

NEW DIGITAL SIGNATURE ALGORITHM EHT

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ABSTRACT. Every public-key encryption/decryption algorithm where the set of possible plain-texts is identical to the set of possible cipher-texts may be converted into a digital signature algorithm. That is quite different in the lattice (code)-based public-key cryptography. The decryption algorithm on a random input produces a valid plain-text, that is a signature, with a negligible probability. That explains why it is so difficult to construct a new secure and efficient lattice-based digital signature system. Though several solutions are known and taking part in the NIST Post Quantum Standardisation Process there is still a need to construct digital signature algorithms based on new principles. In this work, a new and efficient digital signature algorithm is suggested. Its design is simple and transparent. Its security is based on the hardness of an approximate closest vector problem in the maximum norm for some q -ary lattices. The complexity parameters are comparable with those of the round 3 NIST digital signature candidates.

1. INTRODUCTION

Digital signatures are an important area of applications for public-key cryptography. Every public-key encryption/decryption algorithm, where the set of possible plain-texts is identical to the set of possible cipher-texts, may be converted into a digital signature algorithm. The most notable examples are RSA and Rabin crypto-systems. That is quite different in lattice-based and coding-based cryptography. The cipher-text is there larger than the plain-text as in NTRU and Regev's LWE based crypto-systems. The decryption algorithm on a random input produces a valid plain-text, that is a signature, with a negligible probability. That explains why it is so difficult to construct a new secure and efficient lattice-based digital signature system. Though several algorithms as GGH and some of NTRU-based were broken in [16, 5], yet another NTRU-based signature algorithm variation Falcon is among the finalists of the NIST Post Quantum Standardisation Process, [14]. Similarly, several variations of multivariate algorithms as HFE and TTM were broken [11, 9], and another multivariate signature algorithm Rainbow is among the finalists of the NIST competition. The history of the attacks and relevant countermeasures provides a better understanding of the security of the cryptographic algorithms. However, the countermeasures make the resulting algorithms patchy and non-transparent, one may not feel certain about their security. So there is still a need to construct digital signature algorithms based on new principles.

Recently, a new public key crypto-system EHT was described in [2]. It does not seem possible to use this crypto-system for signatures. In the present work, a new and efficient digital signature algorithm (hash-and-sign) is suggested. The design

Date: March 11, 2022.

Key words and phrases. matrices, linear inequalities, digital signatures, q -ary lattices.

of the signature algorithm is simple and transparent. It has some similarity with the crypto-system in [2]. So we call the signature algorithm EHT too. The security is based on the hardness of an approximate closest vector problem (CVP) for some specific q -ary lattices in the maximum norm. One proves that the signature is uniformly distributed if the hashing algorithm provides a uniform distribution on its outputs.

The complexity parameters are comparable with those of the round 3 NIST digital signature candidates. There are three approaches to the cryptanalysis of the new algorithm: find private key given public key only, find the private key or forge a new signature by analysing a number of valid signatures, and forge signatures without the knowledge of the private key. In particular, we claim that it is hard to forge a valid signature for any given message as one will need to solve a hard CVP problem for some specific q -ary lattice for this. The cryptanalysis is presented in Section 5.

Published digital signature lattice-based constructions typically make use of short lattice bases as private keys and their random non-short perturbations as public keys. That is true for GGH [6], some its modifications as DRS, see [18], and NTRU-based signature algorithms as NTRUSign in [4]. Another approach based on the hardness of the SIS (Short Integer Solution) problem was implemented in [10],[15]. The EHT construction does not use neither short bases of relevant lattices, nor the hardness of the SIS problem.

The EHT digital signature algorithm does not have a so-called security proof, the proof that it stands all attacks by a reduction to an NP-hard problem or some hard computational problem in general lattices, etc. That is not uncommon in the field. For instance, the multivariate signature algorithm Rainbow, which is a NIST finalist, does not have a security proof. Another NIST finalist Falcon provides a reduction to the NTRU problem, which is the shortest vector problem for a very particular lattice.

The author is grateful to Markus Hittmeir for a number of suggestions to improve the presentation.

2. SIGNATURE ALGORITHM

In this section a basic version of the new signature algorithm EHT is explained. The algorithm consists of private and public keys generating algorithms, signature generating and verifying algorithms. They all are presented in this section along with the signature verification proof. The signature for a message M is $x \in \mathbb{F}_q^n$ such that $H(M) = Ax + e$ for some public matrix $A \in \mathbb{F}_q^{kn \times n}$ and vector $e \in \mathbb{F}_q^{kn}$, and where H denotes a hash function. The entries of e represented as integers are bounded in absolute values.

In Section 3, we prove that the signature x is uniformly distributed if the hash function provides a uniform distribution on \mathbb{F}_q^{kn} . So in the random oracle model (the hash function is a random oracle) the signature algorithm itself is a random oracle. That implies that EHT signatures, if the algorithm is taken as a black box, are strong existentially unforgeable in the random oracle model. In Section 5.3 we analyse the security of the signature algorithm if a number of valid signatures is available.

A reduced version is in Section 6 below and some explicit parameters are proposed in Section 7. The underlying hard problem for these parameters is described in Section 8.

2.1. Parameters. Let n, k, λ, c be positive integers, and q be an odd prime, $\lambda \geq 3$, and $2\lambda c + 1 < q$. Also, $h = \text{HASH}(M)$ is a hash value of the message M , where h is encoded by a vector in \mathbb{F}_q^{kn} .

2.2. Private Key. The private key consists of three matrices T, B, C .

- (1) The matrix T is an integer $kn \times n$ matrix in a column echelon form

$$T = \begin{pmatrix} t_{11} & 0 & \dots & 0 \\ t_{21} & 0 & \dots & 0 \\ \dots & & & \\ t_{k1} & 0 & \dots & 0 \\ * & t_{12} & \dots & 0 \\ * & t_{22} & \dots & 0 \\ \dots & & & \\ * & t_{k2} & \dots & 0 \\ \dots & & & \\ * & * & \dots & t_{1n} \\ * & * & \dots & t_{2n} \\ \dots & & & \\ * & * & \dots & t_{kn} \end{pmatrix},$$

where entries $t_{1j}, t_{2j}, \dots, t_{kj}$ are called diagonal. The other entries of T denoted by $*$ may be randomly generated. Each tuple $[t_{1j}, t_{2j}, \dots, t_{kj}]$ has to satisfy the following property: for any integer b_1, b_2, \dots, b_k there is an integer u such that

$$(1) \quad \begin{aligned} |(b_1 - t_{1j} u) \bmod q| &\leq c, \\ |(b_2 - t_{2j} u) \bmod q| &\leq c, \\ &\dots, \\ |(b_k - t_{kj} u) \bmod q| &\leq c. \end{aligned}$$

Let, for instance, $q = 61, k = 3, c = 8$. There is only one tuple $[t_1, t_2, t_3] = [1, 4, 15]$ modulo q up to a permutation of entries, multiplication the tuple by non-zero residues and changing the sign of the entries such that for any integers b_1, b_2, b_3 the system of inequalities $|(b_1 - t_1 u) \bmod 61| \leq 8, |(b_2 - t_2 u) \bmod 61| \leq 8, |(b_3 - t_3 u) \bmod 61| \leq 8$ has a solution u .

- (2) The matrix $C = (C_{ij})$ is an integer $kn \times kn$ -matrix, the 1-norm of the rows C_i of which satisfy $\|C_i\|_1 = \sum_{j=1}^{kn} |C_{ij}| \leq \lambda$. To define C one may take $C = P_1 + P_2 + \dots + P_\lambda \bmod q$, where P_i are permutation matrices of size $kn \times kn$. Experimentally, such C is invertible over rationals with high probability if $\lambda > 2$. We assume that C is invertible modulo q . For $\lambda = 2$ one can achieve $C = P_1 + P_2$ of full rank by Lemma 3, see Appendix (Section 11). However, we do not recommend $\lambda = 2$ due to a weakness found in Section 5.
- (3) The matrix B is an arbitrary integer $n \times n$ -matrix invertible modulo q .

Theorem 1. For every integer vector $a = (a_1, a_2, \dots, a_{kn})$ there exist an integer vector $y = (y_1, y_2, \dots, y_n)$ and an integer vector $z = (z_1, z_2, \dots, z_{kn})$, where $|z_i| \leq c$ for every $i = 1, \dots, kn$, such that $a \equiv Ty + z \pmod{q}$.

Proof. We show how to compute iteratively the entries y_j and $z_{(j-1)k+1}, \dots, z_{jk}$ for $j = 1, \dots, n$. For $j = 1$ we set

$$\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_k \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_k \end{pmatrix}.$$

and $y_1 = u$, where u is a solution to the system of inequalities (1). Then

$$\begin{pmatrix} z_1 \\ z_2 \\ \dots \\ z_k \end{pmatrix} \equiv \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_k \end{pmatrix} - \begin{pmatrix} t_{11} \\ t_{21} \\ \dots \\ t_{k1} \end{pmatrix} y_1 \pmod{q}.$$

The entries of the left hand side vector are bounded by c in absolute value by (1). Let T_j be a sub-matrix of T of size $k \times j$ in the rows $jk + 1, jk + 2, \dots, jk + k$ and columns $1, \dots, j$, where $1 \leq j \leq n - 1$. The entries of T_j are denoted by $*$ in the definition of T . For $j > 1$ we set

$$\begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_k \end{pmatrix} \equiv \begin{pmatrix} a_{(j-1)k+1} \\ a_{(j-1)k+2} \\ \dots \\ a_{jk} \end{pmatrix} - T_{j-1} \begin{pmatrix} y_1 \\ \dots \\ y_{j-1} \end{pmatrix} \pmod{q}.$$

Then $y_j = u$, where u is a solution to the system of inequalities (1). So

$$\begin{pmatrix} z_{(j-1)k+1} \\ z_{(j-1)k+2} \\ \dots \\ z_{jk} \end{pmatrix} \equiv \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_k \end{pmatrix} - \begin{pmatrix} t_{1j} \\ t_{2j} \\ \dots \\ t_{kj} \end{pmatrix} y_j \pmod{q}$$

and the entries of the left hand side vector are bounded by c in absolute value. Therefore for every $1 \leq j \leq n$,

$$\begin{pmatrix} a_{(j-1)k+1} \\ a_{(j-1)k+2} \\ \dots \\ a_{jk} \end{pmatrix} \equiv \begin{pmatrix} * & \dots & * & t_{1j} \\ * & \dots & * & t_{2j} \\ * & \dots & * & \dots \\ * & \dots & * & t_{kj} \end{pmatrix} \begin{pmatrix} y_1 \\ \dots \\ y_{j-1} \\ y_j \end{pmatrix} + \begin{pmatrix} z_{(j-1)k+1} \\ z_{(j-1)k+2} \\ \dots \\ z_{jk} \end{pmatrix} \pmod{q}.$$

So $a \equiv Ty + z \pmod{q}$, where $z = (z_1, z_2, \dots, z_{kn})$ and $|z_i| \leq c$. The statement is proved. \square

2.3. Public Key. The public key is an integer $kn \times n$ matrix $A \equiv CTB \pmod{q}$.

2.4. Signature Generation. To sign the message M one computes $h = \text{HASH}(M)$. Let $a = (a_1, a_2, \dots, a_{kn})$ such that $a \equiv C^{-1}h \pmod{q}$. The vectors

$$y = (y_1, y_2, \dots, y_n), \quad z = (z_1, z_2, \dots, z_{kn}),$$

such that $a \equiv Ty + z \pmod{q}$ and $|z_i| \leq c$ are then computed according to Theorem 1, where each y_j is taken uniformly from the set of solutions to (1) for appropriate b_1, \dots, b_k . The signature is $x \equiv B^{-1}y \pmod{q}$, where $x \in \mathbb{F}_q^n$. We call $e = Cz$ the

error vector for M, x . Given h , the vector y is generally not unique. There may exist messages M which admit several valid signatures.

2.5. Signature Verification. To verify the signature x for M one computes $h = \text{HASH}(M)$ and Ax . Let $e \equiv h - Ax \pmod{q}$, where $e \in \mathbb{F}_q^{kn}$ and such that the entries of $e = (e_1, e_2, \dots, e_{kn})$ are at most $(q-1)/2$ in absolute value. The signature is accepted if $|e_i| \leq \lambda c$ for every $1 \leq i \leq kn$.

2.6. Verification Proof. According to the signature generating algorithm $C^{-1}h \equiv Ty + z \pmod{q}$, where $z = (z_1, z_2, \dots, z_{kn})$ and $|z_j| \leq c$ and $x \equiv B^{-1}y$. Then

$$(2) \quad C^{-1}h \equiv Ty + z \equiv TBx + z \pmod{q}, \quad \text{and} \quad h \equiv Ax + Cz \pmod{q},$$

where the entries of $e = Cz$ are bounded by λc in absolute value. So the signature is accepted.

3. SIGNATURE DISTRIBUTION

In this section we prove that if $h = H(M)$ is distributed uniformly on \mathbb{F}_q^{kn} , then the signature x is uniformly distributed on \mathbb{F}_q^n . Recall that (1) has a solution for every b_1, \dots, b_k . We can there put $t_1 = t_{1j} = 1, t_2 = t_{2j}, \dots, t_k = t_{kj}$ to simplify the notation below. So (1) is equivalent to the following statement. For every tuple of residues b_1, \dots, b_k modulo q there exist u and i_1, \dots, i_k , where $|i_1| \leq c, \dots, |i_k| \leq c$, and u is a residue modulo q , such that $b_1 \equiv u + i_1, b_2 \equiv ut_2 + i_2, \dots, b_k \equiv ut_k + i_k$. Let $A(b_1, \dots, b_k)$ denote the set of such u .

In order to prove that the signature $x = B^{-1}y$ is uniformly distributed it is enough to prove that y is uniformly distributed. According to Theorem 1, it is enough to prove that if b_1, \dots, b_k are generated independently and uniformly at random on residues modulo q and the solution u to (1) is taken uniformly from $A(b_1, \dots, b_k)$, then u is uniformly distributed on residues modulo q . The probability of u is equal to

$$\frac{1}{q^k} \sum_{u \in A(b_1, \dots, b_k)} \frac{1}{|A(b_1, \dots, b_k)|},$$

where the sum runs over all b_1, \dots, b_k such that $u \in A(b_1, \dots, b_k)$. The following lemma implies that this probability is $1/q$.

Lemma 2. $\sum_{u \in A(b_1, \dots, b_k)} \frac{1}{|A(b_1, \dots, b_k)|} = q^{k-1}$.

Proof. The inequalities (1) are equivalent to $b_1 - u \equiv i_1, b_2 - t_2 b_1 \equiv i_2 - t_2 i_1, \dots, b_k - t_k b_1 \equiv i_k - t_k i_1$ modulo q , where $|i_1| \leq c, \dots, |i_k| \leq c$. Let $s = s(a_2, \dots, a_k)$ be the number of solutions i_1, \dots, i_k to

$$(3) \quad |i_1| \leq c, \dots, |i_k| \leq c \quad a_2 \equiv i_2 - t_2 i_1 \pmod{q}, \dots, a_k \equiv i_k - t_k i_1 \pmod{q}.$$

Then

$$|A(b_1, \dots, b_k)| = s(a_2, \dots, a_k),$$

where $a_2 \equiv b_2 - t_2 b_1, \dots, a_k \equiv b_k - t_k b_1$. Moreover, $u \in A(b_1, \dots, b_k)$ if and only if $b_1 = u + i_1$, where i_1, \dots, i_k is a solution to (3). Since a_2, \dots, a_k may take any values, we get

$$\sum_{u \in A(b_1, \dots, b_k)} \frac{1}{|A(b_1, \dots, b_k)|} = \sum_{a_2, \dots, a_k} \sum_{i_1, \dots, i_k} \frac{1}{s(a_2, \dots, a_k)} = q^{k-1},$$

where the last sum is over all the solutions i_1, \dots, i_k to (3).

□

4. COMPLEXITY

One may solve the linear system $Ca = h \pmod q$ for a with Wiedemann's algorithm [19] in at most $\lambda(kn)^2$ additions and $(kn)^2$ multiplications modulo q . Another option is to keep the precomputed matrix C^{-1} modulo q and compute $a \equiv C^{-1}h \pmod q$.

In signature generating the vector y may be computed in around $2ckn + kn^2/2$ multiplications modulo q and the complexity of computing $x \equiv B^{-1}y$ is n^2 multiplications. The signature size is $n\lceil \log_2 q \rceil$ bits. The complexity of verification is essentially kn^2 multiplications modulo q to compute Ax . Remark, that q may be taken relatively small compared with digital signature algorithms from the NIST competition, see Table 1. So the computation is very fast in that case.

For the public key one has to keep the matrix A , that is kn^2 residues modulo q . For the private key one keeps the matrix C^{-1} (or a minimal polynomial for C to apply the Wiedemann algorithm) and the matrices B^{-1}, T . Instead, one may keep a seed and generate B^{-1} and T with this seed if necessary. So the size of the private key may be made negligible.

The complexity parameters are significantly lower for the reduced version of the signature algorithm in Section 6.

5. CRYPTANALYSIS

There are three approaches to the cryptanalysis: find private key given public key only, find private key by analysing a number of valid signatures, and forge signatures without the knowledge of the private key.

5.1. Private Key Recovery. We have not found any efficient method to recover the matrices C, T, B from $A = CTB$ besides searching over C or B according to their definitions. However, if λ and k are small, one may recover around $n/\lambda - n/k\lambda$ rightmost columns of CT and B^{-1} relatively fast.

Really, let b be the rightmost column of the matrix B^{-1} . Then Ab is the rightmost column of CT . As T is in a column echelon form, the rightmost column of CT has at most $k\lambda$ nonzero entries. Let $A(m)$ be a sub-matrix of A in $m > n$ randomly chosen rows. With probability at least $\binom{kn-k\lambda}{m} / \binom{kn}{m}$ we have $A(m)b \equiv 0 \pmod q$ and b is recovered by solving a system of linear equations. One thus recovers the rightmost column of CT with the number of trials at most $\binom{kn}{m} / \binom{kn-k\lambda}{m}$. One now eliminates $k\lambda$ rows from A , where the rightmost column of CT has nonzero entries. Then the second rightmost column of CT is similarly recovered, etc. The l -th right most column of CT is found after $\binom{kn-(l-1)k}{m} / \binom{kn-lk\lambda}{m}$ trials for $l \leq n/\lambda - n/k\lambda$. For larger l the complexity of recovering the columns of CT grows very fast. Therefore, approximately $n/\lambda - n/k\lambda$ right most columns of CT and of B^{-1} may be recovered. However, it is not enough to forge signatures.

5.2. Guessing the Signature. Given a hash value h , one may try small values ($\leq \lambda c$ in absolute value) of some n entries of $e \equiv h - Ax \pmod q$, compute x by solving a system of linear equations and check if all other entries of e are at most λc in absolute value. The success probability is $(\frac{2\lambda c + 1}{q})^{(k-1)n}$.

5.3. Multiple Signatures Analysis. Let $m \geq kn$ messages $M_i, i = 1, \dots, m$ be signed with the same private key and s_i be their signatures respectively. Then

$$(4) \quad h_i - As_i \equiv Cf_i \pmod{q}, i = 1, \dots, m,$$

where Cf_i is the error vector for M_i, s_i and $h_i = \text{HASH}(M_i)$. The entries of f_i are bounded by c . In these equations h_i, A, s_i are public while f_i, C are secret. Let

$$U = [h_1 - As_1, \dots, h_m - As_m] \equiv [Cf_1, \dots, Cf_m] \pmod{q}$$

be a matrix of size $kn \times m$ whose columns are left hand side columns in (4). Let b be a row in $C^{-1} \pmod{q}$ and $v \equiv bU \pmod{q}$. The entries of v are at most c in absolute value as they are some entries of f_i . The vector v belongs to the lattice L generated by the rows of U modulo q . The lattice L is of rank m and of volume $\text{Vol} = q^{m-kn}$. The Euclidean norm of v is at most $c\sqrt{m}$ and L contains at least nk such vectors. Thus BKZ reduction or a sieving algorithm may be tried to recover such v and the rows b of $C^{-1} \pmod{q}$. Recovering the matrices T, B is then easy.

The application is successful if $c\sqrt{m}$ is around the first minimum of L . Otherwise, the lattice may contain too many vectors whose norm is close to the norm of the target vectors v . The first minimum is bounded by $\sqrt{\gamma_m} \text{Vol}^{1/m}$, where γ_m is the Hermit constant for rank- m lattices. For large m we have $\gamma_m \leq (1.745/2\pi e)m$, see [17]. We take the smallest m such that

$$|v| \leq c\sqrt{m} \leq \sqrt{(1.745/2\pi e)m} q^{(m-kn)/m}.$$

So $m \geq \left\lceil \ln q / \ln \left(\frac{q}{c} \sqrt{\frac{1.745}{2\pi e}} \right) \right\rceil kn$. The complexity of sieving is $2^{0.292m+o(m)}$ operations according to [12] with the memory of the same order. As m is very large, this method is inefficient. Block BKZ reduction does not provide any advantage over the plain sieving algorithm in this setting. For instance, for the parameters in Section 7 the size of the optimal block computed according to [3] is again m .

Let $\lambda = 2$ and let $C = P_1 + P_2$ be a sum of two permutation matrices P_1, P_2 such that the permutation $Q = P_1P_2^{-1}$ has two cycles of odd length. Then $C^{-1} = (1/2)K$ over rationals, where the entries of K are $0, \pm 1$ by Lemma 3 in the Appendix. So $K(h_i - As_i) = 2f_i \pmod{q}$ and one may recover K and therefore C faster than in the general case. That is why $\lambda = 2$ is not recommended.

5.4. Forging Signatures by Solving CVP. To forge the signature for a hash value h one is to find a vector e whose entries are bounded by λc in absolute value and $h \equiv Ax + e$ for some vector x . This problem always has a solution for the parameters defining the signature algorithm. Let L be a lattice of rank kn and of volume q^{kn-n} generated by the columns of A modulo q . Thus it is enough to solve an approximate CVP-instance for L in the maximum norm.

The solution of this problem implies a vector in L at the Euclidean distance $\leq \lambda c\sqrt{kn}$ from h . By Gaussian heuristic, see [17], the minimum distance between any h and L is $O(\sqrt{kn}q^{1-1/k})$ for average h . Therefore, to forge signatures one has to solve a CVP-instance for L with a small approximation factor $O(\frac{\lambda c}{q^{1-1/k}})$. The approximate CVP is hard for general lattices of large rank if the approximation factor is small [13]. One may also apply an exact CVP algorithm as in [1] or [7]. It is claimed in [7] that the CVP may be solved in heuristic time $2^{0.292d+o(d)}$ by a lattice sieving algorithm with the same amount of memory, where d is the rank of the lattice. That is not efficient for $d = kn$.

6. REDUCED PUBLIC KEY

According to the cryptanalysis in Section 5.1, around $n/\lambda - n/k\lambda$ rightmost columns of CT and of B^{-1} may be recovered relatively fast given A . Therefore these columns may not be considered secret. One can use that to reduce the size of the public key. Let $1 \leq s \leq n$ be an integer parameter and $r = n - s$. We set

$$C = \begin{pmatrix} C_1 & C_4 \\ C_2 & C_3 \end{pmatrix}, \quad T = \begin{pmatrix} T_1 & 0 \\ T_2 & T_3 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & 0 \\ B_2 & I \end{pmatrix}.$$

The sub-matrices C_1, C_3 are square and of size $ks \times ks$ and $kr \times kr$ respectively. That defines the size of the sub-matrices C_2, C_4 . The sub-matrices T_1, T_3 are of size $ks \times s$ and $kr \times r$ respectively, where T_3 is defined by (7). That is in the matrix T_3 only diagonal entries are non-zero. The sub-matrix B_1 is square and of size $s \times s$, invertible modulo q , and I is an identity matrix of size $r \times r$. We assume that C_3, C_4, T_3 are public while $C_1, C_2, T_1, T_2, B_1, B_2$ are secret. Then

$$A \equiv CTB \equiv \begin{pmatrix} C_1T_1B_1 + C_4(T_2B_1 + T_3B_2) & C_4T_3 \\ C_2T_1B_1 + C_3(T_2B_1 + T_3B_2) & C_3T_3 \end{pmatrix} \equiv \begin{pmatrix} A_1 & A_4 \\ A_2 & A_3 \end{pmatrix}.$$

Suppose C_3 is invertible modulo q . Then

$$(5) \quad A \equiv \begin{pmatrix} A' + C_4C_3^{-1}A_2 & A_4 \\ A_2 & A_3 \end{pmatrix} \pmod{q},$$

where $A' = C'T_1B_1$ and $C' \equiv C_1 - C_4C_3^{-1}C_2$. Let, for instance, $\lambda = 4$. One chooses the rows of C_1, C_2 to be of 1-norm equal to 1 and the rows of C_3, C_4 to be of 1-norm equal to 3. Experimentally, with high probability, C_3 is invertible modulo q and C_3^{-1} is quite dense. So the secret matrix C' is not sparse in that case and it hides the structure of T_1B_1 in the definition of A' . As A is public, one may recover the matrix A' . However, one can not recover any columns of $C'T_1$ and B_1^{-1} from A' as in Section 5.1.

As the matrices A_3, A_4 are sparse and may be made constants, one essentially stores only the entries of A_1, A_2 for the public key. That makes kns residues modulo q .

7. PROPOSED PARAMETERS

Let $(n, k, q, \lambda, c) = (230, 2, 23, 4, 2)$. One may use 256-bit hash algorithm with these parameters. The signature size is 1040 bits. There is only one tuple $[t_1, t_2] = [1, 5]$ modulo 23 up to a permutation, multiplication of the tuple modulo q by non-zero residues and changing sign of the tuple entries such that the system of inequalities (1) for $k = 2, c = 2$ has a solution u for every integer b_1, b_2 . One takes such $[t_1, t_2]$ independently to define the diagonal entries of T , other entries are chosen uniformly at random. For a reduced public key one chooses T such that only diagonal entries are non-zero for $r = n/2$ right most columns according to Section 6. The matrices C, B are chosen randomly according to the definitions in Section 6. The entries of $e = Cz$ are bounded by $\lambda c = 8$ in absolute values. The probability to find x such that every entry of $e \equiv h - Ax \pmod{q}$ is bounded by λc in absolute value is $\left(\frac{2\lambda c + 1}{q}\right)^{(k-1)n} \approx 2^{-100.3}$, see Sections 8 and 5.2. Therefore, one has to solve $2^{100.3}$ linear systems with 230 variables modulo 23 to forge the signature with this method on the average. This seems to be the best attack so far in this setting. The algorithm in [1] solves an instance of CVP in Section 5.4 and

therefore finds e in $2^{134.3}$ operations with memory size $2^{134.3}$, these figures may in fact be larger due to hidden factors. The verification of one signature takes around 10^{-3} of a second on a common computer. Generating of one signature is about four times longer.

7.1. EHT versus NIST signature candidates. We summarise the security and complexity parameters of the new algorithm EHT in Table 1 and put them against those of the NIST 3-rd round digital signature candidates with approximately matching security, see [14]. In Table 1 bits, bytes and kilobytes are abbreviated by b, B and kB respectively. The size of the public key is reduced according to Section 6. The parameters of the new algorithm are comparable with those of the NIST candidates.

TABLE 1. Comparison with NIST 3-rd round candidates

	security	public key	arithm. q	sign.
EHT	116 b	30.2 kB	23	130 B
Dilithium level 2	121 b	1.31 kB	8380417	2420 B
Falcon level 2	120 b	0.897 kB	12289	666 B
Rainbow level 1	147 b	58.8 kB	2^{16}	66 B

8. UNDERLYING PROBLEM

Let $n, k, s, r = n - s$ be positive integers, q be an odd prime, and let $\delta < q$ be a positive real. Let R denote an integer matrix of size $kn \times s$, whose entries modulo q were generated uniformly at random. Also, let K be an integer matrix of size $kn \times r$ and of rank r modulo q . Given an integer vector h of size kn , one asks to find integer vectors x_1, x_2 of size s and r respectively and an integer vector e of size kn such that every entry of e is at most δ in absolute value and

$$(6) \quad Rx_1 + Kx_2 + e \equiv h \pmod{q}.$$

Let's denote the concatenations $A = R|K$ and $x = x_1|x_2$. Then (6) is equivalent to $Ax + e \equiv h \pmod{q}$. That is an instance of the CVP in the maximum norm for the lattice generated by the columns of A modulo q . Heuristically, the problem has a solution for every h if $q^n(2\delta + 1)^{kn} > q^{kn}$.

The security of the new signature algorithm is based on the hardness of solving (6) for some matrices R and K . For the parameters $k = 2, s = r = n/2, \lambda = 4$ and matrices specified in Sections 6, the matrix R may be considered as generated

uniformly according to Section 9 below. To construct the matrix K , let

$$(7) \quad T_3 = \begin{pmatrix} t_{11} & 0 & \dots & 0 \\ t_{21} & 0 & \dots & 0 \\ \dots & & & \\ t_{k1} & 0 & \dots & 0 \\ 0 & t_{12} & \dots & 0 \\ 0 & t_{22} & \dots & 0 \\ \dots & & & \\ 0 & t_{k2} & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & t_{1r} \\ 0 & 0 & \dots & t_{2r} \\ \dots & & & \\ 0 & 0 & \dots & t_{kr} \end{pmatrix}$$

be a matrix of size $kr \times r$ for some non-zero entries t_{ij} specified in Section 2.2. Namely, each tuple $[t_{1j}, t_{2j}, \dots, t_{kj}]$ has to satisfy (1).

Also, let C_3 and C_4 be matrices of size $kr \times kr$ and $ks \times kr$ respectively and whose rows have 1-norm (the sum of the absolute values of the entries in each row) equal to 3. The matrices C_3, C_4, T_3 are public. Then

$$(8) \quad K = \begin{pmatrix} C_4 T_3 \\ C_3 T_3 \end{pmatrix}$$

is a matrix of size $kn \times r$ and of rank r . See Section 9 below for details.

8.1. Sub-problem. Let a matrix K of size $kn \times s$ be defined by (8). Given an integer vector h of length kn find an integer vector z of length s and an integer vector e of length kn such that $Kz + e \equiv h \pmod{q}$ and every entry of e is bounded by δ in absolute value. Heuristically, the problem has a solution if $q^s(2\delta + 1)^{kn} > q^{kn}$. An efficient algorithm to solve this equation implies an efficient algorithm to solve the equation (6).

9. REDUCTION TO THE UNDERLYING PROBLEM

When constructing the public matrix A in Section 6, the matrix T_2 of size $kr \times s$ may be taken uniformly at random. So the matrix A_2 of size $kr \times s$ in (5) is uniformly distributed. The matrix $A' = C'T_1B_1$ of size $ks \times s$ is generated independently of A_2 . It depends on $\frac{ks(s-1)}{2}$ randomly chosen entries of T_1 and on s^2 randomly chosen entries of the invertible B_1 , besides randomly chosen C_1, C_2 . So the matrix $R = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ of size $kn \times s$ depends on $\frac{ks(s-1)}{2} + krs + s^2$ independent residues modulo q and on C_1, C_2 .

Let $k = 2$ and $s = r = n/2$, and $\lambda = 4$. We set the public C_3, C_4 to be a sum of three permutation matrices of size $s \times s$ each. The secret C_1, C_2 are permutation matrices of size $s \times s$.

Then the matrix R of size $2n \times n/2$, that is with n^2 entries depends on $n^2 - n/2$ randomly and independently chosen residues modulo q and besides on randomly chosen C_1, C_2 . As $q^{n^2 - n/2} (\frac{n}{2}!)^2 > q^{n^2}$ for relatively small q , the number of independent parameters for the entries of R is larger than the number of their entries. So,

heuristically, the matrix R is uniformly distributed. By construction, the matrix $K = \begin{pmatrix} A_4 \\ A_3 \end{pmatrix} = \begin{pmatrix} C_4 T_3 \\ C_3 T_3 \end{pmatrix}$ is of full rank $r = n/2$.

The signature x for a hash value h satisfies $Ax + e = h$, where the entries of e are at most λc in absolute value. Let $x = x_1|x_2$, where x_1, x_2 are of size s . Then $Ax + e = h$ implies $Rx_1 + Kx_2 + e = h$. To forge a signature one must solve an instance of the problem in Section 8 with parameters $k = 2, r = s = n/2$ and $\delta = \lambda c = 4c$.

The size of the public key is essentially n^2 residues modulo q . Verification cost is essentially n^2 multiplications modulo q .

10. SOLVING THE UNDERLYING PROBLEM BY GUESSING

There are two guessing type algorithms to find a solution to (6). First, one may guess n small (bounded by δ) entries of e , find $x = x_1|x_2$ by solving a system of linear equations modulo q , then check other $kn - n$ entries of e . If they all are small (bounded by δ), then the solution is found. The probability of success is $\frac{(2\delta+1)^{kn-n}}{q}$.

Second, let x_1 be a random vector of our choice and $h - Rx_1 = h_1|h_2$, where the size of h_1 is ks and the size of h_2 is kr . Also, let $e = e_1|e_2$, where the size of e_1 is ks and the size of e_2 is kr . Assume that C_3 is invertible modulo q . By using Theorem 1, one finds vectors x_2 and f such that the entries of f are small (bounded by $\lfloor \delta/3 \rfloor$) and

$$T_3 x_2 + f = C_3^{-1} h_2.$$

Then $C_3 T_3 x_2 + e_2 = h_2$, where the entries of $e_2 = C_3 f$ are small (bounded by δ) as the 1-norm of the rows of C_3 is equal to 3. The probability that the vector $e_1 = h_1 - C_4 T_3 x_2$ has small entries is $\frac{(2\delta+1)^{ks}}{q}$. So (6) is satisfied for such $x = x_1|x_2$ and e . The success probabilities of the both algorithms are equal for $ks = kn - n$. Therefore, one may set $s = n - n/k$. We conjecture that the equation (6) is hard for K defined by (8), such s and for large n .

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11. APPENDIX

Lemma 3. *Let P_1, P_2 be permutation matrices of size $m \times m$. Then $C = P_1 + P_2$ is of rank m if and only if the permutation $Q = P_2 P_1^{-1}$ has only odd cycles. In this case, $\det C = \pm 2^s$, where s is the number of cycles in Q .*

Proof. The matrix C is of full rank if and only if the system of linear equations $x(P_1 + P_2) = 0$ has only zero solution. The system is equivalent to $x = -xQ$. The latter has a non-zero solution if and only if Q has at least one cycle of even length. That proves the first part of the lemma. To prove the rest, let r_1, \dots, r_s be the lengths of the cycles in Q and $R = C P_1^{-1} = I_m + Q$, where I_m is an identity permutation.

Let r be an odd number and $P = [2, 3, \dots, r, 1]$ be a permutation with exactly one cycle of length r . We consider P as a matrix and get

$$(9) \quad R_r = I_r + P = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

So $\det(R_r) = 2$. There is a permutation U such that

$$(10) \quad U^{-1} C P_1^{-1} U = U^{-1} R U = I_m + U^{-1} Q U = \begin{pmatrix} R_{r_1} & 0 & \dots & 0 \\ 0 & R_{r_2} & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & R_{r_s} \end{pmatrix},$$

where R_{r_i} is defined by (9). Therefore, $\det(R) = 2^s$.

□

For $\lambda = 2$ one may choose a random secret permutation Q of $[1, 2, \dots, kn]$ with two cycles of close odd lengths and a random secret permutation P_1 , and set $P_2 = Q P_1$. Then $C = P_1 + P_2$. The inversion of R_r is a Toeplitz matrix of size

$r \times r$ whose first row is $[1, -1, 1, -1, \dots, 1]$ multiplied by $1/2$. By (10) one can easily compute the inversion of C .

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