The Study of Modulo $2^n$

A General Method To Calculate The Probability Or Correlation Coefficients For Most Statistical Property Of Modulo $2^n$

Zhongfeng Niu

Institute of Information Engineering, Chinese Academy of Sciences, Beijing, China

niuzhongfeng@iie.ac.cn

Abstract. In this paper, we present a new concept named the basic function. By the study of the basic function, we find the $O(n)$-time algorithm to calculate the probability or correlation for some property of Modulo $2^n$, including the difference-linear connective correlation coefficients, the linear approximation correlation coefficients, the differential probability, difference-boomerang connective probability, boomerang connective probability, boomerang-difference connective probability, etc.

Keywords: Modulo addition $2^n$ · Markov chain · basic function · difference-linear connective correlation coefficients · linear approximation correlation coefficients · difference-boomerang connective probability · boomerang connective probability · differential probability · boomerang-difference connective probability

Contents

1 Preliminaries 3
2 The Basic Function 4
   2.1 The motivation and property 4
3 Notion Description About The Basic Function Series 6
4 The $A_a$-set 7
5 The $D_f$-set 12
   5.1 Direct method 13
   5.2 Indirect method 16
   5.3 Instance 17
A Reduce The Redundancy of Matrix 23

Introduction

Since the differential analysis and the linear analysis were proposed [1,2], statistical analysis, which makes use of some significant probability for the block cipher to distinguish from the random permutation, have become one of the research hotspots in the cryptanalysis of block cypher in the last 30 years. And most statistical analysis for block cypher is based on the statistical properties of nonlinear functions [1–5]. Specifically, most statistical properties of nonlinear functions is the probability that a series of boolean vector function with the form $\bigoplus_{i=1}^n f(x)_i$, where each $f(x)_i$ in the $\bigoplus_{i=1}^n f(x)_i$ is constituted by the composite
operation of $\oplus$, the nonlinear function and its inverse, are equal to some given values or the correlation coefficients of a single boolean vector function $\bigoplus_{i=1}^{n} f(x)_i$. In most of block cipher, the construction of nonlinear functions is based on the S-box that it can be regard as the nonlinear function with small scale, especially in the SPN structure and Festial structure, the calculation of statistical properties of nonlinear functions can be completed by calculating the statistical properties of S-box. Due to the S-box with tiny scale, the statistical properties of S-box can be got in a short time by trying all possible values. Thus, for the SPN cipher and the Festial cipher, it almost has no problem to do the statistical analysis, such as differential analysis, linear analysis, boomerang analysis, differential-linear analysis, etc. However, because of the modular addition $2^n$ with large-scale adopted as the nonlinear functions in ARX structure, it is infeasible for the ARX cipher to get the statistical properties of nonlinear functions by trying all possible value. In a word, compared with the SPN structure and the Festial structure, most studies of the utilize of statistical analysis method for the ARX cipher, especially the new method or improvement method proposed in the last decade, have made slow progress due to the above reason.

In order to overcome the above problem, we have to look for the polynomial time algorithms to calculate the statistical properties of modular addition $2^n$. In 2001, Lipmaa, etc [6] and Johan Wallén [7] proposed the polynomial time algorithms to calculate the differential probability and the linear approximation correlation coefficients respectively, which was been about 10 years since the differential attack and the linear analysis were proposed. And in 2013, Schulte-Geers [8] showed that mod $2^n$ is CCZ-equivalent to a quadratic vectorial Boolean function. Based on it, he proposed the explicit formula to calculate the linear approximation correlation coefficients and the differential probability, of which time complex are polynomial. Since then, the CCZ-equivalent relation of mod $2^n$ had been acknowledged as the most powerful method to look for the polynomial time algorithms to calculate the statistical properties of modular addition $2^n$. In addition, a series of new statistical properties, including the difference-linear connective correlation coefficients, difference-boomerang connective probability, boomerang connective probability and boomerang-difference connective probability, etc [1–5], were proposed in order to improve the previous methods. And those methods had better performance in the SPN cipher and the Festial cipher. Unfortunately, according to the study of difference-linear connective correlation coefficients for nonlinear function [9], for any two permutations in the same CCZ-equivalent class, their difference-linear connective correlation coefficients are not in general invariant. It means that the CCZ-equivalent relation can’t be regard as the general method to look for polynomial time algorithms to calculate the statistical properties of modular addition $2^n$. Then, the following questions may be asked naturally:

1. How to find the polynomial time algorithms to calculate such new proposed statistical properties?

2. Does there exits a general method that the explicit formula with polynomial time complex can be got for all of the current statistical properties of modular addition $2^n$ or even the properties that may come up in the future?

**Our contribution**

Firstly, this paper give the $O(n)$-time algorithm to calculate the probability or correlation for newly proposed property of Modulo $2^n$, including the difference-linear connective correlation coefficients, the linear approximation correlation coefficients, the differential probability, difference-boomerang connective probability, boomerang connective probability, boomerang-difference connective probability, etc.

Secondly, for all of the current statistical property used for statistical analysis in block cipher, such as difference-linear connective correlation coefficients, the linear approxima-
tion correlation coefficients, the differential probability, difference-boomerang connective probability, boomerang connective probability, boomerang-difference connective probability, etc, it can be summarized as the probability of $\bigoplus_{j=1}^{n} f(x)_{z,j} = A_j$, $1 \leq j \leq m$ or the correlation coefficients of $\lambda \cdot \bigoplus_{j=1}^{n} f(x)_j$, where $f(x)_{z,j}$, $1 \leq j \leq m$, $1 \leq z \leq n_j$; $f(x)_i$, $1 \leq i \leq n$ are composite operation of $\oplus$, the nonlinear function and its inverse. Based the above fact, we propose the concept of the basic function, which is a composite operation of $\oplus$ and $\oplus$. Then, based on the regular of the basic function, we construct the $Aa$-set and $Df$-set. According to the property of the $Aa$-set and $Df$-set, we can answer the question 2. The general method has been found, which is fit for all of current statistical properties of modular addition $2^n$. As a result, the form of formula for all of current statistical properties of modular addition $2^n$ are similar with the Johan Wallén’s work.

## 1 Preliminaries

In this section, we will introduce some basic knowledge that we will use in the following.

**Definition 1 (Addition modulo $2^n$):** For $x, y \in F_2^n$, define $y \boxplus x = x \oplus y \oplus c_{arry}(x, y)$, where $c_{arry}(x, y) = [c_{n-1}, \ldots, c_0]$. The $i$-th bit $c_i$ is defined as

$$
c_0 = 0 \\
c_{i+1} = (x_i \land y_i) \oplus (x_i \land c_i) \oplus (y_i \land c_i), 0 \leq i \leq n - 1
$$

**Definition 2:** Let $x, y \in F_2^n$, $e \in F_2$, define $y \boxplus x \boxplus e = y \boxplus x \boxplus (0, \ldots, 0, e)$.

Let $c_{arry}^*(x, y) = [c_{n-1}^*, \ldots, c_0^*]$, if we define $i$-th bit $c_i^*$ as

$$
c_0^* = e \\
c_{i+1}^* = (x_i \land y_i) \oplus (x_i \land c_i^*) \oplus (y_i \land c_i^*), 0 \leq i \leq t - 1
$$

And the purpose of defining $c_{arry}^*(x, y)$ is to make $y \boxplus x \boxplus e$ have the same form as $y \boxplus x$:

**Theorems 1:** For $x, y \in F_2^n$, $e \in F_2$, define $y \boxplus x \boxplus e = x \oplus y \oplus c_{arry}^*(x, y)$.

**Proof.** If $e = 0$, then the theorem holds.

If $e = 1$, let $c_i = c_{arry}(x, y)[i]$, $c_i^* = c_{arry}(x \oplus y, (0, \ldots, 0, e))[i]$, for $0 \leq i \leq n$; then

$$
c_0^* = 0 \\
c_1^* = x_0 \oplus y_0 \\
c_{i+1}^* = (x_i \land y_i \land c_i) \land c_i^*, 2 \leq i \leq n - 1
$$

Obviously, $c_0^* \land c_0 = 0$, $c_1^* \land c_1 = x_0 \land y_0 \land (x_0 \oplus y_0) = (x_0 \land y_0) \oplus (x_0 \land y_0) = 0$.

Next, we will proof $c_1^* \land c_i = 0$, for $0 \leq i \leq n - 1$, by introduction. Supposed that $c_1^* \land c_i = 0$ for $1 \leq i \leq k$. Then, for $c_{k+1}^* \land c_{k+1}$, we have

$$
c_{k+1}^* = ((x_k \land y_k \land c_k) \land c_{k+1}^*) \land ((x_k \land y_k) \land (x_k \land c_k) \land (y_k \land c_k)) \\
= ((x_k \land y_k) \land c_{k+1}^*) \land ((x_k \land y_k) \land ((x_k \land y_k) \land c_k)) \\
= (x_k \land y_k) \land (x_k \land y_k) \land c_{k+1}^* \\
= (x_k \land y_k) \land (x_k \land y_k) \land c_{k} = 0
$$
Thus, for $2 \leq i \leq n - 1$, we have $c^i_{i+1} = (x_i \oplus y_i \oplus c_i) \land c^i_0 = (x_i \oplus y_i) \land c^i_1$, and

\[
c^0_0 = c_0 \oplus c^0_0 \oplus 1 = 1 \\
c^1_1 = c_1 \oplus c^1_1 \\
c^i_{i+1} = c^i_{i+1} \land c^{i+1}
\]

\[
= (x_i \land y_i) \oplus (x_i \land (c^i_1 \oplus c_i)) \oplus (y_i \land (c^i_1 \oplus c_i)), 2 \leq i \leq n - 1
\]

Notice that $\text{carry}_2^x(y, y) = \text{carry}(x, y) \oplus \text{carry}(x \oplus y, (0, \cdots, 0, e)) \oplus (0, \cdots, 0, e)$.

Thus, $y \boxplus x \oplus (0, \cdots, 0, e) = x \oplus y \oplus \text{carry}_2^x(x, y)$.

From the definition of $\boxplus$, we can see that the subtraction modulo $2^n$ ($\boxplus$) can be converted into the addition modulo $2^n$ ($\boxminus$):

**Theorems 2:** $y \boxminus x = (x \oplus (1, \cdots, 1)) \boxplus y \oplus 1$.

**Proof.** Notice that $(1, \cdots, 1) = -1 \mod 2^n$ and $\text{carry}(x \oplus (1, \cdots, 1), x) = 0^n$, then

\[
(x \oplus (1, \cdots, 1)) \boxplus x = (1, \cdots, 1) = -1 \mod 2^n,
\]

which is equal to

\[
(x \oplus (1, \cdots, 1)) \boxminus 1 = -x \mod 2^n.
\]

Thus, $y \boxminus x = y - x \mod 2^n = (x \oplus (1, \cdots, 1)) \boxplus y \oplus 1$.

Combing with the theorems 3, the $y \boxplus x$ have the same form as $y \boxminus x$:

**Corollary 1:** $y \boxplus x = (x \oplus (1, \cdots, 1)) \oplus y \oplus \text{carry}_2^x(x \oplus (1, \cdots, 1), y)$.

## 2 The Basic Function

### 2.1 The motivation and property

In the cryptanalysis of block ciphers, the scholars proposed many analysis methods based on some statistics properties of the nonlinear function in the cipher. And these statistics properties can be summarized as the probability of $\bigoplus_{j=1}^{m} f(x)_j = A_j, 1 \leq j \leq m$ or the correlation coefficients of $\lambda \cdot (\bigoplus_{j=1}^{n} f(x)_j)$, where $f(x)_j, 1 \leq j \leq m, 1 \leq z \leq n_j$; $f(x)_j, 1 \leq i \leq n$ are composite operation of $\oplus$, the nonlinear function and its inverse. In the ARX cipher, the sole nonlinear function is the $\boxminus$ and its inverse can be converted into the composite operation of $\boxplus$ and $\boxplus$. Thus, we can define the basic function as composite operation of $\boxplus$ and $\boxminus$:

**Definition 3(The Basic Function):** Supposed that $x, y \in F_2^n, E_k = (e_0, e_1, \cdots, e_{k-1}) \in F_2^n, \alpha_0, \alpha_1, \cdots, \alpha_{k-1} \in F_2^n, \beta_0, \beta_1, \cdots, \beta_{k-1} \in F_2^n$. Let $A_k = (\alpha_0, \alpha_2, \cdots, \alpha_{k-1}), B_k = (\beta_0, \beta_2, \cdots, \beta_{k-1})$, where the element of $A_k, B_k$ is n-dimension vector. Then the $f(x, y)_{E_k, A_k, B_k}$ is called basic function with $k$ order, if $f(x, y)_{E_k, A_k, B_k}$ satisfies the follow the form:

\[
(x_0, y_0) = (x, y) \\
(x_{i+1}, y_{i+1}) = (x_i \oplus \alpha_i, (x_i \oplus \alpha_i) \oplus (y_i \oplus \beta_i) \oplus e_i), 0 \leq i \leq k - 1
\]

Then, for $0 \leq i \leq k - 1$, $(x_i \oplus \alpha_i, (x_i \oplus \alpha_i) \oplus (y_i \oplus \beta_i) \oplus e_i)$ is called the $(i + 1)$-th round function, and $\text{carry}_{2^i}^x(x_i \oplus \alpha_i, y_i \oplus \beta_i)$ is called the carry function of $(i + 1)$-th round function in $f(x, y)_{E_k, A_k, B_k}$.

According to the definition 3, we can see that the relation between the $f(x, y)_{E_{k-1}, A_{k-1}, B_{k-1}}$ and the $f(x, y)_{E_k, A_k, B_k}$:
Remark 3: According to the definition of \( f(x, y)E_k.A_k.B_k \), the \( f(x, y)E_k.A_k.B_k \) can be written as:

\[
\begin{align*}
(x_{k-1}, y_{k-1}) &= f(x, y)E_{k-1}.A_{k-1}.B_{k-1} \\
(x_k, y_k) &= (x_{k-1} \oplus \alpha_{k-1}, (x_{k-1} \oplus \alpha_{k-1}) \oplus (y_{k-1} \oplus \beta_{k-1}) \oplus e_{k-1}) \\
f(x, y)E_k.A_k.B_k &= (x_k, y_k)
\end{align*}
\]

where \( E_{k-1} = E_k[0 : k-2] = (e_0, e_1, \cdots e_{k-2}), A_{k-1} = A_k[0 : k-2] = (\alpha_0, \alpha_2, \cdots \alpha_{k-2}), B_{k-1} = B_k[0 : k-2] = (\beta_0, \beta_2, \cdots \beta_{k-2}) \).

Definition 4: For \( X = X^2 || X^1 \in F_2^n \), where \( X^2 \in F_2^p, X^1 \in F_2^p, p + q = n \). Define \( X = X^2 || X^1 = X^2 \cdot 2^p + X^1 \).

Then, according to the definition of Addition modulo \( 2^n \), we can see that the basic function can be divided into two basic functions:

**Theorem 3:** For any \( X^{i+1}, Y^{i+1}, \alpha_0^{i+1}, \alpha_1^{i+1}, \cdots, \alpha_k^{i+1}, \beta_0^{i+1}, \beta_1^{i+1}, \cdots, \beta_k^{i+1} \in F_2^{k+1} \) and \( E_k = (e_0, e_1, \cdots, e_k) \in F_2^k \). Let \( Y^{i+1} = Y^{i+1}[0 : i \cdot t - 1], Y^{i+1} = Y^{i+1}[i : t : (i+1) \cdot t - 1], X^{i+1} = X^{i+1}[0 : i \cdot t - 1], X^{i+1} = X^{i+1}[i : t : (i+1) \cdot t - 1], \) then the \( k \) order basic function \( f(X^{i+1}, Y^{i+1})_{E_k.A_k^{i+1}.B_k^{i+1}} \) can be written as:

\[
f(X^{i+1}, Y^{i+1})_{E_k.A_k^{i+1}.B_k^{i+1}} = f(X^{i+1}, Y^{i+1})_{A_k^{i+1}.B_k^{i+1}} + f(X^{i+1}, Y^{i+1})_{E_k.A_k^{i+1}.B_k^{i+1}}
\]

where

\[
\begin{align*}
A_k^i &= [\alpha_0^i, \alpha_1^i, \cdots, \alpha_{i-1}^i], \quad 0 \leq m \leq k-1, \alpha_m^i = \alpha_m^{i+1}[0 : i \cdot t - 1]; \\
B_k^i &= [\beta_0^i, \beta_1^i, \cdots, \beta_{i-1}^i], \quad 0 \leq m \leq k-1, \beta_m^i = \beta_m^{i+1}[0 : i \cdot t - 1]; \\
C_k^i &= [\alpha_0^i, \alpha_1^i, \cdots, \alpha_{i-1}^i], \quad 0 \leq m \leq k-1, \alpha_m^i = \alpha_m^{i+1}[i \cdot t : (i+1) \cdot t - 1]; \\
D_k^i &= [\beta_0^i, \beta_1^i, \cdots, \beta_{i-1}^i], \quad 0 \leq m \leq k-1, \beta_m^i = \beta_m^{i+1}[i \cdot t : (i+1) \cdot t - 1]; \\
M_k^i &= [c_0^i, c_1^i, \cdots, c_{k-1}^i], \quad 0 \leq m \leq k-1, c_m^i = carry_{m_i}(X_m^{i+1} \oplus \alpha_m^{i+1}, Y_m^{i+1} \oplus \beta_m^{i+1})[i \cdot t]; \\
S_k^i &= [s_0^i, s_1^i, \cdots, s_{k-1}^i], \quad 0 \leq m \leq k-1, s_m^i = carry_{m_i}(X_m^{i+1} \oplus \alpha_m^{i+1}, Y_m^{i+1} \oplus \beta_m^{i+1})[i \cdot t].
\end{align*}
\]

Moreover, \( S_k^i = M_k^{i+1} \).

**Proof.** Notice that when order \( k = 1 \), according to the Definition 2, \( (X^{i+1} \oplus \alpha_0^{i+1}) \oplus (Y^{i+1} \oplus \beta_0^{i+1}) \oplus e_0 \) can be written as:

\[
(X^{i+1} \oplus \alpha_0^{i+1}) \oplus (Y^{i+1} \oplus \beta_0^{i+1}) \oplus e_0 = ((X^{i+1} \oplus \alpha_0^{i+1}) \oplus (Y^{i+1} \oplus \beta_0^{i+1}) \oplus c_0^i) 2^{i \cdot t} + (X^{i+1} \oplus \alpha_0^{i+1}) \oplus (Y^{i+1} \oplus \beta_0^{i+1}) \oplus e_0
\]

Thus, when order \( k = 1 \), the theorem holds.

Supposed that when order \( m \leq k \), the theorem holds.

When order \( m = k + 1 \), according to the definition of \( carry_{m_i}(x, y) \), the value of the \( i \cdot t - th \) bit of \( carry_{m_i}(X_k^{i+1} \oplus \alpha_k^{i+1}, Y_k^{i+1} \oplus \beta_k^{i+1})[i \cdot t] \) is only rely on the first \( i \cdot t - 1 \) bits of \( X_k^{i+1} \oplus \alpha_k^{i+1} \) and \( Y_k^{i+1} \oplus \beta_k^{i+1} \), namely,

\[
\begin{align*}
carry_{m_i}(X_k^{i+1} \oplus \alpha_k^{i+1}, Y_k^{i+1} \oplus \beta_k^{i+1})[i \cdot t] &= carry_{m_i}(X_k^{i+1} \oplus \alpha_k^{i+1}, Y_k^{i+1} \oplus \beta_k^{i+1})[i \cdot t] \\
&= c_k^i
\end{align*}
\]
Thus,
\[
\begin{align*}
    f(X^{i, i+1}, Y^{i, i+1})_{E_{i+1}, A_{i+1}, B_{i+1}^{i+1}} &= (X^{i, i+1} \oplus \alpha^{i, i+1}_k) \sqcup (Y^{i, i+1} \oplus \beta^{i, i+1}_k) \sqcup e_k \\
    &= ((X^{j,i}_k \oplus \alpha^{2,i}_k) \sqcup (Y^{j,i}_k \oplus \beta^{2,i}_k) \sqcup e_k) \cdot 2^{i+t} + (X^{j,i}_k \oplus \alpha^{1,i}_k) \sqcup (Y^{j,i}_k \oplus \beta^{1,i}_k) \sqcup e_k
\end{align*}
\]

On the other hand, due to the assumption of induction, we have:
\[
\begin{align*}
    (X^{j,i}_k, X^{j,i+1}_k) &= (X^{j,i}_k, X^{j,i}_k) \cdot 2^{i+t} + (X^{j,i}, X^{j,i}) \\
    &= f(X^{j,i}, Y^{j,i})_{E_{i+1}, A_{i+1}, B_{i+1}^{i+1}}
\end{align*}
\]

Thus,
\[
\begin{align*}
    f(X^{i, i+1}, Y^{i, i+1})_{E_{i+1}, A_{i+1}, B_{i+1}^{i+1}} &= (X^{i, i+1} \oplus \alpha^{i, i+1}_k) \sqcup (Y^{i, i+1} \oplus \beta^{i, i+1}_k) \sqcup e_k \\
    &= ((X^{j,i}_k \oplus \alpha^{2,i}_k) \sqcup (Y^{j,i}_k \oplus \beta^{2,i}_k) \sqcup e_k) \cdot 2^{i+t} + (X^{j,i}_k \oplus \alpha^{1,i}_k) \sqcup (Y^{j,i}_k \oplus \beta^{1,i}_k) \sqcup e_k \\
    &= f(X^{j,i}, Y^{j,i})_{E_{i+1}, A_{i+1}, B_{i+1}^{i+1}}
\end{align*}
\]

When \( m = k + 1 \), the theorem holds. \( \square \)

**Remark 4:** Obviously, for any \( E_k, A_k^{i+1}, B_k^{i+1} \), when \( X^{1,i+1}, Y^{1,i+1} \) are given, then \( M_k^{i+1} \) are uniquely identified.

### 3 Notion Description About The Basic Function Series

In order to reduce the redundancy of the article, we will introduce some notion description about the given \( z \) basic function \( \{f(X, Y)_{E_k, A_k, B_k} : 1 \leq m \leq z\} \), where \( X, Y \in F_2^t \), which will be frequently adopted in the following proof.

For \( 1 \leq m \leq z \), let \( \alpha^{1,q}_{0,m}, \alpha^{1,q}_{1,m}, \ldots, \alpha^{1,q}_{k_m-1,m}, \beta^{1,q}_{0,m}, \beta^{1,q}_{1,m}, \ldots, \beta^{1,q}_{k_m-1,m} \in F_2^t \), \( E_m = (c_0, c_1, \ldots, c_{k_m-1}) \in F_2^{k_m} \). In addition, for \( 1 \leq m \leq z \), let
\[
\begin{align*}
    X^{1,q} &= X; Y^{1,q} = Y; \\
    A_m^{q} &= A_m = [\alpha^{1,q}_{0,m}, \alpha^{1,q}_{1,m}, \ldots, \alpha^{1,q}_{k_m-1,m}]; \\
    B_m^{q} &= B_m = [\beta^{1,q}_{0,m}, \beta^{1,q}_{1,m}, \ldots, \beta^{1,q}_{k_m-1,m}].
\end{align*}
\]

And for \( 1 \leq i \leq q - 1 \), define:
\[
\begin{align*}
    Y^{i,q} &= Y^{i+1,q}[0 : i \cdot t - 1] = Y^{i,q}[0 : i \cdot t - 1]; \\
    Y^{2,i} &= Y^{i+1,q}[i \cdot t : (i + 1) \cdot t - 1] = Y^{i,q}[i \cdot t : (i + 1) \cdot t - 1]; \\
    X^{i,q} &= X^{i+1,q}[0 : i \cdot t - 1] = X^{i,q}[0 : i \cdot t - 1]; \\
    X^{2,i} &= X^{i+1,q}[i \cdot t : (i + 1) \cdot t - 1] = X^{i,q}[i \cdot t : (i + 1) \cdot t - 1]; \\
    \alpha^{2,i}_{j,m} &= \alpha^{i+1,q}_{j,m}[i \cdot t : (i + 1) \cdot t - 1] = \alpha^{i,q}_{j,m}[i \cdot t : (i + 1) \cdot t - 1], \text{ where } 0 \leq j \leq k_m - 1, 1 \leq m \leq z; \\
    \beta^{2,i}_{j,m} &= \beta^{i+1,q}_{j,m}[i \cdot t : (i + 1) \cdot t - 1] = \beta^{i,q}_{j,m}[i \cdot t : (i + 1) \cdot t - 1], \text{ where } 0 \leq j \leq k_m - 1, 1 \leq m \leq z; \\
    \alpha^{1,i}_{j,m} &= \alpha^{i+1,q}_{j,m}[0 : i \cdot t - 1] = \alpha^{i,q}_{j,m}[0 : i \cdot t - 1], \text{ where } 0 \leq j \leq k_m - 1, 1 \leq m \leq z; \\
    \beta^{1,i}_{j,m} &= \beta^{i+1,q}_{j,m}[0 : i \cdot t - 1] = \beta^{i,q}_{j,m}[0 : i \cdot t - 1], \text{ where } 0 \leq j \leq k_m - 1, 1 \leq m \leq z.
\end{align*}
\]
Beside this, for $1 \leq i \leq q$, $1 \leq m \leq z$, let

\[
(X_{0,m}^{1,i}, y_{0,m}^{1,i}) = (X^{1,i}, Y^{1,i});
\]

\[
(X_{j+1,m}^{1,i}, y_{j+1,m}^{1,i}) = (X_{j,m}^{1,i} \oplus \alpha_{j,m}^{1,i}, (X_{j,m}^{1,i} \oplus \alpha_{j,m}^{1,i}) \oplus (Y_{j,m}^{1,i} \oplus \beta_{j,m}^{1,i}) \oplus e_{j,m}), \text{ where } 0 \leq j \leq k_m - 1.
\]

and for $1 \leq i \leq q$, let

\[
A_i^m = [\alpha_{0,m}^{1,i}, \alpha_{1,m}^{1,i}, \ldots, \alpha_{k_m-1,m}^{1,i}], \text{ where } 1 \leq m \leq z;
\]

\[
B_i^m = [\beta_{0,m}^{1,i}, \beta_{1,m}^{1,i}, \ldots, \beta_{k_m-1,m}^{1,i}], \text{ where } 1 \leq m \leq z.
\]

then, we have $(X_{k_m,j}^{1,i}, Y_{k_m,j}^{1,i}) = f(X^{1,i}, Y^{1,i})_{E_m, A_m^{i+1}, B_m^{i+1}}$, for $1 \leq m \leq z$.

Secondly, for $1 \leq i \leq q$, we define:

\[
c_{j,m} = \text{carry}^*(X_j^{1,i} \oplus \alpha_{j,m}^{1,i}, Y_j^{1,i} \oplus \beta_{j,m}^{1,i})[i \cdot t], \text{ where } 0 \leq j \leq k_m - 1, 1 \leq m \leq z;
\]

\[
M_m^i = [c_{0,m}^i, c_{1,m}^i, \ldots, c_{k_m-1,m}^i], \text{ where } 1 \leq m \leq z.
\]

and for $1 \leq i \leq q - 1$, define:

\[
s_{j,m} = \text{carry}^*(X_j^{2,i} \oplus \alpha_{j,m}^{2,i}, Y_j^{2,i} \oplus \beta_{j,m}^{2,i})[t], \text{ where } 0 \leq j \leq k_m - 1, 1 \leq m \leq z;
\]

\[
S_m^i = [s_{0,m}^i, s_{1,m}^i, \ldots, s_{k_m-1,m}^i], \text{ where } 1 \leq m \leq z.
\]

According to the Remark 4, we have:

**Remark 6:** For $1 \leq m \leq r_j$, $1 \leq i \leq q - 1$, $M_m^{i+1} = S_{km,j}^i$.

In addition, for convenience to the follow following discussion, for $1 \leq i \leq q - 1$, let

\[
T_0^m = A_{m}^{i}, \text{ where } 1 \leq m \leq z;
\]

\[
D_0^m = B_{m}^{i}, \text{ where } 1 \leq m \leq z;
\]

\[
T_i^m = [\alpha_{0,m}^{2,i}, \alpha_{1,m}^{2,i}, \ldots, \alpha_{k_m-1,m}^{2,i}], \text{ where } 1 \leq m \leq z;
\]

\[
D_i^m = [\beta_{0,m}^{2,i}, \beta_{1,m}^{2,i}, \ldots, \beta_{k_m-1,m}^{2,i}], \text{ where } 1 \leq m \leq z.
\]

### 4 The Aa-set

**Definition 6:** For $X^{1,i+1}, Y^{1,i+1} \in F_2^{(i+1)t}$, $0 \leq i \leq q - 1$, given any $z$ basic function \(\{f(X, Y)_{E_m, A_m^{i}, B_m^{i}}; 1 \leq m \leq z\}\), where $X, Y \in F_2^{(i+1)t}$. We define the Aa-set with $Out, In$ as

\[
A_{m}^{i+1}(A_{j}^{i+1}, B_{j}^{i+1}, 1 \leq j \leq z)
\]

\[
= \{(X^{1,i+1}, Y^{1,i+1}) | X^{1,i+1}, Y^{1,i+1} \in F_2^{(i+1)t}, Out = (M_1^{i+1}, \ldots, M_z^{i+1})\}
\]

where $Out, In \in F_2^d, d = \sum_{i=0}^{z} k_i, In = (E_{k_1}, \ldots, E_{k_z})$.

Let $d = \sum_{i=0}^{z} k_i$, then there are $2^d$ possible results for $(M_1^{i+1}, \ldots, M_z^{i+1})$, thus

**Property 3:** For $X^{1,i+1}, Y^{1,i+1} \in F_2^{(i+1)t}$, $0 \leq i \leq q - 1$, given any $z$ basic function \(\{f(X, Y)_{E_m, A_m^{i}, B_m^{i}}; 1 \leq m \leq z\}\), where $X, Y \in F_2^{(i+1)t}$. Then, for all $Out \in F_2^d$, the Aa-set with $Out, In$ satisfies:

\[
\bigcup_{Out \in F_2^d} A_{out}^{i+1}(A_{j}^{i+1}, B_{j}^{i+1}, 1 \leq j \leq z) = \{(X^{1,i+1}, Y^{1,i+1}) | X^{1,i+1}, Y^{1,i+1} \in F_2^{(i+1)t}\}
\]

According to the **Property 2**, we have:
Property 4: For $X_{1,i+1}, Y_{1,i+1} \in F_2^{(i+1)}$, $0 \leq i \leq q - 1$, given any $z$ basic function \( f(X,Y)_{E_{km}} \), \( A_{E_{km}} = B_{E_{km}} \), $1 \leq m \leq z \}, \) where $X,Y \in F_2^{q,t}$. For any $Out_1, Out_2 \in F_2^q$ satisfied $Out_1 \neq Out_2$, then 

\[
A_{Out_1,In}(A_{j+1}^i, B_{j+1}^i, 1 \leq j \leq z) \cap A_{Out_2,In}(A_{j+1}^i, B_{j+1}^i, 1 \leq j \leq z) = \emptyset
\]

holds.

From theorem 3, we can see that the Aa-set with $Out$, In can be divided into many subset, and each disjoint subset has the following recursive structure:

Lemma 1: For $X_{1,i+1}, Y_{1,i+1} \in F_2^{(i+1)}$, $1 \leq i \leq q - 1$, given any $z$ basic function \( f(X,Y)_{E_{km}} \), \( A_{E_{km}} = B_{E_{km}} \), $1 \leq m \leq z \}, \) where $X,Y \in F_2^{q,t}$. We define the Aa-set with $Out$, $Mi$, In as 

\[
A_{Out, Mi, In}^{i+1}(A_{j+1}^i, B_{j+1}^i, 1 \leq j \leq z) = \left\{ \begin{align*} 
&X_{1,i}, Y_{1,i} \in A_{Out, Mi, In}^i(A_{j+1}^i, B_{j+1}^i, 1 \leq j \leq z); \\
&M_i = (M_1^i, \ldots, M_j^i); \\
&X_{2,i}, Y_{2,i} \in A_{Out, Mi}^j(T_{j+1}, D_{j+1}), 1 \leq j \leq z). 
\end{align*} \right. 
\]

where $Mi, Out, In \in F_2^q$, $d = \sum_{i=0}^{z} kr_i$, $In = (E_{k_1}, \ldots, E_{k_z})$. Then, 

\[
A_{Out, Mi, In}^{i+1}(A_{j+1}^i, B_{j+1}^i, 1 \leq j \leq z) = \bigcup_{Mi \in F_2^q} A_{Out,Mi,In}^{i+1}(A_{j+1}^i, B_{j+1}^i, 1 \leq j \leq z)
\]

holds, and for $Mi_1 \neq Mi_2$, $Mi_1, Mi_2 \in F_2^q$, satisfy 

\[
A_{Out,Mi_1,In}^{i+1}(A_{j+1}^i, B_{j+1}^i, 1 \leq j \leq z) \cap A_{Out,Mi_2,In}^{i+1}(A_{j+1}^i, B_{j+1}^i, 1 \leq j \leq z) = \emptyset
\]

Proof. Firstly, due to $A_{j+1}^i, B_{j+1}^i$ can be decided according to the definition from the $A_{j+1}^i, B_{j+1}^i$. And from the property 4, we know that for $Mi_1 \neq Mi_2$, 

\[
A_{Mi_1,In}(A_{j+1}^i, B_{j+1}^i, 1 \leq j \leq z) \cap A_{Mi_2,In}^{i+1}(A_{j+1}^i, B_{j+1}^i, 1 \leq j \leq z) = \emptyset
\]

Thus, 

\[
A_{Out,Mi_1,In}(A_{j+1}^i, B_{j+1}^i, 1 \leq j \leq z) \cap A_{Out,Mi_2,In}^{i+1}(A_{j+1}^i, B_{j+1}^i, 1 \leq j \leq z) = \emptyset
\]

Secondly, from theorem 3, for $1 \leq j \leq z$, we have: 

\[
f(X_{1,i+1}, Y_{1,i+1})_{E_1,A_{j+1}^i,B_{j+1}^i} = f(X_{2,i}, Y_{2,i})_{M_1, T_1, D_1, 2+i^t} \cdot 2+i^t + f(X_{1,i}, Y_{1,i})_{E_2,A_{j+1}^i,B_{j+1}^i}.
\]

Thus, 

\[
A_{Out, In}^{i+1}(A_{j+1}^i, B_{j+1}^i, 1 \leq j \leq z) =f(X_{1,i+1}, Y_{1,i+1})_{E_1,A_{j+1}^i,B_{j+1}^i}, Out = (M_1^{i+1}, \ldots, M_{z+1}^{i+1})
\]

\[
= \bigcup_{Mi \in F_2^q} \left\{ \begin{align*} 
&X_{1,i}, Y_{1,i} \in F_2^{(i+1)}; Out = (M_1^i, \ldots, M_z^i); \\
&M_i = (M_1^i, \ldots, M_z^i); \\
&X_{2,i}, Y_{2,i} \in F_2^q; Out = (S_1^i, \ldots, S_z^i).
\end{align*} \right.
\]

\[
= \bigcup_{Mi \in F_2^q} \left\{ \begin{align*} 
&X_{1,i}, Y_{1,i} \in A_{Out, Mi, In}^i(A_{j+1}^i, B_{j+1}^i, 1 \leq j \leq z); \\
&M_i = (M_1^i, \ldots, M_j^i); \\
&X_{2,i}, Y_{2,i} \in A_{Out, Mi}^j(T_{j+1}, D_{j+1}), 1 \leq j \leq z).
\end{align*} \right.
\]

\[
= \bigcup_{Mi \in F_2^q} A_{Out, Mi, In}^{i+1}(A_{j+1}^i, B_{j+1}^i, 1 \leq j \leq z)
\]

Theorem 4: For $X^{i+1}, Y^{i+1} \in F_2^{(i+1)+i}, 0 \leq i \leq q - 1$, given any z basic function \( \{ f(X, Y) \} \mathcal{E}_{k_m, \mathcal{A}^{(i+1)+i}_{k_m}, \mathcal{B}^{(i+1)+i}_{k_m}}; 1 \leq m \leq z \} \), where $X, Y \in F_2^{(i+1)+i}$. Supposed that $\gamma, \lambda, v, w \in F_2^{(i+1)+i}$, \( Out, In, Middle \in F_2^q \), where $d = \sum_{i=0}^{z-1} k_i$. $In = (E_{k_1}, \cdots E_{k_i})$. Define the correlation coefficients of $h(X^{i+1}, Y^{i+1})$ as:

$$
Cor_{In}^{i+1} \left( \gamma, \lambda, v, w, A_j^{i+1}, B_j^{i+1} \right) = \frac{1}{2^{2(i+1)+i}} \sum_{X, Y \in F_2^{(i+1)+i}} (-1)^{\gamma, \lambda, v, w} h(X^{i+1}, Y^{i+1}) \oplus X^{i+1} \oplus Y^{i+1}
$$

where $h(X^{i+1}, Y^{i+1}) = \bigoplus_{m=1}^{z-1} f(X^{i+1}, Y^{i+1}) \mathcal{E}_{k_m, \mathcal{A}^{(i+1)+i}_{k_m}, \mathcal{B}^{(i+1)+i}_{k_m}}$.

Additionally, define the correlation coefficients of $h(X^{i+1}, Y^{i+1})$ over the Aa-set with $Out, In$ as:

$$
Cor_{Out, In}^{i+1} \left( \gamma, \lambda, v, w, A_j^{i+1}, B_j^{i+1} \right) = \frac{1}{2^{2(i+1)+i}} \sum_{X, Y \in F_2^{(i+1)+i}} (-1)^{\gamma, \lambda, v, w} h(X^{i+1}, Y^{i+1}) \oplus X^{i+1} \oplus Y^{i+1}
$$

And for $1 \leq i \leq q - 1$, define the correlation coefficients of $h(X^{i+1}, Y^{i+1})$ over the Aa-set with $Out, Mi, In$ as:

$$
Cor_{Out, Mi, In}^{i+1} \left( \gamma, \lambda, v, w, A_j^{i+1}, B_j^{i+1} \right) = \frac{1}{2^{2(i+1)+i}} \sum_{X, Y \in F_2^{(i+1)+i}} (-1)^{\gamma, \lambda, v, w} h(X^{i+1}, Y^{i+1}) \oplus X^{i+1} \oplus Y^{i+1}
$$

From the property of the Aa-set, we can see that the correlation coefficients of $h(X^{i+1}, Y^{i+1})$ is equal to the sum of the $h(X^{i+1}, Y^{i+1})$'s correlation coefficients over the Aa-set with $Out, In$ for all $Out \in F_2^q$. And the correlation coefficients of $h(X^{i+1}, Y^{i+1})$ over the Aa-set with $Out, In$ have the following recursive structure:

Theorem 4: For $X^{i+1}, Y^{i+1} \in F_2^{(i+1)+i}, 0 \leq i \leq q - 1$, given any z basic function \( \{ f(X, Y) \} \mathcal{E}_{k_m, \mathcal{A}^{(i+1)+i}_{k_m}, \mathcal{B}^{(i+1)+i}_{k_m}}; 1 \leq m \leq z \} \), where $X, Y \in F_2^{(i+1)+i}$. Supposed that $\gamma, \lambda, v, w \in F_2^{(i+1)+i}$, \( Out, In, Middle \in F_2^q \), where $d = \sum_{i=0}^{z-1} k_i$. $In = (E_{k_1}, \cdots E_{k_i})$. Then:

$$
Cor_{In}^{i+1} \left( \gamma, \lambda, v, w, A_j^{i+1}, B_j^{i+1} \right) = \sum_{Out \in F_2^q} Cor_{Out, In}^{i+1} \left( \gamma, \lambda, v, w, A_j^{i+1}, B_j^{i+1} \right)
$$
And for $1 \leq i \leq q - 1$, 
\[
\begin{align*}
\text{Cor}^{i+1}_{\text{Out}, \text{In}} \left( A^{i+1}_j, B^{i+1}_j, 1 \leq j \leq z \right) &= \sum_{Mi \in P^d_2} \text{Cor}^{i+1}_{\text{Out}, \text{Mi}, \text{In}} \left( A^{i+1}_j, B^{i+1}_j, 1 \leq j \leq z \right) \\
&= \sum_{Mi \in P^d_2} \text{Cor}^{i+1}_{\text{Out}, \text{Mi}} \left( T^{i+1}_j, D^{i+1}_j, 1 \leq j \leq z \right) \times \text{Cor}^{j}_{\text{Mi}, \text{In}} \left( A^{j+1}_i, B^{j+1}_i, 1 \leq j \leq z \right)
\end{align*}
\]
where
\[
\begin{align*}
\gamma^1 &= \gamma[0 : i \cdot t - 1], \quad \lambda^1 = \lambda[0 : i \cdot t - 1], \quad v^1 = v[0 : i \cdot t - 1], \quad w^1 = w[0 : i \cdot t - 1], \\
\gamma^2 &= \gamma[i \cdot t : (i + 1) \cdot t - 1], \quad \lambda^2 = \lambda[i \cdot t : (i + 1) \cdot t - 1], \quad v^2 = v[i \cdot t : (i + 1) \cdot t - 1], \quad w^2 = w[i \cdot t : (i + 1) \cdot t - 1].
\end{align*}
\]

Proof. According to the property 3 and property 4, we have:
\[
\begin{align*}
\text{Cor}^{i+1}_{\text{In}} \left( A^{i+1}_j, B^{i+1}_j, 1 \leq j \leq z \right) &= \frac{1}{2^{2(t+1)\ell}} \sum_{(X_{1,i+1}, Y_{1,i+1}) \in F^d_{i+1}} (-1)^{(\gamma, \lambda)} h(X^{1,i+1} Y^{1,i+1} \otimes X^{1,i+1} \otimes w Y^{1,i+1}) \\
&= \frac{1}{2^{2(t+1)\ell}} \sum_{(X_{1,i+1}, Y_{1,i+1}) \in \bigcup_{Out \in P^d_2} \text{An}^{i+1}_{Out, \text{In}} (A^{i+1}_j, B^{i+1}_j, 1 \leq j \leq z)} (-1)^{(\gamma, \lambda)} h(X^{1,i+1} Y^{1,i+1} \otimes X^{1,i+1} \otimes w Y^{1,i+1}) \\
&= \frac{1}{2^{2(t+1)\ell}} \sum_{Out \in P^d_2} \text{Cor}^{i+1}_{\text{In}, \text{Out}} (A^{i+1}_j, B^{i+1}_j, 1 \leq j \leq z)
\end{align*}
\]
Likely, from lemma 1, we can also get
\[
\begin{align*}
\text{Cor}^{i+1}_{\text{In}, \text{Out}} (A^{i+1}_j, B^{i+1}_j, 1 \leq j \leq z) &= \sum_{Mi \in P^d_2} \text{Cor}^{i+1}_{\text{In}, \text{Mi}, \text{Out}} (A^{i+1}_j, B^{i+1}_j, 1 \leq j \leq z)
\end{align*}
\]
Secondly, from theorem 3, for $1 \leq j \leq z$, we have
\[
\begin{align*}
f(X^{1,i+1}, Y^{1,i+1})_{E_j A^{i+1}_j B^{i+1}_j} &= f(X^{2,i}, Y^{2,i})_{M^i j, T^i j, D^i j, 2^t} + f(X^{1,i}, Y^{1,i})_{E_j A^{j}_j B^{j}_j} \\
\text{Then}
\begin{align*}
h(X^{1,i+1}, Y^{1,i+1}) &= \bigoplus_{j=1}^{z} h(X^{1,i+1}, Y^{1,i+1})_{E_j A^{i+1}_j B^{i+1}_j} \\
&= \bigoplus_{j=1}^{z} f(X^{2,i}, Y^{2,i})_{M^i j, T^i j, D^i j, 2^t} + \bigoplus_{j=1}^{z} f(X^{1,i}, Y^{1,i})_{E_j A^{j}_j B^{j}_j} \\
&= h(X^{1,i}, Y^{1,i}) \cdot 2^t + h(X^{2,i}, Y^{2,i})
\end{align*}
\]
It can be concluded that:

\[
\text{Cor}^{i+1}_{\text{In}, \text{M}, \text{Out}} \left( A^i_j, B^i_j, 1 \leq j \leq z \right) \\
= \frac{1}{2^{2it}} \sum_{(X^{i,i}, Y^{i,i}) \in A^i_{\text{M}, \text{In}}(A^i_j, B^i_j, 1 \leq j \leq z)} (-1)^{(\gamma, \lambda) \cdot h(X^{i,i}, Y^{i,i})} V^{i,i} \sum_{X, Y} \leq 1 \leq z \times 1^{2it} \sum_{(X^{i,i}, Y^{i,i}) \in A^i_{\text{M}, \text{In}}(T^i_j, D^i_j, 1 \leq j \leq z)} (-1)^{(\gamma, \lambda) \cdot h(X^{i,i}, Y^{i,i})} V^{i,i} \sum_{X, Y} \leq 1 \leq z.
\]

Thus,

\[
\text{Cor}^{i+1}_{\text{Out}, \text{M}, \text{In}} \left( \gamma, \lambda, v, w, \right) \\
= \sum_{M \in \mathcal{F}^2_i} \text{Cor}^{i+1}_{\text{Out}, \text{M}, \text{In}} \left( \gamma, \lambda, v, w, 1 \leq j \leq z \right) \\
= \sum_{M \in \mathcal{F}^2_i} \text{Cor}^{i+1}_{\text{Out}, \text{M}, \text{In}} \left( \gamma, \lambda, v, w, T^i_j, D^i_j, 1 \leq j \leq z \right) \times \text{Cor}^{i+1}_{\text{M}, \text{In}} \left( \gamma, \lambda, v, w, 1 \leq j \leq z \right)
\]

Given any \(z\) basic function \(\{f(X, Y)_{E_{km}, A^i_{km}, B^i_{km}}; 1 \leq m \leq z\}\), where \(X, Y \in F_2^q\). For \(0 \leq i \leq q - 1\), supposed that matrix \(M^{a_i} \in R^{2^{it} \times 2^t}\), of which elements define as:

\[
M^{a_i}[\text{Out}][\text{In}] = \text{Cor}^{i+1}_{\text{Out}, \text{In}} \left( \gamma, \lambda, v, w, 1 \leq j \leq z \right)
\]

where \(\text{Out}, \text{In} \in F_2^d\), \(\gamma, \lambda, v, w \in F_2^q\), \(\gamma = \gamma[i : t : (i+1) \cdot t - 1], \lambda = \lambda[i : t : (i+1) \cdot t - 1], v = v[i : t : (i+1) \cdot t - 1], w = w[i : t : (i+1) \cdot t - 1]\).

Then, we have:

**Theorem 6:** Given any \(z\) basic function \(\{f(X, Y)_{E_{km}, A^i_{km}, B^i_{km}}; 1 \leq m \leq z\}\), where \(X, Y \in F_2^q\). For \(\gamma, \lambda, v, w \in F_2^q\), \(\text{In} \in F_2^d\), where \(d = \sum_{i=0}^{q-1} k_i\), \(\text{In} = (E_{k_1}, \cdots E_{k_z})\). Then,

\[
\text{Cor}^q_{\text{In}} \left( \gamma, \lambda, v, w, 1 \leq j \leq z \right) = \prod_{i=0}^{q-1} M^{a_i} Q^{i}_{\text{In}}
\]

where \(1 \leq i \leq q - 1, L = (1, 1, \cdots, 1) \in F_2^d, Q_{\text{In}} \in F_2^d\), of which sole nonzero component satisfies \(Q[\text{In}] = 1\).

**Proof.** For \(1 \leq i \leq q\), define the vector \(\text{Base}^{i}_{\text{In}} \in F_2^q\) as follow:

\[
\text{Base}^{i}_{\text{In}}[\text{Out}] = \text{Cor}^{i+1}_{\text{Out}, \text{In}} \left( \gamma, \lambda, v, w, 1 \leq j \leq z \right)
\]

where \(\text{Out} \in F_2^q, \gamma = \gamma[0 : i \cdot t - 1], \lambda = \lambda[0 : i \cdot t - 1], v = v[0 : i \cdot t - 1], w = w[0 : i \cdot t - 1].\)

According to the **lemma 1**, we have:

\[
\text{Cor}^q_{\text{In}} \left( \gamma, \lambda, v, w, 1 \leq j \leq z \right) = L \cdot \text{Base}^{i}_{\text{In}}
\]
And by the definition, we see that $\text{Base}^i = M_a^0 \cdot Q^T$.
In addition, according to the lemma 1, for $1 \leq i \leq q - 1$, $0 \leq \text{Out} \leq 2^d - 1$, we have
\[
\text{Base}^i_{In}[\text{Out}] = \sum_{\text{Out} \in F^d} M_a^i[\text{Out}][\text{Out}_1] \cdot \text{Base}^i_{In}[\text{Out}_1]
\]
Namely,
\[
\text{Base}^i_{In} = M_a^i \cdot \text{Base}^i_{In}
\]
Thus, the theorem holds.

\[\square\]

5 The $Df$-set

Recall the law of total probability, given a series of subset $\{A_i, 1 \leq i \leq n\}$ of the total space, where $\bigcup_{i=1}^{n} A_i$ is to equal the total space and each $A_i$ are disjoint with each other, then for any event $B$, its probability $P(B)$ is equal to $\sum_{i=0}^{n} P(B \cap A_i)$. Notice that the the Aa-set with $Out$, $In$ satisfy the above property, thus the same idea can be adopted to do the following study.

**Definition 8:** For $X^1_{i+1}, Y^1_{i+1} \in F^{(i+1)-t}$, $0 \leq i \leq q - 1$, given any $z$ basic function $\{f(X, Y)_{E_k, A_m, A_m^0, B_m^0} : 1 \leq m \leq z\}$, where $X, Y \in F^q_t$. Supposed that $G \in F^{n \times z}$, $C = (J, N) \in (F^{n \times (i+1)-t}, F^{n \times (i+1)-t})$, $Out, In \in F^d_2$, where $d = \sum_{i=0}^{q} k_i$, $In = (E_k_1, \cdots E_k_n)$. We define the $Df$-set with $In$ and the $Df$-set with $Out$, $In$ respectively:

\[
\begin{align*}
Df_{Out, In}^{i+1,n}(C, G, A_{j+1}^{i+1}, B_{j+1}^{i+1}; 1 \leq j \leq z) &= \left\{ (X^1_{i+1}, Y^1_{i+1}) \bigg| X^1_{i+1}, Y^1_{i+1} \in F^{(i+1)-t}_{2} \right. \\
h_s(X^1_{i+1}, Y^1_{i+1}) &= C[s], 1 \leq s \leq n.
\end{align*}
\]

\[
\begin{align*}
Df_{In}^{i+1,n}(C, G, A_{j+1}^{i+1}, B_{j+1}^{i+1}; 1 \leq j \leq z) &= \left\{ (X^1_{i+1}, Y^1_{i+1}) \bigg| X^1_{i+1}, Y^1_{i+1} \in F^{(i+1)-t}_{2} \right. \\
h_s(X^1_{i+1}, Y^1_{i+1}) &= C[s], 1 \leq s \leq n.
\end{align*}
\]

\[
\begin{align*}
Df_{Out, In}^{i+1,n}(C, G, A_{j+1}^{i+1}, B_{j+1}^{i+1}; 1 \leq j \leq z) &= \left\{ (X^1_{i+1}, Y^1_{i+1}) \bigg| X^1_{i+1}, Y^1_{i+1} \in F^{(i+1)-t}_{2} \right. \\
h_s(X^1_{i+1}, Y^1_{i+1}) &= C[s], 1 \leq s \leq n.
\end{align*}
\]

\[
\begin{align*}
Df_{Out, In}^{i+1,n}(C, G, A_{j+1}^{i+1}, B_{j+1}^{i+1}; 1 \leq j \leq z) &= (F^{n \times (i+1)-t}_2 \times F^{n \times (i+1)-t}_2) - Df_{In}^{i+1,n}(C, G, A_{j+1}^{i+1}, B_{j+1}^{i+1}; 1 \leq j \leq z)) \cap A_{Out, In}^{i+1,n}(A_{j+1}^{i+1}, B_{j+1}^{i+1}; 1 \leq j \leq z)
\end{align*}
\]

where
\[
h_s(X^1_{i+1}, Y^1_{i+1}) = \bigoplus_{m=1}^{n} G[s, m] \cdot f(X^1_{i+1}, Y^1_{i+1})_{E_k, A_m^{i+1}, B_m^{i+1}}, 1 \leq s \leq n.
\]

According to the property 3 and property 4 we have:

**Property 5:** For $X^1_{i+1}, Y^1_{i+1} \in F^{(i+1)-t}_{2}$, $0 \leq i \leq q - 1$, given any $z$ basic function $\{f(X, Y)_{E_k, A_m, A_m^0, B_m^0} : 1 \leq m \leq z\}$, where $X, Y \in F^q_t$. Supposed that $G \in F^{n \times z}$, $C = (J, N) \in (F^{n \times (i+1)-t}_2, F^{n \times (i+1)-t}_2)$, $Out, In \in F^d_2$, where $d = \sum_{i=0}^{q} k_i$, $In = (E_k_1, \cdots E_k_n)$. Then

\[
\bigcup_{Out \in F^d_2} Df_{Out, In}^{i+1,n}(C, G, A_{j+1}^{i+1}, B_{j+1}^{i+1}; 1 \leq j \leq z) = Df_{In}^{i+1,n}(C, G, A_{j+1}^{i+1}, B_{j+1}^{i+1}; 1 \leq j \leq z)
\]
According to the Property 2, we have:

**Property 6:** For $X^{i,+1}, Y^{i,+1} \in F^{(i+1)-t}_{2}$, $0 \leq i \leq q - 1$, given any $z$ basic function \( \{f(X,Y)_{E_{m,A_{m}},B_{m}}; 1 \leq m \leq z\} \), when $X,Y \in F^{q-t}_{2}$. Supposed that $G \in F^{n \times z}_{2}$, $C = (J,N) \in (F^{n \times (i+1)-t}_{2}, F^{n \times (i+1)-t}_{2})$, $Out, In \in F^{d}_{2}$, where $d = \sum_{i=0}^{z} k_i$, $In = (E_{k_1}, \ldots, E_{k_z})$.

If any two \( \hat{Out}_1, \hat{Out}_2 \in F^{d}_{2} \) satisfy $Out_1 \neq Out_2$, then

$$D_{f^{i+1,n}_{Out,In}}(C,G,A_{j}^{i+1},B_{j}^{i+1}, 1 \leq j \leq z) \cap D_{f^{i+1,n}_{Out,In}}(C,G,A_{j}^{i+1},B_{j}^{i+1}, 1 \leq j \leq z) = \emptyset$$

As same as the law of total probability, we can get the relation between the Df-set with $In$ and the Df-set with $Out$, $In$:

**Theorem 5:** For $X^{i,+1}, Y^{i,+1} \in F^{(i+1)-t}_{2}$, $0 \leq i \leq q - 1$, given any $z$ basic function \( \{f(X,Y)_{E_{m,A_{m}},B_{m}}; 1 \leq m \leq z\} \), when $X,Y \in F^{q-t}_{2}$. Supposed that $G \in F^{n \times z}_{2}$, $C = (J,N) \in (F^{n \times (i+1)-t}_{2}, F^{n \times (i+1)-t}_{2})$, $Out, In \in F^{d}_{2}$, where $d = \sum_{i=0}^{z} k_i$, $In = (E_{k_1}, \ldots, E_{k_z})$.

Then,

$$\sum_{Out \in F^{d}_{2}} \#D_{f^{i+1,n}_{Out,In}}(C,G,A_{j}^{i+1},B_{j}^{i+1}, 1 \leq j \leq z) = \#D_{f^{i+1,n}_{In}}(C,G,A_{j}^{i+1},B_{j}^{i+1}, 1 \leq j \leq z)$$

### 5.1 Direct method

According to the Lemma 1 and Theorem 3, it can be concluded that the Df-set with $Out$, $In$ have the following recursive structure:

**Lemma 2:** For $X^{i,+1}, Y^{i,+1} \in F^{(i+1)-t}_{2}$, $0 \leq i \leq q - 1$, given any $z$ basic function \( \{f(X,Y)_{E_{m,A_{m}},B_{m}}; 1 \leq m \leq z\} \), when $X,Y \in F^{q-t}_{2}$. Supposed that $G \in F^{n \times z}_{2}$, $C = (J,N) \in (F^{n \times (i+1)-t}_{2}, F^{n \times (i+1)-t}_{2})$, $Out, In \in F^{d}_{2}$, where $d = \sum_{i=0}^{z} k_i$, $In = (E_{k_1}, \ldots, E_{k_z})$.

Let $J^1 = J[0 : i \cdot t - 1], J^2 = J[i : i \cdot t : (i+1) \cdot t - 1], N^1 = N[0 : i \cdot t - 1], N^2 = N[i : t : (i+1) \cdot t - 1], C^1 = (J^1, N^1), C^2 = (J^2, N^2)$.

Define

$$D_{f^{i+1,n}_{Out,In}}(C,G,A_{j}^{i+1},B_{j}^{i+1}, 1 \leq j \leq z) = \left\{\begin{array}{l}
(X^{1,i} || X^{2,i} ; Y^{1,i} || Y^{2,i}) \\
X^{1,i}, Y^{1,i} \in D_{f^{i,n}_{Out,In}}(C^1,G,A_{j}^{i+1},B_{j}^{i+1}, 1 \leq j \leq z) \\
X^{2,i}, Y^{2,i} \in D_{f^{i,n}_{Out,In}}(C^2,G,T_{j}^{i},D_{j}^{i}, 1 \leq j \leq z)
\end{array}\right.$$ 

Then,

$$D_{f^{i+1,n}_{Out,In}}(C,G,A_{j}^{i+1},B_{j}^{i+1}, 1 \leq j \leq z) = \bigcup_{M \in F^{d}_{2}} D_{f^{i+1,n}_{Out,M,In}}(C,G,A_{j}^{i+1},B_{j}^{i+1}, 1 \leq j \leq z)$$

holds. And for any two $M_{i1}, M_{i2} \in F^{d}_{2}$, where $M_{i1} \neq M_{i2}$, satisfy

$$D_{f^{i+1,n}_{Out,M_{i1},In}}(C,G,A_{j}^{i+1},B_{j}^{i+1}, 1 \leq j \leq z) \cap D_{f^{i+1,n}_{Out,M_{i2},In}}(C,G,A_{j}^{i+1},B_{j}^{i+1}, 1 \leq j \leq z) = \emptyset$$

**Proof.** According to the Theorem 3, we have

$$f(X^{i+1}, Y^{i+1})_{E_{k_j},A_{j}^{i+1},B_{j}^{i+1}} = f(X^{i+1}, Y^{i+1})_{E_{k_j},A_{j}^{i+1},B_{j}^{i+1}} + f(X^{2,i}, Y^{2,i})_{M_{i1},T_{j}^{i},D_{j}^{i}, 2^{i+t}} + f(X^{1,i}, Y^{1,i})_{E_{k_j},A_{j}^{i+1},B_{j}^{i+1}}.$$
Thus, for $1 \leq s \leq n$:

$$
\begin{align*}
h_s(X^{1,i+1}, Y^{1,i+1}) &= \bigoplus_{m=1}^{z} G[s, m] \ast f(X^{1,i+1}, Y^{1,i+1})_{E_{km}, A^i_{m+1}, B^s_m} \\
&= \bigoplus_{M=1}^{z} G[s, m] \ast f(X^{2,i}, Y^{2,i})_{M^i_m, T^i_m, D^i_m, 2^{i-t}} + \bigoplus_{m=1}^{z} G[s, m] \ast f(X^{1,1}, Y^{1,1})_{E_{km}, A^i_{m+1}, B^s_m} \\
&= h_s(X^{1,i}, Y^{1,i}) \cdot 2^{i-t} + h_s(X^{2,i}, Y^{2,i})
\end{align*}
$$

It means that:

$$
\begin{align*}
Df^{i+1,n}_{in}(C, G, A^{i+1}_{j}, B^s_j, 1 \leq j \leq z) \\
= \begin{cases}
(X^{2,i}||X^{1,i}, Y^{2,i}||Y^{1,i}) & X^{1,i}, Y^{1,i} = f^{i+1,n}_{in}(C^{i}, G, A^{i+1}_{j}, B^s_j, 1 \leq j \leq z) \\
& h_s(X^{1,i}, Y^{1,i}) = C^i_1, \ldots, C^i_2, h_s(X^{2,i}, Y^{2,i}) = C^i_2, 1 \leq s \leq n.
\end{cases}
\end{align*}
$$

Then,

$$
\begin{align*}
Df^{i+1,n}_{in}(C, G, A^{i+1}_{j}, B^s_j, 1 \leq j \leq z) \cap A^{i+1}_{Out,Mi,In}(A^{i+1}_{j}, B^{i+1}_{j}, 1 \leq j \leq z) \\
= \begin{cases}
(X^{2,i}||X^{1,i}, Y^{2,i}||Y^{1,i}) & X^{1,i}, Y^{1,i} = f^{i+1,n}_{in}(C^{i}, G, A^{i+1}_{j}, B^s_j, 1 \leq j \leq z) \\
& X^{2,i}, Y^{2,i} = A^{i+1}_{Out,Mi,In}(T^{i}_{j}, D^{i}_{j}, 1 \leq j \leq z) \\
& M^i_1 = (M^{i+1}_1, \ldots, M^{i+1}_z).
\end{cases}
\end{align*}
$$

It can be concluded that

$$
\begin{align*}
Df^{i+1,n}_{in}(C, G, A^{i+1}_{j}, B^{i+1}_{j}, 1 \leq j \leq z) \cap A^{i+1}_{Out,Mi,In}(A^{i+1}_{j}, B^{i+1}_{j}, 1 \leq j \leq z) \\
= Df^{i+1,n}_{Out,Mi,In}(C, G, A^{i+1}_{j}, B^{i+1}_{j}, 1 \leq j \leq z)
\end{align*}
$$

From Lemma 1, for any two $M_{i1}, M_{i2} \in P^d_f$, where $M_{i1} \neq M_{i2}$, satisfy

$$
A^{i+1}_{Out,Mi,In}(A^{i+1}_{j}, B^{i+1}_{j}, 1 \leq j \leq z) \cap A^{i+1}_{Out,Mi,In}(A^{i+1}_{j}, B^{i+1}_{j}, 1 \leq j \leq z) = \emptyset.
$$

Thus,

$$
\begin{align*}
Df^{i+1,n}_{Out,Mi,In}(C, G, A^{i+1}_{j}, B^{i+1}_{j}, 1 \leq j \leq z) \cap Df^{i+1,n}_{Out,Mi,In}(C, G, A^{i+1}_{j}, B^{i+1}_{j}, 1 \leq j \leq z) = \emptyset
\end{align*}
$$

Secondly, due to

$$
\begin{align*}
A^{i+1}_{Out,In}(A^{i+1}_{j}, B^{i+1}_{j}, 1 \leq j \leq z) = \bigcup_{Mi \in P^d_f} A^{i+1}_{Out,Mi,In}(A^{i+1}_{j}, B^{i+1}_{j}, 1 \leq j \leq z)
\end{align*}
$$

holds in Lemma 1, we can get:

$$
\begin{align*}
Df^{i+1,n}_{Out,In}(C, G, A^{i+1}_{j}, B^{i+1}_{j}, 1 \leq j \leq z) \\
= Df^{i+1,n}_{in}(C, G, A^{i+1}_{j}, B^{i+1}_{j}, 1 \leq j \leq z) \bigcup A^{i+1}_{Out,In}(A^{i+1}_{j}, B^{i+1}_{j}, 1 \leq j \leq z) \\
= Df^{i+1,n}_{in}(C, G, A^{i+1}_{j}, B^{i+1}_{j}, 1 \leq j \leq z) \bigcup \bigcup_{Mi \in P^d_f} A^{i+1}_{Out,Mi,In}(A^{i+1}_{j}, B^{i+1}_{j}, 1 \leq j \leq z)
\end{align*}
$$

□
According to the above recursive structure, we can calculate the order of the $D_f$-set with $Out, In$ based on the following recurrence relation:

**Corollary 2:** For $1 \leq i \leq q - 1$, we get

$$
\#D_{i+1,n}^{Out, In}(C, G, A_j^{i+1}, B_j^{i+1}, 1 \leq j \leq z) = \sum_{M \in F_2^q} \#D_{i,n}^{Out, M}(C_j^2, G, T_j^2, D_j^2, 1 \leq j \leq z) \\
\times \#D_{i,n}^{M, In}(C_j^1, G, A_j^1, B_j^1, 1 \leq j \leq z).
$$

holds.

Given any $z$ basic function $\{f(X, Y)_{E_{k_m}, A_{k_m}^{i_m}, B_{k_m}^{i_m}} : 1 \leq m \leq z\}$, where $X, Y \in F_2^q$. For $0 \leq i \leq q - 1$, supposed that matrix $Md^i \in R^{2^q \times 2^q}$, of which elements define as:

$Md^i[Out][In_1] = \frac{1}{2^{2q^2}} \#D_{i,n}^{Out, In_1}(C_1^i, G, T_1^2, D_1^2, 1 \leq j \leq z)$

where $Out, In_1 \in F_2^q$, $G \in F_2^{n \times z}$, $C = (J, N) \in (F_2^{n \times z}, F_2^{n \times q})$, $J_2 = J[i \cdot t : (j + 1) \cdot t - 1]$, $N_2 = N[i \cdot t : (j + 1) \cdot t - 1]$, $C_j^2 = (J, N_2^2)$. Then, we have:

**Theorem 7:** Given any $z$ basic function $\{f(X, Y)_{E_{k_m}, A_{k_m}^{i_m}, B_{k_m}^{i_m}} : 1 \leq m \leq z\}$, where $X, Y \in F_2^q$. For $G \in F_2^{n \times z}$, $C = (J, N) \in (F_2^{n \times z}, F_2^{n \times q})$, $In \in F_2^q$, where $d = \sum_{i=0}^{z} k_i$, $In = (E_{k_1}, \cdots, E_{k_z})$. Then,

$$
\frac{1}{2^{2q^2}} \cdot \#D_{i,n}^{Out}(C, G, A_j^i, B_j^i, 1 \leq j \leq z) = L \prod_{i=0}^{q-1} Md^i \cdot Q_{In}^T
$$

where $L = (1, 1, \cdots, 1) \in F_2^d$, $Q_{In} \in F_2^d$ of which the sole nonzero component satisfies $Q[In] = 1$.

**Proof.** For $1 \leq i \leq q$, define vector $Num^i_{In} \in F_2^d$ as follow:

$$
Num^i_{In}[Out] = \frac{1}{2^{2q^2}} \cdot D_{i,n}^{Out, In}(C_1^i, G_0, A_j^i, B_j^i, 1 \leq j \leq z)
$$

where $Out \in F_2^q$.

According to the theorem 5 we have:

$$
\frac{1}{2^{2q^2}} \cdot D_{i,n}^{Out}(C, G, A_j^i, B_j^i, 1 \leq j \leq z) = L \cdot Num^i_{In}
$$

And by the definition, we see that $Num^i_{In} = Md^i \cdot Q_{In}^T$.

In addition, according to the corollary 2, for $1 \leq i \leq q - 1$, $Out \in F_2^d$, we have

$$
Num^{i+1}_{In}[Out] = \sum_{Out_1 \in F_2^d} Md^i[Out][Out_1] \cdot Num^i_{In}[Out_1]
$$

Namely,

$$
Num^{i+1}_{In} = Md^i \cdot Num^i_{In}
$$

Thus, the theorem holds.
5.2 Indirect method

In this part, we will use the Markov chain to complete the proof of the Theorem 7:

**Theorem 7:** Given any $z$ basic function $\{f(X,Y)_{E_{km}, A_{km}, B_{km}} | 1 \leq m \leq z\}$, where $X, Y \in F_2^n$. For $G \in F_2^n \times z$, $C = (J, N) \in (F_2^n \times q)^t$, $In \in F_2^d$, where $d = \sum_{i=0}^z k_i$, $In = \langle E_{k_1}, \cdots, E_{k_z} \rangle$. Then,

$$\frac{1}{22^q} \cdot \#Df^I_{In}(C, G, A^q_j, B^q_j, 1 \leq j \leq z) = L \prod_{i=0}^{q-1} M^d_i \mathcal{Q}^{T}_I n$$

where $L = (1, 1, \cdots, 1) \in F_2^d$, $Q_{In} \in F_2^d$, of which the sole nonzero component satisfies $Q[In] = 1$.

**Proof.** Supposed that $(X^{1,i+1}, Y^{1,i+1} \in Df^{i+1,n}_{Out, In}(C, G, A^{i+1}_j, B^{i+1}_j, 1 \leq j \leq z)$ represents the state $T^{i+1,n}_{Out, In}(C^{i+1}_j, G, A_j^{i+1}, B_j^{i+1}, 1 \leq j \leq z)$, where $0 \leq i \leq q - 1$. And supposed that $(X^{1,i}, Y^{1,i+1} \in Df^{i+1,n}_{Out, In}(C, G, A^{i+1}_j, B^{i+1}_j, 1 \leq j \leq z)^* \text{ represents the state } F^{i+1,n}_{Out, In}(C^{i+1}_1, G, A_j^{i+1}, B_j^{i+1}, 1 \leq j \leq z)$, where $0 \leq i \leq q - 1$. Thus, we can find that any two of the above states are disjoint and the union of all the above state is $\Omega$ according to the Property 5-6.

Secondly, for $1 \leq s \leq n$, due to

$$h_s(X^{1,i+1}, Y^{1,i+1}) = \sum_{m=1}^{z} G[s, m] \cdot f(X^{1,i+1}, Y^{1,i+1})_{E_{km}, A^{i+1}_m, B^{i+1}_m} = \sum_{M=1}^{z} G[s, m] \cdot f(X^{2,i}, Y^{2,i})_{M^m, T^m, D^m} \cdot 2^{i+1} + \sum_{m=1}^{z} G[s, m] \cdot f(X^{1,i}, Y^{1,i})_{E_{km}, A^{i}_m, B^{i}_m} = h_s(X^{1,i}, Y^{1,i}) \cdot 2^{i+1} + h_s(X^{2,i}, Y^{2,i})$$

If we know the value of $X^{1,i}, Y^{1,i}$, then the probability of which sate $(X^{1,i+1}, Y^{1,i+1})$ belongs to is depend on the state that $X^{1,i}, Y^{1,i}$ belongs to. Thus, we can use it to build a Markov chain.

For $1 \leq i \leq q - 1$, $O_1, O_2 \in F_2^n$:

$$Pr(T^{i+1,n}_{Out, In}(C^{i+1}_j, G, A^{i+1}_j, B^{i+1}_j, 1 \leq j \leq z)|T^{i,n}_{Out, In}(C^{i}_j, G, A^{i}_j, B^{i}_j, 1 \leq j \leq z)) = M^d_i[O_1][O_2]$$

$$Pr(T^{i+1,n}_{Out, In}(C^{i+1}_j, G, A^{i+1}_j, B^{i+1}_j, 1 \leq j \leq z)|F^{i+1,n}_{Out, In}(C^{i+1}_j, G, A^{i+1}_j, B^{i+1}_j, 1 \leq j \leq z)) = 0$$

Thus, the one step transform matrix form $i$ to $i + 1$ is:

$$[M^d_i, O_1]$$

Then, the $q - 1$ step transform matrix is

$$\prod_{i=1}^{q-1} M^d_i, O_1$$

Define vector $V_{In} \in F_2^d$ as follow:

$$V_{In}[Out] = Df^{1,n}_{Out, In}(C^1_j, G^0, A^1_j, B^1_j, 1 \leq j \leq z)$$

where $Out \in F_2^d$. Then $V_{In} = M^d \cdot \mathcal{Q}^{T}_I n$. According to the property of Markov chain, we get

$$Pr((X^{1,i+1}, Y^{1,i+1}) \in Df^{i+1,n}_{In}(C, G, A^{q}_j, B^{q}_j, 1 \leq j \leq z)) = L \prod_{i=1}^{q-1} M^d_i \cdot V_{In} = L \prod_{i=0}^{q-1} M^d_i \cdot \mathcal{Q}^{T}_I n$$

□
Likely, if we use the Markov chain, we get another form of Theorem 6:
Given any $z$ basic function $\{f(X,Y)_{E_k},A_{k_1},B_{k_m}; 1 \leq m \leq z\}$, where $X, Y \in F_2^t$. For $0 \leq i \leq q - 1$, supposed that matrix $M^0, M^1 \in R^{2^d \times 2^d}$, of which elements define as:

$$
M^0[Out][In] = \frac{#A^q_{Out,In}(D_j, D_j, 1 \leq j \leq z)}{2^{t+1}} + \frac{1}{2} Cor^q_{Out,In}(\frac{\gamma_j^2, \lambda_j, V_j, W_j, 1 \leq j \leq z}{2^{t+1}})
$$

$$
M^1[Out][In] = \frac{#A^q_{Out,In}(D_j, D_j, 1 \leq j \leq z)}{2^{t+1}} - \frac{1}{2} Cor^q_{Out,In}(\frac{\gamma_j^2, \lambda_j, V_j, W_j, 1 \leq j \leq z}{2^{t+1}})
$$

where $Out, In \in F_2^q, \gamma, \lambda, v, w \in F_2^q$, $\bar{v} = \gamma[i \cdot t : (i + 1) \cdot t - 1], \lambda = \lambda[i \cdot t : (i + 1) \cdot t - 1], \bar{w} = w[i \cdot t : (i + 1) \cdot t - 1]$. Then, for $0 \leq i \leq q - 1$, define the matrix $M_i$ as follow:

$$
\begin{bmatrix}
M^0, M^1
M^1, M^0
\end{bmatrix}
$$

Theorem 8: Given any $z$ basic function $\{f(X,Y)_{E_k},A_{k_1},B_{k_m}; 1 \leq m \leq z\}$, where $X, Y \in F_2^t$. For $0 \leq i \leq q - 1$, define the matrix $M_i$ as follow:

$$
Cor^q_{In}(\frac{\gamma_j^q, \lambda_j, V_j, W_j, 1 \leq j \leq z}{2^{t+1}}) = (L, O) \cdot \prod_{i=0}^{q-1} M_i \cdot (Q_{In}, Q_{In})^T
$$

where $L = (1, 1, \cdots, 1), O = (0, 0, \cdots, 0) \in F_2^{2^d}, Q_{In} \in F_2^{2^d}$ of which the sole nonzero component satisfies $Q_{In} = 1$.

Remark 7: Supposed that $m = q$ and $t = 1$ in theorem 6 and theorem 7, if $2^m \gg 2^d$ and $m \gg 2^d$, then the time complex of calculating the $Cor^q_{In}(\frac{\gamma_j^q, \lambda_j, V_j, W_j, 1 \leq j \leq z}{2^{t+1}})$ and $Df^m_{In}(C, G, A_j, B_j, 1 \leq j \leq z)$ is about $O(m)$.

5.3 Instance

For $F : (x, y) \overset{E}{\rightarrow} (x, x \oplus y)$, it can be treated as 1-order basic function. Besides this, its inverse function $F^{-1}$ is $(x, y) \overset{E^{-1}}{\rightarrow} (x, x \oplus y)$. According to the corollary 1, it can be conversed into $(x, y) \overset{F^{-1}}{\rightarrow} (x, (x \oplus (1, 1, \cdots, 1)) \oplus y \oplus 1)$. For $\alpha, \beta, \gamma, \lambda \in F_2^q$, let $E = [1, 0], B_1 = [0, \lambda], A_1 = [0, \gamma], B_2 = [\beta, \lambda], A_2 = [\alpha, \gamma]$. Then, for 2-order basic function $f(x, y)_{E,A_1,B_1}$ and $f(x, y)_{E,A_2,B_2}$, the following equation holds:

$$
\begin{align*}
\frac{f(x, y)_{E,A_1,B_1}}{F^{-1}(F(x, y) \oplus (\gamma, \lambda))} \\
\frac{f(x, y)_{E,A_2,B_2}}{F^{-1}(F(x \oplus \alpha, y \oplus \beta) \oplus (\gamma, \lambda))}
\end{align*}
$$

Thus, according to the theorem 6 and the theorem 7, we have:

1. The formula for calculating the boomerang connective probability and its variant :

Corollary 3(BCT): Let $F$ be $(x, y) \overset{E}{\rightarrow} (x, x \oplus y)$, an element of BCT $[4]$ defined by $BCT(\alpha, \beta, \gamma, \lambda) = \#((x, y) \mid x, y \in F_2^q, F^{-1}(F(x, y) \oplus (\gamma, \lambda))) \oplus F^{-1}(F(x \oplus \alpha, y \oplus \beta) \oplus (\gamma, \lambda)) = (\alpha, \beta) \cdot 2^{-2n}$

where $\alpha, \beta, \gamma, \lambda \in F_2^{2n}$. Then, for $0 \leq i \leq n - 1$, let $Out = (o_1, o_2, o_3, o_4) \in F_2^4, In = (e_1, e_2, e_3, e_4) \in F_2^4, d[i] = [\alpha[i], \beta[i], \gamma[i], \lambda[i]], L = (1, 1, \cdots, 1) \in F_2^{10}, Q = (0, 0, 0, 0, 1, 0, \cdots, 0) \in F_2^{10}$. Beside this, for $0 \leq i \leq n - 1$, supposed that the matrix

Zhongfeng Niu 17
\[ M_d[i] \in F_2^{16 \times 16}, \text{ of which elements define as:} \]

\[
M_d[i][Out][I^n] = Df_{Out,I^n}^{1,1}((\alpha[i],\beta[i]),(0,0),\{(0,1),(0,\gamma[i]),(0,\lambda[i])\},\{(0,1),(\alpha[i],\gamma[i]),(\beta[i],\lambda[i])\})
\]

\[
= \# \left\{ (x,y) \left| \begin{array}{c}
e_4 + e_3 + e_1 + e_2 = 0,\ carry_{y_1}(y,x)[1] = o_1, \\
carry_{y_2}(y \oplus x + e_1 + \gamma[i],x \oplus 1 + \gamma[i])[1] = o_2, \\
carry_{y_3}(y \oplus \beta[i],x \oplus \alpha[i])[1] = o_3, \\
carry_{y_4}(y \oplus \beta[i] \oplus x \oplus \alpha[i] + e_4 + \lambda[i],x \oplus 1 + \alpha[i] \oplus \gamma[i])[1] = o_4;
\end{array} \right. \right\}
\]

where \(Out,I^n \in F_2^4\).

Thus, we have

\[
BCT(\alpha,\beta,\gamma,\lambda) = 2^{-2n} \cdot L \prod_{i=0}^{n-1} M_d[i] Q^T
\]

**Corollary 4 (\(BCT^1\)):** Let \(F\) be \((x,y) \rightarrow (x,x \oplus y)\), an element of \(BCT^1[10]\) defined as

\[
BCT^1(\alpha,\beta,\gamma,\lambda,\theta,\zeta) = \# \{ (x,y) | x,y \in F_2^n, F^{-1}(F(x,y) \oplus (\gamma,\lambda)) \oplus F^{-1}(F(x \oplus \alpha, y \oplus \beta) \oplus (\gamma,\lambda)) = (\theta,\zeta) \} \cdot 2^{-2n}
\]

where \(\alpha,\beta,\gamma,\lambda,\theta,\zeta \in F_2^{2n}\). Then, for \(0 \leq i \leq n - 1\), let \(Out = (o_1,o_2,o_3,o_4) \in F_2^4\), \(I^n = (e_1,e_2,e_3,e_4) \in F_2^4\), \(d[i] = (\alpha[i],\beta[i],\gamma[i],\lambda[i],\theta[i],\zeta[i]) \in F_2^6\), \(L = (1,1,\ldots,1) \in F_2^{16}\), \(Q = (0,0,0,0,0,0,0,\ldots,0) \in F_2^{16} \). Beside this, for \(0 \leq i \leq n - 1\), supposed that the matrix \(M_d[i] \in F_2^{16 \times 16}\), of which elements define as:

\[
M_d[i][Out][I^n] = Df_{Out,I^n}^{1,1}((\theta[i],\zeta[i]),(0,0),\{(0,1),(0,\gamma[i]),(0,\lambda[i])\},\{(0,1),(\alpha[i],\gamma[i]),(\beta[i],\lambda[i])\})
\]

\[
= \# \left\{ (x,y) \left| \begin{array}{c}
\alpha[i] = \theta[i], \\
carry_{y_1}(y \oplus x + e_1 + \gamma[i],x \oplus 1 + \gamma[i])[1] = o_1, \\
carry_{y_2}(y \oplus x + e_1 + \lambda[i],x \oplus 1 + \gamma[i])[1] = o_2, \\
carry_{y_3}(y \oplus \beta[i],x \oplus \alpha[i])[1] = o_3, \\
carry_{y_4}(y \oplus \beta[i] \oplus x \oplus \alpha[i] + e_4 + \lambda[i],x \oplus 1 + \alpha[i] \oplus \gamma[i])[1] = o_4;
\end{array} \right. \right\}
\]

where \(Out,I^n \in F_2^4\).

Thus, we have:

\[
BCT^1(\alpha,\beta,\gamma,\lambda,\theta,\zeta) = 2^{-2n} \cdot L \prod_{i=0}^{n-1} M_d[i] Q^T
\]

**Corollary 5(\(BCT^2\)):** Let \(F\) be \((x,y) \rightarrow (x,x \oplus y)\), an element of \(BCT^2[10]\) defined as

\[
BCT^2(\alpha,\beta,\gamma,\lambda,\theta,\zeta) = \# \{ (x,y) | x,y \in F_2^n, F^{-1}(F(x,y) \oplus (\theta,\zeta)) \oplus F^{-1}(F(x \oplus \alpha, y \oplus \beta) \oplus (\gamma,\lambda)) = (\alpha,\beta) \} \cdot 2^{-2n}
\]

where \(\alpha,\beta,\gamma,\lambda,\theta,\zeta \in F_2^{2n}\). Then, for \(0 \leq i \leq n - 1\), let \(Out = (o_1,o_2,o_3,o_4) \in F_2^4\), \(I^n = (e_1,e_2,e_3,e_4) \in F_2^4\), \(d[i] = (\alpha[i],\beta[i],\gamma[i],\lambda[i],\theta[i],\zeta[i]) \in F_2^6\), \(L = (1,1,\ldots,1) \in F_2^{16}\), \(Q = (0,0,0,0,0,0,0,\ldots,0) \in F_2^{16} \). Beside this, for \(0 \leq i \leq n - 1\), supposed that the
Thus, we have:

\[
M_{d[i]}[Out][In] = D_{f_{Out,I,n}}^{1,1}((α[i], β[i]), (0, 0), \{(0, 1), (0, \overline{θ[i]}), (0, ζ[i])\}, \{(0, 1), (α[i], \overline{γ[i]}), (β[i], ι[i])\})
\]

\[
= \# \begin{cases} 
\lambda[i] + θ[i] + γ[i] + ζ[i] + ε_2 + ε_1 + ε_0 = 0, \\
θ[i] + γ[i] = 0, carry_{y1} (y, x)[1] = \alpha_1, \\
carry_{y2} (y + x + ε_1 + ζ[i], x + 1 + \theta [i])[1] = \alpha_2, \\
carry_{y3} (y + β[i], x + ε[i])[1] = \alpha_3, \\
carry_{y4} (y + β[i] + x + α[i] + ε_2 + ε_1 + λ[i], x + 1 + α[i] + γ[i])[1] = \alpha_4; \\
\end{cases}
\]

where \(carry_{y1}(x, y)[1] = (x \land y) \oplus (x \land e) \oplus (e \land y); x, y \in F_2.\)

Thus, we have:

\[
BCT^1(α, β, γ, λ, θ, ζ) = 2^{-2n} \cdot L \prod_{i=0}^{n-1} M_{d[i]} Q^T
\]

3. The formula for calculating the difference-boomerang connective probability and the inverse difference-boomerang probability, respectively:

**Corollary 6-1:** Let \(F \) be \((x, y) \rightarrow (x, x \oplus y),\) an element of DBT \([5]\) defined by

\[
DBT(α, β, γ, λ, θ, ζ) = \# \begin{cases} 
x, y \in F_2^n, F (x, y) \oplus F (x \oplus α, y \oplus β) = (θ, ζ), \\
x \in F_2^n, F (x, y) = (θ, ζ), \\
\end{cases}
\]

where \(α, β, γ, λ, θ, ζ \in F_2^n.\) Then, for \(0 \leq i \leq n - 1, \) let \(Out = (o_1, o_2, o_3, o_4) \in F_2^4, In = (ε_1, ε_2, ε_3, ε_4) \in F_2^4, d[i] = (α[i], β[i], γ[i], λ[i], θ[i], ζ[i]) \in F_2^6, L = (1, 1, \cdots, 1) \in F_2^{16}, Q = (0, 0, 0, 0, 1, 0, 0, \cdots, 0) \in F_2^{16}.\) Beside this, for \(0 \leq i \leq n - 1, \) supposed that the matrix \(M_{d[i]} \in F_2^{16 \times 16},\) of which elements define as:

\[
M_{d[i]}[Out][In] = D_{f_{Out,I,n}}^{1,1}((α[i], β[i]), (0, 0), \{(0, 1), (0, \overline{θ[i]}), (0, ζ[i])\}, \{(0, 1), (α[i], \overline{γ[i]}), (β[i], ι[i])\})
\]

\[
= \# \begin{cases} 
α[i] = θ[i], α[i] \oplus β[i] \oplus ζ[i] \oplus ε_2 + ε_1 = 0, \\
e_2 + ε_1 + ε_0 = 0, carry_{y1} (y, x)[1] = \alpha_1, \\
carry_{y2} (y + x + ε_1 + ζ[i], x + 1 + \theta [i])[1] = \alpha_2, \\
carry_{y3} (y + β[i], x + ε[i])[1] = \alpha_3, \\
carry_{y4} (y + β[i] + x + α[i] + ε_2 + ε_1 + λ[i], x + 1 + α[i] + γ[i])[1] = \alpha_4; \\
\end{cases}
\]

where \(carry_{y1}(x, y)[1] = (x \land y) \oplus (x \land e) \oplus (e \land y); x, y \in F_2.\)

Thus, we have:

\[
DBT(α, β, γ, λ, θ, ζ) = 2^{-2n} \cdot L \prod_{i=0}^{n-1} M_{d[i]} Q^T
\]

**Corollary 6-2:** Let \(F \) be \((x, y) \rightarrow (x, x \oplus y),\) an element of DBT \([5]\) defined by

\[
DBT(α, β, γ, λ, θ, ζ) = \# \begin{cases} 
x, y \in F_2^n, F^{-1}(x, y) \oplus F^{-1} (x \oplus α, y \oplus β) = (θ, ζ), \\
x \in F_2^n, F^{-1}(x, y) = (θ, ζ), \\
\end{cases}
\]

where \(α, β, γ, λ, θ, ζ \in F_2^n.\) Then, for \(0 \leq i \leq n - 1, \) let \(Out = (o_1, o_2, o_3, o_4) \in F_2^4, In = (ε_1, ε_2, ε_3, ε_4) \in F_2^4, d[i] = (α[i], β[i], γ[i], λ[i], θ[i], ζ[i]) \in F_2^6, L = (1, 1, \cdots, 1) \in F_2^{16}, Q = (0, 0, 0, 0, 1, 0, 0, \cdots, 0) \in F_2^{16}.\)
$Q = (0, 0, 0, 0, 0, 0, 0, 0, 1, 0, \cdots, 0) \in F_2^{16}$. Beside this, for $0 \leq i \leq n - 1$, supposed that the matrix $M_{d[i]} \in F_2^{16 \times 16}$, of which elements define as:

$$M_{d[i]}[Out][In] = Df_{Out, In}^{1,1}(\alpha[i], \beta[i], \gamma[i]) = \# \left\{ (x, y) \mid \alpha[i] = \theta[i], \alpha[i] \oplus \beta[i] \oplus \gamma[i] \oplus e_1 \oplus e_3 = 0, \quad e_4 \oplus e_3 \oplus e_1 \oplus e_2 = 0, \quad carry_{e_1} (y, x \oplus 1)[1] = o_1, \quad carry_{e_2} (y \oplus x \oplus 1 \oplus e_1 \oplus \lambda[i], x \oplus \gamma[i])[1] = o_2, \quad carry_{e_3} (y \oplus \beta[i], x \oplus 1 \oplus \alpha[i])[1] = o_3, \quad where \ carry_{e_4} (x, y)[1] = (x \land y) \oplus (x \land e) \oplus (e \land y) ; \ x, y, \in F_2. \right\}$$

where $Out, In \in F_2^4$.

Then, we have:

$$DBT(\alpha, \beta, \gamma, \lambda, \theta, \zeta) = 2^{-2n} \cdot L \prod_{i=0}^{n-1} M_{d[i]} Q^T$$

3. The formula for calculating the difference probability:

**Corollary 7:** (DDT) Let $S$ be $S(x, y) = x \oplus y$, an element of DDT [1] defined by

$$DDT(\alpha, \beta, \Delta) = \# \{ (x, y) \mid x, y \in F_2^{16}, S(x \oplus \alpha, y \oplus \beta) \oplus S(x, y) = \Delta \} \cdot 2^{-2n}$$

where $\alpha, \beta, \Delta \in F_2^n$. Then, for $0 \leq i \leq n - 1$, let $Out = (o_1, o_3) \in F_2^2, In = (e_1, e_3), L = (1, 1, 1, 1), Q = (1, 0, 0, 0) \in F_2^4, d[i] = (\alpha[i], \beta[i], \Delta[i]) \in F_2^4$. Beside this, for $0 \leq i \leq n - 1$, supposed that the matrix $M_{d[i]} \in F_2^{16 \times 16}$, of which elements define as:

$$M_{d[i]}[Out, In] = Df_{Out, In}^{1,1}(\alpha[i], \Delta[i], (0, 0), \{(1), (0), (0)\}, \{(1), (\alpha[i]), (\beta[i])\})$$

$$= \# \left\{ (x, y) \mid \alpha[i] \oplus \beta[i] \oplus \Delta[i] \oplus e_1 \oplus e_3 = 0, \quad carry_{e_1} (y, x)[1] = o_1, \quad carry_{e_2} (y \oplus \beta[i], x \oplus \alpha[i])[1] = o_3, \quad where \ carry_{e_4} (x, y)[1] = (x \land y) \oplus (x \land e) \oplus (e \land y) ; \ x, y, \in F_2. \right\}$$

where $Out, In \in F_2^2$.

Thus, we have:

$$DDT(\alpha, \beta, \Delta) = 2^{-2n} \cdot L \prod_{i=0}^{n-1} M_{d[i]} Q^T$$

4. The formula for calculating difference-linear connective correlation coefficients:

**Corollary 8:** (DLCT) Let $S$ be a $S(x, y) = x \oplus y$, an element of DLCT [4] defined by

$$DLCT(\alpha, \beta, \lambda) = 2^{-2n} \cdot \sum_{x, y \in F_2^4} (-1)^{\lambda \land (S(x \oplus \alpha, y \oplus \beta) \oplus S(x, y))}$$

where $\alpha, \beta, \lambda \in F_2^n$. Then, for $0 \leq i \leq n - 1$, let $Out = (o_1, o_3) \in F_2^2, In = (e_1, e_3), L = (1, 1, 1, 1), Q = (1, 0, 0, 0) \in F_2^4, a[i] = (\alpha[i], \beta[i], \lambda[i]) \in F_2^4$. Beside this, for $0 \leq i \leq n - 1$,
supposed that the matrix \( M_{a[i]} \in F^4_2 \times 4 \), of which elements define as :

\[
M_{a[i]}[Out][In] = \text{Cor}^1_{Out, In} \left( \begin{array}{c}
0, \lambda[i], 0, 0, \\
\{ (0), (0) \}, \{ (\alpha[i]), (\beta[i]) \}
\end{array} \right)
\]

\[
= \sum_{(x,y) \in \text{Set}_{a[i], Out, In}} (-1)^{\lambda[i]} (S(x \oplus \alpha[i], y \oplus \beta[i]) \oplus S(x, y)) \oplus e_1 \oplus e_2
\]

where \( \text{Set}_{a[i], Out, In} = \left\{ (x, y) \mid \begin{array}{c}
carry_{c_1} (y, x) [1] = o_1, \\
carry_{c_2} (y \oplus \beta [i], x \oplus \alpha [i]) [1] = o_3,
\end{array} \right. \)

where \( \text{car}ry_{c}(x, y)[1] = (x \land y) \oplus (x \land e) \oplus (e \land y) \); \( x, y \in F_2 \).

Thus, we have:

\[
DLCT(\alpha, \beta, \lambda) = L \prod_{i=0}^{n-1} M_{a[i]} Q^T
\]

5. The formula for calculating the linear approximation correlation coefficients (LAT):

**Corollary 9 (LAT):** Let \( S \) be \( S(x, y) = x \oplus y \), an element of LAT \([2]\) defined by

\[
LAT(\mu, \omega, \lambda) = 2^{-2n} \cdot \sum_{x,y \in F_2^n} (-1)^{\mu \cdot x \oplus \omega \cdot y \oplus \lambda \cdot S(x, y)}
\]

where \( \mu, \omega, \lambda \in F_2^n \). Then, for \( 0 \leq i \leq n - 1 \), let \( Out \in F_2, In \in F_2, L = (1, 1), Q = (1, 0) \), \( a[i] = (\mu[i], \omega[i], \lambda[i]) \in F_2^3 \). Beside this, for \( 0 \leq i \leq n - 1 \), supposed that the matrix \( M_{a[i]} \in F^2_2 \times 2 \), of which elements define as :

\[
M_{a[i]}[Out][In] = \text{Cor}^1_{Out, In} \left( \begin{array}{c}
0, \lambda[i], \mu[i], \omega[i], \\
\{ (0), (0) \}, \{ (\alpha[i]), (\beta[i]) \}
\end{array} \right)
\]

\[
= \sum_{(x,y) \in \text{Set}_{a[i], Out, In}} (-1)^{\mu[i]} (x \oplus \omega[i] y \oplus \lambda[i]) (S(x, y)) \oplus In)
\]

where \( \text{Set}_{a[i], Out, In} = \left\{ (x, y) \mid \begin{array}{c}
carry_{c_1} (x, y) [1] = Out, \\
where carry_{c}(x, y)[1] = (x \land y) \oplus (x \land e) \oplus (e \land y) ; \ x, y \in F_2, \right. \)

Thus, we have:

\[
LAT(\mu, \omega, \lambda) = L \prod_{i=0}^{n-1} M_{a[i]} Q^T
\]

According to the Theorem 8 in Appendix-A, after reducing the redundancy of matrix, we have the formulas for calculating the boomerange-difference connective probability and the variant of difference-boomerange connective probability, respectively:

**Corollary 10 (BDT):** Let \( F \) be \( (x, y) \stackrel{F}{\rightarrow} (x, x \oplus y) \), an element of BDT \([10]\) defined as

\[
BDT(\alpha, \beta, \gamma, \lambda, \theta, \zeta) = \# \left\{ (x, y) \mid \begin{array}{c}
x, y \in F_2^n, (x, y) \oplus F^{-1}(F(x \oplus \alpha, y \oplus \beta)) = (\theta, \zeta)
\end{array} \right. \}
\]

\[
= 2^{-2n}
\]

where \( \alpha, \beta, \gamma, \lambda, \theta, \zeta \in F_2^n \). Then, for \( 0 \leq i \leq n - 1 \), let \( Out = (o_1, o_2, o_3, o_4, o_5) \in F_2^5 \), \( In = (e_1, e_2, e_3, e_4, e_5) \in F_2^5 \), \( L = (1, 1, \cdots, 1) \in F_2^{32} \), \( d[i] = (\alpha[i], \beta[i], \gamma[i], \lambda[i], \theta[i], \zeta[i]) \in F_2^{32} \),
Thus, we have:

$$
F^2_2, Q = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, \cdots, 0) \in F^{32}_2.
$$

Besides this, for $0 \leq i \leq n - 1$, supposed that the matrix $M_{d[i]} \in F^{32 \times 32}$, of which elements define as:

$$
M_{d[i]}[Out][In] = \# \begin{pmatrix}
\alpha[i] = \theta[i], 1 \oplus \beta[i] \oplus \zeta[i] \oplus e_5 \oplus e_3 = 0, \\
e_4 \oplus e_3 \oplus e_1 \oplus e_2 = 0, \ carry_{e_4}(y, x)[1] = o_1, \\
carry_{e_3}(y \oplus e_1 \oplus \lambda[i], x \oplus 1 \oplus \gamma[i])[1] = o_2, \\
carry_{e_2}(y \oplus e_3 \oplus e_4 \oplus e_1 \oplus e_2 = 0, \ carry_{e_3}(y \oplus e_2 \oplus e_4 \oplus e_1 \oplus e_2 = 0, \ carry_{e_2}(y \oplus \lambda[i], x \oplus \alpha[i])[1] = o_3, \\
carry_{e_1}(y \oplus e_3 \oplus e_4 \oplus e_1 \oplus e_2 = 0, \ carry_{e_2}(y \oplus \lambda[i], x \oplus \alpha[i])[1] = o_4;
\end{pmatrix}
$$

where $Out, In \in F_2^3$.

Thus, we have:

$$
BDT(\alpha, \beta, \gamma, \lambda, \theta, \zeta, e_1, e_2, e_3, e_4, e_5) = 2^{-2n} \cdot L \prod_{i=0}^{n-1} M_{d[i]} Q^T
$$

**Corollary 11 (DBT¹):** Let $F$ be $(x, y) \xrightarrow{E} (x, x \boxplus y)$, an element of $DBT$ [10] defined as

$$
DBT^1(\alpha, \beta, \gamma, \lambda, \theta, \zeta, e_1, e_2, e_3, e_4, e_5) \in F_2^4, L = (1, 1, \cdots, 1) \in F_2^{16}, Q = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \in F_2^{16},
$$

$\ de_d[i] = (\alpha[i], \beta[i], \gamma[i], \lambda[i], \theta[i], \zeta[i], e_1, e_2, e_3, e_4, e_5) \in F_2^{16}$. Beside this, for $0 \leq i \leq n - 1$, supposed that the matrix $M_{d[i]} \in F_2^{16 \times 16}$, of which elements define as:

$$
M_{d[i]}[Out][In] = \# \begin{pmatrix}
\alpha[i] = \theta[i], 1 \oplus \beta[i] \oplus \zeta[i] \oplus e_5 \oplus e_3 = 0, \eta[i] = \theta[i], \\
\psi[i] = \theta[i] \oplus \zeta[i] \oplus e_5 \oplus e_3 \oplus e_1 \oplus e_2 = 0, \ carry_{e_3}(y, x)[1] = o_1, \\
carry_{e_2}(y \oplus e_1 \oplus \lambda[i], x \oplus 1 \oplus \gamma[i])[1] = o_2, \\
carry_{e_3}(y \oplus e_3 \oplus e_4 \oplus e_1 \oplus e_2 = 0, \ carry_{e_3}(y \oplus \lambda[i], x \oplus \alpha[i])[1] = o_3, \\
carry_{e_1}(y \oplus e_3 \oplus e_4 \oplus e_1 \oplus e_2 = 0, \ carry_{e_2}(y \oplus \lambda[i], x \oplus \alpha[i])[1] = o_4;
\end{pmatrix}
$$

where $Out, In \in F_2^4$.

Thus, we have:

$$
DBT^1(\alpha, \beta, \gamma, \lambda, \theta, \zeta, e_1, e_2, e_3, e_4, e_5) = 2^{-2n} \cdot L \prod_{i=0}^{n-1} M_{d[i]} Q^T
$$

**References**


A Reduce The Redundancy of Matrix

Definition 9: For any 2 basic function series \( \{f(X,Y)_{E_{k_{m,1}}, d_{k_{m,1}}, B_{k_{m,1}}^{2}, 1 \leq m \leq r_1}\} \)
and \( \{f(X,Y)_{E_{k_{m,2}}, A_{k_{m,2}}, B_{k_{m,2}}^{2}, 1 \leq m \leq r_2}\} \), if there exist a \( k_1 \), such that \( k_1 \geq 0 \),
\( E_{k_{r_1}, d_{k_{r_1}}}, A_{k_{r_1}}, B_{k_{r_1}}^{2}, 1 \leq k_1 \leq 1 \cdot \deg \) \( k_1 \)
then we called that the two basic function series are similar. And the degree of similarity \( \deg \) for the two basic function series is defined as
\[
\deg = \max \{k + 1 | k \geq 0, E_{k_{m,1}} [0 : k] = E_{k_{m,2}} [0 : k], A_{k_{m,1}} [0 : k] = A_{k_{m,2}} [0 : k], B_{k_{m,1}}^{2} [0 : k] = B_{k_{m,2}}^{2} [0 : k] \}
\]

Beside this, if two basic function series are not similar, we define \( \deg = 0 \).

Definition 10: Given any \( z \) number \( r_1, \cdots, r_z \), supposed that \( 1 \leq j_1 \leq j_2 \leq z \), \( \deg \leq r_{j_1}, r_{j_2} \), where \( d = \sum_{i=1}^{z} r_i \), then we can define
\[
Sim_{\deg}^{r_{j_2}} (V = F^d, V = V_{j_1} [0 : d], V_{j_2} [0 : d], V_{i} \in F_2^n, 1 \leq i \leq z)
\]
Define the bijection \( F : Sim_{\deg}^{r_{j_2}} \rightarrow F^{d-x} \) as:
\[
F(V_{j_1}, V_{j_2}, \cdots, V_{j_1}, \cdots, V_{j_2}, \cdots, V_{i}) = (V_{j_1}, V_{j_2}, \cdots, V_{j_1}, \cdots, V_{j_2} [\deg : r_{j_2} - 1], \cdots, V_{i})
\]

Theorem 8: For any positive integer \( q, t, z, n \), given any \( z \) basic function series
\( \{f(X^q, Y^q)_{E_{k_{m,1}}, A_{k_{m,1}}, B_{k_{m,1}}^{2}, 1 \leq m \leq r_1}\} \), \( 1 \leq j \leq z \); then we can get \( T_{k_{r_1}, j_1} \cdot D_{k_{r_1}, j_1} \) for \( A_{k_{m,1}}, B_{k_{m,1}}^{2} \), \( 1 \leq j \leq z \), \( 0 \leq j \leq q - 1 \). And there are two basic function series, of which code are \( j_1, j_2 \) respectively, are similar. Then we define the degree of similarity for the two basic function series is \( deg \). Supposed that \( G_0 \in F_2^{q \times 2} \),
\( G_1 \in F_2^{n \times r_1}, 1 \leq j \leq z \); \( C = (J, N) \in (F_2^{q \times t}, F_2^{n \times q - 1}) \), \( Out, In_1 \in F_2^{d}, In \in Sim_{\deg}^{r_{j_2}} \),
where \( d = \sum_{i=0}^{z} k_{r_i} \cdot In = (E_{k_{r_1}, j_1}, \cdots, E_{k_{r_1}, j_2}) \). Let \( C = (J, N) = C_{d}^{\frac{1}{q} \cdot (J_{q}, N_{q})} \). We have:
\[
D_{J_{n}}^{J_1}(C, G_0, G_1, A_{k_{r_1, j_1}}^{2}, B_{k_{r_1, j_2}}^{2}, 1 \leq j \leq z) = L \prod_{i=0}^{q-1} Md^i \cdot Q^T
\]
where \( L = (1, 1, 1, \cdots, 1) \in F_2^{d-x}, Q \in F_2^{d-x} \), of which the sole nonzero component satisfies \( Q[F(\text{In})] = 1 \), and \( Md^i \in R^{(d-x) \times (d-x)} \) satisfying \( Md^i[\text{out}, \text{in}] = \)
Theorem 9: For any positive integer $q, t, z, n$, given any $z$ basic function series \( \{f(X^{1,q}, Y^{1,q})_{E}\}_{m,j} \), $A^{k}_{m,j}$, $B^{k}_{m,j}$, $1 \leq m \leq r_{j}$, $X^{1,q}, Y^{1,q} \in F^{q}_{2}$, $1 \leq j \leq z$; then we can get $T^{q}_{k_{r_{j}}}, D^{q}_{k_{r_{j}}}$ from $A^{k}_{m,j}, B^{k}_{m,j}$, where $1 \leq j \leq z$, $0 \leq j < q - 1$. And there are two basic function series of which code are $j_{1}, j_{2}$ respectively, are similar. Then we define the degree of similarity for the two basic function series is deg. Supposed that $G_{j} \in F^{q}_{2}$, $1 \leq j \leq z$; $\gamma, \lambda, v, w \in F^{q}_{2}$, $Out, In_{1} \in F^{q}_{2}$, $In \in Sim^{j_{1},j_{2}}$, where $d = \sum_{i=0}^{z} \gamma_{i}$, $In = (E_{k_{r_{1}}}, \cdots E_{k_{r_{z}}})$. Let $\lambda^{q}_{1} = \gamma$, $\gamma^{q}_{1} = \gamma$, $V^{q}_{1} = v$, $W^{q}_{1} = w$. We have:

\[
\text{Cor}_{In}^{q} \left( A^{q}_{k_{r_{j}}}, B^{q}_{k_{r_{j}}}, G_{j}, 1 \leq j \leq z \right) = L \prod_{i=0}^{q-1} \text{Ma}^{i} Q^{T},
\]

where $L = (1, 1, \cdots, 1) \in F^{q-1}_{2}$, $Q \in F^{q-1}_{2}$, of which the sole nonzero component satisfies $Q[F(\text{In})] = 1$, and $\text{Ma}^{i} \in R^{(d-deg) \times (d-deg)}$ satisfying $\text{Ma}^{i}[\text{out}, \text{in}] = 0$.

Proof. According to the remark 5, for $1 \leq i \leq q$, we have:

\[
\text{Cor}_{Out, In}^{q} \left( A^{q}_{k_{r_{j}}}, B^{q}_{k_{r_{j}}}, G_{j}, 1 \leq j \leq z \right) = 0,
\]

when $i \notin Sim^{j_{1},j_{2}}$ and $Out \in Sim^{j_{1},j_{2}}$.

Thus, for $1 \leq i \leq q$, when $In \in Sim^{j_{1},j_{2}}$,

\[
\text{Cor}_{In}^{q} \left( A^{q}_{k_{r_{j}}}, B^{q}_{k_{r_{j}}}, G_{j}, 1 \leq j \leq z \right) = \sum_{Out \in Sim^{j_{1},j_{2}}} \text{Cor}_{Out, In}^{q} \left( A^{q}_{k_{r_{j}}}, B^{q}_{k_{r_{j}}}, G_{j}, 1 \leq j \leq z \right)
\]

By above, the theorem holds. $\Box$
And for \(1 \leq i \leq q - 1\), when \(In \in \text{Sim}^{j_1, j_2}_{\text{deg}}\),

\[
Cor_{Out, In}^{i+1} \left( \gamma_{1,i}^{1}, \lambda_{1,i}^{1}, V_{1,i}^{1}, W_{1,i}^{1}, A_{k_{r,j,j}}^{i+1}, B_{k_{r,j,j}}^{i+1}, G_{j}, 1 \leq j \leq z \right)
\]

\[
= \sum_{M_i \in \text{Sim}^{j_1, j_2}_{\text{deg}}} Cor_{Out, M_i}^{1} \left( \gamma_{2,i}^{1}, \lambda_{2,i}^{1}, V_{2,i}^{1}, W_{2,i}^{1}, T_{k_{r,j,j}}^{i}, D_{k_{r,j,j}}^{i}, G_{j}, 1 \leq j \leq z \right) \times Cor_{M_i, In}^{i} \left( \gamma_{1,i}^{1}, \lambda_{1,i}^{1}, V_{1,i}^{1}, W_{1,i}^{1}, A_{k_{r,j,j}}^{i}, B_{k_{r,j,j}}^{i}, G_{j}, 1 \leq j \leq z \right)
\]

Next, use the bijection \(F\) to transform \(\text{Sim}^{j_1, j_2}_{\text{deg}}\) into \(F_2^{d-\text{deg}}\) and do the similar way like the theorem 6. Then, the theorem holds. \(\square\)