

On the Dual of Generalized Bent Functions[†]

Jiixin Wang*, Fang-Wei Fu

Abstract

In this paper, we study the dual of generalized bent functions $f : V_n \rightarrow \mathbb{Z}_{p^k}$ where V_n is an n -dimensional vector space over \mathbb{F}_p and p is an odd prime, k is a positive integer. It is known that weakly regular generalized bent functions always appear in pairs since the dual of a weakly regular generalized bent function is also a weakly regular generalized bent function. The dual of non-weakly regular generalized bent functions can be generalized bent or not generalized bent. By generalizing the construction of [6], we obtain an explicit construction of generalized bent functions for which the dual can be generalized bent or not generalized bent. We show that the generalized indirect sum construction method given in [15] can provide a secondary construction of generalized bent functions for which the dual can be generalized bent or not generalized bent. By using the knowledge on ideal decomposition in cyclotomic field, we prove that $f^{**}(x) = f(-x)$ if f is a generalized bent function and its dual f^* is also a generalized bent function. For any non-weakly regular generalized bent function f which satisfies that $f(x) = f(-x)$ and its dual f^* is generalized bent, we give a property and as a consequence, we prove that there is no self-dual generalized bent function $f : V_n \rightarrow \mathbb{Z}_{p^k}$ if $p \equiv 3 \pmod{4}$ and n is odd. For $p \equiv 1 \pmod{4}$ or $p \equiv 3 \pmod{4}$ and n is even, we give a secondary construction of self-dual generalized bent functions. In the end, we characterize the relations between the generalized bentness of the dual of generalized bent functions and the bentness of the dual of bent functions, as well as the self-duality relations between generalized bent functions and bent functions by the decomposition of generalized bent functions.

Index Terms

Generalized bent functions; dual; self-dual; (non)-weakly regular; Walsh transform

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I. INTRODUCTION

Throughout this paper, let p be an odd prime, \mathbb{Z}_{p^k} be the ring of integers modulo p^k , \mathbb{F}_p^n be the vector space of the n -tuples over \mathbb{F}_p , \mathbb{F}_{p^n} be the finite field with p^n elements and V_n be an n -dimensional vector space over \mathbb{F}_p where k, n are positive integers.

Boolean bent functions introduced by Rothaus [14] play an important role in cryptography, coding theory, sequences and combinatorics. In 1985, Kumar *et al.* [9] generalized Boolean bent functions to bent functions over finite fields of odd characteristic. Due to the importance of bent functions, they have been studied extensively. There is an exhaustive survey [1] and a book [11] for bent functions and generalized bent functions. Recently, the notion of Boolean generalized bent functions has been generalized to generalized bent functions from V_n to \mathbb{Z}_{p^k} [12].

A function f from V_n to \mathbb{Z}_{p^k} is called a generalized p -ary function, or simply p -ary function when $k = 1$. The Walsh transform of a generalized p -ary function $f : V_n \rightarrow \mathbb{Z}_{p^k}$ is the complex valued function on V_n defined as

$$W_f(a) = \sum_{x \in V_n} \zeta_{p^k}^{f(x)} \zeta_p^{-\langle a, x \rangle}, a \in V_n \quad (1)$$

where $\zeta_{p^k} = e^{\frac{2\pi\sqrt{-1}}{p^k}}$ is the complex primitive p^k -th root of unity and $\langle a, x \rangle$ denotes a (nondegenerate) inner product in V_n . And f can be recovered by the inverse transform

$$\zeta_{p^k}^{f(x)} = \frac{1}{p^n} \sum_{a \in V_n} W_f(a) \zeta_p^{\langle a, x \rangle}, x \in V_n. \quad (2)$$

The classical representations of V_n are \mathbb{F}_p^n with $\langle a, x \rangle = a \cdot x$ and \mathbb{F}_{p^n} with $\langle a, x \rangle = Tr_1^n(ax)$, where $a \cdot x$ is the usual dot product over \mathbb{F}_p^n , $Tr_1^n(\cdot)$ is the absolute trace function. When $V_n = V_{n_1} \times \cdots \times V_{n_s}$ ($n = \sum_{i=1}^s n_i$), let $\langle a, b \rangle = \sum_{i=1}^s \langle a_i, b_i \rangle$ where $a = (a_1, \dots, a_s)$, $b = (b_1, \dots, b_s) \in V_n$.

The generalized p -ary function $f : V_n \rightarrow \mathbb{Z}_{p^k}$ is called a generalized p -ary bent function, or simply p -ary bent function when $k = 1$ if $|W_f(a)| = p^{\frac{n}{2}}$ for any $a \in V_n$. In [12], the authors have shown that the Walsh transform of a generalized p -ary bent function $f : V_n \rightarrow \mathbb{Z}_{p^k}$ satisfies that for any $a \in V_n$,

$$W_f(a) = \begin{cases} \pm p^{\frac{n}{2}} \zeta_{p^k}^{f^*(a)} & \text{if } p \equiv 1 \pmod{4} \text{ or } n \text{ is even,} \\ \pm \sqrt{-1} p^{\frac{n}{2}} \zeta_{p^k}^{f^*(a)} & \text{if } p \equiv 3 \pmod{4} \text{ and } n \text{ is odd} \end{cases} \quad (3)$$

where f^* is a function from V_n to \mathbb{Z}_{p^k} and is called the dual of f . We call f self-dual if $f^* = f$. In the sequel, if f^* is also a generalized bent function, let f^{**} denote the dual of f^* .

If $f : V_n \rightarrow \mathbb{Z}_{p^k}$ is a generalized bent function with dual f^* , define

$$\mu_f(a) = p^{-\frac{n}{2}} \zeta_{p^k}^{-f^*(a)} W_f(a), a \in V_n \quad (4)$$

and

$$\varepsilon_f(a) = \xi^{-1} \mu_f(a), a \in V_n \quad (5)$$

where $\xi = 1$ if $p \equiv 1 \pmod{4}$ or n is even and $\xi = \sqrt{-1}$ if $p \equiv 3 \pmod{4}$ and n is odd. By (3), it is easy to see that ε_f is a function from V_n to $\{\pm 1\}$. If μ_f is a constant function, then f is called weakly regular, otherwise f is called non-weakly regular. In particular, if $\mu_f(a) = 1, a \in V_n$, f is called regular.

There are some works on the dual of p -ary bent functions [3], [5], [6], [13]. It is known that weakly regular generalized bent functions always appear in pairs since the dual of a weakly regular generalized bent function is also a weakly regular generalized bent function [12]. For non-weakly regular bent functions, the dual can be bent or not bent. In [5], Çeşmelioglu *et al.* analysed the first known construction of non-weakly regular bent functions [2] and showed that this construction yields bent functions for which the dual is bent if using some proper weakly regular plateaued functions or bent functions whose dual is also bent as building blocks. In [6], Çeşmelioglu *et al.* provided the first explicit construction of non-weakly regular bent functions for which the dual is not bent. As a consequence, the dual of non-weakly regular generalized bent functions can be generalized bent or not generalized bent. For instance, if $f : V_n \rightarrow \mathbb{F}_p$ is a non-weakly regular bent function whose dual is (not) bent, then obviously $p^{k-1}f : V_n \rightarrow \mathbb{Z}_{p^k}$ is a non-weakly regular generalized bent function whose dual is (not) generalized bent. In [5], the authors also analysed the self-duality of p -ary bent functions from V_n to \mathbb{F}_p . They characterized quadratic self-dual bent functions. For $p \equiv 1 \pmod{4}$, they proposed a secondary construction of self-dual bent functions, which can be used to construct non-quadratic self-dual bent functions by using (non)-quadratic self-dual bent functions as building blocks. However, for $p \equiv 3 \pmod{4}$ and n is even, they only illustrated that there exist ternary non-quadratic self-dual bent functions by considering special ternary bent functions. For $p \equiv 3 \pmod{4}$ and n is odd, they showed that there is no weakly regular self-dual bent function and they pointed out that there exist non-weakly regular self-dual bent functions. Indeed, they pointed out that the non-weakly regular

bent function $g_3(x) = Tr_1^3(x^{22} + x^8)$ from \mathbb{F}_{3^3} to \mathbb{F}_3 is self-dual. But in fact, g_3 is not self-dual. In this paper, we will prove that there is no self-dual generalized bent function $f : V_n \rightarrow \mathbb{Z}_{p^k}$ if $p \equiv 3 \pmod{4}$ and n is odd. For weakly regular bent functions f with dual f^* , it is known that $f^{**}(x) = f(-x)$ holds where f^{**} denotes the dual of f^* . Recently, for non-weakly regular bent functions f whose dual f^* is also bent, Özbudak and Pelen [13] showed that $f^{**}(x) = f(-x)$ also holds by studying the value distributions of the dual of non-weakly regular bent functions.

In this paper, we study the dual of generalized bent functions from V_n to \mathbb{Z}_{p^k} . By generalizing the construction of Corollary 2 of [6], we obtain an explicit construction of generalized bent functions for which the dual can be generalized bent or not generalized bent. Note that in [6], a sufficient condition is given to construct non-weakly regular bent functions whose dual is not bent, however, a sufficient condition for the dual of the constructed non-weakly regular bent function to be bent is not given. We will give a sufficient condition to illustrate that the construction of Corollary 2 of [6] can also be used to construct non-weakly regular bent functions whose dual is bent. We will show that the generalized indirect sum construction method given in [15] can provide a secondary construction of generalized bent functions for which the dual can be generalized bent or not generalized bent. For generalized bent functions f whose dual f^* is also generalized bent, different from the proof method in [13], we prove $f^{**}(x) = f(-x)$ by using the knowledge on ideal decomposition in cyclotomic field, which seems more concise. For any non-weakly regular generalized bent function f which satisfies that $f(x) = f(-x)$ and the dual f^* is generalized bent, we give a property and as a consequence, we prove that there is no self-dual generalized bent function $f : V_n \rightarrow \mathbb{Z}_{p^k}$ if $p \equiv 3 \pmod{4}$ and n is odd. For $p \equiv 1 \pmod{4}$ or $p \equiv 3 \pmod{4}$ and n is even, we give a secondary construction of self-dual generalized bent functions. In the end, by the decomposition of generalized bent functions, we characterize the relations between the generalized bentness of the dual of generalized bent functions and the bentness of the dual of bent functions, as well as the self-duality relations between generalized bent functions and bent functions.

The rest of the paper is organized as follows. In Section II, we present some needed results on generalized bent functions and number fields. In Section III, we give constructions of generalized bent functions whose dual can be generalized bent or not generalized bent (Theorems 1 and 2). In Section IV, for generalized bent functions f whose dual f^* is also generalized bent, by using the knowledge on ideal decomposition in cyclotomic field, we prove that $f^{**}(x) = f(-x)$

holds (Theorem 3). In Section V, we prove that there is no self-dual generalized bent function $f : V_n \rightarrow \mathbb{Z}_{p^k}$ if $p \equiv 3 \pmod{4}$ and n is odd (Theorem 4). In Section VI, we give a secondary construction of self-dual generalized bent functions $f : V_n \rightarrow \mathbb{Z}_{p^k}$ if $p \equiv 1 \pmod{4}$ or n is even (Theorem 5). In Section VII, we characterize the relations between the generalized bentness of the dual of generalized bent functions and the bentness of the dual of bent functions, as well as the self-duality relations between generalized bent functions and bent functions (Theorem 6). In Section VIII, we make a conclusion.

II. PRELIMINARIES

In this section, we present some needed results on generalized bent functions and number fields.

A. Some Results on Generalized Bent Functions

Let $f : V_n \rightarrow \mathbb{Z}_{p^k}$ be a generalized bent function, then $W_f(a) = \mu_f(a)p^{\frac{n}{2}}\zeta_{p^k}^{f^*(a)}$ for any $a \in V_n$ where $\mu_f(a) = \xi\varepsilon_f(a)$, $\xi = 1$ if $p \equiv 1 \pmod{4}$ or n is even and $\xi = \sqrt{-1}$ if $p \equiv 3 \pmod{4}$ and n is odd, $\varepsilon_f : V_n \rightarrow \{\pm 1\}$, f^* is the dual of f . By the inverse transform (2), we have

$$\begin{aligned}\zeta_{p^k}^{f(x)} &= \frac{1}{p^n} \sum_{a \in V_n} W_f(a) \zeta_p^{\langle a, x \rangle} \\ &= \frac{1}{p^n} \sum_{a \in V_n} \xi \varepsilon_f(a) p^{\frac{n}{2}} \zeta_{p^k}^{f^*(a)} \zeta_p^{\langle a, x \rangle} \\ &= \xi p^{-\frac{n}{2}} \sum_{a \in V_n} \varepsilon_f(a) \zeta_{p^k}^{f^*(a)} \zeta_p^{\langle a, x \rangle}.\end{aligned}$$

Hence for any $x \in V_n$,

$$\xi \sum_{a \in V_n} \varepsilon_f(a) \zeta_{p^k}^{f^*(a)} \zeta_p^{\langle a, x \rangle} = p^{\frac{n}{2}} \zeta_{p^k}^{f(x)}. \quad (6)$$

In this paper, let η be the multiplicative quadratic character of \mathbb{F}_{p^n} , that is,

$$\eta(x) = \begin{cases} 1 & \text{if } x \in \mathbb{F}_{p^n}^* \text{ is a square element,} \\ -1 & \text{if } x \in \mathbb{F}_{p^n}^* \text{ is a non-square element.} \end{cases}$$

Let $\alpha \in \mathbb{F}_{p^n}^*$. Then the function $f : \mathbb{F}_{p^n} \rightarrow \mathbb{F}_p$ defined by $f(x) = \text{Tr}_1^n(\alpha x^2)$ is a weakly regular bent function and $W_f(a) = (-1)^{n-1} \varepsilon^n \eta(\alpha) p^{\frac{n}{2}} \zeta_p^{\text{Tr}_1^n(-\frac{a^2}{4\alpha})}$ for any $a \in \mathbb{F}_{p^n}$ where $\varepsilon = 1$ if $p \equiv 1 \pmod{4}$ and $\varepsilon = \sqrt{-1}$ if $p \equiv 3 \pmod{4}$ (see [8]).

In [12], the authors characterized the relations between generalized bent functions and bent functions by the decomposition of generalized bent functions. The following lemma follows from Theorem 16 and its proof of [12].

Lemma 1 ([12]). *Let $k \geq 2$ be an integer. Let $f : V_n \rightarrow \mathbb{Z}_{p^k}$ with the decomposition $f(x) = \sum_{i=0}^{k-1} f_i(x)p^{k-1-i} = f_0(x)p^{k-1} + \bar{f}_1(x)$ where f_i is a function from V_n to \mathbb{F}_p for any $0 \leq i \leq k-1$ and $\bar{f}_1(x) = \sum_{i=1}^{k-1} f_i(x)p^{k-1-i}$ is a function from V_n to $\mathbb{Z}_{p^{k-1}}$. Then f is a generalized bent function if and only if $g_{f,F} \triangleq f_0 + F(f_1, \dots, f_{k-1})$ is a bent function for any function $F : \mathbb{F}_p^{k-1} \rightarrow \mathbb{F}_p$. Furthermore, if f is a generalized bent function, then $f^*(x) = f_0^*(x)p^{k-1} + \lambda(x)$ and $g_{f,F}^*(x) = f_0^*(x) + F(\lambda_1(x), \dots, \lambda_{k-1}(x))$ where $\lambda(x) = \sum_{i=1}^{k-1} \lambda_i(x)p^{k-1-i}$, $\lambda_i : V_n \rightarrow \mathbb{F}_p$ and $\lambda : V_n \rightarrow \mathbb{Z}_{p^{k-1}}$ satisfies that for any $a \in V_n$, $\sum_{x \in V_n: \bar{f}_1(x) = \lambda(a)} \zeta_p^{f_0(x) - \langle a, x \rangle} = W_{f_0}(a)$ and $\sum_{x \in V_n: \bar{f}_1(x) = v} \zeta_p^{f_0(x) - \langle a, x \rangle} = 0$ for any $v \in \mathbb{Z}_{p^{k-1}}$ with $v \neq \lambda(a)$.*

B. Some Results on Number Fields

In this subsection, we give some results on number fields. For more results on number fields, we refer to [7].

Suppose L/K is a number field extension. Let \mathcal{O}_L and \mathcal{O}_K be the ring of integers in L and K respectively. Any nonzero ideal I of \mathcal{O}_L can be uniquely (up to the order) expressed as

$$I = \mathfrak{B}_1^{e_1} \dots \mathfrak{B}_g^{e_g}$$

where $\mathfrak{B}_1, \dots, \mathfrak{B}_g$ are distinct (nonzero) prime ideals of \mathcal{O}_L and $e_i \geq 1$ ($1 \leq i \leq g$). And when $I = \mathfrak{p}\mathcal{O}_L$ where \mathfrak{p} is a nonzero prime ideal of \mathcal{O}_K , $\mathfrak{p} = \mathfrak{B}_i \cap \mathcal{O}_K$ for any $1 \leq i \leq g$.

In this paper, we consider cyclotomic field $L = \mathbb{Q}(\zeta_{p^k})$. Then $\mathcal{O}_L = \mathbb{Z}[\zeta_{p^k}]$ and $\{\zeta_p^i \zeta_{p^k}^j : 0 \leq i \leq p-2, 0 \leq j \leq p^{k-1}-1\}$ is an integral basis of \mathcal{O}_L . For prime ideal $p\mathbb{Z}$ of \mathbb{Z} , $(p\mathbb{Z})\mathcal{O}_L = p\mathcal{O}_L$ and

$$p\mathcal{O}_L = ((1 - \zeta_{p^k})\mathcal{O}_L)^{\varphi(p^k)}$$

where $(1 - \zeta_{p^k})\mathcal{O}_L$ is a prime ideal of \mathcal{O}_L and φ is the Euler function.

For any integer $j \in \mathbb{Z}$, let $v_p(j) \in \mathbb{Z}$ denote the p -adic valuation of j , that is, $p^{v_p(j)} \mid j$ and $p^{v_p(j)+1} \nmid j$.

III. CONSTRUCTIONS OF GENERALIZED BENT FUNCTIONS WHOSE DUAL IS (NOT) GENERALIZED BENT

In this section, we give constructions of generalized bent functions whose dual can be generalized bent or not generalized bent.

The following theorem gives an explicit construction of generalized bent functions for which the dual can be generalized bent or not generalized bent, which is a generalization of Corollary 2 of [6]. Note that in Corollary 2 of [6], a sufficient condition is given to construct non-weakly regular bent functions whose dual is not bent, however, a sufficient condition for the dual of the constructed non-weakly regular bent function to be bent is not given. In the following theorem, we also give a sufficient condition to illustrate that the construction of [6] can also be used to construct non-weakly regular bent functions whose dual is bent.

Theorem 1. *Let $m \geq 2$ be an integer. Let $\alpha, \beta \in \mathbb{F}_{p^m}^*$ satisfy $1 + i\alpha + j\beta \neq 0$ for any $i, j \in \mathbb{F}_p$. Let $F(x, y_1, y_2) = p^{k-1}(Tr_1^m(x^2) + (y_1 + Tr_1^m(\alpha x^2))(y_2 + Tr_1^m(\beta x^2))) + g(y_2 + Tr_1^m(\beta x^2))$, $(x, y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_p \times \mathbb{F}_p$, where g is an arbitrary function from \mathbb{F}_p to \mathbb{Z}_{p^k} . Then*

1) *F is a generalized bent function and it is weakly regular if and only if $\eta(1 + i\alpha + j\beta) = 1$ for any $i, j \in \mathbb{F}_p$.*

2) *The dual $F^*(x, y_1, y_2) = p^{k-1}(Tr_1^m(-\frac{x^2}{4(1+y_1\alpha+y_2\beta)}) - y_1y_2) + g(y_1)$ and F^* is a generalized bent function if and only if*

$$\left| \sum_{y_1, y_2 \in \mathbb{F}_p} \eta(1 + y_1\alpha + y_2\beta) \zeta_p^{g(y_1)} \zeta_p^{-y_1y_2 + b_1y_1 + b_2y_2} \right| = p \quad \text{for any } b_1, b_2 \in \mathbb{F}_p. \quad (7)$$

In particular, if $\eta(1 + i\alpha + j\beta) = \eta(1 + i\alpha)$ for any $i, j \in \mathbb{F}_p$ and there exist $i_1, i_2 \in \mathbb{F}_p$ such that $\eta(1 + i_1\alpha) \neq \eta(1 + i_2\alpha)$, then F^ is a non-weakly regular generalized bent function.*

Proof: 1) For any $(a, b_1, b_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_p \times \mathbb{F}_p$, we have

$$\begin{aligned} W_F(a, b_1, b_2) &= \sum_{x \in \mathbb{F}_{p^m}, y_1, y_2 \in \mathbb{F}_p} \zeta_p^{F(x, y_1, y_2)} \zeta_p^{-Tr_1^m(ax) - b_1y_1 - b_2y_2} \\ &= \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{Tr_1^m(x^2) - Tr_1^m(ax)} \sum_{y_1, y_2 \in \mathbb{F}_p} \zeta_p^{g(y_2 + Tr_1^m(\beta x^2))} \zeta_p^{(y_1 + Tr_1^m(\alpha x^2))(y_2 + Tr_1^m(\beta x^2)) - b_1y_1 - b_2y_2} \\ &= \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{Tr_1^m(\gamma_b x^2 - ax)} \sum_{z_1, z_2 \in \mathbb{F}_p} \zeta_p^{z_1z_2 - b_1z_1 - b_2z_2} \zeta_p^{g(z_2)} \end{aligned}$$

$$= W_{Tr_1^m(\gamma_b x^2)}(a) W_{p^{k-1}z_1 z_2 + g(z_2)}(b_1, b_2)$$

where $\gamma_b = 1 + b_1\alpha + b_2\beta$ and in the third equality we use the change of variables $z_1 = y_1 + Tr_1^m(\alpha x^2)$, $z_2 = y_2 + Tr_1^m(\beta x^2)$. Since $\gamma_b \neq 0$, then $Tr_1^m(\gamma_b x^2)$ is a bent function and $W_{Tr_1^m(\gamma_b x^2)}(a) = (-1)^{m-1} \epsilon^m \eta(\gamma_b) p^{\frac{m}{2}} \zeta_p^{Tr_1^m(-\frac{a^2}{4\gamma_b})}$ where $\epsilon = 1$ if $p \equiv 1 \pmod{4}$ and $\epsilon = \sqrt{-1}$ if $p \equiv 3 \pmod{4}$. Since $p^{k-1}z_1 z_2 + g(z_2)$ is a generalized Maiorana-McFarland bent function, then $W_{p^{k-1}z_1 z_2 + g(z_2)}(b_1, b_2) = p \zeta_p^{-p^{k-1}b_1 b_2 + g(b_1)}$. Hence F is generalized bent and

$$W_F(a, b_1, b_2) = \mu_F(a, b_1, b_2) p^{\frac{m+2}{2}} \zeta_p^{F^*(a, b_1, b_2)} \quad (8)$$

where $\mu_F(a, b_1, b_2) = (-1)^{m-1} \epsilon^m \eta(\gamma_b)$ and its dual $F^*(a, b_1, b_2) = p^{k-1}(Tr_1^m(-\frac{a^2}{4\gamma_b}) - b_1 b_2) + g(b_1)$.

From (8), it is obviously that F is weakly regular if and only if $\eta(\gamma_b) = \eta(1 + b_1\alpha + b_2\beta)$, $b_1, b_2 \in \mathbb{F}_p$ are the same. By $\eta(1) = 1$, we have F is weakly regular if and only if $\eta(\gamma_b) = 1$ for any $b_1, b_2 \in \mathbb{F}_p$.

2) By $F^*(x, y_1, y_2) = p^{k-1}(Tr_1^m(-\frac{x^2}{4\gamma_y}) - y_1 y_2) + g(y_1)$ where $\gamma_y = 1 + y_1\alpha + y_2\beta$, for any $(a, b_1, b_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_p \times \mathbb{F}_p$ we have

$$\begin{aligned} & W_{F^*}(a, b_1, b_2) \\ &= \sum_{y_1, y_2 \in \mathbb{F}_p} \zeta_p^{-y_1 y_2 - b_1 y_1 - b_2 y_2} \zeta_p^{g(y_1)} \sum_{x \in \mathbb{F}_{p^m}} \zeta_p^{Tr_1^m(-\frac{x^2}{4\gamma_y} - ax)} \\ &= \sum_{y_1, y_2 \in \mathbb{F}_p} \zeta_p^{-y_1 y_2 - b_1 y_1 - b_2 y_2} \zeta_p^{g(y_1)} (-1)^{m-1} \epsilon^m p^{\frac{m}{2}} \eta(-\frac{1}{4\gamma_y}) \zeta_p^{Tr_1^m(\gamma_y a^2)} \\ &= \lambda \sum_{y_1, y_2 \in \mathbb{F}_p} \eta(\gamma_y) \zeta_p^{-y_1 y_2 - (b_1 - Tr_1^m(a^2\alpha))y_1 - (b_2 - Tr_1^m(a^2\beta))y_2} \zeta_p^{g(y_1)} \end{aligned}$$

where $\lambda = (-1)^{m-1} \epsilon^m \eta(-1) p^{\frac{m}{2}} \zeta_p^{Tr_1^m(a^2)}$, hence it is easy to see that F^* is generalized bent if and only if (7) holds. In particular, if $\eta(1 + i\alpha + j\beta) = \eta(1 + i\alpha)$ for any $i, j \in \mathbb{F}_p$, then for any $b_1, b_2 \in \mathbb{F}_p$,

$$\begin{aligned} & \sum_{y_1, y_2 \in \mathbb{F}_p} \eta(1 + y_1\alpha + y_2\beta) \zeta_p^{-y_1 y_2 + b_1 y_1 + b_2 y_2} \zeta_p^{g(y_1)} \\ &= \sum_{y_1 \in \mathbb{F}_p} \eta(1 + y_1\alpha) \zeta_p^{b_1 y_1} \zeta_p^{g(y_1)} \sum_{y_2 \in \mathbb{F}_p} \zeta_p^{(b_2 - y_1)y_2} \\ &= \eta(1 + b_2\alpha) p \zeta_p^{b_1 b_2} \zeta_p^{g(b_2)}, \end{aligned}$$

thus (7) holds and F^* is generalized bent. Furthermore, if there exist $i_1, i_2 \in \mathbb{F}_p$ such that $\eta(1 + i_1\alpha) \neq \eta(1 + i_2\alpha)$, it is easy to see that F^* is non-weakly regular. ■

Remark 1. When $k = 1$, $g = 0$ and $1, \alpha, \beta \in \mathbb{F}_{p^m}$ are linearly independent over \mathbb{F}_p , the above construction reduces to Corollary 2 of [6], which is the first explicit construction of bent functions whose dual is not bent. As for general $\alpha, \beta \in \mathbb{F}_{p^m}^*$ with $1 + i\alpha + j\beta \neq 0, i, j \in \mathbb{F}_p$, the condition (7) does not seem to hold in general, one can easily obtain non-weakly regular generalized bent functions whose dual is not generalized bent. But we emphasize that if $\alpha, \beta \in \mathbb{F}_{p^m}^*$ with $1 + i\alpha + j\beta \neq 0, i, j \in \mathbb{F}_p$ satisfy that $\eta(1 + i\alpha + j\beta) = \eta(1 + i\alpha)$ for any $i, j \in \mathbb{F}_p$ and $\eta(1 + i\alpha), i \in \mathbb{F}_p$ are not all the same, then the function constructed by the above construction is a non-weakly regular generalized bent function whose dual is also generalized bent.

We give two examples of non-weakly regular generalized bent functions by using Theorem 1 for which the dual of the first example is not generalized bent and the dual of the second example is generalized bent.

Example 1. Let $p = 5$, $m = 2$, $k = 3$. Let z be the primitive element of \mathbb{F}_{5^2} with $z^2 + 4z + 2 = 0$. Let $\alpha = \beta = z$, $g : \mathbb{F}_5 \rightarrow \mathbb{Z}_{5^3}$ be defined as $g(x) = x^3$. Then one can verify that the function F constructed by Theorem 1 is a non-weakly regular generalized bent function and its dual F^* is not generalized bent.

Example 2. Let $p = 3$, $m = 5$, $k = 2$. Let z be the primitive element of \mathbb{F}_{3^5} with $z^5 + 2z + 1 = 0$. Let $\alpha = z^{10}, \beta = z^{47}$, $g : \mathbb{F}_3 \rightarrow \mathbb{Z}_{3^2}$ be defined as $g(x) = x$. Then one can verify that $\eta(1 + j\beta) = 1, \eta(1 + \alpha + j\beta) = \eta(1 + 2\alpha + j\beta) = -1$ for any $j \in \mathbb{F}_3$, hence the function F constructed by Theorem 1 is a non-weakly regular generalized bent function and its dual F^* is generalized bent.

Now we show that the generalized indirect sum construction method given in [15] can provide a secondary construction of generalized bent functions for which the dual can be generalized bent or not generalized bent. For the sake of completeness, we give the proof of the following lemma, which can be obtained by Theorem 5 of [15].

Lemma 2 ([15]). Let $f_i(i \in \mathbb{F}_p^t) : V_r \rightarrow \mathbb{Z}_{p^k}$ be generalized bent functions. Let $g_s(0 \leq s \leq t) : V_m \rightarrow \mathbb{F}_p$ be bent functions which satisfy that for any $j = (j_1, \dots, j_t) \in \mathbb{F}_p^t$, $G_j \triangleq (1 - j_1 - \dots -$

$j_t)g_0 + j_1g_1 + \dots + j_tg_t$ is a bent function and $G_j^* = (1 - j_1 - \dots - j_t)g_0^* + j_1g_1^* + \dots + j_tg_t^*$ and $\mu_{G_j}(y) = u, y \in V_m$ where μ_{G_j} is defined by (4) and u is a constant independent of j . Let g be an arbitrary function from \mathbb{F}_p^t to \mathbb{Z}_{p^k} . Then $F(x, y) = f_{(g_0(y)-g_1(y), \dots, g_0(y)-g_t(y))}(x) + p^{k-1}g_0(y) + g(g_0(y) - g_1(y), \dots, g_0(y) - g_t(y)), (x, y) \in V_r \times V_m$ is a generalized bent function.

Proof: For any $(a, b) \in V_r \times V_m$, we have

$$\begin{aligned}
& W_F(a, b) \\
&= \sum_{x \in V_r, y \in V_m} \zeta_{p^k}^{f_{(g_0(y)-g_1(y), \dots, g_0(y)-g_t(y))}(x) + g(g_0(y)-g_1(y), \dots, g_0(y)-g_t(y))} \zeta_p^{g_0(y) - \langle a, x \rangle - \langle b, y \rangle} \\
&= \sum_{i_1, \dots, i_t \in \mathbb{F}_p} \sum_{y: g_0(y) - g_j(y) = i_j, 1 \leq j \leq t} \sum_{x \in V_r} \zeta_{p^k}^{f_{(i_1, \dots, i_t)}(x) + g(i_1, \dots, i_t)} \zeta_p^{g_0(y) - \langle a, x \rangle - \langle b, y \rangle} \\
&= p^{-t} \sum_{i_1, \dots, i_t \in \mathbb{F}_p} \zeta_{p^k}^{g(i_1, \dots, i_t)} W_{f_{(i_1, \dots, i_t)}}(a) \sum_{y \in V_m} \zeta_p^{g_0(y) - \langle b, y \rangle} \sum_{j_1 \in \mathbb{F}_p} \zeta_p^{(i_1 - (g_0 - g_1)(y))j_1} \\
&\quad \dots \sum_{j_t \in \mathbb{F}_p} \zeta_p^{(i_t - (g_0 - g_t)(y))j_t} \\
&= p^{-t} \sum_{i_1, \dots, i_t \in \mathbb{F}_p} \zeta_{p^k}^{g(i_1, \dots, i_t)} W_{f_{(i_1, \dots, i_t)}}(a) \sum_{j_1, \dots, j_t \in \mathbb{F}_p} \zeta_p^{i_1j_1 + \dots + i_tj_t} W_{G_{(j_1, \dots, j_t)}}(b).
\end{aligned}$$

As for any $j_1, \dots, j_t \in \mathbb{F}_p$, $G_{(j_1, \dots, j_t)} \triangleq (1 - j_1 - \dots - j_t)g_0 + j_1g_1 + \dots + j_tg_t$ is a bent function and $G_{(j_1, \dots, j_t)}^* = (1 - j_1 - \dots - j_t)g_0^* + j_1g_1^* + \dots + j_tg_t^*$ and $\mu_{G_{(j_1, \dots, j_t)}}(y) = u, y \in V_m$ where u is a constant independent of (j_1, \dots, j_t) , we have

$$\begin{aligned}
& W_F(a, b) \\
&= up^{\frac{m}{2}} p^{-t} \sum_{i_1, \dots, i_t \in \mathbb{F}_p} \zeta_{p^k}^{g(i_1, \dots, i_t)} W_{f_{(i_1, \dots, i_t)}}(a) \sum_{j_1, \dots, j_t \in \mathbb{F}_p} \zeta_p^{i_1j_1 + \dots + i_tj_t + (1 - j_1 - \dots - j_t)g_0^*(b) + j_1g_1^*(b) + \dots + j_tg_t^*(b)} \\
&= up^{\frac{m}{2}} p^{-t} \zeta_p^{g_0^*(b)} \sum_{i_1, \dots, i_t \in \mathbb{F}_p} \zeta_{p^k}^{g(i_1, \dots, i_t)} W_{f_{(i_1, \dots, i_t)}}(a) \sum_{j_1 \in \mathbb{F}_p} \zeta_p^{(g_1^*(b) - g_0^*(b) + i_1)j_1} \dots \sum_{j_t \in \mathbb{F}_p} \zeta_p^{(g_t^*(b) - g_0^*(b) + i_t)j_t} \\
&= up^{\frac{m}{2}} \zeta_p^{g_0^*(b)} \zeta_{p^k}^{g(g_0^*(b) - g_1^*(b), \dots, g_0^*(b) - g_t^*(b))} W_{f_{(g_0^*(b) - g_1^*(b), \dots, g_0^*(b) - g_t^*(b))}}(a).
\end{aligned} \tag{9}$$

Hence, by (9), it is easy to see that $F : V_r \times V_m \rightarrow \mathbb{Z}_{p^k}$ is a generalized bent function if $f_i, i \in \mathbb{F}_p^t$ are generalized bent functions from V_r to \mathbb{Z}_{p^k} . \blacksquare

Based on Lemma 2, we give the following theorem.

Theorem 2. *With the same notation as Lemma 2. The dual of the generalized bent function*

constructed by Lemma 2 is a generalized bent function if and only if for any $y \in V_m$, the dual of $f_{(g_0(y)-g_1(y), \dots, g_0(y)-g_t(y))}$ is a generalized bent function.

Proof: By (9), we have that the dual of the generalized bent function F constructed by Lemma 2 is $F^*(x, y) = f_{(g_0^*(y)-g_1^*(y), \dots, g_0^*(y)-g_t^*(y))}(x) + p^{k-1}g_0^*(y) + g(g_0^*(y) - g_1^*(y), \dots, g_0^*(y) - g_t^*(y))$. Note that for a weakly regular bent function h with dual h^* , h^* is also a weakly regular bent function and $h^{**}(x) = h(-x)$, $\mu_{h^*} = \mu_h^{-1}$. Since for any $j = (j_1, \dots, j_t) \in \mathbb{F}_p^t$, $G_j = (1-j_1-\dots-j_t)g_0+j_1g_1+\dots+j_tg_t$ is a bent function and $G_j^* = (1-j_1-\dots-j_t)g_0^*+j_1g_1^*+\dots+j_tg_t^*$ and $\mu_{G_j}(y) = u$, $y \in V_m$ where u is a constant independent of j , we have G_j^* is a bent function and $G_j^{**}(y) = G_j(-y) = (1-j_1-\dots-j_t)g_0(-y) + j_1g_1(-y) + \dots + j_tg_t(-y) = (1-j_1-\dots-j_t)g_0^{**}(y) + j_1g_1^{**}(y) + \dots + j_tg_t^{**}(y)$ and $\mu_{G_j^*}(y) = u^{-1}$, $y \in V_m$, that is, $g_s^*(0 \leq s \leq t)$ also satisfy the condition of Lemma 2. Since F^* has the same form as F and $g_s^*(0 \leq s \leq t)$ satisfy the condition of Lemma 2, by (9), for any $(a, b) \in V_r \times V_m$ we have

$$\begin{aligned} W_{F^*}(a, b) &= u^{-1} p^{\frac{m}{2}} \zeta_p^{g_0^{**}(b)} \zeta_{p^k}^{g(g_0^{**}(b)-g_1^{**}(b), \dots, g_0^{**}(b)-g_t^{**}(b))} W_{f_{(g_0^{**}(b)-g_1^{**}(b), \dots, g_0^{**}(b)-g_t^{**}(b))}}^*(a) \\ &= u^{-1} p^{\frac{m}{2}} \zeta_p^{g_0(-b)} \zeta_{p^k}^{g(g_0(-b)-g_1(-b), \dots, g_0(-b)-g_t(-b))} W_{f_{(g_0(-b)-g_1(-b), \dots, g_0(-b)-g_t(-b))}}^*(a). \end{aligned}$$

If for any $y \in V_m$, $f_{(g_0(y)-g_1(y), \dots, g_0(y)-g_t(y))}^*$ is generalized bent, then obviously F^* is generalized bent. Suppose F^* is not generalized bent. If there exists $y \in V_m$ such that $f_{(g_0(y)-g_1(y), \dots, g_0(y)-g_t(y))}^*$ is not generalized bent, let $a \in V_r$ with $|W_{f_{(g_0(y)-g_1(y), \dots, g_0(y)-g_t(y))}}^*(a)| \neq p^{\frac{r}{2}}$ and $b = -y$, then $|W_{F^*}(a, b)| \neq p^{\frac{r+m}{2}}$ and F^* is not generalized bent, which is a contradiction. Hence, F^* is a generalized bent function if and only if for any $y \in V_m$, the dual of $f_{(g_0(y)-g_1(y), \dots, g_0(y)-g_t(y))}$ is a generalized bent function. \blacksquare

When $k = 1, t = 1, m = 2$ and $g = 0$, $g_0(y) = y_1y_2, g_1(y) = y_1y_2 - y_2, y = (y_1, y_2) \in \mathbb{F}_p \times \mathbb{F}_p$, Theorem 2 reduces to Theorem 3 of [6]. The corresponding function of Theorem 3 of [6] is in the Generalized Maiorana-McFarland bent functions class (see [4]). In [15], the authors showed that the canonical way to construct Generalized Maiorana-McFarland bent functions can be obtained by the generalized indirect sum construction method. In [15], the authors also showed that p -ary PS_{ap} bent functions

$$g_s(y) = Tr_1^m(\alpha_s G(y_1 y_2^{p^m-2})), y = (y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}, 0 \leq s \leq t$$

satisfy the condition of Lemma 2 where $m \geq t + 1$, $\alpha_0, \dots, \alpha_t \in \mathbb{F}_{p^m}$ are linearly independent over \mathbb{F}_p and G is a permutation over \mathbb{F}_{p^m} with $G(0) = 0$.

Since the above $g_s (0 \leq s \leq t)$ satisfy the condition of Lemma 2 and $\{(g_0(y) - g_1(y), \dots, g_0(y) - g_t(y)), y = (y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}\} = \mathbb{F}_p^t$, by Theorem 2 we have the following corollary.

Corollary 1. *Let $f_i (i \in \mathbb{F}_p^t) : V_r \rightarrow \mathbb{Z}_{p^k}$ be generalized bent functions. Let $g_s (0 \leq s \leq t) : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \rightarrow \mathbb{F}_p$ be defined as $g_s(y) = \text{Tr}_1^m(\alpha_s G(y_1 y_2^{p^m-2}))$, $y = (y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ where $m \geq t + 1$, $\alpha_0, \dots, \alpha_t \in \mathbb{F}_{p^m}$ are linearly independent over \mathbb{F}_p and G is a permutation over \mathbb{F}_{p^m} with $G(0) = 0$. Let g be an arbitrary function from \mathbb{F}_p^t to \mathbb{Z}_{p^k} . Then the dual of the generalized bent function $F : V_r \times \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \rightarrow \mathbb{Z}_{p^k}$ defined as $F(x, y) = f_{(g_0(y) - g_1(y), \dots, g_0(y) - g_t(y))}(x) + p^{k-1} g_0(y) + g(g_0(y) - g_1(y), \dots, g_0(y) - g_t(y))$, $(x, y) = (x, y_1, y_2) \in V_r \times \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ is generalized bent if and only if for any $i \in \mathbb{F}_p^t$, the dual of f_i is generalized bent.*

IV. A PROPERTY OF GENERALIZED BENT FUNCTIONS WHOSE DUAL IS GENERALIZED BENT

In this section, let $f : V_n \rightarrow \mathbb{Z}_{p^k}$ be a generalized bent function whose dual f^* is generalized bent. By using the knowledge on ideal decomposition in cyclotomic field, we will prove $f^{**}(x) = f(-x)$. For the case of bent functions $f : V_n \rightarrow \mathbb{F}_p$ whose dual f^* is bent, Özbudak and Pelen [13] have shown that $f^{**}(x) = f(-x)$ holds by studying the value distributions of the dual of non-weakly regular bent functions. Compared with the proof method in [13], our proof method seems more concise. Before proof, we need a lemma.

Lemma 3. *Let $L = \mathbb{Q}(\zeta_{p^k})$, $\mathcal{O}_L = \mathbb{Z}[\zeta_{p^k}]$. Then for any $1 \leq j \leq p^k - 1$, $(1 + \zeta_{p^k}^j)\mathcal{O}_L = \mathcal{O}_L$ and $(1 - \zeta_{p^k}^j)\mathcal{O}_L = ((1 - \zeta_{p^k})\mathcal{O}_L)^{p^s}$ where $s = v_p(j)$.*

Proof: For any $1 \leq j \leq p^k - 1$, by

$$\frac{1}{1 + \zeta_{p^k}^j} = \frac{1 - \zeta_{p^k}^j}{1 - \zeta_{p^k}^{2j}} = \frac{1 - \zeta_{p^k}^{2lj}}{1 - \zeta_{p^k}^{2j}} = \sum_{i=0}^{l-1} \zeta_{p^k}^{2ij} \in \mathcal{O}_L$$

where $l \in \mathbb{Z}_{p^k}$ satisfies $2l \equiv 1 \pmod{p^k}$, we have that $1 + \zeta_{p^k}^j$ is a unit of \mathcal{O}_L , that is, $(1 + \zeta_{p^k}^j)\mathcal{O}_L = \mathcal{O}_L$.

For any $1 \leq j \leq p^k - 1$, let $s = v_p(j)$. Then $0 \leq s < k$ and $\gcd(j, p^k) = p^s$. Let $M = \mathbb{Q}(\zeta_{p^{k-s}})$, then $\mathcal{O}_M = \mathbb{Z}[\zeta_{p^{k-s}}]$ and M is a subfield of L since $\zeta_{p^{k-s}} = \zeta_{p^k}^{p^s}$. By

$$\frac{1 - \zeta_{p^{k-s}}^{\frac{j}{p^s}}}{1 - \zeta_{p^{k-s}}} = \sum_{i=0}^{\frac{j}{p^s}-1} \zeta_{p^{k-s}}^i \in \mathcal{O}_M \subseteq \mathcal{O}_L$$

and

$$\frac{1 - \zeta_{p^{k-s}}}{1 - \zeta_{p^{k-s}}^{\frac{j}{p^s}}} = \frac{1 - \zeta_{p^{k-s}}^{\frac{j}{p^s}t}}{1 - \zeta_{p^{k-s}}^{\frac{j}{p^s}}} = \sum_{i=0}^{t-1} \zeta_{p^{k-s}}^{i\frac{j}{p^s}} \in \mathcal{O}_M \subseteq \mathcal{O}_L$$

where $t \in \mathbb{Z}_{p^{k-s}}$ satisfies $\frac{j}{p^s}t \equiv 1 \pmod{p^{k-s}}$, we have that $\frac{1 - \zeta_{p^{k-s}}^{\frac{j}{p^s}}}{1 - \zeta_{p^{k-s}}}$ is a unit of \mathcal{O}_L . Note that t exists since $\gcd(\frac{j}{p^s}, p^{k-s}) = 1$. Hence $(1 - \zeta_{p^k}^j)\mathcal{O}_L = (1 - \zeta_{p^{k-s}}^{\frac{j}{p^s}})\mathcal{O}_L = (1 - \zeta_{p^{k-s}})\mathcal{O}_L$. By $p\mathcal{O}_M = ((1 - \zeta_{p^{k-s}})\mathcal{O}_M)^{\varphi(p^{k-s})}$ and $(p\mathcal{O}_M)\mathcal{O}_L = p\mathcal{O}_L = ((1 - \zeta_{p^k})\mathcal{O}_L)^{\varphi(p^k)}$, we have $((1 - \zeta_{p^{k-s}})\mathcal{O}_L)^{\varphi(p^{k-s})} = ((1 - \zeta_{p^k})\mathcal{O}_L)^{\varphi(p^k)}$. By the uniqueness of the decomposition of $(1 - \zeta_{p^{k-s}})\mathcal{O}_L$, we have $(1 - \zeta_{p^{k-s}})\mathcal{O}_L = ((1 - \zeta_{p^k})\mathcal{O}_L)^{\frac{\varphi(p^k)}{\varphi(p^{k-s})}} = ((1 - \zeta_{p^k})\mathcal{O}_L)^{p^s}$. Hence, $(1 - \zeta_{p^k}^j)\mathcal{O}_L = ((1 - \zeta_{p^k})\mathcal{O}_L)^{p^s}$. \blacksquare

Now we prove $f^{**}(x) = f(-x)$ by using Lemma 3. In the subsequent of this paper, $L = \mathbb{Q}(\zeta_{p^k})$, $\mathcal{O}_L = \mathbb{Z}[\zeta_{p^k}]$ unless otherwise stated.

Theorem 3. *Let $f : V_n \rightarrow \mathbb{Z}_{p^k}$ be a generalized bent function whose dual f^* is also a generalized bent function. Then $f^{**}(x) = f(-x)$, $x \in V_n$, where f^{**} is the dual of f^* .*

Proof: Consider the left-hand side of Equation (6),

$$\begin{aligned} & \xi \sum_{x \in V_n} \varepsilon_f(x) \zeta_{p^k}^{f^*(x)} \zeta_p^{\langle a, x \rangle} \\ &= \xi \sum_{x \in V_n: \varepsilon_f(x)=1} \zeta_{p^k}^{f^*(x)} \zeta_p^{\langle a, x \rangle} - \xi \sum_{x \in V_n: \varepsilon_f(x)=-1} \zeta_{p^k}^{f^*(x)} \zeta_p^{\langle a, x \rangle} \\ &= 2\xi \sum_{x \in V_n: \varepsilon_f(x)=1} \zeta_{p^k}^{f^*(x)} \zeta_p^{\langle a, x \rangle} - \xi \sum_{x \in V_n} \zeta_{p^k}^{f^*(x)} \zeta_p^{\langle a, x \rangle} \\ &= 2\xi \sum_{x \in V_n: \varepsilon_f(x)=1} \zeta_{p^k}^{f^*(x)} \zeta_p^{\langle a, x \rangle} - \xi W_{f^*}(-a), \end{aligned}$$

where $\xi = 1$ if $p \equiv 1 \pmod{4}$ or n is even and $\xi = \sqrt{-1}$ if $p \equiv 3 \pmod{4}$ and n is odd, $\varepsilon_f : V_n \rightarrow \{\pm 1\}$ is defined by (5). Since f^* is a generalized bent function, we have $W_{f^*}(-a) = \xi \varepsilon_{f^*}(-a) p^{\frac{n}{2}} \zeta_{p^k}^{f^{**}(-a)}$ where $\varepsilon_{f^*} : V_n \rightarrow \{\pm 1\}$. Thus, for any $a \in V_n$ we have

$$\xi \sum_{x \in V_n} \varepsilon_f(x) \zeta_{p^k}^{f^*(x)} \zeta_p^{\langle a, x \rangle} = 2\xi \sum_{x \in V_n: \varepsilon_f(x)=1} \zeta_{p^k}^{f^*(x)} \zeta_p^{\langle a, x \rangle} - \xi^2 \varepsilon_{f^*}(-a) p^{\frac{n}{2}} \zeta_{p^k}^{f^{**}(-a)}. \quad (10)$$

Let $X_a = \xi \sum_{x \in V_n: \varepsilon_f(x)=1} \zeta_{p^k}^{f^*(x)} \zeta_p^{\langle a, x \rangle}$. By (6) and (10), we have

$$2X_a = p^{\frac{n}{2}} \zeta_{p^k}^{f(a)} (1 + \xi^2 \varepsilon_{f^*}(-a) \zeta_{p^k}^{f^{**}(-a) - f(a)}), a \in V_n. \quad (11)$$

Suppose there exists $a \in V_n$ such that $f^{**}(-a) \neq f(a)$, that is, $\Delta_a \triangleq f^{**}(-a) - f(a) \neq 0$.

1) When n is even, then $\xi = 1$. Note that in this case, $X_a \in \mathcal{O}_L$. By (11) and Lemma 3, we have

$$(2\mathcal{O}_L)(X_a\mathcal{O}_L) = \begin{cases} ((1 - \zeta_{p^k})\mathcal{O}_L)^{\varphi(p^k)\frac{n}{2}} & \text{if } \varepsilon_{f^*}(-a) = 1, \\ ((1 - \zeta_{p^k})\mathcal{O}_L)^{\varphi(p^k)\frac{n}{2} + p^{vp}(\Delta_a)} & \text{if } \varepsilon_{f^*}(-a) = -1. \end{cases}$$

Since $\frac{1}{2} \notin \mathcal{O}_L$, we have $2\mathcal{O}_L \neq \mathcal{O}_L$. Indeed, if $\frac{1}{2} \in \mathcal{O}_L$, then by $\{\zeta_{p^k}^0, \dots, \zeta_{p^k}^{\varphi(p^k)-1}\}$ is an integer basis of \mathcal{O}_L , $\frac{1}{2} = \sum_{i=0}^{\varphi(p^k)-1} a_i \zeta_{p^k}^i$ where $a_i \in \mathbb{Z}$, that is, $\sum_{i=0}^{\varphi(p^k)-1} 2a_i \zeta_{p^k}^i = \zeta_{p^k}^0$. This equation deduces $a_0 = \frac{1}{2}$, $a_i = 0$ ($1 \leq i \leq \varphi(p^k) - 1$), which contradicts $a_0 \in \mathbb{Z}$. Then by the uniqueness of the decomposition of $2\mathcal{O}_L$, we have

$$2\mathcal{O}_L = ((1 - \zeta_{p^k})\mathcal{O}_L)^t \quad (12)$$

for some positive integer t . From (12), we have $2\mathbb{Z} = \mathbb{Z} \cap (1 - \zeta_{p^k})\mathcal{O}_L$. And from $p\mathcal{O}_L = ((1 - \zeta_{p^k})\mathcal{O}_L)^{\varphi(p^k)}$, we have $p\mathbb{Z} = \mathbb{Z} \cap (1 - \zeta_{p^k})\mathcal{O}_L$. Hence, $2\mathbb{Z} = p\mathbb{Z}$, which is a contradiction since p is an odd prime. So in this case, $f^{**}(-a) = f(a)$ for any $a \in V_n$.

2) When n is odd, by multiplying both sides of (11) by \sqrt{p} , we obtain

$$2X_a\sqrt{p} = p^{\frac{n+1}{2}} \zeta_{p^k}^{f(a)} (1 + \xi^2 \varepsilon_{f^*}(-a) \zeta_{p^k}^{f^{**}(-a) - f(a)}). \quad (13)$$

Recall that when n is odd, $\xi = 1$ if $p \equiv 1 \pmod{4}$ and $\xi = \sqrt{-1}$ if $p \equiv 3 \pmod{4}$. Note that $\xi^2 \in \{\pm 1\}$ and $X_a\sqrt{p} \in \mathcal{O}_L$ since $\xi\sqrt{p} = \sum_{i \in \mathbb{F}_p^*} \eta(i) \zeta_p^i \in \mathbb{Z}[\zeta_p]$ by a well known result on Gauss sums (see [10]). By (13) and Lemma 3, we have

$$(2\mathcal{O}_L)((X_a\sqrt{p})\mathcal{O}_L) = \begin{cases} ((1 - \zeta_{p^k})\mathcal{O}_L)^{\varphi(p^k)\frac{n+1}{2}} & \text{if } \xi^2 \varepsilon_{f^*}(-a) = 1, \\ ((1 - \zeta_{p^k})\mathcal{O}_L)^{\varphi(p^k)\frac{n+1}{2} + p^{vp}(\Delta_a)} & \text{if } \xi^2 \varepsilon_{f^*}(-a) = -1. \end{cases}$$

Since $2\mathcal{O}_L \neq \mathcal{O}_L$, by the uniqueness of the decomposition of $2\mathcal{O}_L$, we have $2\mathcal{O}_L = ((1 - \zeta_{p^k})\mathcal{O}_L)^t$ for some positive integer t . Then with the same argument as 1), we have $2\mathbb{Z} = p\mathbb{Z}$, which is a contradiction. So in this case, $f^{**}(-a) = f(a)$ for any $a \in V_n$. \blacksquare

V. NONEXISTENCE OF SELF-DUAL GENERALIZED BENT FUNCTION IF $p \equiv 3 \pmod{4}$ AND n IS ODD

In this section, we will show that there is no self-dual generalized bent function $f : V_n \rightarrow \mathbb{Z}_{p^k}$ if $p \equiv 3 \pmod{4}$ and n is odd. Note that in [5], for $p \equiv 3 \pmod{4}$ and n is odd, the authors showed that there is no weakly regular self-dual bent function and they pointed out that there exist non-weakly regular self-dual bent functions. Indeed, they pointed out that the non-weakly regular bent function $g_3(x) = Tr_1^3(x^{22} + x^8)$ from \mathbb{F}_{3^3} to \mathbb{F}_3 is self-dual. But in fact, it is easy to verify that g_3 is not self-dual by using MAGMA and there is no self-dual bent function from V_n to \mathbb{F}_p if $p \equiv 3 \pmod{4}$ and n is odd according to the theorem of this section.

Let $f : V_n \rightarrow \mathbb{Z}_{p^k}$ be a generalized bent function satisfying $f(x) = f(-x), x \in V_n$. For any $a \in V_n$, we have

$$\begin{aligned} W_f(a) &= \sum_{x \in V_n} \zeta_{p^k}^{f(x)} \zeta_p^{-\langle a, x \rangle} \\ &= \sum_{x \in V_n} \zeta_{p^k}^{f(-x)} \zeta_p^{\langle a, x \rangle} \\ &= \sum_{x \in V_n} \zeta_{p^k}^{f(x)} \zeta_p^{\langle a, x \rangle} \\ &= W_f(-a), \end{aligned}$$

where in the second equation we use the change of variable $x \mapsto -x$ and in the third equation we use $f(x) = f(-x)$. By $W_f(a) = W_f(-a)$ and $W_f(a) = \xi \varepsilon_f(a) p^{\frac{n}{2}} \zeta_{p^k}^{f^*(a)}$ where $\xi = 1$ if $p \equiv 1 \pmod{4}$ or n is even and $\xi = \sqrt{-1}$ if $p \equiv 3 \pmod{4}$ and n is odd, $\varepsilon_f : V_n \rightarrow \{\pm 1\}$, we have

$$\varepsilon_f(a) = \varepsilon_f(-a), f^*(a) = f^*(-a). \quad (14)$$

Note that for any generalized bent function $f : V_n \rightarrow \mathbb{Z}_{p^k}$ satisfying $f(x) = f(-x)$, Equation (14) holds.

For any $a \in V_n$, let

$$S_a = \sum_{x \in V_n: \varepsilon_f(x)=1} \zeta_{p^k}^{f^*(x)} \zeta_p^{\langle a, x \rangle}, \quad T_a = \sum_{x \in V_n: \varepsilon_f(x)=-1} \zeta_{p^k}^{f^*(x)} \zeta_p^{\langle a, x \rangle}.$$

Suppose the dual f^* is generalized bent. Then by the definitions of S_a and T_a , we have $S_a + T_a = W_{f^*}(-a) = \xi \varepsilon_{f^*}(-a) p^{\frac{n}{2}} \zeta_{p^k}^{f^{**}(-a)}$ where $\varepsilon_{f^*} : V_n \rightarrow \{\pm 1\}$. Since $f^*(x) = f^*(-x)$, we have $\varepsilon_{f^*}(-a) = \varepsilon_{f^*}(a)$. By Theorem 3, we have $f^{**}(-a) = f(a)$. Hence, we obtain

$$S_a + T_a = \xi \varepsilon_{f^*}(a) p^{\frac{n}{2}} \zeta_{p^k}^{f(a)}. \quad (15)$$

By (6), we have

$$S_a - T_a = \xi^{-1} p^{\frac{n}{2}} \zeta_{p^k}^{f(a)}. \quad (16)$$

By (15) and (16), we have

$$S_a = p^{\frac{n}{2}} \zeta_{p^k}^{f(a)} \frac{\xi \varepsilon_{f^*}(a) + \xi^{-1}}{2}, \quad T_a = p^{\frac{n}{2}} \zeta_{p^k}^{f(a)} \frac{\xi \varepsilon_{f^*}(a) - \xi^{-1}}{2}. \quad (17)$$

Based on the above analysis and (17), we obtain the following proposition. For weakly regular generalized bent functions, the following proposition is obvious. But for non-weakly regular generalized bent functions, the following proposition is not obvious.

Proposition 1. *Let $f : V_n \rightarrow \mathbb{Z}_{p^k}$ be a generalized bent function which satisfies that $f(x) = f(-x)$, $x \in V_n$ and the dual f^* is a generalized bent function. Let $\varepsilon_f, \varepsilon_{f^*} : V_n \rightarrow \{\pm 1\}$ be defined by (5). Then*

- 1) *If $p \equiv 1 \pmod{4}$ or n is even, then $\varepsilon_{f^*}(0) = \varepsilon_f(0)$;*
- 2) *If $p \equiv 3 \pmod{4}$ and n is odd, then $\varepsilon_{f^*}(0) = -\varepsilon_f(0)$.*

Proof: Let $g_0 : V_n \rightarrow \mathbb{F}_p$ and $\bar{g}_1 : V_n \rightarrow \mathbb{Z}_{p^{k-1}}$ satisfy $f^* = g_0 p^{k-1} + \bar{g}_1$. Note that when $k = 1$, $\mathbb{Z}_{p^{k-1}} = \{0\}$ and $g_0 = f^*$, $\bar{g}_1 = 0$.

1) If $p \equiv 1 \pmod{4}$ or n is even, then $\xi = 1$. By (17), we have $T_a = 0$ if $\varepsilon_{f^*}(a) = 1$ and $S_a = 0$ if $\varepsilon_{f^*}(a) = -1$. Suppose $\varepsilon_{f^*}(0) = -\varepsilon_f(0)$. Without loss of generality, assume $\varepsilon_f(0) = 1$. Then $\varepsilon_{f^*}(0) = -1$ and $S_0 = 0$. By the definition of S_0 , we have $\sum_{x \in V_n: \varepsilon_f(x)=1} \zeta_{p^k}^{f^*(x)} = 0$. For any $i_0 \in \mathbb{F}_p, \bar{i}_1 \in \mathbb{Z}_{p^{k-1}}$, let

$$A_{i_0, \bar{i}_1} = \{x \in V_n : \varepsilon_f(x) = 1 \text{ and } g_0(x) = i_0, \bar{g}_1(x) = \bar{i}_1\} \quad (18)$$

and N_{i_0, \bar{i}_1} denote the size of A_{i_0, \bar{i}_1} . Then we have

$$\sum_{i_0 \in \mathbb{F}_p} \sum_{\bar{i}_1 \in \mathbb{Z}_{p^{k-1}}} N_{i_0, \bar{i}_1} \zeta_p^{i_0} \zeta_{p^k}^{\bar{i}_1} = 0.$$

By $\sum_{i=0}^{p-1} \zeta_p^i = 0$ and the above equation we obtain

$$\sum_{i_0=0}^{p-2} \sum_{\bar{i}_1 \in \mathbb{Z}_{p^{k-1}}} (N_{i_0, \bar{i}_1} - N_{p-1, \bar{i}_1}) \zeta_p^{i_0} \zeta_{p^k}^{\bar{i}_1} = 0,$$

which deduces $N_{i_0, \bar{i}_1} = N_{p-1, \bar{i}_1}$ for any $i_0 \in \mathbb{F}_p, \bar{i}_1 \in \mathbb{Z}_{p^{k-1}}$ since $\{\zeta_p^i \zeta_{p^k}^j : 0 \leq i \leq p-2, 0 \leq j \leq p^{k-1}-1\}$ is an integral basis of $\mathbb{Z}[\zeta_{p^k}]$. In particular, $N_{g_0(0), \bar{g}_1(0)} = N_{i_0, \bar{g}_1(0)}$ for any $i_0 \in \mathbb{F}_p$.

By (14), we have $f^*(x) = f^*(-x)$ and $\varepsilon_f(x) = \varepsilon_f(-x)$. From $f^*(x) = f^*(-x)$, we have $g_0(x) = g_0(-x)$ and $\bar{g}_1(x) = \bar{g}_1(-x)$. And by $\varepsilon_f(x) = \varepsilon_f(-x)$, we have $x \in A_{i_0, \bar{i}_1}$ if and only if $-x \in A_{i_0, \bar{i}_1}$. Note that $0 \in A_{g_0(0), \bar{g}_1(0)}$ since $\varepsilon_f(0) = 1$. Hence, $N_{g_0(0), \bar{g}_1(0)}$ is odd and N_{i_0, \bar{i}_1} is even if $(i_0, \bar{i}_1) \neq (g_0(0), \bar{g}_1(0))$, which contradicts $N_{g_0(0), \bar{g}_1(0)} = N_{i_0, \bar{g}_1(0)}, i_0 \in \mathbb{F}_p$. Hence $\varepsilon_{f^*}(0) = \varepsilon_f(0)$.

2) If $p \equiv 3 \pmod{4}$ and n is odd, then $\xi = \sqrt{-1}$. By (17), we have $S_a = 0$ if $\varepsilon_{f^*}(a) = 1$ and $T_a = 0$ if $\varepsilon_{f^*}(a) = -1$. Suppose $\varepsilon_{f^*}(0) = \varepsilon_f(0)$. Without loss of generality, assume $\varepsilon_f(0) = 1$. Then $\varepsilon_{f^*}(0) = 1$ and $S_0 = 0$. By the definition of S_0 , we have $\sum_{x \in V_n: \varepsilon_f(x)=1} \zeta_{p^k}^{f^*(x)} = 0$. For any $i_0 \in \mathbb{F}_p, \bar{i}_1 \in \mathbb{Z}_{p^{k-1}}$, let A_{i_0, \bar{i}_1} be defined by (18) and N_{i_0, \bar{i}_1} denote the size of A_{i_0, \bar{i}_1} . Then with the same argument as 1), we have that $N_{g_0(0), \bar{g}_1(0)} = N_{i_0, \bar{g}_1(0)}$ for any $i_0 \in \mathbb{F}_p$ and $N_{g_0(0), \bar{g}_1(0)}$ is odd, N_{i_0, \bar{i}_1} is even if $(i_0, \bar{i}_1) \neq (g_0(0), \bar{g}_1(0))$, which is a contradiction. Hence, $\varepsilon_{f^*}(0) = -\varepsilon_f(0)$. ■

By using Proposition 1, we obtain the following theorem.

Theorem 4. *There is no self-dual generalized bent function $f : V_n \rightarrow \mathbb{Z}_{p^k}$ if $p \equiv 3 \pmod{4}$ and n is odd.*

Proof: Let $f : V_n \rightarrow \mathbb{Z}_{p^k}$ be a self-dual generalized bent function. Since f is a generalized bent function whose dual is generalized bent, we have $f^{**}(x) = f(-x)$ by Theorem 3. And since f is self-dual, that is, $f^* = f$, we have $f(x) = f(-x)$. By $f^* = f$ and Proposition 1, we have $\varepsilon_f(0) = -\varepsilon_f(0)$ if $p \equiv 3 \pmod{4}$ and n is odd. Hence $\varepsilon_f(0) = 0$, which contradicts $\varepsilon_f(0) \in \{\pm 1\}$. Therefore, there is no self-dual generalized bent function $f : V_n \rightarrow \mathbb{Z}_{p^k}$ if $p \equiv 3 \pmod{4}$ and n is odd. ■

VI. A SECONDARY CONSTRUCTION OF SELF-DUAL GENERALIZED BENT FUNCTIONS IF

$$p \equiv 1 \pmod{4} \text{ OR } n \text{ IS EVEN}$$

In this section, for $p \equiv 1 \pmod{4}$ or n is even, we give a secondary construction of self-dual generalized bent functions $f : V_n \rightarrow \mathbb{Z}_{p^k}$. First, we give a lemma.

Lemma 4. *Let m be a positive integer and m be even if $p \equiv 3 \pmod{4}$. Let $a \in \mathbb{F}_{p^m}^*$ be an arbitrary element. Let $\alpha, \beta \in \{\pm z^{\frac{m-1}{4}}\}$ where z is a primitive element of \mathbb{F}_{p^m} . Let*

$f_i (i \in \mathbb{F}_p) : V_r \rightarrow \mathbb{Z}_{p^k}$ be generalized bent functions. Let $g_0 : \mathbb{F}_{p^m} \times \mathbb{F}_{p^m} \rightarrow \mathbb{F}_p$ be defined as $g_0(y_1, y_2) = \text{Tr}_1^m(\frac{\beta}{2}(y_1^2 + y_2^2))$, $(y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$. Let $h : \mathbb{F}_{p^m} \rightarrow \mathbb{F}_p$ and $g : \mathbb{F}_p \rightarrow \mathbb{Z}_{p^k}$ be arbitrary functions. Then the function $F(x, y_1, y_2) = f_{h(a\alpha y_1 + ay_2)}(x) + p^{k-1}g_0(y_1, y_2) + g(h(a\alpha y_1 + ay_2))$ is a generalized bent function from $V_r \times \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ to \mathbb{Z}_{p^k} and its dual $F^*(x, y_1, y_2) = f_{h(-\beta(a\alpha y_1 + ay_2))}(x) + p^{k-1}g_0(y_1, y_2) + g(h(-\beta(a\alpha y_1 + ay_2)))$.

Proof: First note that $4 \mid (p^m - 1)$. Indeed, if $p \equiv 1 \pmod{4}$, then $p^m \equiv 1 \pmod{4}$ and if $p \equiv 3 \pmod{4}$ and m is even, then $p^m \equiv (-1)^m \equiv 1 \pmod{4}$, that is, $4 \mid (p^m - 1)$.

By Lemma 2 and (9), we only need to prove that g_0 and g_1 defined by $g_1(y_1, y_2) = g_0(y_1, y_2) - h(a\alpha y_1 + ay_2)$, $(y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$ satisfy the condition of Lemma 2 and $g_0^*(y_1, y_2) = g_0(y_1, y_2)$, $g_1^*(y_1, y_2) = g_0(y_1, y_2) - h(-\beta(a\alpha y_1 + ay_2))$, $(y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$. For any $(b_1, b_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$, we have

$$\begin{aligned} & W_{g_1}(b_1, b_2) \\ &= \sum_{y_1, y_2 \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}_1^m(\frac{\beta}{2}(y_1^2 + y_2^2)) - h(a\alpha y_1 + ay_2) - \text{Tr}_1^m(b_1 y_1 + b_2 y_2)} \\ &= \sum_{z_1, z_2 \in \mathbb{F}_{p^m}} \zeta_p^{\text{Tr}_1^m(-\frac{\alpha\beta}{2a^2} z_1 z_2) - h(z_2) - \text{Tr}_1^m(\frac{b_1 - \alpha b_2}{2a} z_1 + \frac{-\alpha b_1 + b_2}{2a} z_2)} \\ &= W_{\text{Tr}_1^m(-\frac{\alpha\beta}{2a^2} z_1 z_2) - h(z_2)}\left(\frac{b_1 - \alpha b_2}{2a}, \frac{-\alpha b_1 + b_2}{2a}\right) \end{aligned} \tag{19}$$

where in the second equation we use the change of variables $z_1 = ay_1 + a\alpha y_2$, $z_2 = a\alpha y_1 + ay_2$ and $\alpha^2 = -1$. Since $\text{Tr}_1^m(-\frac{\alpha\beta}{2a^2} z_1 z_2) - h(z_2)$ is an Maiorana-McFarland bent function, we have

$$\begin{aligned} & W_{\text{Tr}_1^m(-\frac{\alpha\beta}{2a^2} z_1 z_2) - h(z_2)}\left(\frac{b_1 - \alpha b_2}{2a}, \frac{-\alpha b_1 + b_2}{2a}\right) \\ &= p^m \zeta_p^{\text{Tr}_1^m(\frac{2a^2}{\alpha\beta} \cdot \frac{b_1 - \alpha b_2}{2a} \cdot \frac{-\alpha b_1 + b_2}{2a}) - h(-\frac{2a^2}{\alpha\beta} \cdot \frac{b_1 - \alpha b_2}{2a})} \\ &= p^m \zeta_p^{\text{Tr}_1^m(\frac{\beta}{2}(b_1^2 + b_2^2)) - h(-\beta(a\alpha b_1 + ab_2))} \end{aligned} \tag{20}$$

where in the last equation we use $\alpha^2 = \beta^2 = -1$. Note that h is arbitrary. Hence, by (19) and (20) we have that g_0, g_1 are regular bent functions and $g_0^*(y_1, y_2) = g_0(y_1, y_2)$, $g_1^*(y_1, y_2) = g_0(y_1, y_2) - h(-\beta(a\alpha y_1 + ay_2))$, $(y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$.

By (19) and (20), $(1 - i)g_0(y_1, y_2) + ig_1(y_1, y_2) = g_0(y_1, y_2) - ih(a\alpha y_1 + ay_2)$ is a regular bent function and $((1 - i)g_0 + ig_1)^*(y_1, y_2) = g_0(y_1, y_2) - ih(-\beta(a\alpha y_1 + ay_2))$, $(1 - i)g_0^*(y_1, y_2) + ig_1^*(y_1, y_2) = (1 - i)g_0(y_1, y_2) + ig_0(y_1, y_2) - ih(-\beta(a\alpha y_1 + ay_2)) = g_0(y_1, y_2) - ih(-\beta(a\alpha y_1 + ay_2))$ for any $i \in \mathbb{F}_p$, that is, g_0, g_1 satisfy the condition of Lemma 2. \blacksquare

The following Theorem gives a secondary construction of self-dual generalized bent functions.

Theorem 5. *With the same notation as Lemma 4. The function F constructed by Lemma 4 is a self-dual generalized bent function if any one of the following conditions is satisfied:*

1) $p \equiv 1 \pmod{4}$, $f_i (i \in \mathbb{F}_p)$ are self-dual generalized bent functions satisfying $f_i = f_j$ if $i = j\beta^a$ for some $0 \leq a \leq 3$, $h : \mathbb{F}_{p^m} \rightarrow \mathbb{F}_p$ is defined as $h(x) = \text{Tr}_1^m(x)$, $g : \mathbb{F}_p \rightarrow \mathbb{Z}_{p^k}$ is an arbitrary function satisfying $g(y) = g(y')$ if $y = y'\beta^b$ for some $0 \leq b \leq 3$.

2) $p \equiv 1 \pmod{4}$ or m is even, $f_i (i \in \mathbb{F}_p)$ are self-dual generalized bent functions satisfying $f_i = f_{-i}$, $h : \mathbb{F}_{p^m} \rightarrow \mathbb{F}_p$ is defined as $h(x) = \text{Tr}_1^m(x^2)$, $g : \mathbb{F}_p \rightarrow \mathbb{Z}_{p^k}$ is an arbitrary function satisfying $g(y) = g(-y)$.

3) $p \equiv 1 \pmod{4}$ or m is even, $f_i (i \in \mathbb{F}_p)$ are self-dual generalized bent functions, $h : \mathbb{F}_{p^m} \rightarrow \mathbb{F}_p$ is defined as $h(x) = \text{Tr}_1^m(x^4)$, $g : \mathbb{F}_p \rightarrow \mathbb{Z}_{p^k}$ is an arbitrary function.

Proof: 1) If $p \equiv 1 \pmod{4}$, that is, $4 \mid (p-1)$, we have $\beta^{p-1} = (\pm z^{\frac{p-1}{4}})^{p-1} = (z^{\frac{p-1}{4}})^{p^m-1} = 1$, that is, $\beta \in \mathbb{F}_p$. When $h(x) = \text{Tr}_1^m(x)$, for any $(y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$, $h(-\beta(a\alpha y_1 + ay_2)) = \text{Tr}_1^m(-\beta(a\alpha y_1 + ay_2)) = -\beta \text{Tr}_1^m(a\alpha y_1 + ay_2) = -\beta h(a\alpha y_1 + ay_2)$. Since $f_i (i \in \mathbb{F}_p)$ are self-dual generalized bent functions and $f_i = f_j$ if $i = j\beta^a$ for some $0 \leq a \leq 3$, $\beta^2 = -1$, we have $f_{h(-\beta(a\alpha y_1 + ay_2))}^* = f_{h(-\beta(a\alpha y_1 + ay_2))} = f_{-\beta h(a\alpha y_1 + ay_2)} = f_{h(a\alpha y_1 + ay_2)}$. Since $g(y) = g(y')$ if $y' = y\beta^b$ for some $0 \leq b \leq 3$ and $\beta^2 = -1$, we have $g(h(-\beta(a\alpha y_1 + ay_2))) = g(-\beta h(a\alpha y_1 + ay_2)) = g(h(a\alpha y_1 + ay_2))$. Hence, it is easy to see that the generalized bent function F constructed by Lemma 4 satisfies $F = F^*$, that is, F is a self-dual generalized bent function.

2) When $h(x) = \text{Tr}_1^m(x^2)$, for any $(y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$, since $\beta^2 = -1$, we have $h(-\beta(a\alpha y_1 + ay_2)) = \text{Tr}_1^m((-\beta(a\alpha y_1 + ay_2))^2) = \text{Tr}_1^m(-(a\alpha y_1 + ay_2)^2) = -\text{Tr}_1^m((a\alpha y_1 + ay_2)^2) = -h(a\alpha y_1 + ay_2)$. Since $f_i (i \in \mathbb{F}_p)$ are self-dual generalized bent functions and $f_i = f_{-i}$, we have $f_{h(-\beta(a\alpha y_1 + ay_2))}^* = f_{h(-\beta(a\alpha y_1 + ay_2))} = f_{-h(a\alpha y_1 + ay_2)} = f_{h(a\alpha y_1 + ay_2)}$. Since $g(y) = g(-y)$, we have $g(h(-\beta(a\alpha y_1 + ay_2))) = g(-h(a\alpha y_1 + ay_2)) = g(h(a\alpha y_1 + ay_2))$. Hence, it is easy to see that the generalized bent function F constructed by Lemma 4 satisfies $F = F^*$, that is, F is a self-dual generalized bent function.

3) When $h(x) = \text{Tr}_1^m(x^4)$, for any $(y_1, y_2) \in \mathbb{F}_{p^m} \times \mathbb{F}_{p^m}$, since $\beta^4 = 1$, we have $h(-\beta(a\alpha y_1 + ay_2)) = \text{Tr}_1^m((-\beta(a\alpha y_1 + ay_2))^4) = \text{Tr}_1^m((a\alpha y_1 + ay_2)^4) = h(a\alpha y_1 + ay_2)$. Since $f_i (i \in \mathbb{F}_p)$ are self-dual generalized bent functions, we have $f_{h(-\beta(a\alpha y_1 + ay_2))}^* = f_{h(-\beta(a\alpha y_1 + ay_2))} = f_{h(a\alpha y_1 + ay_2)}$. For an arbitrary function $g : \mathbb{F}_p \rightarrow \mathbb{Z}_{p^k}$, $g(h(-\beta(a\alpha y_1 + ay_2))) = g(h(a\alpha y_1 + ay_2))$. Hence, it

is easy to see that the generalized bent function F constructed by Lemma 4 satisfies $F = F^*$, that is, F is a self-dual generalized bent function. ■

Remark 2. *One can verify that Theorem 3 of [5] is a special case of the above case 1) with $k = 1, m = 1$ and $g = 0$. In [5], for $p \equiv 3 \pmod{4}$ and n is even, the authors only illustrated that there exist ternary non-quadratic self-dual bent functions from V_n to \mathbb{F}_p by considering special ternary bent functions. Theorem 5 can be used to construct non-quadratic self-dual bent functions from V_n to \mathbb{F}_p for $p \equiv 3 \pmod{4}$ and even integer $n \geq 6$ by using (non)-quadratic self-dual bent functions as building blocks.*

We give two examples by using Theorem 5.

Example 3. *Let $p = 5, k = 2, r = 1, m = 1$. Let $\alpha = \beta = 2$ and $a = 1$. Let $f_i(x) = 5x^2, x \in \mathbb{F}_5, i = 0, 2, 3, 4$ and $f_1(x) = 20x^2, x \in \mathbb{F}_5$. Then $f_i (i \in \mathbb{F}_5)$ are self-dual generalized bent functions. Let $h(x) = Tr_1^m(x^4) = x^4, x \in \mathbb{F}_5$. Let $g : \mathbb{F}_5 \rightarrow \mathbb{Z}_{5^2}$ be defined as $g(y) = 2y^2, y \in \mathbb{F}_5$. Then the generalized bent function constructed by Lemma 4 is $F(x, y_1, y_2) = f_{(2y_1+y_2)^4}(x) + 5(y_1^2 + y_2^2) + 2((2y_1 + y_2)^4 \pmod{5})^2, (x, y_1, y_2) \in \mathbb{F}_5^3$ and it is a self-dual generalized bent function according to 3) of Theorem 5.*

Example 4. *Let $p = 7, k = 1, r = m = 2$. Let z be the primitive element of \mathbb{F}_{7^2} with $z^2 + 6z + 3 = 0$. Let $\alpha = \beta = z^{12}$ and $a = z$. Let $f_0(x) = Tr_1^2(4z^{12}x^2), f_1(x) = f_6(x) = Tr_1^2(3z^{12}x^2) + 1, f_2(x) = f_5(x) = Tr_1^2(3z^{12}x^2) + 2, f_3(x) = f_4(x) = Tr_1^2(3z^{12}x^2) + 3, x \in \mathbb{F}_{7^2}$. Then $f_i (i \in \mathbb{F}_7)$ are quadratic self-dual bent functions. Let $h(x) = Tr_1^2(x^2), x \in \mathbb{F}_{7^2}$. Let $g = 0$. Then the bent function constructed by Lemma 4 is $F(x, y_1, y_2) = f_{Tr_1^2((z^{13}y_1 + zy_2)^2)}(x) + Tr_1^2(4z^{12}(y_1^2 + y_2^2)), (x, y_1, y_2) \in \mathbb{F}_{7^2} \times \mathbb{F}_{7^2} \times \mathbb{F}_{7^2}$. It is a self-dual bent function according to 2) of Theorem 5 and it is easy to verify that it is non-quadratic.*

VII. RELATIONS BETWEEN THE DUAL OF GENERALIZED BENT FUNCTIONS AND THE DUAL OF BENT FUNCTIONS

In this section, we characterize the relations between the generalized bentness of the dual of generalized bent functions and the bentness of the dual of bent functions, as well as the self-duality relations between generalized bent functions and bent functions. The main result is the following theorem:

Theorem 6. Let $k \geq 2$ be an integer. Let $f : V_n \rightarrow \mathbb{Z}_{p^k}$ be a generalized bent function with the decomposition $f(x) = \sum_{i=0}^{k-1} f_i(x)p^{k-1-i} = f_0(x)p^{k-1} + \bar{f}_1(x)$ where f_i is a function from V_n to \mathbb{F}_p for any $0 \leq i \leq k-1$ and $\bar{f}_1(x) = \sum_{i=1}^{k-1} f_i(x)p^{k-1-i}$ is a function from V_n to $\mathbb{Z}_{p^{k-1}}$. For any function $F : \mathbb{F}_p^{k-1} \rightarrow \mathbb{F}_p$, define $g_{f,F} = f_0 + F(f_1, \dots, f_{k-1})$. Then

- 1) f^* is generalized bent if and only if for any function $F : \mathbb{F}_p^{k-1} \rightarrow \mathbb{F}_p$, $g_{f,F}^*$ is bent.
- 2) f is self-dual generalized bent if and only if for any function $F : \mathbb{F}_p^{k-1} \rightarrow \mathbb{F}_p$, $g_{f,F}$ is self-dual bent.

Proof: 1) First, by Lemma 1, $g_{f,F}$ is bent for any $F : \mathbb{F}_p^{k-1} \rightarrow \mathbb{F}_p$. And by Lemma 1, we have $f^*(x) = f_0^*(x)p^{k-1} + \lambda(x)$ and $g_{f,F}^*(x) = f_0^*(x) + F(\lambda_1(x), \dots, \lambda_{k-1}(x))$ where $\lambda = \sum_{i=1}^{k-1} \lambda_i p^{k-1-i}$, $\lambda_i : V_n \rightarrow \mathbb{F}_p$ and $\lambda : V_n \rightarrow \mathbb{Z}_{p^{k-1}}$ satisfies that for any $a \in V_n$, $\sum_{x \in V_n: \bar{f}_1(x)=\lambda(a)} \zeta_p^{f_0(x)-\langle a, x \rangle} = W_{f_0}(a)$ and $\sum_{x \in V_n: \bar{f}_1(x)=v} \zeta_p^{f_0(x)-\langle a, x \rangle} = 0$ for any $v \neq \lambda(a)$. Hence, by Lemma 1, f^* is generalized bent if and only if for any function $F : \mathbb{F}_p^{k-1} \rightarrow \mathbb{F}_p$, $g_{f,F}^*$ is bent.

2) Suppose f is self-dual generalized bent, that is, $f = f^*$. By $f = \sum_{i=0}^{k-1} f_i p^{k-1-i}$ and $f^* = f_0^* p^{k-1} + \sum_{i=1}^{k-1} \lambda_i p^{k-1-i}$, we have $f_0 = f_0^*$, $f_i = \lambda_i$, $1 \leq i \leq k-1$. As $g_{f,F}^* = f_0^* + F(\lambda_1, \dots, \lambda_{k-1})$, $g_{f,F} = f_0 + F(f_1, \dots, f_{k-1}) = g_{f,F}^*$, that is, $g_{f,F}$ is self-dual bent for any function $F : \mathbb{F}_p^{k-1} \rightarrow \mathbb{F}_p$. Suppose for any function $F : \mathbb{F}_p^{k-1} \rightarrow \mathbb{F}_p$, $g_{f,F}$ is self-dual bent, that is, $g_{f,F} = g_{f,F}^*$ for any function $F : \mathbb{F}_p^{k-1} \rightarrow \mathbb{F}_p$. Let $F = 0$, we obtain $f_0 = f_0^*$. Let $F = F_i$, $1 \leq i \leq k-1$ where $F_i(x_1, \dots, x_{k-1}) = x_i$, we obtain $f_i = \lambda_i$, $1 \leq i \leq k-1$. Hence, $f^* = f$, that is, f is self-dual generalized bent. ■

By Theorem 6, if f_0^* is not bent (resp., f_0 is not self-dual bent), then obviously f^* is not generalized bent (resp., f is not self-dual generalized bent). But the inverses are not true. We illustrate this with the following two examples.

Example 5. Let z be the primitive element of \mathbb{F}_{3^5} with $z^5 + 2z + 1 = 0$. Let $f : \mathbb{F}_{3^5} \times \mathbb{F}_3 \times \mathbb{F}_3 \rightarrow \mathbb{Z}_{3^2}$ be defined as $f = 3f_0 + f_1$ where $f_i : \mathbb{F}_{3^5} \times \mathbb{F}_3 \times \mathbb{F}_3 \rightarrow \mathbb{F}_3$, $i = 0, 1$, $f_0(x, y_1, y_2) = Tr_1^5(x^2) + (y_1 + Tr_1^5(z^{47}x^2))(y_2 + Tr_1^5(z^{10}x^2))$ and $f_1(x, y_1, y_2) = y_2 + Tr_1^5(z^{10}x^2)$. Then f is a generalized bent function constructed by Theorem 1. One can verify that f^* is not generalized bent, but f_0^* is bent.

Example 6. Let $f : \mathbb{F}_5^2 \rightarrow \mathbb{Z}_{5^2}$ be defined as $f = 5f_0 + f_1$ where $f_0(x_1, x_2) = x_1^2 + x_2^2$, $(x_1, x_2) \in \mathbb{F}_5^2$ and $f_1(x_1, x_2) = 2x_1 + x_2$. Then f is a generalized bent function and $f^* = 5g_0 + g_1$ where

$g_0(x_1, x_2) = x_1^2 + x_2^2, (x_1, x_2) \in \mathbb{F}_5^2$ and $g_1(x_1, x_2) = x_1 + 3x_2, (x_1, x_2) \in \mathbb{F}_5^2$. Hence, f is not self-dual generalized bent, but f_0 is self-dual bent.

VIII. CONCLUSION

In this paper, we study the dual of generalized bent functions $f : V_n \rightarrow \mathbb{Z}_{p^k}$ where V_n is an n -dimensional vector space over \mathbb{F}_p and p is an odd prime, k is a positive integer. We give an explicit construction of generalized bent functions whose dual can be generalized bent or not generalized bent. We show that the generalized indirect sum construction method given in [15] can provide a secondary construction of generalized bent functions for which the dual can be generalized bent or not generalized bent. For generalized bent functions f whose dual f^* is generalized bent, by ideal decomposition in cyclotomic field, we prove $f^{**}(x) = f(-x)$. For generalized bent functions f which satisfy that $f(x) = f(-x)$ and its dual f^* is generalized bent, we give a property and as a consequence, we prove that there is no self-dual generalized bent function from V_n to \mathbb{Z}_{p^k} if $p \equiv 3 \pmod{4}$ and n is odd. For other cases, we give a secondary construction of self-dual generalized bent functions. In the end, we characterize the relations between the generalized bentness of the dual of generalized bent functions and the bentness of the dual of bent functions, as well as the self-duality relations between generalized bent functions and bent functions.

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