

Limitations on Uncloneable Encryption and Simultaneous One-Way-to-Hiding

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Abstract. We study uncloneable quantum encryption schemes for classical messages as recently proposed by Broadbent and Lord [BL20]. We focus on the information-theoretic setting and give several limitations on the structure and security of these schemes: Concretely, 1) We give an explicit cloning-indistinguishable attack that succeeds with probability $\frac{1}{2} + \mu/16$ where μ is related to the largest eigenvalue of the resulting quantum ciphertexts. 2) For a uniform message distribution, we partially characterize the scheme with the minimal success probability for cloning attacks. 3) Under natural symmetry conditions, we prove that the rank of the ciphertext density operators has to grow at least logarithmically in the number of messages to ensure uncloneable security. 4) The *simultaneous* one-way-to-hiding (O2H) lemma is an important technique in recent works on uncloneable encryption and quantum copy protection. We give an explicit example which shatters the hope of reducing the multiplicative “security loss” constant in this lemma to below $9/8$.

1 Introduction

The linearity of quantum mechanics, leading to features like the “quantum no-cloning theorem” and the imperfect distinguishability of non-orthogonal quantum states, has opened up several opportunities in cryptography. In fact, the very first protocols in quantum information processing where cryptographic applications capitalizing on the mentioned features of quantum theory, like Wiesner’s quantum money [Wie83] and the BB84 key exchange protocol [BB84] (for a survey of other cryptographic applications, see [BS16]). Another application of the fact that quantum information cannot be copied, called *uncloneable encryption*, was recently explored by Broadbent and Lord [BL20]: Using a secret key, Alice encrypts a classical message into a *quantum* ciphertext, which is passed to a cloner, Eve, who copies the ciphertext into two quantum registers. These quantum registers are then provided to two separated parties, Bob and Charlie. Alice also provides the secret key to Bob and Charlie who attempt to guess the message. The adversaries Eve, Bob, and Charlie win if and only if Bob and Charlie both correctly decrypt the message. Given a message drawn from some (possibly adversarially chosen) distribution p over a set of size M , the adversaries can always win with the maximum probability that p assigns to any message, by outputting the same fixed message as their guess. The goal is to devise an encryption scheme such that 1) a legitimate receiver can recover the message from the ciphertext and the secret key; 2) the probability that the adversaries win does not significantly exceed the trivial success probability. Following [BL20], we call the latter requirement *uncloneable-security*, which is a strengthening of the confidentiality notion using the same message distribution (e.g. one-wayness if p is the uniform distribution, and IND if p is uniform on an adversarially chosen pair of messages).¹ Studying this notion of security is motivated by cryptographic applications such as quantum money and the prevention of storage attack by classical adversaries² [BL20]. Furthermore, uncloneable encryption is fundamentally related to quantum copy-protection [Aar09, ALL⁺20, ALP20, CMP20].

The adversaries can win with probability one if the ciphertext is classical and zero-error decryption of the ciphertext is possible, by simply copying the classical ciphertext and share copies between Bob

¹ Gottesman studied a different definition of uncloneable encryption called *quantum tamper-detection* [Got03].

² In this scenario, an eavesdropper Eve without quantum memory is forced to immediately measure any intercepted quantum ciphertexts, without knowing the key (yet). Uncloneable security implies that these classical measurement outcomes cannot be used to determine the message even after learning the key, because otherwise, copying these classical outcomes and handing them to Bob and Charlie would violate uncloneable security.

and Charlie. Broadbent and Lord [BL20] constructed two schemes with quantum ciphertexts. First, they studied the scheme in which each bit is randomly encoded in a BB84 basis determined by a secret key. The optimal probability of winning was shown to be $\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)^n$ when the message is n random bits. Second, they constructed a scheme based on a random oracle to which Alice, Bob, and Charlie have quantum access. They prove that when a message is uniformly distributed over a set of size M and the number of queries made by Bob and Charlie is polynomially bounded, the optimal probability of winning is bounded by $\frac{9}{M} + \text{negl}(\lambda)$ where λ is a security parameter.

The security proof of the scheme introduced in [BL20], as well as the security proof of the copy-protection scheme presented in [CMP20], are based on a “simultaneous” variant of the so-called one-way-to-hiding (O2H) lemma, originally introduced by Unruh [Unr15] (see [AHU19,BHH⁺19,KSS⁺20] for variations and improvements). A variant of that result implies that given a quantum algorithm \mathcal{A} having quantum oracle access to a random function $H : \mathcal{X} \rightarrow \mathcal{Y}$, the probability that the algorithm correctly finds $H(x)$ for a fixed input $x \in \mathcal{X}$ is upper-bounded by $\frac{1}{|\mathcal{Y}|} + q\sqrt{p}$ where q is the number of queries made by \mathcal{A} and p is the probability of obtaining x when measuring the input register of the oracle for a randomly chosen query. In the “simultaneous” version of this problem, two non-communicating parties with shared entanglement, run quantum algorithms \mathcal{A} and \mathcal{B} with quantum oracle access to the same random function $H : \mathcal{Y} \rightarrow \mathcal{X}$. In [BL20, Lemma 21], it has been shown that the probability that both algorithms correctly output $H(x)$ for a fixed x is upper-bounded by

$$\frac{9}{|\mathcal{Y}|} + \text{poly}(q_{\mathcal{A}}, q_{\mathcal{B}})\sqrt{p}, \quad (1)$$

where $q_{\mathcal{A}}$ and $q_{\mathcal{B}}$ are the number of queries made by \mathcal{A} and \mathcal{B} , respectively, poly is a polynomial, and p is the probability that measuring the input registers of both algorithms at two independently chosen queries returns x on both sides.

1.1 Our Contributions

In this article, we explore the fundamental information-theoretic limits of uncloneable encryption. In particular, we prove the following four results.

1. When the message is chosen uniformly at random to be either an adversarially chosen message or a default one, we construct in Section 3 an explicit cloning attack with probability of success $\frac{1}{2} + \frac{\mu}{16}$ where μ is the maximum (over all messages) of the average (with respect to the key) of the maximum eigenvalue of the corresponding ciphertext (see Corollary 3.3). This bound implies that in order to have the probability of success for all attacks limited to at most $\frac{1}{2} + \text{negl}(\lambda)$ (as desired in [BL20, Definition 11]), the quantity μ should be negligible in λ .
2. When the message is uniformly distributed over all messages, under some symmetry conditions on the ciphertexts, we show in Section 5 that there exists an attack with the probability of winning $\Omega\left(\sqrt{\frac{\log M}{dM}}\right)$ where M is the number of messages and d is the dimension of the Hilbert space corresponding to ciphertexts (see Theorem 5.1). This lower bound implies that under our symmetry conditions, to ensure that the probability of winning is $O\left(\frac{1}{M}\right)$ for all adversaries (as desired in [BL20, Definition 8]), the rank of the ciphertext density operators has to grow at least as $\Omega(\log M)$.
3. Fixing the number of messages M and the dimension of the ciphertext Hilbert space, we partially characterize the encryption scheme that minimizes the optimal adversarial winning probability when the message is uniformly distributed over all possible messages (Section 4). Our characterization involves an optimization over all probability distributions over a finite set. In addition, we formulate a natural conjecture, that would allow removing the symmetry requirement from 2.
4. Finally, in Section 6, we employ some of the insights from 1. above in the context of the simultaneous O2H lemma. Here, an important open question is whether the factor 9 is an artifact of the proof technique used in [BL20], or whether a probability of success of $\frac{1}{|\mathcal{Y}|} + \text{poly}(q_{\mathcal{A}}, q_{\mathcal{B}})\sqrt{p}$ is, in fact, possible? Put differently, do there actually exist algorithms \mathcal{A} and \mathcal{B} that simultaneously succeed in guessing $H(x)$ with non-trivial probability while not allowing simultaneous extraction of x from their queries?

In this work, we answer the above question. We provide an example with $p = 0$ (so simultaneous query-based extraction never succeeds), $\mathcal{Y} = \{0, 1\}$ but \mathcal{A} and \mathcal{B} both output $H(0)$ with probability $9/16$, which is strictly larger than the trivial $\frac{1}{2}$. This example illustrates that the simultaneous setting is fundamentally different from the single-party setting.

2 Notation and Preliminaries

Let \mathbb{N} denote the set of positive integers. For $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \dots, n\}$.

With some abuse of notation, we denote the Hilbert space corresponding to the quantum system A by A itself. $|A|$ denotes the dimension of A . We also denote the tensor product of A and B by AB . $\mathbf{1}_A$ is the identity operator over A . Let $\mathcal{L}(A)$ denote the set of all linear operators from A to A , $\mathcal{D}(A)$ denote the set of all density operators over A and $\mathcal{U}(A)$ denote the set of all unitary operators over A . A quantum channel from the quantum system A to the quantum system B is a linear trace-preserving completely positive map from $\mathcal{L}(A)$ to $\mathcal{L}(B)$. A positive operator-valued measure (POVM) over quantum system A is a collection $\{P_x\}_{x \in \mathcal{X}}$, where \mathcal{X} is a finite set, P_x is a positive operator in $\mathcal{L}(A)$ for all $x \in \mathcal{X}$, and $\sum_{x \in \mathcal{X}} P_x = \mathbf{1}_A$. For a density matrix $\rho_A \in \mathcal{D}(A)$, $\lambda_{\max}(\rho_A)$ denotes the maximum eigenvalue of ρ_A .

The *uniformly spherical measure* and *Haar measure* are defined over the unit sphere in A and $\mathcal{U}(A)$, respectively, as in [Wat18, Chapter 7].

2.1 Uncloneable Encryption

We recall the definition of an encryption scheme that encrypts a classical message and a classical key to a quantum ciphertext.

Definition 2.1 ([BL20], Definition 4). *A quantum encryption of classical messages scheme (QECM) is a triplet of algorithms $\mathcal{E} = (\text{KeyGen}, \text{Enc}, \text{Dec})$ described as follows:*

- The key generation algorithm KeyGen samples a classical key $k \in \mathcal{K}$ from the key space \mathcal{K} with distribution P_K .
- The encryption algorithm $\text{Enc}_k(m)$ takes as inputs the classical key k and classical message $m \in \mathcal{M}$ from the message set \mathcal{M} and produces a quantum ciphertext $\rho_A \in \mathcal{D}(A)$.
- The decryption algorithm $\text{Dec}_k(\rho_A)$ takes as inputs the classical key k and quantum ciphertext ρ_A and returns the classical message $m \in \mathcal{M}$.

We note that our definition of a QECM differs slightly from [BL20]. In particular, we do not include a security parameter in our definition of a QECM, because we only study information-theoretic security in this article and therefore do not impose any computational assumptions on the adversary. Our results hold for any fixed underlying parameter of the scheme.

Correctness is defined in the natural way.

Definition 2.2. *A QECM $\mathcal{E} = (\text{KeyGen}, \text{Enc}, \text{Dec})$ is (perfectly) correct if for all k produced by KeyGen , and for all messages $m \in \mathcal{M}$, it holds that*

$$\Pr[\text{Dec}_k(\text{Enc}_k(m)) = m] = 1. \tag{2}$$

As in [BL20], we study two flavors of uncloneable security.

Uncloneable Security for Uniformly Distributed Messages A uniformly distributed message is encrypted using a secret key. The adversary then “clones” the ciphertext into two quantum registers and passes each register on to a separate party, called Bob and Charlie. Bob and Charlie then learn the secret key and attempt to individually decrypt the message. They are successful if they simultaneously decrypt the correct value of the message.

Definition 2.3 (Cloning Attack, [BL20], Definition 7). *Let $\mathcal{E} = (\text{KeyGen}, \text{Enc}, \text{Dec})$ be a QECM scheme. A cloning attack against \mathcal{E} is a triple $\mathcal{A} = (\mathcal{N}_{A \rightarrow BC}, \{P_m^k\}, \{Q_m^k\})$ such that*

- The quantum channel $\mathcal{N}_{A \rightarrow BC}$ describes the adversary’s cloning operation.

- For every possible key k , $\{P_m^k\}_{m \in \mathcal{M}}$ is Bob’s POVM on B to guess the message m .
- For every possible key k , $\{Q_m^k\}_{m \in \mathcal{M}}$ is Charlie’s POVM on C to guess the message m .

The success probability of a cloning attack \mathcal{A} against encryption scheme \mathcal{E} with uniform messages is

$$p_{\text{win-unif}}(\mathcal{E}; \mathcal{A}) \triangleq \frac{1}{|\mathcal{M}|} \sum_m \mathbb{E}_{k \leftarrow \text{KeyGen}} (\text{tr} (P_m^k \otimes Q_m^k \mathcal{N}_{A \rightarrow BC}(\text{Enc}_k(m)))) \quad (3)$$

We also define the optimal probability of winning as

$$p_{\text{win-unif}}^*(\mathcal{E}) \triangleq \sup_{\mathcal{A}} p_{\text{win-unif}}(\mathcal{E}; \mathcal{A}). \quad (4)$$

where the supremum is taken over all cloning attacks \mathcal{A} .

Uncloneable-Indistinguishable Security Here, the adversary chooses a message $m_1 \in \mathcal{M}$. The encrypted message is uniformly distributed over the set $\{m_0, m_1\}$ for a fixed $m_0 \in \mathcal{M}$. The rest of the definition of an attack \mathcal{A} is similar to in the previous section.

Definition 2.4 (Cloning-indistinguishability Attack, [BL20], Definition 10). Let $\mathcal{E} = (\text{KeyGen}, \text{Enc}, \text{Dec})$ be a QECM scheme and fix $m_0 \in \mathcal{M}$. A cloning-indistinguishability attack against \mathcal{E} and m_0 is a quadruple $\mathcal{A} = (m_1, \mathcal{N}_{A \rightarrow BC}, \{P_b^k\}, \{Q_b^k\})$ such that

- m_1 is a message in $\mathcal{M} \setminus \{m_0\}$.
- The quantum channel $\mathcal{N}_{A \rightarrow BC}$ describes the adversary’s cloning operation.
- For every possible key k , $\{P_b^k\}_{b \in \{0,1\}}$ is Bob’s POVM on B to guess the message m_b .
- For every possible key k , $\{Q_b^k\}_{b \in \{0,1\}}$ is Charlie’s POVM on C to guess the message m_b .

The success probability of a cloning-indistinguishability attack \mathcal{A} against encryption scheme \mathcal{E} is

$$p_{\text{win-ind}}(\mathcal{E}; \mathcal{A}) \triangleq \frac{1}{2} \sum_{b \in \{0,1\}} \mathbb{E}_{k \leftarrow \text{KeyGen}} (\text{tr} (P_b^k \otimes Q_b^k \mathcal{N}_{A \rightarrow BC}(\text{Enc}_k(m_b))))). \quad (5)$$

We also define the optimal probability of winning as

$$p_{\text{win-ind}}^*(\mathcal{E}) \triangleq \sup_{\mathcal{A}} p_{\text{win-ind}}(\mathcal{E}; \mathcal{A}). \quad (6)$$

3 Simultaneous Guessing and Uncloneable-Indistinguishable Security

We consider the situation where two parties Bob and Charlie have quantum side information about a classical random variable X belonging to set \mathcal{X} , and they simultaneously try to guess X by local measurements.

Definition 3.1. For a classical-quantum-quantum (cqq) state $\rho_{XBC} = \sum_{x \in \mathcal{X}} P_X(x) |x\rangle\langle x| \otimes \rho_{BC}^x$, the simultaneous guessing probability is defined as

$$p_{\text{guess}}(X|B; C)_\rho \triangleq \sup_{\{P_x\}_{x \in \mathcal{X}}, \{Q_x\}_{x \in \mathcal{X}}} \sum_{x \in \mathcal{X}} P_X(x) \text{tr}((P_x \otimes Q_x) \rho_{BC}^x) \quad (7)$$

One of our central objects of study in this article is the following “cloning operation:”

$$V_{A \rightarrow BC} : |\phi\rangle \mapsto \frac{1}{\sqrt{2}} (|\perp\rangle_B \otimes |\phi\rangle_C + |\phi\rangle_B \otimes |\perp\rangle_C), \quad (8)$$

where $|\perp\rangle$ is a unit vector orthogonal to A . Intuitively, $V_{A \rightarrow BC}$ distributes the input state to B and C “in superposition.”

Let ρ, σ be perfectly distinguishable states, for example $\rho = |0\rangle\langle 0|, \sigma = |1\rangle\langle 1|$. We now consider the task where for a random bit $X \in \{0, 1\}$, Bob and Charlie have to simultaneously distinguish the following two cases:

- if $X = 0$: $V\rho V^\dagger$ is handed to Bob and Charlie,
- if $X = 1$: $V\sigma V^\dagger$ is handed to Bob and Charlie.

The following lemma gives a non-trivial lower bound on their simultaneous guessing probability of X for this task. In particular, for pure states like $\rho = |0\rangle\langle 0|$, $\sigma = |1\rangle\langle 1|$, we obtain a lower bound of $\frac{1}{2} + \frac{1}{16} = \frac{9}{16}$. At first sight, it seems counterintuitive that Bob and Charlie are able to succeed with probability strictly higher than $\frac{1}{2}$. One might think that after applying the cloning operation $V_{A \rightarrow BC}$, the state is either with Bob or with Charlie, so the other party will succeed with probability at most $\frac{1}{2}$. However, as one can see from the explicit simultaneous guessing strategy that we construct in the proof of the lemma, Bob and Charlie can exploit the quantum coherence of the state after applying $V_{A \rightarrow BC}$ to achieve a simultaneous guessing probability strictly larger than $\frac{1}{2}$.

Lemma 3.2. *Let $\rho, \sigma \in \mathcal{D}(A)$ such that $\rho\sigma = 0$ and define $\omega_{XBC} \triangleq \frac{1}{2}|0\rangle\langle 0|_X \otimes V\rho V^\dagger + \frac{1}{2}|1\rangle\langle 1|_X \otimes V\sigma V^\dagger$. We have*

$$p_{\text{guess}}(X|B; C)_\omega \geq \frac{1}{2} + \frac{\max(\lambda_{\max}(\rho), \lambda_{\max}(\sigma))}{16} \quad (9)$$

Proof. We consider eigen-decompositions

$$\rho = \sum_{i \in [|A|]} \lambda_i |a_i\rangle\langle a_i|, \quad \sigma = \sum_{i \in [|A|]} \mu_i |b_i\rangle\langle b_i|, \quad (10)$$

such that $\lambda_1 \geq \dots \geq \lambda_{|A|}$ and $\mu_1 \geq \dots \geq \mu_{|A|}$. We set $|\phi\rangle \triangleq \sqrt{1-\alpha}|a_1\rangle + \sqrt{\alpha}|\perp\rangle$ for some parameter $\alpha \in [0, 1]$ (to be determined below) and

$$\Pi \triangleq |\phi\rangle\langle\phi| + \sum_{i=2}^{|A|} |a_i\rangle\langle a_i|. \quad (11)$$

Π is a projector and one can verify the following equalities by straightforward calculations:

$$\langle\perp|\Pi|\perp\rangle = \alpha \quad (12)$$

$$\langle a_1|\Pi|a_1\rangle = 1 - \alpha \quad (13)$$

$$\langle a_1|\Pi|\perp\rangle = \langle\perp|\Pi|a_1\rangle = \sqrt{\alpha(1-\alpha)} \quad (14)$$

$$\langle a_i|\Pi|a_i\rangle = 1 \quad \forall i \in \{2, \dots, |A|\} \text{ such that } \lambda_i > 0 \quad (15)$$

$$\langle a_i|\Pi|\perp\rangle = \langle\perp|\Pi|a_i\rangle = 0 \quad \forall i \in \{2, \dots, |A|\} \text{ such that } \lambda_i > 0. \quad (16)$$

It holds that $\lambda_i \mu_j |\langle a_i|b_j\rangle|^2 = 0$ for all i and j since $\rho\sigma = 0$. Hence $\Pi|b_j\rangle = 0$ for all j with $\mu_j > 0$ and

$$\langle b_j|(\mathbf{1} - \Pi)|b_j\rangle = 1 \quad (17)$$

$$\langle b_j|(\mathbf{1} - \Pi)|\perp\rangle = \langle\perp|(\mathbf{1} - \Pi)|b_j\rangle = 0, \quad (18)$$

for all j with $\mu_j > 0$.

Bob and Charlie both use the POVM $\{\Pi, \mathbf{1} - \Pi\}$ as their local guessing strategies for X . By definition of $p_{\text{guess}}(X|B; C)_\omega$, we have

$$p_{\text{guess}}(X|B; C)_\omega \geq \frac{1}{2} (\text{tr}((\Pi \otimes \Pi)V\rho V^\dagger) + \text{tr}((\mathbf{1} - \Pi) \otimes (\mathbf{1} - \Pi)V\sigma V^\dagger)). \quad (19)$$

The first term on the right-hand side of Eq.(19) is

$$\text{tr}((\Pi \otimes \Pi)V\rho V^\dagger) = \sum_{i \in [|A|]} \lambda_i \text{tr}((\Pi \otimes \Pi)V|a_i\rangle\langle a_i|V^\dagger) \quad (20)$$

$$= \frac{1}{2} \sum_{i \in [|A|]} \lambda_i \text{tr}((\Pi \otimes \Pi)(|a_i\rangle\langle a_i| \otimes |\perp\rangle\langle\perp| + |\perp\rangle\langle\perp| \otimes |a_i\rangle\langle a_i| + |a_i\rangle\langle\perp| \otimes |\perp\rangle\langle a_i| + |\perp\rangle\langle a_i| \otimes |a_i\rangle\langle\perp|)) \quad (21)$$

$$\stackrel{(a)}{=} 2\lambda_1\alpha(1-\alpha) + \alpha \sum_{i=2}^{|A|} \lambda_i \quad (22)$$

$$= \alpha + \lambda_1\alpha(1-2\alpha), \quad (23)$$

where (a) follows by using Eq. (12)-(16). Similarly applying Eq. (17) and (18) yields that

$$\text{tr}(((\mathbf{1} - \Pi) \otimes (\mathbf{1} - \Pi))V\sigma V^\dagger) = 1 - \alpha. \quad (24)$$

Combining Eq. (19), (23), and (24) and setting $\alpha \triangleq 1/4$, we obtain that

$$p_{\text{guess}}(X|B;C)_\omega \geq \frac{1}{2} + \frac{\lambda_1}{16}. \quad (25)$$

Finally, without loss of generality we can assume that $\lambda_1 \geq \mu_1$, and therefore, $\lambda_1 = \max(\lambda_{\max}(\rho), \lambda_{\max}(\sigma))$. \square

Applying the above lemma to the setting of uncloneable-indistinguishable encryption, we obtain the following corollary.

Corollary 3.3. *For any correct (see Definition 2.2) QECCM scheme \mathcal{E} and arbitrary $m_0 \in \mathcal{M}$, there exists an uncloneable-indistinguishable attack \mathcal{A} against \mathcal{E} and m_0 for which it holds that*

$$p_{\text{win-ind}}(\mathcal{E}; \mathcal{A}) \geq \frac{1}{2} + \frac{\max_{m \in \mathcal{M}} \mathbb{E}_{k \leftarrow \text{KeyGen}}(\lambda_{\max}(\text{Enc}_k(m)))}{16} \quad (26)$$

Proof. Let $m_1 \triangleq \text{argmax}_{m \in \mathcal{M} \setminus \{m_0\}} \mathbb{E}_{k \leftarrow \text{KeyGen}}(\lambda_{\max}(\text{Enc}_k(m)))$. We consider the uncloneable-indistinguishable attack $(m_1, V_{A \rightarrow BC}, \{\Pi^k, \mathbf{1} - \Pi^k\}, \{\Pi^k, \mathbf{1} - \Pi^k\})$ where the projector Π^k is Π defined in the proof of Lemma 3.2 for $\rho = \text{Enc}_k(m_0)$ and $\sigma = \text{Enc}_k(m_1)$. The claim then follows directly from the lemma. \square

4 An Optimal Scheme

In this section, we provide a partial answer to the following question: For QECCM schemes with finite message set $\mathcal{M} = [M]$ and a d -dimensional ciphertext space A , which QECCM scheme $\mathcal{E} = (\text{KeyGen}, \text{Enc}, \text{Dec})$ minimizes $p_{\text{win-unif}}^*(\mathcal{E})$? It turns out that the best QECCM schemes in terms of uncloneable security are of a simple form, formally defined in Definition 4.1. Intuitively, the optimal scheme maps every classical message to a completely mixed state over a subspace of A , so that different messages are mapped to states with disjoint support to ensure correctness. The key additionally specifies a Haar-random unitary to hide the message.

For simplicity of the proof, we work with Haar-random unitaries in this section, which is a continuous distribution and results in infinite-sized keys. In practice, one would want to work with finite key sizes and pick the unitary from a suitably chosen two-design instead.

Definition 4.1. *Fix an orthonormal basis $a = (|a_1\rangle, \dots, |a_d\rangle)$ for A and a random variable T taking values in $\mathcal{T} \triangleq \{(t_1, \dots, t_M) \in \mathbb{N}^M : \sum_m t_m \leq d\}$. We define a QECCM scheme $\widetilde{\mathcal{E}}_{a,T} = (\widetilde{\text{KeyGen}}, \widetilde{\text{Enc}}, \widetilde{\text{Dec}})$ as follows. The key generation $\widetilde{\text{KeyGen}}$ outputs a pair $K = (T, U)$ where U is a random unitary over A distributed according to the Haar measure and independent of T . Furthermore, we define*

$$\widetilde{\text{Enc}}_{((t_1, \dots, t_M), u)}(m) \triangleq u \left(\frac{1}{t_m} \sum_{i=\sum_{j=1}^{m-1} t_j}^{\sum_{j=1}^m t_j} |a_i\rangle\langle a_i| \right) u^\dagger. \quad (27)$$

Decryption $\widetilde{\text{Dec}}$ is defined by applying u^\dagger , measuring in basis a and identifying the message m .

The following theorem shows that QECCM schemes of the form above are optimal in terms of uncloneable security for uniform messages.

Theorem 4.2. *For any correct (see Definition 2.2) QECCM scheme $\mathcal{E} = (\text{KeyGen}, \text{Enc}, \text{Dec})$, it holds that*

$$p_{\text{win-unif}}^*(\mathcal{E}) \geq \inf_T p_{\text{win-unif}}^*(\widetilde{\mathcal{E}}_{a,T}), \quad (28)$$

where the infimum is taken over all probability distributions over T and a is an arbitrary orthonormal basis for A .

Proof. We first sketch the proof, which has two steps. First, for the QECM \mathcal{E} , we augment the key with a random unitary U and apply it to the ciphertext in the encryption procedure. This still results in a correct QECM with improved uncloneable security. When U is distributed according to Haar measure, the basis that ciphertexts were initially encoded in will be forgotten, and only their spectrum matters. Second, we apply a random permutation of the eigenvectors of each ciphertext. This extra randomness is not necessary to decrypt the message but makes the spectrum of each ciphertext flat.

We now provide the complete proof. Due to correctness of the QECM \mathcal{E} , the ciphertext density matrices $\text{Enc}_k(1), \dots, \text{Enc}_k(M)$ are mutually orthogonal for a fixed k . Thus, there exists an orthonormal basis $(|e_1\rangle, \dots, |e_d\rangle)$ and real numbers $\{\lambda_{m,i} : m \in \mathcal{M}, i \in [d]\}$ both depending on k such that

$$\text{Enc}_k(m) = \sum_i \lambda_{m,i} |e_i\rangle\langle e_i| \quad (29)$$

and $\lambda_{m,i}\lambda_{m',i} = 0$ for all distinct m, m' and all i . Let $\mathcal{S}_m \triangleq \{i : \lambda_{m,i} > 0\}$. Without loss of generality we can assume that $\mathcal{S}_m = \left\{i : \sum_{j=1}^{m-1} |\mathcal{S}_j| + 1 \leq i \leq \sum_{j=1}^m |\mathcal{S}_j|\right\}$. Additionally, we define $t(k) \triangleq (|\mathcal{S}_1|, \dots, |\mathcal{S}_M|)$ and T as $t(K)$ where $K \leftarrow \text{KeyGen}$. It is sufficient to show that for any orthonormal basis $a = (|a_1\rangle, \dots, |a_d\rangle)$, $\text{P}_{\text{win-unif}}^*(\mathcal{E}) \geq \text{P}_{\text{win-unif}}^*(\widetilde{\mathcal{E}}_{a,T})$ where $\widetilde{\mathcal{E}}_{a,T}$ is defined in Definition 4.1. For a function $g : \mathcal{M} \rightarrow \mathcal{D}(BC)$ (think of it as encryption followed by a cloning operation), we define

$$\Delta(g) \triangleq \sup_{\{P_m\}_{m \in \mathcal{M}}, \{Q_m\}_{m \in \mathcal{M}}} \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} \text{tr}((P_m \otimes Q_m)g(m)), \quad (30)$$

where the supremum is taken over all pairs of POVMs $\{P_m\}_{m \in \mathcal{M}}, \{Q_m\}_{m \in \mathcal{M}}$. We can write

$$\text{P}_{\text{win-unif}}^*(\mathcal{E}) = \sup_{B,C, \mathcal{N}_{A \rightarrow BC}} \mathbb{E}_{k \leftarrow \text{KeyGen}}(\Delta(\mathcal{N}_{A \rightarrow BC} \circ \text{Enc}_k)). \quad (31)$$

For a fixed unitary u , we denote by \mathcal{M}_u the quantum channel mapping ρ to $u\rho u^\dagger$. We have

$$\sup_{B,C, \mathcal{N}_{A \rightarrow BC}} \mathbb{E}_{k \leftarrow \text{KeyGen}}(\Delta(\mathcal{N}_{A \rightarrow BC} \circ \text{Enc}_k)) = \sup_{B,C, \mathcal{N}_{A \rightarrow BC}} \mathbb{E}_{k \leftarrow \text{KeyGen}}(\Delta(\mathcal{N}_{A \rightarrow BC} \circ \mathcal{M}_u \circ \text{Enc}_k)) \quad (32)$$

because $\mathcal{N}_{A \rightarrow BC} \mapsto \mathcal{N}_{A \rightarrow BC} \circ \mathcal{M}_u$ is bijection. Therefore, if U is distributed according to Haar measure over $U(A)$, it holds that

$$\sup_{B,C, \mathcal{N}_{A \rightarrow BC}} \mathbb{E}_{k \leftarrow \text{KeyGen}}(\Delta(\mathcal{N}_{A \rightarrow BC} \circ \text{Enc}_k)) = \mathbb{E}_U \left(\sup_{B,C, \mathcal{N}_{A \rightarrow BC}} \mathbb{E}_{k \leftarrow \text{KeyGen}}(\Delta(\mathcal{N}_{A \rightarrow BC} \circ \mathcal{M}_U \circ \text{Enc}_k)) \right) \quad (33)$$

$$\geq \sup_{B,C, \mathcal{N}_{A \rightarrow BC}} \mathbb{E}_{k \leftarrow \text{KeyGen}}(\mathbb{E}_U(\Delta(\mathcal{N}_{A \rightarrow BC} \circ \mathcal{M}_U \circ \text{Enc}_k))). \quad (34)$$

Let Λ_m be the set of permutations π of $[d]$ such that $\pi(i) = i$ for all $i \notin \mathcal{S}_m$. We denote by V_π the unitary on A corresponding to the permutation π defined by

$$V_\pi |e_i\rangle = |e_{\pi(i)}\rangle. \quad (35)$$

Let π_m be uniformly distributed over Λ_m for all $m \in \mathcal{M}$. For any fixed unitary v and for U distributed according to Haar measure over $U(A)$, Uv is also distributed according to Haar measure. We hence have

$$\mathbb{E}_U(\Delta(\mathcal{N}_{A \rightarrow BC} \circ \mathcal{M}_U \circ \text{Enc}_k)) = \mathbb{E}_{\pi_1, \dots, \pi_M} \left(\mathbb{E}_U \left(\Delta \left(\mathcal{N}_{A \rightarrow BC} \circ \mathcal{M}_{UV_{\pi_1} \dots V_{\pi_M}} \circ \text{Enc}_k \right) \right) \right) \quad (36)$$

$$= \mathbb{E}_{\pi_1, \dots, \pi_M} \left(\mathbb{E}_U \left(\Delta \left(\mathcal{N}_{A \rightarrow BC} \circ \mathcal{M}_U \circ \mathcal{M}_{V_{\pi_1}} \circ \dots \circ \mathcal{M}_{V_{\pi_M}} \circ \text{Enc}_k \right) \right) \right) \quad (37)$$

$$\stackrel{(a)}{\geq} \mathbb{E}_U \left(\Delta \left(\mathcal{N}_{A \rightarrow BC} \circ \mathcal{M}_U \circ \left(\mathbb{E}_{\pi_1, \dots, \pi_M} \left(\mathcal{M}_{V_{\pi_1}} \circ \dots \circ \mathcal{M}_{V_{\pi_M}} \circ \text{Enc}_k \right) \right) \right) \right). \quad (38)$$

where (a) follows from the convexity of $\Delta(\cdot)$. For a fixed $m \in \mathcal{M}$, we have

$$\mathbb{E}_{\pi_1, \dots, \pi_M} \left((\mathcal{M}_{V_{\pi_1}} \circ \dots \circ \mathcal{M}_{V_{\pi_M}} \circ \text{Enc}_k)(m) \right) \stackrel{(b)}{=} \mathbb{E}_{\pi_m} (\mathcal{M}_{\pi_m} (\text{Enc}_k(m))) \quad (39)$$

$$= \mathbb{E}_{\pi_m} \left(\sum_i \lambda_{m,i} V_{\pi_m} |e_i\rangle \langle e_i| V_{\pi_m}^\dagger \right) \quad (40)$$

$$= \mathbb{E}_{\pi_m} \left(\sum_i \lambda_{m,i} |e_{\pi_m(i)}\rangle \langle e_{\pi_m(i)}| \right) \quad (41)$$

$$= \mathbb{E}_{\pi_m} \left(\sum_i \lambda_{m, \pi_m^{-1}(i)} |e_i\rangle \langle e_i| \right) \quad (42)$$

$$= \sum_i \mathbb{E}_{\pi_m} \left(\lambda_{m, \pi_m^{-1}(i)} |e_i\rangle \langle e_i| \right) \quad (43)$$

$$= \sum_{i \in \mathcal{S}_m} \frac{1}{|\mathcal{S}_m|} |e_i\rangle \langle e_i|, \quad (44)$$

where (b) follows since for $m \neq m'$, $\mathcal{M}_{\pi_{m'}}(\rho) = \rho$ for any ρ with support in the span of $\{|e_i\rangle : i \in \mathcal{S}_m\}$. Upon defining unitary v_k as $v_k|a_i\rangle = |e_i\rangle$, $\forall i \in [d]$, we can re-write Eq. (44) as

$$\mathbb{E}_{\pi_1, \dots, \pi_M} \left((\mathcal{M}_{V_{\pi_1}} \circ \dots \circ \mathcal{M}_{V_{\pi_M}} \circ \text{Enc}_k)(m) \right) = v_k \left(\sum_{i \in \mathcal{S}_m} \frac{1}{|\mathcal{S}_m|} |a_i\rangle \langle a_i| \right) v_k^\dagger = \mathcal{M}_{v_k} \left(\sum_{i \in \mathcal{S}_m} \frac{1}{|\mathcal{S}_m|} |a_i\rangle \langle a_i| \right). \quad (45)$$

Combining Eq. (38) and Eq. (45), we obtain

$$\mathbb{E}_U (\Delta (\mathcal{N}_{A \rightarrow BC} \circ \mathcal{M}_U \circ \text{Enc}_k)) \geq \mathbb{E}_U \left(\Delta \left(\mathcal{N}_{A \rightarrow BC} \circ \mathcal{M}_U \circ \mathcal{M}_{v_k} \circ \widehat{\text{Enc}}_{t(k)} \right) \right) \quad (46)$$

$$= \mathbb{E}_U \left(\Delta \left(\mathcal{N}_{A \rightarrow BC} \circ \mathcal{M}_{U v_k} \circ \widehat{\text{Enc}}_{t(k)} \right) \right) \quad (47)$$

$$= \mathbb{E}_U \left(\Delta \left(\mathcal{N}_{A \rightarrow BC} \circ \mathcal{M}_U \circ \widehat{\text{Enc}}_{t(k)} \right) \right), \quad (48)$$

where

$$\widehat{\text{Enc}}_t(m) \triangleq \frac{1}{t_m} \sum_{i=\sum_{j=1}^{m-1} t_j}^{\sum_{j=1}^m t_j} |a_i\rangle \langle a_i|. \quad (49)$$

Putting Eq. (31), Eq. (34), and Eq. (48) together, we have

$$P_{\text{win-unif}}^*(\mathcal{E}) \geq \sup_{B, C, \mathcal{N}_{A \rightarrow BC}} \mathbb{E}_{k \leftarrow \text{KeyGen}} \left(\mathbb{E}_U \left(\Delta \left(\mathcal{N}_{A \rightarrow BC} \circ \mathcal{M}_U \circ \widehat{\text{Enc}}_{t(k)} \right) \right) \right) \quad (50)$$

$$= \sup_{B, C, \mathcal{N}_{A \rightarrow BC}} \mathbb{E}_T \left(\mathbb{E}_U \left(\Delta \left(\mathcal{N}_{A \rightarrow BC} \circ \mathcal{M}_U \circ \widehat{\text{Enc}}_T \right) \right) \right) \quad (51)$$

$$\stackrel{(c)}{=} P_{\text{win-unif}}^* \left(\widetilde{\mathcal{E}}_{a, T} \right) \quad (52)$$

where (c) follows because $\mathcal{M}_u \circ \widehat{\text{Enc}}_t(m) = \widetilde{\text{Enc}}_{(t, u)}(m)$. This completes the proof. \square

We remark that if the decryption algorithm, which can be specified by a POVM on the ciphertext space, is projective, but the QECM scheme is only *approximately correct*, Equation (28) can be proven as well, up to an additive error proportional to the square root of the correctness error. This is the case because such a scheme can be made perfectly correct by modifying the encryption algorithm to prepare the post-measurement state of a successful decryption measurement instead of the original ciphertext, which only differs from the original encryption algorithm up to a small error bounded by the gentle measurement lemma [Win99].

We leave it as an open problem to characterize the optimal probability distribution of T , but we conjecture that a deterministic T that splits the space evenly is optimal.

Conjecture 4.3. Let d and M be such that $d = M\ell$ for $\ell \in \mathbb{N}$. Optimal uncloneable security is achieved for $\mathcal{E}_{a, T}$ when $T = (\ell, \dots, \ell) \in \mathbb{N}^M$ with probability one.

5 Uncloneable Security for Uniformly Distributed Message

We prove a lower bound on $p_{\text{win-unif}}^*(\mathcal{E})$ for QECM schemes \mathcal{E} whose ciphertexts have small rank.

Theorem 5.1. *Let \mathcal{E} be a correct (see Definition 2.2) QECM scheme satisfying the following conditions:*

1. P_K is the uniform distribution over $[\kappa]$ for $\kappa \in \mathbb{N}$.
2. For all k and for all m , $\text{Enc}_k(m) = \frac{\Pi_m^k}{\ell}$ where Π_m^k is a projection and $\ell = |A|/|\mathcal{M}|$

Then there exists an absolute constant $c > 0.151$ such that

$$p_{\text{win-unif}}^*(\mathcal{E}) \geq c \sqrt{\frac{\log |\mathcal{M}|}{|\mathcal{M}| |A|}}. \quad (53)$$

Remark 5.2. Although Assumption 2 in Theorem 5.1 seem restricting, according to Theorem 4.1, we can always assume that $\text{Enc}_k(m) = \frac{\Pi_m^k}{\text{tr}(\Pi_m^k)}$ for projection Π_m^k and for all m and k . Furthermore, if Conjecture 4.3 holds, we can assume that $\text{tr}(\Pi_m^k) = |\mathcal{M}|/|A|$ as required in Theorem 5.1.

The proof of Theorem 5.1 is based on a connection between uncloneable encryption and so-called monogamy of entanglement games (MEG) introduced and studied in [TFKW13]. This connection was used in [BL20] to prove upper-bounds on the success probability of cloning attacks on a QECM. A MEG \mathcal{G} has three players Alice, Bob, and Charlie and is described by a set \mathcal{M} , a random variable K with distribution P_K , a Hilbert space A , and a POVM $\{F_m^k\}_{m \in \mathcal{M}}$ for each realization k of K . Bob and Charlie prepare a tripartite quantum state ρ_{ABC} and pass sub-system A to Alice. Bob and Charlie keep sub-systems B and C , respectively, and no communication is allowed between them after state preparation. Alice performs POVM $\{F_m^k\}_{m \in \mathcal{M}}$ on sub-system A when $K = k$ and provides K to both Bob and Charlie. Bob and Charlie then perform POVMs $\{P_m^k\}_{m \in \mathcal{M}}$ and $\{Q_m^k\}_{m \in \mathcal{M}}$, respectively, on their corresponding sub-system. Bob and Charlie win if all three parties obtain the same outcome from their measurements. The probability of winning is

$$\mathbb{E}_{k \leftarrow P_K} \left(\sum_m \text{tr}(F_m^k \otimes P_m^k \otimes Q_m^k \rho_{ABC}) \right). \quad (54)$$

We denote by $p_{\text{win}}^*(\mathcal{G})$ the maximum probability of winning that Bob and Charlie can achieve for a fixed game \mathcal{G} . In the next two lemmas, we establish lower-bounds on the probability of winning a MEG.

Lemma 5.3. *Let \mathcal{G} be a MEG such that K is distributed uniformly over $[\kappa]$ for an even integer $\kappa \in \mathbb{N}$. We have*

$$p_{\text{win}}^*(\mathcal{G}) \geq \sup_{\mu, |\phi\rangle} \sqrt{\frac{1}{\kappa |\mathcal{M}|} \sum_{k=1}^{\kappa/2} \langle \phi | F_{\mu(k)}^k | \phi \rangle}, \quad (55)$$

where the supremum is over all unit vectors $|\phi\rangle$ and all functions $\mu : [\kappa/2] \rightarrow \mathcal{M}$.

Proof. See Appendix A. □

Lemma 5.4. *Let \mathcal{G} be a MEG such that 1) K takes values in $[\kappa]$ for an even integer $\kappa \in \mathbb{N}$, 2) F_m^k is a projection for all m and k , and 3) $F_m^k F_{m'}^k = 0$ for all distinct m and m' and for all k . We then have*

$$\sup_{\mu, |\phi\rangle} \sum_{k=1}^{\kappa/2} \langle \phi | F_{\mu(k)}^k | \phi \rangle \geq 0.0228 \frac{\kappa \log |\mathcal{M}|}{|A|}, \quad (56)$$

where the supremum is taken over all unit vectors $|\phi\rangle$ and all functions $\mu : [\kappa/2] \rightarrow \mathcal{M}$.

Proof. See Appendix B. □

We are now ready to prove Theorem 5.1. Given a QECM scheme \mathcal{E} satisfying the conditions in Theorem 5.1, we define a MEG \mathcal{G} with the same \mathcal{M} , K , and A . We also fix an orthonormal basis $(|e_1\rangle, \dots, |e_d\rangle)$ and set

$$F_m^k \triangleq (\Pi_m^k)^T = \sum_{i=1}^d \sum_{j=1}^d \langle e_i | \Pi_m^k | e_j \rangle |e_j\rangle \langle e_i|. \quad (57)$$

We define the Choi isomorphism J that maps a quantum channel $\mathcal{N}_{A \rightarrow BC}$ to a density operator

$$\frac{1}{d} \sum_{i=1}^d \sum_{j=1}^d |e_i\rangle \langle e_j| \otimes \mathcal{N}_{A \rightarrow BC}(|e_i\rangle \langle e_j|). \quad (58)$$

Fix Hilbert spaces B , C and POVMs $\{P_m^k\}_{m \in \mathcal{M}}$ and $\{Q_m^k\}_{m \in \mathcal{M}}$ for each k over B and C , respectively. We have

$$\frac{1}{\kappa} \sum_{k \in [\kappa]} \sum_{m \in \mathcal{M}} \text{tr}(F_m^k \otimes P_m^k \otimes Q_m^k J(\mathcal{N}_{A \rightarrow BC})) \quad (59)$$

$$= \frac{1}{\kappa} \sum_{k \in [\kappa]} \sum_{m \in \mathcal{M}} \text{tr} \left(F_m^k \otimes P_m^k \otimes Q_m^k \left(\frac{1}{d} \sum_{i=1}^d \sum_{j=1}^d |e_i\rangle \langle e_j| \otimes \mathcal{N}_{A \rightarrow BC}(|e_i\rangle \langle e_j|) \right) \right) \quad (60)$$

$$= \frac{1}{d\kappa} \sum_{k \in [\kappa]} \sum_{m \in \mathcal{M}} \sum_{i=1}^d \sum_{j=1}^d \langle e_j | F_m^k | e_i \rangle \text{tr}(P_m^k \otimes Q_m^k \mathcal{N}_{A \rightarrow BC}(|e_i\rangle \langle e_j|)) \quad (61)$$

$$= \frac{1}{d\kappa} \sum_{k \in [\kappa]} \sum_{m \in \mathcal{M}} \text{tr}(P_m^k \otimes Q_m^k \mathcal{N}_{A \rightarrow BC}(\Pi_m^k)) \quad (62)$$

$$= \frac{1}{|\mathcal{M}| \kappa} \sum_{m \in \mathcal{M}} \sum_k \text{tr}(P_m^k \otimes Q_m^k \mathcal{N}_{A \rightarrow BC}(\text{Enc}_k(m))). \quad (63)$$

Here we have used Assumption 2. in the last step. Since the Choi isomorphism is a bijection between the set of quantum channels and the set of density operators, we have

$$\sup_{\mathcal{N}_{A \rightarrow BC}} \frac{1}{|\mathcal{M}| \kappa} \sum_{m \in \mathcal{M}} \sum_k \text{tr}(P_m^k \otimes Q_m^k \mathcal{N}_{A \rightarrow BC}(\text{Enc}_k(m))) \quad (64)$$

$$= \sup_{\rho_{ABC}} \frac{1}{\kappa} \sum_k \sum_m \text{tr}(F_m^k \otimes P_m^k \otimes Q_m^k \rho_{ABC}). \quad (65)$$

Therefore $p_{\text{win}}^*(\mathcal{G}) = p_{\text{win-unif}}^*(\mathcal{E})$. Applying Lemma 5.3 and Lemma 5.4, we obtain the result.

6 Counter-Example for Simultaneous O2H

The so-called one-way-to-hiding (O2H) lemma [Unr15] is an important tool in the analysis of the quantum random-oracle model. Informally, it states that if an algorithm has an advantage over random guessing in determining which of two quantum-accessible oracles it has query access to, there is a reduction that outputs an input on which the two oracles differ. The latter process is called *extraction*. In [BL20], a *simultaneous* variant of this lemma has been presented (Lemma 21). In this setting, the starting point is two non-communicating agents that receive (in general entangled) quantum inputs and interact with an oracle. If the two agents simultaneously succeed in producing an output of the oracle corresponding to an input where the two possible oracles differ, then that input can be simultaneously extracted by each agent. In [CMP20], another variant of the simultaneous O2H lemma was shown. Here, like in the single-party O2H lemma, it is only required that the two agents distinguish the two possible oracles. The simultaneous success probability in the distinguishing task, however, needs to be close to 1 to guarantee a non-trivial success probability forextraction. It is an interesting open question whether a simultaneous analogue of Unruh's O2H lemma exists that gives a non-trivial extraction guarantee whenever two agents as described above simultaneously succeed at distinguishing two oracles.

In the following, we provide a counterexample, answering the above question in the negative. More precisely, for a random function $H : \{0, 1\} \rightarrow \{0, 1\}$, we exhibit an input state $|\psi\rangle_{BC}$ and algorithms \mathcal{A}_B and \mathcal{A}_C such that the following holds. *i*) When provided with the registers B , and C , of $|\psi\rangle$, respectively, as input, the two algorithms both output $H(0)$ simultaneously with probability $> 1/2$, and *ii*) the two O2H extractors never succeed simultaneously.

Before presenting the counterexample, let us formally state the simultaneous O2H lemma that is implicitly proven in [CMP20].

Lemma 6.1. *For $L \in \{B, C\}$, let q_L be a nonnegative integer, U_L a unitary and $\{\pi_L^y\}_{y \in \{0,1\}^n}$ be a projective measurement. Let further $|\psi\rangle$ be a unit vector and $x \in \{0, 1\}^\lambda$. Then, we have*

$$\mathbb{E}_H \left\| \Pi^{H(x)} \left((U_B O_B^H)^{q_B} \otimes (U_C O_C^H)^{q_C} \right) |\psi\rangle \right\|^2 \leq \frac{9}{2^n} + (3q_B q_C + 2) q_B q_C \sqrt{M} \quad (66)$$

where $\Pi^{H(x)} = \pi_B^{H(x)} \otimes \pi_C^{H(x)}$ and

$$M = \mathbb{E}_k \mathbb{E}_\ell \mathbb{E}_H \left\| \left(|x\rangle\langle x|_{B_Q} \otimes |x\rangle\langle x|_{C_Q} \right) \left((U_B O_B^H)^k \otimes (U_C O_C^H)^\ell \right) |\psi\rangle \right\|^2 \quad (67)$$

with k, ℓ , and H being uniformly distributed over $\{0, \dots, q_B - 1\}$, $\{0, \dots, q_C - 1\}$, and $\{h : \{0, 1\}^\lambda \rightarrow \{0, 1\}^n\}$, respectively.

The important thing to note here is, that the trivially achievable left-hand side for Equation (66), that does not require querying the oracle, is 2^{-n} . The inequality in Equation (66), however, provides a non-trivial extraction guarantee only once the left-hand side is strictly larger than $9 \cdot 2^{-n}$. This discrepancy also prevents a straight-forward generalization to a full search-to-decision-style simultaneous O2H lemma, and causes the weak bound in the non-trivial generalization in this direction presented in [CMP20]. Note also that the above lemma can be generalized to apply to algorithms that guess only a function of the output $H(x)$, see Lemma 19 in [CMP20]. Therefore, a version of the above theorem with the factor 9 replaced by a constant $c < 2$ would directly imply a search-to-decision variant.

We continue to present our counterexample that shows that Lemma 6.1 has no chance of being true when replacing the factor 9 with any constant $c < \frac{9}{8} = 1.125$.

Theorem 6.2. *For $n = 1$ and $q_B = q_C = 1$, there exist U_L and π_L for $L \in \{B, C\}$ such that, with the notation from Lemma 6.1,*

$$\mathbb{E}_H \left\| \Pi^{H(x)} (U_B O_B^H \otimes U_C O_C^H) |\psi\rangle \right\|^2 = \frac{9}{16}, \quad (68)$$

but $M = 0$.

Proof. Let the registers B and C have two qubits each, where the first one corresponds to the input to H , the output of H will be XORred to the second qubit. Define $|\psi\rangle_{BC} = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle_B \otimes |1\rangle|+\rangle_C + |1\rangle|+\rangle_B \otimes |0\rangle|0\rangle_C)$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and set $U_B = U_C = \mathbf{1}$. After the two queries, the joint state is

$$|\psi_1\rangle_{BC} = (O_B^H \otimes O_C^H) |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|H(0)\rangle_B \otimes |1\rangle|+\rangle_C + |1\rangle|+\rangle_B \otimes |0\rangle|H(0)\rangle_C). \quad (69)$$

Setting $|\phi\rangle = |0\rangle|H(0)\rangle$ and $|\perp\rangle = |1\rangle|+\rangle$, we observe that

$$|\psi_1\rangle_{BC} = V_{A \rightarrow BC} |\phi\rangle_A \quad (70)$$

where $V_{A \rightarrow BC}$ was defined in Equation (8). We employ Lemma 3.2 with $\rho = |00\rangle\langle 00|$ and $\sigma = |01\rangle\langle 01|$ to conclude that there exist π_L^b for $L \in \{B, C\}$, $b = 0, 1$ such that

$$\mathbb{E}_H \left\| \Pi^{H(0)} (U_B O_B^H \otimes U_C O_C^H) |\psi\rangle \right\|^2 = \frac{1}{2} + \frac{1}{16} = \frac{9}{16}. \quad (71)$$

However, the extraction measurement never succeeds. Indeed, there is only one query to “choose from” on each side, and the computational basis measurement of the first qubits of B and C of the query input state $|\psi\rangle_{BC}$ defined above never returns $(0, 0)$, hence $M = 0$. \square

7 Open Problems

One major question is whether there exists a sequence of QECCMs $\{\mathcal{E}_\lambda\}_{\lambda \in \mathbb{N}}$ such that

$$\lim_{\lambda \rightarrow \infty} p_{\text{win-ind}}^*(\mathcal{E}_\lambda) = \frac{1}{2} \text{ or } \lim_{\lambda \rightarrow \infty} |\mathcal{M}_\lambda| p_{\text{win-unif}}^*(\mathcal{E}_\lambda) = 1. \quad (72)$$

Analyzing the performance of the scheme defined in Definition 4.1 might answer this question. Another open question is the validity of Conjecture 4.3, which simplifies the characterization of the optimal scheme and justifies the assumptions that we made in Theorem 5.1.

Regarding simultaneous O2H, while our results show that the constant 9 on the right-hand side of Eq. (66) cannot be replaced by 1, finding the optimal constant could be of interest both for uncloneable encryption and quantum copy-protection applications. Extensions of simultaneous O2H to more than two parties are natural to study and might also have natural applications in quantum copy protection.

Acknowledgements

We would like to thank Michael Walter for useful discussions. CM was funded by a NWO VENI grant (Project No. VI.Veni.192.159). MT and CS were supported by a NWO VIDI grant (Project No. 639.022.519).

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A Proof of Lemma 5.3

In this section, we prove Lemma 5.3, which we recall for the reader's convenience.

Lemma A.1. *Let \mathcal{G} be a MEG such that K is distributed uniformly over $[\kappa]$ for an even integer $\kappa \in \mathbb{N}$. We have*

$$p_{\text{win}}^*(\mathcal{G}) \geq \sup_{\mu, |\phi\rangle} \sqrt{\frac{1}{\kappa |\mathcal{M}|} \sum_{k=1}^{\kappa/2} \langle \phi | F_{\mu^{(k)}}^k | \phi \rangle}, \quad (55)$$

where the supremum is over all unit vectors $|\phi\rangle$ and all functions $\mu : [\kappa/2] \rightarrow \mathcal{M}$.

We consider Hilbert spaces B and C spanned by orthonormal bases $\{|m\rangle : m \in \mathcal{M}\}$. We also set $P_m^k = Q_m^k = |m\rangle\langle m|$. We then have

$$p_{\text{win}}^*(\mathcal{G}) \geq \sup_{\rho_{ABC}} \sum_k \frac{1}{\kappa} \sum_m \text{tr}(F_m^k \otimes |m\rangle\langle m| \otimes |m\rangle\langle m| \rho_{ABC}) \quad (73)$$

$$= \frac{1}{\kappa} \left\| \sum_k \sum_m F_m^k \otimes |m\rangle\langle m| \otimes |m\rangle\langle m| \right\| \quad (74)$$

$$= \frac{1}{\kappa} \max_m \left\| \sum_k F_m^k \right\|. \quad (75)$$

Now note that permuting $\{F_m^k : m \in \mathcal{M}\}$ depending on k does not change $p_{\text{win}}^*(\mathcal{G})$. Therefore, we have

$$p_{\text{win}}^*(\mathcal{G}) \geq \max_{\mu: \mathcal{K} \rightarrow \mathcal{M}} \frac{1}{\kappa} \left\| \sum_k F_{\mu^{(k)}}^k \right\|. \quad (76)$$

We then have

$$\left\| \sum_k F_{\mu^{(k)}}^k \right\|^2 \geq \left\| \sum_k F_{\mu^{(k)}}^k | \phi \rangle \right\|^2 \quad (77)$$

$$= \left\| \sum_{k=1}^{\kappa/2} F_{\mu^{(k)}}^k | \phi \rangle \right\|^2 + \sum_{k=\kappa/2+1}^{\kappa} \left(\left\| F_{\mu^{(k)}}^k | \phi \rangle \right\|^2 + 2\text{Re} \left\langle \phi \left| \left(F_{\mu^{(k)}}^k \right)^\dagger \left(\sum_{j=1}^{k-1} F_{\mu^{(j)}}^j \right) \right| \phi \right\rangle \right) \quad (78)$$

$$\geq 2 \sum_{k=\kappa/2+1}^{\kappa} \text{Re} \left\langle \phi \left| \left(F_{\mu^{(k)}}^k \right)^\dagger \left(\sum_{j=1}^{k-1} F_{\mu^{(j)}}^j \right) \right| \phi \right\rangle \quad (79)$$

Let M be a uniform random variable on \mathcal{M} . For all $k \geq \kappa/2 + 1$ and all $\mu : [\kappa] \rightarrow \mathcal{M}$, we have

$$\mathbb{E} \left[\text{Re} \left\langle \phi \left| \left(F_M^k \right)^\dagger \left(\sum_{j=1}^{k-1} F_{\mu^{(j)}}^j \right) \right| \phi \right\rangle \right] = \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} \text{Re} \left\langle \phi \left| \left(F_m^k \right)^\dagger \left(\sum_{j=1}^{k-1} F_{\mu^{(j)}}^j \right) \right| \phi \right\rangle \quad (80)$$

$$= \frac{1}{|\mathcal{M}|} \text{Re} \left\langle \phi \left| \left(\sum_{j=1}^{k-1} F_{\mu^{(j)}}^j \right) \right| \phi \right\rangle \quad (81)$$

$$= \frac{1}{|\mathcal{M}|} \sum_{j=1}^{k-1} \langle \phi | F_{\mu^{(j)}}^j | \phi \rangle \quad (82)$$

$$\geq \frac{1}{|\mathcal{M}|} \sum_{j=1}^{\kappa/2} \langle \phi | F_{\mu^{(j)}}^j | \phi \rangle. \quad (83)$$

It follows that for all $k \geq \kappa/2 + 1$ there exists $\mu(k)$ such that

$$\operatorname{Re} \left\{ \langle \phi | (F_{\mu(k)}^k)^\dagger \left(\sum_{j=1}^{k-1} F_{\mu(j)}^j \right) | \phi \rangle \right\} \geq \frac{1}{|\mathcal{M}|} \sum_{j=1}^{\kappa/2} \langle \phi | F_{\mu(j)}^j | \phi \rangle. \quad (84)$$

and thus

$$2 \sum_{k=\kappa/2+1}^{\kappa} \operatorname{Re} \left\{ \langle \phi | (F_{\mu(k)}^k)^\dagger \left(\sum_{j=1}^{k-1} F_{\mu(j)}^j \right) | \phi \rangle \right\} \geq \frac{\kappa}{|\mathcal{M}|} \sum_{j=1}^{\kappa/2} \langle \phi | F_{\mu(j)}^j | \phi \rangle. \quad (85)$$

Combining Eq. (76), Eq. (79), and Eq. (85) yields the result.

B Proof of Lemma 5.4

In this section, we prove Lemma 5.4, which we recall for the reader's convenience.

Lemma B.1. *Let \mathcal{G} be a MEG such that 1) K takes values in $[\kappa]$ for an even integer $\kappa \in \mathbb{N}$, 2) F_m^k is a projection for all m and k , and 3) $F_m^k F_{m'}^k = 0$ for all distinct m and m' and for all k . We then have*

$$\sup_{\mu, |\phi\rangle} \sum_{k=1}^{\kappa/2} \langle \phi | F_{\mu(k)}^k | \phi \rangle \geq 0.0228 \frac{\kappa \log |\mathcal{M}|}{|A|}, \quad (86)$$

where the supremum is taken over all unit vectors $|\phi\rangle$ and all functions $\mu : [\kappa/2] \rightarrow \mathcal{M}$.

Proof. We assume for simplicity that $\mathcal{M} = [M]$ and $d \triangleq |A|$. Let $|\psi\rangle$ be distributed according to uniformly spherical measure on the unit sphere in A . We have

$$\sup_{\mu, |\phi\rangle} \sum_{k=1}^{\kappa/2} \langle \phi | F_{\mu(k)}^k | \phi \rangle \geq \mathbb{E}_{|\psi\rangle} \left(\max_{\mu(1), \dots, \mu(\kappa/2)} \sum_{k=1}^{\kappa/2} \|F_{\mu(k)}^k |\psi\rangle\|^2 \right) \quad (86)$$

$$= \mathbb{E}_{|\psi\rangle} \left(\sum_{k=1}^{\kappa/2} \max_{\mu(k)} \|F_{\mu(k)}^k |\psi\rangle\|^2 \right) \quad (87)$$

$$= \sum_{k=1}^{\kappa/2} \mathbb{E}_{|\psi\rangle} \left(\max_m \|F_m^k |\psi\rangle\|^2 \right). \quad (88)$$

Hence, it is enough to show that $\mathbb{E}_{|\psi\rangle} \left(\max_m \|F_m^k |\psi\rangle\|^2 \right) \geq 0.0457 \frac{\log M}{d}$ for any fixed $k \in [\kappa/2]$. By our assumption on POVM $\{F_m^k\}_{m \in \mathcal{M}}$, there exists an orthonormal basis $(|e_1\rangle, \dots, |e_d\rangle)$ such that $F_m^k = \sum_{i \in \mathcal{I}_m} |e_i\rangle\langle e_i|$ where $(\mathcal{I}_1, \dots, \mathcal{I}_M)$ forms a partition of $[d]$. Let (a_1, \dots, a_d) and (b_1, \dots, b_d) be two independent sequences of iid standard normal random variables. Define

$$|\tilde{\psi}\rangle = \sum_i (a_i + jb_i) |e_i\rangle \quad (89)$$

$$|\psi\rangle = |\tilde{\psi}\rangle / \|\tilde{\psi}\rangle\| \quad (90)$$

Then, $|\psi\rangle$ is distributed according to uniformly spherical measure. We also have

$$\|F_m^k |\psi\rangle\|^2 = \frac{\sum_{i \in \mathcal{I}_m} (a_i^2 + b_i^2)}{\sum_{i=1}^d (a_i^2 + b_i^2)} \quad (91)$$

Let (X_1, \dots, X_M) be independent random variables such that $X_m \sim \text{Erlang}(\text{tr}(F_m^k), 1/2)$ (See Appendix C for the definition and properties of Erlang distribution). We then have

$$\mathbb{E}_{|\psi\rangle} \left(\max_m \|F_m^k |\psi\rangle\|^2 \right) = \mathbb{E} \left(\frac{\max_{m \in \mathcal{M}} X_m}{\sum_{m' \in \mathcal{M}} X_{m'}} \right) \quad (92)$$

$$\stackrel{(a)}{\geq} 0.0457 \frac{\log M}{\sum_{m=1}^M \text{tr}(F_m^k)} \quad (93)$$

$$= 0.0457 \frac{\log M}{d}, \quad (94)$$

where (a) follows from Lemma C.2. \square

C The Erlang distribution

In this appendix, we gather some properties of the [Erlang distribution](#) used in our work.

Definition C.1. *Erlang(k, λ) is a probability distribution over $[0, \infty)$ characterized by two parameters $k \in \mathcal{N}$ and $\lambda > 0$ and defined by its probability density function as*

$$f(x) \triangleq \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}. \quad (95)$$

When $X \sim \text{Erlang}(k, \lambda)$, we have

$$\mathbb{E}(X) = k/\lambda \quad (96)$$

$$\Pr[X \leq x] = 1 - \sum_{i=0}^{k-1} \frac{e^{-\lambda x} (\lambda x)^i}{i!}. \quad (97)$$

Moreover, if X_1, \dots, X_{2k} are independent standard normal random variables, then $\sum_{i=1}^{2k} |X_i|^2$ has Erlang($k, \frac{1}{2}$) distribution.

Lemma C.2. *Let X_1, \dots, X_n be independent such that $X_i \sim \text{Erlang}(k_i, \lambda)$. We have*

$$\mathbb{E} \left(\frac{\max_{i \in [n]} X_i}{\sum_{i=1}^n X_i} \right) \geq c \frac{\log n}{\sum_{i=1}^n k_i}. \quad (98)$$

for some absolute constant $c > \frac{1-e^{-1}-0.5}{2 \log(e)} \approx 0.0457$.

Proof. We have for all $a, b > 0$

$$\mathbb{E} \left(\frac{\max_{i \in [n]} X_i}{\sum_{i=1}^n X_i} \right) \geq \frac{a}{b} \Pr \left[\max_{i \in [n]} X_i \geq a \text{ and } \sum_{i=1}^n X_i \leq b \right] \quad (99)$$

$$\stackrel{(i)}{\geq} \frac{a}{b} \left(1 - \Pr \left[\max_{i \in [n]} X_i < a \right] - \Pr \left[\sum_{i=1}^n X_i > b \right] \right) \quad (100)$$

$$\stackrel{(ii)}{\geq} \frac{a}{b} \left(1 - \Pr \left[\max_{i \in [n]} X_i < a \right] - \frac{\sum_{i=1}^n \mathbb{E}(X_i)}{b} \right) \quad (101)$$

$$= \frac{a}{b} \left(1 - \Pr \left[\max_{i \in [n]} X_i < a \right] - \frac{\sum_{i=1}^n k_i}{b\lambda} \right) \quad (102)$$

where (i) follows from the union bound and (ii) follows from Markov's inequality. Furthermore, by the independence of X_1, \dots, X_n , we have

$$\Pr \left[\max_{i \in [n]} X_i < a \right] = \prod_{i=1}^n \Pr[X_i < a] \quad (103)$$

$$= \prod_{i=1}^n \left(1 - e^{-\lambda a} \sum_{j=0}^{k_i-1} \frac{(\lambda a)^j}{j!} \right) \quad (104)$$

$$\leq (1 - e^{-\lambda a})^n \quad (105)$$

$$= e^{\ln(1 - e^{-\lambda a})n} \quad (106)$$

$$\leq e^{-e^{-\lambda a}n}. \quad (107)$$

Setting $a = \ln n / \lambda$ and $b = \frac{2 \sum_i k_i}{\lambda}$, we obtain that

$$\mathbb{E} \left(\frac{\max_{i \in [n]} X_i}{\sum_{i=1}^n X_i} \right) \geq \frac{1 - e^{-1} - 0.5 \ln n}{2 \sum k_i} \quad (108)$$

$$= \frac{1 - e^{-1} - 0.5 \log n}{2 \log(e) \sum k_i}. \quad (109)$$

□