

# Building MPCitH-based Signatures from MQ, MinRank, Rank SD and PKP

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**Abstract.** The MPC-in-the-Head paradigm is a useful tool to build practical signature schemes. Many such schemes have been already proposed, relying on different assumptions. Some are relying on standard symmetric primitives like AES, some are relying on MPC-friendly primitives like LowMC or Rain, and some are relying on well-known hard problems like the syndrome decoding problem.

This work focuses on the third type of MPCitH-based signatures. Following the same methodology as the work of Feneuil, Joux and Rivain (CRYPTO'22), we apply the MPC-in-the-Head paradigm to several problems: the multivariate quadratic problem, the MinRank problem, the rank syndrome decoding problem, and the permuted kernel problem. Our goal is to study how this paradigm behaves for each of those problems.

For the multivariate quadratic problem, our scheme outperforms slightly the existing schemes when considering large fields (as  $\mathbb{F}_{256}$ ), and for the permuted kernel problem, we obtain larger sizes. Even if both schemes do not outperform the existing ones according to the communication cost, they are highly parallelizable and compatible with some MPC-in-the-Head techniques while the former proposals were not.

Moreover, we propose two efficient MPC protocols to check that the rank of a matrix over a field  $\mathbb{F}_q$  is upper bounded by a public constant. The first one relies on the rank decomposition while the second one relies on  $q$ -polynomials. We then use them to build signature schemes relying on the MinRank problem and the rank syndrome decoding problem. Those schemes outperform the former schemes, achieving sizes below 6 KB (while using only 256 parties for the MPC protocol).

**Keywords:** zero-knowledge proofs, post-quantum signatures, MPC-in-the-head

## 1 Introduction

The MPC-in-the-Head paradigm [IKOS07] is a versatile framework to design zero-knowledge proofs of knowledge, by relying on secure multi-party computation (MPC) techniques. After sharing the secret witness, the prover emulates “in her head” an MPC protocol with  $N$  parties and commits each party’s view independently. The verifier then challenges the prover to reveal the views of a random subset of parties. By the privacy of the MPC protocol, nothing is revealed about the witness, which implies the zero-knowledge property. On the other hand, a malicious prover needs to cheat for at least one party, which shall be discovered by the verifier with high probability, hence ensuring the soundness property.

Combined with the Fiat-Shamir transform [FS87], the MPCitH paradigm provides a useful tool for building practical signatures. The security of the resulting scheme only depends on the security of commitment/hash functions and the security of a one-way function. The choice of this one-way function is left to the signature designers. A first research track [ARS<sup>+</sup>15,DKR<sup>+</sup>21] consists in designing MPC-friendly primitives and in using them with the MPC-in-the-Head paradigm to get short signatures. This methodology has the disadvantage of requiring deep cryptanalysis of the introduced primitives. Another strategy would be to use standard symmetric primitives like AES as security assumptions for the MPCitH-based signatures, but it tends to produce larger signatures. As a last option, we can rely on a hard problem that exists for a long time

and thus is well understood. For example, [FJR22] succeeds in designing an efficient signature scheme using the syndrome decoding problem (over the Hamming weight), which is one of the oldest problems of code-based cryptography. The case of the syndrome decoding problem has been covered, but a natural question would be

Which performances can we have when using  
the MPC-in-the-Head paradigm with other hard problems?

Some articles [Wan22,FJR21,BG22,FMRV22] already apply this paradigm to hard problems (multivariate quadratic problem, MinRank problem, subset sum problem, ...). One of the drawbacks of almost all the schemes is that, when there is no structure to exploit, they need to rely on *protocols with helpers* [Beu20]. This technique introduced by [KKW18] and formalized by [Beu20] is quite powerful, but suffers from a high computational cost. As a consequence, the number of parties involved in the MPC protocol must stay low to have a practical scheme (in practice, many works take 32 as a limit for the number of parties), preventing achieving smaller sizes. Recently, [BG22] succeeds in leveraging the structure when considering structured hard problems (as the ideal rank syndrome decoding problem) and thus succeeds in achieving smaller sizes by removing the helper from [FJR21].

The present work aims to complete the state of the art of the MPC-in-the-Head applied to hard problems. Table 1 overviews schemes producing the shortest signatures for some hard problems.

Hard Problem	Best scheme	Achieved sizes
Multivariate Quadratic	Over $\mathbb{F}_4$ , [Wan22]	8.4 – 9.4 KB
	Over $\mathbb{F}_{256}$ , our work	6.9 – 8.3 KB
Min Rank	Our work	5.4 – 7.0 KB
Permuted Kernel	[BG22]	8.6 – 9.7 KB
Subset Sum	[FMRV22]	21.1 – 33.2 KB
Syndrome Decoding ( <i>Hamming</i> )	[FJR22]	Over $\mathbb{F}_2$ , 10.9 – 15.6 KB
		Over $\mathbb{F}_{256}$ , 8.3 – 11.5 KB
Syndrome Decoding ( <i>Rank</i> )	Our work	5.8 – 7.2 KB

Table 1: State of the art of the MPCiH-based signatures, including this work.

*Our contribution.* In this article, we consider several hard problems for which we propose new zero-knowledge proofs using the MPC-in-the-Head paradigm.

First, we propose a new zero-knowledge proof of knowledge for the *multivariate quadratic* problem. The resulting signature scheme outperforms [Wan22] only when the base field is large enough (*e.g.*  $\mathbb{F}_{256}$ ).

Secondly, we propose two efficient MPC protocols which take as input a matrix  $M \in \mathbb{F}_q^{n \times m}$  and which check that the rank of  $M$  is upper bounded by  $r$ , where  $r$  is a public positive integer:

- the first one decomposes  $M$  as a product  $TR$  where  $T \in \mathbb{F}_q^{n \times r}$  and  $R \in \mathbb{F}_q^{r \times m}$ , and uses an MPC protocol that checks the correctness of a matrix multiplication;
- the second one relies on the fact that the rows of  $M$  (represented as elements of  $\mathbb{F}_{q^m}$ ) are roots of a  $q$ -polynomial of degree  $q^r$  and on the fact that computing a  $q$ -polynomial is efficient in MPC while exploiting the linearity of the Frobenius endomorphism  $v \mapsto v^q$ .

We then use those protocols to build efficient signatures relying on the *MinRank* problem or on the *rank syndrome decoding* problem. Our schemes outperform all the previous proposals, by achieving sizes below 7 KB. They also outperform the [BG22]’s proposals which use structured problems (as the ideal rank syndrome decoding problem) to achieve small sizes.

Finally, we propose a new zero-knowledge proof of knowledge for the *permuted kernel* problem. The existing proposals are already quite efficient, achieving sizes below 10 KB [BG22]. They all rely on permutations, which is quite natural since the problem itself uses permutations. However, securely implementing

permutations is a tricky exercise. Our proposal achieves larger sizes, but uses no permutation at all. Our proposal is also compatible with the techniques proposed by [FR22] (as fast signature verification) and those proposed by [AGH<sup>+</sup>23] (which improves the running times of the MPCitH-based schemes relying on additive sharings), while the previous proposals for PKP are not.

*Paper organization.* The paper is organized as follows: In Section 2, we introduce the necessary background on the MPC-in-the-Head paradigm. We present our general methodology in Section 3. Then we apply it to the *multivariate quadratic* problem in Section 4, to the *MinRank* problem and the *rank syndrome decoding* problem in Section 5, and to the *permuted kernel* problem in Section 6. Finally, we discuss the computational cost of the obtained schemes in Section 7.

## 2 Preliminaries

Throughout the paper,  $\mathbb{F}_q$  shall denote the finite field with  $q$  elements. For any  $m \in \mathbb{N}^*$ , the integer set  $\{1, \dots, m\}$  is denoted  $[m]$ . For a probability distribution  $D$ , the notation  $s \leftarrow D$  means that  $s$  is sampled from  $D$ . For a finite set  $S$ , the notation  $s \leftarrow S$  means that  $s$  is uniformly sampled at random from  $S$ .

In this paper, we shall use the standard cryptographic notions of (honest verifier) zero-knowledge proof of knowledge and secure multiparty computation protocols (in the semi-honest model). We refer to [FR22] for the formal definition of those notions.

### 2.1 The MPC-in-the-Head Paradigm

The MPC-in-the-Head (MPCitH) paradigm introduced in [IKOS07] offers a way to build zero-knowledge proofs from secure multi-party computation (MPC) protocols. Let us assume we have an MPC protocol in which  $N$  parties  $\mathcal{P}_1, \dots, \mathcal{P}_N$  securely and correctly evaluate a function  $f$  on a secret input  $x$  with the following properties:

- the secret  $x$  is encoded as a sharing  $\llbracket x \rrbracket$  and each  $\mathcal{P}_i$  takes a share  $\llbracket x \rrbracket_i$  as input;
- the function  $f$  outputs ACCEPT or REJECT;
- the views of  $t$  parties leak no information about the secret  $x$ .

We can use this MPC protocol to build a zero-knowledge proof of knowledge of an  $x$  for which  $f(x)$  evaluates to ACCEPT. The prover proceeds as follows:

- she builds a random sharing  $\llbracket x \rrbracket$  of  $x$ ;
- she simulates locally (“in her head”) all the parties of the MPC protocol;
- she sends commitments to each party’s view, *i.e.* party’s input share, secret random tape and sent and received messages, to the verifier;
- she sends the output shares  $\llbracket f(x) \rrbracket$  of the parties, which should correspond to ACCEPT.

Then the verifier randomly chooses  $t$  parties and asks the prover to reveal their views. After receiving them, the verifier checks that they are consistent with an honest execution of the MPC protocol and with the commitments. Since only  $t$  parties are opened, revealed views leak no information about the secret  $x$ , while the random choice of the opened parties makes<sup>3</sup> the cheating probability upper bounded by  $(N - t)/N$ , thus ensuring the soundness of the zero-knowledge proof.

All MPC protocols described in this article fit the model described in [FR22], meaning that the parties take as input an additive sharing  $\llbracket x \rrbracket$  of the secret  $x$  (one share per party) and that they compute one or several rounds in which they perform three types of actions:

**Receiving randomness:** the parties receive a random value  $\varepsilon$  from a randomness oracle  $\mathcal{O}_R$ . When calling this oracle, all the parties get the same random value  $\varepsilon$ .

<sup>3</sup> We implicitly assume here that the communication between parties is broadcast.

**Receiving hint:** the parties can receive a sharing  $\llbracket \beta \rrbracket$  (one share per party) from a hint oracle  $\mathcal{O}_H$ . The hint  $\beta$  can depend on the witness  $w$  and the previous random values sampled from  $\mathcal{O}_R$ .

**Computing & broadcasting:** the parties can locally compute  $\llbracket \alpha \rrbracket := \llbracket \varphi(v) \rrbracket$  from a sharing  $\llbracket v \rrbracket$  where  $\varphi$  is an  $\mathbb{F}$ -linear function, then broadcast all the shares  $\llbracket \alpha \rrbracket_1, \dots, \llbracket \alpha \rrbracket_N$  to publicly reconstruct  $\alpha := \varphi(v)$ . The function  $\varphi$  can depend on the previous random values  $\{\varepsilon^i\}_i$  from  $\mathcal{O}_R$  and on the previous broadcasted values.

We refer to [FR22] for the detailed transformation of such MPC protocols into zero-knowledge proofs of knowledge and for the resulting performance.

### 3 Methodology

In each of the following sections, we focus on a specific hard problem which is supposed quantum-resilient:

- Section 4: Multivariate Quadratic Problem;
- Section 5.2: Min Rank Problem;
- Section 5.3: Syndrome Decoding in the *rank* metric;
- Section 6: Permuted Kernel Problem.

For each of them, we will use the MPC-in-the-Head paradigm to build a new zero-knowledge protocol. To proceed, we will first describe the MPC protocol we use. This MPC protocol will fit the model described in [FR22] and will satisfy the following properties:

- it takes as input an additive sharing of a candidate solution of the studied problem, and eventually an additive sharing of auxiliary data;
- the MPC parties get (only once) a common random value from an oracle  $\mathcal{O}_R$ ;
- when the tested solution is valid (*i.e.* a solution of the studied hard problem) and when the auxiliary data are genuinely computed, the MPC protocol always outputs ACCEPT; otherwise, it outputs ACCEPT with probability at most  $p$  (over the randomness of  $\mathcal{O}_R$ ), where  $p$  is called the *false positive rate*;
- the views of all the parties except one leak no information about the candidate solution.

By applying the MPC-in-the-Head paradigm to this MPC protocol, we get a 5-round zero-knowledge proof of knowledge of a solution of the studied problem (see [FR22, Theorem 2] with the privacy threshold  $\ell := N - 1$ ), with soundness error

$$\frac{1}{N} + \left(1 - \frac{1}{N}\right) \cdot p$$

where  $N$  is the number of parties involved in the multi-party computation. We do not exhibit the obtained proof of knowledge since the transformation is standard. We refer the reader to [FR22] for a detailed explanation of how to concretely apply the MPC-in-the-Head paradigm.

To obtain a signature scheme, we apply the Fiat-Shamir transform [FS87] to the previous protocol. Since this protocol has 5 rounds, the security of the resulting scheme should take into account the attack of [KZ20]. More precisely, the forgery cost of the signature scheme is given by

$$\text{cost}_{\text{forge}} := \min_{\tau_1, \tau_2: \tau_1 + \tau_2 = \tau} \left\{ \frac{1}{\sum_{i=\tau_1}^{\tau} \binom{\tau}{i} p^i (1-p)^{\tau-i}} + N^{\tau_2} \right\}$$

where  $\tau$  is the number of parallel executions.

*Remark 1.* For the permuted kernel problem, the MPC protocol we propose slightly differs from the above description. The parties call the oracle  $\mathcal{O}_R$  twice (instead of once). The resulting scheme is a 7-round proof (not a 5-round proof) with the same soundness error as before. However, the forgery cost is not the same (see Section 6).

Finally, we compare the resulting scheme with all the former schemes which are non-interactive identification schemes based on the same security assumption. To proceed, we first list all these schemes with their formulae of the forgery security and of the communication cost. Since some quantities occur several times, we define some notations to ease the readability. For the forgery cost, we introduce the two following notations:

- $\varepsilon_{\text{helper}}(\tau, M, \varepsilon)$  is the soundness error of a protocol with helper [Beu20] when the helper entity is emulated by a cut-and-choose phase.  $M$  is the total number of repetitions in the cut-and-choose phase,  $\varepsilon$  is the soundness of the unitary protocol relying on the helper, and  $\tau$  is the number of repetitions of this unitary protocol. We have

$$\varepsilon_{\text{helper}}(\tau, M, \varepsilon) := \max_{M-\tau \leq k \leq M} \left\{ \frac{\binom{k}{M-\tau}}{\binom{M}{M-\tau}} \cdot \varepsilon^{k-(M-\tau)} \right\}.$$

- $\text{KZ}(p_1, p_2)$  is the forgery cost of [KZ20] for a 5-round protocol<sup>4</sup>. We have

$$\text{KZ}(p_1, p_2) := \min_{\tau_1, \tau_2: \tau_1 + \tau_2 = \tau} \left\{ \frac{1}{\sum_{i=\tau_1}^{\tau} \binom{\tau}{i} p_1^i (1-p_1)^{\tau-i}} + \frac{1}{p_2^{\tau_2}} \right\}$$

For the communication cost (*i.e.* the signature size), we introduce the following notations:

- $\mu_{\text{seed}}$  is the cost of sending a  $\lambda$ -bit seed;
- $\mu_{\text{dig}}$  is the cost of sending a  $2\lambda$ -bit commitment/hash digest;
- $\mu_{\text{helper}}$  is the cost (per repetition) of using the helper technique of [Beu20], this cost satisfies

$$\mu_{\text{helper}} \leq (\mu_{\text{seed}} + \mu_{\text{dig}}) \cdot \log_2 \left( \frac{M}{\tau} \right)$$

where  $M$  is the number of repetitions involved in the cut-and-choose phase emulating the helper. It corresponds to the cost of revealing  $M - \tau$  leaves among  $M$  in a seed tree, with the cost of sending the authentication paths of  $\tau$  leaves among  $M$  in a Merkle tree.

- $\mu_{\text{MPCitH}}$  is the fixed cost (per repetition) of using the MPC-in-the-Head paradigm, we have

$$\mu_{\text{MPCitH}} = \mu_{\text{seed}} \cdot \log_2 N + \mu_{\text{dig}}.$$

It corresponds to the cost of revealing all the leaves but one in a seed tree of  $N$  leaves (plus a commitment digest).

Then, to get a numerical comparison, we select one or two instances of the studied hard problem and we compare all these schemes for these precise instances. To proceed, we need to select the parameters of the schemes when relevant. The signature schemes based on the MPC-in-the-Head paradigm have as parameter the number  $N$  of parties involved in the multi-party computation. When taking a small  $N$ , we get a faster scheme, but when taking a large  $N$ , we get shorter signature sizes. To have a fair comparison between the different schemes, we will always take the same  $N$ :

- when the protocol relies on a helper, we take  $N = 8$  to have a fast scheme and  $N = 32$  to have short sizes.
- otherwise, we take  $N = 32$  to have a fast scheme and  $N = 256$  to have short sizes.

### 3.1 Matrix Multiplication Checking Protocol

In our constructions, we need an MPC protocol that checks that three matrices  $X, Y, Z$  satisfy  $Z = X \cdot Y$ . We describe in Figure 1 such a protocol  $\Pi_{\text{MM}}^\eta$  which has a positive parameter  $\eta$ . This protocol is a matrix variant of the multiplication checking protocol of [BN20] (optimized in [KZ22]).

<sup>4</sup> in the case where the verifier can not perform some checks after receiving the first response (see [KZ20] for details).

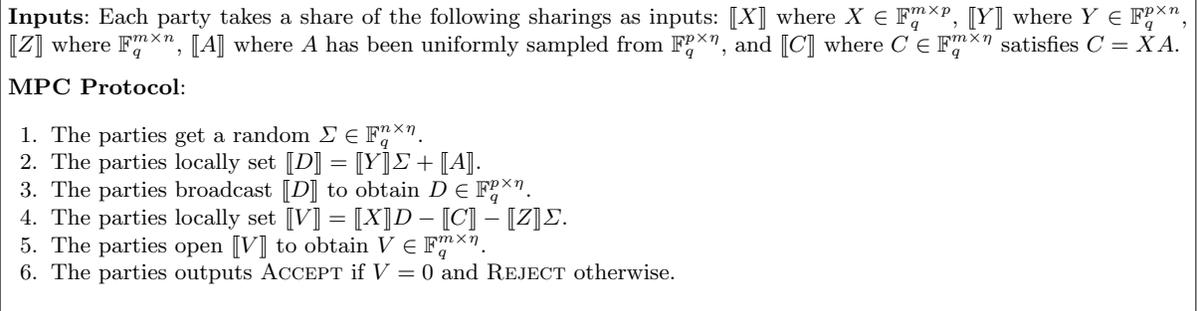


Fig. 1: The MPC protocol  $\Pi_{MM}^\eta$  which checks that  $Z = X \cdot Y$  (MM stands for Matrix Multiplication).

**Lemma 1.** *If  $Z = X \cdot Y$  and if  $C$  are genuinely computed, then  $\Pi_{MM}^\eta$  always outputs ACCEPT. If  $Z \neq X \cdot Y$ , then  $\Pi_{MM}^\eta$  outputs ACCEPT with probability at most  $\frac{1}{q^\eta}$ .*

*Proof.* We have

$$\begin{aligned}
 V &= XD - C - Z\Sigma \\
 &= X(Y\Sigma + A) - C - Z\Sigma \\
 &= (XY - Z)\Sigma - (C - XA).
 \end{aligned}$$

If  $Z = XY$  and  $C = XA$ ,  $V$  is equal to zero and thus the parties will always output ACCEPT. In contrast, if  $Z \neq XY$ , then there exists  $(i^*, j^*) \in [m] \times [n]$  such that  $Z_{i^*, j^*} - (X \cdot Y)_{i^*, j^*} \neq 0$ . Given  $k \in \{1, \dots, \eta\}$ ,  $\Sigma_{j^*, k}$  is uniformly sampled in  $\mathbb{F}_q$  and then  $((Z - X \cdot Y)\Sigma)_{i^*, k}$  is uniformly random in  $\mathbb{F}_q$  (because one term of the term is uniformly random). Thus, the probability that  $V$  is zero is at most the probability that  $(Z - X \cdot Y)\Sigma$  is equal to  $(C - XA)$  on the row  $i^*$  whereas the row  $i^*$  of  $(Z - X \cdot Y)\Sigma$  is uniformly random in  $\mathbb{F}_q^\eta$ , *i.e.* the probability that  $V$  is zero (at row  $i^*$ ) is at most  $\frac{1}{q^\eta}$ . □

### 3.2 MPCitH Optimizations

It is often possible to optimize the communication cost of a scheme relying on the MPC-in-the-Head paradigm. The common optimization tricks are the following:

- Except for the last party, the input share of a party can be derived from a seed using a pseudo-random generator. Thus, when we need to reveal the input share, we just need to reveal a seed. In practice, a prover must reveal the input shares of  $N - 1$  parties, so it would imply revealing  $N - 1$  seeds. To save more communication, we can generate the seeds using a tree structure, decreasing the number of revealed seeds to  $\log_2(N)$  (see [KKW18, Sec. 2.3] for details).
- We do not need to reveal shares for shared random values (as  $A$  in Figure 1) since they can be entirely derived from the seeds of the previous point.
- We do not need to reveal shares for shared publicly-known values (see [KZ22, Sec. 2.4] for details). For example, we do not need to reveal the share of  $V$  broadcast by the hidden party in Figure 1. Indeed, this share can be deduced from the shares of the other parties and knowing that  $V$  must be equal to zero (otherwise the verification fails).

## 4 Proof of Knowledge for $\mathcal{MQ}$

We want to build a zero-knowledge proof of knowledge for the *multivariate quadratic problem*:

**Definition 1 (Multivariate Quadratic Problem - Matrix Form).** *Let  $(q, m, n)$  be positive integers. The multivariate quadratic problem with parameters  $(q, m, n)$  is the following problem:*

Let  $(A_i)_{i \in [m]}$ ,  $(b_i)_{i \in [m]}$ ,  $x$  and  $y$  be such that:

1.  $x$  is uniformly sampled from  $\mathbb{F}_q^n$ ,
2. for all  $i \in [m]$ ,  $A_i$  is uniformly sampled from  $\mathbb{F}_q^{n \times n}$ ,
3. for all  $i \in [m]$ ,  $b_i$  is uniformly sampled from  $\mathbb{F}_q^n$ ,
4. for all  $i \in [m]$ ,  $y_i$  is defined as  $y_i := x^T A_i x + b_i^T x$ .

From  $((A_i)_{i \in [m]}, (b_i)_{i \in [m]}, y)$ , find  $x$ .

The prover wants to convince the verifier that she knows  $x \in \mathbb{F}_q^n$  such that

$$\begin{cases} y_1 = x^T A_1 x + b_1^T x \\ \vdots \\ y_m = x^T A_m x + b_m^T x \end{cases}$$

To proceed, she will rely on the MPC-in-the-Head paradigm: she will first share the secret vector  $x$  and then use an MPC protocol which verifies that this vector satisfies the above relations.

*MPC Protocol.* Instead of checking the  $m$  relations separately, we batch them into a linear combination where coefficients  $\gamma_1, \dots, \gamma_m$  are uniformly sampled in the field extension  $\mathbb{F}_{q^n}$ . The MPC protocol will check that

$$\sum_{i=1}^m \gamma_i (y_i - x^T A_i x - b_i^T x) = 0. \quad (1)$$

If one of the relations was not satisfied, then Equation (1) would be satisfied only with a probability  $\frac{1}{q^n}$ . We can write the equality as

$$\begin{aligned} \sum_{i=1}^m \gamma_i (y_i - b_i^T x) &= \sum_{i=1}^m \gamma_i (x^T A_i x) \\ &= x^T \left( \sum_{i=1}^m \gamma_i A_i \right) x \\ &= \langle x, w \rangle \quad \text{where } w := \left( \sum_{i=1}^m \gamma_i A_i \right) x \end{aligned}$$

By defining  $z := \sum_{i=1}^m \gamma_i (y_i - b_i^T x)$  and  $w := (\sum_{i=1}^m \gamma_i A_i) x$ , proving Equation (1) is equivalent to proving that

$$z = \langle x, w \rangle.$$

And to prove the above equality, we can rely on the subprotocol  $\Pi_{\text{MM}}$  described in Section 3.1 (assuming that all the scalars live in  $\mathbb{F}_{q^n}$ ). Thus, the MPC protocol proceeds as follows:

1. The parties get random  $\gamma_1, \dots, \gamma_m \in \mathbb{F}_{q^n}$ .
2. The parties locally set  $\llbracket z \rrbracket = \sum_{i=1}^m \gamma_i (y_i - b_i^T \llbracket x \rrbracket)$ .
3. The parties locally set  $\llbracket w \rrbracket = (\sum_{i=1}^m \gamma_i A_i) \llbracket x \rrbracket$ .
4. The parties execute the protocol  $\Pi_{\text{MM}}$  to check that  $z = \langle w, x \rangle$ .

Since this sub-protocol  $\Pi_{\text{MM}}$  produces false positive events with a rate of  $\frac{1}{q^n}$ , if  $x$  does not satisfy the  $\mathcal{MQ}$  relations, the complete MPC protocol outputs ACCEPT only with a probability of at most

$$\frac{1}{q^n} + \left(1 - \frac{1}{q^n}\right) \frac{1}{q^n} = \frac{2}{q^n} - \frac{1}{q^{2n}}.$$

The complete MPC protocol is described in Figure 2.

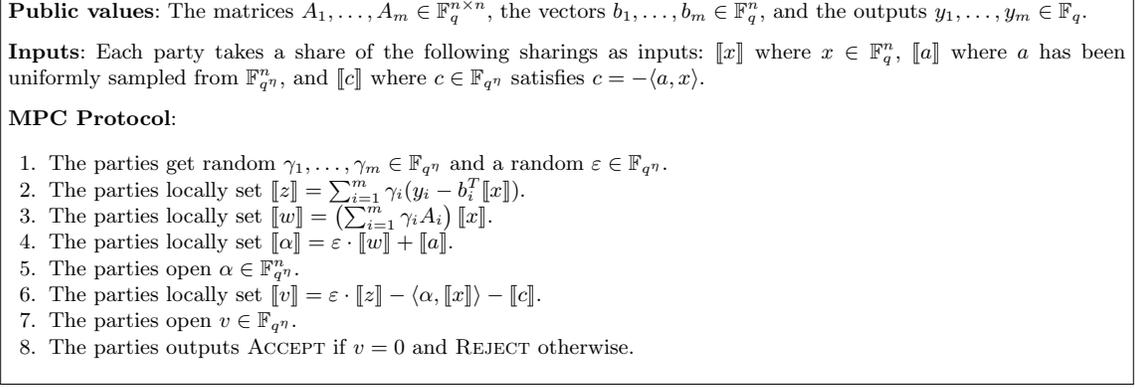


Fig. 2: An MPC protocol that verifies that the given input corresponds to a solution of an  $\mathcal{MQ}$  problem.

*Proof of Knowledge.* Using the MPC-in-the-Head paradigm (see Section 2.1), we transform the above MPC protocol into an interactive zero-knowledge proof of knowledge which enables us to convince a verifier that a prover knows the solution of a  $\mathcal{MQ}$  problem. The soundness error of the resulting protocol is

$$\varepsilon := \frac{1}{N} + \left(1 - \frac{1}{N}\right) \left(\frac{2}{q^n} - \frac{1}{q^{2n}}\right).$$

By repeating the protocol  $\tau$  times, we get a soundness error of  $\varepsilon^\tau$ . To obtain a soundness error of  $\lambda$  bits, we can take  $\tau = \left\lceil \frac{-\lambda}{\log_2 \varepsilon} \right\rceil$ . We can transform the interactive protocol into a non-interactive argument/signature thanks to the Fiat-Shamir transform [FS87]. According to [KZ20], the security of the resulting scheme is

$$\text{cost}_{\text{forge}} := \min_{\tau_1, \tau_2: \tau_1 + \tau_2 = \tau} \left\{ \frac{1}{\sum_{i=\tau_1}^{\tau} \binom{\tau}{i} p^i (1-p)^{\tau-i}} + N^{\tau_2} \right\}$$

where  $p := \frac{2}{q^n} - \frac{1}{q^{2n}}$ .

The communication cost of the scheme (in bits) is

$$4\lambda + \tau \cdot \left( \underbrace{n \cdot \log_2(q)}_x + \underbrace{n \cdot \eta \cdot \log_2(q)}_\alpha + \underbrace{\eta \cdot \log_2(q)}_c + \underbrace{\lambda \cdot \log_2 N + 2\lambda}_{\text{MPCitH}} \right)$$

where  $\lambda$  is the security level,  $\eta$  is a scheme parameter and  $\tau$  is computed such that the soundness error is of  $\lambda$  bits in the interactive case and such that  $\text{cost}_{\text{forge}}$  is of  $\lambda$  bits in the non-interactive case. The public key corresponds to the coefficients of the  $\mathcal{MQ}$  equations (namely  $(A_i)_{i \in [m]}$  and  $(b_i)_{i \in [m]}$ ) which can be represented by a  $\lambda$ -bit seed and the vector  $y \in \mathbb{F}_q^m$ . Its size is thus  $\lambda + m \log_2 q$  bits.

*Performance and comparison.* In what follows, we compare our scheme with the state of the art on two  $\mathcal{MQ}$  instances:

**Instance 1.** Multivariate Quadratic equations over a small field:

$$(q, m, n) = (4, 88, 88),$$

**Instance 2.** Multivariate Quadratic equations over a larger field:

$$(q, m, n) = (256, 40, 40).$$

Both of these instances are believed to correspond to a security of 128 bits [BMSV22].

We provide in Tables 2 and 3 a complete comparison of our scheme with the state of the art. In the comparison, we put MQ-DSS [CHR<sup>+</sup>16] which corresponds to the non-interactive version of the 5-round identification scheme of [SSH11]. For the sake of completeness, we also put how the 3-round identification scheme of [SSH11] would perform when applying the Fiat-Shamir transform on it.

Over a small field, the Mesquite [Wan22] scheme has the smallest communication cost, even if our scheme produces competitive signature sizes. Over a larger field, we can produce signature sizes close to 7 KB, and thus we outperform all the former schemes.

*Remark 2.* In contrast with the former state of the art, the communication cost of our scheme is independent of the number  $m$  of  $\mathcal{MQ}$  relations.

Scheme Name	Security	Signature Size
[SSH11] (3 rounds)	$(3/2)^\tau$	$\mu_{\text{dig}} + \tau [2\mu_{\text{var}} + \mu_{\text{out}} + 2\mu_{\text{dig}}]$
MQ-DSS [CHR <sup>+</sup> 16]	$\text{KZ}(\frac{1}{q}, \frac{1}{2})$	$2\mu_{\text{dig}} + \tau [2\mu_{\text{var}} + \mu_{\text{out}} + 2\mu_{\text{dig}}]$
MUDFISH [Beu20]	$\varepsilon_{\text{helper}}(\tau, M, \frac{1}{q'})^{-1}$	$\mu_{\text{dig}} + \tau [2\mu_{\text{var}} + \mu_{\text{out}} + 2\mu_{\text{seed}} + \mu_{\text{dig}} \cdot \log_2(q') + \mu_{\text{helper}}]$
Mesquite [Wan22]	$\varepsilon_{\text{helper}}(\tau, M, \frac{1}{N})^{-1}$	$\mu_{\text{dig}} + \tau [\mu_{\text{var}} + \mu_{\text{out}} + \mu_{\text{MPCitH}} + \mu_{\text{helper}}]$
Our scheme	$\text{KZ}(\frac{2}{q^\eta} - \frac{1}{q^{2\eta}}, \frac{1}{N})$	$2\mu_{\text{dig}} + \tau [(1 + \eta) \cdot \mu_{\text{var}} + \eta \cdot \log_2 q + \mu_{\text{MPCitH}}]$

Table 2: Sizes of the signatures relying on the  $\mathcal{MQ}$  problem (restricting to the schemes using the FS heuristics). The used notations are:  $\mu_{\text{var}} := n \log_2 q$ ,  $\mu_{\text{out}} := m \log_2 q$ , plus all the notations defined in Section 3.

Instance	Protocol Name	Variant	Parameters				Signature Size	Public Key Size
			$N$	$M$	$\tau$	$\eta$		
$q = 4$ $m = 88$ $n = 88$	[SSH11] (3 rounds)	-	-	219	-	-	28 502 B	38 B
	MQ-DSS [CHR <sup>+</sup> 16]	-	-	316	-	-	41 444 B	
	MUDFISH [Beu20]	-	4	191	68	-	14 640 B	
	Mesquite [Wan22]	Fast	8	187	49	-	9 578 B	
		Short	32	389	28	-	<b>8 609 B</b>	
	Our scheme	Fast	32	-	40	6	10 764 B	
Short		256	-	25	8	9 064 B		
$q = 256$ $m = 40$ $n = 40$	[SSH11] (3 rounds)	-	-	219	-	-	40 328 B	56 B
	MQ-DSS [CHR <sup>+</sup> 16]	-	-	156	-	-	28 768 B	
	MUDFISH [Beu20]	Fast	8	176	51	-	15 958 B	
		Short	16	250	36	-	13 910 B	
	Mesquite [Wan22]	Fast	8	187	49	-	11 339 B	
		Short	32	389	28	-	9 615 B	
Our scheme	Fast	32	-	36	2	8 488 B		
	Short	256	-	25	2	<b>7 114 B</b>		

Table 3: Sizes of the signatures relying on the  $\mathcal{MQ}$  problem (restricting to the schemes using the FS heuristics). Numerical comparison.

## 5 Proofs of Knowledge for MinRank and Rank SD

In this section, we propose arguments of knowledge for the MinRank problem (Section 5.2) and the Rank SD problem (Section 5.3). But before that, in Section 5.1, we propose two efficient MPC protocols which check that a matrix  $M$  has a rank of at most  $r$ .

In what follows, we denote  $\text{wt}_R(M)$  the rank of a matrix  $M$ .

### 5.1 Matrix Rank Checking Protocols

We want to build MPC protocols which check that a matrix has rank at most  $r$ . Such MPC protocols will be used for arguments of knowledge with the MPC-in-the-Head paradigm. We propose two protocols:

- the first one relies on the rank decomposition of matrices. It has the advantage of being quite *simple*, but its false positive rate is *large*.
- the second one relies on linearized polynomials. It has the advantage of having a *very small* false positive rate, but it sometimes requires to handle field extensions of *large degrees*.

**Using Rank Decomposition.** Let us design an MPC protocol which checks that a matrix  $M \in \mathbb{F}^{m \times n}$  has a rank of at most  $r$ , *i.e.*  $\text{wt}_R(M) \leq r$ . To proceed, we will rely on the *rank decomposition*:

$$\begin{aligned} & \text{a matrix } M \in \mathbb{F}_q^{n \times m} \text{ has a rank of at most } r \\ & \text{if and only if there exists } T \in \mathbb{F}_q^{n \times r} \text{ and } R \in \mathbb{F}_q^{r \times m} \text{ such that } M = TR. \end{aligned}$$

In practice, our MPC protocol that we will denote  $\Pi_{RC-RD}^\eta$  takes as input such matrices  $T$  and  $R$  (in addition to  $M$ ) and simply executes the matrix multiplication checking protocol  $\Pi_{MM}^\eta$  (see Section 3.1), for some positive integer  $\eta$ .

**Theorem 1.** *If  $\text{wt}_R(M) \leq r$  and if  $T, R$  are genuinely computed, then  $\Pi_{RC-RD}^\eta$  always outputs ACCEPT. If  $\text{wt}_R(M) > r$ , then  $\Pi_{RC-RD}$  outputs ACCEPT with probability at most  $\frac{1}{q^\eta}$ . More precisely, if  $\text{wt}_R(M) = w + \delta$  with  $\delta \geq 1$ , then  $\Pi_{RC-RD}^\eta$  outputs ACCEPT with probability at most  $\frac{1}{q^{\delta-\eta}}$ .*

*Proof.* The final broadcast matrix  $V$  in  $\Pi_{MM}^\eta$  satisfies

$$V = (TR - M)\Sigma - (C - TA)$$

where matrices  $A$  and  $C$  have been built before receiving the random  $\Sigma$ . We have

$$\begin{aligned} \text{wt}_R(M - TR) &\geq \text{wt}_R(M) - \text{wt}_R(TR) \\ &\geq (r + \delta) - r = \delta \end{aligned}$$

It means that  $TR - M$  has at least  $\delta$  non-zero coefficients  $(i_1, j_1), \dots, (i_\delta, j_\delta)$  which are over  $\delta$  different rows and over  $\delta$  different columns, *i.e.*

$$\forall k_1, k_2 \in [\delta], (i_{k_1} \neq i_{k_2}) \wedge (j_{k_1} \neq j_{k_2}).$$

Let us consider  $k \in [\delta]$ . The  $j_k$ th row of  $\Sigma$  is uniformly sampled in  $\mathbb{F}_q^\eta$  and thus the  $i_k$ th row of  $(M - TR)\Sigma$  is uniformly random in  $\mathbb{F}_q^\eta$  (because one term of the sum is uniformly random). Thus, the probability that the  $i_k$ th row of  $V$  is zero is the probability that  $(M - TR)\Sigma$  is equal to  $(C - TA)$  on the row  $i_k$  whereas the row  $i_k$  of  $(M - TR)\Sigma$  is uniformly random in  $\mathbb{F}_q^\eta$ , *i.e.* the probability that the  $i_k$ th row of  $V$  is zero is  $\frac{1}{q^\eta}$ . By taking a union bound over all  $k$ , we get that the probability that  $V$  is zero is at most  $\frac{1}{q^{\delta-\eta}}$ .  $\square$

**Using Linearized Polynomials.** In what follows, we represent a matrix of  $\mathbb{F}_q^{m \times n}$  as an element of  $(\mathbb{F}_q^m)^n$ . We want to design an MPC protocol which checks that a matrix  $M = (x_1, \dots, x_n) \in (\mathbb{F}_{q^m})^n$  has a rank of at most  $r$ . Equivalently, it means that all  $x_i$  belongs to an  $\mathbb{F}_q$ -linear subspace  $U$  of  $\mathbb{F}_{q^m}$  of dimension  $r$ . Let us define the polynomial  $L_U(X)$  as

$$L_U(X) := \prod_{u \in U} (X - u) \in \mathbb{F}_{q^m}[X].$$

The degree of  $L_U$  is  $q^r$  since  $U$  has  $q^r$  elements. Showing that  $\text{wt}(M) \leq r$  can be done by showing that all  $x_i$ 's are roots of  $L_U$ .

According to [LN96, Theorem 3.52],  $L_U$  is a  $q$ -polynomial over  $\mathbb{F}_{q^m}$ , meaning that it is of the form

$$L_U(X) = X^{q^r} + \sum_{i=0}^{r-1} \beta_i X^{q^i}.$$

Such polynomials are convenient for multi-party computation since the Frobenius endomorphism  $X \mapsto X^q$  is a linear application in field extensions of  $\mathbb{F}_q$  and thus it is communication-free to compute  $\llbracket x^q \rrbracket, \llbracket x^{q^2} \rrbracket, \dots$  from  $\llbracket x \rrbracket$ .

The core idea of the rank checking protocol is to check that  $L_U(x_1) = L_U(x_2) = \dots = L_U(x_n) = 0$ . To proceed, the MPC protocol will batch these checkings by uniformly sampling  $\gamma_1, \dots, \gamma_n \in \mathbb{F}_{q^m}$  and checking that

$$\sum_{j=1}^n \gamma_j \cdot L_U(x_j) = 0. \quad (2)$$

If one  $x_i$  is not a root of the polynomial  $L_U$ , then Equation (2) is satisfied only with probability  $\frac{1}{q^m}$ . Let us rewrite the left term of (2):

$$\begin{aligned} \sum_{j=1}^n \gamma_j \cdot L_U(x_j) &= \sum_{j=1}^n \gamma_j \cdot \left( x_j^{q^r} + \sum_{i=0}^{r-1} \beta_i x_j^{q^i} \right) \\ &= \underbrace{\sum_{j=1}^n \gamma_j \cdot x_j^{q^r}}_{:= -z} + \sum_{i=0}^{r-1} \beta_i \cdot \underbrace{\sum_{j=1}^n \gamma_j x_j^{q^i}}_{:= w_i} \end{aligned}$$

By defining  $z := -\sum_{j=1}^n \gamma_j \cdot x_j^{q^r}$  and  $w_i := \sum_{j=1}^n \gamma_j x_j^{q^i}$  for  $i \in \{0, \dots, r-1\}$ , proving Equation (2) is equivalent to proving

$$z = \langle \beta, w \rangle.$$

Our MPC protocol that we will denote  $\Pi_{\text{RC-LP}}^\eta$  takes as input  $\llbracket x_1 \rrbracket, \dots, \llbracket x_n \rrbracket$  and  $\llbracket L_U \rrbracket := X^{q^r} + \sum_{i=0}^{r-1} \llbracket \beta_i \rrbracket X^{q^i}$  proceeds as follows:

1. The parties get random  $\gamma_1, \dots, \gamma_n \in \mathbb{F}_{q^{m \cdot \eta}}$ .
2. The parties locally set  $\llbracket z \rrbracket = -\sum_{j=1}^n \gamma_j \llbracket x_j \rrbracket^{q^r}$ .
3. The parties locally set  $\llbracket w_i \rrbracket = \sum_{j=1}^n \gamma_j \llbracket x_j \rrbracket^{q^i}$  for all  $i \in \{0, \dots, r-1\}$ .
4. The parties execute the protocol  $\Pi_{\text{MM}}$  to check that  $z = \langle \beta, w \rangle$  over  $\mathbb{F}_{q^{m \cdot \eta}}$ .

**Theorem 2.** *If  $\text{wt}_R(M) \leq r$  and if  $L_U$  are genuinely computed, then  $\Pi_{\text{RC-LP}}^\eta$  always outputs ACCEPT. If  $\text{wt}_R(M) > r$ , then  $\Pi_{\text{RC-LP}}^\eta$  outputs ACCEPT with probability at most  $\frac{1}{q^{m \cdot \eta}} + \left(1 - \frac{1}{q^{m \cdot \eta}}\right) \frac{1}{q^{m \cdot \eta}}$ .*

*Proof.*  $\llbracket L_U \rrbracket$  is a  $q$ -polynomial over  $\mathbb{F}_{q^m}$  of degree exactly  $q^r$ . It means that its number of roots is at most  $q^r$ . According to [LN96, Theorem 3.50], the roots form a  $\mathbb{F}_q$ -linear subspace  $V$  of the field extension  $\mathbb{F}_{q^s}$  of  $\mathbb{F}_{q^m}$ . Since  $\mathbb{F}_{q^m}$  is also a linear subspace of  $\mathbb{F}_{q^s}$ ,  $V \cap \mathbb{F}_{q^m}$  is a linear subspace of  $\mathbb{F}_{q^s}$  (and of  $\mathbb{F}_{q^m}$ ). Its dimension is at most  $r$  (since it has at most  $q^r$  elements). If  $\text{wt}_R(M) > r$ , there exist  $i^*$  such that

$$L_U(x_{i^*}) \neq 0.$$

We then have two options resulting in  $\Pi_{\text{RC-LP}}^\eta$  outputting ACCEPT:

- Either  $\sum_{j=1}^n \gamma_j \cdot L_U(x_j) = 0$ , which occurs with probability  $\frac{1}{q^{m \cdot \eta}}$ ;
- Or  $\sum_{j=1}^n \gamma_j \cdot L_U(x_j) \neq 0$ , i.e.  $z \neq \langle \beta, w \rangle$  and  $\Pi_{\text{MM}}$  outputs ACCEPT, which occurs with probability  $\frac{1}{q^{m \cdot \eta}}$  since  $\Pi_{\text{MM}}$  has a false positive rate of  $\frac{1}{q^{m \cdot \eta}}$ .

□

## 5.2 Proof of Knowledge for MinRank

We want to build a zero-knowledge proof of knowledge for the *MinRank problem*:

**Definition 2 (MinRank Problem).** Let  $(q, m, n, k)$  be positive integers. The *MinRank problem* with parameters  $(q, m, n, k)$  is the following problem:

Let  $M_0, M_1, \dots, M_k, E$  and  $x$  such that:

- $x$  is uniformly sampled from  $\mathbb{F}_q^k$ ,
- for all  $i \in [k]$ ,  $M_i$  is uniformly sampled from  $\mathbb{F}_q^{n \times m}$ ,
- $E$  is uniformly sampled from  $\{E \in \mathbb{F}_q^{n \times m} : \text{wt}_R(E) \leq w\}$ ,
- $M_0$  is defined as  $M_0 = E - \sum_{i=1}^k x_i M_i$ .

From  $(M_0, M_1, \dots, M_k)$ , find  $x$ .

The prover wants to convince the verifier that she knows such an  $x$ . To proceed, the prover will first share the secret vector  $x$  and then use an MPC protocol which verifies that this vector satisfies the above property.

*MPC Protocol.* We want to build an MPC protocol which takes as input (a sharing of)  $x$  and which outputs

$$\begin{cases} \text{ACCEPT} & \text{if } \text{wt}_R(E) \leq r \\ \text{REJECT} & \text{otherwise.} \end{cases}$$

where  $E := M_0 + \sum_{i=1}^k x_i M_i$ .

Given  $\llbracket x \rrbracket$ , the parties can locally build  $\llbracket E \rrbracket$  as  $M_0 + \sum_{i=1}^k \llbracket x_i \rrbracket M_i$ . It remains to check that  $\llbracket E \rrbracket$  corresponds to the sharing of a matrix of rank at most  $r$ . It can be done using one of the two rank checking protocols described in Section 5.1:  $\Pi_{\text{RC-RD}}^\eta$  relying on the rank decomposition or  $\Pi_{\text{RC-LP}}^\eta$  relying on linearized polynomials, for some parameter  $\eta$ .

The complete MPC protocol is described in Figure 3 when relying on the rank decomposition and in Figure 4 when relying on linearized polynomials. In the second case, the rows of the matrix  $E$  are rewritten as elements of  $\mathbb{F}_q^m$ , but when  $m \neq n$ , it can be more convenient to work on the columns (depending of the values of  $m$  and  $n$ ).

**Public values:**  $M_0, M_1, \dots, M_k \in \mathbb{F}_q^{n \times m}$ .

**Inputs:** Each party takes a share of the following sharings as inputs:  $\llbracket x \rrbracket$  where  $x \in \mathbb{F}_q^k$ ,  $\llbracket T \rrbracket$  where  $T \in \mathbb{F}_q^{n \times r}$ ,  $\llbracket R \rrbracket$  where  $R \in \mathbb{F}_q^{r \times m}$ ,  $\llbracket a \rrbracket$  where  $a$  has been uniformly sampled from  $\mathbb{F}_q^{r \times \eta}$ , and  $\llbracket c \rrbracket \in \mathbb{F}_q^{n \times \eta}$ , such that  $M_0 + \sum_{i=1}^k x_i M_i = TR$  and  $c = Ta$ .

**MPC Protocol:**

1. The parties get a random  $\Sigma \in \mathbb{F}_q^{m \times \eta}$ .
2. The parties locally set  $\llbracket E \rrbracket = M_0 + \sum_{i=1}^k \llbracket x_i \rrbracket M_i$ .
3. The parties locally set  $\llbracket \alpha \rrbracket = \llbracket R \rrbracket \Sigma + \llbracket a \rrbracket$ .
4. The parties open  $\alpha \in \mathbb{F}_q^{r \times \eta}$ .
5. The parties locally set  $\llbracket v \rrbracket = \llbracket T \rrbracket \alpha - \llbracket c \rrbracket - \llbracket E \rrbracket \Sigma$ .
6. The parties open  $v \in \mathbb{F}_q^{n \times \eta}$ .
7. The parties outputs ACCEPT if  $v = 0$  and REJECT otherwise.

Fig. 3: An MPC protocol based on the *rank decomposition* technique ( $\Pi_{\text{RC-RD}}$ ) which verifies that the given input corresponds to a solution of a MinRank problem.

**Public values:**  $M_0, M_1, \dots, M_k \in \mathbb{F}_q^{n \times m}$ .

**Inputs:** Each party takes a share of the following sharings as inputs:  $\llbracket x \rrbracket$  where  $x \in \mathbb{F}_q^k$ ,  $\llbracket L_U \rrbracket := X^{q^r} + \sum_{i=0}^{r-1} \llbracket \beta_i \rrbracket X^{q^i}$  where  $L_U(X) := \prod_{u \in U} (X - u) \in \mathbb{F}_{q^m}[X]$ ,  $\llbracket a \rrbracket$  where  $a$  has been uniformly sampled from  $\mathbb{F}_{q^{m \cdot \eta}}$ , and  $\llbracket c \rrbracket \in \mathbb{F}_{q^{m \cdot \eta}}$ , such that  $c = -\langle \beta, a \rangle$ .

**MPC Protocol:**

1. The parties get random  $\gamma_1, \dots, \gamma_n \in \mathbb{F}_{q^{m \cdot \eta}}$ .
2. The parties get a random  $\varepsilon \in \mathbb{F}_{q^{m \cdot \eta}}$ .
3. The parties locally set  $\llbracket E \rrbracket = M_0 + \sum_{i=1}^k \llbracket x_i \rrbracket M_i$ .
4. The parties locally write the rows of  $\llbracket E \rrbracket$  as elements  $(e_1, \dots, e_m)$  of  $\mathbb{F}_{q^m}$ .
5. The parties locally set  $\llbracket z \rrbracket = -\sum_{j=1}^n \gamma_j \llbracket e_j \rrbracket^{q^r}$ .
6. The parties locally set  $\llbracket w_i \rrbracket = \sum_{j=1}^n \gamma_j \llbracket e_j \rrbracket^{q^i}$  for all  $i \in \{0, \dots, r-1\}$ .
7. The parties locally set  $\llbracket \alpha \rrbracket = \varepsilon \cdot \llbracket w \rrbracket + \llbracket a \rrbracket$ .
8. The parties open  $\alpha \in \mathbb{F}_{q^{m \cdot \eta}}$ .
9. The parties locally set  $\llbracket v \rrbracket = \varepsilon \cdot \llbracket z \rrbracket - \langle \alpha, \llbracket \beta \rrbracket \rangle - \llbracket c \rrbracket$ .
10. The parties open  $v \in \mathbb{F}_{q^{m \cdot \eta}}$ .
11. The parties outputs ACCEPT if  $v = 0$  and REJECT otherwise.

Fig. 4: An MPC protocol based on the technique using *linearized polynomials* ( $\Pi_{\text{RC-LP}}$ ) which verifies that the given input corresponds to a solution of a MinRank problem.  $U$  is a  $\mathbb{F}_q$ -linear subspace of  $\mathbb{F}_{q^m}$  of dimension  $r$  which contains the rows  $(e_1, \dots, e_n)$  of  $E := M_0 + \sum_{i=1}^k x_i M_i \in \mathbb{F}_q^{n \times m}$  represented as elements of  $\mathbb{F}_{q^m}$ .

*Proof of Knowledge.* Using the MPC-in-the-Head paradigm (see Section 2.1), we transform the above MPC protocol into an interactive zero-knowledge proof of knowledge which enables us to convince a verifier that a prover knows the solution of a MinRank problem. The soundness error of the resulting protocol is

$$\varepsilon := \frac{1}{N} + \left(1 - \frac{1}{N}\right) p_\eta$$

where  $p_\eta := \frac{1}{q^\eta}$  when using  $\Pi_{\text{RC-RD}}^\eta$  and  $p_\eta := \frac{2}{q^{m \cdot \eta}} - \frac{1}{q^{2 \cdot m \cdot \eta}}$  when using  $\Pi_{\text{RC-LP}}^\eta$ . By repeating the protocol  $\tau$  times, we get a soundness error of  $\varepsilon^\tau$ . To obtain a soundness error of  $\lambda$  bits, we can take  $\tau = \left\lceil \frac{-\lambda}{\log_2 \varepsilon} \right\rceil$ . We can transform the interactive protocol into a non-interactive proof/signature thanks to the Fiat-Shamir transform [FS87]. According to [KZ20], the security of the resulting scheme is

$$\text{cost}_{\text{forge}} := \min_{\tau_1, \tau_2: \tau_1 + \tau_2 = \tau} \left\{ \frac{1}{\sum_{i=\tau_1}^{\tau} \binom{\tau}{i} p_\eta^i (1-p_\eta)^{\tau-i}} + N^{\tau_2} \right\}.$$

When using  $\Pi_{\text{RC-RD}}$ , the communication cost of the scheme (in bits) is

$$4\lambda + \tau \cdot \left( \underbrace{\binom{k}{x}}_x + \underbrace{r \times m}_R + \underbrace{r \times n}_T + \underbrace{r \times \eta}_\alpha + \underbrace{n \times \eta}_c \right) \cdot \log_2 q + \underbrace{\lambda \cdot \log_2 N + 2\lambda}_{\text{MPCitH}}$$

where  $\lambda$  is the security level,  $r$  is a scheme parameter and  $\tau$  is computed such that the soundness error is of  $\lambda$  bits in the interactive case and such that  $\text{cost}_{\text{forge}}$  is of  $\lambda$  bits in the non-interactive case.

And when using  $\Pi_{\text{RC-LP}}$ , the communication cost of the scheme (in bits) is

$$4\lambda + \tau \cdot \left( \underbrace{\binom{k}{x}}_x + \underbrace{r \times m}_{L_U} + \underbrace{r \times m \times \eta}_\alpha + \underbrace{m \times \eta}_c \right) \cdot \log_2 q + \underbrace{\lambda \cdot \log_2 N + 2\lambda}_{\text{MPCitH}}$$

The public key corresponds to the  $k+1$  matrices  $M_0, \dots, M_k$ . The matrices  $M_1, \dots, M_k$  can be represented by a  $\lambda$ -bit seed and [BESV22] showed that we can generate  $M_0$  such that it requires only  $(mn - k) \log_2 q$  bits to send. The size of the public key is thus  $\lambda + (mn - k) \log_2 q$  bits.

*Performance and comparison.* In what follows, we compare our scheme with the state of the art on the MinRank instance [BESV22]:

$$(q, m, n, k, r) = (16, 16, 16, 142, 4),$$

which targets the security level that corresponds to the NIST category I. We provide in Tables 4 and 5 a complete comparison of our scheme with the state of the art. To provide a fair comparison, we propose two variants for [Cou01] and [SINY22]: the first one corresponds to the scheme as described in the original article and the second one is an optimized version. This optimized version includes the following tricks:

- Instead of revealing all the commitments during the first round, the prover just sends a hash digest of them. Then, to enable the verifier to recompute this digest, the prover just needs to send the commitment digests that the verifier can not compute herself.
- The random combination used in the schemes (usually denoted  $\beta$ ) is derived from a seed. Then, instead of sending the coefficients of  $\beta$ , the prover can just send this seed. Moreover, this seed and the masks involved in the schemes (usually denoted  $T$ ,  $S$  and  $X$ ) are also derived from a *common* seed.
- Instead of revealing two matrices such that the difference is of rank (at most)  $r$ , the prover sends one of the matrices and directly the difference (which is cheaper to send), and thus the verifier can deduce the non-sent matrix.

In the comparison, we put how [BG22, Section 2] would perform if we apply the same technique for MinRank problem ([BG22] does not consider the MinRank problem in their article).

First, let us remark that [SINY22] presents no advantage compared to [Cou01]. The soundness error of each iteration is  $1/2$  instead of  $2/3$ , but each iteration is more expensive. The achieved communication cost is thus equivalent to [Cou01]. [BESV22] is a protocol with helper [Beu20]. The components in the proof transcript are the same as for [Cou01] (and [SINY22]), but it succeeds in achieving a bit smaller signature size just by sending a smaller number of seeds and digests. The MPC-in-the-Head paradigm enables to obtain much smaller sizes. Using techniques from [BG22], the resulting size is around 10 KB. In an independent work, [ARZV22] recently proposes a new scheme using techniques which are similar to our protocol with  $\Pi_{\text{RC-RD}}$ : they are working on another matrix relation<sup>5</sup> but use a less efficient matrix multiplication checking protocol. They succeed in producing signatures with sizes below 8 KB. Our scheme with  $\Pi_{\text{RC-RD}}$  achieves similar sizes as [ARZV22], but our scheme with  $\Pi_{\text{RC-LP}}$  outperforms all the previous ones achieving sizes below 6 KB. For the sake of completeness, we put in the comparison tables how [ARZV22] would perform if we use  $\Pi_{\text{MM}}$  as a subroutine.

### 5.3 Proof of Knowledge for Rank SD

We want to build a zero-knowledge proof of knowledge for the *rank syndrome decoding problem*:

**Definition 3 (Rank Syndrome Decoding Problem - Standard Form).** *Let  $\mathbb{F}_{q^m}$  be the finite field with  $q^m$  elements. Let  $(n, k, r)$  be positive integers such that  $k \leq n$ . The rank syndrome decoding problem with parameters  $(q, m, n, k, r)$  is the following problem:*

*Let  $H$ ,  $x$  and  $y$  be such that:*

1.  $H$  is uniformly sampled from  $\{(H' | I_{n-k}), H' \in \mathbb{F}_{q^m}^{(n-k) \times n}\}$ ,
2.  $x$  is uniformly sampled from  $\{x \in \mathbb{F}_{q^m}^n : \text{wt}_R(x) \leq r\}$ ,
3.  $y$  is built as  $y := Hx$ .

*From  $(H, y)$ , find  $x$ .*

*Remark 3.* The rank  $\text{wt}_R(x)$  of an element  $x$  of  $\mathbb{F}_{q^m}^n$  is the dimension of the  $\mathbb{F}_q$ -linear subspace spanned by  $x_1, \dots, x_n$ . Equivalently, it is the rank of the matrix  $M$  for which the rows are  $x_1, \dots, x_n$  represented as vectors of  $\mathbb{F}_q^m$ .

<sup>5</sup> They express the  $m - r$  last columns *w.r.t.* the  $r$  first ones.

Scheme Name	Security	Signature Size
[Cou01]	$(3/2)^\tau$	$3\tau \cdot \mu_{\text{dig}} + \tau \left[ \frac{2}{3}\mu_{\text{mat}} + \frac{2}{3}\mu_{\text{combi}} + \frac{2}{3}\mu_{\text{seed}} \right]$
[Cou01], opt.	$(3/2)^\tau$	$\mu_{\text{dig}} + \tau \left[ \frac{1}{3}(\mu_{\text{mat}} + \mu_{\text{rank}} + \mu_{\text{combi}} + 2\mu_{\text{seed}}) + \mu_{\text{dig}} \right]$
[SINY22]	$2^\tau$	$6\tau \cdot \mu_{\text{dig}} + \tau \left[ \mu_{\text{mat}} + \frac{1}{2}\mu_{\text{combi}} + \frac{10}{4}\mu_{\text{seed}} \right]$
[SINY22], opt.	$2^\tau$	$\mu_{\text{dig}} + \tau \left[ \frac{1}{2}(\mu_{\text{mat}} + \mu_{\text{rank}} + \mu_{\text{combi}} + 3\mu_{\text{seed}}) + 2\mu_{\text{dig}} \right]$
[BESV22]	$\varepsilon_{\text{helper}}(\tau, M, \frac{1}{2})^{-1}$	$\mu_{\text{dig}} + \tau \left[ \frac{1}{2}(\mu_{\text{mat}} + \mu_{\text{rank}} + \mu_{\text{combi}} + \mu_{\text{seed}}) + \mu_{\text{dig}} + \mu_{\text{helper}} \right]$
[BG22]	$\varepsilon_{\text{helper}}(\tau, M, \frac{1}{N})^{-1}$	$\mu_{\text{dig}} + \tau \left[ \mu_{\text{combi}} + \mu_{\text{rank}} + \mu_{\text{MPCitH}} + \mu_{\text{helper}} \right]$
[ARZV22]	$\text{KZ}(\frac{1}{q^n}, \frac{1}{N})$	$2\mu_{\text{dig}} + \tau \left[ \mu_{\text{combi}} + (n^2 + 2rn - r^2) \log_2 q + \mu_{\text{MPCitH}} \right]$
[ARZV22]+ $I_{\text{MM}}$	$\text{KZ}(\frac{1}{q^n}, \frac{1}{N})$	$2\mu_{\text{dig}} + \tau \left[ \mu_{\text{combi}} + (r(n-r) + \eta(n-2r)) \log_2 q + \mu_{\text{MPCitH}} \right]$
Our scheme (RD)	$\text{KZ}(\frac{1}{q^n}, \frac{1}{N})$	$2\mu_{\text{dig}} + \tau \left[ \mu_{\text{combi}} + \mu_{\text{rank}} + \eta(n+r) \log_2 q + \mu_{\text{MPCitH}} \right]$
Our scheme (LP)	$\text{KZ}(\frac{2}{q^{m\eta}} - \frac{1}{q^{2m\eta}}, \frac{1}{N})$	$2\mu_{\text{dig}} + \tau \left[ \mu_{\text{combi}} + rm \log_2 q + \eta(r+1)m \log_2 q + \mu_{\text{MPCitH}} \right]$

Table 4: Sizes of the signatures relying on the MinRank problem (restricting to the schemes using the FS heuristics). The used notations are:  $\mu_{\text{mat}} := mn \log 2q$ ,  $\mu_{\text{rank}} := r(m+n) \log_2 q$ ,  $\mu_{\text{combi}} := k \log_2 q$ , plus all the notations defined in Section 3.

Instance	Protocol Name	Variant	Parameters				Signature Size	Public Key Size
			$N$	$M$	$\tau$	$\eta$		
$q = 16$ $m = 16$ $n = 16$ $k = 142$ $r = 4$	[Cou01]	-	-	-	219	-	52 430 B	73 B
		Optimized	-	-	219	-	28 575 B	
	[SINY22]	-	-	-	128	-	50 640 B	
		Optimized	-	-	128	-	28 128 B	
	[BESV22]	-	-	256	128	-	26 405 B	
	[BG22]	Fast	8	187	49	-	13 644 B	
		Short	32	389	28	-	10 937 B	
	[ARZV22]	Fast	32	-	28	-	10 116 B	
		Short	256	-	18	-	7 422 B	
	[ARZV22]+ $I_{\text{MM}}$	Fast	32	-	33	9	8 155 B	
		Short	256	-	19	9	6 277 B	
	Our scheme (RD)	Fast	32	-	33	5	9 288 B	
		Short	256	-	19	9	7 122 B	
	Our scheme (LP)	Fast	32	-	28	1	7 204 B	
	Short	256	-	18	1	<b>5 518 B</b>		

Table 5: Comparison of the signatures relying on the MinRank problem (restricting to the schemes using the FS heuristics). Numerical comparison.

The prover wants to convince the verifier that she knows such an  $x$ , *i.e.* a vector  $x \in \mathbb{F}_q^n$  such that  $y = Hx$  and  $\text{wt}_R(x) \leq r$ . Previous works propose proofs of knowledge where the constraint on the weight is an equality, but it is sometimes easier to just prove an inequality (see [FJR22] for the case of the Hamming weight). To proceed, the prover will first share the secret vector  $x$  and then use an MPC protocol that verifies that this vector satisfies the above property.

*Remark 4.* In the above definition, the parity-check matrix is in *standard form*. It does not decrease the hardness of the problem (since the transformation into a standard form is a polynomial transformation), but it enables us to simplify the construction we propose.

*MPC Protocol.* We want to build an MPC protocol which takes as input (a sharing of)  $x$  and which outputs

$$\begin{cases} \text{ACCEPT} & \text{if } y = Hx \text{ and } \text{wt}_R(x) \leq r \\ \text{REJECT} & \text{otherwise.} \end{cases}$$

Since  $H$  is in standard form, the equality  $y = Hx$  implies that  $x$  can be written as

$$x = \begin{pmatrix} x_A \\ y - H'x_A \end{pmatrix}$$

for some  $x_A \in \mathbb{F}_q^k$ . Therefore, we will build an MPC protocol which takes as input (a sharing of)  $x_A$  and which outputs

$$\begin{cases} \text{ACCEPT} & \text{if } \text{wt}_R(x) \leq r \text{ where } x := \begin{pmatrix} x_A \\ y - H'x_A \end{pmatrix} \\ \text{REJECT} & \text{otherwise.} \end{cases}$$

Given  $\llbracket x_A \rrbracket$ , the parties can locally build  $\llbracket x_B \rrbracket$  as  $\llbracket x_B \rrbracket := y - H'\llbracket x_A \rrbracket$ , and so they can deduce a sharing  $\llbracket x \rrbracket$  of  $x$  (simply by concatenating the shares of  $\llbracket x_A \rrbracket$  with the shares of  $\llbracket x_B \rrbracket$ ). It remains to check that  $\llbracket x \rrbracket$  corresponds to the sharing of a vector of  $\mathbb{F}_q^m$  of rank at most  $r$ . The latter can be done using one of the two rank checking protocols described in Section 5.1:  $\Pi_{\text{RC-RD}}^\eta$  relying on the rank decomposition or  $\Pi_{\text{RC-LP}}^\eta$  relying on linearized polynomials, for some parameter  $\eta$ .

The complete MPC protocol is described in Figure 5 when relying on the rank decomposition and in Figure 6 when relying on linearized polynomials.

**Public values:**  $H = (H' | I_{n-k}) \in \mathbb{F}_q^{(n-k) \times n}$  and  $y \in \mathbb{F}_q^{n-k}$ .

**Inputs:** Each party takes a share of the following sharings as inputs:  $\llbracket x_A \rrbracket$  where  $x_A \in \mathbb{F}_q^k$ ,  $\llbracket T \rrbracket$  where  $T \in \mathbb{F}_q^{m \times r}$ ,  $\llbracket R \rrbracket$  where  $R \in \mathbb{F}_q^{r \times m}$ ,  $\llbracket a \rrbracket$  where  $a$  has been uniformly sampled from  $\mathbb{F}_q^{r \times \eta}$ , and  $\llbracket c \rrbracket$  where  $c \in \mathbb{F}_q^{n \times \eta}$ , such that  $c = TR$  and  $X = TR$  where  $X$  is the matrix form of  $x$ .

**MPC Protocol:**

1. The parties get a random  $\Sigma \in \mathbb{F}_q^{m \times \eta}$ .
2. The parties locally set  $\llbracket x_B \rrbracket = y - H'\llbracket x_A \rrbracket$ .
3. The parties locally write  $\llbracket x \rrbracket := (\llbracket x_A \rrbracket, \llbracket x_B \rrbracket)$  as a matrix  $\llbracket X \rrbracket$ .
4. The parties locally set  $\llbracket \alpha \rrbracket = \llbracket R \rrbracket \Sigma + \llbracket a \rrbracket$ .
5. The parties open  $\alpha \in \mathbb{F}_q^{r \times \eta}$ .
6. The parties locally set  $\llbracket v \rrbracket = \llbracket T \rrbracket \alpha - \llbracket c \rrbracket - \llbracket X \rrbracket \Sigma$ .
7. The parties open  $v \in \mathbb{F}_q^{m \times \eta}$ .
8. The parties outputs ACCEPT if  $v = 0$  and REJECT otherwise.

Fig. 5: An MPC protocol based on the *rank decomposition* technique ( $\Pi_{\text{RC-RD}}^\eta$ ) which verifies that the given input corresponds to a solution of a rank syndrome decoding problem.

*Proof of Knowledge.* Using the MPC-in-the-Head paradigm (see Section 2.1), we transform the above MPC protocol into an interactive zero-knowledge proof of knowledge which enables us to convince a verifier that a prover knows the solution of a rank syndrome decoding problem. The soundness error of the resulting protocol is

$$\varepsilon := \frac{1}{N} + \left(1 - \frac{1}{N}\right) p_\eta$$

where  $p_\eta := \frac{1}{q^\eta}$  when using  $\Pi_{\text{RC-RD}}^\eta$  and  $p_\eta := \frac{2}{q^{m \cdot \eta}} - \frac{1}{q^{2 \cdot m \cdot \eta}}$  when using  $\Pi_{\text{RC-LP}}^\eta$ . By repeating the protocol  $\tau$  times, we get a soundness error of  $\varepsilon^\tau$ . To obtain a soundness error of  $\lambda$  bits, we can take  $\tau = \left\lceil \frac{-\lambda}{\log_2 \varepsilon} \right\rceil$ . We can transform the interactive protocol into a non-interactive proof/signature thanks to the Fiat-Shamir transform [FS87]. According to [KZ20], the security of the resulting scheme is

$$\text{cost}_{\text{forge}} := \min_{\tau_1, \tau_2: \tau_1 + \tau_2 = \tau} \left\{ \frac{1}{\sum_{i=\tau_1}^{\tau} \binom{\tau}{i} p_\eta^i (1 - p_\eta)^{\tau-i}} + N^{\tau_2} \right\}.$$

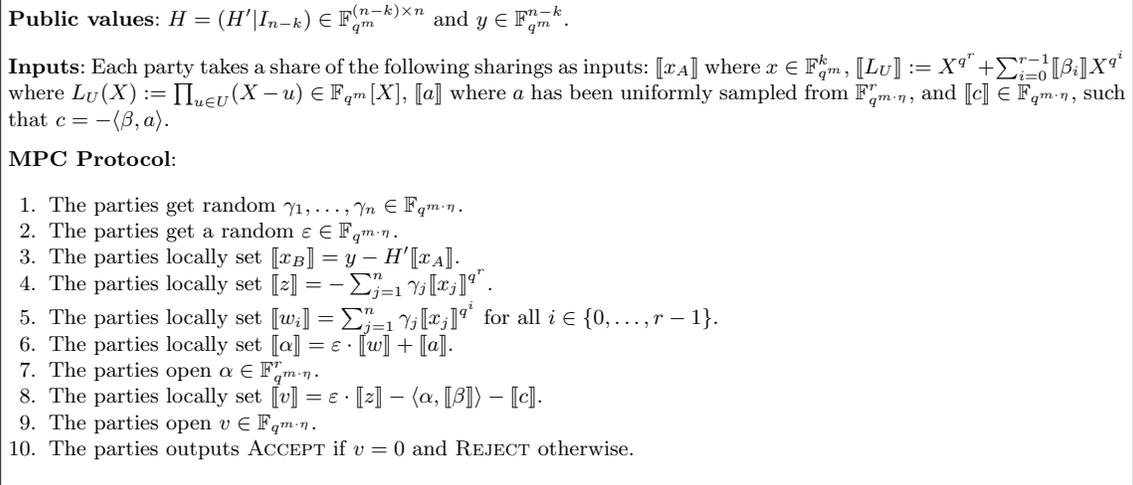


Fig. 6: An MPC protocol based on the technique using *linearized polynomials* ( $\Pi_{\text{RC-LP}}$ ) which verifies that the given input corresponds to a solution of a rank syndrome decoding problem.  $U$  is a  $\mathbb{F}_q$ -linear subspace  $U$  of  $\mathbb{F}_{q^m}$  of dimension  $r$  which contains  $x_1, \dots, x_n$ .

When using  $\Pi_{\text{RC-RD}}$ , the communication cost of the scheme (in bits) is

$$4\lambda + \tau \cdot \left( \underbrace{(k \cdot m)}_{x_A} + \underbrace{r \times m}_R + \underbrace{r \times n}_T + \underbrace{r \times \eta}_\alpha + \underbrace{n \times \eta}_c \right) \cdot \log_2 q + \underbrace{\lambda \cdot \log_2 N + 2\lambda}_{\text{MPCitH}}$$

where  $\lambda$  is the security level,  $\eta$  is a scheme parameter and  $\tau$  is computed such that the soundness error is of  $\lambda$  bits in the interactive case and such that  $\text{cost}_{\text{forge}}$  is of  $\lambda$  bits in the non-interactive case.

And when using  $\Pi_{\text{RC-LP}}$ , the communication cost of the scheme (in bits) is

$$4\lambda + \tau \cdot \left( \underbrace{(k \cdot m)}_{x_A} + \underbrace{r \times m}_{L_U} + \underbrace{r \times m \times \eta}_\alpha + \underbrace{m \times \eta}_c \right) \cdot \log_2 q + \underbrace{\lambda \cdot \log_2 N + 2\lambda}_{\text{MPCitH}}$$

The public key corresponds to the matrix  $H$  which can be represented by a  $\lambda$ -bit seed and the vector  $y \in \mathbb{F}_{q^m}^{n-k}$ . Its size is thus  $\lambda + m(n-k) \log_2 q$  bits.

*Performance and comparison.* In what follows, we compare our scheme with the state of the art on the Rank Syndrome Decoding instance [BG22]:

$$(q, m, n, k, r) = (2, 32, 30, 14, 9),$$

which targets a 128-bit security level. We provide in Tables 6 and 7 a complete comparison of our scheme with the state of the art. To get a more complete comparison, we include the schemes [Ste94], [Vér96] and [FJR21] which can be easily adapted for the rank metric (by replacing the permutations by rank isometries). Moreover, we put in Table 7 the achieved performance of [BG22] when relying on the structured rank syndrome decoding problem (the parameters of the structured problem come from the original article).

The first schemes [Ste94] and [Vér96] can achieve signature sizes of around 30 KB (let us remark that some optimization tricks have been used to achieve these sizes). Then, using the MPC-in-the-Head technique of the “shared permutation”, [FJR21] and [BG22] divide this size by half, achieving communication cost around 15 KB (13 – 19 KB). Finally, our new schemes outperform all these schemes by achieving sizes around 6 – 11 KB. The scheme using a  $q$ -polynomial even outperforms the [BG22]’s proposals<sup>6</sup> which rely on structured rank syndrome decoding problems.

Scheme Name	Security	Signature Size
[Ste94]	$(3/2)^\tau$	$\mu_{\text{dig}} + \tau \left[ \frac{1}{3}(2\mu_{\text{mat}} + \mu_{\text{rank}} + 2\mu_{\text{seed}}) + \mu_{\text{dig}} \right]$
[Vér96]	$(3/2)^\tau$	$\mu_{\text{dig}} + \tau \left[ \frac{1}{3}(\mu_{\text{mat}} + \mu_{\text{ptx}} + \mu_{\text{rank}} + 2\mu_{\text{seed}}) + \mu_{\text{dig}} \right]$
[FJR21]	$\varepsilon_{\text{helper}}(\tau, M, \frac{1}{N})^{-1}$	$\mu_{\text{dig}} + \tau [\mu_{\text{mat}} + \mu_{\text{ptx}} + \mu_{\text{rank}} + \mu_{\text{MPCitH}} + \mu_{\text{helper}}]$
[BG22]	$\varepsilon_{\text{helper}}(\tau, M, \frac{1}{N})^{-1}$	$\mu_{\text{dig}} + \tau [\mu_{\text{mat}} + \mu_{\text{rank}} + \mu_{\text{MPCitH}} + \mu_{\text{helper}}]$
Our scheme (RD)	$\text{KZ}(\frac{1}{q^\eta}, \frac{1}{N})$	$2\mu_{\text{dig}} + \tau [\mu_{\text{ptx}} + \mu_{\text{rank}} + \eta(n+r) \log_2 q + \mu_{\text{MPCitH}}]$
Our scheme (LP)	$\text{KZ}(\frac{2}{q^{m-\eta}} - \frac{1}{q^{2m-\eta}}, \frac{1}{N})$	$2\mu_{\text{dig}} + \tau [\mu_{\text{ptx}} + rm \log_2 q + \eta(r+1)m \log_2 q + \mu_{\text{MPCitH}}]$

Table 6: Sizes of the signatures relying on the rank syndrome decoding problem (restricting to the schemes using the FS heuristics). The used notations are:  $\mu_{\text{mat}} := mn \log_2 2q$ ,  $\mu_{\text{rank}} := r(m+n) \log_2 q$ ,  $\mu_{\text{ptx}} := mk \log_2 q$ , plus all the notations defined in Section 3.

Instance	Protocol Name	Variant	Parameters				Signature Size	Public Key Size
			$N$	$M$	$\tau$	$\eta$		
$q = 2$ $m = 31$ $n = 30$ $k = 15$ $r = 9$	Stern [Ste94]	-	-	-	219	-	31 358 B	75 B
	Véron [Vér96]	-	-	-	219	-	27 115 B	
	[FJR21]	Fast	8	187	49	-	19 328 B	
		Short	32	389	28	-	14 181 B	
	[BG22]	Fast	8	187	49	-	15 982 B	
		Short	32	389	28	-	12 274 B	
	Our scheme (RD)	Fast	32	-	33	19	11 000 B	
		Short	256	-	21	24	8 543 B	
Our scheme (LP)	Fast	32	-	30	1	7 376 B		
	Short	256	-	20	1	<b>5 899 B</b>		
Ideal RSD	[BG22]	Fast	32	-	37	-	12 607 B	95 B
		Short	256	-	26	-	10 126 B	
Ideal RSL	[BG22]	Fast	32	-	27	-	9 392 B	410 B
		Short	256	-	17	-	6 754 B	

Table 7: Sizes of the signatures relying on the rank syndrome decoding problem (restricting to the schemes using the FS heuristics). Numerical comparison.

*Remark 5.* Let us focus on the zero-knowledge proof relying on linearized polynomials. Thanks to the structure of the MPC protocol, it is possible to use Shamir’s secret sharings over  $\mathbb{F}_{q^m}$  instead of additive sharings (even if the base field is  $\mathbb{F}_q$  due to the  $\mathbb{F}_q$ -linearity of the Frobenius endomorphism). We describe in Appendix A how the MPC protocol behaves when using Shamir’s secret sharing. As explained in [FR22], using such sharings reduces the computational cost of emulating the MPC protocol and enables to have fast signature verifications. The fact that we can share values over  $\mathbb{F}_{q^m}$  implies that we can use techniques from [FR22], with a very large number  $N$  of parties ( $N$  is upper bounded by  $q^m$ ). This is not true for the zero-knowledge proof relying on the rank decomposition, or for both zero-knowledge proofs about the MinRank problem. For those proofs, the number  $N$  of parties would be upper bounded by  $q$ , which is small when considering concrete instances.

*Remark 6.* It is possible to transform a proof of knowledge for rank syndrome decoding (RSD) problem into a proof of knowledge for sum-rank syndrome decoding (SRSD) problem. The latter consists, given  $(H, y)$ , in finding  $x \in \mathbb{F}^n$  such that  $y = Hx$  and

$$\text{wt}_{SR}(x) \leq r$$

where  $\text{wt}_{SR}((x_1, \dots, x_{n/\ell})) := \sum_{i=0}^{n/\ell} \text{wt}_R(x_i)$  with  $\ell$  a SRSD parameter and with  $x_i \in \mathbb{F}^\ell$ . Let us denote  $X, X_1, \dots, X_{\frac{n}{\ell}}$  the matrix form of  $x, x_1, \dots, x_{\frac{n}{\ell}}$ . Proving that  $x$  (or equivalently  $X$ ) satisfies  $\text{wt}_{SR}(x) \leq r$

<sup>6</sup> These sizes are larger than the ones in [BG22] because they take  $N = 1024$ , but here to have a fair comparison with the other schemes, we take  $N = 256$ .

can be done by proving that the matrix

$$\begin{pmatrix} X_1 & & & \\ & X_2 & & \\ & & \ddots & \\ & & & X_{\frac{n}{r}} \end{pmatrix}$$

has a rank of at most  $r$ . Thus all proofs for RSD can be used for SRSD, but they must handle a large matrix. We propose in Appendix B another MPC protocol to check that  $\text{wt}_{SR}(x) \leq r$ , which does not rely on the above transformation. The core idea of this protocol is to transform  $x$  into a vector  $d$  (using  $H_{MM}$ ) such that

$$\text{wt}_{SR}(x) = \text{wt}_H(d).$$

Then using the MPC protocol of [FJR22], we can check that  $\text{wt}_H(d) \leq r$  and thus we get the desired inequality.

## 6 Proof of Knowledge for Permuted Kernel Problem

We want to build a zero-knowledge proof of knowledge for the *permuted kernel problem*:

**Definition 4 (Inhomogenous Permuted Kernel Problem).** Let  $\mathbb{F}_q$  be the finite field with  $q$  elements. Let  $m$  and  $n$  be positive integers. The permuted kernel problem with parameters  $(q, m, n)$  is the following problem:

Let  $H, y, v$  and  $\sigma$  be such that:

1.  $H$  is uniformly sampled from  $\mathbb{F}_q^{m \times n}$ ,
2.  $v$  is uniformly sampled from  $\mathbb{F}_q^n$ ,
3.  $\sigma$  is a random permutation of  $[n]$ ,
4.  $y$  is built as  $y := H\sigma(v)$ .

From  $(H, y, v)$ , find  $\sigma$ .

The prover wants to convince the verifier that she knows a permutation  $\sigma$  such that  $y = H\sigma(v)$ . Sharing the permutation seems the natural strategy, all the previous works adopt it. However, implementing permutations in a secure way (secure against timing and cache attacks) is a tricky exercise. We propose here a new proof of knowledge which has a larger communication cost, but which has the advantage of not relying on permutations. To proceed, the prover will first share the secret vector  $x := \sigma(v)$  and then use an MPC protocol which verifies that this vector satisfies the desired property.

*MPC Protocol.* We want to build an MPC protocol which takes as input (a sharing of)  $x := \sigma(v)$  and which outputs

$$\begin{cases} \text{ACCEPT} & \text{if } y = Hx \text{ and } \exists \sigma : x = \sigma(v) \\ \text{REJECT} & \text{otherwise.} \end{cases}$$

Proving that  $y = Hx$  is easy since it is linear. The hard part is to prove that there exists a permutation between  $x$  and  $v$ , without using any permutation. To proceed, we will check that the two following polynomials are equal:

$$P(X) = (X - x_1) \dots (X - x_n) \quad \text{and} \quad Q(X) = (X - v_1) \dots (X - v_n).$$

If they are equal, it means that they have the same roots, and thus we can deduce that  $x := (x_1, \dots, x_n)$  and  $v := (v_1, \dots, v_n)$  are equal up to the order of their coordinates. In practice, to check that  $P(X)$  and  $Q(X)$  are equal, we will rely on the Schwartz-Zippel Lemma: we sample a random evaluation point  $\xi$  in the field extension  $\mathbb{F}_{q^{\eta_1}}$  (for some positive integer  $\eta_1$ ) and we check that  $P(\xi)$  is equal to  $Q(\xi)$ . If the two polynomials are not equal, the probability to get  $P(\xi) = Q(\xi)$  is upper bounded by

$$\frac{n}{|\mathbb{F}_{q^{\eta_1}}|}$$

since  $n$  is the degree of  $P(X) - Q(X)$ . Thus, the MPC protocol will compute  $P(\xi) = (\xi - x_1) \dots (\xi - x_n)$  from a sharing  $\llbracket x \rrbracket$  of  $x$  and will compare the result with  $Q(\xi)$ .

Given an evaluation point  $\xi$ , let us denote

$$\begin{aligned} s_1 &:= (\xi - x_1) \\ s_2 &:= (\xi - x_1)(\xi - x_2) \\ &\vdots \\ s_n &:= (\xi - x_1)(\xi - x_2) \dots (\xi - x_n). \end{aligned}$$

The MPC protocol could proceed as follows:

1. The parties get a random evaluation point  $\xi \in \mathbb{F}_{q^{\eta_1}}$ .
2. The parties get as hints  $\llbracket s_1 \rrbracket, \dots, \llbracket s_{n-1} \rrbracket$  (which depend on  $\xi$ ).
3. The parties execute a multiplication checking protocol to check that

$$\forall i \in \{1, \dots, n-1\}, s_i \cdot x_{i+1} = s_{i+1}$$

where  $s_n := Q(\xi)$ .

However, all existing multiplication checking protocols induce a communication cost that depends on the bit size of the multiplication triples. In what follows, we assume that  $n$  is even. Let us define

$$\begin{aligned} t_1 &:= x_1 \cdot x_2 \\ t_2 &:= x_3 \cdot x_4 \\ &\vdots \\ t_{n/2} &:= x_{n-1} \cdot x_n \end{aligned}$$

To save communication, the MPC protocol we consider will proceed as follows:

1. The parties get a random evaluation point  $\xi \in \mathbb{F}_{q^{\eta_1}}$ .
2. The parties get as hints  $\llbracket t_1 \rrbracket, \dots, \llbracket t_{n/2} \rrbracket$  which live in  $\mathbb{F}_q$ .
3. The parties get as hints  $\llbracket s_4 \rrbracket, \llbracket s_6 \rrbracket, \dots, \llbracket s_{n-2} \rrbracket$  which live in  $\mathbb{F}_{q^{\eta_1}}$ .
4. The parties execute a multiplication checking protocol to check that

$$\forall i \in \{1, \dots, n/2\}, x_{2i-1} \cdot x_{2i} = t_i.$$

5. The parties execute a multiplication checking protocol to check that

$$\forall i \in \{2, \dots, n/2\}, s_{2i-2} \cdot (\xi^2 - (x_{2i-1} + x_{2i})\xi + t_i) = s_{2i}$$

where  $s_2 := (\xi^2 - (x_1 + x_2)\xi + t_1)$  and  $s_n := Q(\xi)$ .

Since  $t_i$ 's bitsize is  $\eta_1$  times smaller than  $s_i$ 's, the communication cost of this MPC protocol is smaller than the previous one.

The MPC protocol is completely described in Figure 7. As batch multiplication checking protocol, we use the MPC protocol  $\Pi_{\text{BMC}}$  described in Figure 8 (inspired from [BDK<sup>+</sup>21]).

*Proof of Knowledge.* Using the MPC-in-the-Head paradigm (see [FR22, Theorem 2]), we transform the above MPC protocol into an interactive 7-round zero-knowledge proof of knowledge which enables us to convince a verifier that a prover knows the solution of a permuted kernel problem. The soundness error of the resulting protocol is

$$\varepsilon := \frac{1}{N} + \left(1 - \frac{1}{N}\right) p_{\eta_1, \eta_2}$$

**Public values:**  $H \in \mathbb{F}_q^{m \times n}$ ,  $y \in \mathbb{F}_q^m$  and  $v \in \mathbb{F}_q^n$ .

**Inputs:** Each party takes a share of the following sharings as inputs:  $\llbracket x \rrbracket$  where  $x \in \mathbb{F}_q^n$ .

**MPC Protocol:**

1. The parties get a random  $\xi \in \mathbb{F}_{q^{\eta_1}}$ .
2. The parties get as hints  $\llbracket t_1 \rrbracket, \dots, \llbracket t_{n/2} \rrbracket$  where

$$\forall i \in \{1, \dots, \frac{n}{2}\}, t_i = x_{2i-1} \cdot x_{2i}.$$

3. The parties get as hints  $\llbracket s_4 \rrbracket, \llbracket s_6 \rrbracket, \dots, \llbracket s_{n-2} \rrbracket$  where

$$\forall i \in \{2, \dots, \frac{n}{2} - 1\}, s_{2i} = s_{2i-2} \cdot (\xi - x_{2i-1}) \cdot (\xi - x_{2i})$$

with  $s_2 := (\xi - x_1)(\xi - x_2)$ .

4. The parties execute in parallel the MPC protocols

$$\llbracket v_1 \rrbracket \leftarrow \Pi_{\text{BMC}}^{\eta_1, \eta_2} \left( \begin{array}{c} \llbracket x_1 \rrbracket, \llbracket x_2 \rrbracket, \llbracket t_1 \rrbracket \\ \llbracket x_3 \rrbracket, \llbracket x_4 \rrbracket, \llbracket t_2 \rrbracket \\ \vdots \\ \llbracket x_{n-1} \rrbracket, \llbracket x_n \rrbracket, \llbracket t_{\frac{n}{2}} \rrbracket \end{array} \right)$$

and

$$\llbracket v_2 \rrbracket \leftarrow \Pi_{\text{BMC}}^{\eta_2} \left( \begin{array}{c} \llbracket s_2 \rrbracket, \quad \xi^2 - (\llbracket x_3 \rrbracket + \llbracket x_4 \rrbracket) \cdot \xi + \llbracket t_2 \rrbracket, \quad \llbracket s_4 \rrbracket \\ \llbracket s_4 \rrbracket, \quad \xi^2 - (\llbracket x_5 \rrbracket + \llbracket x_6 \rrbracket) \cdot \xi + \llbracket t_3 \rrbracket, \quad \llbracket s_6 \rrbracket \\ \vdots \\ \llbracket s_{n-4} \rrbracket, \xi^2 - (\llbracket x_{n-3} \rrbracket + \llbracket x_{n-2} \rrbracket) \cdot \xi + \llbracket t_{\frac{n}{2}-1} \rrbracket, \llbracket s_{n-2} \rrbracket \\ \llbracket s_{n-2} \rrbracket, \quad \xi^2 - (\llbracket x_{n-1} \rrbracket + \llbracket x_n \rrbracket) \cdot \xi + \llbracket t_{\frac{n}{2}} \rrbracket, \quad Q(\xi) \end{array} \right)$$

where  $\llbracket s_2 \rrbracket = \xi^2 - (\llbracket x_1 \rrbracket + \llbracket x_2 \rrbracket) \cdot \xi + \llbracket t_1 \rrbracket$ .

5. The parties open  $v_1$  and  $v_2$ .
6. The parties locally set  $\llbracket v_0 \rrbracket = H[x]$ , and they open  $v_0$ .
7. The parties outputs ACCEPT if  $v_0 = y$ ,  $v_1 = 0$  and  $v_2 = 0$ , and REJECT otherwise.

Fig. 7: An MPC protocol which verifies that the given input corresponds to a solution of a permuted kernel problem.

where

$$p_{\eta_1, \eta_2} := 1 - \left(1 - \frac{n}{q^{\eta_1}}\right) \left(1 - \frac{n-1}{q^{\eta_1 \cdot \eta_2}}\right) \left(1 - \frac{1}{q^{\eta_1 \cdot \eta_2}}\right).$$

By repeating the protocol  $\tau$  times, we get a soundness error of  $\varepsilon^\tau$ . To obtain a soundness error of  $\lambda$  bits, we can take  $\tau = \left\lceil \frac{-\lambda}{\log_2 \varepsilon} \right\rceil$ . We can transform the interactive protocol into a non-interactive proof/signature thanks to the Fiat-Shamir transform [FS87]. According to [KZ20] (adapted for 7-round proof), the security of the resulting scheme is

$$\text{cost}_{\text{forge}} := \min_{\tau_1 + \tau_2 + \tau_3 = \tau} \left\{ \frac{1}{\text{SPMF}(\tau, \tau_1, p_1)} + \frac{1}{\text{SPMF}(\tau - \tau_1, \tau_2, p_2)} + N^{\tau_2} \right\}$$

where  $\text{SPMF}(\tau, \tau', p) := \sum_{i=\tau'}^{\tau} \binom{\tau}{i} p^i (1-p)^{\tau-i}$  and

$$p_1 := \frac{n}{q^{\eta_1}},$$

$$p_2 := 1 - \left(1 - \frac{n-1}{q^{\eta_1 \cdot \eta_2}}\right) \left(1 - \frac{1}{q^{\eta_1 \cdot \eta_2}}\right).$$

**Inputs:** Each party takes a share of the following sharings as inputs:

$$\begin{pmatrix} \llbracket r_1 \rrbracket, \llbracket s_1 \rrbracket, \llbracket t_1 \rrbracket \\ \vdots \\ \llbracket r_n \rrbracket, \llbracket s_n \rrbracket, \llbracket t_n \rrbracket \end{pmatrix}$$

**MPC Protocol:**

1. The parties get as hints  $\llbracket a \rrbracket$ ,  $\llbracket b \rrbracket$  and  $\llbracket c \rrbracket$  where  $a$  and  $b$  are uniformly random in  $\mathbb{K}$  and  $c = a \cdot b$ .
2. The parties locally build the polynomials  $\llbracket R \rrbracket$  and  $\llbracket S \rrbracket$  such that

$$\forall i \in [n], \begin{cases} \llbracket R \rrbracket(\gamma_i) = \llbracket r_i \rrbracket \\ \llbracket S \rrbracket(\gamma_i) = \llbracket s_i \rrbracket \end{cases}.$$

3. The parties get as hints  $\llbracket t_{n+1} \rrbracket, \dots, \llbracket t_{2n-1} \rrbracket$  where

$$\forall i \in [n-1], t_{n+i} = (R \cdot S)(\gamma_{n+i}).$$

4. The parties locally build the polynomial  $\llbracket T \rrbracket$  such that

$$\forall i \in [2n-1], \llbracket T \rrbracket(\gamma_i) = \llbracket t_i \rrbracket.$$

5. The parties get random  $r, \varepsilon \in \mathbb{K}$ .
6. The parties locally set  $\llbracket \alpha \rrbracket = \varepsilon \cdot \llbracket R \rrbracket(r) + \llbracket a \rrbracket$  and  $\llbracket \beta \rrbracket = \llbracket S \rrbracket(r) + \llbracket b \rrbracket$ .
7. The parties open  $\alpha, \beta \in \mathbb{K}$ .
8. The parties locally set  $\llbracket v \rrbracket = \varepsilon \cdot \llbracket T \rrbracket(r) - \llbracket c \rrbracket + \alpha \cdot \llbracket b \rrbracket + \beta \cdot \llbracket a \rrbracket - \alpha \cdot \beta$ .

Fig. 8: An MPC protocol  $\Pi_{\text{BMC}}^\eta$  which verifies that, for all  $i \in [n]$ ,  $r_i \cdot s_i = t_i$ , where all  $(r_i, s_i, t_i)$ 's belong to a field  $\mathbb{F}$ . Let us denote  $\mathbb{K}$  the field extension of degree  $\eta$ .  $\gamma_1, \dots, \gamma_{2n-1}$  are *distinct* elements of  $\mathbb{F}$  (we assume that  $|\mathbb{F}| \geq 2n-1$ ).

The communication cost<sup>7</sup> of the scheme (in bits) is

$$4\lambda + \tau \cdot \left( \underbrace{(n-m)}_{x_A} \cdot \log_2 q + \mu_{\text{misc}} + \underbrace{\lambda \cdot \log_2 N + 2\lambda}_{\text{MPCitH}} \right)$$

with

$$\mu_{\text{misc}} := \underbrace{\left( \frac{n}{2} + \eta_1 \left( \frac{n}{2} - 2 \right) \right)}_{t_1, \dots, t_{\frac{n}{2}}, s_4, \dots, s_{n-2}} + \underbrace{\left( \frac{n}{2} - 1 \right) + \eta_1 \left( \frac{n}{2} - 2 \right)}_T + \underbrace{5\eta_1 \eta_2}_{\alpha, \beta, c} \cdot \log_2 q$$

where  $\lambda$  is the security level,  $(\eta_1, \eta_2)$  are scheme parameters and  $\tau$  is computed such that the soundness error is of  $\lambda$  bits in the interactive case and such that  $\text{cost}_{\text{forge}}$  is of  $\lambda$  bits in the non-interactive case.

*Performance and comparison.* In what follows, we compare our scheme with the state of the art on the permuted kernel instance [Beu20]:

$$(q, n, m) = (997, 61, 28).$$

We provide in Tables 8 and 9 a comparison of our scheme with the state of the art. To get a more complete comparison, we include the schemes [Ste94], [Vér96] and [FJR21] which can be easily adapted for the permuted kernel problem.

The first schemes [Ste94] and [Vér96] can achieve signature sizes of around 20 – 25 KB (let us remark that some optimization tricks have been used to achieve these sizes). Then, using a *protocol with helper*, Beullens [Beu20] reduces the sizes to around 15 KB (12–18 KB). Thanks to their MPC-in-the-Head technique

<sup>7</sup> The formula of this cost assumes that the matrix is in standard form (as in Section 5.3). We omit this detail in Figure 7 for the sake of simplicity.

of the “shared permutation”, [FJR21] achieves similar performance. [BG22] then succeeds to remove the helper from [FJR21] by leveraging the linearity of the permuted kernel problem and thus currently has the best sizes from the state of the art (9 – 10 KB). Our scheme has similar signature sizes as [Beu20] and [FJR21], and is outperformed by [BG22]. However, our scheme presents several advantages:

- instead of using permutations, our scheme works on polynomials, which is easier to securely implement;
- our scheme is more parallelizable since all the parties run computation *in parallel*, whilst the parties in [FJR21] and [BG22] run computation *in series*;
- our scheme is more compatible with existing MPC-in-the-Head techniques. As example, our scheme is compatible with techniques from [FR22] (like fast signature verification) and from [AGH<sup>+</sup>23] (see next section), while the previous schemes based on PKP were not.

Scheme Name	Security	Signature Size
[Sha90]	$\text{KZ}(\frac{1}{q}, \frac{1}{2})$	$\tau [2\mu_{\text{dig}} + \mu_{\text{mask}} + \mu_{\text{small}}]$
[Ste94]	$(3/2)^\tau$	$\mu_{\text{dig}} + \tau [\frac{1}{3}(2\mu_{\text{mask}} + \mu_{\text{small}} + 2\mu_{\text{seed}}) + \mu_{\text{dig}}]$
[Vér96]	$(3/2)^\tau$	$\mu_{\text{dig}} + \tau [\frac{1}{3}(\mu_{\text{mask}} + \mu_{\text{ptx}} + \mu_{\text{small}} + 2\mu_{\text{seed}}) + \mu_{\text{dig}}]$
PKP-DSS [BFK <sup>+</sup> 19]	$\text{KZ}(\frac{1}{q-1}, \frac{1}{2})$	$\mu_{\text{dig}} + \tau \cdot [\mu_{\text{mask}} + \mu_{\text{dig}} + 2\mu_{\text{seed}}]$
SUSHYFISH [Beu20]	$\varepsilon_{\text{helper}}(\tau, M, \frac{1}{q'})^{-1}$	$\mu_{\text{dig}} + \tau [\mu_{\text{mask}} + \mu_{\text{small}} + 2\mu_{\text{seed}} + \mu_{\text{dig}} \cdot \log_2(q') + \mu_{\text{helper}}]$
[FJR21]	$\varepsilon_{\text{helper}}(\tau, M, \frac{1}{N})^{-1}$	$\mu_{\text{dig}} + \tau [\mu_{\text{mask}} + \mu_{\text{ptx}} + \mu_{\text{small}} + \mu_{\text{MPCitH}} + \mu_{\text{helper}}]$
[BG22]	$\text{KZ}(\frac{1}{q-1}, \frac{1}{N})$	$\mu_{\text{dig}} + \tau [\mu_{\text{mask}} + \mu_{\text{small}} + \mu_{\text{MPCitH}} + \mu_{\text{helper}}]$
Our scheme	$\text{KZ}_3(p_1, p_2, \frac{1}{N})$	$2\mu_{\text{dig}} + \tau [\mu_{\text{ptx}} + \mu_{\text{misc}} + \mu_{\text{MPCitH}}]$ where $\mu_{\text{misc}} := ((n-1)(\eta_1+1) + \eta_1(5\eta_2-3)) \log_2 q$

Table 8: Sizes of the signatures relying on the permuted kernel problem (restricting to the schemes using the FS heuristics). The used notations are:  $\mu_{\text{mask}} := n \log_2 2q$ ,  $\mu_{\text{small}} := n \log_2 n$ ,  $\mu_{\text{ptx}} := (n-m) \log_2 q$ , plus all the notations defined in Section 3.

Instance	Protocol Name	Variant	Parameters					Signature Size
			$N$	$M$	$\tau$	$\eta_1$	$\eta_2$	
$q = 997$ $n = 61$ $m = 38$	Shamir [Sha90]	-	-	-	149	-	-	27 746 B
	Stern [Ste94]	-	-	-	219	-	-	23 848 B
	Véron [Vér96]	-	-	-	219	-	-	21 272 B
	PKP-DSS [BFK <sup>+</sup> 19]	-	-	-	149	-	-	20 961 B
	SUSHYFISH [Beu20]	Fast	4	191	68	-	-	18 448 B
		Short	128	916	20	-	-	12 145 B
	[FJR21]	Fast	8	187	49	-	-	15 420 B
		Short	32	389	28	-	-	11 947 B
	[BG22]	Fast	32	-	42	-	-	9 896 B
		Short	256	-	31	-	-	<b>8 813 B</b>
Our scheme	Fast	32	-	41	2	2	16 373 B	
	Short	256	-	24	3	2	12 816 B	

Table 9: Sizes of the signatures relying on the permuted kernel problem (restricting to the schemes using the FS heuristics). Numerical comparison.

## 7 Running times

To provide a fair comparison of our work with the state of the art, we need to give an estimation of the computational performances of our proposals. The best way to proceed would be to have optimized implementations for them, but producing such implementations requires dedicated work for each of the proposed signature schemes. Since the code of those schemes would be similar except for the part about the MPC protocols, we decided to develop a unified MPC-in-the-Head library<sup>8</sup>. The idea is to factorize as much as possible the common code of the MPCitH-based signatures. As long as they respect the expected API, a user just needs to implement

- the code which generates an instance of the hard problem with its solution,
- the computation of a party in the MPC protocol.

Then they can rely on the library to get the desired signature scheme. Thanks to this library, we were able to estimate the running times of the schemes proposed in this article.

Until recently, the only way to implement an MPCitH-based proof system was by emulating all the parties of the underlying MPC protocol, implying that we would need to emulate  $N$  times a party per repetition. The recent work [AGH<sup>+</sup>23] changes this drastically. The authors suggest generating the input shares of the parties in a correlated way using a hypercube approach. This optimization enables us to emulate only  $1 + \log_2(N)$  parties per repetition. For example, in Section 4, we propose to take  $\tau = 25$  and  $N = 256$  for the “short” trade-off of our scheme. Without the optimization of [AGH<sup>+</sup>23], we would need to emulate  $\tau \cdot N = 6400$  times a party per signing. With it, we just need to emulate  $\tau \cdot (1 + \log_2 N) = 225$  times a party, reducing the computational cost of the MPC emulation by a factor of 28.

We included the [AGH<sup>+</sup>23] optimization in the library. The obtained signing times are given in Table 10, except for the scheme relying on the permuted kernel problem. We put the running time of [AGH<sup>+</sup>23] for SDitH in the table, but to provide a fairer comparison with the other schemes, we reimplement it using our library and give the achieved performances. In our implementations, the pseudo-randomness is generated using AES in counter mode, the hash function is instantiated with SHA3, and the MPC challenge (*i.e.* the challenge provided by  $\mathcal{O}_R$ , see Section 2.1) is sampled using SHAKE. We benchmarked our schemes on a 3.8 GHz Intel Core i7-10700K CPU with the support of AVX2 and AES instructions. All the reported timings were measured on this CPU while disabling Intel Turbo Boost.

In our benchmarks, we decompose the running time of our schemes in six parts: the expansion of the seed trees, the commitments of the input shares, the expansion of the input shares from seeds, the remaining operations to prepare input shares (*e.g.* the computation of the shares of the “main” parties of the hypercube technique), the emulation of the MPC protocol and the rest of the computation.

We optimized the factorized code which mainly relies on symmetric primitives. For example, we rely on fourfold calls of Keccak (for SHA3) using AVX instructions. However, the arithmetic parts used by the MPC protocols have *not been optimized*, since it would require dedicated work for each scheme (and is out of the scope of this article).

In Table 10, we did not give the running times for the key generation and the signature verification. For all these schemes, the key generation is fast since it only consists in generating a random instance of the underlying hard problem. It usually takes less than 0.5 ms. Moreover, for all the MPCitH-based schemes relying on additive sharings, the verification time is similar (slightly smaller) to the signing time since the verifier must re-emulate the MPC protocol (as the prover) except for one party (to keep the zero-knowledge property).

Here is an analysis of the obtained running times:

- **Tree Expansion:** it consists in deriving  $N$  seeds from a master seed using the structure of a binary tree. This operation only depends on the number of parties  $N$ , and it is repeated at each repetition (*i.e.*  $\tau$  times). Thus, when we fix  $N$ , the computation contribution is linear in  $\tau$ . It can be observed from the benchmark: when  $N = 32$ , it takes  $0.0073 \cdot \tau$  ms, and when  $N = 256$  it takes  $0.055 \cdot \tau$  ms.

<sup>8</sup> This library is available at <https://github.com/CryptoExperts/libmpcith>.

Scheme	Tree Expansion	Commitment	Randomness Expansion	Share Preparation	MPC Emulation	Misc	Total signing time		Size
	in ms						in ms	in Mc	in bytes
<i>Variant "Short" - 256 parties (N = 256)</i>									
SDitH [FJR22]	-	-	-	-	-	-	3-7*	-	8 459
	0.93	0.97	0.61	0.29	4.57	0.41	7.78*	30	
MQ over $\mathbb{F}_{256}$	1.37	1.42	0.53	0.40	6.25	0.59	10.56	40	7 114
MQ over $\mathbb{F}_{251}$	1.37	1.42	1.24	1.77	2.17	0.59	8.56	33	7 114
MinRank (with RD)	1.06	1.11	1.52	0.51	3.75	0.44	8.39	32	7 122
MinRank (with LP)	0.99	1.05	1.12	0.45	13.23	0.38	17.22	65	5 518
Rank SD (with RD)	1.16	1.22	0.69	0.27	2.36	0.42	6.12	23	8 543
Rank SD (with LP)	1.10	1.14	0.51	0.24	3.72	0.38	7.09	27	5 899
<i>Variant "Fast" - 32 parties (N = 32)</i>									
SDitH [FJR22]	-	-	-	-	-	-	1.3-3.8*	-	11 835
	0.20	0.22	0.12	0.04	5.35	0.17	6.10*	23	
MQ over $\mathbb{F}_{256}$	0.26	0.28	0.10	0.05	6.9	0.24	7.83	30	8 488
MQ over $\mathbb{F}_{251}$	0.26	0.28	0.22	0.23	2.15	0.28	3.42	13	8 488
MinRank (with RD)	0.24	0.27	0.28	0.07	2.68	0.16	3.70	14	9 288
MinRank (with LP)	0.20	0.23	0.21	0.12	13.63	0.15	14.54	55	7 204
Rank SD (with RD)	0.24	0.27	0.13	0.07	2.30	0.18	3.19	12	11 000
Rank SD (with LP)	0.22	0.24	0.09	0.03	3.71	0.12	4.41	17	7 376

Table 10: Benchmark of our implementations of the proposed signature schemes (128 bits of security). All the timings are given in milliseconds, except those in the column “in Mc” which are given in megacycles. Timings with \* correspond to the implementation of [AGH<sup>+</sup>23], while timings with ♦ correspond to our own implementation of SDitH using the library. The verification is around 5 – 10% faster than the signing.

- **Commitment:** it consists in committing the input shares of  $N$  parties. In practice, it consists in committing a  $\lambda$ -bit seed for all the parties except the last one. The cost of committing the entire input share of the last party tends to be negligible compared to the cost of committing  $N - 1$  seeds. Thus, the computation contribution of the commitments is roughly linear in  $N \cdot \tau$ . From the benchmark, we get that it takes  $0.0575 \cdot \tau$  ms when  $N = 256$  and  $0.0082 \cdot \tau$  ms when  $N = 32$  (committing a seed with a salt takes around 220 nanoseconds).
- **Randomness Expansion:** it consists in expanding seeds to get input shares. The computational cost depends on the number  $\tau$  of repetitions, the size of the input shares, and the field from which elements should be sampled. When the field is an extension of  $\mathbb{F}_2$ , the sampling can be efficient. However, sampling in another field is less efficient since we need to deal with rejection. It explains why the cost of this step is larger for MQ over  $\mathbb{F}_{251}$  than for MQ over  $\mathbb{F}_{256}$ .
- **Share Preparation:** it consists in getting the input share of the last party from the other ones and in computing the shares of the “main” parties of the hypercube technique (see [AGH<sup>+</sup>23] for details). It depends on  $\tau$ , the size of the input shares, and the additive law of the underlying field. This step is very efficient when working in characteristic two since the addition is the bitwise XOR. When working in prime fields, we need to deal with reduction.
- **MPC Emulation:** it consists in emulating the MPC protocols. Thanks to the hypercube technique, it consists in emulating  $1 + \log_2(N)$  parties by repetition. The important point to remark here is that the choice of  $N$  does not impact a lot the emulation cost. It comes from that  $\tau \approx \frac{\lambda}{\log_2(N)}$ , so the total computation cost of the emulation corresponds<sup>9</sup> to the cost of emulating  $\tau \cdot (1 + \log_2(N)) \approx \lambda + \frac{\lambda}{\log_2(N)}$  parties.
- **Misc:** it corresponds to the rest of the signing computation (decompression of the public key, building the signature, ...).

In this article, we propose two MPC protocols to check that a matrix has a small rank: one based on rank decomposition (RD), and one based on  $q$ -polynomials (LP). The second protocol leads to smaller signature

<sup>9</sup> We omit here that  $\tau$  is larger than  $\frac{\lambda}{\log_2(N)}$  to be secure against the forgery attack of [KZ20], but the conclusion would be the same.

sizes, but it tends to be less efficient in running time since it involves computation in a field extension. From the benchmark, we can observe that both protocols give similar running times when applied to the rank syndrome decoding problem. However, when applied to MinRank, the MPC protocol based on  $q$ -polynomials gives a slow scheme. As explained previously, the arithmetics of the implementations have not been optimized. The scheme “MinRank (with LP)” suffers from this lack of optimizations<sup>10</sup>.

## 8 Conclusion

In this work, we studied how the MPC-in-the-Head paradigm behaves for the multivariate quadratic problem, the MinRank problem, the rank syndrome decoding problem and the permuted kernel problem.

While a straight application of this paradigm to the *permuted kernel problem* seems to produce schemes with limited performances, it enables to reduce communication cost when considering the *multivariate quadratic problem* on larger fields as  $\mathbb{F}_{256}$ .

The main contribution of this work is to reduce the task of proving the low rank of a matrix to proving that some field elements are roots of a  $q$ -polynomial. Such polynomials are MPC-friendly thanks to the linearity of the Frobenius endomorphism. Using this reduction, we can produce signatures relying on the *MinRank problem* and on the *rank syndrome decoding problem* with sizes below 6 KB.

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<sup>10</sup> Let us also remark that the difference of performance between both implementations of SDitH mainly comes from that our arithmetic of  $\mathbb{F}_{256}$  has not been optimized.

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## – Supplementary Material –

### A Using Shamir’s Secret Sharings in the Proof of Knowledge for Rank SD

Let us focus on the MPC protocol described in Figure 6. This protocol checks that a vector corresponds to a solution of a rank syndrome decoding problem, by using a  $q$ -polynomial. In what follows, we describe how the MPC protocol behaves when replacing additive sharings by Shamir’s secret sharings over  $\mathbb{F}_{q^m}$ .

To share a secret value  $v \in \mathbb{F}_{q^m}$ , the  $(\ell + 1, N)$ -Shamir’s secret sharing scheme proceeds as follows:

- sample  $r_1, \dots, r_\ell$  uniformly in  $\mathbb{F}_{q^m}$ ,
- build the polynomial  $P$  as  $P(X) = v + \sum_{i=1}^{\ell} r_i X^i$ ,
- build the shares  $\llbracket v \rrbracket_i$  as evaluations  $P(e_i)$  of  $P$  for each  $i \in \{1, \dots, N\}$ , where  $e_1, \dots, e_N$  are public non-zero distinct points of  $\mathbb{F}_{q^m}$ .

From a sharing  $\llbracket v \rrbracket$  of  $v$ , the parties can easily build a sharing of  $v^q$ : they just need to compute

$$\llbracket v^q \rrbracket_i \leftarrow \llbracket v \rrbracket_i^q$$

for all  $i$ . However, the parties’ evaluation points of  $\llbracket v^q \rrbracket$  are not  $e_1, \dots, e_N$ , but they are  $e_1^q, \dots, e_N^q$ . Indeed, we have

$$\begin{aligned} P(X)^q &= \left( v + \sum_{i=1}^{\ell} r_i X^i \right)^q \\ &= v^q + \sum_{i=1}^{\ell} r_i^q X^{q \cdot i} \\ &= v^q + \sum_{i=1}^{\ell} r_i^q (X^q)^i \\ &= P'(X^q), \quad \text{where } P' := v^q + \sum_{i=1}^{\ell} r_i^q X^i. \end{aligned}$$

Thus for all  $i$ , we get

$$\llbracket v \rrbracket_i^q = P(e_i)^q = P'(e_i^q) = \llbracket v^q \rrbracket_i$$

if  $P'$  is the polynomial which encodes  $\llbracket v^q \rrbracket$ .

Adding two sharings is possible if and only if *those two sharings have the same parties’ evaluation points*. The MPC protocol described in Figure 6 satisfies this property, enabling us to replace the additive sharings by Shamir’s secret sharings over  $\mathbb{F}_{q^m}$ . If we denote  $e_1, \dots, e_N$  the parties’ evaluation points of  $\llbracket x_A \rrbracket$ , then

- for all  $i \in \{0, \dots, r - 1\}$ , the parties’ evaluation points for  $\llbracket w_i \rrbracket$ ,  $\llbracket a_i \rrbracket$  and  $\llbracket \alpha_i \rrbracket$  are  $e_1^{q^i}, \dots, e_N^{q^i}$ ,
- the parties’ evaluation points for  $\llbracket \beta \rrbracket$ ,  $\llbracket z \rrbracket$  and  $\llbracket c \rrbracket$  are  $e_1^{q^r}, \dots, e_N^{q^r}$ .

### B Proof of Knowledge for Sum-Rank SD

We want to build a zero-knowledge proof of knowledge for the *sum-rank syndrome decoding problem*:

**Definition 5 (Sum-Rank Syndrome Decoding Problem).** Let  $\mathbb{F}_{q^m}$  be the finite field with  $q^m$  elements. Let  $(n, k, \ell, r)$  be positive integers such that  $k \leq n$  and  $\ell \mid n$ . We define the sum-rank weight  $\text{wt}_{SR}(x)$  of an element of  $\mathbb{F}_{q^m}^n$  as

$$\text{wt}_{SR}(x) := \sum_{i=1}^{n/\ell} \text{wt}_R(x_i),$$

with  $x := (x_1, \dots, x_{\frac{n}{\ell}})$ . The sum-rank syndrome decoding problem with parameters  $(q, m, n, k, \ell, r)$  is the following problem:

Let  $H$ ,  $x$  and  $y$  be such that:

1.  $H$  is uniformly sampled from  $\{(H' \mid I_{n-k}), H' \in \mathbb{F}_{q^m}^{(n-k) \times n}\}$ ,
2.  $x$  is uniformly sampled from  $\{x \in \mathbb{F}_{q^m}^n : \text{wt}_{SR}(x) \leq r\}$ ,
3.  $y$  is built as  $y := Hx$ .

From  $(H, y)$ , find  $x$ .

The prover wants to convince the verifier that she knows such an  $x$ , i.e. a vector  $x \in \mathbb{F}_{q^m}^n$  such that  $y = Hx$  and  $\text{wt}_{SR}(x) \leq r$ . To proceed, the prover will first share the secret vector  $x$  and then use an MPC protocol which verifies that this vector satisfies the above property.

*MPC Protocol.* As in Section 5.3,  $H$  is in standard form and we split the secret

$$x := \begin{pmatrix} x_A \\ y - H'x_A \end{pmatrix}.$$

We want to build an MPC protocol which takes as input (a sharing of)  $x_A$  and which outputs

$$\begin{cases} \text{ACCEPT} & \text{if } \sum_{i=1}^{n/\ell} \text{wt}_R(x_i) \leq r \text{ where } \begin{pmatrix} x_1 \\ \vdots \\ x_{\frac{n}{\ell}} \end{pmatrix} := \begin{pmatrix} x_A \\ y - H'x_A \end{pmatrix} \\ \text{REJECT} & \text{otherwise.} \end{cases}$$

For each chunk  $x_i \in \mathbb{F}_{q^m}^\ell$  with  $i \in [\frac{n}{\ell}]$ , let us define the binary vector  $d_i \in \{0, 1\}^\ell$  as

$$\forall j \in [\ell], (d_i)_j := \begin{cases} 0 & \text{if } (x_i)_j \in \text{Vect}_{\mathbb{F}_q}((x_i)_1, \dots, (x_i)_{j-1}) \\ 1 & \text{otherwise} \end{cases}$$

and let us remark that there exists a lower triangular matrix  $T_i \in \mathbb{F}_q^{\ell \times \ell}$  with the form  $\begin{pmatrix} 1 & & (0) \\ * & 1 & \\ * & * & \ddots \\ * & * & * & 1 \end{pmatrix}$  such

that

$$d_i \circ x_i = T_i x_i$$

where  $\circ$  is the component-wise multiplication. The matrix  $T_i$  corresponds to the process of removing dependencies in  $x_i$ . We have

$$\text{wt}_H(d_i) \geq \text{wt}_H(d_i \circ x_i) = \text{wt}_R(d_i \circ x_i)$$

since each non-zero coordinates of  $d_i \circ x_i$  are independent by definition of  $d_i$ . Moreover, we have

$$\text{wt}_R(d_i \circ x_i) = \text{wt}_R(T_i x_i) = \text{wt}_R(x_i)$$

since  $T_i$  is invertible. By defining  $d := (d_1, \dots, d_{\frac{n}{\ell}})$ , the MPC protocol will check that  $\text{wt}_H(d) \leq r$ , and since

$$\sum_{i=1}^{\frac{n}{\ell}} \text{wt}_R(x_i) = \sum_{i=1}^{\frac{n}{\ell}} \text{wt}_R(d_i \circ x_i) \leq \sum_{i=1}^{\frac{n}{\ell}} \text{wt}_H(d_i) = \text{wt}_H(d)$$

the desired inequality would be checked. In order to check  $\text{wt}_H(d) \leq w$ , we will use the protocol of [FJR22].

To sum up, to check the weight inequality, the MPC protocol takes as input the vectors  $d_1, \dots, d_{\frac{n}{\ell}}$  and the matrices  $T_1, \dots, T_{\frac{n}{\ell}}$  (in addition to  $x_A$ ) and proceeds as follows:

1. The parties locally build  $\llbracket x \rrbracket$  as

$$\begin{pmatrix} \llbracket x_A \rrbracket \\ y - H' \llbracket x_A \rrbracket \end{pmatrix}.$$

2. The parties execute the [FJR22]'s protocol to check that  $\text{wt}_H(d) \leq r$ .
3. For  $i \in \{1, \dots, \frac{n}{\ell}\}$ , the parties check that  $d_i \circ x_i = T_i x_i$  as follows:
  - The parties locally set  $\llbracket D_i \rrbracket \in \mathbb{F}_q^{\ell \times \ell}$  as a diagonal matrix for which the diagonal is the vector  $\llbracket d_i \rrbracket$ .
  - The parties executes the protocol  $\Pi_{\text{MM}}^\eta$  to check that  $(D_i - T_i)x_i = 0$ .

The MPC protocol is completely described in Figure 9.

*Proof of Knowledge.* Using the MPC-in-the-Head paradigm (see Section 2.1), we transform the above MPC protocol into an interactive zero-knowledge proof of knowledge which enables to convince a verifier that a prover knows the solution of a sum-rank syndrome decoding problem. The soundness error of the resulting protocol is

$$\varepsilon := \frac{1}{N} + \left(1 - \frac{1}{N}\right) \max\left(\frac{1}{q^\eta}, \delta_{\eta_1, \eta_2}\right)$$

where  $\delta_{\eta_1, \eta_2}$  is the false positive rate of [FJR22]. By repeating the protocol  $\tau$  times, we get a soundness error of  $\varepsilon^\tau$ . To obtain a soundness error of  $\lambda$  bits, we can take  $\tau = \left\lceil \frac{-\lambda}{\log_2 \varepsilon} \right\rceil$ . We can transform the interactive protocol into a non-interactive proof / signature thanks to the Fiat-Shamir transform [FS87]. According to [KZ20], the security of the resulting scheme is

$$\text{cost}_{\text{forge}} := \min_{\tau_1, \tau_2: \tau_1 + \tau_2 = \tau} \left\{ \frac{1}{\sum_{i=\tau_1}^{\tau} \binom{\tau}{i} p^i (1-p)^{\tau-i}} + N^{\tau_2} \right\}$$

where  $p := \max\left(\frac{1}{q^\eta}, \delta_{\eta_1, \eta_2}\right)$ .

The communication cost of the scheme (in bits) is

$$4\lambda + \tau \cdot \left( \underbrace{(k \cdot m)}_{x_A} + \underbrace{\frac{n(\ell-1)\ell}{2}}_{T_1, \dots} + \underbrace{\frac{n(\ell-1)}{\ell}}_{d_1, \dots} + \underbrace{\frac{n}{\ell}(\eta \cdot \ell) + \eta \cdot \min\{m, \ell-1\}}_{c, \alpha_1, \dots} \right) + \underbrace{(2r\eta_1 + 3\eta_1\eta_2)}_{\text{SDitH}} \cdot \log_2 q + \underbrace{\lambda \cdot \log_2 N + 2\lambda}_{\text{MPCitH}}$$

where  $\lambda$  is the security level,  $r$  is a scheme parameter and  $\tau$  is computed such that the soundness error is of  $\lambda$  bits in the interactive case and such that  $\text{cost}_{\text{forge}}$  is of  $\lambda$  bits in the non-interactive case.

**Public values:**  $H = (H'|I_{n-k}) \in \mathbb{F}_q^{(n-k) \times n}$  and  $y \in \mathbb{F}_q^{n-k}$ .

**Inputs:** Each party takes a share of the following sharings as inputs:

- $\llbracket x_A \rrbracket$  where  $x \in \mathbb{F}_q^k$ ,
- $\llbracket d \rrbracket, \llbracket T_1 \rrbracket, \dots, \llbracket T_{\frac{n}{\ell}} \rrbracket$  where  $d \in \{0, 1\}^n$  and  $T_1 \dots, T_{\frac{n}{\ell}} \in \mathbb{F}_q^{\ell \times \ell}$  such that

$$\forall i \in \{1, \dots, \frac{n}{\ell}\}, d_i \circ x_i = T_i x_i$$

- $\llbracket a_1 \rrbracket, \dots, \llbracket a_{\frac{n}{\ell}} \rrbracket$  where  $a_1, \dots, a_{\frac{n}{\ell}} \in \mathbb{F}_q^{\eta \times \ell}$ .
- $\llbracket c \rrbracket$  where  $c \in \mathbb{F}_q^{\eta}$  such that  $c = \sum_{i=1}^{\frac{n}{\ell}} a_i x_i$
- $\llbracket Q \rrbracket$  where  $Q = \prod_{d_i \neq 0} (X - \gamma_i) \in \mathbb{F}_q^{\eta_1}[X]$
- $\llbracket P \rrbracket$  where  $P \in \mathbb{F}_q^{\eta_1}[X]$  satisfies  $SQ = FP$  with  $F(X) := \prod_{i \in [n]} (X - \gamma_i)$  and  $S$  the unique polynomial of degree  $n - 1$  such that  $S(\gamma_i) = d_i$  for all  $i \in [n]$ .
- $\llbracket a' \rrbracket, \llbracket b' \rrbracket, \llbracket c' \rrbracket$  where  $a', b', c' \in \mathbb{F}_q^{\eta_1 \eta_2}$  such that  $c' = a' \cdot b'$ .

$\llbracket T \rrbracket$  where  $T \in \mathbb{F}_q^{n \times w}$  and  $\llbracket R \rrbracket$  where  $R \in \mathbb{F}_q^{w \times wm}$ , such that  $X = TR$  where  $X$  is the matrix form of  $x$ .

**MPC Protocol:**

1. The parties get random  $\Sigma_1, \dots, \Sigma_{\frac{n}{\ell}} \in \mathbb{F}_q^{\eta \times \ell}$ .
2. The parties get random  $r, \varepsilon' \in \mathbb{F}_q^{\eta_1 \eta_2}$ .
3. The parties locally compute  $\llbracket S \rrbracket$  by interpolation such that  $\forall j, \llbracket S(\gamma_j) \rrbracket = \llbracket d_j \rrbracket \in \mathbb{F}_q$ .
4. The parties locally compute  $\llbracket S(r) \rrbracket, \llbracket Q(r) \rrbracket$  and  $\llbracket P(r) \rrbracket$ .
5. The parties locally set  $\llbracket \alpha' \rrbracket = \varepsilon' \cdot \llbracket Q(r) \rrbracket + \llbracket a' \rrbracket$  and  $\llbracket \beta' \rrbracket = \llbracket S(r) \rrbracket + \llbracket b' \rrbracket$ .
6. The parties open  $\alpha'$  and  $\beta'$ .
7. The parties locally set  $\llbracket v' \rrbracket = \varepsilon' \cdot \llbracket (F \cdot P)(r) \rrbracket - \llbracket c' \rrbracket + \alpha' \cdot \llbracket b' \rrbracket + \beta' \cdot \llbracket a' \rrbracket - \alpha' \cdot \beta'$ .
8. The parties locally set  $\llbracket x_B \rrbracket = y - H' \llbracket x_A \rrbracket$ .
9. The parties locally set  $\llbracket x \rrbracket = (\llbracket x_A \rrbracket, \llbracket x_B \rrbracket)$ .
10. For  $i \in \{1, \dots, \frac{n}{\ell}\}$ ,
  - The parties locally write  $\llbracket d_i \rrbracket$  as a diagonal matrix  $\llbracket D_i \rrbracket \in \mathbb{F}_q^{\ell \times \ell}$ .
  - The parties locally set  $\llbracket \alpha_i \rrbracket = \Sigma_i (\llbracket D_i \rrbracket - \llbracket T_i \rrbracket) + \llbracket a_i \rrbracket$ .
  - The parties open  $\alpha_i \in \mathbb{F}_q^{\eta \times \ell}$ .
11. The parties locally set  $\llbracket v \rrbracket = \sum_{i=1}^{\frac{n}{\ell}} \alpha_i \llbracket x_i \rrbracket - \llbracket c \rrbracket$ .
12. The parties outputs ACCEPT if  $v = 0$  and  $v' = 0$ , and REJECT otherwise.

Fig. 9: An MPC Protocol that verifies that the given input corresponds to a solution of a sum-rank syndrome decoding problem.  $\gamma_1, \dots, \gamma_n$  are distinct points of  $\mathbb{F}_q^{\eta_1}$