

# Invariants for EA- and CCZ-equivalence of APN and AB functions

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## Abstract

An  $(n, m)$ -function is a mapping from  $\mathbb{F}_2^n$  to  $\mathbb{F}_2^m$ . Such functions have numerous applications across mathematics and computer science, and in particular are used as building blocks of block ciphers in symmetric cryptography. The classes of APN and AB functions have been identified as cryptographically optimal with respect to providing resistance against two of the most powerful known cryptanalytic attacks, namely differential and linear cryptanalysis. The classes of APN and AB functions are directly related to optimal objects in many other branches of mathematics, and have been a subject of intense study since at least the early 90's. Finding new constructions of these functions is hard; one of the most significant practical issues is that any tentatively new function must be proven inequivalent to all the known ones. Testing equivalence can be significantly simplified by computing invariants, i.e. properties that are preserved by the appropriate equivalence relation. In this paper, we survey the known invariants for CCZ- and EA-equivalence, with a particular focus on their utility in distinguishing between inequivalent instances of APN and AB functions. We evaluate each invariant with respect to how easy it is to implement in practice, how efficiently it can be calculated on a computer, and how well it can distinguish between distinct EA- and CCZ-equivalence classes.

## 1 Introduction

A vectorial Boolean function, or  $(n, m)$ -function, is any mapping from the vector space  $\mathbb{F}_2^n$  over the finite field  $\mathbb{F}_2 = \{0, 1\}$ , to the vector space  $\mathbb{F}_2^m$ , where  $n$  and  $m$  are arbitrary natural numbers. In the particular case when  $m = 1$ , we refer to  $(n, 1)$ -functions simply as Boolean functions. In this sense, an  $(n, m)$ -function can be seen as an  $m$ -dimensional vector of  $(n, 1)$ -functions, hence the name. Vectorial Boolean functions are natural objects that have many applications within computer science and mathematics. Perhaps the simplest way to appreciate their utility is to observe that an  $(n, m)$ -function can be interpreted as an operation that accepts  $n$  bits as input, and returns  $m$  bits as output. Since virtually all data can be encoded as sequences of bits, this means that any transformation that operates on data of any kind can be naturally represented in terms of Boolean functions and vectorial Boolean functions. In particular, this is how all data is represented on an electronic computer; and one need look no further than the indicator function of a subset, or the incidence matrix of a graph to find examples of mathematical structures that can be encoded

using binary functions. The spectrum of applications of vectorial Boolean functions includes areas as diverse as set theory, artificial intelligence, and algorithm design.

Vectorial Boolean functions also play a crucial role in cryptography, where they are used as building blocks of both stream and block ciphers. In both cases, the properties of the functions used directly influence the ultimate strength of the encryption; and thus, investigating the cryptographic properties of  $(n, m)$ -functions, and finding concrete instances of such functions that provide good resistance against various cryptanalytic attacks is an important topic of research. A prominent example is the Rijndael block cipher, selected as the Advanced Encryption Standard (AES) by the US National Institute of Standards and Technology (NIST), which is one of the most secure and easily the most popular block cipher to date [27, 28]. One of the major factors contributing to the security of Rijndael is a strong vectorial Boolean function that lies at the core of the encryption. In general, if a particular form of cryptanalysis succeeds because of a cryptographically weak function, this is due to the function having some undesirable property that can be exploited by the attacker. Researchers have identified several properties and statistics that measure the resistance of a function to different kinds of attack. This quantification of the cryptographic strength of vectorial Boolean functions has the great advantage that it allows the security of a given function to be measured in an objective and systematic way; and, what is of equal importance, provides researchers in the field with a concrete goal, namely to find functions attaining the optimum value of these properties.

Two of the most powerful attacks against modern block ciphers are the so-called differential cryptanalysis [5] and linear cryptanalysis [40]. The statistics that measure the resistance of a function  $F$  to these two attacks are called differential uniformity (denoted  $\Delta_F$ ) and nonlinearity (denoted  $\mathcal{NL}(F)$ ), respectively. The differential uniformity of a function should be as low as possible, and the nonlinearity should be as high as possible in order to resist these two attacks. In the case of  $(n, m)$ -functions with  $n = m$  (which is arguably the most practically significant and, therefore, well-studied case), the classes of  $(n, n)$ -functions that achieve optimum differential uniformity and nonlinearity are called almost perfect nonlinear (APN) and almost bent (AB) functions, respectively. These two classes were introduced in the early 90's [3, 41, 42], and have been the subject of intense study ever since. The fact that these functions are cryptographically optimal implies that they have very little structure or patterns that can be exploited by a malicious attacker; unfortunately, this also makes the study and construction of such functions difficult, both in theory and in practice. This is witnessed by the fact that a number of long-standing questions and problems on APN and AB functions remain open to this day. The reader is referred to [23] for a comprehensive overview of the background and results in the area, including a list of open problems.

One promising vector of attack for resolving some of these problems is to find new instances of APN and AB functions. In this respect, it must be noted that any AB function is necessarily APN; the converse does not hold in general, although any quadratic APN function over  $\mathbb{F}_{2^n}$  with odd  $n$  is AB [24]. This means that AB functions provide the optimum resistance to both differential and linear cryptanalysis; and that searching for APN function is a natural way to search for AB functions as well. The number of  $(n, n)$ -functions is astronom-

ical even for small values of  $n$ : there are  $(2^n)^{2^n}$  such functions for any natural number  $n$ , and so for values as small as  $n = 6$ , an exhaustive search is out of the question at the current level of computational technology. On the one hand, this makes it necessary to use more sophisticated combinations of mathematical characterizations and computational methods in order to find new APN functions; on the other hand, the large number of  $(n, n)$ -functions suggests that the total number of APN functions (despite them being a rather special class of functions) also grows exponentially with the dimension  $n$ , and their enumeration and classification quickly become infeasible.

To make the classification of some given class of objects manageable, a typical approach is to only classify them up to some suitable notion of equivalence that preserves the properties of interest to our study. In the case of APN and AB functions, these are the differential uniformity and the nonlinearity. At present, there are several known notions of equivalence on vectorial Boolean functions that preserve both of these properties. The most general known relation of this type is called Carlet-Charpin-Zinoviev equivalence (after the names of its inventors) [24], or CCZ-equivalence for short; results on APN and AB functions in the literature are typically given up to CCZ-equivalence. A less general, but also very frequently used equivalence relation is the extended affine equivalence, or EA-equivalence for short. Although it is known to be strictly less general than CCZ-equivalence (even if we allow taking inverses of permutations in addition to EA-equivalence) [19], it has been shown that CCZ- and EA-equivalence coincide in the case of quadratic functions [48]; since the vast majority of known APN functions are quadratic, this makes the study of EA-equivalence and its properties almost as useful as those of CCZ-equivalence in practice.

Classifying functions up to CCZ-equivalence (or EA-equivalence) dispenses, at least partly, with the problem of their overwhelming number, but it raises another issue, namely: how to prove the equivalence or inequivalence of two given functions. This is a matter of great practical importance in the study of APN functions, since any “new” function obtained from a computational search or theoretical construction needs to be compared for equivalence against representatives from the equivalence classes of all known APN functions; this “new” function is then genuinely new only in the case that it is not equivalent to any of these representatives. Despite the definitions of both EA-equivalence and CCZ-equivalence being natural and simple, testing whether two given functions are equivalent is a very hard problem. Showing the inequivalence of two functions theoretically is only possible in some special cases, and is still very laborious; see e.g. [16], which contains a theoretical proof of inequivalence to the Gold, Kasami, and inverse power APN functions of an infinite family of APN binomials. Computationally testing equivalence is quite difficult too: to the best of our knowledge, there is currently no algorithm that can decide the CCZ-equivalence of any two functions directly from the definition; instead, one typically uses tests relying on isomorphism of linear codes [34, 10]. These tests have a number of shortcomings: they are difficult to implement (provided one does not have a working implementation of a test for linear code isomorphism); the computational time and memory consumption is significant; and, at least in the case of some implementations, these tests can give false negatives. Despite this, the linear code test remains the only option for testing CCZ-equivalence in the general case. Other algorithms, operating from first principles, have been published for dealing with specialized cases of CCZ- and EA-equivalence, such

as in the case of affine and linear equivalence [6], and the so-called restricted EA-equivalence [21, 43]; and, more recently, an algorithm relying on invariants for testing EA-equivalence for even dimensions [37].

The classification of functions into equivalence classes can be significantly simplified and sped-up by means of invariants. An invariant is a property or statistic that is preserved under a given equivalence relation. For instance, if  $p$  is an integer-valued statistic computable for any  $(n, n)$ -function  $F$ , and is also a CCZ-invariant, then for any  $F, G : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$  that are CCZ-equivalent, we must have  $p(F) = p(G)$ . Invariants can facilitate the classification in several ways. Suppose that a tentative new instance of an APN function is obtained via some construction or computer search. First, the values of different invariants for a potential new function can be computed, and compared with those values for representatives from the known classes of APN functions. If the set of values for the new function does not match that of any known representative, then we can immediately conclude that the discovered function is indeed new, and no further tests need to be performed. If, however, the set of values does coincide with that of one or more of the known representatives, then only the representatives with that exact same set of values need to be tested for equivalence against the newly discovered function; this will typically be a significantly smaller set of functions. Most invariants are numerical values, which makes it easier to verify that the computation has produced a meaningful result, and precludes the possibility of false positive or false negatives. Finally, some invariants have a natural interpretation, and describe some property or structure of the function, which can be significant and useful in contexts other than deciding equivalence. Constantly updated tables of the known invariants for all known CCZ-inequivalent APN representatives can be found online at [1].

In this paper, we survey the known invariants for  $(n, n)$ -functions, with an emphasis on their utility for the classification of APN and AB functions. We evaluate each invariant with respect to several desirable properties. Ideally, we would like an invariant to be:

- simple: in other words, that it shouldn't require any complicated algorithms or special software for its computation; ideally, an invariant would be easily implementable on any general purpose programming language without specialized tools or knowledge;
- efficient: that the invariant can be computed quickly, and without using too much memory for a reasonable range of dimensions  $n$ ;
- useful: that it should take many different values for APN functions from distinct equivalence classes.

## 2 Preliminaries

Let  $n, m$  be natural numbers. An  $(n, m)$ -**function**, or **vectorial Boolean function** is a mapping between the vector spaces  $\mathbb{F}_2^n$  and  $\mathbb{F}_2^m$  over the finite field with two elements,  $\mathbb{F}_2$ . In the following, we concentrate on the case  $n = m$ , but we remark in passing that  $(n, 1)$ -functions are called simply **Boolean functions** (as opposed to vectorial Boolean functions). An  $(n, m)$ -function  $F$  can be seen as a vector  $F = (f_1, f_2, \dots, f_m)$  of Boolean  $(n, 1)$ -functions, each function  $f_i(x)$

$x$	$F(x)$
000	000
001	101
010	110
011	111
100	100
101	011
110	001
111	010

Table 1: Truth table of a  $(3, 3)$ -function

giving the value of the  $i$ -th coordinate of  $F(x)$  for  $x \in \mathbb{F}_2^n$ . The functions  $f_i$  are called the **coordinate functions** of  $F$ . The **component functions** of  $F$  are all non-zero linear combinations of its coordinate functions. For  $b \in \mathbb{F}_2^m$ , the component function corresponding to the linear combination defined by  $b$  is denoted by  $F_b$ ; that is,  $F_b = \sum_{i=1}^m b_i f_i$ , where  $b = (b_1, b_2, \dots, b_m)$  and  $f_i$  are the coordinate functions of  $F$ . As we shall see in Section 2.2, some cryptographic properties of an  $(n, m)$ -function  $F$  are expressed in terms of the Hamming distance between its components and certain other functions. We recall that the **Hamming distance**  $d_H(F, G)$  between two  $(n, m)$ -functions  $F$  and  $G$  is the number of inputs  $x \in \mathbb{F}_2^n$  on which the values of  $F$  and  $G$  are distinct, i.e.  $d_H(F, G) = \#\{x \in \mathbb{F}_2^n : F(x) \neq G(x)\}$ .

In the discussion of some of the invariants, we will use the notion of a multiset. Intuitively, a multiset is an unordered collection of elements in which the same element can occur multiple times. We will refer to the number of times that an element  $e$  occurs in a multiset  $M$  as the **multiplicity** of  $e$  in  $M$ , and will refer to the collection of the multiplicities of all elements in  $M$  as the **multiplicities of  $M$** .

## 2.1 Representation of vectorial Boolean functions

A vectorial Boolean function  $F$  from  $\mathbb{F}_2^n$  to  $\mathbb{F}_2^m$  can be represented in several different ways. Perhaps the simplest is the so-called truth-table representation, which is simply an exhaustive list of the values  $F(x)$  for all possible inputs  $x \in \mathbb{F}_2^n$ . The truth table for a  $(3, 3)$ -function is given as an example in Table 1. The truth-table representation is rather efficient for implementing vectorial Boolean functions on a computer; in fact, the straightforward implementations of most invariants work best if the input  $(n, n)$ -function is given as a truth-table. Nonetheless, it has a number of serious shortcomings that make other representations preferable: the size of the table grows exponentially with  $n$  and  $m$ ; it is difficult to observe any properties of the function from its truth table without performing non-trivial computations; and it is difficult to express infinite families and constructions via truth tables.

Any  $(n, m)$ -function can be uniquely represented as a multivariate polynomial of the form

$$F(x_1, x_2, \dots, x_n) = \sum_{I \subseteq \{1, 2, \dots, n\}} a_I \prod_{i \in I} x_i,$$

where  $a_I \in \mathbb{F}_2^m$  for all  $I \subseteq \{1, 2, \dots, n\}$  and the variables  $x_1, x_2, \dots, x_n$  take values in  $\mathbb{F}_2$ . This representation is known as the **algebraic normal form (ANF)** of  $F$ . In some cases, the ANF allows for a significantly more compact representation of an  $(n, n)$ -function (when the dimensions  $n$  and  $m$  are large, so that the size of the truth table become prohibitive; while the number of terms with a non-zero coefficient in the ANF is small). The degree of the ANF (as a multivariate polynomial) is called the **algebraic degree** of  $F$ ; due to the uniqueness of the ANF, this notion is well defined. The algebraic degree has cryptographic significance (it must be large in order to resist higher-order differential attacks) and, as well shall see later, it is invariant under EA-equivalence (but not under CCZ-equivalence).

A function of algebraic degree 1, 2, or 3 is called **affine**, **quadratic**, or **cubic**, respectively. Any affine  $(n, n)$ -function  $A$  satisfies

$$A(x) + A(y) + A(z) = A(x + y + z)$$

for any  $x, y, z \in \mathbb{F}_2^n$ , and thus this notion coincides with the usual definition of affinity. If an affine function  $L$  satisfies  $L(0) = 0$  so that

$$L(x) + L(y) + L(x + y)$$

for any  $x, y \in \mathbb{F}_2^n$ , it is called **linear**. Affine and linear functions as defined here behave in the same way as they do over any vector space, and so all familiar notions and principles from linear algebra can be carried over to this case.

For example, the function given by its truth-table in Table 1 has the ANF

$$F(x_1, x_2, x_3) = (0, 1, 1)x_1x_2 + (0, 1, 0)x_1x_3 + (1, 0, 0)x_2x_3 + (1, 0, 0)x_1 + (1, 1, 0)x_2 + (1, 0, 1)x_3. \quad (1)$$

We can immediately see that its algebraic degree is 2; in other words, this is a quadratic function. In this particular case, the size of the ANF is not significantly less than that of the truth-table; but as the dimension  $n$  increases, the disparity between the size of the two representations becomes more pronounced.

Let  $\mathbb{F}_{2^n}$  denote the finite field with  $2^n$  elements for some natural number  $n$ . Recall that  $\mathbb{F}_{2^n}$  can be represented as an  $n$ -dimensional vector space over the prime field  $\mathbb{F}_2$ . Thus,  $\mathbb{F}_2^n$  can be identified with  $\mathbb{F}_{2^n}$ , and  $(n, n)$ -functions can be seen as mapping from  $\mathbb{F}_{2^n}$  to itself. This allows any  $(n, n)$ -function to be represented as a univariate polynomial of the form

$$F(x) = \sum_{i=0}^{2^n-1} a_i x^i,$$

where  $a_i \in \mathbb{F}_{2^n}$  for  $0 \leq i \leq 2^n - 1$ . This is known as the **univariate representation** of  $F$ ; just like the ANF, it always exists and is uniquely defined. The algebraic degree can be obtained directly from the univariate representation as the maximum binary weight of an exponent  $i$  with a non-zero coefficient  $a_i$ . The **binary weight** (also called 2-weight) of a natural number  $i$ , denoted by  $w_2(i)$ , is the number of distinct powers of 2 in its binary decomposition; that is, if we write  $i$  as  $i = \sum_j a_j 2^j$  for  $a_j \in \{0, 1\}$ , then  $w_2(i) = \sum_j a_j$ . Equivalently,  $w_2(i)$  is the number of non-zero bits in the binary representation of  $i$  (for instance,  $w_2(11) = 3$  since 11 can be written as  $2^3 + 2^1 + 2^0$ , or as 1011 in binary). The algebraic degree of  $F$  can be then

$$\deg(F) = \max\{w_2(i) : 0 \leq i \leq 2^n - 1, a_i \neq 0\}.$$

Family	Exponent	Conditions	Algebraic degree	Source
Gold	$2^i + 1$	$\gcd(i, n) = 1$	2	[35, 41]
Kasami	$2^{2^i} - 2^i + 1$	$\gcd(i, n) = 1$	$i + 1$	[36, 39]
Welch	$2^t + 3$	$n = 2t + 1$	3	[30]
Niho	$2^t + 2^{t/2} - 1, t \text{ even}$ $2^t + 2^{(3t+1)/2} - 1, t \text{ odd}$	$n = 2t + 1$	$(t + 2)/2$ $t + 1$	[29]
Inverse	$2^{2^t} - 1$	$n = 2t + 1$	$n - 1$	[41, 3]
Dobbertin	$2^{4i} + 2^{3i} + 2^{2i} + 2^i - 1$	$n = 5i$	$i + 3$	[31]

Table 2: Known infinite families of APN power functions over  $\mathbb{F}_{2^n}$

We also remark that the component functions of  $F$  can be expressed using the absolute trace function  $\text{Tr} : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$  as the Boolean functions  $F_b : x \mapsto \text{Tr}(bF(x))$  for all non-zero  $b \in \mathbb{F}_{2^n}$ . We recall that the absolute trace is defined by

$$\text{Tr}(x) = x + x^2 + x^{2^2} + \dots + x^{2^{n-1}}.$$

The univariate representation is by far the most widely used at the moment. A decisive reason for this is that many of the known APN functions (including infinite constructions and families) have a very simple form under this representation; a classic example is the Gold function  $x^3$ , which is known to be APN over  $\mathbb{F}_{2^n}$  for any natural number  $n$  [35, 41]. Indeed, the (3, 3)-function represented by Table 1 and the ANF in (1) is precisely  $x^3$  over  $\mathbb{F}_{2^3}$ . At the time of writing, most known infinite constructions of APN functions are expressed in the univariate representation; there are also a few constructions based on the so-called bivariate representation (that we do not treat here, but briefly discuss below). A list of the known infinite families of APN monomials is given in Table 2, and a list of the known infinite polynomial families is given in Table 3.

The above is by no means an exhaustive list of known representations of vectorial Boolean functions. Nonetheless, they provide sufficient context for the sequence, and we shall not go into any further details on representation of  $(n, n)$ -functions. We will, however, mention the bivariate representation, in which a  $(2n, 2n)$ -function can be represented as pair of polynomials  $(F_1(x, y), F_2(x, y))$  in two variables (see e.g. [23], p.47); the representation of a quadratic function by means of a so-called quadratic APN matrix (QAM) [50]; and the representation of a function by means of the values of its first-order derivatives [45].

## 2.2 APN and AB functions

Let  $F$  be an  $(n, m)$ -function for some natural number  $n$ , and denote by  $\delta_F(a, b)$  the number of solutions  $x \in \mathbb{F}_{2^n}$  to the equation

$$F(x + a) + F(x) = b \tag{2}$$

for some  $a, b \in \mathbb{F}_{2^n}$ . Note that the equation expresses the difference between two outputs of  $F$  whose corresponding inputs are at distance  $a$ . If for some given  $a$ , some value of  $b$  is significantly more likely to occur than all others, an attacker can exploit this to derive a correlation between the input and output of the function. This is the basic idea of differential cryptanalysis, which is one of the most powerful known attacks against block ciphers [5]. In order to be

ID	Functions	Conditions	Source
F1- F2	$x^{2^s+1} + u^{2^k-1}x^{2^{ik}+2^{mk+s}}$	$n = pk, \gcd(k, 3) = \gcd(s, 3k) = 1, p \in \{3, 4\}, i = sk \bmod p, m = p - i, n \geq 12, u$ primitive in $\mathbb{F}_{2^n}^*$	[16]
F3	$sx^{q+1} + x^{2^i+1} + x^{q(2^i+1)} + cx^{2^i q+1} + c^q x^{2^i+q}$	$q = 2^m, n = 2m, \gcd(i, m) = 1, c \in \mathbb{F}_{2^n}, s \in \mathbb{F}_{2^n} \setminus \mathbb{F}_q, X^{2^i+1} + cX^{2^i} + c^q X + 1$ has no solution $x$ s.t. $x^{q+1} = 1$	[14]
F4	$x^3 + a^{-1}\text{Tr}_n(a^3 x^9)$	$a \neq 0$	[17]
F5	$x^3 + a^{-1}\text{Tr}_n^3(a^3 x^9 + a^6 x^{18})$	$3 n, a \neq 0$	[18]
F6	$x^3 + a^{-1}\text{Tr}_n^3(a^6 x^{18} + a^{12} x^{36})$	$3 n, a \neq 0$	[18]
F7- F9	$ux^{2^s+1} + u^{2^k}x^{2^{-k}+2^{k+s}} + vx^{2^{-k}+1} + wu^{2^k+1}x^{2^s+2^{k+s}}$	$n = 3k, \gcd(k, 3) = \gcd(s, 3k) = 1, v, w \in \mathbb{F}_{2^k}, vw \neq 1, 3 (k+s), u$ primitive in $\mathbb{F}_{2^n}^*$	[8]
F10	$(x + x^{2^m})^{2^k+1} + u'(ux + u^{2^m}x^{2^m})^{(2^k+1)2^i} + u(x + x^{2^m})(ux + u^{2^m}x^{2^m})$	$n = 2m, m \geq 2$ even, $\gcd(k, m) = 1$ and $i \geq 2$ even, $u$ primitive in $\mathbb{F}_{2^n}^*$ , $u' \in \mathbb{F}_{2^m}$ not a cube	[51]
F11	$a^2x^{2^{2m+1}+1} + b^2x^{2^{2m+1}+1} + ax^{2^{2m}+2} + bx^{2^{2m}+2} + (c^2 + c)x^3$	$n = 3m, m$ odd, $L(x) = ax^{2^{2m}} + bx^{2^m} + cx$ satisfies the conditions of Lemma 8 of [13]	[13]
F12	$u(u^q x + x^q u)(x^q + x) + (u^q x + x^q u)^{2^{2i}+2^{3i}} + a(u^q x + x^q u)^{2^{2i}}(x^q + x)^{2^i} + b(x^q + x)^{2^i+1}$	$q = 2^m, n = 2m, \gcd(i, m) = 1, x^{2^i+1} + ax + b$ has no roots in $\mathbb{F}_{2^m}$	[47]
F13	$x^3 + a(x^{2^i+1})^{2^k} + bx^{3 \cdot 2^m} + c(x^{2^i+m+2^m})^{2^k}$	$n = 2m = 10, (a, b, c) = (\beta, 1, 0, 0), i = 3, k = 2, \beta$ primitive in $\mathbb{F}_{2^2}$ $n = 2m, m$ odd, $3 \nmid m, (a, b, c) = (\beta, \beta^2, 1), \beta$ primitive in $\mathbb{F}_{2^2}, i \in \{m-2, m, 2m-1, (m-2)^{-1} \bmod n\}$	[20]

Table 3: Known infinite families of quadratic APN polynomials over  $\mathbb{F}_{2^n}$

resistant to such attacks, the number of solutions to (2) should be as uniform as possible over all possible choices of  $b \in \mathbb{F}_{2^n}$  for any fixed non-zero  $a \in \mathbb{F}_{2^n}$ . The **differential uniformity** of  $F$ , denoted by  $\Delta_F$ , is the largest value of  $\delta_F(a, b)$  over all choice of  $0 \neq a \in \mathbb{F}_{2^n}$  and  $b \in \mathbb{F}_{2^n}$ . Symbolically:

$$\Delta_F = \max\{\delta_F(a, b) : a, b \in \mathbb{F}_{2^n}, a \neq 0\}.$$

The differential uniformity should be as low as possible in order to resist differential cryptanalysis, and since  $x + a$  is a solution to (2) whenever  $x$  is, the differential uniformity of any  $(n, n)$ -function can be no lower than two. A function is called **almost perfect nonlinear (APN)** if it achieves this trivial lower bound with equality.

The **differential spectrum**  $\mathcal{D}_F$  of  $F$  is the multiset of the values of  $\delta_F(a, b)$  over all  $a, b \in \mathbb{F}_{2^n}$  with  $a \neq 0$ ; that is,

$$\mathcal{D}_F = \{\delta_F(a, b) : a, b \in \mathbb{F}_{2^n}, a \neq 0\}.$$

Clearly, a function is APN if and only if its differential spectrum consists of the two values 0 and 2. The differential spectrum is, in fact, invariant under CCZ-equivalence, but it is practically useless for the purpose of distinguishing inequivalent APN functions. It does, however, take a much more prominent role as an invariant of the ortho-derivatives of quadratic APN functions, as described in Section 4.5.

We remark that the function  $D_a F(x) = F(x+a) + F(x)$  is called the (discrete first-order) **derivative** of  $F$  in direction  $a \in \mathbb{F}_{2^n}$ . An APN function can be equivalently defined as a function all of whose derivatives  $D_a F$  for  $a \neq 0$  are 2-to-1 functions.

Another powerful attack against block ciphers is linear cryptanalysis [40], which attempts to approximate the behavior of a function by means of linear functions. Intuitively, a function  $F$  should be as far away from all linear functions as possible in order to be resistant to this attack. The nonlinearity of a Boolean function  $f : \mathbb{F}_{2^n} \rightarrow \mathbb{F}_2$  is defined as the minimum Hamming distance between  $f$  and any affine  $(n, 1)$ -function. This notion can then be naturally generalized to the case of vectorial Boolean functions through the nonlinearity of their component functions. The **nonlinearity** of an  $(n, n)$ -function  $F$ , denoted by  $\mathcal{NL}(F)$ , is defined as the minimum Hamming distance between any component function of  $F$ , and any affine  $(n, 1)$ -function. The nonlinearity should be as high as possible in order to resist linear cryptanalysis, and it has been shown [25, 46] that it satisfies

$$\mathcal{NL}(F) \leq 2^{n-1} - 2^{(n-1)/2}$$

for any  $(n, n)$ -function  $F$ . The functions that attain this upper bound with equality are called **almost bent (AB)** functions. Note that AB functions exist only for odd values of  $n$ . In the case of even  $n$ , functions with nonlinearity  $2^{n-1} - 2^{n/2}$  are known, and they are conjectured to be optimal with respect to nonlinearity; nonetheless, the exact upper bound on the nonlinearity for  $(2n, 2n)$ -functions remains an open question.

It is also known that any AB function is necessarily APN. The converse does not hold in general, although it is known that any quadratic  $(n, n)$ -function is AB when  $n$  is odd [24]. Thus, AB functions provide optimal resistance against both linear and differential cryptanalysis. In particular, constructing new instances and families of APN functions is a natural approach to finding new constructions of AB functions.

Besides providing the best possible resistance to differential and linear cryptanalysis, respectively, APN and AB functions correspond to optimal objects in other areas of mathematics and computer science, including coding theory, combinatorics, projective geometry, and sequence design. Developments in the study of cryptographic vectorial Boolean functions therefore naturally lead to progress in other areas; and results and methods used in these related fields of study can be applied to the investigation of APN and AB functions. In essence, the natural definitions of APN and AB functions make them significant, universal objects that transcend the immediate practical needs of cryptography and have a much broader relevance.

The algebraic degree, as mentioned previously, should be high in order to resist higher-order differential attacks. In addition, there are quite a few other statistics and properties for measuring the cryptographic strength of a function

against various kinds of attacks. Since in this paper we mostly focus on invariants with respect to the classification of APN and AB functions, we do not go into further details here. We refer the reader to [23] for a more comprehensive treatment of the subject.

### 2.3 Equivalence relations

Equivalence relations on  $(n, n)$ -functions that preserve the differential uniformity and nonlinearity are used to reduce the number of functions that need to be studied and classified. Currently, CCZ-equivalence is the most general known equivalence relation preserving these properties, and so classification and computational results on APN and AB function are typically given up to CCZ-equivalence.

The notion of the CCZ-equivalence of two functions is expressed in terms of their graphs. Let  $F$  and  $G$  be  $(n, n)$ -functions for some natural number  $n$ . The graph of  $F$  is the set  $\Gamma_F = \{(x, F(x)) : x \in \mathbb{F}_{2^n}\}$ . Note that for any  $(n, n)$ -function  $F$ , its graph  $\Gamma_F$  is contained in the set  $\mathbb{F}_{2^n}^2$  of pairs of elements from  $\mathbb{F}_{2^n}$ . The latter can be identified with  $\mathbb{F}_{2^{2n}}$ , and thus we can assume that  $\Gamma_F$  is contained in  $\mathbb{F}_{2^{2n}}$ . Then  $F$  and  $G$  are said to be **Carlet-Charpin-Zinoviev equivalent**, or **CCZ-equivalent** for short, if there is an affine permutation  $A$  of  $\mathbb{F}_{2^{2n}}$  mapping  $\Gamma_F$  to  $\Gamma_G$ , i.e.

$$\{A(x) : x \in \Gamma_F\} = \Gamma_G.$$

In Section 3, we will concentrate on properties that are left invariant by CCZ-equivalence. For the time being, we remark that CCZ-equivalence preserves neither the algebraic degree, nor the bijectivity of the function. This is noteworthy, as both of these can (and have) been used constructively. In 2010, John Dillon constructed the only currently known instance of an APN  $(n, n)$ -permutation for even  $n$  by traversing the CCZ-equivalence class of a known (non-bijective) APN function over the same field [11]. In a similar vein, most of the known instances of APN functions listed in the literature are quadratic. As pointed out in Section 2.2, it is desirable for the algebraic degree to be high in order to resist higher-order differential attacks. Traversing the CCZ-class of a quadratic APN function may yield functions of higher algebraic degree that are CCZ-equivalent to it (and hence also APN).

A special case of CCZ-equivalence is the so-called extended affine equivalence, or EA-equivalence for short. Two  $(n, n)$ -functions  $F$  and  $G$  are said to be **EA-equivalent** if there exist affine  $(n, n)$ -functions  $A_1, A_2, A$  with  $A_1$  and  $A_2$  bijective such that

$$A_1 \circ F \circ A_2 + A = G. \tag{3}$$

CCZ-equivalence is strictly more general than EA-equivalence combined with taking inverses of permutations [19]. Nonetheless, the two equivalence relations coincide in the case when both  $F$  and  $G$  are quadratic; that is, if  $F$  and  $G$  are quadratic  $(n, n)$ -functions, then  $F$  and  $G$  are CCZ-equivalent if and only if they are EA-equivalent [48]. In practice, this makes EA-equivalence (and hence, properties that are invariant under EA-equivalence) almost as useful as CCZ-equivalence in the case of APN functions, since all known APN instances are equivalent to quadratic functions or monomials, with only a single known exception for  $n = 6$  [33].

Further specializations of EA-equivalence can be obtained by imposing additional restrictions on  $A_1$ ,  $A_2$  and  $A$ . If  $A = 0$  in (3), we say that  $F$  and  $G$  are **affine equivalent**. If  $A = 0$  and  $A_1(0) = A_2(0) = 0$  (so that  $A_1$  and  $A_2$  are linear instead of merely affine), we say that  $F$  and  $G$  are **linear equivalent**. Further restrictions have been investigated, such as the so-called restricted EA-equivalence [21, 43]. These relations are of limited interest in the APN case, and so we mostly concentrate on the notions of EA- and CCZ-equivalence.

For the sake of completeness, we also mention a special kind of equivalence that can be defined for two power functions. If  $F(x) = x^{e_1}$  and  $G(x) = x^{e_2}$  are  $(n, n)$ -functions for some natural numbers  $e_1$  and  $e_2$ , then we say that  $F$  and  $G$  are **cyclotomic equivalent** if there exists a natural number  $k$  such that  $2^k e_1 = e_2 \pmod{2^n - 1}$  or  $2^k (e_1)^{-1} = e_2 \pmod{2^n - 1}$ , where  $e_1^{-1}$  is the multiplicative inverse of  $e_1$  modulo  $2^n - 1$  (if it exists). It is known that if two power functions are CCZ-equivalent, they are necessarily cyclotomic equivalent. This greatly reduces the complexity of deciding equivalence in the case of power functions since, unlike EA- and CCZ-equivalence, testing cyclotomic equivalence is simple and amounts to solving modular equations.

## 2.4 A note on the computational results

In the following Sections 3 and 4, we survey the most frequently used invariants for classifying APN and AB functions up to CCZ- and EA-equivalence, and report on some computational results for measuring how quickly these invariants can be computed, and how well they can distinguish between distinct CCZ- and EA-equivalence classes. In this section, we describe the computational equipment that we used and the sets of functions that we tested the invariants on.

Computations in C and Python were performed on a HP EliteDesk 800 G2 SFF computer, with a quad core 3.2 GHz processor with 15 GB of memory. Computations in Magma were performed on a server with 56 3.2 GHz cores and 500 GB of memory.

All the experiments that we conduct are for dimensions  $n \geq 6$  since the classification of APN functions for  $n < 6$  is already complete [9]. In the case of  $n = 6$ , we use the 14 CCZ-inequivalent representatives from the switching classes [33]; 13 of them are quadratic and represent all CCZ-classes of quadratic functions over  $\mathbb{F}_{2^6}$  [32], while the remaining function is the only known example of an APN instance CCZ-inequivalent to monomial and quadratic functions. For  $n = 7$ , we use the list of 390 functions from [49, 50] along with the newly found quadratic APN function from [38]; as shown in [38], the quadratic representatives therein encompass all possible CCZ-classes of quadratic APN functions over  $\mathbb{F}_{2^7}$ . For  $n = 8$ , we use the 8181 functions from [49]; we note that recently more than 12 000 new inequivalent APN instances have been discovered in  $\mathbb{F}_{2^8}$  [2], but we do not involve these in the computations since the goal is to give an empiric idea of how efficient the various invariants are in distinguishing distinct CCZ-classes (rather than to do a complete classification); a constantly updated list of invariants for the known functions is available at [1]. For  $n = 9$  and  $n = 10$ , we take CCZ-inequivalent representatives from the known infinite APN families along with the new functions found in [2]; this is necessary, since some invariants tend to take a lot of different values among the known APN instances, but only one or two values for instances belonging to the currently

known APN families. Since no classification of APN functions is available for dimensions  $n \geq 11$ , we restrict ourselves to only computing invariants for a few select functions in order to estimate the computation time, without making any claims about the distinguishing power of the invariants in those dimensions.

### 3 Invariants under CCZ-equivalence

#### 3.1 Trivial invariants

The differential uniformity (and, more generally, the differential spectrum) and nonlinearity are invariant under CCZ-equivalence, but this does not help with the classification of APN and AB functions, since by definition they have a fixed value of the differential uniformity and nonlinearity, respectively.

#### 3.2 The extended Walsh spectrum

##### 3.2.1 Definition

The Walsh transform  $W_F : \mathbb{F}_2^n \rightarrow \mathbb{Z}$  is an integer-valued function that can be associated with any  $(n, n)$ -function  $F$ . More precisely, the **Walsh transform** of  $F$  is defined as

$$W_F(a, b) = \sum_{x \in \mathbb{F}_2^n} (-1)^{b \cdot F(x) + a \cdot x}, \quad (4)$$

where “ $\cdot$ ” denotes a scalar product, or dot product on  $\mathbb{F}_2^n$  (or, equivalently, on  $\mathbb{F}_2^n$ ). The properties of the Walsh transform do not depend on the concrete choice of the scalar product. There are at least two frequently used choices:

- if the multiplicands  $a = (a_1, a_2, \dots, a_n)$  and  $b = (b_1, b_2, \dots, b_n)$  are viewed as  $n$ -dimensional binary vectors in  $\mathbb{F}_2^n$ , the product can be defined as  $a \cdot b = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$ , with addition and multiplication being over  $\mathbb{F}_2$ ;
- the product can also be defined as  $a \cdot b = \text{Tr}(ab)$ , where  $\text{Tr} : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$  is the absolute trace function from  $\mathbb{F}_2^n$  to the prime field  $\mathbb{F}_2$ ; in this case, the operands  $a$  and  $b$  are multiplied in the finite field, and then this product is mapped onto  $\mathbb{F}_2$  via the trace.

The former realization tends to be more suitable for implementing the Walsh transform in a general-purpose programming language, while the latter is typically used in theoretical proofs and constructions.

The values  $W_F(a, b)$  of the Walsh transform are referred to as **Walsh coefficients**. The **Walsh spectrum** of an  $(n, n)$ -function  $F$  is the multiset of the values of its Walsh coefficients for all possible  $a, b \in \mathbb{F}_2^n$ . The **extended Walsh spectrum**  $\mathcal{W}_F$  of  $F$  is the multiset of the absolute values of all of the Walsh coefficients of  $F$ , that is

$$\mathcal{W}_F = \{|W_F(a, b)| : a, b \in \mathbb{F}_2^n\}.$$

The extended Walsh spectrum is invariant under CCZ-equivalence.

As we will see in the evaluation below, the extended Walsh spectrum is not a very useful invariant in terms of its distinguishing power. Nonetheless,

it is a good idea to compute it as a first step in analyzing a function, since the computation of several of the following invariants either require, or are facilitated by, knowledge of all values of the Walsh transform. Furthermore, it does become useful in distinguishing inequivalent quadratic APN functions when applied to their ortho-derivatives rather than to the functions themselves; ortho-derivatives are described in Section 4.5.

### 3.2.2 Evaluation

The Walsh transform can be implemented easily on any programming language; its computation from (4) requires nothing more complicated than basic arithmetic operations (addition, modulation, and XOR, and possibly finite field multiplication depending on how the scalar product is implemented). On algebra systems such as *Magma* [7] that include built-in functionality for computations over finite fields, both implementations can be easily realized.

Depending on the implementation, the Walsh transform can be computed very efficiently. A straightforward implementation in C taking an  $(n, n)$ -function's truth table as input needs around 20 seconds to compute the entire Walsh spectrum for  $n = 10$ ; further measurements are provided in Table 4 below. The memory consumption is negligible.

$n$	6	7	8	9	10	11	12
time (s)	0.023	0.076	0.391	2.863	22.566	171.602	1410.009
total	14	491	21 115	46	21	-	-
values	2	2	6	2	2	-	-

Table 4: Time for computing the extended Walsh spectrum in C

Unfortunately, the extended Walsh spectrum does not do a good job of distinguishing between inequivalent functions. Among all known APN instances  $F$  over  $\mathbb{F}_{2^n}$  for odd values of  $n$ , there are only two possible values of  $W_F$ : all known APN functions except for the inverse power function  $x^{2^n-2}$  have the so-called Gold-like spectrum (that is, the same as the Gold function  $x^3$ ); and the inverse power function has an extended Walsh spectrum that is distinct from that of any other known APN function.

For  $n = 6$  and  $n = 10$ , we observe a similar partition. More precisely, all known APN functions over  $\mathbb{F}_{2^6}$  have a Gold-like spectrum, except for one (function 2.5 from [33]); and all functions over  $\mathbb{F}_{2^{10}}$  have a Gold-like spectrum, except for the Dobbertin power function  $x^{339}$ . For  $n = 8$ , the situation is a bit more varied. We know more than 24 000 CCZ-inequivalent APN functions over  $\mathbb{F}_{2^8}$  [2], and these have six distinct values of the extended Walsh spectrum. Still, the vast majority of the known APN functions have a Gold-like spectrum; and it is remarkable that the same is true for all known APN functions (regardless of the dimension  $n$ ) that have been classified into infinite families.

## 3.3 Invariants from associated designs

### 3.3.1 Definition

An **incidence structure** is a triple  $(\mathcal{P}, \mathcal{B}, \mathcal{I})$ , where  $\mathcal{P} = \{p_1, p_2, \dots, p_m\}$  is a set of points,  $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$  is a set of blocks, and  $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$  is an incidence

relation. Typically, we assume that the blocks in  $\mathcal{B}$  are subsets of  $\mathcal{P}$ , and the incidence relation  $\mathcal{I}$  is set membership. The notion of an incidence structure is quite natural and general; the field of combinatorial design theory studies incidence structures called block designs. We can associate to any incidence structure a so-called **incidence matrix**, which is a binary  $m \times n$  matrix  $M$  representing the incidence relation  $\mathcal{I}$ . More precisely, for any  $i, j$  in the range  $1 \leq i \leq m$  and  $1 \leq j \leq n$ , the element  $M_{i,j}$  on the  $i$ -th row and  $j$ -th column of  $M$  is equal to 1 if  $(p_i, b_j) \in \mathcal{I}$ ; and is equal to 0 otherwise. We refer the reader to [4] or [26] for more background on incidence structures and combinatorial designs.

Given any  $(n, n)$ -function  $F$ , two designs can be associated with it [33]. In both cases, the set of points is simply  $\mathbb{F}_{2^n}^2$ , that is, the set of all pairs of elements from the finite field. The first design is denoted by  $dev(G_F)$ , and its blocks are of the form

$$\{(x + a, F(x) + b) : x \in \mathbb{F}_{2^n}\}$$

for  $a, b \in \mathbb{F}_{2^n}$ . The second design is denoted by  $dev(D_F)$ , and its blocks are the sets

$$\{(x + y + a, F(x) + F(y) + b) : x, y \in \mathbb{F}_{2^n}\}$$

for  $a, b \in \mathbb{F}_{2^n}$ . The rank of the incidence matrix of  $dev(G_F)$  is called the  $\Gamma$ -**rank** of  $F$ , and the rank of the incidence matrix of  $dev(D_F)$  is called the  $\Delta$ -**rank** of  $F$ . The  $\Gamma$ - and  $\Delta$ -rank are shown to be invariant under CCZ-equivalence, and are two of the currently most widely used invariants in practice.

The orders of the automorphism groups of  $dev(G_F)$  and  $dev(D_F)$  are also CCZ-invariant, but their computation is only feasible for small dimensions, and so they are not quite as useful as the  $\Gamma$ - and  $\Delta$ -rank. Nonetheless, these automorphism groups give rise to a CCZ-invariant that can be quite useful in practice: the order of the so-called multiplier group. The **multiplier group** is the subgroup of the automorphism group of  $dev(G_F)$  consisting of automorphisms of a special form; its order, denoted by  $\mathcal{M}(G_F)$ , is invariant under CCZ-equivalence, and can be computed much more efficiently than the order of the full automorphism group.

### 3.3.2 Evaluation

The definition of these invariants is conceptually simple, although it does require familiarity with the notion of a combinatorial design and its incidence matrix (and automorphism group, in the case of the  $\mathcal{M}(G_F)$ ). In the case of the  $\Gamma$ - and  $\Delta$ -rank, the incidence matrix corresponding to  $dev(G_F)$  and  $dev(D_F)$  can be constructed quite easily. The main issue is the computation of its rank, which requires a sophisticated implementation in order to be efficiently computable in practice. In both cases, we need to compute the rank of a binary  $2^{2n} \times 2^{2n}$  matrix (whose construction takes significant time and which occupies a lot of memory, especially for higher dimensions), and a straightforward implementation of say Gaussian elimination is not nearly fast enough. One is thus forced to rely on software libraries or algebra systems (such as *Magma*) that implement efficient algorithms for computing the rank of a matrix; or, otherwise, to implement such highly non-trivial algorithms oneself.

Computing the  $\Gamma$ - and  $\Delta$ -rank is fairly efficient for small dimensions, but the time complexity grows exponentially; for  $n = 10$ , computing a single  $\Gamma$ -

rank can take more than a week. A summary of computation times is given in Table 5. The real bottleneck, however, is the memory consumption; the 500 GB of memory available on our server suffice only for computing  $\Gamma$ -ranks up to dimension 10, and  $\Delta$ -ranks up to dimension 9.

$n$	6	7	8	9	10
$\Gamma$ -rank	2	15	138	4229	899024
$\Delta$ -rank	-	-	-	-	-

Table 5: Computation times (in seconds) for the  $\Gamma$ - and  $\Delta$ -rank

Despite these shortcomings, the  $\Gamma$ -rank,  $\Delta$ -rank, and the order  $\mathcal{M}(G_F)$  of the multiplier group are rather useful invariants, as they can take on a lot of distinct values for the known functions. Table 6 gives a summary of the number of distinct values that the  $\Gamma$ -rank,  $\Delta$ -rank, and  $\mathcal{M}(G_F)$  can take individually, and the number of distinct combinations of values that they can take on together. Note that computing  $\Delta$ -ranks for  $n \geq 10$  and computing any of the invariants from this section for  $n \geq 11$  is currently impossible due to insufficient memory on our server.

$n$	no. of functions	$\Gamma$ -ranks	$\Delta$ -ranks	$\mathcal{M}(G_F)$	combinations
6	14	9	3	7	11
7	490	14	6	5	20
8	8192	24	11	10	49
9	11	10	8	5	10
10	16	15	-	7	15

Table 6: Number of distinct design invariants for some known APN functions

### 3.4 The distance invariant

#### 3.4.1 Definition

A lower bound on the Hamming distance between an APN function  $F$  and the closest APN function to it in terms of Hamming distance is shown in [15]. The value of this lower bound is calculated from the minimum value contained in a multiset  $\Pi_F$  of natural numbers that can be associated with any  $(n, n)$ -function  $F$ . The multiset  $\Pi_F$  is shown to be invariant under CCZ-equivalence in the case of APN functions; that is, if  $F$  and  $G$  are CCZ-equivalent APN functions, then  $\Pi_F = \Pi_G$ . One can, however, easily find counterexamples to  $\Pi_F$  being invariant in the case when  $F$  and  $G$  are not APN.

The multiset  $\Pi_F$  is defined as follows. Let  $F$  be an  $(n, n)$ -function for some natural number  $n$ . For any  $b, c \in \mathbb{F}_{2^n}$ , we define a set

$$\pi_F(b, c) = \{a \in \mathbb{F}_{2^n} : (\exists x \in \mathbb{F}_{2^n}) F(x) + F(a+x) + F(a+c) = b\}.$$

That is,  $\pi_F(b, c)$  contains all directions  $a \in \mathbb{F}_{2^n}$  for which the derivative  $D_a F(x) = F(x+a) + F(x)$  maps to  $F(a+c) + b$ . The multiset  $\Pi_F$  then consists of the cardinalities of  $\pi_F(b, c)$  for all possible choices of  $b, c \in \mathbb{F}_{2^n}$ ; symbolically:

$$\Pi_F = \{\#\pi_F(b, c) : b, c \in \mathbb{F}_{2^n}\}.$$

As indicated above, if  $F$  and  $G$  are APN and CCZ-equivalent, then we must necessarily have  $\Pi_F = \Pi_G$ . The lower bound on the Hamming distance mentioned previously can be computed as  $\lceil (\min \Pi_F)/3 \rceil + 1$ , where  $\min \Pi_F$  is the smallest value in  $\Pi_F$ . It is also shown in [15] that the lower bound is not, in general, tight for  $n \geq 5$ . It is remarkable that while this lower bound is CCZ-invariant, the exact value of the Hamming distance to the closest APN function is not.

If  $F$  is quadratic, it is shown that

$$\{\#\pi_F(b, c) : b \in \mathbb{F}_{2^n}\} = \{\#\pi_F(b, c') : b \in \mathbb{F}_{2^n}\}$$

for any  $c, c' \in \mathbb{F}_{2^n}$ . It is thus sufficient to compute only the reduced multiset

$$\Pi_F^0 = \{\#\pi_F(b, 0) : b \in \mathbb{F}_{2^n}\},$$

which then completely determines the full multiset  $\Pi_F$ . Recall that the quadratic case is by far the most practically significant, as almost all known APN instances are quadratic, or at least CCZ-equivalent to quadratic ones. This reduction is then quite valuable, as it allows us to reduce the time for computing this invariant by a factor of  $2^n$ .

### 3.4.2 Evaluation

Computing  $\Pi_F$  (or  $\Pi_F^0$ ) is simple as it only requires summing finite field elements (which can be implemented via binary XOR on most programming languages) and counting how many times a certain value occurs.

The computation time is quite fast, especially in the case of quadratic functions when only the reduced multiset  $\Pi_F^0$  has to be computed. Table 7 shows some example computation times (in seconds) of a straightforward C implementation for functions over  $\mathbb{F}_{2^n}$  for  $5 \leq n \leq 11$ . The bottom two rows give the total number of functions on which the invariant was tested and the number of distinct values that  $\Pi_F$  takes over those functions, respectively.

$n$	$\Pi_F^0$	$\Pi_F$	all	values
5	0.002	0.064	3	2
6	0.003	0.192	14	5
7	0.004	0.512	491	2
8	0.004	1.024	8181	6669
9	0.005	2.56	11	2
10	0.031	31.744	21	4
11	0.066	135.168	13	2

Table 7: Computation times and number of distinct values of  $\Pi_F$  for some known APN instances

As indicated by Table 7, this invariant does not do a very good job of distinguishing inequivalent functions for odd dimensions. Indeed, for all odd  $n$  in the range  $5 \leq n \leq 11$ , we observe only two values of  $\Pi_F$ ; one is attained by the inverse power function, while all the remaining known APN functions take the second value. This is particularly remarkable for  $n = 7$ , where we know a large number of CCZ-inequivalent APN instances. Furthermore, as shown in [9, 38], the considered representatives for  $n = 5$  and  $n = 7$  cover all CCZ-classes of

quadratic APN functions over  $\mathbb{F}_{2^5}$  and  $\mathbb{F}_{2^7}$ , respectively; we can thus conclude, that all quadratic APN functions over  $\mathbb{F}_{2^5}$  and  $\mathbb{F}_{2^7}$  have a Gold-like value of  $\Pi_F$ .

In the case of even dimensions, on the other hand,  $\Pi_F$  proves to be quite useful for distinguishing between CCZ-equivalence classes. For  $n = 8$ , we get 6669 distinct values for the tested 8181 functions from [49]. What is particularly noteworthy, is that instances over even dimensions from the known infinite APN families always take the same value of  $\Pi_F$  as the Gold function  $x^3$ . All the other values of  $\Pi_F$  that we observe correspond to instances discovered by computational searches that have not been classified into infinite families yet. Thus, the inverse power function  $x^{2^n-2}$ , which is APN over  $\mathbb{F}_{2^n}$  with  $n$  odd, is currently the only known infinite family to have a value of  $\Pi_F$  distinct from that of the Gold function.

## 4 Invariants under EA-equivalence

Despite EA-equivalence being strictly less general than CCZ-equivalence, the two notions coincide in the case of quadratic functions. Since almost all known APN functions are CCZ-equivalent to monomials or to quadratic functions, this makes tests and invariants for EA-equivalence as useful as ones for the more general CCZ-equivalence in practice.

### 4.1 Trivial invariants

Since EA-equivalence is a particular case of CCZ-equivalence, any two functions that are EA-equivalent are also CCZ-equivalent. Consequently, all invariants described in Section 3 remain invariants under EA-equivalence as well.

The algebraic degree of any  $(n, n)$ -function is invariant under EA-equivalence since the composition of an arbitrary function  $F$  with an affine function does not change the algebraic degree of  $F$ . Since the majority of known APN functions are quadratic, the algebraic degree is a nearly useless invariant for classifying APN functions.

### 4.2 Number of subspaces in the set of bent components

#### 4.2.1 Definition

Recall that a Boolean  $(n, 1)$ -function  $f$  is called bent if its non-linearity equals  $2^{n-1} - 2^{n/2-1}$ . Equivalently,  $f$  is bent if and only if its Walsh transform satisfies  $W_f(a) = \pm 2^{n/2}$  for all  $a \in \mathbb{F}_{2^n}$ . Given an  $(n, n)$ -function  $F$ , we can define the set  $\mathcal{S}_F$  of those elements  $b \in \mathbb{F}_{2^n}$  for which the component function  $F_b = \text{Tr}(bF(x))$  is not bent; symbolically, we can write

$$\mathcal{S}_F = \{b \in \mathbb{F}_{2^n} : (\exists a \in \mathbb{F}_{2^n}) W_F(a, b) \neq \pm 2^{n/2}\},$$

or, equivalently,

$$\mathcal{S}_F = \{b \in \mathbb{F}_{2^n} : (\exists a \in \mathbb{F}_{2^n}) W_F(a, b) = 0\}.$$

If we now denote by  $n_F(i)$  the number of  $i$ -dimensional linear subspaces contained in  $\mathcal{S}_F$ , the value  $n_F(i)$  is an EA-invariant for any natural number  $i$ ; that is, if  $F$  and  $G$  are EA-equivalent, then  $n_F(i) = n_G(i)$  for all  $i$  [12]. This

is easy to see from the fact that if  $G = A_1 \circ F \circ A_2 + A$ , with all involved functions defined as in (3), then  $b \in \mathcal{S}_F$  if and only if  $A'_1(b) \in \mathcal{S}_G$ , where  $A'_1(x)$  is the adjoint operator of  $A_1(x) + A_1(0)$ . Clearly, if  $n_F(i) = 0$  for some  $i$ , then  $n_F(j) = 0$  for all  $j \geq i$  as well. This allows us to compute  $\mathcal{S}_F$  and the values of  $n_F(i)$  for  $i = 1, 2, 3, \dots$  until an  $i'$  is found for which  $n_F(i') = 0$ , and to use the vector  $N_F = (n_F(1), n_F(2), \dots, n_F(i' - 1))$  for distinguishing between EA-classes.

Note that this invariant is only useful in the case of even dimensions. In the case of odd  $n$ , any quadratic APN  $(n, n)$ -function is AB, so that we have  $\{W_F(a, b) : a \in \mathbb{F}_{2^n}\} = \{0, \pm 2^{(n+1)/2}\}$  for any  $0 \neq b \in \mathbb{F}_{2^n}$ . Consequently,  $\mathcal{S}_F = \mathbb{F}_{2^n}$ , and  $N_F$  has no distinguishing power.

#### 4.2.2 Evaluation

Implementing  $N_F$  as an invariant is simple and easily doable in any general-purpose programming language. As observed in Section 3.2, the Walsh transform can be readily implemented via standard arithmetic and logic operations, and the entire Walsh spectrum of a given function can be computed quite quickly. In this case, the entire Walsh spectrum does not need to be computed (for every  $b \in \mathbb{F}_{2^n}$ , we can stop computing  $W_F(a, b)$  as soon as we find an  $a \in \mathbb{F}_{2^n}$  which witnesses that  $F_b$  is not bent), so the computation of  $\mathcal{S}_F$  is even faster. In particular, if the Walsh spectrum or extended Walsh spectrum has already been computed for the given function (say, in the process of computing some other invariant),  $\mathcal{S}_F$  can be recovered almost immediately.

Computing the number  $n_F(i)$  of subspaces of a given dimension  $i$  is the most computationally heavy part of the calculation, and there does not appear to be any obvious method for doing so save for an exhaustive search. Such an exhaustive search does not require anything more complicated than performing finite field (or, equivalently, vector space) addition via XOR, and verifying that a set of elements is closed under addition.

The memory consumption is also very modest, although the computation time does increase exponentially with the dimension. Table 8 shows the time (in seconds) for computing  $N_F$  for  $x^3$  over  $\mathbb{F}_{2^n}$  for all even dimensions  $n$  in the range  $6 \leq n \leq 12$  on a simple implementation in C.

$n$	6	8	10	12
time	0.004	0.168	11.508	1151.773
number of functions	14	8181	-	-
values	6	641	-	-

Table 8: Computation times and number of distinct values for  $N_F$  for some known APN instances

### 4.3 The thickness spectrum

#### 4.3.1 Definition

The notion of the thickness of a vector space is introduced in [22]. Given an  $(n, n)$ -function  $F$ , we begin by finding the set  $\mathcal{Z}_F$  of its Walsh zeros, that is, the

pairs  $(a, b) \in \mathbb{F}_2^{2n}$  on which the Walsh transform of  $F$  evaluate to zero:

$$\mathcal{Z}_F = \{(a, b) : a, b \in \mathbb{F}_2^n, W_F(a, b) = 0\}.$$

The **thickness** of any subspace  $V \subseteq \mathcal{Z}_F$  is defined as the dimension of its projection onto  $\{(0, x) : x \in \mathbb{F}_2^n\}$ . Equivalently, if  $V$  is an  $n$ -dimensional space and  $L$  is a linear permutation of  $\mathbb{F}_2^{2n}$  mapping  $\{(x, 0) : x \in \mathbb{F}_2^n\}$  to  $V$ , we can write  $L$  in matrix form as

$$L = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The thickness of  $V$  is then the rank of  $c$ .

Let  $i$  be any natural number, and let us denote by  $t_F(i)$  the number of  $n$ -dimensional subspaces of  $\mathcal{Z}_F$  that have thickness  $i$ . The **thickness spectrum** of  $F$  is then the vector  $T_F = (t_F(1), t_F(2), \dots, t_F(i))$ , where  $i$  is the smallest natural number for which  $t_F(i) = 0$ . As with the number  $n_F(i)$  of subspaces in the set of non-bent components, if  $t_F(i) = 0$  for some  $i$ , then  $t_F(j) = 0$  for all  $j \geq i$  as well, so the previous definition is justified. The thickness spectrum can then be shown to be invariant under EA-equivalence.

### 4.3.2 Evaluation

The computation of the thickness spectrum involves computing  $\mathcal{Z}_F$  (which essentially involves computing the Walsh spectrum of  $F$ ), then going through all  $n$ -dimensional subspaces of  $\mathcal{Z}_F$  and computing the thickness of each subspace. As with the computation of  $N_F$  in Section 4.2, there is no obviously better way to find these subspaces than by an exhaustive search. On the one hand, this makes the implementation conceptually simple: after finding the  $n$ -dimensional subspaces, it is trivial to project them onto  $\{(0, x) : x \in \mathbb{F}_2^n\}$  by restricting the coordinates on the left-hand side to zero; and computing the dimension of the projections amounts to counting the number of elements that they contain.

On the other hand, the time for computing  $T_F$  is longer than that for computing  $N_F$ , since we have to work over a  $2n$ -dimensional (instead of  $n$ -dimensional) vector space. It is intuitively clear that if  $k < l < n$  are natural numbers, then finding all  $k$ -dimensional subspaces in a set of elements from  $\mathbb{F}_2^{2n}$  is easier than finding all  $l$ -dimensional subspaces. In the case of  $N_F$ , one might restrict to the computation of  $n_F(i)$  for only small values of  $i$ , and still hope to obtain a useful invariant. In the case of  $T_F$ , all  $n$ -dimensional subspaces have to be found before any further computations can take place. Table 9 gives a summary of the time needed for computing the thickness spectrum of  $x^3$  over  $\mathbb{F}_2^n$  with  $6 \leq n \leq 10$  using the SboxU library [44].

$n$	6	7	8	9	10
time	0.86	1.12	0.92	188.174	8.91
total	14	-	8181	46	21
values	8	-	185	40	7

Table 9: Computation time and number of values of the thickness spectrum for some known APN instances

## 4.4 A family of invariants based on zero sums

### 4.4.1 Definition

An algorithm for computationally testing two given  $(n, n)$ -functions  $F$  and  $G$  for EA-equivalence is developed in [37]. The algorithm is based on computing two multisets, one for  $F$  and one for  $G$ , which contain some information about the mapping  $A_1$  from (3). More precisely, each element of  $\mathbb{F}_{2^n}$  occurs in each of the two multisets with a certain multiplicity, and it is shown that  $x$  and  $A_1(x)$  must have the same multiplicity for any  $x \in \mathbb{F}_{2^n}$  if (3) holds. In particular, the multiplicities of the two associated multisets must be the same, i.e. they are invariant under EA-equivalence.

To make this more precise, we let  $k$  be a natural number, and define

$$\Sigma_F^k = \{F(x_1) + F(x_2) + \dots + F(x_k) : x_1, x_2, \dots, x_k \in \mathbb{F}_{2^n}, x_1 + x_2 + \dots + x_k = 0\};$$

in other words,  $\Sigma_F^k$  is the multiset of the sums of  $F$  on all  $k$ -tuples of elements adding up to 0. If  $F$  and  $G$  are EA-equivalent, and  $k$  is even, then the multiplicities of  $\Sigma_F^k$  and  $\Sigma_G^k$  are the same. Note that it does not matter whether these  $k$ -tuples are taken to be ordered, unordered, or whether we allow repetitions among their elements; all of these variations lead to what is essentially the same invariant.

The smallest possible value for which the invariants make sense is  $k = 4$  (for  $k = 2$ , the multiset  $\Sigma_F^2$  would consist only of zeros). The multiplicities of  $\Sigma_F^k$  can always be computed via the Walsh transform; if  $m_F^k(s)$  denotes the multiplicity of  $s \in \mathbb{F}_{2^n}$  in  $\Sigma_F^k$ , then

$$m_F^k(s) = \frac{1}{2^{2n}} \sum_{a, b \in \mathbb{F}_{2^n}} (-1)^{b \cdot s} W_F^k(a, b). \quad (5)$$

This method for computing the multiplicities has the advantage that its time complexity does not depend on the choice of  $k$ , which only affects the power to which the Walsh coefficients in (5) have to be raised.

It must be noted that in the case of APN functions, the  $\Sigma_F^k$  invariant is essentially the same as the  $\Pi_F$  invariant described in Section 3.4; that is, if  $F$  and  $G$  are both APN, then  $\Pi_F = \Pi_G$  if and only if  $\Sigma_F^k$  and  $\Sigma_G^k$  have the same multiplicities. An advantage of  $\Sigma_F^k$  is that it remains invariant for functions of any differential uniformity, while  $\Pi_F$  is invariant only in the case of APN functions;  $\Sigma_F^k$  also provides information about what the EA-equivalence between the two tested functions might look like, while  $\Pi_F$  does not. On the other hand,  $\Pi_F$  is invariant under CCZ-equivalence (and it is easy to find counterexamples showing that  $\Sigma_F^k$  is not), and provides a lower bound on the distance between  $F$  and the closest APN function.

### 4.4.2 Evaluation

One of the advantages of the algorithm in [37] is that it does not require any complicated mathematical or algorithmic machinery, and can be implemented from first principles (as opposed to the linear code test for CCZ-equivalence). Regardless of whether the computation is done from the definition, or via the Walsh transform, it only requires finite field (or vector space) addition, which can be represented as XOR and counting multiplicities of elements from the

finite field. If the Walsh spectrum of  $F$  is available, computing the multiplicities of  $\Sigma_F^k$  via (5) is quite straightforward.

As noted above,  $\Pi_F$  and  $\Sigma_F^4$  have the same distinguishing power in the case of APN functions. In particular, these invariants are only useful in the case of even dimensions, in which case they can take a lot of distinct values. Once again, representatives from the known infinite families of APN functions (with the notable exception of the inverse power function over odd dimensions) have the same value of  $\Sigma_F^4$  as the Gold function  $x^3$ .

## 4.5 Ortho-derivatives

### 4.5.1 Definition

The concept of an ortho-derivative is introduced in [44], and appears to be quite useful for partitioning APN functions into EA-equivalence classes. Given a quadratic  $(n, n)$ -function  $F$ , an **ortho-derivative** of  $F$  is any  $(n, n)$ -function  $\pi_F$  such that

$$\pi_F(a) \cdot (F(x) + F(a+x) + F(a) + F(0)) = 0 \quad (6)$$

for all  $x \in \mathbb{F}_{2^n}$ . In this sense,  $\pi_F$  is orthogonal to all values of the expression  $F(x+a) + F(x) + F(a) + F(0) = D_a F(x) + D_a F(0)$ . Note that ortho-derivatives can be defined for any  $(n, n)$ -function  $F$ ; however,  $F$  is APN if and only if  $\pi_F(a)$  is uniquely defined for all non-zero  $a \in \mathbb{F}_{2^n}$ . Thus, in the APN case, a unique  $(n, n)$ -function  $\pi_F$  can be associated with any APN  $(n, n)$ -function  $F$ . Furthermore, it is known that if  $F$  and  $G$  are two EA-equivalent  $(n, n)$ -functions via  $A_1 \circ F \circ A_2 + A = G$ , then

$$\pi_G = A_1^{-1} \circ \pi_F \circ A_2;$$

thus, the ortho-derivatives of two EA-equivalent APN functions are EA-equivalent themselves. The advantage is that the ortho-derivatives are much more “varied” than the APN functions themselves; as pointed out in [44], it seems that the algebraic degree of  $\pi_F$  is  $n-2$  whenever  $F$  is an APN  $(n, n)$ -function. This is intuitively one of the reasons that the ortho-derivatives can take on many different values for the various invariants under EA-equivalence and CCZ-equivalence, and the latter become immensely more useful for partitioning functions into EA-classes.

### 4.5.2 Evaluation

Computing the truth table of the ortho-derivative itself is a very easy task; the simplest way involves guessing the value of  $\pi_F(a)$  for each  $0 \neq a \in \mathbb{F}_{2^n}$ . For every possible candidate for the value of  $\pi_F(a)$ , it suffices to verify that (6) is satisfied. Computing (6), on other hand, requires nothing more than vector space addition and the implementation of the scalar product. The truth table of  $\pi_F$  can thus be reconstructed very quickly, even for relatively large dimensions. Table 10 gives some sample running times for computing the ortho-derivative of  $x^3$  using SboxU.

Once the ortho-derivative is computed, it remains to apply some of the previously examined invariants, such as the Walsh spectrum and differential spectrum to distinguish between the EA-classes of ortho-derivatives. The number

$n$	6	7	8	9	10	11	12
time	0	0.00021	0.0006	0.002	0.34	0.22	0.84
functions	14	488	-	-	-	-	-
values	14	488	-	-	-	-	-

Table 10: Computation time for finding the TT of  $\pi_F$ , and number of values of the Walsh and differential spectra of  $\pi_F$  for some known APN instances

of values in Table 10 refers to the number of distinct combinations of the Walsh spectrum and differential spectrum observed among the ortho-derivatives. Note that the ortho-derivatives themselves are not APN, and so the differential spectrum becomes a useful invariant. In fact, as indicated by the data in Table 10, this is sufficient to distinguish between all currently known classes of quadratic APN functions. Thus, a partitioning by means of the ortho-derivatives appears to have the same strength in practice as an EA-equivalence test. Nonetheless, invariants can only be used to disprove the equivalence between two functions; and so, if two  $(n, n)$ -functions  $F$  and  $G$  have the same invariants for  $\pi_F$  and  $\pi_G$ , this does not constitute a proof of their EA-equivalence. It would thus be very interesting to find examples of EA-inequivalent functions whose ortho-derivatives have the same spectrum of invariant values.

## 5 Conclusion

We have surveyed some known invariants on  $(n, n)$ -functions under CCZ- and EA-equivalence. The list of invariants is not meant to be complete, but aims to include all invariants that are commonly used in the classification of APN and AB functions in practice. Each of the invariants is evaluated in terms of how easy it is to implement in a general-purpose programming language, how efficient it is to compute, and how well it can distinguish between distinct equivalence classes of  $(n, n)$ -functions.

The considered CCZ-invariants include:

- the differential uniformity and non-linearity (trivial);
- the extended Walsh spectrum  $\mathcal{W}_F$ ;
- the invariants from the associated combinatorial designs  $dev(G_F)$  and  $dev(D_F)$ , viz. the  $\Gamma$ -rank,  $\Delta$ -rank, and order of the multiplier group, as well as the orders of the automorphism groups of  $dev(G_F)$  and  $dev(D_F)$ ;
- the multiset  $\Pi_F$  used in the computation of the lower bound on the Hamming distance between two APN functions (invariant only for APN functions).

The considered EA-invariants include:

- all CCZ-invariants;
- the algebraic degree  $\deg(F)$ ;
- the number  $n_F(i)$  of  $i$ -dimensional subspaces in the set  $\mathcal{S}_F$  of non-bent components of  $F$ ;

- the number  $t_F(i)$  of  $n$ -dimensional subspaces of thickness  $i$  in the set  $\mathcal{Z}_F$  of Walsh zeros of  $F$ ;
- the multiplicities of the elements in the multiset  $\Sigma_F^k$  for all even  $k$ , i.e. the number of times that each element of  $\mathbb{F}_{2^n}$  can be expressed as a sum  $F(x_1) + F(x_2) + \cdots + F(x_k)$  with  $x_1 + x_2 + \cdots + x_k = 0$ .

For each invariant, we have used a straightforward implementation in a suitable programming language (typically C in the case of the invariants that can be defined and computed from first principles, and Magma when more sophisticated mathematical structures or algorithms are involved) and have given example running times over the range of dimensions  $n$  in which the invariant can be reasonably used in practice. We have also counted how many different values each invariant can take over some of the known APN instances in each dimension, and have remarked on the further properties and significance of some of the invariants.

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