_{\text{ABA}})$-hybrid model, except with error probability of at most $\frac{n}{|P|}$. The protocol makes $O(M \cdot n^3)$ calls to $F_{\text{VSS}}$ and $O(n^3)$ calls to $F_{\text{ABA}}$, and additionally incurs a communication of $O((M \cdot |Z| \cdot n^3 + |Z| \cdot n^4) \cdot (|F|))$ bits.

**Proof.** The communication complexity and the number of calls to $F_{\text{VSS}}$ and $F_{\text{ABA}}$ simply follows from the communication complexity of $\Pi_{\text{RandMultCI}}$ and the fact that there might be $O(n)$ instances of $\Pi_{\text{RandMultCI}}$ in the protocol. This is because from Lemma C.23, if any instance of $\Pi_{\text{RandMultCI}}$ fails, then at least one new corrupt party is globally discarded and included in $GD$. Once all the corrupt parties are included in $GD$, then from Claim C.19 the next instance of $\Pi_{\text{RandMultCI}}$ is bound to give the correct output.

We next prove the security. For the ease of explanation, we consider the case where only one multiplication-triple is generated in $\Pi_{\text{StatTriples}}$; i.e. $M = 1$. The proof can easily be modified for any general $M$.

Let $Adv$ be an arbitrary adversary, attacking the protocol $\Pi_{\text{StatTriples}}$ by corrupting a set of parties $Z^* \subseteq Z$, and let $Env$ be an arbitrary environment. We show the existence of a simulator $S_{\text{StatTriples}}$ (Fig 22), such that for any $Z^* \subseteq Z$, the outputs of the honest parties and the view of the adversary in the protocol $\Pi_{\text{StatTriples}}$ is indistinguishable from the outputs of the honest parties and the view of the adversary in an execution in the ideal world involving $S_{\text{StatTriples}}$ and $F_{\text{Triples}}$, except with probability at most $\frac{n}{|P|}$.

The high level idea of the simulator is very similar to that of the simulator for the protocol $\Pi_{\text{PerTriples}}$ (see the proof of Theorem 3.4). Throughout the simulation, the simulator itself performs the role of the ideal functionalities $F_{\text{VSS}}$ and $F_{\text{ABA}}$ whenever required and performs the role of the honest parties, exactly as per the steps of the protocol. In each iteration, the simulator simulates the actions of honest parties during the underlying instance of $\Pi_{\text{RandMultCI}}$ by playing the role of the honest parties with random inputs. Once the simulator finds any iteration of $\Pi_{\text{RandMultCI}}$ to be successful, the simulator learns the secret-sharing of the output triple of that iteration and sends the shares of this triple, corresponding to the corrupt parties to $F_{\text{Triples}}$, on the behalf of $Adv$.
for \( F_{ABA} \) and \( F_{VSS} \) for \( \text{Adv} \) during the underlying instances of \( \Pi_{\text{BasicMult}} \), by itself performing the role of \( F_{ABA} \) and \( F_{VSS} \). Next, based on whether the instance is successful or not, simulator does the following.

- **If during the instance** \( \Pi_{\text{RandMultCI}}(P, Z, S, GD, \text{iter}) \), simulator has set \( \text{flag}^{(i)}_{\text{iter}} = 0 \), corresponding to any \( P_i \notin Z^* \): In this case, let \( (\tilde{a}_{\text{iter}}, \tilde{b}_{\text{iter}}, \tilde{c}_{\text{iter}}) \) be the output of the honest parties from the instance of \( \Pi_{\text{RandMultCI}} \). The simulator then sets \( \{(\tilde{a}_{\text{iter}})_q, (\tilde{b}_{\text{iter}})_q, (\tilde{c}_{\text{iter}})_q\} \) to be the shares corresponding to the parties in \( Z^* \) and goes to the step labelled Interaction with \( F_{\text{Triples}} \).

- **If during the instance** \( \Pi_{\text{RandMultCI}}(P, Z, S, GD, \text{iter}) \), simulator has set \( \text{flag}^{(i)}_{\text{iter}} = 1 \), corresponding to any \( P_i \notin Z^* \): In this case, the simulator sets \( \text{iter} = \text{iter} + 1 \) and goes to step labelled Detectable Triple Generation.

  - Interaction with \( F_{\text{Triples}} \): Let \( \{(\tilde{a})_q, (\tilde{b})_q, (\tilde{c})_q\}_{S \cap Z^* \neq \emptyset} \) be the shares set by the simulator corresponding to the parties in \( Z^* \). The simulator sends \( \text{shares, sid, } \{(\tilde{a})_q, (\tilde{b})_q, (\tilde{c})_q\}_{S \cap Z^* \neq \emptyset} \) to \( F_{\text{Triples}} \), on the behalf of \( \text{Adv} \).

Figure 22: Simulator for the protocol \( \Pi_{\text{StatTriples}} \) where \( \text{Adv} \) corrupts the parties in set \( Z^* \subset Z \)

We now prove a series of claims which will help us to finally prove the theorem. We first show that the view generated by \( S_{\text{StatTriples}} \) for \( \text{Adv} \) is identically distributed to \( \text{Adv} \)'s view during the real execution of \( \Pi_{\text{StatTriples}} \).

**Claim C.24.** The view of \( \text{Adv} \) in the simulated execution with \( S_{\text{PerTriples}} \) is identically distributed as the view of \( \text{Adv} \) in the real execution of \( \Pi_{\text{StatTriples}} \).

**Proof.** In both the real as well as simulated execution, the parties run an instance of \( \Pi_{\text{RandMultCI}} \) for each iteration \( \text{iter} \), where in the simulated execution, the role of the honest parties is played by the simulator, including the role of \( F_{\text{VSS}} \) and \( F_{\text{ABA}} \). Now, in either execution, if \( \text{flag}^{(i)}_{\text{iter}} \) is set to 0 during some iteration \( \text{iter} \) corresponding to any honest \( P_i \), then from Lemma C.23, the view of \( \text{Adv} \) will be independent of the underlying triple and hence, will be identically distributed in both the executions. Else, in both executions, at least one new corrupt party gets discarded and the parties proceed to the next iteration. Hence, the view of \( \text{Adv} \) in both executions is identically distributed.

We now show that conditioned on the view of \( \text{Adv} \), the output of honest parties is identically distributed in the real execution of \( \Pi_{\text{StatTriples}} \) involving \( \text{Adv} \), as well as in the ideal execution involving \( S_{\text{StatTriples}} \) and \( F_{\text{Triples}} \).

**Claim C.25.** Conditioned on the view of \( \text{Adv} \), the output of the honest parties is identically distributed in the real execution of \( \Pi_{\text{StatTriples}} \) involving \( \text{Adv} \) and in the ideal execution involving \( S_{\text{StatTriples}} \) and \( F_{\text{Triples}} \), except with probability at most \( \frac{n}{|\mathbb{F}|} \).

**Proof.** Consider an arbitrary view \( \text{View} \) of \( \text{Adv} \), generated as per some execution of \( \Pi_{\text{StatTriples}} \). From Lemma C.23 in the real execution of \( \Pi_{\text{StatTriples}} \), during each iteration, all honest parties either obtain shares of a random multiplication triple, or discard a new maliciously-corrupt party. Since \( |Z^*| < n \), it will take less than \( n \) iterations to discard all the maliciously-corrupt parties. Furthermore, once all parties in \( Z^* \) are discarded, from Claim C.19 the next instance of \( \Pi_{\text{RandMultCI}} \) will output a secret-shared multiplication-triple for the honest parties. Consequently, within \( n \) iterations, there will be some iteration \( \text{iter} \), such that all honest parties \( P_i \) eventually set \( \text{flag}^{(i)}_{\text{iter}} \) to 0 and output a secret-shared triple \( (\tilde{a}_{\text{iter}}, \tilde{b}_{\text{iter}}, \tilde{c}_{\text{iter}}) \). Moreover, from the union bound, it follows that except with probability at most \( \frac{n}{|\mathbb{F}|} \), the triple \( (\tilde{a}_{\text{iter}}, \tilde{b}_{\text{iter}}, \tilde{c}_{\text{iter}}) \) will be a multiplication-triple. Furthermore, from Lemma C.23 the triple will be randomly distributed over \( \mathbb{F} \).

To complete the proof, we show that conditioned on the shares \( \{((\tilde{a}_{\text{iter}})_q, (\tilde{b}_{\text{iter}})_q, (\tilde{c}_{\text{iter}})_q)\}_{S_q \cap Z^* \neq \emptyset} \) (which are determined by \( \text{View} \)), the honest parties output a secret-sharing of some random multiplication-triple in the simulated execution, which is consistent with the shares \( \{((\tilde{a}_{\text{iter}})_q, (\tilde{b}_{\text{iter}})_q, (\tilde{c}_{\text{iter}})_q)\}_{S_q \cap Z^* \neq \emptyset} \). However, this simply follows from the fact that in the simulated execution, \( S_{\text{StatTriples}} \) sends the shares \( \{((\tilde{a}_{\text{iter}})_q, (\tilde{b}_{\text{iter}})_q, (\tilde{c}_{\text{iter}})_q)\}_{S_q \cap Z^* \neq \emptyset} \) to \( F_{\text{Triples}} \) on the behalf of the parties in \( Z^* \), and as an output, \( F_{\text{Triples}} \) generates a random secret-sharing of some random multiplication-triple consistent with these shares.

\[ \square \]
D  MPC Protocol in the Pre-Processing Model

The perfectly-secure AMPC protocol \( \Pi_{\text{AMPC}} \) in the \((\mathcal{F}_{\text{Triples}},\mathcal{F}_{\text{VSS}},\mathcal{F}_{\text{ABA}})\)-hybrid model is presented in Fig 23. The high level idea behind the protocol is already discussed in Section 5. The protocol has a pre-processing phase where secret-shared random multiplication triples are generated, an input phase where each party verifiably generates a secret-sharing of its input for the function \( f \) and a common subset of input-providers is selected, and a circuit-evaluation phase where the circuit is securely evaluated and the function output is publicly reconstructed.

In the protocol, all honest parties may not be reconstructing the function-output at the same “time” and different parties may be at different phases of the protocol, as the protocol is executed asynchronously. Consequently, a party upon reconstructing the function-output, cannot afford to terminate immediately, as its presence and participation might be needed for the completion of various phases of the protocol by other honest parties. A standard trick to get around this problem in the AMPC protocols \([20, 21, 14]\) is to have an additional termination phase, whose code is executed concurrently throughout the protocol to check if a party can “safely” terminate the protocol with the function output.

\[ \text{Protocol } \Pi_{\text{AMPC}} \]

Set the sharing specification as \( S = \{S_1, \ldots, S_h\} \overset{\text{def}}{=} \{P \setminus Z | Z \in \mathcal{Z}\} \), where \( \mathcal{Z} \) is the adversary structure \( Z \).

**Pre-Processing Phase**

1. Send (triples, \( \text{sid}, P_i \)) to the functionality \( \mathcal{F}_{\text{Triples}} \).
2. Request output from \( \mathcal{F}_{\text{Triples}} \) until an output (tripleshare, \( \text{sid}, \{[a^{(i)}]_q, [b^{(i)}]_q, [c^{(i)}]_q\} \), \( i \in \{1, \ldots, M\}, P_i \in S_q \)) is received from \( \mathcal{F}_{\text{Triples}} \).

**Input Phase**

Once the output from \( \mathcal{F}_{\text{Triples}} \) is received, then proceed as follows.

- **Secret-sharing of the Inputs and Collecting Shares of Other Inputs:**
  
  1. Upon having the input \( x^{(i)} \) for the function \( f \), randomly select the shares \( x_1^{(i)}, \ldots, x_h^{(i)} \in F \), subject to the condition that \( x^{(i)} = x_1^{(i)} + \ldots + x_h^{(i)} \). Send (dealer, \( \text{sid}_j, P_i, (x_1^{(i)}, \ldots, x_h^{(i)}) \)) to \( \mathcal{F}_{\text{VSS}} \), where \( \text{sid}_j \overset{\text{def}}{=} \text{sid}_i \).
  
  2. For \( j = 1, \ldots, n \), request for output from \( \mathcal{F}_{\text{VSS}} \) with \( \text{sid}_j \) corresponding to the dealer \( P_j \), until an output is received.

- **Selecting Common Input-Providers:**
  
  1. If (share, \( \text{sid}_j, P_j, \{[x^{(j)}]_q\}, P_i \in S_q \)) is received from \( \mathcal{F}_{\text{VSS}} \) with \( \text{sid}_j \), then send (vote, \( \text{sid}_j, 1 \)) to \( \mathcal{F}_{\text{ABA}} \) with \( \text{sid}_j \), where \( \text{sid}_j \overset{\text{def}}{=} \text{sid}_j \| i \).
  
  2. For \( j = 1, \ldots, n \), keep requesting for output from \( \mathcal{F}_{\text{ABA}} \) with \( \text{sid}_j \), until an output is received.
  
  3. If there exists a set of parties \( \mathcal{GP}_j \), such that \( P \setminus \mathcal{GP}_j \in \mathcal{Z} \) and (decide, \( \text{sid}_j, 1 \)) is received from \( \mathcal{F}_{\text{ABA}} \) with \( \text{sid}_j \), corresponding to each \( P_j \in \mathcal{GP}_j \), then send (vote, \( \text{sid}_j, 0 \)) to every \( \mathcal{F}_{\text{ABA}} \) with \( \text{sid}_j \), for which no input has been provided yet.
  
  4. Once (decide, \( \text{sid}_j, v_j \)) is received from \( \mathcal{F}_{\text{ABA}} \) with \( \text{sid}_j \), for every \( j \in \{1, \ldots, n\} \), set \( \mathcal{CS} = \{P_j : v_j = 1\} \).
  
  5. Wait until (share, \( \text{sid}_j, P_j, \{[x^{(j)}]_q\} \), \( P_i \in S_q \)) is received from \( \mathcal{F}_{\text{VSS}} \) for every \( P_j \in \mathcal{CS} \). For every \( P_j \notin \mathcal{CS} \), participate in an instance of the protocol \( \Pi_{\text{PerDefSh}} \) with public input 0 to generate a default secret-sharing of 0.

**Circuit-Evaluation Phase**

Evaluate each gate \( g \) in the circuit according to the topological ordering as follows, depending upon the type of \( g \).

- **Addition Gate:** If \( g \) is an addition gate with inputs \( x, y \) and output \( z \), then corresponding to every \( S_q \) such that \( P_i \in S_q \), set \([z]_q = [x]_q + [y]_q \) as the share corresponding to \( z \). Here \( \{[x]_q\}_{P_i \in S_q} \) and \( \{[y]_q\}_{P_i \in S_q} \) are \( P_i \)'s shares corresponding to gate-inputs \( x \) and \( y \) respectively.

- **Multiplication Gate:** If \( g \) is the \( \ell \)th multiplication gate with inputs \( x, y \) and output \( z \), where \( \ell \in \{1, \ldots, M\} \), then do the following:
1. Corresponding to every $S_q$ such that $P_i \in S_q$, set $[d^{(t)}]_q \triangleq [x]_q - [a^{(t)}]_q$ and $[e^{(t)}]_q \triangleq [y]_q - [b^{(t)}]_q$, where $\{[x]_q\}_{P_i \in S_q}$ and $\{[y]_q\}_{P_i \in S_q}$ are $P_i$'s shares corresponding to gate-inputs $x$ and $y$ respectively and $\{([a^{(t)}]_q, [b^{(t)}]_q, [e^{(t)}]_q)\}_{P_i \in S_q}$ are $P_i$'s shares corresponding to the $\ell$th multiplication-triple.

2. Participate in instances of $\Pi_{PerRec}$ with shares $\{[d^{(t)}]_q\}_{P_i \in S_q}$ and $\{[e^{(t)}]_q\}_{P_i \in S_q}$ to publicly reconstruct $d^{(t)}$ and $e^{(t)}$, where $d^{(t)} \triangleq x - a^{(t)}$ and $e^{(t)} \triangleq y - b^{(t)}$.

3. Upon reconstructing $d^{(t)}$ and $e^{(t)}$, corresponding to every $S_q$ such that $P_i \in S_q$, set $[z]_q \triangleq d^{(t)} \cdot e^{(t)} + d^{(t)} \cdot [b^{(t)}]_q + e^{(t)} \cdot [a^{(t)}]_q + [e^{(t)}]_q$. Set $\{[z]_q\}_{P_i \in S_q}$ as the shares corresponding to $z$.

- **Output Gate:** If $g$ is the output gate with output $y$, then participate in an instance of $\Pi_{PerRec}$ with shares $\{[y]_q\}_{P_i \in S_q}$ to publicly reconstruct $y$.

**Termination Phase**

Concurrently execute the following steps during the protocol:

1. If the circuit-output $y$ is computed, then send (ready, sid, $P_i$, $y$) to every party in $P$.
2. If the message (ready, sid, $P_j$, $y$) is received from a set of parties $A$ such that $Z$ satisfies $Q^{(1)}(A, Z)$ condition, then send (ready, sid, $P_i$, $y$) to every party in $P$.
3. If the message (ready, sid, $P_j$, $y$) is received from a set of parties $W$ such that $P \setminus W \subseteq Z$, then output $y$ and terminate.

---

"Thus $S$ is $Z$-private.

\[ b \text{The notation } \text{sid} \text{ is used here to distinguish among the } n \text{ different calls to } F_{VSS}. \]

\[ c \text{The notation } \text{sid} \text{ is used here to distinguish among the } n \text{ different calls to } F_{ABA}. \]

Figure 23: The perfectly-secure AMPC protocol in the $(F_{Triples}, F_{VSS}, F_{ABA})$-hybrid model. The public inputs of the protocol are $P$, ckt and $Z$. The above steps are executed by every $P_i \in P$.

Intuitively, protocol $\Pi_{AMPC}$ eventually terminates as the set $CS$ is eventually decided. This is because even if the corrupt parties do not secret-share their inputs, the inputs of all honest parties are eventually secret-shared. Once $CS$ is decided, the evaluation of each gate will be eventually completed: while the addition gates are evaluated non-interactively, the evaluation of multiplication gates requires reconstructing the corresponding masked gate-inputs which is eventually completed due to the reconstruction protocols. The privacy of the inputs of the honest parties in $CS$ will be maintained as the sharing specification $S$ is $Z$-private. Moreover, the inputs of the corrupt parties in $CS$ will be independent of the inputs of the honest parties in $CS$, as inputs are secret-shared via calls to $F_{VSS}$. Finally, correctness holds since each gate is evaluated correctly. We next rigorously formalize this intuition by giving a formal security proof and show that the protocol $\Pi_{AMPC}$ is perfectly-secure, if the parties have access to ideal functionalities $F_{Triples}$, $F_{VSS}$ and $F_{ABA}$.

**Theorem 5.1** Protocol $\Pi_{AMPC}$ UC-securely realizes the functionality $F_{AMPC}$ for securely computing $f$ (see Fig 23 in Appendix A) with perfect security in the $(F_{Triples}, F_{VSS}, F_{ABA})$-hybrid model, in the presence of a static malicious adversary characterized by an adversary-structure $Z$, satisfying the $Q^{(3)}(P, Z)$ condition. The protocol makes one call to $F_{Triples}$ and $O(n)$ calls to $F_{VSS}$ and $F_{ABA}$ and additionally incurs a communication of $O(M \cdot |Z| \cdot n^2 \log |\mathbb{F}|)$ bits, where $M$ is the number of multiplication gates in the circuit $ckt$ representing $f$.

**Proof:** The communication complexity in the $(F_{Triples}, F_{VSS}, F_{ABA})$-hybrid model follows from the fact that for evaluating each multiplication gate, the parties need to run 2 instances of the reconstruction protocol $\Pi_{PerRec}$.

For security, let $Adv$ be an arbitrary real-world adversary corrupting the set of parties $Z^* \subseteq Z$ and let $Env$ be an arbitrary environment. We show the existence of a simulator $S_{AMPC}$, such that the output of honest parties and the view of the adversary in an execution of the real protocol with $Adv$ is identical to the output in an execution with $S_{AMPC}$ involving $F_{AMPC}$ in the ideal model. This further implies that $Env$ cannot distinguish between the two executions. The steps of the simulator are given in Fig 24.

The high level idea of the simulator is as follows. During the simulated execution, the simulator itself performs the role of the ideal functionalities $F_{Triples}$, $F_{VSS}$ and $F_{ABA}$ whenever required. Performing the role of $F_{Triples}$ allows the simulator to learn the secret-sharing of all the multiplication-triples. During the input phase, whenever
Adv secret-shares any value through \( \mathcal{F}_{\text{VSS}} \) on the behalf of a corrupt party, the simulator records this on the behalf of the corrupt party. This allows the simulator to learn the function-input of the corresponding corrupt party. On the other hand, for the honest parties, the simulator picks arbitrary values as their function-inputs and simulates the secret-sharing of those input values using random shares, as per \( \mathcal{F}_{\text{VSS}} \). To select the common input-providers during the simulated execution, the simulator itself performs the role of \( \mathcal{F}_{\text{ABA}} \) and simulates the honest parties as per the steps of the protocol and \( \mathcal{F}_{\text{ABA}} \). This allows the simulator to learn the common subset of input-providers \( CS \), which the simulator passes to the functionality \( \mathcal{F}_{\text{AMPC}} \). Notice that the function-inputs for each corrupt party in \( CS \) will be available with the simulator. This is because for every corrupt party \( P_i \) which is added to \( CS \), at least one honest party \( P_j \) should participate with input 1 in the corresponding call to \( \mathcal{F}_{\text{ABA}} \). This implies that the honest party \( P_j \) must have received the shares \( P_j \) sent to \( \mathcal{F}_{\text{VSS}} \) from \( \mathcal{F}_{\text{VSS}} \). Since in the simulation, the role of \( \mathcal{F}_{\text{VSS}} \) is played by the simulator, it implies that the full vector of shares provided by \( P_j \) to \( \mathcal{F}_{\text{VSS}} \) will be known to the simulator. Hence, along with \( CS \), the simulator can send the corresponding function-inputs of the corrupt parties in \( CS \) to \( \mathcal{F}_{\text{AMPC}} \). Upon receiving the function-output, the simulator simulates the steps of the honest parties for the gate evaluations as per the protocol. Finally, for the output gate, the simulator arbitrarily computes a secret-sharing of the function-output \( y \) received from \( \mathcal{F}_{\text{AMPC}} \), which is consistent with the shares which corrupt parties hold for the output-gate sharing. Then, on the behalf of the honest parties, the simulator sends the shares corresponding to the above sharing of \( y \) during the public reconstruction of \( y \). This ensures that in the simulated execution, Adv learns the function-output \( y \). For the termination phase, the simulator sends \( y \) on the behalf of honest parties.

### Simulation \( S_{\text{AMPC}} \)

\( S_{\text{AMPC}} \) constructs virtual real-world honest parties and invokes the real-world adversary Adv. The simulator simulates the view of Adv, namely its communication with Env, the messages sent by the honest parties, and the interaction with various functionalities. In order to simulate Env, the simulator \( S_{\text{AMPC}} \) forwards every message it receives from Env to Adv and vice-versa. The simulator then simulates the various phases of the protocol as follows.

#### Pre-Processing Phase

**Simulating the call to \( \mathcal{F}_{\text{Triples}} \):** The simulator simulates the steps of \( \mathcal{F}_{\text{Triples}} \) by itself playing the role of \( \mathcal{F}_{\text{Triples}} \). Namely, it receives the shares corresponding to the parties in \( Z^* \) for each multiplication-triple from Adv and then randomly generates secret-sharing of \( M \) random multiplication-triples \( \{(a_i^{(j)}, b_i^{(j)}, c_i^{(j)})\}_{j=1,\ldots,M} \) consistent with the provided shares. At the end of simulation of this phase, the simulator will know the entire vector of shares corresponding to the secret-sharing of all multiplication-triples.

**Input Phase**

- The simulator simulates the operations of the honest parties during the input phase, by randomly picking \( \overline{x}^{(j)} \) as the input, for every \( P_i \notin Z^* \), selecting random shares \( \overline{x}_1^{(j)}, \ldots, \overline{x}_h^{(j)} \) such that \( \overline{x}^{(j)} = \overline{x}_1^{(j)} + \ldots + \overline{x}_h^{(j)} \), and setting \( [\overline{x}^{(j)}]_q = \overline{x}_q^{(j)} \), for \( q = 1, \ldots, h \). When Adv requests output from \( \mathcal{F}_{\text{VSS}} \) with \( \text{sid}_j \) on the behalf of any party \( P_i \in Z^* \), then the simulator responds with an output \( \text{(share, sid}_j, P_i, \{[\overline{x}^{(j)}]_q\}_{q \in \mathbb{S}}) \) on the behalf of \( \mathcal{F}_{\text{VSS}} \).
- Whenever Adv sends \( (\text{dealert, sid}_j, P_i, (x_1^{(i)}, \ldots, x_h^{(i)})) \) to \( \mathcal{F}_{\text{VSS}} \) on the behalf of any \( P_i \in Z^* \), the simulator records the input \( x^{(i)} = x_1^{(i)} + \ldots + x_h^{(i)} \) on the behalf of \( P_i \) and sets \( [x^{(i)}]_q = [x_1^{(i)}]_q, \ldots, [x_h^{(i)}]_q \).
- When the simulation reaches the “Selecting Common Input-Providers” stage, the simulator simulates the interface of \( \mathcal{F}_{\text{ABA}} \) to Adv by itself performing the role of \( \mathcal{F}_{\text{ABA}} \). When the first honest party completes the simulated input phase, \( S_{\text{AMPC}} \) learns the set \( CS \).

**Interaction with \( \mathcal{F}_{\text{AMPC}} \):** Once the simulator learns \( CS \), it sends the input values \( x^{(i)} \) that it has recorded on the behalf of each \( P_i \in (Z^* \cap CS) \), and the set of input-providers \( CS \) to \( \mathcal{F}_{\text{AMPC}} \). Upon receiving the output \( y \) from \( \mathcal{F}_{\text{AMPC}} \), the simulator starts the simulation of circuit-evaluation phase.

#### Circuit-Evaluation Phase

The simulator simulates the evaluation of each gate \( g \) in the circuit in topological order as follows:

- **Addition Gate:** Since this step involves local computation, the simulator does not have to simulate any messages on the behalf of the honest parties. The simulator locally adds the secret-sharing corresponding to the gate-inputs and obtains the secret-sharing corresponding to the gate-output.
- **Multiplication Gate**: If $g$ is the $\ell^{th}$ multiplication gate in the circuit, then the simulator takes the complete secret-sharing of the $\ell^{th}$ multiplication triple $(\hat{a}^\ell, \hat{b}^\ell, \hat{c}^\ell)$ and computes the messages of the honest parties as per the steps of the protocol (by considering the secret-sharing of the above multiplication-triple and the secret-sharing of the gate-inputs), and sends them to Adv on behalf of the honest parties as part of the instances of $\Pi_{\text{PerRec}}$ protocol. Once the simulation of the circuit-evaluation phase is done, the simulator will know the secret-sharing corresponding to the gate-output.

- **Output Gate**: Let $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_h)$ be the secret-sharing corresponding to the output gate, available with $S_{\text{AMPC}}$ during the simulated circuit-evaluation. The simulator then randomly selects shares $\tilde{y}_1, \ldots, \tilde{y}_h$ such that $\tilde{y}_1 + \ldots + \tilde{y}_h = y$ and $\tilde{y}_q = \tilde{y}_q$ corresponding to every $S_q \in S$ where $S_q \cap Z^* \neq \emptyset$. Then, as part of the instance of $\Pi_{\text{PerRec}}$ protocol to reconstruct the function output, the simulator sends the shares $\{\tilde{y}_q\}_{S_q \in S}$ to Adv on behalf of the honest parties.

### Termination Phase

The simulator sends a ready message for $y$ to Adv on the behalf of $P_i \notin Z^*$, if in the simulated execution, $P_i$ has computed $y$.

**Figure 24**: Simulator for the protocol $\Pi_{\text{AMPC}}$ where Adv corrupts the parties in set $Z^* \in \mathbb{Z}$

We next prove a sequence of claims, which helps us to show that the joint distribution of the honest parties and the view of Adv is identical in both the real, as well as the ideal-world. We first claim that in any execution of $\Pi_{\text{AMPC}}$, a set $CS$ is eventually generated. This automatically implies that the honest parties eventually possess a secret-sharing of $M$ random multiplication-triples generated by $F_{\text{Triples}}$, as well as a secret-sharing of the inputs of the parties in $CS$.

**Claim D.1.** In any execution of $\Pi_{\text{AMPC}}$, a set $CS$ is eventually generated, such that for every $P_j \in CS$, there exists some $x^{(j)}$ held by $P_j$ which is eventually secret-shared.

**Proof.** As the proof of this claim is similar to the proof of Claim B.32, we skip the formal proof.

We next show that the view generated by $S_{\text{AMPC}}$ for Adv is identically distributed to Adv’s view during the real execution of $\Pi_{\text{AMPC}}$.

**Claim D.2.** The view of Adv in the simulated execution with $S_{\text{AMPC}}$ is identically distributed to the view of Adv in the real execution of $\Pi_{\text{AMPC}}$.

**Proof.** It is easy to see that the view of Adv during the pre-processing phase is identically distributed in both the executions. This is because in both the executions, Adv receives no messages from the honest parties and the steps of $F_{\text{Triples}}$ are executed by the simulator itself in the simulated execution. Namely, in both the executions, Adv’s view consists of the shares of $M$ random multiplication-triples corresponding to the parties in $Z^*$. So, let us fix these shares. Conditioned on these shares, during the input phase, Adv learns the shares $\{[x^{(j)}]_q\}_{P_j \notin Z^*,(S_q \cap Z^*) \neq \emptyset}$ during the real execution corresponding to the parties $P_j \notin Z^*$. In the simulated execution, it learns the shares $\{[\tilde{x}^{(j)}]_q\}_{P_j \notin Z^*,(S_q \cap Z^*) \neq \emptyset}$. Since the sharing specification $S$ is $Z$-private and the vector of shares $(x_1^{(j)}, \ldots, x_h^{(j)})$ as well as $(\tilde{x}_1^{(j)}, \ldots, \tilde{x}_h^{(j)})$ are randomly chosen, it follows that the distribution of the shares $\{[x^{(j)}]_q\}_{P_j \notin Z^*,(S_q \cap Z^*) \neq \emptyset}$ as well as $\{[\tilde{x}^{(j)}]_q\}_{P_j \notin Z^*,(S_q \cap Z^*) \neq \emptyset}$ is identical and independent of both $x^{(j)}$ as well as $\tilde{x}^{(j)}$, so let us fix these shares. Since the role of $F_{\text{ABA}}$ is played by the simulator itself, it follows easily that the view of Adv during the selection of the set $CS$ is identically distributed in both the real as well as the simulated execution.

During the evaluation of linear gates, no communication is involved. During the evaluation of multiplication gates, in the simulated execution, the simulator will know the secret-sharing associated with gate-inputs and also the secret-sharing of the associated multiplication-triple. Hence, the simulator correctly sends the shares corresponding to the values $d^{(\ell)}$ and $e^{(\ell)}$ as per the protocol on the behalf of the honest parties. Moreover, the values $d^{(\ell)}$ and $e^{(\ell)}$ will be randomly distributed for Adv in both the executions, since the underlying multiplication-triple is randomly distributed, conditioned on the shares of the corrupt parties. Thus, Adv’s view during the evaluation of multiplications gates is identically distributed in both the executions.
For the output gate, the shares received by \( \text{Adv} \) in the real execution from the honest parties correspond to a secret-sharing of the function-output \( y \). From the steps of \( S_{\text{AMPC}} \), it is easy to see that the same holds even in the simulated execution, as \( S_{\text{AMPC}} \) sends to \( \text{Adv} \) shares corresponding to a secret-sharing of \( y \), which are consistent with the shares held by \( \text{Adv} \). Hence, \( \text{Adv} \)'s view is identically distributed in both the executions during the evaluation of output gate. Finally, it is easy to see that \( \text{Adv} \)'s view is identically distributed in both the executions during the termination phase. This is because in both the executions, every honest party who has obtained the function output \( y \), sends a ready message for \( y \).

We next claim that conditioned on the view of \( \text{Adv} \) (which is identically distributed in both the executions from the last claim), the output of the honest parties is identically distributed in both the worlds.

**Claim D.3.** Conditioned on the view of \( \text{Adv} \), the output of the honest parties is identically distributed in the real execution of \( \Pi_{\text{AMPC}} \) involving \( \text{Adv} \), as well as in the ideal execution involving \( S_{\text{AMPC}} \) and \( F_{\text{AMPC}} \).

**Proof.** Let \( \text{View} \) be an arbitrary view of \( \text{Adv} \), and let \( CS \) be the set of input-providers determined by \( \text{View} \) (from Claim D.1 such a set \( CS \) is bound to exist). Moreover, according to \( \text{View} \), for every \( P_i \in CS \), there exists some input \( x^{(i)} \) such that the parties hold a secret-sharing of \( x^{(i)} \). Furthermore, from Claim D.2 if \( P_i \in Z^* \) then the corresponding secret-sharing is included in \( \text{View} \). For \( P_i \notin Z^* \), the corresponding \( x^{(i)} \) is uniformly distributed conditioned on the shares of \( x^{(i)} \) available with \( \text{Adv} \) as determined by \( \text{View} \). Let us fix the \( x^{(i)} \) values corresponding to the parties in \( CS \) and denote the vector of values \( x^{(i)} \), where \( x^{(i)} = 0 \) if \( P_i \notin CS \), by \( \vec{x} \).

It is easy to see that in the ideal-world, the output of the honest parties is \( y \), where \( y \triangleq f(\vec{x}) \). This is because \( S_{\text{AMPC}} \) provides the identity of \( CS \) along with the inputs \( x^{(i)} \) corresponding to \( P_i \in (CS \cap Z^*) \) to \( F_{\text{AMPC}} \). We now show that the honest parties eventually output \( y \) even in the real-world. For this, we argue that all the values during the circuit-evaluation phase of the protocol are correctly secret-shared. Since the evaluation of linear gates needs only local computation, it follows that the output of the linear gates will be correctly secret-shared. During the evaluation of a multiplication gate, the honest parties will hold a secret-sharing of the corresponding \( d^{(l)} \) and \( e^{(l)} \) values, as during the pre-processing phase, all the multiplication-triples are generated in a secret-shared fashion, since they are computed and distributed by \( F_{\text{Triples}} \). Since \( S \) satisfies the \( Q^{(2)}(S, Z) \) condition, the honest parties eventually get \( d^{(l)} \) and \( e^{(l)} \) through the instances of \( \Pi_{\text{PerRec}} \). This automatically implies that the honest parties eventually hold a secret-sharing of \( y \) and reconstruct it correctly, as \( y \) is reconstructed through an instance of \( \Pi_{\text{PerRec}} \). Hence, during the termination phase, every honest party will eventually send a ready message for \( y \), while the parties in \( Z^* \) may send a ready message for \( y' \neq y \). Since \( Z^* \in Z \), it follows that no honest party ever sends a ready message for \( y' \). Hence no honest party ever outputs \( y' \), as it will never receive the required number of ready messages for \( y' \). Since the ready messages of the honest parties for \( y \) are eventually delivered to every honest party, it follows that eventually, all honest parties receive sufficiently many ready messages to obtain some output, even if the corrupt parties does not send the required messages.

Now let \( P_i \) be the *first honest* party to terminate the protocol with some output. From the above arguments, the output has to be \( y \). This implies that \( P_i \) receives ready messages for \( y \) from a set of parties \( P \setminus Z \), for some \( Z \in Z \). Let \( H \) be the set of *honest* parties whose ready messages are received by \( P_i \). It is easy to see that \( H \notin Z \), as otherwise, \( Z \) does not satisfy the \( Q^{(3)}(P, Z) \) condition. The ready messages of the parties in \( H \) are eventually delivered to every honest party and hence, *each* honest party (including \( P_i \)) eventually executes step 2 of the termination phase and sends a ready message for \( y \). It follows that the ready messages of *all* honest parties \( P \setminus Z^* \) are eventually delivered to every honest party (irrespective of whether \( \text{Adv} \) sends all the required messages), guaranteeing that all honest parties eventually obtain the output \( y \).

The theorem now follows from Claims D.1 D.3.

\( \Box \)