

# Efficient Online-friendly Two-Party ECDSA Signature

Haiyang Xue

The University of Hong Kong  
State Key Laboratory of Information  
Security, Institute of Information  
Engineering, Chinese Academy of  
Sciences  
haiyangxc@gmail.com

Man Ho Au

The University of Hong Kong  
allenau@cs.hku.hk

Xiang Xie

Shanghai Key Laboratory of  
Privacy-Preserving Computation  
xiexiang@matrizelements.com

Tsz Hon Yuen

The University of Hong Kong  
thyuen@cs.hku.hk

Handong Cui

The University of Hong Kong  
hdcui@cs.hku.hk

## ABSTRACT

Two-party ECDSA signatures have received much attention due to their widespread deployment in cryptocurrencies. Depending on whether or not the message is required, we could divide two-party signing into two different phases, namely, offline and online. Ideally, the online phase should be made as lightweight as possible. At the same time, the cost of the offline phase should remain similar to that of a normal signature generation. However, the existing two-party protocols of ECDSA are not optimal: either their online phase requires decryption of a ciphertext, or their offline phase needs at least two executions of multiplicative-to-additive conversion which dominates the overall complexity. This paper proposes an online-friendly two-party ECDSA with a lightweight online phase and a single multiplicative-to-additive function in the offline phase. It is constructed by a novel design of a *re-sharing* of the secret key and a *linear sharing* of the nonce. Our scheme significantly improves previous protocols based on either oblivious transfer or homomorphic encryption. We implement our scheme and show that it outperforms prior online-friendly schemes (i.e., those have lightweight online cost) by a factor of roughly 2 to 9 in both communication and computation. Furthermore, our two-party scheme could be easily extended to the 2-out-of- $n$  threshold ECDSA.

## KEYWORDS

ECDSA; threshold signature; two-party signature; blockchain; zero-knowledge proof

## 1 INTRODUCTION

Threshold digital signatures [13, 14] allow distributed signing among  $n$  parties such that a given message is signed if and only if at least  $t + 1$  of the participants agree to sign it. Threshold signature has attracted a lot of attention in the academic community and industry recently, possibly due to its applications in blockchain and cryptocurrencies. A decentralized key management framework of ECDSA [11], the most commonly used signature scheme in blockchain, is urgently required to protect the secret key and hence the cryptocurrency.

**Threshold ECDSA.** ECDSA [11] is the Elliptical Curve (EC) version of the Digital Signature Algorithm (DSA). The core part of ECDSA

involves computing

$$s = k^{-1}(H(m) + xr) \quad (1)$$

where  $x$  is the secret key,  $k$  is the secret nonce, and  $r$  is the public nonce (i.e., the  $x$ -coordinate of  $k \cdot P$  where  $P$  is the EC group generator). The main obstacle of constructing a threshold version of ECDSA is to compute, in a distributed way,  $s$  satisfying Equation 1, where  $k$  and  $x$  are secrets shared among the participants. In detail, given the shares of  $k$  and  $x$ , computing the shares of  $k^{-1}$  and  $k^{-1}x$  are the most expensive part of the entire signing procedure. (We remark that, if not specified, the computation in this section is over  $\mathbb{Z}_q$  where  $q$  is the EC group order.)

A large number of practical protocols aim to address this obstacle in recent years. Lindell [25] proposed a very efficient two-party signing protocol, which utilizes multiplicative shares of  $k$ ,  $x$ , and the additively homomorphic encryption scheme Paillier. The resulting signing procedure only involves homomorphic operations and a decryption operation, and thus is practical. Other follow-up works such as [6, 8, 15, 16, 19, 26, 33] handle the above obstacles with the help of the *multiplicative-to-additive* functionality (denoted as MtA hereafter). Specifically, with  $a$  and  $b$  as inputs, the MtA functionality securely computes and outputs  $\alpha$  and  $\beta$  such that  $\alpha + \beta = ab$ . Thus it helps two parties to transform shares of  $k$ ,  $x$  into additive share of  $k^{-1}$  and  $k^{-1}x$ , or to transform shares of  $k^{-1}$ ,  $x$  into additive share of  $k^{-1}x$  and computing  $k \cdot P$ . Then, each participant could easily compute the additive share of  $s$  from  $r$  and their shares of  $k^{-1}$  and  $k^{-1}x$ . After the reveal of additive share of  $s$ , the signing procedure simply sums them up and outputs the signature after verifying its correctness.

However, realizing the MtA functionality always involves complicated zero-knowledge proof systems. This makes MtA the heaviest part of threshold signature schemes. A typical threshold signing invokes the MtA functionality 2 to 4 times. Specifically, given additive shares of  $k = k_1 + k_2$  and  $x = x_1 + x_2$ , two calls of MtA are required to compute shares of  $k^{-1}$ , and another two MtA are needed to compute additive shares of  $k^{-1}x$ . The same holds when given additive shares of  $k^{-1}$  and  $x$  to jointly compute  $k \cdot P$  and additive share of  $k^{-1}x$ .

**Online and Offline Signing.** In real-world applications, it is extremely useful to divide an MPC protocol into *offline* and *online* phases. In the offline phase, the message-independent part of the

**Table 1: The offline/online cost of two-party (case of) ECDSA. Enc (resp. Dec) is an execution of Paillier/CL encryption (resp. decryption). “NI” indicates non-interactive online signing. The online phase of “Fast” schemes requires extra dozens of EC point multiplications besides the verification.**

Schemes	$x$	$k$ or $k^{-1}$	Offline	(NI) Online
[7, 25]	$x_1 x_2$	$k_1 k_2$	Enc	(✓) Dec
[15] 2-of-2	$x_1 x_2$	$k_1 k_2$	2MtA	(✓) optimal
[15] 2-of- $n$	$x_1 + x_2$	$k_1 k_2$	3MtA	(✓) optimal
[26]	$x_1 + x_2$	$k_1 + k_2$	2MtA	(×) MtA
[8, 19]	$x_1 + x_2$	$k_1 + k_2$	4MtA	(×) Fast
[6, 16]	$x_1 + x_2$	$k_1 + k_2$	4MtA	(✓) optimal
<b>2ECDSA</b>	$x_1 + x_2$	$k_1(r_1 + k_2)$	1MtA	(✓) optimal

protocol is computed. This is also known as pre-processing or pre-computation and can utilize idle CPU time. This phase could be run several times and in batches. Output of the offline phase is stored for use in the online phase when the message to be processed is known. This separation is commonly discussed in the literature. For example, general-purpose MPC protocols could run the offline phase to generate a batch of random Beaver triples [3] to be used later to handle multiplication gates in the online phase.

As for threshold signatures, several works [6, 15, 16] have already considered splitting the signing process of threshold ECDSA into offline and online phases. In threshold signature, verifying the final signature is inevitable as it allows the detection of malicious behaviors. Thus, we say that the online phase of a threshold signature is *optimal* if it is non-interactive and its cost is the same as a single signature verification. The scheme is said to be *online-friendly* if its online phase is optimal. Existing works of threshold ECDSA either have their online phase [7, 25, 33] requiring a decryption of an additively homomorphic encryption or their offline phase [6, 15, 16, 19, 26, 33] needing 2 to 4 calls of MtA functionality. Table 1 shows a summary of these works.

Starting from [27], several competitive two-party ECDSA protocols [7, 25, 33] utilize multiplicative sharing in combination with additively homomorphic encryption. Concretely, the signing key  $x$  and the nonce  $k$  are shared multiplicatively as  $x_1 x_2$  and  $k_1 k_2$ . Two parties jointly compute  $k^{-1}x$  from multiplicative shares with the additively homomorphic property of the underlying encryption scheme. In spite of Paillier encryption as in [25] or Castagnos and Laguillaumie (CL) encryption as in [7, 33], the final online phase of this method requires a transfer and the decryption of the ciphertext. Although the entire signing procedure is relatively practical, the online phase is still heavy and not optimal.

As shown in Table 1, previous protocols require either a relatively slow online computation, or 2 to 4 calls of MtA in the offline phase when the online phase is optimal. The sub-protocol MtA dominates the overall complexity since it is either computationally expensive (for those based on homomorphic encryption [8, 26]) or needs a very large amount of communication (for those based on oblivious transfer [15]). For example, a single call of Paillier-based MtA needs 14 Paillier encryption/exponentiations and that of OT-based MtA requires communication of approximately 90 KB of data.

In this paper, we focus on the two-party ECDSA. Our goal is giving a construction to achieve (nearly) optimal online performance with minimal number of calls, i.e., a single call, to the MtA functionality.

## 1.1 Our Contribution

We propose an online-friendly two-party ECDSA, 2ECDSA, such that its online computation is nearly optimal and its offline phase just needs a single call of MtA.

- (1) The online phase of our protocol is non-interactive and optimal. It only requires transmitting a single element, and its computation cost is dominated by the verification of the resulting signature. Our offline phase runs in three-pass with a single MtA, thus has significant improvement over [15] and the two-party cases of [6, 8, 16, 19, 26, 33]. Two novel techniques are developed, which may be of independent interest.
  - (a) The *linear sharing* of the nonce  $k = k_1(r_1 + k_2)$ , where  $k_1, r_1$  are chosen by party  $P_1$  and  $k_2$  is chosen by the other party,  $P_2$ . It is different from the existing simple additive or multiplicative sharing of the nonce.
  - (b) The *re-sharing* of the secret  $x$  by using  $k_2$  in the offline phase. New share  $x'_1$  (resp.  $x'_2$ ) of the signing key is chosen by  $P_1$  (resp.  $P_2$ ), such that  $x = x'_1 k_2 + x'_2$  (which is also a linear function).

These techniques enable us to construct an online-friendly 2ECDSA with a single MtA. Details are given in Sec. 1.2.

- (2) We provide an implementation of our two-party 2ECDSA protocol in Rust, with instantiation of the MtA functionality from Paillier encryption, CL encryption, and oblivious transfer. We give an efficiency comparison with all previous two-party ECDSA and the two-party case of threshold ECDSA. Details are shown in Table 2. On the premise of preserving fast online computation, our scheme reduces the offline cost of Paillier-based protocol to 226 ms and 6.3 KB, and further to 141 ms and 4.1 KB based on Paillier-EC assumption. For CL-based instantiation, the complexity of the offline phase is 1386 ms and 1.7 KB. Based on oblivious transfer (OT), the offline cost of our scheme is 2.6 ms and 90.9 KB.
- (3) Applying 2-out-of- $n$  Shamir secret-sharing [30] to  $n$ -party additively shared secret during key generation, our two-party 2ECDSA could be easily extended to the 2-out-of- $n$  ECDSA. The general scheme’s signing phase has the same complexity as 2ECDSA.

## 1.2 Technical Overview

Recall that the main bottleneck of previous schemes is due to the use of complex protocols for the two parties to compute  $k^{-1}(H(m) + rx)$  such that  $x = x_1 + x_2$  and  $k = k_1 + k_2$  (resp.  $x = x_1 x_2$  and  $k = k_1 k_2$ ) in the case of additive sharing (resp. multiplicative sharing). In other words, the bottleneck is inherent in the multiple executions of MtA for these shares.

**Starting point.** We start from a simple combination of multiplicative sharing of  $k$  (i.e.  $k = k_1 k_2$ ) and additive sharing of  $x$  (i.e.  $x = x_1 + x_2$ ). This has been utilized by Doerner et al. [15] to handle

**Table 2: Comparison of signing with two-party protocols and two-party case of threshold ECDSA from Paillier, OT and CL respectively. These concrete numbers are based on computational security parameter  $\lambda = 128$  and statistical security parameter 80. For those Paillier-based schemes, E represents a Paillier exponentiation over  $\mathbb{Z}_{N^2}$  (we approximately count 3 operations of mod  $N$  as one operation of mod  $N^2$ . Some of E are single “short” exponentiation), while for CL-based schemes E means an exponentiation over CL group. M refers to elliptic curve point multiplication which is almost free compared with E. In the communication column, EC points, CL group elements, and ring elements of Paillier are encoded by  $\kappa$ ,  $7\kappa$  (only for  $\kappa = 256$ ), and  $2\ell_N$  bits respectively. Estimating in parentheses includes the constant overhead with the standard security recommendation, i.e.  $\kappa = 256$  and  $\ell_N = 3072$ . “Paillier-EC” means that Paillier-EC assumption is applied to eliminate a zero-knowledge proof.**

Signing Protocols	Computation		Communication		Passes
	offline	online	offline	online	
LNR18 [26]	28E + 157M (461ms)	14E + 121M (302ms)	$32\ell_N + 67\kappa$ (12KB)	$16\ell_N + 51\kappa$ (6.6KB)	8
GG18 [19]	42E + 40M (1237ms)	17M (3ms)	$40\ell_N + 18\kappa$ (15.5KB)	$9\kappa$ (288B)	9
CGGMP20 [6]	208E + 44M (2037ms)	2M (0.2ms)	$118\ell_N + 20\kappa$ (44KB)	$\kappa$ (32B)	4
2ECDSA (Paillier)	14E + 11M (226ms)	2M (0.2ms)	$16\ell_N + 11\kappa$ (6.3KB)	$\kappa$ (32B)	3
Lin17 [25] (Paillier-EC)	2E + 8M (34ms)	1E + 2M (8ms)	$12\kappa$ (192B)	$2\ell_N$ (768B)	3
GG18 [19] (Paillier-EC)	18E + 40M (360ms)	17M (3ms)	$16\ell_N + 18\kappa$ (6.6KB)	$9\kappa$ (288B)	9
2ECDSA (Paillier-EC)	8E + 14M (141ms)	2M (0.2ms)	$10\ell_N + 12\kappa$ (4.1KB)	$\kappa$ (32B)	3
CCLST19 [7]	4E + 8M (475ms)	1E + 2M (190ms)	$6\kappa$ (208B)	$14\kappa$ (505B)	3
CCLST20 [8]	28E + 8M (3316ms)	17M (3ms)	$140\kappa$ (4.5KB)	$9\kappa$ (288B)	8
YCX21 [33]	28E + 8M (4550ms)	17M (3ms)	$140\kappa$ (4.5KB)	$9\kappa$ (288B)	8
2ECDSA (CL)	11E + 11M (1386ms)	2M (0.2ms)	$53\kappa$ (1.7KB)	$\kappa$ (32B)	3
DKLS18 [15]	13M (2.9ms)	2M (0.2ms)	$16\kappa^2$ (169.8KB)	$\kappa$ (32B)	2
DKLS19 [16]	13M (3.7ms)	2M (0.2ms)	$20\kappa^2$ (180KB)	$\kappa$ (32B)	7
2ECDSA (OT)	11M (2.6ms)	2M (0.2ms)	$8\kappa^2$ (90.9KB)	$\kappa$ (32B)	3

2-out-of- $n$  threshold scheme, albeit, in a rather inefficient way. To jointly compute  $s = H(m)/k_1k_2 + r(x_1 + x_2)/k_1k_2$ , they adopt three MtA to export the additive shares of  $1/k_1 \cdot 1/k_2$ ,  $x_1/k_1 \cdot 1/k_2$ , and  $1/k_1 \cdot x_2/k_2$  respectively. Thus, their two-party case of 2-out-of- $n$  scheme is worse than their direct two-party scheme where only two MtA are required.

**The first attempt: Re-sharing of the secret.** We resolve this problem with a *re-sharing* of secret  $x$  using a share of nonce  $k_2$ . Concretely, in the offline phase the secret  $x = x_1 + x_2$  is re-shared to  $x'_1, x'_2$  using one MtA such that

$$x_1 + x_2 = x'_1k_2 + x'_2.$$

In the online phase,  $P_2$  computes  $s_2 = k_2^{-1}(H(m) + rx'_2)$ , and  $P_1$  could derive the signature component  $s$  from  $s_2, k_1$  and  $x'_1$  since

$$\begin{aligned} s &= k_1^{-1}(s_2 + rx'_1) \\ &= k_1^{-1}k_2^{-1} [H(m) + r(x'_2 + k_2x'_1)] \\ &= k^{-1}(H(m) + rx). \end{aligned}$$

The offline phase re-shares the secret  $x = x_1 + x_2$  into  $x = x'_1k_2 + x'_2$ , with a single MtA. Specifically,  $P_1$  chooses a random  $x'_1 \leftarrow \mathbb{Z}_q$ , and then  $P_1$  and  $P_2$  invoke MtA with input  $x'_1$  and  $k_2$  respectively, and receive shares  $t_A$  and  $t_B$  such that  $t_A + t_B = x'_1k_2$ . Then  $P_1$  masks  $x_1$  with  $t_A$  and sends  $cc = t_A - x_1$  to  $P_2$ .  $P_2$  could

extract its new share  $x'_2$  from  $cc, x_2$ , and  $t_B$ , since

$$x'_2 = x_1 + x_2 - x'_1k_2 = (t_A - cc) + x_2 - x'_1k_2 = -t_B - cc + x_2.$$

The resulting scheme is online-friendly and requires a single MtA in the offline phase. Unfortunately, this solution is insecure and a malicious adversary may cheat.

**Attack on the first attempt.** We show that a malicious  $P_2$  can obtain  $x_1$  in the previous scheme. Observe that

$$x_1 = t_A - cc = x'_1k_2 - t_B - cc,$$

where  $k_2, t_B, cc$  are known to  $P_2$ . The malicious  $P_2$  can set  $k_2 = 0$  as the input of MtA and learn  $P_1$ 's secret  $x_1 = -t_B - cc$ .

**Our solution: Linear sharing of the nonce.** A simple solution to rule out the attack of  $k_2 = 0$  is to add a zero-knowledge proof of  $k_2 \neq 0$  for  $P_2$ , but it would be quite expensive. Instead, we apply a re-randomization method to solve this problem. Now, the re-sharing of the secret is changed to  $x = x'_1(k_2 + r_1) + x'_2$ , where  $r_1$  is chosen by  $P_1$  and could be given to  $P_2$ . To achieve that,  $P_1$  now masks  $x_1$  with  $t_A$  and a random  $r_1$  by setting  $cc = t_A + r_1x'_1 - x_1$ . Now  $x_1 = t_A + r_1x'_1 - cc = x'_1(k_2 + r_1) - t_B - cc$ . If  $r_1$  is chosen by  $P_1$  after  $k_2$  is chosen by the (malicious)  $P_2$ ,  $P_2$  learns nothing about  $x_1$  since  $k_2 + r_1 = 0$  with probability at most  $1/q$ .

In order to support this change in the re-sharing of the secret, we need to view  $P_2$ 's share of the nonce as  $(k_2 + r_1)$ . Interestingly, while we require  $r_1$  to be chosen by  $P_1$  (in order to withstand the above

attack),  $(k_2 + r_1)^{-1}$  can be computed by  $P_2$  itself (otherwise, we need another round of MtA). Hence, we change the multiplicative sharing of the nonce  $k$  into a linear function  $k_1(r_1 + k_2)$ . In the offline phase,  $P_1$  picks random  $r_1$  and sends it to  $P_2$ . Later in the online phase,  $(k_2 + r_1)^{-1}$  can be computed by  $P_2$ . This is the reason why we use a linear sharing of the nonce.

Additional consistency check is needed when setting up  $k \cdot P$  by  $P_1$  and  $P_2$ . Details are given in Sec. 3.

### 1.3 Extension and Instantiations

**Extending to 2-out-of- $n$  Access Structures.** Our two-party 2ECDSA is presented using 2-out-of-2 additive share of the private key  $x$ . It could be easily extended to a 2-out-of- $n$  protocol using Shamir secret-sharing, in a way similar to Gennaro and Goldfeder [19]. A similar approach has been described in [26] and [6]. We also let  $x$  be the additive share of each party's contribution  $x_i$ , i.e.  $x = x_1 + \dots + x_n$ . It is natural to use a 2-out-of- $n$  Shamir secret-sharing [30] to convert it into a 2-out-of- $n$  shares of  $x$  in a verifiable manner.

Here, we give a brief overview. For details, please refer to Appendix A. Each  $P_i$  chooses a linear function  $f_i$  such that  $f_i(0) = x_i$  and sends  $x_i^{(j)} = f_i(j)$  to  $P_j$  for all  $j \in [1, n]$ . Then, every two parties  $P_i, P_j$  could recover  $x_k$  (for every  $k \in [1, n]$ ) via interpolating from  $x_k^{(i)}$  and  $x_k^{(j)}$ , i.e.,  $x_k = \Lambda_i x_k^{(i)} + \Lambda_j x_k^{(j)}$  where  $\Lambda_i, \Lambda_j$  are Lagrange coefficients. With Shamir's secret sharing, any two parties  $P_i, P_j$  could generate  $\Lambda_i \sum_{k=1}^n x_k^{(i)}$  and  $\Lambda_j \sum_{k=1}^n x_k^{(j)}$  such that  $x = \Lambda_i \sum_{k=1}^n x_k^{(i)} + \Lambda_j \sum_{k=1}^n x_k^{(j)}$ . They could further invoke the signing of 2ECDSA directly with  $\Lambda_i \sum_{k=1}^n x_k^{(i)}$  and  $\Lambda_j \sum_{k=1}^n x_k^{(j)}$  as their additive shares of  $x$ . The general 2-out-of- $n$  ECDSA has the same signing protocol with 2ECDSA.

**On the instantiations of MtA.** Existing constructions of MtA from OT, Paillier encryption, and CL-encryption can be directly applied to our 2ECDSA.

Doerner et al. [15] proposed an OT-based MtA from Simplest OT [10] and KOS [23] OT-extension protocols. Their MtA is computationally efficient, while the communication cost is rather high (e.g., 90 KB for 128 bits security). Although improved method was applied, their two-party ECDSA requires a communication cost of at least 168 KB. Integrating the OT-based MtA into our scheme, we obtain the first OT-based two-party signature with communication cost less than 100 KB at 128-bit security.

Paillier-based MtA is first proposed in [27] and improved by [6, 19, 26]. The main issue when using Paillier cryptosystem for MtA is that it operates over  $\mathbb{Z}_N$  rather than  $\mathbb{Z}_q$ , where  $q$  is the EC group order. As a result, expensive zero-knowledge range proofs are required. There are several solutions to simply or even remove these proofs, such as range proof with slack [26], and a non-standard Paillier-EC assumption [19] (refer to Appendix C.3). Using range proof with slack and Paillier-EC assumption, a Paillier-based MtA requires 8 Paillier exponentiations and a transmission of  $10 \log N$  bits. Depending on Paillier-EC assumption is applied or not, we propose two Paillier-based schemes.

Castagnos et al. [7] replace Paillier encryption with Castagnos and Laguillaumie [9] encryption over class group. The key feature of CL-encryption is that it allows instantiations where the message

space is exactly  $\mathbb{Z}_q$ . However, this kind of MtA requires new zero-knowledge proofs performed on unknown order groups, which is the heaviest part of all these constructions. Follow-up works [8, 33] further improve the underlying zero-knowledge proof system.

We note that there are other instantiations from noisy Reed-Solomon encodings (RS) [20] and Ring-LWE [2]. However, these constructions are not very suitable for the parameters related to ECDSA. We leave it as the future work to improve the underlying MtA protocol with these techniques.

### 1.4 Related Works and Discussion

**1.4.1 Related works.** Efficient constructions of threshold ECDSA fall into the following three categories. For more details, please refer to [1].

**Paillier-based Schemes.** Following [27], Lindell [25] proposed a competitive two-party ECDSA utilizing multiplicative sharing of secret and nonce in combination with Paillier encryption. Lindell's scheme has the best overall efficiency while its online phase needs to perform a Paillier decryption. Later, Lindell et al. [26], and Gennaro and Goldfeder [19] proposed a full threshold ECDSA protocol with additive shares. They both require at least 8 communication rounds and their online phase is interactive. Recently, Canetti et al. [6] proposed an online-friendly three-pass threshold ECDSA at the cost of extra overhead.

**CL-based Schemes.** Castagnos et al. [7] addressed the problem of relying on non-standard assumption in [25] by replacing Paillier encryption with CL-encryption [9] which allows the message space to match that of the signature space ( $\mathbb{Z}_q$ ). Castagnos et al. [8] further extended their work to full threshold by following Gennaro and Goldfeder's blueprint. Very recently, Yuen et al. [33] improves the underlying zero-knowledge proof of Castagnos et al.'s protocol, thus reduces the overall bandwidth and running time. However, the online phase of these schemes either requires the computation of decryption or is interactive.

**OT-based Schemes.** The OT-based schemes are online-friendly and do not require extra assumptions. Doerner et al. [15] used multiplicative sharing of the signing key and the nonce as [7, 25], and achieved fast online computation with the help of two MtA from the oblivious transfer. They also generalized their work to 2-of- $n$  cases. Later, they [16] proposed a full threshold scheme. These schemes are very fast in signing time, while the communication cost is the bottleneck.

**1.4.2 Discussion.** We further discuss issues related to round complexity and which party obtain the final signature in existing approaches.

**On the Communication Rounds of Signing.** As shown in Table 2, our scheme and [7, 25] require only 3 rounds (i.e., passes). All the other protocols except [15] require more communication rounds. However, as mentioned by [15, Sec. III], it relied on the generic group model (GGM) [31] to achieve this round reduction. Otherwise, 4 rounds are needed.

**On the Generation of Final Signature.** There are two commonly used syntax regarding the output of the protocol, namely, 1) each party generates a "signature share" and anyone (including a third party) could compute the final signature from the signature shares;

2) only one party obtains the final signature. Notable examples of the former include [6, 8, 16, 19, 26], and the latter include [7, 15, 25] as well as our scheme. The obstacle in adding this property to our scheme and [7, 15, 25] is that the nonce is shared multiplicatively, and thus signature component “s” cannot be reconstructed from a simple linear combination of signature shares.

## 1.5 Paper Organization.

The rest of paper is organised as follows. We review preliminaries in section 2. Then we propose our protocol and prove its security in section 3. In section 4, we show several instantiations of 2ECDSA. Finally, section 5 presents a comprehensive analysis and comparison with existing schemes.

## 2 PRELIMINARY

### 2.1 The ECDSA Signature

Let  $\mathbb{G}$  be an elliptic curve group of order  $q$  with base point (generator)  $P$ . The algorithm makes use of the hash function  $H$ . Curve coordinates and scalars are represented in  $\kappa = \log q$  bits. The ECDSA scheme works as follows [11].

- (1) **Keygen**( $1^\kappa$ ): on input  $1^\kappa$ 
  - Choose a random  $x \leftarrow \mathbb{Z}_q$ , set  $x$  as the private key.
  - Compute  $Q = x \cdot P$ , and set  $Q$  as the public key.
- (2) **Sign**( $x, m$ ): on input sign key  $x$  and message  $m$ 
  - Choose a random  $k \leftarrow \mathbb{Z}_q$ , compute  $R = (r_x, r_y) = k \cdot P$ .
  - Compute  $r = r_x \bmod q$  and  $s = k^{-1}(H(m) + rx) \bmod q$ .
  - Output  $(r, s)$  as the signature.
- (3) **Verify**( $m; (r, s)$ ) calculates  $(r_x, r_y) = R = s^{-1}H(m) \cdot P + s^{-1}r \cdot Q$  and outputs 1 if and only if  $r = r_x \bmod q$ .

It is well known that for every valid signature  $(r, s)$ , the pair  $(r, -s)$  is also a valid signature. To make  $(r, s)$  unique, in this paper, we mandate that the “smaller” of  $\{s, -s\}$  is the output.

### 2.2 Ideal Functionality for Two-Party ECDSA

The ideal functionality  $\mathcal{F}_{\text{ECDSA}}$  for two-party ECDSA is shown in Figure 1. It consists of two functions, namely, a key generation function **Keygen**, called once, and a signing function **Sign**, called an arbitrary number of times under the generated key.

### 2.3 The Multiplicative-to-Additive (MtA) Functionality

The  $\mathcal{F}_{\text{MtA}}$  functionality, listed in Figure 2, is parameterized by the group order  $q$ . It runs with two parties,  $P_1$  and  $P_2$ , who may participate in the Setup phase once, and the Multiplication phases as many times as they wish.  $\mathcal{F}_{\text{MtA}}$  runs and outputs  $\alpha, \beta$ , from two parties inputs, respectively,  $a$  and  $b$ , under the restriction that  $\alpha + \beta = ab \bmod q$ .

It could be instantiated from OT [15], Paillier encryption [19, 26], CL encryption [8], etc. Please refer to Sec. 4 for more details.

Consider an elliptic curve group  $\mathbb{G}$  of order  $q$  with generator  $P$ , then:

**Keygen:** On receiving **Keygen**( $\mathbb{G}, P, q$ ) from  $P_1$  and  $P_2$

- Generate key pair  $(Q, x)$  where  $x \leftarrow \mathbb{Z}_q$  and  $Q = x \cdot P$ .
- Choose a hash function  $H : \{0, 1\}^* \rightarrow \{0, 1\}^\kappa$ .
- Send  $Q$  and  $H$  to  $P_1$  and  $P_2$ .
- Store  $(\mathbb{G}, P, q, H, x, Q)$  and ignore further calls.

**Sign:** On receiving **Sign**( $\text{sid}, m$ ) from both  $P_1$  and  $P_2$ , where keys have been generated from a call to **Keygen** and  $\text{sid}$  has not been used

- Choose  $k \leftarrow \mathbb{Z}_q$  and compute  $R = (r_x, r_y) = k \cdot P$ .
- Compute  $r = r_x \bmod q$  and  $s = k^{-1}(H(m) + rx) \bmod q$ .
- Send  $(r, s)$  to both  $P_1$  and  $P_2$ .
- Store **(Complete, sid)** in the memory.

**Figure 1: The functionality  $\mathcal{F}_{\text{ECDSA}}$  for two-party ECDSA signature.**

**Setup:** On receiving **(setup)** from  $P_1$  and  $P_2$

- Store and send **(setup-complete)** to  $P_1$  and  $P_2$ .

**Multiplication:** On receiving **(input, sid,  $a \in \mathbb{Z}_q$ )** from  $P_1$ , **(input, sid,  $b \in \mathbb{Z}_q$ )** from  $P_2$  where  $\text{sid}$  has not been used, if **(setup-complete)** exists,

- Sample  $\alpha \in \mathbb{Z}_q$  and compute  $\beta = ab - \alpha \bmod q$ .
- Send **(output-1, sid,  $\alpha$ )** to  $P_1$
- Send **(output-2, sid,  $\beta$ )** to  $P_2$ .

**Figure 2: The functionality  $\mathcal{F}_{\text{MtA}}$  of multiplicative-to-additive protocol.**

### 2.4 Zero-Knowledge Proof

Let  $\mathcal{R}$  be a polynomial-time-decidable binary relation. The corresponding language  $L$  consists of statement  $x$  such that there exists witness  $w$  and  $(x, w) \in \mathcal{R}$ . We specify  $L$  as an NP language.

An interactive proof consists of an interactive prover algorithm  $P$  and a verifier algorithm  $V$  that runs in PPT time. We call  $(P, V)$  an interactive proof for relation  $\mathcal{R}$  if it has the completeness and soundness properties. Completeness means that for every  $x \in L$ ,  $\langle P, V \rangle(x)$  is always 1. Soundness means that for every  $x \notin L$  and every prover  $P^*$ ,  $\Pr[\langle P^*, V \rangle(x) = 1]$  is negligible. When the soundness holds for computationally bounded provers, the system is usually called an “argument”. In this paper, both proof and argument are collectively referred to as proof.

*Definition 2.1 (zero-knowledge).* Let  $(P, V)$  be an interactive proof for some language  $L$ .  $(P, V)$  is zero knowledge if for every PPT verifier  $V^*$  there exists a PPT simulator  $\text{Sim}$  such that the two ensembles  $\{\text{View}_{V^*}^P(x)\}_{x \in L}$  and  $\{\text{Sim}(x)\}_{x \in L}$  are identical.

We could also define statistical (resp. computational) zero-knowledge, if the two ensembles are statistically (resp. computationally) indistinguishable.

*Definition 2.2 (proof-of-knowledge).* Let  $\zeta : \{0, 1\}^* \rightarrow [0, 1]$  be a function.  $(P, V)$  is a proof of knowledge for relation  $\mathcal{R}$  with knowledge error  $\zeta$  if the following properties are satisfied:

- *Completeness*: If  $P$  and  $V$  follow the protocol on input  $x$  and private input  $w$  to  $P$  where  $(x, w) \in \mathcal{R}$ , then  $V$  always accepts.
- *Knowledge Soundness*: there exists a probabilistic oracle machine  $\text{Ext}$  such that for every prover function  $P^*$  and every  $x \in L$ ,  $\text{Ext}$  satisfies the following: Denote  $\epsilon(x)$  the probability that  $V$  accepts on input  $x$  after interacting with  $P^*$ . If  $\epsilon(x) > \zeta(x)$ , on input  $x$  with access to  $P^*$ ,  $\text{Ext}$  runs in expected polynomial time and outputs a string  $w$  such that  $(x, w) \in \mathcal{R}$  with probability at least  $\epsilon(x) - \zeta(x)$ .

We remark that every zero-knowledge proof of knowledge (ZKPoK) in this paper is transformed to non-interactive [4] using the Fiat-Shamir paradigm [18] in the random oracle model.

**ZKPoK of Discrete Logarithm.** Define the relation

$$\mathcal{R}_{DL} := \{(R, P, x) \mid R, P \in \mathbb{G}, R = x \cdot P\},$$

where the parameters of elliptic curve  $(\mathbb{G}; P; q)$  are implicit public parameters. We use the standard Schnorr proof [29] for  $\mathcal{R}_{DL}$ , apply the Fiat-Shamir [18] transformation to get a non-interactive ZKPoK. Denote by  $\text{nizkPoK}$  the proof generator and  $\text{Verifzk}$  the verify algorithm for  $\mathcal{R}_{DL}$ .

### 3 TWO-PARTY SIGNATURES FOR ECDSA

In this section, we present a two-party protocol 2ECDsa. We first describe the distributed key generation phase that is executed once, followed by the signing phase which may run multiple times. The process is also illustrated in Fig. 3.

Let  $\text{MtA}$  be the multiplicative-to-additive functionality,  $\text{nizkPoK}$  be the NIZK proof for discrete logarithm relation  $\mathcal{R}_{DL}$ . Let  $H : \{0, 1\}^* \rightarrow \{0, 1\}^K$  be a hash function.

Although we make a logical presentation of the signing procedure in four phases, in the actual protocol they are intertwined. We could reorder the messages in the instantiations such that messages of  $\text{MtA}$  from  $P_2$  to  $P_1$  come first, followed by all messages from  $P_1$  to  $P_2$ , which results in a 3-pass signing protocol.

**Distributed Key Generation Phase Keygen** $(\mathbb{G}, P, q)$ .

Given the joint input  $(\mathbb{G}, P, q)$  and security parameter  $\lambda$ :

- (1)  $P_1$ 's first message:
  - $P_1$  chooses a random  $x_1 \leftarrow \mathbb{Z}_q$ , and computes  $Q_1 = x_1 \cdot P$  and  $\text{nizk1} = \text{nizkPoK}(Q_1, x_1)$
  - $P_1$  sends  $f_1 = H(Q_1, \text{nizk1})$  to  $P_2$
- (2)  $P_2$ 's first message:
  - $P_2$  chooses a random  $x_2 \leftarrow \mathbb{Z}_q$ , and computes  $Q_2 = x_2 \cdot P$  and  $\text{nizk2} = \text{nizkPoK}(Q_2, x_2)$
  - $P_2$  sends  $Q_2, \text{nizk2}$  to  $P_1$
- (3)  $P_1$ 's second message:
  - On receiving  $Q_2, \text{nizk2}$  from  $P_2$ ,  $P_1$  verifies  $\text{nizk2}$ . If  $\text{Verifzk}(\text{nizk2})=0$ , abort
  - Else,  $P_1$  sends  $Q_1, \text{nizk1}$  to  $P_2$
- (4)  $P_2$ 's verification:
  - On receiving  $Q_1, \text{nizk1}$  from  $P_1$ ,  $P_2$  verifies  $\text{nizk1}$ .
  - If  $f_1 \neq H(Q_1, \text{nizk1})$  or  $\text{Verifzk}(\text{nizk1})=0$ , abort.
- (5) **Compute output**:
  - $P_1$  computes  $Q = Q_1 + Q_2$ , stores  $(Q, x_1, Q_1, Q_2)$ .
  - $P_2$  computes  $Q = Q_2 + Q_1$ , stores  $(Q, x_2, Q_1, Q_2)$ .

**Distributed Sign Phase**  $\text{Sign}(\text{sid}, m)$ . They begin with the session id  $\text{sid}$ ,  $m$  the message to be signed, and additive share of secret key. The protocol is divided into 4 logical steps, and only the last one is online.

- $P_1$  has  $(Q, x_1, Q_1, Q_2)$  as the output of  $\text{Keygen}$ , message  $m$  and session id  $\text{sid}$ .
- $P_2$  has  $(Q, x_2, Q_1, Q_2)$  as the output of  $\text{Keygen}$ , message  $m$  and session id  $\text{sid}$ .

(1) **Commitment of  $P_2$ 's nonce**:

- $P_2$  chooses a random  $k_2 \leftarrow \mathbb{Z}_q$ , and computes  $R_2 = k_2 \cdot P$  with  $\text{nizk3} = \text{nizkPoK}(R_2, k_2)$
- $P_2$  computes and sends  $f_2 = H(R_2, \text{nizk3})$  to  $P_1$ .

(2) **MtA and Consistency Check**:

- $P_1$  chooses a random  $x'_1 \leftarrow \mathbb{Z}_q$ , and computes  $Q'_1 = x'_1 \cdot P$ .
- $P_1$  and  $P_2$  invoke the  $\text{MtA}$  functionality with input  $x'_1$  and  $k_2$  respectively and receives  $t_A, t_B$  such that

$$t_A + t_B = x'_1 k_2 \pmod{q}.$$

- $P_1$  chooses a random  $r_1 \leftarrow \mathbb{Z}_q$ , and computes  $cc := t_A + x'_1 r_1 - x_1 \pmod{q}$  and sends  $(r_1, cc)$  to  $P_2$ .
- $P_2$  checks the consistency by checking

$$(t_B + cc) \cdot P \stackrel{?}{=} (r_1 + k_2) \cdot Q'_1 - Q_1.$$

- if the consistency check passes,  $P_2$  computes

$$x'_2 = x_2 - (t_B + cc) \pmod{q}.$$

(3) **Nonce Key Exchange**:

- $P_1$  chooses a random  $k_1 \leftarrow \mathbb{Z}_q$ , computes and sends  $R_1 = k_1 \cdot P$  with  $\text{nizk4} = \text{nizkPoK}(R_1, k_1)$  to  $P_2$ .
- $P_2$  aborts if  $\text{Verifzk}(\text{nizk4})=0$ , otherwise sends  $(R_2, \text{nizk3})$  to  $P_1$  and computes  $R = (r_x, r_y) = (k_2 + r_1) \cdot R_1$ , and  $r = r_x \pmod{q}$ .
- $P_1$  aborts if  $f_2 \neq H(R_2, \text{nizk3})$  or  $\text{Verifzk}(\text{nizk3})=0$ , otherwise computes  $R = (r_x, r_y) = k_1 \cdot R_2 + k_1 r_1 \cdot P$  and  $r = r_x \pmod{q}$ .

(4) **Online Signature**:

- Given  $m$ ,  $P_2$  computes  $h = H(m)$  and sends

$$s_2 = (k_2 + r_1)^{-1} (h + r x'_2) \pmod{q}$$

to  $P_1$ .

- On receiving  $s_2$ ,  $P_1$  computes

$$s = k_1^{-1} (s_2 + r x'_1) \pmod{q}.$$

- $P_1$  aborts if  $\text{Verify}(m; (r, s)) = 0$ , else returns  $(r, s)$  as the final signature.

*Correctness.* By the definition of  $x'_2$ , we have

$$\begin{aligned} x'_2 &= x_2 - (t_B + cc) \\ &= x_2 - (t_B + t_A + x'_1 r_1 - x_1) \\ &= x_2 - (x'_1 k_2 + x'_1 r_1) + x_1 \end{aligned}$$

over  $\mathbb{Z}_q$ , thus  $x'_1 (r_1 + k_2) + x'_2 = x_1 + x_2 \pmod{q}$ .

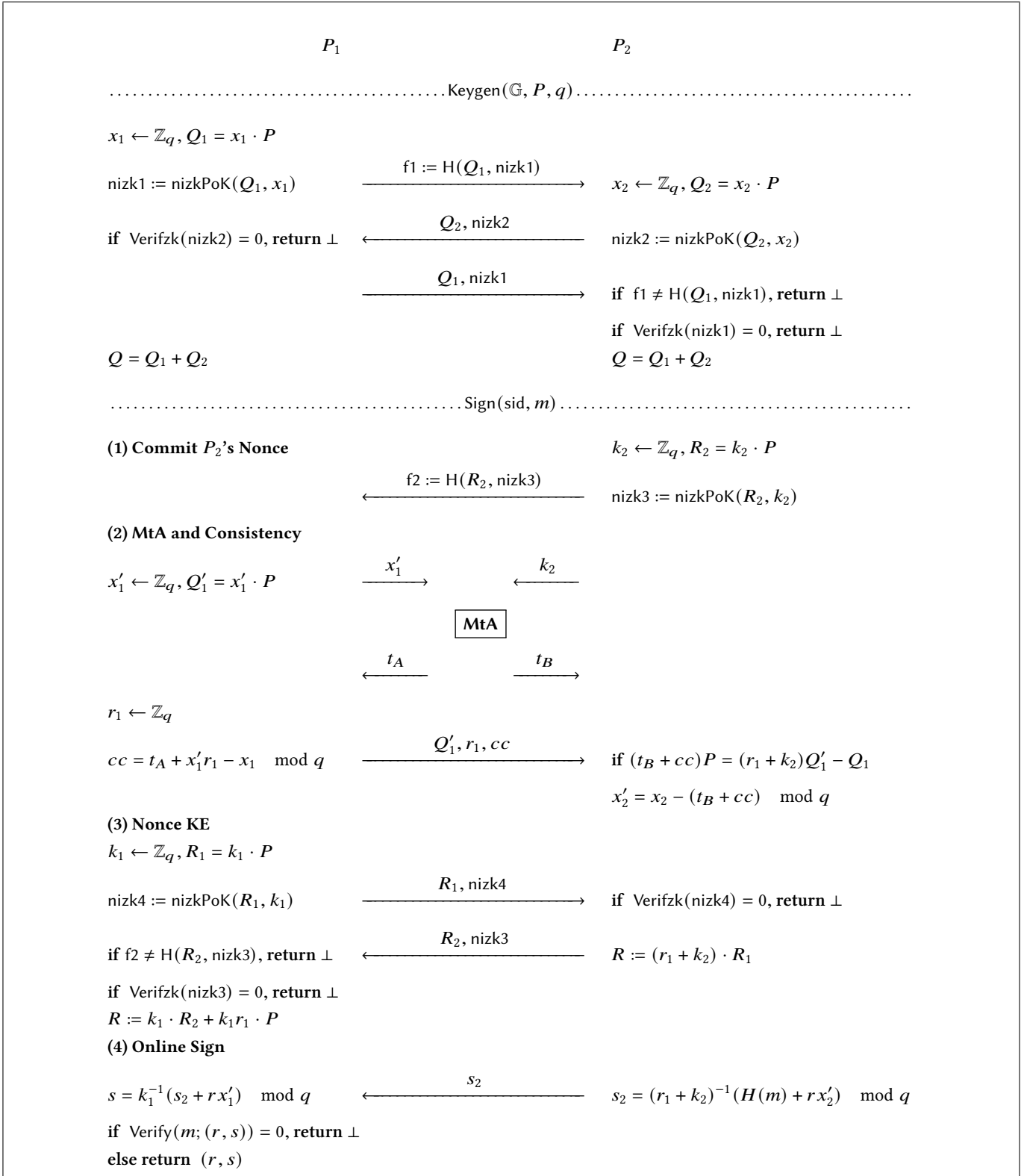


Figure 3: Two-party protocol 2ECDSA from a single MtA.

Let  $k := k_1(r_1 + k_2) \pmod q$ , then  $R = k \cdot P$ . We have

$$\begin{aligned} s &= k_1^{-1}(s_2 + rx'_1) \\ &= k_1^{-1} \left[ (r_1 + k_2)^{-1}(H(m) + rx'_2) + rx'_1 \right] \\ &= k_1^{-1}(r_1 + k_2)^{-1} [H(m) + r(x'_2 + x'_1 r_1 + x'_1 k_2)] \\ &= k_1^{-1}(r_1 + k_2)^{-1} [H(m) + r(x_1 + x_2)] \\ &= k_1^{-1}(r_1 + k_2)^{-1}(H(m) + rx) \end{aligned}$$

over  $\mathbb{Z}_q$ . Thus,  $(r, s)$  is a valid signature of  $m$ .

### 3.1 Security of 2ECDSA

**THEOREM 3.1.** *The two-party 2ECDSA protocol in Figure 3 securely computes  $\mathcal{F}_{\text{ECDSA}}$  in the random oracle model in the presence of a malicious static adversary under the real/ideal definition. Concretely, there exists a simulator for the scheme such that any probabilistic polynomial time adversary, who corrupted  $P_1$  or  $P_2$ , can distinguish a real execution of the protocol from a simulated one with only negligible probability.*

The simulator  $\mathcal{S}$  could only access to an ideal functionality  $\mathcal{F}_{\text{ECDSA}}$  for computing ECDSA signatures. All  $\mathcal{S}$  learns in the ideal world is the public key  $Q$  generated in the key generation phase and several signatures for messages of its choice in the signature phase. In the real world, the adversary, having either corrupted  $P_1$  or  $P_2$  will also see all the interactions with the non-corrupted party. Thus  $\mathcal{S}$  must be able to simulate the adversary's view of these interactions, while only knowing the expected output. The proof proceeds in two cases: the adversary corrupts  $P_1$ , and the adversary corrupts  $P_2$ .

**$\mathcal{S}$  simulates  $P_2$ -Corrupted  $P_1$ .** We first show that if adversary  $\mathcal{A}$  corrupts  $P_1$ , there exists simulator  $\mathcal{S}$  such that the output distribution of  $\mathcal{S}$  is indistinguishable from  $\mathcal{A}$ 's view in the real execution of the protocol.

Simulator  $\mathcal{S}$  maintains a hash list  $L_h$  for  $H$ . On any query  $y$  to  $H$ , if  $\exists(y, h_y) \in L_h$ , return  $h_y$ , else return  $h_y \leftarrow \{0, 1\}^\kappa$  and add  $(y, h_y)$  to  $L_f$ .

#### Key Generation Phase.

- (1) Given input  $\text{Keygen}_{\mathbb{G}, P, q}$ ,  $\mathcal{S}$  sends  $\text{Keygen}_{\mathbb{G}, P, q}$  to  $\mathcal{F}_{\text{ECDSA}}$  and receives back the public key  $Q$ .
- (2)  $\mathcal{S}$  invokes  $\mathcal{A}$  upon input  $\text{Keygen}_{\mathbb{G}, P, q}$  and receives  $f_1$ .
  - if there exists  $(Q_1 || \text{nizk1}, f_1) \in L_h$ , check  $\text{Verifzk}(\text{nizk1})$ . If  $\text{Verifzk}(\text{nizk1}) = 1$ , extract  $x_1$  with  $\text{Ext}$ . If  $Q_1 = x_1 \cdot P$ , compute  $Q_2 = Q - Q_1$ .
  - otherwise, choose random  $Q_2$ .

With the help of zero knowledge simulator  $\text{Sim}$ ,  $\mathcal{S}$  computes  $\text{nizk2}$ , the non-interactive proof of knowledge for the discrete log of  $Q_2$ .

- (3)  $\mathcal{S}$  sends  $Q_2, \text{nizk2}$  to  $\mathcal{A}$ .
- (4)  $\mathcal{S}$  receives  $Q_1, \text{nizk1}$  from  $\mathcal{A}$ . If  $(Q_1 || \text{nizk1}, f_1) \notin L_h$ , abort.  $\mathcal{S}$  simulates  $P_2$  aborting if  $\text{Verifzk}(\text{nizk1}) = 0$  or  $Q_1 \neq x_1 \cdot P$ .
- (5)  $\mathcal{S}$  sends continue to  $\mathcal{F}_{\text{ECDSA}}$  for  $P_2$  to receive output and stores  $(Q, x_1, Q_1, Q_2)$ .

To pass the check of  $f_1 = H(Q_1, \text{nizk1})$ ,  $\mathcal{A}$  must have queried  $(Q_1, \text{nizk1})$  to the random oracle. Thus, the difference between the real execution and the ideal execution simulated by  $\mathcal{S}$  is the generation of  $Q_2$  and  $\text{nizk2}$ . In the real execution,  $Q_2 = x_2 \cdot P$  where

$x_2 \leftarrow \mathbb{Z}_q$ , and  $\text{nizk2} = \text{nizkPoK}(Q_2, x_2)$ , while in the later  $Q_2 = Q - Q_1$  and  $\text{nizk2} \leftarrow \text{Sim}(Q_2, P)$  where  $Q$  is returned by  $\mathcal{F}_{\text{ECDSA}}$ .  $\text{Ext}$  extracts  $x_1$  with knowledge error  $1/q$ . Since  $\mathcal{F}_{\text{ECDSA}}$  samples  $Q$  uniformly at random from  $\mathbb{G}$ , conditional on the extraction of  $x_1$ , the distribution of  $Q_2$  in both cases is identical. Since  $\text{Sim}$  perfectly simulate the proof, the distribution of  $\text{nizk2}$  is also identical.

#### Signing Phase.

- Given input  $\text{Sign}(\text{sid}, m)$ ,  $\mathcal{S}$  sends  $\text{Sign}(\text{sid}, m)$  to  $\mathcal{F}_{\text{ECDSA}}$  and receives signature  $(r, s)$ .
  - Using the verification procedure,  $\mathcal{S}$  recovers  $R$  from  $(r, s)$ .
- (1) Commitment:  $\mathcal{S}$  invokes  $\mathcal{A}$  with input  $\text{Sign}(\text{sid}, m)$  and sends a random string  $f_2 \leftarrow \{0, 1\}^\kappa$  to  $\mathcal{A}$ .
  - (2) MtA and consistency check:
    - $\mathcal{S}$  interacts with  $\mathcal{A}$  on behave of  $\mathcal{F}_{\text{MtA}}$  and in doing so receives  $\mathcal{A}$ 's input  $x'_1$  and output shares  $t_A$ .
    - On receiving  $Q'_1, r_1, cc$  from  $\mathcal{A}$ ,  $\mathcal{S}$  checks the consistency by verifying  $Q'_1 \stackrel{?}{=} x'_1 \cdot P$  and  $cc \stackrel{?}{=} t_A + x'_1 r_1 - x_1 \pmod q$ , and simulates  $P_2$  aborting if these equations do not hold.
  - (3) Nonce key exchange:
    - Upon receiving  $(R_1, \text{nizk4})$  from  $\mathcal{A}$ ,  $\mathcal{S}$  simulates  $P_2$  aborting if  $\text{Verifzk}(\text{nizk4}) = 0$ . Else  $\mathcal{S}$  extracts  $k_1$  utilizing  $\text{Ext}$  algorithm.
    - $\mathcal{S}$  computes  $R_2 = k_1^{-1} \cdot R - r_1 \cdot P$ , generates  $\text{nizk3}$  by querying zero knowledge simulator  $\text{Sim}$ , and adds  $(R_2 || \text{nizk3}, f_2)$  to the hash list  $L_h$ .
  - (4) Online Signature:  $\mathcal{S}$  computes  $s_2 = k_1 s - x'_1 r \pmod q$  where  $x'_1$  and  $k_1$  are extracted from  $\text{nizk1}$  and  $\text{nizk4}$  respectively, and sends  $R_2, \text{nizk3}, s_2$  to  $\mathcal{A}$ .

The difference between a real execution and the simulation is how  $R_2$  and  $s_2$  are computed, and the consistency of  $Q'_1$  and  $x'_1$  is check. In the simulation,  $R_2$  is  $k_1^{-1} \cdot R - r_1 \cdot P$  whereas in the real execution  $R_2 = k_2 \cdot P$  where  $k_2 \leftarrow \mathbb{Z}_q$ . Since  $\mathcal{F}_{\text{ECDSA}}$  samples  $R$  uniformly at random from  $\mathbb{G}$ , the distribution in both cases is identical.  $\text{Ext}$  extracts  $k_1$  with knowledge error  $1/q$ . Conditional on the correctness of  $k_1$ , in the simulation

$$s_2 = k_1 s - x'_1 r = (r_1 + k_2)^{-1}(H(m) + rx'_2) \pmod q$$

which is identical to that in the real execution. Conditions  $Q'_1 = x'_1 \cdot P$  and  $cc = t_A + x'_1 r_1 - x_1 \pmod q$  in the simulation are equivalent to  $(t_B + cc) \cdot P = (r_1 + k_2) \cdot Q'_1 - Q_1$  in the real game.

This implies that the view of a corrupted  $P_1$  in the real execution is indistinguishable from that of the simulation, i.e., the advantage of any PPT adversary who corrupts  $P_1$  to distinguish the real execution and simulated execution given by  $\mathcal{S}$  is negligible.

**$\mathcal{S}$  simulates  $P_1$ -Corrupted  $P_2$ .** We show that if an adversary  $\mathcal{A}$  corrupts  $P_2$ , there exists a simulator  $\mathcal{S}$  such that the output distribution of  $\mathcal{S}$  is indistinguishable with  $\mathcal{A}$ 's view in the real execution of the protocol. Simulator  $\mathcal{S}$  maintains a hash list  $L_h$  for  $H$ . On any query  $y$  to  $H$ , if  $\exists(y, h_y) \in L_h$ , return  $h_y$ , else return  $h_y \leftarrow \{0, 1\}^\kappa$  and add  $(y, h_y)$  to  $L_h$ .

#### Key Generation Phase.

- (1) Given input  $\text{Keygen}_{\mathbb{G}, P, q}$ ,  $\mathcal{S}$  sends  $\text{Keygen}_{\mathbb{G}, P, q}$  to  $\mathcal{F}_{\text{ECDSA}}$  and receives back the public key  $Q$ .



- (2)  $\mathcal{S}$  invokes  $\mathcal{A}$  with input  $\text{Keygen}(\mathbb{G}, P, q)$  and sends a random string  $f1 \leftarrow \{0, 1\}^K$  to  $\mathcal{A}$ .
- (3) Upon receiving  $(Q_2, \text{nizk2})$  from  $\mathcal{A}$ ,  $\mathcal{S}$  computes  $Q_1 = Q - Q_2$  and generates  $\text{nizk1}$  with the help of zero knowledge simulator  $\text{Sim}$ .
- (4)  $\mathcal{S}$  adds  $(Q_1 || \text{nizk1}, f1)$  to the hash list  $L_h$ .
- (5)  $\mathcal{S}$  sends  $Q_1, \text{nizk1}$  to  $\mathcal{A}$  and stores  $(x_2, Q, Q_1, Q_2)$ .

The difference between real execution and ideal execution simulated by  $\mathcal{S}$  is the generation of  $Q_1$  and  $\text{nizk1}$ . In the real execution,  $Q_1 = x_1 \cdot P$  where  $x_1 \leftarrow \mathbb{Z}_q$ , and  $\text{nizk1} = \text{nizkPoK}(Q_1, x_1)$ , while in the later  $Q_1 = Q - Q_2$  and  $\text{nizk1} \leftarrow \text{Sim}(Q_1, P)$  where  $Q$  is returned by  $\mathcal{F}_{\text{ECDSA}}$ .  $\text{Ext}$  extracts  $x_2$  with knowledge error  $1/q$ . Since  $\mathcal{F}_{\text{ECDSA}}$  samples  $Q$  uniformly at random from  $\mathbb{G}$ , conditional on the extraction of  $x_2$ , the distribution of  $Q_1$  in both cases is identical. Since  $\text{Sim}$  perfectly simulate the proof, the distribution of  $\text{nizk1}$  is also identical.

#### Signing Phase.

- Given input  $\text{Sign}(\text{sid}, m)$ ,  $\mathcal{S}$  sends  $\text{Sign}(\text{sid}, m)$  to  $\mathcal{F}_{\text{ECDSA}}$  and receives signature  $(r, s)$ .
  - Using the verification procedure,  $\mathcal{S}$  recovers  $R$  from  $(r, s)$ .
- (1) Commitment:  $\mathcal{S}$  invokes  $\mathcal{A}$  with input  $\text{Sign}(\text{sid}, m)$  and receives  $f2$  from  $\mathcal{A}$ .
    - if  $\exists (R_2 || \text{nizk3}, f2) \in L_h$ ,  $\mathcal{S}$  extracts  $k_2$  such that  $R_2 = k_2 \cdot P$  with the help of knowledge extractor (if the proof is accepted). Then  $\mathcal{S}$  samples a random  $r_1 \leftarrow \mathbb{Z}_q$  and computes  $R_1 = (r_1 + k_2)^{-1} \cdot R$ .
    - otherwise, samples a random  $r_1 \leftarrow \mathbb{Z}_q$  and a random point  $R_1$  and generates  $\text{nizk4}$  with the help of zero knowledge simulator  $\text{Sim}$
  - (2) MtA and consistency check:
    - $\mathcal{S}$  interacts with  $\mathcal{A}$  on behave of  $\mathcal{F}_{\text{MtA}}$  and in doing so receives its input  $k'_2$  and output shares  $t_B$ .
    - $\mathcal{S}$  samples a random  $cc \leftarrow \mathbb{Z}_q$ , computes
$$Q'_1 = (k'_2 + r_1)^{-1} [(t_B + cc) \cdot P + Q_1],$$
and sends  $(Q'_1, r_1, cc)$  to  $\mathcal{A}$ .
  - (3) Nonce key exchange:  $\mathcal{S}$  computes  $R_1 = (r_1 + k_2)^{-1} \cdot R$ , generates  $\text{nizk4}$  by querying zero knowledge simulator  $\text{Sim}$ , and sends them to  $\mathcal{A}$ .
  - (4) Online signature: Upon receiving  $R_2, \text{nizk3}, s_2$  from  $\mathcal{A}$ ,  $\mathcal{S}$  checks the proof of  $\text{NIZK3}$ , and whether  $f2 = H(R_2, \text{nizk3})$  and

$$s_2 \cdot (r_1 \cdot P + R_2) \stackrel{?}{=} h \cdot P + r \cdot (x_2 - t_B - cc) \cdot P.$$

If the checks pass,  $\mathcal{S}$  returns  $(r, s)$  as the final signature, else aborts.

The difference between a real execution and the simulation is how  $R_1$  and  $Q'_1$  are computed, and the condition that  $(r, s)$  is output or not.

In the simulation,  $R_1$  is  $R_1 = (r_1 + k_2)^{-1} \cdot R$  whereas in the real execution  $R_1 = k_1 \cdot P$  where  $k_1 \leftarrow \mathbb{Z}_q$ . Since  $\mathcal{F}_{\text{ECDSA}}$  samples  $R$  uniformly at random from  $\mathbb{G}$ , the distribution in both cases is identical. In the simulation,  $Q'_1$  is  $(k'_2 + r_1)^{-1} [(t_B + cc) \cdot P + Q_1]$  whereas in the real execution  $Q'_1 = x'_1 \cdot P$  for  $x'_1 \leftarrow \mathbb{Z}_q$ . Since  $r_1$  and  $cc$  are sampled randomly and the consistency check always passes, the distribution in both cases is identical.

In the real execution, the Verify algorithm checks that  $s \cdot R \stackrel{?}{=} h \cdot P + r \cdot Q$ , i.e., implicitly checks

$$s_2(r_1 + k_2) \stackrel{?}{=} h + r(x_2 - t_B - cc) \pmod{q}. \quad (2)$$

Since

$$s_2 \cdot (r_1 \cdot P + R_2) = h \cdot P + r(x_2 - t_B - cc) \cdot P \quad (3)$$

holds if and only if Equation 2 is correct, the condition to output  $(r, s)$  in both the real and simulated case is identical.

This implies that the view of a corrupted  $P_2$  in the real execution is indistinguishable with that of the simulation, i.e., the advantage of any PPT adversary who corrupts  $P_2$  to distinguish the real execution and simulated execution given by  $\mathcal{S}$  is negligible.

## 4 INSTANTIATIONS OF MTA AND THEIR APPLICATIONS TO 2ECDSA

There are several constructions of MtA which could be directly applied to our 2ECDSA.

### 4.1 MtA from Oblivious Transfer

Following Gilboa's semi-honest oblivious transfer [21], and motivated by two-party ECDSA, Doerner et al. [15] proposed a MtA against malicious adversary. Their construction is based upon Simplest OT [10] and KOS [23] OT-extension protocols. Please refer to Appendix B for a detailed description. Their MtA is computationally very efficient, while the communication is rather large. To compute a MtA, at least  $8\kappa^2$  bits must be transferred. Concretely, when  $\kappa = 256$ , the communication of a single execution of MtA is  $\approx 90$  KB.

Their OT-based MtA could be applied to 2ECDSA. The online phase of both [15] and our protocols are extremely lightweight: sending a single field element and computing two elliptic curve point multiplications. In our protocol, a single MtA is required while the two-party ECDSA of [15] needs two. Thus, our solution requires roughly half of the communication and computation of [15]. A concrete comparison is given in Sec. 5.

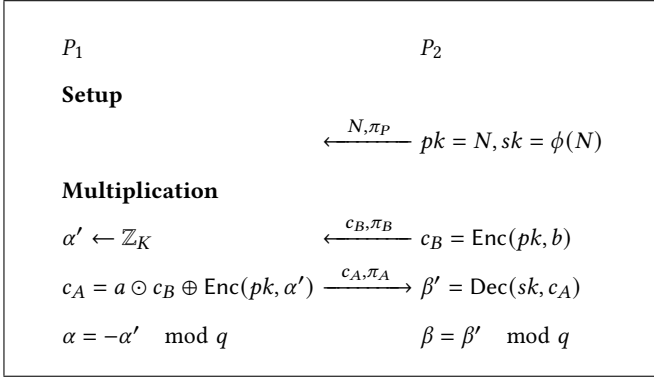
### 4.2 MtA from Paillier

Gennaro and Goldfeder [19] presented a Multiplicative-to-Additive conversion protocol from Paillier encryption, which has appeared many times before [12, 25, 27]. Lindell et al. [26] also proposed a similar protocol and call it private multiplication. We recall it and explain its application to our 2ECDSA.

Let  $pk = N$  and  $sk = \phi(N)$  be the public and secret keys of Paillier encryption. Denote the encryption of message  $m$  by  $\text{Enc}(pk, m)$ , the decryption of ciphertext  $c$  by  $\text{Dec}(sk, c)$ . Denote by  $c_1 \oplus c_2 = c_1 c_2 \pmod{N^2}$  the addition of the plaintext in ciphertexts  $c_1$  and  $c_2$ , and by  $a \odot c = c^a \pmod{N^2}$  the multiplication of the plaintext in  $c$  by a scalar  $a$ .

The Paillier-based MtA between  $P_1$  with input  $a$  and  $P_2$  with input  $b$  works in Figure 4 where

- $\pi_P$  is a ZKPoK for  $\{(N; \phi(N)) \mid \gcd(N, \phi(N)) = 1\}$ ;
- $\pi_B$  is a ZKPoK for  $\{(c_B; b) \mid c_B = \text{Enc}(pk, b) \wedge b \in \mathbb{Z}_q\}$ ;
- $\pi_A$  is a ZKPoK for  $\{(c_A, c_B; a, a') \mid c_A = a \odot c_B \oplus \text{Enc}(pk, a') \wedge a \in \mathbb{Z}_q \wedge a' \in \mathbb{Z}_K\}$ .



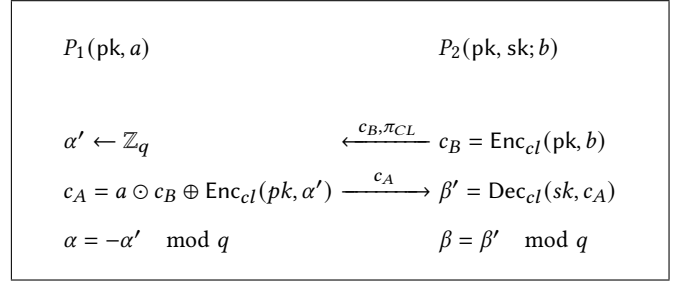
**Figure 4: The Paillier-based MtA from [19].  $K$  is the parameter to be determined.**

Please refer to Appendix C for the full description of Paillier-based MtA and these ZKPoK protocols (w/o slack regarding soundness).

As noted by previous works [6, 19, 25, 26],  $\pi_A$  and  $\pi_B$  are very expensive. Several works have been done to reduce or even remove these proofs. Lindell [25], Gennaro and Goldfeder [19] suggested to eliminate  $\pi_A$  by relying on a non-standard Paillier-EC assumption (Appendix C.3). Lindell et al. [26], Gennaro and Goldfeder [19], and Canetti et al. [6] found that it is enough to use proofs with a lot of slack regarding soundness. The slack version of  $\pi_A$  and  $\pi_B$  roughly cost computing 6 Paillier exponentiations and sending  $6 \log N$  bits. Thus, the overall cost of Paillier-based MtA is computing approximately 14 Paillier exponentiations and sending  $16 \log N$  bits. Under the Paillier-EC assumption, it could be further reduced to 8 Paillier exponentiations and  $10 \log N$  bits respectively.

**4.2.1 Applying Paillier-based MtA to 2ECDSA.** We propose two versions of Paillier-based 2ECDSA depending on whether or not MtA relies on Paillier-EC assumption. When applying Paillier-EC assumption, a proof-of-knowledge for the discrete logarithm of  $Q'_1$  should be added against corrupted  $P_1$ . (In order to extract  $x'_1$  to querying the oracle in Paillier-EC assumption.) Actually, Gennaro and Goldfeder [19] also proposed another non-standard assumption to further eliminate  $\pi_B$ , and their technique could also be applied to our protocol. Nevertheless, to simplify the analysis, we do not include it in this paper and just analyze Gennaro and Goldfeder’s scheme based on Paillier-EC.

Lindell’s two-party protocol [25] has the best overall performance. Nevertheless, their online phase is comparatively expensive, i.e., sending a Paillier ciphertext and computing Paillier decryption. The online phase of the two-party case of Lindell’s threshold scheme [26] is worst, since 14 Paillier exponentiations are required. The online phase of [19] is fast, while their offline phase requires at least computing 18 Paillier exponentiations and sending  $16 \ell_N$  bits depending on Paillier-EC is required or not. The online phase of the two-party case of Canetti et al. [6] and our scheme are extremely lightweight. However, the offline phase of [6] needs 204 Paillier exponentiations. Our scheme relies on 8 or 14 Paillier exponentiations and sending  $10 \ell_N$  or  $16 \ell_N$  bits depending on whether Paillier-EC is required. A concrete comparison is given in Sec. 5.



**Figure 5: The CL-based MtA extracted from [7, 8].**

### 4.3 MtA with CL-encryption Achieving Game-based Security

To address the problem of non-standard assumption in Paillier-based MtA, Castagnos et al. [7, 8] replaces Paillier with CL encryption [9] which allows instantiations where the message space is exactly  $\mathbb{Z}_q$ .

Let  $(\hat{s}, f, \hat{g}_q, g_q, \hat{G}, F, q)$  be public parameters of CL encryption (as defined in [8]), where  $F = \langle f \rangle$  is the subgroup of  $\hat{G}$  with order  $q$ ,  $g_q$  is a random element in  $\langle \hat{g}_q \rangle$ . Denote by  $sk, pk = g_q^{sk}$  secret-public key pair. CL encryption  $\text{Enc}_{cl}(pk, m, r)$  computes  $(c_1, c_2) = (g_q^r, pk^r f^m)$  as the encryption of  $m \in \mathbb{Z}_q$  with randomness  $r \leftarrow [0, S]$  (for some  $S$ ), and  $\text{Dec}_{cl}(sk, c_1, c_2)$  computes  $\log_f(c_2/c_1^{sk})$  to decrypt. We also denote by  $\oplus$  the addition of the plaintext in two ciphertexts, and by  $\odot$  the scalar multiplication on ciphertext.

We abstract CL-based MtA from Castagnos et al.’s protocol [8].  $P_2$  generates public-secret keypair and sends  $pk$  to  $P_1$  in the setup phase. The CL-based MtA protocol is presented in Fig. 5, where  $\pi_{CL}$  is a ZKPoK for relation

$$\{ (pk, c_1, c_2; m, r) \mid pk \in \hat{G}, r \in [0, S], c_1 = g_q^r \wedge c_2 = pk^r f^m \}.$$

Please refer to Appendix D for the proof  $\pi_{CL}$ .

The CL-encryption requires 2 exponentiations on the class group while the decryption requires 1. The prover of  $\pi_{CL}$  needs 2 exponentiations while the verifier computes 4 exponentiations. The size of  $\pi_{CL}$  is about a CL ciphertext. Thus, the overall cost of CL-based MtA is sending 3 CL ciphertexts and computing 9 exponentiations over the class group.

Note that the scheme of Figure 5 dose not provide full simulation-based security when  $P_1$  is corrupted. We remark that simulation based security could be achieved by adding an expensive ZKPoK for the affine operation of  $c_A$  on  $c_B$ . In our specific usage, we choose the same strategy as Castagnos et al. [8, Sec. 2.2] to achieve a weaker game-based threshold unforgeability given in Appendix D.2, thus do not add this ZKPoK.

**4.3.1 Applying CL-based MtA to 2ECDSA.** Instantiating MtA by CL will result in a CL-based 2ECDSA<sup>1</sup>. In both [7, 8] and our protocol, CL encryption/decryption and/or  $\pi_{CL}$  dominate the overall complexity.

Although the overall cost of [7] is best, a CL ciphertext should be transferred and a decryption computation is required making the online phase computationally expensive. In particular, it is more

<sup>1</sup> $\pi_{CL}$  requires a setup step to guarantee that  $g_q$  is a random element of the subgroup  $\langle \hat{g}_q \rangle$ . It is executed in the key generation phase.

than 50 times slower compared with [8, 33], and nearly 1000 times slower than ours in the online phase. The online phase of two-party case of [8, 33] is fast (although it is interactive), while it costs 28 exponentiations and 140 $\kappa$  bits (when  $\kappa = 256$ ). Our online phase is non-interactive and extremely fast, and our offline cost is just 11 exponentiations and 53 $\kappa$  bits. A concrete comparison is given in Sec. 5.

## 5 IMPLEMENTATION AND COMPARISON

In this section, we give a comprehensive implementation and comparison of two-party ECDSA from Paillier, OT and CL-encryption. Although [6, 8, 16, 19, 26] support threshold larger than two, we only consider their performance in two-party case.

We benchmark our implementation using Rust on a MacBook Pro 13-inch 2019 with Intel Core i5 @ 1.4 GHz CPU and 16 GB 2133 MHz LPDDR3 RAM running macOS Mojave v10.14.5. For simplicity, we evaluate the protocols in a single laptop and consider only the computation time. We remark that the comparison will further favor our scheme if latency is taken into account. The reason is, as shown in Sec. 1.4, our scheme has fewer communication rounds than other schemes. The results are the median time of running 100 times. All benchmarks were taken over curve secp256k1 which is recommended by NIST [24] and is the curve used in Bitcoin, among many other blockchains and cryptocurrencies. We use SHA-256 to instantiate the hash functions, random oracles and/or the PRG.

The code for our CL-based protocol is available at <https://github.com/LatticeX-Foundation/CG-MPC-ECDSA>. The code for Paillier and OT-based schemes will be available soon.

Overall comparison is shown in Figure 6 and Table 2. The detailed comparisons are given in the following subsections.

### 5.1 Paillier-based Schemes

We set security parameters  $\kappa = 256$  (achieving 128-bits computational security), 80 bits statistical security, and 80 bits soundness error for our Paillier-based 2ECDSA. The underlying MtA is that in Appendix C. Lin17 [25] uses 40-bits statistical security and soundness error. LNR18 [26] and CGGMP20 [6] set the same statistical and soundness parameters with us. GG18 [19] sets 128-bits computational security, 256-bit statistical security and 128-bits soundness error. We note that these works recommend 2048-bit module  $N$ . To make a fair comparison with CL and OT-based schemes, we set a 3072-bit module in the implementation and comparison.

We implement two versions of 2ECDSA and GG18 (one with and one without Paillier-EC), LNR18, and CGGMP20 based on the elementary code of ZenGo [32] and run ZenGo’s code on Lin17. The results of computation and communication on signing are presented in Table 3. The key generation of our scheme has a similar cost with GG18, LNR18, and CGGMP20.

Without Paillier-EC assumption, our scheme improves GG18’s offline complexity by a factor of 5 for computation and a factor of 2 for communication. Our implementation outperforms CGGMP20’s work by a factor of 9 for computation and a factor of 8 for communication. Without Paillier-EC assumption, we improve GG18’s computation by a factor of 2 and its communication by a factor of 1.6.

**Table 3: Cost comparison of Paillier-based schemes.**

Schemes	Computation		Communication	
	Offline	Online	Offline	Online
LNR18 [26]	461ms	302ms	12.1KB	6.6KB
GG18 [19]	1237ms	3ms	15.5KB	288B
CGGMP20 [6]	2037ms	0.2ms	44KB	32B
2ECDSA (Paillier)	226ms	0.2ms	6.3KB	32B
Lin17 [25] (Paillier-EC)	34ms	8ms	192B	768B
GG18 [19] (Paillier-EC)	360ms	3ms	6.6KB	288B
2ECDSA (Paillier-EC)	141ms	0.2ms	4.1KB	32B

**Table 4: Cost comparison of OT-based schemes.**

Schemes	Computation		Communication	
	Offline	Online	Offline	Online
DKLS18 [15]	2.9ms	0.2ms	169.8KB	32B
DKLS19 [16]	3.7ms	0.2ms	180KB	32B
2ECDSA (OT)	2.6ms	0.2ms	90.9KB	32B

### 5.2 OT-based Schemes

We set security parameters  $\kappa = 256$  (achieving 128-bits computational security), 80-bits statistical security and soundness error as DKLS18-19 [15, 16] did. We implement OT-based 2ECDSA using MtA of [28] and run their code on DKLS18 and DKLS19 protocols. The results of computation and communication on signing are reported in Table 4. The key generation of our scheme has the same complexity as DKLS19.

The offline communication of our scheme is 90.9 KB, which outperforms DKLS19-20 by a factor of roughly 2.

### 5.3 CL-based Schemes

For CL-based instantiations, we set 128-bits computational security, take 80-bits statistical distance and soundness error as YCX21 [33] did. CCLST19 [7] used 40-bits statistical and soundness security, and CCLST20 [8] utilized 128-bits for soundness error and 80-bits for statistical distance. We implement CL-based 2ECDSA and two-party schemes of CCLST20 and YCX21 and run ZenGo’s code on CCLST19. The cost comparison is reported in Table 5. The key generation of our scheme has the same complexity with [8].

The offline complexity of our scheme is 1.7KB and 1386ms. CCLST20’s offline phase needs at least 2 $\times$  cost than our work. The offline phase of YCX21 requires 2 $\times$  communication and 3 $\times$  computation than our scheme.

## 6 CONCLUSION

We propose an online-friendly two-party 2ECDSA such that its online computation is extremely fast and its offline phase requires only a single execution of MtA. Our scheme could be efficiently instantiated with constructions of MtA from Paillier, CL encryptions and oblivious transfer. Furthermore, our scheme can be easily extended to the more general case of 2-out-of- $n$ .

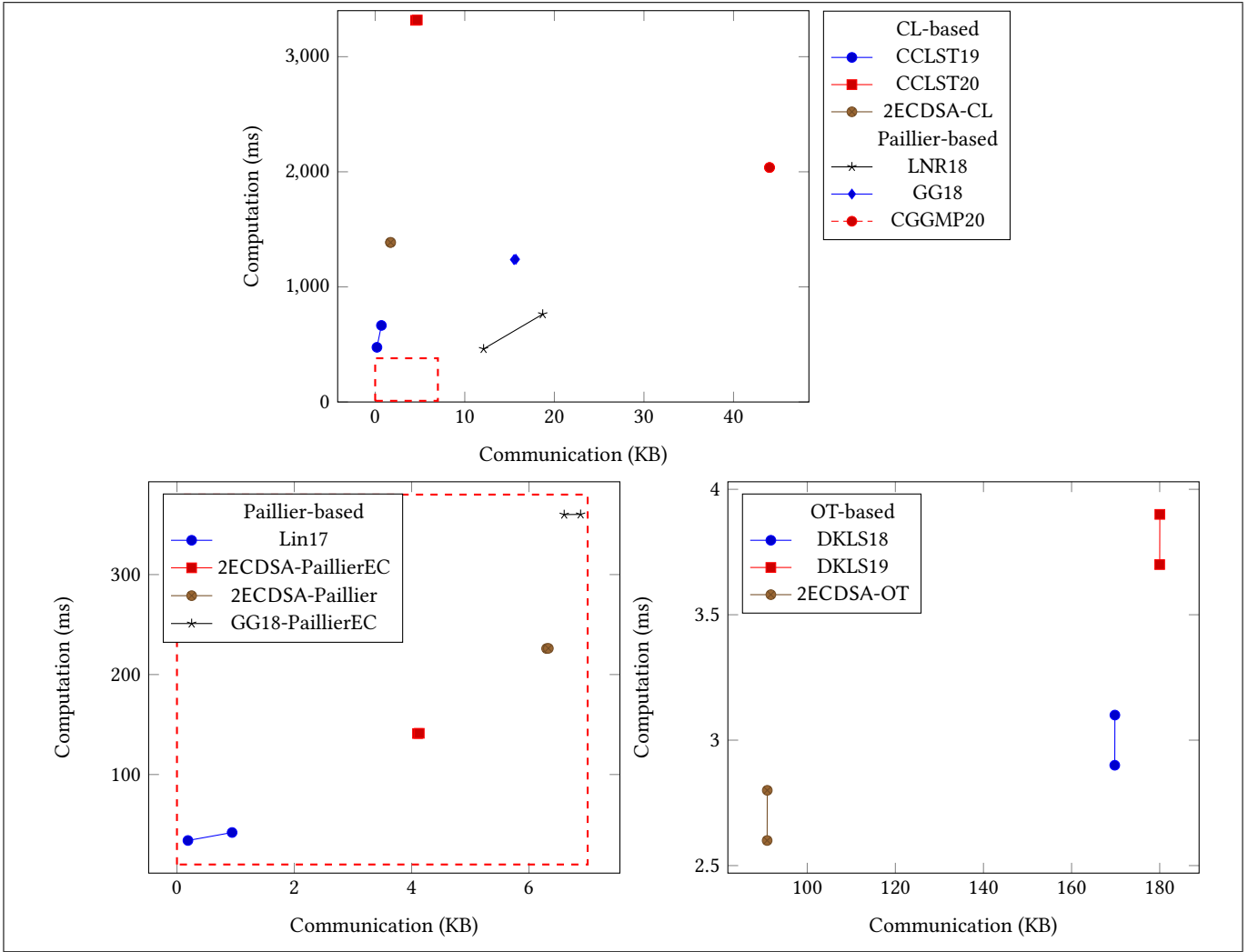


Figure 6: Cost comparison of all two-party schemes. For every scheme, the south-west point is the offline cost, while north-east point is the overall cost. Thus, the distance between these two points indicates the online complexity. The dashed red rectangles share the same area.

Table 5: Cost comparison of CL-based schemes.

Schemes	Computation		Communication	
	Offline	Online	Offline	Online
CCLST19 [7]	475ms	190ms	505B	208B
CCLST20 [8]	3316ms	3ms	4.5KB	288B
YCX21 [33]	4550ms	3ms	4.5KB	288B
2ECDSA (CL)	1386ms	0.2ms	1.7KB	32B

Our work focuses on the two-party ECDSA. We believe the idea of this work will lead to improvements of the full threshold ECDSA, and we leave this for further work.

## ACKNOWLEDGMENTS

We would like to thank Xuyang Song and Xueli Wang for their help in the experiments. Haiyang Xue is supported by the National Natural Science Foundation of China (No. 62172412), the National Key Research and Development Program of China (No. 2020YFB1807502). Man Ho Au is supported by the National Natural Science Foundation of China (No. 61972332), the Research Grant Council of Hong Kong (GRF Project 15211120).

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## A KEY GENERATION PROTOCOL FOR 2-OUT-OF- $n$ ACCESS STRUCTURES

**Distributed Key Generation Phase  $\text{Keygen}(\mathbb{G}, P, q)$ .** Given the joint input  $(\mathbb{G}, P, q)$ :

- Phase 1: Each player  $P_i$  selects  $x_i \leftarrow \mathbb{Z}_q$  and computes  $X_i = x_i \cdot P$  and a ZKPoK of discrete logarithm using  $x_i$ . Then,  $P_i$  broadcasts a commitment of  $X_i$  and corresponding ZKPoK.
- Phase 2: Upon receiving commitments from other parties,  $P_i$  decommits  $X_i$  and corresponding ZKPoK. Then,  $P_i$  checks all the received commitments and the ZKPoKs.
- Phase 3: Each party  $P_i$  runs  $(2, n)$  Feldman-VSS [17] using his secret  $x_i$ . Specifically, let  $\{x_1^{(i)}, \dots, x_n^{(i)}\}$  be the secret sharing of  $x_i$ ,  $P_i$  adds the private shares received from Feldman-VSS, i.e.,  $v_i = \sum_{j=1}^n x_i^{(j)} \pmod q$ . Note that  $V_i = v_i \cdot P$  is publicly computable.
- Phase 4: Each player  $P_i$  proves in zero-knowledge that it knows  $v_i$ , the discrete logarithm of  $V_i$ . Each player sets  $Q = \sum_{i=1}^n X_i$  to be the public key, otherwise aborts. Note that  $x = \sum_{i=1}^n u_i \pmod q$  be the secret key.

Assuming that  $P_i$  and  $P_j$  are the two parties to sign  $m$ , they could compute  $\Lambda_i v_i$ ,  $\Lambda_j v_j$  respectively (such that  $x = \Lambda_i v_i + \Lambda_j v_j$ ), where  $\Lambda_i$  and  $\Lambda_j$  are Lagrange coefficients.

## B MTA FROM OBLIVIOUS TRANSFER

We recall the MtA from OT proposed in [15].

Let  $\mathcal{F}_{\text{OTe}}^\ell$  be the Correlated OT-extension functionality that allows arbitrarily many Correlated OT instances to be executed in batches of size  $\ell$ . The input of the receiver is a vector of choice bits while the sender’s input is a vector of correlated elements. The functionality samples  $\ell$  random pads and sends them to the sender. To the receiver, it sends the pads if the sender’s corresponding choice bits were 0, otherwise the sum of the pads and their corresponding correlations. Please refer to [15, Sec. IV and Appendix A] for the concrete definition and instantiation.

The OT-based MtA is constructed in the  $\mathcal{F}_{\text{OTe}}^\ell$  hybrid model. It is parameterized by the statistical security parameter  $s$ , the curve order  $q$ , and  $\kappa = |q|$ . Let  $\mathbf{g} = \mathbf{g}^G \parallel \mathbf{g}^R$  be a coefficient vector where  $\mathbf{g}^G$  satisfies  $\mathbf{g}_i^G = 2^{i-1}$ , and  $\mathbf{g}^R$  is a public random vector. Assume the input of Alice and Bob is  $a, b \in \mathbb{Z}_q$  respectively, they execute the following protocol to export  $\alpha, \beta$  such that  $\alpha + \beta = ab$ .

### Encoding:

- Bob samples  $\gamma \leftarrow \{0, 1\}^{\kappa+2s}$ , and encodes its input as  $\mathbf{b} = \text{Bits}(b - \langle \mathbf{g}^R, \gamma \rangle) \parallel \gamma$ .
- Alice samples  $\hat{a} \leftarrow \mathbb{Z}_q$  and sets  $\mathbf{a} = \{a \parallel \hat{a}\}_{j \in [1, 2\kappa+2s]}$ .

### Multiplication:

- Alice who plays as sender and Bob as receiver, invokes functionality  $\mathcal{F}_{\text{OTe}}^\ell$  with their encoded input where  $\ell = 2\kappa + 2s$ . They receive as outputs, respectively,  $\{t_{A_j} \parallel \hat{t}_{A_j}\}_{j \in [1, 2\kappa+2s]}$  and  $\{t_{B_j} \parallel \hat{t}_{B_j}\}_{j \in [1, 2\kappa+2s]}$
- Alice and Bob generate two shared random values by calling the random oracle, i.e.,  $(\chi, \hat{\chi}) \leftarrow \text{RO}(\text{transcript})$ .
- Alice computes and sends  $\mathbf{r} = \{\chi t_{A_j} + \hat{\chi} \hat{t}_{A_j}\}_{j \in [1, 2\kappa+2s]}$ ,  $u = \chi a + \hat{\chi} \hat{a}$  to Bob.
- Bob aborts if  $\chi t_{B_j} + \hat{\chi} \hat{t}_{B_j} \neq \mathbf{b}_j u - r_j$  for any  $j \in [1, 2\kappa + 2s]$ .
- Alice and Bob compute their output shares respectively, i.e.,  $\alpha = \sum_{j \in [1, 2\kappa+2s]} \mathbf{g}_j t_{A_j}$ , and  $\beta = \sum_{j \in [1, 2\kappa+2s]} \mathbf{g}_j t_{B_j}$ .

## C MTA FROM PAILLIER

We recall the MtA protocol from Paillier encryption, which has appeared many times before [6, 12, 19, 25–27].

Paillier is a well-known additive homomorphic encryption scheme. Let  $pk = N$  and  $sk = \phi(N)$  be the public and secret keys of Paillier encryption respectively. Denote the encryption of message  $m$  with randomness  $r$  by  $\text{Enc}(pk, m, r) = r^N(1+N)^m \pmod{N^2}$ , the decryption of ciphertext  $c$  by  $\text{Dec}(sk, c)$ . For simplicity, we may eliminate the randomness  $r$  and denote encryption as  $\text{Enc}(pk, m)$ . We denote by  $c_1 \oplus c_2 = c_1 c_2 \pmod{N^2}$  the addition of the plaintext in ciphertexts  $c_1$  and  $c_2$ , and by  $a \odot c = c^a \pmod{N^2}$  the multiplication of the plaintext in  $c$  by a scalar  $a$ .

Zero-knowledge proofs for the following relation are needed for the Paillier-based MtA.

**Proof correct Paillier keypair generation** Define the relation

$$\mathcal{R}_p = \{(N, \phi(N)) \mid \gcd(N, \phi(N)) = 1\}$$

of valid Paillier public keys. We remark that standard Paillier is defined for  $N = pq$  with  $p, q$  prime. As did in [25], we only require  $\gcd(N, \phi(N)) = 1$  since all we need is the additive homomorphic property. The proof  $\text{ZKPoK}_{\mathcal{R}_p}$  for  $\mathcal{R}_p$  was proposed in [22, Sec.3.3] and [25, Appendix A] which (for a soundness error  $2^{-t}$ ) requires  $3t$  Paillier encryption and  $4t$  GCD computation. Please refer to [25, Appendix A] for the description of  $\text{ZKPoK}_{\mathcal{R}_p}$ .

**ZKPoK for Paillier Encryption with Range Proof.** Define the relation

$$\mathcal{R}_{\text{PWR}} := \{(N, q, c; m, r) \mid c = r^N(1+N)^x \pmod{N^2} \wedge x \in \mathbb{Z}_q\}$$

of Paillier encryption with range proof. Lindell [25, Appendix A] proposed a rather expensive ZKPoK for  $\mathcal{R}_{\text{PWR}}$  based on a proof from [5, Sec.1.2.2]. Later, Lindell et al. [26], Gennaro and Goldfeder [19], and Canetti et al. [6] gave an efficient proof with slack under Strong-RSA assumption and proved that the proof with slack is enough for MtA. Please refer to C.1 for a ZKPoK for  $\mathcal{R}_{\text{PWR}}$  with slack.

**ZKPoK for Range-Bounded Affine Operation.** Define the relation  $\mathcal{R}_{\text{AffRan}}$  as

$$\{(pk, q, c_A, c_B; a, \alpha) \mid c_A = a \odot c_B \oplus \text{Enc}(pk, \alpha) \wedge a \in \mathbb{Z}_q \wedge \alpha \in \mathbb{Z}_K\}$$

of Paillier encryption with range proof. We may also define the relation  $\mathcal{R}'_{\text{AffRan}}$  as

$$\{(N, q, c_A, c_B; a, \alpha) \mid c_A = (c_B)^a(1+N)^\alpha \pmod{N^2} \wedge a \in \mathbb{Z}_q \wedge \alpha \in \mathbb{Z}_K\}.$$

Let  $\text{ZKPoK}_{\mathcal{R}'_{\text{AffRan}}}$  (resp.  $\text{ZKPoK}_{\mathcal{R}_{\text{AffRan}}}$ ) be the ZKPoK proof for  $\mathcal{R}'_{\text{AffRan}}$  (resp.  $\mathcal{R}_{\text{AffRan}}$ ). Please refer to C.2 for this proof with slack.

With those ZKPoKs, the Paillier-based MtA works as in the following. We remark that all the ZKPoKs are transferred to non-interactive using the Fiat-Shamir paradigm [18]. Its security relies on the semantic security of Paillier encryption. As noted by [19, 25],  $\pi_A$  could be eliminated under the non-standard Paillier-EC assumption which is recalled in C.3.

In the setup phase,

- $P_2$  generates a Paillier key-pair  $pk = N, sk = \phi(N)$ .
- $P_2$  computes a  $\text{ZKPoK}_{\mathcal{R}_p}$  proof for the correctness of Paillier public key.

- $P_2$  sends  $pk, \text{ZKPoK}_{\mathcal{R}_p}$  to  $P_1$  who will verify the proof.

In the multiplication phase, let  $a, b \in \mathbb{Z}_q$  be the input of  $P_1$  and  $P_2$  respectively.

- $P_2$  initiates the protocol
  - Compute  $c_B = \text{Enc}(pk, b)$  and a  $\text{ZKPoK}_{\mathcal{R}_{\text{PWR}}}$  proof  $\pi_B$ .
  - Send  $(c_B, \pi_B)$  to  $P_1$ .
- On receiving  $c_B, \pi_B$ ,  $P_1$  do the following
  - Verify  $\pi_B$ , and abort if it fails.
  - Choose  $\alpha' \leftarrow \mathbb{Z}_K$ . Set output  $\alpha = -\alpha' \pmod{q}$ .
  - Compute the ciphertext  $c_A = a \odot c_B \oplus \text{Enc}(pk, \alpha')$ , and a  $\text{ZKPoK}_{\mathcal{R}_{\text{AffRan}}}$  proof  $\pi_A$  with witness  $a, \alpha'$ . (or Compute  $c_A = a \odot c_B(1+N)^{\alpha'} \pmod{N^2}$ , and a  $\text{ZKPoK}_{\mathcal{R}'_{\text{AffRan}}}$  proof  $\pi_A$ .)
  - Send  $(c_A, \pi_A)$  to  $P_2$ .
- Upon receiving  $(c_A, \pi_A)$ ,  $P_2$  do the following
  - Verify  $\pi_A$ , and abort if it fails.
  - Compute  $\beta' = \text{Dec}(sk, c_A)$  and output  $\beta = \beta' \pmod{q}$ .

### C.1 Range Proof for Paillier with Slack under Strong-RSA Assumption

We recall the range proof for Paillier with slack, i.e. the following relation  $\mathcal{R}_{\text{PWR}}$  proposed in [26, Sec.6.2.6].

$$\mathcal{R}_{\text{PWR}} := \{(N, q, c; m, r) \mid c = r^N(1+N)^x \pmod{N^2} \wedge x \in \mathbb{Z}_q\}.$$

The soundness only guarantees that  $x \in [-2^{t+\ell}q, 2^{t+\ell}q]$ . As analyzed in [26, Sec. 6.2], the following range proof with slack is enough for Paillier-based MtA by adding  $2^{t+\ell}q$  to  $a$  and setting  $K = 2^{t+\ell+s}q^2$ .

Let  $t, \ell, s$  be security parameters. In the implementation, we set  $t = s = 128$  and  $\ell = 80$  for 80 bits statistical security. Let  $N_0$  be the modulus for Pedersen, and  $g, h \in QR_{N_0}$  be random. The Pedersen commitment of  $x$  with randomness  $\rho$  is  $g^x h^\rho \pmod{N_0}$ . It is assumed that the Paillier modulus  $N$  was generated by the Prover and the Pedersen parameters  $(N_0, g, h)$  were generated by the Verifier. We instruct the parties, in the setup phase, to prove in zero-knowledge that all the parameters were generated correctly.

In the setup phase, the verifier  $V$  sends the prover  $P$  parameters  $N_0, g, h$  and zero-knowledge proof that all the parameters are generated correctly.

- Prover's first message:  $P$  chooses  $\rho, \gamma, \tau \leftarrow \mathbb{Z}_{N_0}$ ,  $\alpha, a \in \mathbb{Z}_{2^{t+\ell}q}$ ,  $\beta \leftarrow \mathbb{Z}_N$  and computes  $\tilde{C} = g^x h^\rho \pmod{N_0}$ ,  $A = \beta^N(1+N)^\alpha \pmod{N^2}$ ,  $B = g^\alpha h^\gamma \pmod{N_0}$ , and  $D = g^a h^\tau \pmod{N_0}$ . Then Prover sends  $(\tilde{C}, A, B, D)$  to  $V$ .
- Challenge:  $V$  chooses  $e \in \{0, 1\}^{2t}$  and  $e_1 \in \mathbb{Z}_{2^t}$  and sends them to  $P$ .
- Prover's second message:  $P$  computes  $z_1 = \alpha + ex$  (over the integer),  $z_2 = \beta r^e \pmod{N}$ ,  $z_3 = \gamma + \rho e$ ,  $z_4 = a + x e_1$ ,  $z_5 = \tau + e_1 \rho$ .  $P$  sends  $(z_1, z_2, z_3, z_4, z_5)$  to  $V$ .
- Verification: Accept if and only if
  - $z_2^N(1+N)^{z_1} = A(C)^e \pmod{N^2}$
  - $g^{z_1} h^{z_3} = B(\tilde{C})^e \pmod{N_0}$
  - $z_4 \in [2^t q, 2^{t+\ell} q]$
  - $g^{z_4} h^{z_5} = D(\tilde{C})^{e_1} \pmod{N_0}$

**Table 6: The CL encryption scheme.**

$\text{Enc}_{cl}(\text{pk}, m)$	$\text{Dec}_{cl}(\text{sk}, c_1, c_2)$
Pick $r \leftarrow [0, S]$ Return $(g_q^r, \text{pk}^r f^m)$	Compute $M = c_2 / c_1^{\text{sk}}$ Return $\log_f(M)$

## C.2 Proof of Paillier-Pedersen Range-Bounded Affine Operation

We recall the proof for affine operation over Paillier ciphertext in given range, i.e. for relation  $\mathcal{R}'_{\text{AffRan}}$  proposed in [26, Sec.6.2.7].  $\mathcal{R}'_{\text{AffRan}}$  is defined as

$$\{(N, q, c_A, c_B; a, \alpha) \mid c_A = (c_B)^a (1+N)^\alpha \pmod{N^2} \\ \wedge a \in \mathbb{Z}_q \wedge \alpha \in \mathbb{Z}_{q^2 2^{t+\ell+s}}\}$$

Let  $t, \ell, s$  be security parameters. In the implementation, we set  $t = s = 128$  and  $\ell = 80$  for 80 bits statistical security. The proof works as follows. In the setup phase, the verifier  $V$  sends the prover  $P$  parameters  $N_0, g, h$  and zero-knowledge proof that all the parameters are generated correctly. (In our specific usage to MtA,  $N_0 = N$  since the verifier generates  $N$ .)

- Prover's first message:  $P$  chooses  $b \in \mathbb{Z}_{2^{t+\ell}q}, \beta \leftarrow \mathbb{Z}_{2^{2t+2\ell+s}q^2}, \rho_1, \rho_2, \rho_3, \rho_4 \leftarrow \mathbb{Z}_N$ .  $P$  computes  $A = c_B^b (1+N)^\beta \pmod{N^2}, B_1 = g^b h^{\rho_1} \pmod{N}, B_2 = g^\beta h^{\rho_2} \pmod{N}, B_3 = g^a h^{\rho_3} \pmod{N}, B_4 = g^\alpha h^{\rho_4} \pmod{N}$ .  $P$  sends  $(A, B_1, B_2, B_3, B_4)$  to  $V$ .
- Challenge:  $V$  sends a random  $e \in \mathbb{Z}_{2^t}$  to  $P$ .
- Prover's second message:  $P$  computes and sends  $z_1 = b + ea, z_2 = \beta + e\alpha, z_3 = \rho_1 + e\rho_3, z_4 = \rho_2 + e\rho_4$  to  $V$ .
- Verification: Accept if and only if
  - $z_1 \in [2^t q, 2^{t+\ell} q]$
  - $z_2 \in [2^{2t+\ell+s} q^2, 2^{2t+2\ell+s} q^2]$
  - $c_B^{z_1} (1+N)^{z_2} = A (c_A)^e \pmod{N^2}$
  - $g^{z_3} h^{z_4} = B_1 (B_3)^e \pmod{N}$
  - $g^{z_2} h^{z_4} = B_2 (B_4)^e \pmod{N}$

## C.3 Paillier-EC Assumption

The Paillier-EC assumption states the security of encryption even the adversary is given a restricted oracle. Concretely, it says that any PPT adversary  $\mathcal{A}$  can only win the following experiment (i.e. the experiment outputs 1) with  $1/2$  plus a negligible probability.

- Generate a Paillier key pair  $(pk, sk)$
- Choose  $w_0, w_1 \leftarrow \mathbb{Z}_q$  and compute  $Q_0 = w_0 \cdot P$
- Choose a random  $b \leftarrow \{0, 1\}$  and compute  $c = \text{Enc}(pk, w_b)$
- Let  $b' = \mathcal{A}^{O(\cdot, \cdot)}(pk, c, Q_0)$ , where  $O(c, a, b) = 1$  iff  $\text{Dec}(c) = a + bw_b \pmod{q}$  and the oracle  $O$  holds once it returns 0.
- The experiment returns 1 if and only if  $b' = b$ .

## D THE CL-ENCRYPTION, ZKPOK FOR CL-ENCRYPTION AND GAME-BASED THRESHOLD UNFORGEABILITY.

We recall the CL encryption [9] and ZKPoK for CL encryption from [8].

Let  $(\tilde{s}, f, \hat{g}_q, g_q, \hat{G}, F, q)$  be public parameters of CL encryption (as defined in [8]), where  $F = \langle f \rangle$  is the subgroup of  $\hat{G}$  with order  $q$ ,  $g_q$  is a random element in  $\langle \hat{g}_q \rangle$ . Denote by  $\text{sk}, \text{pk} = g_q^{\text{sk}}$  secret-public key pair. CL encryption and decryption works as Table 6, where  $S$  is an integer to be determined.

We denote by  $c \oplus \hat{c}$  the addition of the plaintext in ciphertexts  $c$  and  $\hat{c}$ , and by  $a \odot c$  the multiplication of the plaintext in  $c$  by a scalar  $a$ .

### D.1 ZKPoK for correctness of CL encryption

We recall the zero-knowledge argument of knowledge for correctness of ciphertext from [8]. Define relation

$$R_{\text{CL}} := \{(\text{pk}, c_1, c_2; m, r) \mid \text{pk} \in \hat{G}, r \in [0, S], \\ c_1 = g_q^r \wedge c_2 = \text{pk}^r f^m\}.$$

Let  $C$  be the challenge space and  $C = |C|$ . In the setup, the prover should send public parameters to verifier and proves  $g_q$  is a random element of  $\langle \hat{g}_q \rangle$  (using the proof given in [8, Sec. 3.2]).

- Prover's first message:  $P$  chooses  $r_1 \leftarrow [0, 2^{80}\tilde{s}C], r_1 \leftarrow \mathbb{Z}_q$ , computes  $t_1 = g_q^{r_1}, t_2 = \text{pk}^{r_1} f^{r_1}$ .  $P$  sends  $(t_1, t_2)$  to  $V$ .
- Challenge:  $V$  sends a random  $e \leftarrow C$  to  $P$ .
- Prover's second message:  $P$  computes and sends  $z_1 = r_1 + er, z_2 = r_1 + em$  to  $V$ .
- Verification: Accept if and only if
  - $z_1 \in [0, 2^{40}\tilde{s}C(2^{40} + 1)]$
  - $z_2 \in \mathbb{Z}_q$
  - $g_q^{z_1} = t_1 c_1^e$
  - $\text{pk}^{z_2} f^{z_2} = t_2 (c_2)^e$

### D.2 Game-based Threshold Unforgeability

We recall the definition of game-based threshold unforgeability given in [8, Sec. 2.2], and specify it to two-party case.

*Definition D.1 (Two-party signature unforgeability).* Consider a two-party signature (Keygen, Sign, Verify), and a PPT algorithm  $\mathcal{A}$ , having corrupted one of two parties, and which is given the view of two-party protocols Keygen and Sign on input messages of its choice (chosen adaptively) as well as signatures on these messages. Let  $M$  be the set of aforementioned messages. The two-party signature is said to be unforgeable under chosen message attack if for any such  $\mathcal{A}$  the probability that  $\mathcal{A}$  can produce a signature on a message  $m \notin M$  is negligible.