SIMULTANEOUS DIAGONALIZATION OF INCOMPLETE
MATRICES AND APPLICATIONS

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Abstract. We consider the problem of recovering the entries of diagonal matrices \( \{U_a\}_a \) for \( a = 1, \ldots, t \) from multiple “incomplete” samples \( \{W_a\}_a \) of the form \( W_a = PU_aQ \), where \( P \) and \( Q \) are unknown matrices of low rank.

We devise practical algorithms for this problem depending on the ranks of \( P \) and \( Q \). This problem finds its motivation in cryptanalysis: we show how to significantly improve previous algorithms for solving the approximate common divisor problem and breaking CLT13 cryptographic multilinear maps.

1. Introduction

1.1. Problem Statement. This work considers the following computational problem from linear algebra.

Definition 1.1 (Problems A, B, C, D). Let \( n \geq 2, t \geq 2 \) and \( 2 \leq p, q \leq n \) be integers. Let \( \{U_a : 1 \leq a \leq t\} \) be diagonal matrices in \( \mathbb{Q}^{n \times n} \). Let \( \{W_a : 1 \leq a \leq t\} \) be matrices in \( \mathbb{Q}^{p \times q} \) and \( W_0 \in \mathbb{Q}^{p \times q} \) such that \( W_0 \) has full rank and there exist matrices \( P \in \mathbb{Q}^{p \times n} \) of full rank \( p \) and \( Q \in \mathbb{Q}^{n \times q} \) of full rank \( q \), such that \( W_0 = P \cdot Q \) and \( W_a = P \cdot U_a \cdot Q \) for \( 1 \leq a \leq t \). We distinguish the following cases:

(\( \mathbb{A} \)) \( p = n \) and \( q = n \)

(\( \mathbb{B} \)) \( p = n \) and \( q < n \)

(\( \mathbb{C} \)) \( p < n \) and \( q = n \)

(\( \mathbb{D} \)) \( p < n \) and \( q = p \)

In each of the four cases, the problem states as follows:

(1) Given the matrices \( \{W_a : 0 \leq a \leq t\} \), compute \( \{(u_{1,i}, \ldots, u_{t,i}) : 1 \leq i \leq n\} \), where for \( 1 \leq a \leq t \), \( u_{a,1}, \ldots, u_{a,n} \in \mathbb{Q} \) are the diagonal entries of matrices \( \{U_a : 1 \leq a \leq t\} \) as above.

(2) Determine whether the solution is unique.

Problem A is straightforward for any \( t \geq 1 \) by simultaneous diagonalization of \( W_0^{-1}W_a = Q^{-1}U_aQ \) for every \( a \). Problems \( \mathbb{B} \) and \( \mathbb{C} \) are equivalent in view of their symmetry in \( p \) and \( q \), and any algorithm for one solves the other upon transposing. Therefore, we shall devise algorithms for \( \mathbb{C} \) and \( \mathbb{D} \) only. We refer to the matrices \( \{W_a\}_a \) as “incomplete”, as the low rank matrices \( P \) and/or \( Q \) “steal” information. Of interest is the case when \( p \) is much smaller than \( n \). We remark that Problem A is an underlying problem in previous works [CP19, CHL\textsuperscript{+}15] in cryptanalysis.

Key words and phrases. Linear Algebra, Cryptanalysis, Approximate-Common-Divisor Problem, Multilinear Maps in Cryptography.
1.2. Our Contributions. Mainly, we provide efficient algorithms for Problems $C$ and $D$ of Def. 1.1 and show how to minimize the parameters $p$ and $t$ with respect to $n$. We further propose two concrete applications of our algorithms in cryptography. We believe that our algorithms are of independent interest and hope that more applications are to be found.

**Algorithms for Problems $C$ and $D$.** Our approach to Problem $C$ is to use the invertibility of $Q$ and write $W_a = PU_aQ = PQ Q^{-1}U_aQ = W_aZ_a$ with $Z_a = Q^{-1}U_aQ$, for every $1 \leq a \leq t$. As $W_0$ is not invertible, we cannot recover $Z_a$ directly. However we interpret this as a system of linear equations to solve for $\{Z_a\}_a$. This system is, in general, underdetermined and does not yield the matrices $\{Z_a\}_a$ uniquely. However, exploiting the special feature that $\{Z_a\}_a$ commute among each other leads to additional linear equations. This enables to recover $\{Z_a\}_a$ uniquely, and simultaneous diagonalization eventually yields the diagonal entries of $\{U_a\}_a$. We determine exact bounds on the parameters to ensure that we have at least as many linear equations as variables; we obtain that $p$ and $t$ can be set as $O(\sqrt{n})$. Our algorithm is heuristic only, but performs well in practice.

We reduce Problem $D$ to Problem $C$ by “augmenting” $Q$ with extra columns so that it becomes invertible. In this case, we show that $p$ can be close to $2n/3$. We refer to Sec. 3 and 4 for a complete description of our algorithms and provide the results of practical experiments in Sec. 6.

**Improved algorithm for an approximate common divisor problem.** Approximate common divisor problems have gained a lot of interest and different variants have been investigated. In [CH13], Cohn and Heninger study generalizations of the approximate common divisor problem via lattices. A simple version including only a single prime number is studied in [GGM16]. A lattice cryptanalysis of the single-prime version is described in [VDGHV10]. In this work we consider the multi-prime version (CRT-ACD Problem) from [CP19], which is a factorization problem with constraints based on Chinese Remaindering.

We improve the two-step algorithm by [CP19]. Namely, we remark that [CP19] relies on solving a certain instance of Problem $A$. By solving an appropriate instance of Problem $C$ instead, we obtain a quadratic improvement in the number of input samples. Namely, letting $n$ be the number of secret primes in the public modulus $M$, we can factor $M$ given only $O(\sqrt{n})$ input samples, whereas [CP19] uses $O(n)$. We therefore achieve complete factorization of the public modulus while limiting the input size drastically.

**Improved cryptanalysis of CLT13 Multilinear Maps.** In 2013, [GGH13] described the first construction of cryptographic multilinear maps, and since then, many important applications in cryptography were found. A similar construction over the integers was described in [CLT13] and a third construction based on the LWE Problem was proposed [GGH15]. In the last years, many attacks against these constructions appeared. The most devastating are the so-called “zeroizing attack”, exploiting the availability of low-level encodings of zero. The algorithm [CHL+15] recovers all secret parameters of [CLT13] in the multiparty Diffie-Hellman key exchange. Similar attacks have been described against GGH13 and GGH15, see [HL16] [CLLT16].
Our third contribution is therefore to improve the cryptanalysis of Cheon et al. against CLT13 when fewer encodings are public. Namely, we rely on solving some instance of Problem \textit{A}. By solving instances of Problems \textit{C} or \textit{D} instead, we can lower the number of public encodings required for the cryptanalysis. Specifically, for a composite modulus $x_0$ of $n$ primes, we obtain improved algorithms using only $O(\sqrt{n})$ encodings of zero (compared to $n$ in \cite{CHL+15}), or in total $4n/3$ encodings (compared to $2n + 2$ in \cite{CHL+15}). We confirm our results with practical experiments in Sec. 6.

2. Notations and Preliminary Remarks

2.1. Notation. For $n \in \mathbb{Z}_{\geq 1}$, let $[n]$ be the set $\{1, \ldots, n\}$. For a set $R$ and $r, s \in \mathbb{Z}_{\geq 1}$, we let $R^{r \times s}$ be the set of $r \times s$ matrices with entries in $R$. For $A \in R^{r \times s}$ and $B \in R^{r \times s'}$, $[A|B] \in R^{r \times (s+s')}$ is the matrix obtained by concatenating the columns of $A$ and $B$. We let $1_n$ be the identity matrix in dimension $n \in \mathbb{Z}_{\geq 1}$. For a set $S$, its cardinality is denoted by $|S|$. We shall make some important considerations about Def. 1.1.

(i) Let $\{W_a\}_a$ be as in Def. 1.1, $\pi \in \mathfrak{S}_n$ be a permutation with associated matrix $A_{\pi} \in \{0,1\}^{n \times n}$ and $D$ any invertible diagonal $n \times n$ matrix. Then $P' = PDA_{\pi}$ and $Q' = A_{\pi}^{-1}D^{-1}Q$ satisfy $W_0 = P'Q'$ and $W_a = P'U_a'Q'$ for all $a \in [t]$, where $U_a' = A_{\pi}^{-1}U_aA_{\pi}$ is obtained from $U_a$ by permuting its diagonal entries via $\pi$. Thus, $P', \{U_a'\}_a$ and $Q'$ satisfy the same problem. For this reason, we only ask to recover the set $\{(u_{1,i}, \ldots, u_{t,i}) : 1 \leq i \leq n\}$ in Def. 1.1.

(ii) If $t = 1$ in Problem \textit{C}, then the problem is not solvable because its solution is not unique. Namely, we write $W_1 = W_0Z_1$, where $Z_1 = Q^{-1}U_1Q$ is diagonalizable with eigenvalues the diagonal entries of $U_1$. But also, for every $v \in \ker(W_0)$ one has $W_1 = W_0(Z_1 + vw_1^T)$ for some $w_1 \in \mathbb{Q}^n$. Now, $Z_1$ and $Z_1 + vw_1^T$ likely have different eigenvalues which means that the solution is not unique.

(iii) There are cases when the problem is clearly not solvable for $p < n$. For example, if $P = [p_{0x(n-p)}]$ then for all $a$ the matrix $PU_a$ only involves the first $p$ diagonal entries of $U_a$ and the information on the remaining $n - p$ is lost. These cases will not occur for "generic" or "random" instances of the problem.

(iv) If a matrix $W_0 = PQ$ is not available as input (we call it a "special input" here), then one can recover ratios of diagonal entries of the matrices $\{U_a\}_a$, if $t \geq 3$. Namely, defining $P' = PU_1$ and assuming that $U_1$ is invertible, one obtains $W_0' := P'Q = W_1'$ and for $2 \leq a \leq t$, $W_a' := P'(U_aU_1^{-1})Q = W_a$. Running the algorithm on input $\{W_a' : 0 \leq a \leq t - 1\}$ reveals the tuples of diagonal entries of the matrices $U_aU_1^{-1}$ for $1 \leq a \leq t - 1$. We will use this approach in Sec. 5.2.3 to improve the (CLT13) multilinear map cryptanalysis.

(v) For simplicity, we have stated Def. 1.1 over $\mathbb{Q}$. More generally, we can consider matrices over a field $K$ with exact linear algebra (e.g. solving linear systems, diagonalizing matrices, etc.). Our algorithms apply to that case.
3. An Algorithm for Problem $\mathbb{C}$

We describe an algorithm to solve Problem $\mathbb{C}$ of Def. 1.1.

3.1. Description. Consider integers $n, t \geq 2$ and $2 \leq p < n$ and an instance of Problem $\mathbb{C}$. We remark that it is enough to solve the following problem.

**Definition 3.1 (Problem $\mathbb{C}'$).** Let integers $n, t \geq 2$ and $2 \leq p < n$. Given

- a matrix $V \in \mathbb{Q}^{p \times n}$ of rank $p$ and a basis matrix $E \in \mathbb{Q}^{n \times (n-p)}$ of ker($V$),
- a set of matrices $\{Y_a : a \in [t]\} \subseteq \mathbb{Q}^{n \times n}$

compute matrices $\{X_a : a \in [t]\} \subseteq \mathbb{Q}^{(n-p) \times n}$, such that the matrices $Y_a + EX_a$ for $a \in [t]$ commute with each other.

**Proposition 3.2.** Let $\{W_a : 0 \leq a \leq t\}$ as in Problem $\mathbb{C}$. Let $E \in \mathbb{Q}^{n \times (n-p)}$ be a basis matrix of the kernel of $W_0$. Let $W_0^+$ be a right-inverse of $W_0$. Define $V = W_0$ and $Y_a = W_0^+ W_a$ for $a \in [t]$. Assume that Problem $\mathbb{C}'$ is uniquely solvable for the input matrices $V, E$ and $\{Y_a : a \in [t]\}$.

Then Problem $\mathbb{C}$ is uniquely solvable for the input matrices $\{W_a : 0 \leq a \leq t\}$. Moreover, the matrix $Q$ in the assumption of Problem $\mathbb{C}$ is unique up to multiplication by a permutation matrix and an invertible diagonal matrix if at least one of the matrices $\{U_a\}$ has pairwise distinct diagonal entries.

**Proof.** Write $W_0 = PQ$ and $W_a = PU_a Q$ as in Problem $\mathbb{C}$. For all $a \in [t]$, we write $W_a = (PQ)(Q^{-1} U_a Q) = W_0 Z_a$, where $Z_a := Q^{-1} U_a Q$. The matrices $\{Z_a : a \in [t]\}$ commute and are simultaneously diagonalizable. For every $a \in [t]$, $Z_a$ can be written as $Z_a = Y_a + EX_a$ for some $X_a \in \mathbb{Q}^{(n-p) \times n}$ since $W_0 Y_a = W_a$. Since the matrices $\{Z_a\}_a$ commute, $V, E$ and $\{Y_a\}_a$ define a valid input for Problem $\mathbb{C}'$. By assumption, we can compute the matrices $\{X_a\}_a$ by solving Problem $\mathbb{C}'$ and these are unique. From the knowledge of $\{X_a\}_a$, we compute $Z_a = Y_a + EX_a$ for $a \in [t]$. Then these matrices are also unique. Thus the set of tuples of eigenvalues $\{(u_{i,1}, \ldots, u_{i,t}) : 1 \leq i \leq n\}$ is unique and can be computed by simultaneous diagonalization.

For the last part of the statement, assume that we have matrices $P', Q'$, diagonal matrices $\{U'_a\}_a$, which are necessarily of the form $U'_a = A^{-1} U_a A$ for a permutation matrix $A$, such that $W_0 = P'Q'$ and $W_a = P' U'_a Q'$ for every $a$. By uniqueness of the matrices $\{Z_a\}_a$, we have

$$Z_a = Q^{-1} U_a Q = Q'^{-1} U'_a Q' = Q'^{-1} A^{-1} U_a A Q', \quad a \in [t]$$

or, equivalently $U_a (QQ'^{-1} A^{-1}) = (QQ'^{-1} A^{-1}) U_a$ for $a \in [t]$. Thus, $D := QQ'^{-1} A^{-1}$ commutes with the matrices $\{U_a\}_a$ and so is diagonal itself, as one of $\{U_a\}$ has pairwise distinct entries. This gives $Q = DAQ'$ and proves the statement. □

3.1.1. Solving Problem $\mathbb{C}'$. We consider matrices $V, E, \{Y_a\}_a$ as in Problem $\mathbb{C}'$. We want to compute matrices $\{X_a\}_a$ such that the matrices $Z_a = Y_a + EX_a$ commute for all $a \in [t]$, that is, the Jacobi bracket $[Z_a, Z_b] = Z_a Z_b - Z_b Z_a$ is the zero matrix for all $a < b$. Using $Z_a = Y_a + EX_a$, this is equivalent to

$$0 = Y_a Y_b - Y_b Y_a + E \cdot S_{ab} + Y_a E X_b - Y_b E X_a \quad \ldots \quad (3.1)$$

If $W_0$ (of full rank $p$) is defined over the complex numbers, one can take $W_0^* = W_0^T (W_0 W_0^*)^{-1}$ where $W_0^*$ is the conjugate transpose of $W_0$, and $W_0^T$ over the real numbers.
where \( S_{ab} := X_a Y_b + X_a E X_b - X_b Y_a - X_b E X_a \). Left multiplication by \( V \) and \( V E = 0 \) imply \( V Y_a Y_b - V Y_b Y_a + V Y_a E X_b - V Y_b E X_a = 0 \), which is equivalent to

\[
\Delta_{ab} = V Y_a E X_a - V Y_b E X_b , \quad 1 \leq a < b \leq t ,
\]

where \( \Delta_{ab} := V Y_a Y_b - V Y_b Y_a \) is completely explicit in terms of the input matrices. Eq. (3.2) describes a system of linear equations over \( \mathbb{Q} \) in the variables given by the entries of \( X_a \) and \( X_b \). Since \( \Delta_{ab} \) has size \( p \times n \), this gives a system of \( np \) linear equations in the \( 2(n - p)n \) variables given by the entries of \( X_a \) and \( X_b \). Writing (3.2) for every \( (a, b) \in [t]^2 \) with \( a < b \) we obtain a system of \( t(t - 1)/2np \) linear equations and \( t(n - p)n \) variables given by the entries of the matrices \( \{ X_a : a \in [t] \} \).

From what precedes and Prop. 3.2 we deduce the following result.

**Proposition 3.3.** A unique solution to Problem \( \mathcal{C} \) is implied by the existence of a unique solution to the explicit system of linear equations given in (3.2), which is a system of \( \frac{1}{2} t(t - 1)np \) linear equations in \( t(n - p)n \) variables. There are at least as many equations as variables as soon as

\[
\frac{p}{n} \geq \frac{2}{t + 1} .
\]

Since there is no obvious linear dependence in the equations of the system, we heuristically expect, in the generic case, to find a unique solution \( \{ X_a : a \in [t] \} \) under Condition 3.3. This solves Problem \( \mathcal{C}' \), and therefore Problem \( \mathcal{C} \).

### 3.2. Algorithm.

We refer to this algorithm as Algorithm \( \mathcal{A}_C \) in the sequel.

**Input:** A valid input for Problem \( \mathcal{C} \)

**Output:** "Success" or "Fail"; in case of "Success", also output a solution. "Success" means uniqueness of the solution; "Fail" means that no solution was found.

1. Compute a basis matrix \( E \) of \( \ker (W_0) \).
2. Define \( W_0^a = W_0^a (W_0 W_0^a)^{-1} \) and for \( (a, b) \in [t]^2 \) with \( a < b \), compute the matrices \( \Delta_{ab} = W_a W_0^a W_b - W_b W_0^a W_a \).
3. Solve the system of linear equations described in Eq. 3.2.
   3.1. If the solution is not unique, then output "Fail" and break.
   3.2. Otherwise, denote by \( \{ X_a : a \in [t] \} \) the unique solution.
4. Perform simultaneous diagonalization of \( Z_a = W_0^a W_a + E X_a \) for \( a \in [t] \).
5. Output "Success" with the tuples of eigenvalues of the matrices \( \{ Z_a \} \).

### 3.3. Optimization of the parameters.

We find minimal possible (with respect to \( n \)) values for \( t \) and \( p \). In our applications in Sec. 3 we are led to minimize \( p + t \) as a function of \( n \). Following Prop. 3.3 we set \( F_n(t) = p_n(t) + t = \frac{2n}{t+1} + t \) with \( t \in \mathbb{R}_{>0} \) and \( n \in \mathbb{Z}_{\geq 2} \). It is easy to see that \( F_n \) has a minimum at \( t_0 = \sqrt{2n} - 1 \) which gives \( p = p_n(t_0) = \sqrt{2n} \). This shows that minimal values for \( p \) and \( t \) are \( \mathcal{O}(\sqrt{n}) \). This is confirmed practically in Sec. 6.

### 4. An Algorithm for Problem \( \mathcal{D} \)

We now present an algorithm to solve Problem \( \mathcal{D} \) of Def. 1.1.
4.1. Description. Consider integers \( n, t \geq 2 \) and \( 2 \leq p < n \) and an instance of Problem \( \mathcal{D} \). The main idea of our algorithm is a reduction to Problem \( \mathcal{C} \) which can be solved using Algorithm \( \mathcal{A}_\mathcal{C} \). More precisely, we exhibit matrices (that are augmentations of \( \{W_a\}_a \) ) \( W'_a = PQ' \) and \( W'_a = PU_aQ' \) for \( a \in [t] \), for the same diagonal matrices \( \{U_a\}_a \) and for some \( n \times n \) invertible matrix \( Q' \).

4.1.1. Reducing Problem \( \mathcal{D} \) to Problem \( \mathcal{C} \). For \( 1 \leq a, b \leq t \), we define the matrices

\[
\Delta_{ab} = W_a W_0^{-1} W_b - W_b W_0^{-1} W_a.
\]

Note that \( \Delta_{ab} = -\Delta_{ba} \). We have the following lemma.

**Lemma 4.1.** Let \( W_0 = PQ \) and \( W_a = PU_aQ \) for \( a \in [t] \) as in Problem \( \mathcal{D} \). Let \( B = QW_0^{-1} P - 1_n \in \mathbb{Q}^{n \times n} \) and let \( r \) denote its rank. Then:

(i) \( r = n - p \)

(ii) there exist matrices \( V_a \in \mathbb{Q}^{p \times r} \) and \( G_a \in \mathbb{Q}^{r \times p} \) for \( a \in [t] \) such that for all \( 1 \leq a < b \leq t \), one has \( \Delta_{ab} = V_a G_b - V_b G_a \).

**Proof.** (i) Let \( C = QW_0^{-1} P \). Then \( CQ = Q \) and the column-image of \( Q \) is contained in the eigenspace, say \( \mathcal{E} \), of \( C \) for eigenvalue 1. So, \( \mathcal{E} \) has dimension at least \( p \). However, the rank of \( C \) is bounded above by the rank of \( Q \), i.e. by \( p \). Finally, \( \mathcal{E} \) has dimension exactly \( p \) and the rank \( r \) of \( B = C - 1_n \) equals \( n - p \).

(ii) For every \( 1 \leq a, b \leq t \), we can write

\[
\Delta_{ab} = PU_a (QW_0^{-1} P - 1_n) U_b Q - PU_b (QW_0^{-1} P - 1_n) U_a Q
\]

\[
= PU_a BU_b Q - PU_b BU_a Q
\]

since \( U_a \) and \( U_b \) commute. Since \( B \) has rank \( r \), there exist matrices \( B_1 \in \mathbb{Q}^{n \times r} \), \( B_2 \in \mathbb{Q}^{r \times n} \) with \( B = B_1 B_2 \). Setting \( V_a = PU_a B_1 \) and \( G_a = B_2 U_a Q \) gives the claim. \( \square \)

The following properties of the matrix \( B \) defined in Lem. 4.1 are useful.

**Lemma 4.2.** Let \( W_0 = PQ \) and \( W_a = PU_aQ \) for \( a \in [t] \) as in Problem \( \mathcal{D} \). Let \( B \in \mathbb{Q}^{n \times n} \) be the matrix of Lem. 4.1 with respect to \( P \) and \( Q \) and let \( r = n - p \). Let \( B_1 \in \mathbb{Q}^{n \times r} \) and \( B_2 \in \mathbb{Q}^{r \times n} \) be such that \( B = B_1 B_2 \). Then:

(i) \( PB_1 = 0_{p \times r} \)

(ii) The matrix \( Q' := [Q][B_1] \) is an \( n \times n \) invertible matrix.

**Proof.** (i) The matrix \( B_2 \) defines a surjection \( B_2 : \mathbb{Q}^n \to \mathbb{Q}^r \). Thus for every \( x \in \mathbb{Q}^r \), we write \( x = B_2 y \) for some \( y \in \mathbb{Q}^n \) and obtain \( PB_1 x = PB_1 (B_2 y) = (PB)y = 0 \).

(ii) Since \( r = n - p \), \( Q' \) has size \( n \times n \). To show its invertibility, we show that \( \text{im}(Q') \cap \text{im}(B_1) = \{0\} \). Since \( B_2 \) is surjective, the images of \( B_1 \) and \( B_1 B_2 = B \) coincide. Let \( Qx = By \in \text{im}(Q) \cap \text{im}(B_1) \), with \( x \in \mathbb{Q}^r \) and \( y \in \mathbb{Q}^n \). This gives \( Qx = (QW_0^{-1} P - 1_n) y = QW_0^{-1} Py - y \). Thus \( y = QW_0^{-1} Py - Qx = Qz \) with \( z = W_0^{-1} Py - x \). Therefore, \( Qx = By = B(Qz) = 0 \) because \( BQ = 0 \). \( \square \)

We now show that finding matrices \( \{V_a\}_a \) such that there exist \( \{G_a\}_a \) satisfying \( \Delta_{ab} = V_a G_b - V_b G_a \) for every \( a, b \) is sufficient to solve Problem \( \mathcal{D} \). We view these matrices as being complementary to \( \{W_a\}_a \) because they define themselves an instance of Problem \( \mathcal{D}' \) with the same solution as \( \{W_a\}_a \) (see the proof of Lem. 4.1). This allows us to increase the rank of \( Q \). We thus now formulate Problem \( \mathcal{D}' \).
**Definition 4.3** (Problem \( \mathbb{D}' \)). Let \( n, t \geq 2 \) and \( 2 \leq p < n \) be integers. For every \( 1 \leq a, b \leq t \), let \( \Delta_{ab} \in \mathbb{Q}^{p \times p} \) be such that \( \Delta_{ab} = V_a G_b - V_b G_a \) for \( V_a \in \mathbb{Q}^{p \times (n-p)} \) of rank \( n-p \) and \( G_a \in \mathbb{Q}^{(n-p) \times p} \). The problem states as follows: Given the matrices \( \Delta_{ab} \) for all \( 1 \leq a, b \leq t \), compute such matrices \( V_a \) for \( a \in [t] \).

The following proposition links Problem \( \mathbb{D} \) and Problem \( \mathbb{C} \).

**Proposition 4.4.** Let \( W_0 = PQ \) and \( W_a = PU_a Q \) for \( a \in [t] \) be as in Problem \( \mathbb{D} \). For \( 1 \leq a, b \leq t \), let \( \Delta_{ab} \) be the matrices defined in \( 4.4 \). Moreover, assume that

(i) Problem \( \mathbb{D}' \) is uniquely solvable for the input matrices \( \{ \Delta_{ab} : 1 \leq a < b \leq t \} \) and denote by \( \{ V_a : a \in [t] \} \) the unique solution.

(ii) Problem \( \mathbb{C} \) is uniquely solvable for the input matrices \( W_0' = [W_0|0_{p \times (n-p)}] \in \mathbb{Q}^{p \times n} \) and \( W_a' = [W_a|V_a] \in \mathbb{Q}^{p \times n} \) for \( a \in [t] \).

Then Problem \( \mathbb{D} \) is uniquely solvable on input \( \{ V_a : 0 \leq a \leq t \} \) and the unique solution is given by the unique solution to Problem \( \mathbb{C} \) on input \( \{ W_a' : 0 \leq a \leq t \} \).

**Proof.** By Lem. 4.1, there exist \( V_a \in \mathbb{Q}^{p \times t} \) and \( G_a \in \mathbb{Q}^{p \times p} \) for \( a \in [t] \) such that \( \Delta_{ab} = V_a G_b - V_b G_a \) for all \( 1 \leq a < b \leq t \). Therefore, the matrices \( \{ \Delta_{ab} \}_{a,b} \) define an instance of Problem \( \mathbb{D}' \). By assumption (i), we compute the unique solution \( \{ V_a \}_{a} \) for this problem.

Now, let \( W_0' = [W_0|0_{p \times (n-p)}] \in \mathbb{Q}^{p \times n} \) and \( W_a' = [W_a|V_a] \in \mathbb{Q}^{p \times n} \) for \( a \in [t] \). Let \( B = Q W_0'^{-1} P - I_n \) as in Lem. 4.1 of rank \( r = n - p \). Let \( B_1 \in \mathbb{Q}^{n \times r} \) and \( B_2 \in \mathbb{Q}^{r \times n} \) be a rank factorization of \( B \), i.e., \( B = B_1 B_2 \). Letting \( Q' := \{ Q|B_1 \} \in \mathbb{Q}^{p \times n} \), we have \( PQ' = P|Q|B_1 = [W_0|0_{p \times r}] = W_0' \) and, by uniqueness of \( \{ V_a \}_{a} \) (see proof of Lem. 4.1).

\[
PU_a Q' = PU_a Q B_1 = [W_a|V_a] = W_a'
\]

for \( a \in [t] \), as \( PB_1 = 0_{n \times r} \) by Lem. 1.2 (i). The matrix \( Q' \) is invertible by Lem. 4.2 (ii). Therefore, \( W_0' \) and \( \{ W_a' \}_{a} \) define a valid input for Problem \( \mathbb{C} \). By assumption (ii), this problem is uniquely solvable and the solution must be the tuples of diagonal entries of the matrices \( \{ U_a \}_{a} \). This is also a solution to Problem \( \mathbb{D} \) since the matrices \( \{ U_a \}_{a} \) are the same for the input matrices \( \{ W_a \}_{a} \) for Problem \( \mathbb{D} \) and \( \{ W_a' \}_{a} \) for Problem \( \mathbb{C} \).

### 4.1.2. Solving Problem \( \mathbb{D}' \)

In view of Prop. 4.4, it remains to compute matrices \( \{ V_a \}_{a} \) from \( \{ \Delta_{ab} \}_{a,b} \). We achieve this by standard linear algebra, and combining with Algorithm \( A_\mathbb{C} \) describes a full algorithm for Problem \( \mathbb{D} \).

From now on we assume \( t \geq 3 \). Let \( \Delta_{ab} = V_a G_b - V_b G_a \) for \( 1 \leq a, b \leq t \) as in Problem \( \mathbb{D}' \). Let \( r = n - p \) and \( r_{ab} \) be the rank of \( \Delta_{ab} \); clearly, \( r_{ab} \leq \min(2r, p) \). We further assume \( p > 2n/3 \) (equivalently \( 2r < p \)), which is a necessary condition as otherwise the matrices \( \Delta_{ab} \) likely have full rank and thus cannot reveal any information. We define \( \mathbb{K}_{ab} := \ker(\Delta_{ab}) = \mathbb{K}_{ba} \subseteq \mathbb{Q}^p \) and

\[
\mathbb{K}_a = \bigcap_{b \in [t], b \neq a} \mathbb{K}_{ab}, \ a \in [t].
\]

Let \( \mathbb{V}_a \) be the image of the matrix \( V_a \) for \( a \in [t] \). We first argue that, heuristically, \( \mathbb{V}_a \subseteq \mathbb{K}_a \) for every \( b \neq a \). Let \( v \in \mathbb{V}_a \). If there exists \( x \in \mathbb{Q}^p \) such that \( v = V_a G_b x \) and \( V_b G_a x = 0 \) then \( v = \Delta_{ab} x \), i.e., \( v \in \mathbb{K}_{ab} \). Such an element \( x \) must therefore lie in \( \ker(V_a G_a) \). It is easy to see that this intersection is non-empty if \( \ker(V_a G_b) \cap \ker(V_b G_a) = \mathbb{Q}^p \).
Heuristically, as \( \{ V_a \}_a \) have rank \( r \), \( \text{ker}(V_a G_b) + \text{ker}(V_b G_a) \) has dimension at least \( 2(p - r) \); accordingly we can heuristically expect that \( \text{ker}(V_a G_b) + \text{ker}(V_b G_a) = \mathbb{Q}^p \) as soon as \( 2(p - r) > p \), i.e. \( p > 2n/3 \).

We now justify that, heuristically under a suitable parameter selection, \( \mathcal{K}_a = \mathcal{V}_a \) for every \( a \in [t] \). For fixed \( a \in [t] \), we compute \( \mathcal{K}_a \) modulo \( \mathcal{V}_a \) and consider \( \mathcal{K}_{ab} := \mathcal{K}_a / \mathcal{V}_a \subseteq \mathbb{Q}^{p-p} \) for \( b \neq a \). Then \( \mathcal{K}_a = \mathcal{V}_a \) if and only if \( \mathcal{K}_a := \bigcap_{b \neq a} \mathcal{K}_{ab} = \{ 0 \} \). Since \( \mathcal{V}_a \) has dimension \( r \), \( \mathcal{K}_{ab} \) has dimension \( r_{ab} - r \). For every \( b \neq a \), we view \( \mathcal{K}_{ab} \) as the kernel of \( \mathbb{Q}^{p-r} \to \mathbb{Q}^{p-r} / \mathcal{K}_{ab} \), represented by a matrix \( A_{ab} \in \mathbb{Q}^{(p-r_{ab}) \times (p-r)} \). Therefore \( \mathcal{K}_a \) is represented by an augmented matrix \( A_a = [A_{a1} \ldots | A_{a,a-1} | A_{a,a+1} \ldots | A_{at}] \) describing the kernel of \( \mathbb{Q}^{p-r} \to \bigoplus_{b \neq a} \mathbb{Q}^{p-r} / \mathcal{K}_{ab} \). The matrix \( A_a \) has \( \sum_{b \in [t], b \neq a} (p - r_{ab}) \) rows and \( p - r \) columns. Now, \( \mathcal{K}_a = \mathcal{V}_a \) if and only if \( A_a \) has full rank; and heuristically, we expect this to be the case as soon as \( \sum_{b \in [t], b \neq a} (p - r_{ab}) \geq p - r \).

**Remark 4.5.** (i) In fact, we expect that \( \mathcal{K}_a = \mathcal{V}_a \) for every \( a \), if \( (t-1)(p-2r) \geq p - r \), i.e.

\[
\frac{p}{n} \geq \frac{2t - 3}{3t - 5} \quad \text{or, equivalently,} \quad t \geq \frac{2p - n}{3p - 2n} + 1.
\]

(ii) We assumed \( t \geq 3 \) so that the intersections \( \{ \mathcal{K}_a \}_a \) are well-defined. If \( t = 2 \), \( \mathcal{K}_1 \) coincides with the image of \( \Delta_{12} \), which will not reveal \( V_1 \) and \( V_2 \).

We compute bases of \( \{ \mathcal{K}_a \}_a \) by standard linear algebra. For the rest of this section, assume \( \mathcal{K}_a = \mathcal{V}_a \) for every \( a \), and let \( C_a \) be a basis matrix for \( \mathcal{K}_a \). Thus, there exists \( M_a \in \text{GL}_r(\mathbb{Q}) \) such that \( V_a = C_a M_a \). This gives for \( a < b \):

\[
\Delta_{ab} = V_a G_b - V_b G_a = C_a (M_a G_b) - M_b G_a = C_a N_{ab} - C_b N_{ba} = M_a G_b
\]

with \( N_{ab} = M_a G_b \). In (4.7), \( \Delta_{ab} \) and \( C_a, C_b \) are known, which allows to compute \( N^{(ab)} = [N_{ab} | N_{ba}] \) as a solution to \( \Delta_{ab} = [C_a] - [C_b] \cdot N^{(ab)} \). Once \( \{ N_{ab} \}_{a,b} \) are computed, we obtain a system of linear equations over \( \mathbb{Q} \), given by

\[
M_a^{-1} \cdot N_{ab} = G_b, \quad 1 \leq a < b \leq t.
\]

It has \( \frac{1}{2} t(t-1) rp \) equations (there are \( \frac{1}{2} t(t-1) \) choices for pairs \( (a,b) \) and for each pair the matrix equality gives \( rp \) equations) and \( tr^2 + trp = trn \) variables, given by the \( tr^2 \) entries of the matrices \( \{ M_a^{-1} : a \in [t] \} \) and the \( trp \) entries of the matrices \( \{ G_b : b \in [t] \} \). Heuristically, if \( trn \leq \frac{1}{2} t(t-1) rp \), i.e. \( 2n \leq (t-1)p \), the system is expected to have a unique solution. This bound is automatically satisfied if (4.6) holds. This reveals \( \{ M_a : a \in [t] \} \) and thus \( \{ V_a : a \in [t] \} \) by computing \( V_a = C_a M_a \).

**Proposition 4.6.** Assume that \( \mathcal{K}_a = \mathcal{V}_a \) for every \( a \in [t] \) (see Rem. 4.5 (i)). Then, a unique solution to Problem \( \mathcal{W}' \) is implied by the existence of a unique solution to the explicit system of linear equations given in (4.8), which is a system of \( \frac{1}{2} t(t-1)(n-p)p \) linear equations in \( t(n-p)n \) variables. There are at least as many equations as variables as soon as \( p(t-1) \geq 2n \).

4.2. Algorithm. We refer to this Algorithm as Algorithm \( \mathcal{A_B} \) in the sequel.
Our interest is therefore to minimize the cardinality of the set $M$. In conclusion, we expect Algorithm 5.1. Improved algorithm for the CRT-ACD Problem. 

Based on previous works, we describe two applications for our algorithms and obtain significant improvements.

Remark 4.7. Problem $\mathcal{D}$ of Def. 4.1 is symmetric in the sense that $P$ and $Q$ have the same rank. An asymmetric variant consists in having $P$ and $Q$ of ranks $p \neq q$. Our algorithm adapts to that case: if $p < q$, then "cutting" the last $q - p$ columns of $\{W_a\}_a$ means "cutting" the last $q - p$ columns of $Q$, which reduces to the symmetric case. This approach is however not very genuine, as it "cuts" information instead of possibly exploiting it. We leave it open to find a better algorithm.

4.3. Optimization of the parameters. We find minimal possible values for $t$ and $p$ with respect to a given $n$. In Sec. 4.2, we see that it is of interest to minimize $2p + t$ in order to minimize the number of public encodings in $\text{CLT13}$. According to (4.6), the main (heuristic) condition to be ensured is $p \geq \frac{2n}{3} - \frac{n}{2}$. We set $\Delta(t) = 2p_n(t) + t = \frac{2n}{3} - \frac{n}{2} + t$ for $t \in \mathbb{R}_0 \setminus \{5/3\}$ and $n \geq 2$. Then $\Delta_n$ has a minimum at $t_0 = \frac{1}{3} (\sqrt{2n} + 5)$, with $p_n(t_0) = \frac{2}{3} n + \frac{1}{3} \sqrt{2n}$ and $\Delta_n(t_0) = \frac{4}{3} n + \frac{2}{3} \sqrt{2n} + \frac{5}{3}$. In conclusion, we expect Algorithm $\mathcal{A}_D$ to succeed for $p = \lfloor p_n(t_0) \rfloor$ and $t = \lceil t_0 \rceil$.

5. Applications

We describe two applications for our algorithms and obtain significant improvements on previous works.

5.1. Improved algorithm for the CRT-ACD Problem. We consider the following "multi-prime" version of the Approximate Common Divisor Problem $\text{CPI9}$ based on Chinese Remaindering:

Definition 5.1 (CRT-ACD Problem). Let $\eta, \rho, \eta, \rho \in \mathbb{Z}_{> 1}$. Let $p_1, \ldots, p_n$ be distinct $\eta$-bit prime numbers and $M = \prod_{i=1}^n p_i$. Consider a non-empty finite set $S$ of integers in $\mathbb{Z} \cap [0, M)$ such that for every $x \in S$:

$$x \equiv x_i \pmod{p_i}, \quad 1 \leq i \leq n$$

for uniformly distributed integers $x_i \in \mathbb{Z}$ satisfying $|x_i| \leq 2^\rho$.

The CRT-ACD problem states as follows: given the set $S$, the integers $\eta, \rho$ and $M$, factor $M$ completely (i.e. find the prime numbers $p_1, \ldots, p_n$).

Clearly, the larger the set $S$, the more information one can exploit to factor $M$. Our interest is therefore to minimize the cardinality of the set $S$ with respect to $n$. 

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5.1.1. The algorithm of \cite{CP19}. Coron and Pereira propose an algorithm for the case \#\mathcal{S} = n + 1. They proceed in two steps called the "orthogonal lattice attack" following \cite{NS99} and the "algebraic attack" following \cite{CHL15}. We briefly review their algorithm; for a complete description we refer to \cite{CP19} Sec. 4.3.

Let \( \mathcal{S} = \{x_1, \ldots, x_n, y\} \) and \( x = (x_1, \ldots, x_n) \in \mathcal{S}^n \). Then, the vector \( b = (x, y \cdot x) \in \mathbb{Z}^n \) is public, and by the Chinese Remainder Theorem, letting \( x \equiv x^{(i)} \pmod{p_i} \) and \( y \equiv y^{(i)} \pmod{p_i} \) for all \( i \in [n] \), one has \( b \equiv \sum_{i=1}^{n} c_i (x^{(i)}, y^{(i)} x^{(i)}) =: \sum_{i=1}^{n} c_i b^{(i)} \pmod{M} \), for some integers \( c_1, \ldots, c_n \). If the vectors \( \{x^{(i)}\}_i \) are \( \mathbb{R} \)-linearly independent, then so are \( \{b^{(i)}\}_i \) and generate a \( 2n \)-dimensional lattice \( \mathcal{L} \) of rank \( n \). Importantly, by Def. 5.1, the vectors \( \{b^{(i)}\}_i \) are reasonably short vectors (of \( \ell_2 \)-norm approximately \( 2^{n/2} \); and \( \rho \) is considered much smaller than \( \eta \).

The "orthogonal lattice attack" is an algorithm, which on input \( b \), outputs a basis of the completion \( \overline{\mathcal{L}} = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Q} \) of \( \mathcal{L} \), performing lattice reduction on the lattice \( \langle b \rangle^* \) of vectors \( v \in \mathbb{Z}^n \) such that \( \langle v, b \rangle \equiv 0 \pmod{M} \). The parameters are chosen accordingly, and one essentially requires \( 2\rho < \eta \).

Upon finding a basis \( \{b^{(i)}\}_i \) of \( \overline{\mathcal{L}} \), Coron and Pereira proceed with the "algebraic attack". The bases \( \{b^{(i)}\}_i \) of \( \mathcal{L} \) and \( \{b^{(i)}\}_i \) of \( \mathcal{L} \) are related via an unknown invertible base change matrix \( Q \in \mathbb{Q}^{n \times n} \). Letting \( P = [x^{(1)}] \ldots [x^{(n)}] \in \mathbb{Z}^{n \times n} \) with columns \( \{x^{(i)}\}_i \), one obtains matrix relations

\begin{equation}
W_0 = P \cdot Q, \quad W_1 = P \cdot U_1 \cdot Q
\end{equation}

where \( U_1 \) is \( n \times n \) diagonal with entries \( \{y^{(i)}\}_i \). The matrix \( W_0 \) is invertible (over \( \mathbb{Q} \)) and one computes the eigenvalues \( \{y^{(i)}\}_i \) of \( W_1 W_0^{-1} = PU_1 P^{-1} \). Using \( y \equiv y^{(i)} \pmod{p_i} \), one factors \( M \) by computing greatest common divisors.

5.1.2. A naive improvement. There is a naive generalization of \cite{CP19} using only \( \mathcal{O}(\sqrt{n}) \) public instances in \( \mathcal{S} \). However, we argue that this approach gives a worse range of parameters when combined with \cite{CP19}.

For integers \( p \geq 2 \) and \( t \geq 1 \) of size \( \mathcal{O}(\sqrt{n}) \), let \( x = (y_1 \cdot z, \ldots, y_t \cdot z) \in \mathbb{Z}^p \) of dimension \( \mathcal{O}(n) \) for \( y_1, \ldots, y_t \in \mathcal{S} \) and \( z \in \mathcal{S}^p \). This variant reduces \#\mathcal{S} considerably, as \#\mathcal{S} = p + t = \mathcal{O}(\sqrt{n}) \). However, \cite{CP19} requires to construct the vector \( b = (x, y \cdot x) \) for \( y \in \mathcal{S} \). This gives rise to residue vectors \( \{b^{(i)}\}_i \) of approximate \( \ell_2 \)-norm \( 2^{t/2} \) instead of \( 2^{2n/2} \) as in \cite{CP19}. Therefore the stronger condition \( 3\rho < \eta \) will be required for the orthogonal lattice attack to succeed. In our improvement, we would like to lower \#\mathcal{S} while continuing to use \( 2\rho < \eta \), as in \cite{CP19}.

5.1.3. Our improved algorithm. We recognize that \( (5.9) \) defines an instance of Problem A of Def. 1.1 with \( t = 1 \) because \( P \) and \( Q \) have rank \( n \). Our improvement lies in generalizing the vector \( b \) as to obtain an instance of Problem C.

We consider \#\mathcal{S} < \( n + 1 \) and write for convenience \( \mathcal{S} = \{x_1, \ldots, x_p, y_1, \ldots, y_t\} \) with integers \( 2 \leq p < n \) and \( 2 \leq t < n \) satisfying \( 2n \leq (t + 1)p \). We let \( x = (x_1, \ldots, x_p) \in \mathcal{S}^p \) and \( b = (x, y_1 \cdot x, \ldots, y_t \cdot x) \in \mathbb{Z}^{(t+1)p} \). As previously, let \( \{b^{(i)}\}_i \) denote the short residue vectors modulo the primes \( \{p_i\}_i \) and \( x \equiv x^{(i)} \pmod{p_i} \), \( y_a \equiv y^{(i)}_a \pmod{p_i} \) for \( a \in [t] \) and \( i \in [n] \). By the Chinese Remainder Theorem, we observe that \( b \) lies in the lattice \( \mathcal{L} = \bigoplus_{i=1}^{n} \mathbb{Z} b^{(i)} \) modulo \( M \). Namely, there are
integers $c_1, \ldots, c_n$ such that
\[
    b \equiv \sum_{i=1}^{n} c_i \begin{bmatrix} x_i \\ y_{i(1)} \cdot x(i) \\ \vdots \\ y_{i(\ell)} \cdot x(i) \end{bmatrix} =: \sum_{i=1}^{n} c_i b(i) \pmod{M}
\]

As in [CP19], the orthogonal lattice algorithm reveals a basis $\{b(i)\}_i$ of $\mathbb{Z}$ and the $\ell_2$-norm of $\{b(i)\}_i$ is still approximately $2\rho$.

Contrary to (5.9), we now derive matrix equations
\[
    W_0 = P \cdot Q, \quad W_a = P \cdot U_a \cdot Q, \quad a \in [t]
\]
where $P \in \mathbb{Z}^{p \times n}$ has columns $\{x(i)\}_i$ and $\{U_a\}_a$ are $n \times n$ diagonal with entries $\{y_{a(i)}\}_{a,i}$. The matrix $Q$ is a base change matrix from $\{b(i)\}_i$ to $\{b(i)\}_i$. If $W_0$ has rank $p$, Eq. (5.10) now defines a valid input for Problem C of Def. 1.1 and Algorithm $A_C$ from Sec. 3 reveals the diagonal entries $\{y_{a(i)}\}_{a,i}$ of the matrices $\{U_a\}_a$. One can then factor $M$ by computing $\gcd(y_a - y_a^i, M)$.

From Sec. 3.3 we see that $\#S = p + t$ is minimized for $p = \lfloor \sqrt{2n} \rfloor$ and $t + 1 = \lceil \sqrt{2n} \rceil$. Thus, $\#S = 2\lfloor \sqrt{2n} \rfloor = \mathcal{O}(\sqrt{n})$. In summary, letting $n$ be the number of secret primes in the public modulus $M$, we can factor $M$ given only $\mathcal{O}(\sqrt{n})$ input samples, whereas [CP19] uses $\mathcal{O}(n)$.

**Remark 5.2.** We remark that we do not impact the security of the key-exchange from [CP19], as it uses certain encodings of matrices. However, the product of matrices does not commute, so our techniques do not apply to that case.

### 5.2. Improved Cryptanalysis of CLT13 Multilinear Maps

We consider the CLT13 Multilinear Map Scheme by Coron et al., [CLT13]. Cheon et al. [CHL+15] described a polynomial-time attack against the Diffie-Hellman key exchange based on CLT13 when enough encodings of zero are public. Such encodings are for instance public in the rerandomization procedure. It is of interest to investigate this cryptanalysis when only a limited number of such encodings is available. Namely, not every CLT13-based construction necessarily reveals enough such encodings and the attack of Cheon et al. is prevented.

#### 5.2.1. CLT13 Multilinear Maps

The CLT13 Multilinear Map is a construction over the integers based on the notion of graded encoding scheme [GGH13]. Its hardness relies on Chinese Remainder-representations and factorization. We fix an integer $n \geq 2$, thought of as a dimension for CLT13. The message space is $\bigoplus_{i=1}^{n} \mathbb{Z}/g_i \mathbb{Z}$ for some small secret primes $\{g_i\}_i$. The encoding space has a graded structure and supports homomorphic addition and multiplication. It is defined over $\bigoplus_{i=1}^{n} \mathbb{Z}/p_i \mathbb{Z}$ for large secret primes $\{p_i\}_i$ with public product $x_0 = \prod_i p_i$. More precisely, an encoding of a message $m = (m_i)_i \in \bigoplus_{i=1}^{n} \mathbb{Z}/g_i \mathbb{Z}$ at level $k \in [\kappa]$ (where $\kappa$ denotes the multilinearity degree) is an integer $c$ such that $c \equiv (r_i g_i + m_i) \cdot z^{-k} \pmod{p_i}$ for all $i \in [n]$ where $z \in (\mathbb{Z}/x_0 \mathbb{Z})^\times$ and $r_i$ is a random "small" noise. By the Chinese Remainder Theorem $c$ is computed modulo $x_0$. For encodings $c$ at the last level $\kappa$, a public zero-testing procedure allows to test if $c$ encodes zero. This procedure works by computing $\omega(c) := p_{z^t} \cdot c$ for a public parameter $p_{z^t} \in \mathbb{Z}/x_0 \mathbb{Z}$. Then $c$ encodes the zero message if $\omega$ is "small" compared to $x_0$. In [CLT13], one actually defines...
a vector of \( n \) zero-test parameters \( \{ p_{zt,i} : i \in [n] \} \) to define a proper zero-testing. For the precise parameter setting, we refer to [CLT13 Sec. 3.1].

5.2.2. Cryptanalysis. The algorithm from [CHL+15] reveals all secret parameters given sufficiently many encodings of zero. We briefly recall the attack here, and for simplicity of exposition, assume \( \kappa = 3 \). Consider sets \( \mathcal{A} = \{ \alpha_j : j \in [n] \} \), \( \mathcal{B} = \{ \beta_1, \beta_2 \} \) and \( \mathcal{C} = \{ \gamma_k : k \in [n] \} \) of encodings at level 1 and where all encodings in \( \mathcal{A} \) encode zero. Therefore, there are \#(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}) = 2n + 2 \) encodings in total. In the previous notation, we write \( \alpha_j \equiv \alpha_{ji}/z \pmod{p_i} \), \( \beta_a \equiv \beta_{ai}/z \pmod{p_i} \) and \( \gamma_k \equiv \gamma_{ki}/z \pmod{p_i} \) for all \( i, j, k \in [n] \) and \( a \in [2] \). Because the products \( \alpha_j \beta_a \gamma_k \) encode zero at level 3, correct zero-testing ensures that the zero-test equations \( \omega_{jk}^{(a)} = p_{zt}(\alpha_j \beta_a \gamma_k) \), given by

\[
\omega_{jk}^{(a)}(\alpha_j \beta_a \gamma_k) = \begin{bmatrix} \beta_{ai} & \cdots & \beta_{aj} \\ \vdots & \ddots & \vdots \\ \beta_{an} & \cdots & \beta_{aj} \end{bmatrix} \begin{bmatrix} \gamma_{k1} \\ \vdots \\ \gamma_{kn} \end{bmatrix}
\]

for certain explicit integers \( p_{zt,i} \) for \( i \in [n] \) defining the zero-test parameter, hold over \( \mathbb{Z} \) instead of \( \mathbb{Z}/x_0 \mathbb{Z} \). Writing these relations out for all indices \((j,k) \in [n]^2\), the \( n \times n \) matrices \( W_a := (\omega_{jk}^{(a)})_{j,k \in [n]} \) for \( a = 1, 2 \) satisfy

\[
W_a = P \cdot U_a \cdot Q
\]

for secret matrices \( P, Q \) of full rank \( n \) (corresponding to encodings of \( \mathcal{A} \) and \( \mathcal{C} \), respectively) and diagonal matrices \( \{ U_a \}_a \) containing the elements \( \{ \beta_{ai} : i \in [n] \} \). If at least one of \( W_1, W_2 \) is invertible over \( \mathbb{Q} \) (say \( W_2 \)), the attacker computes the eigenvalues \( \{ \beta_{1i}/\beta_{2i} : i \in [n] \} \) of \( W_1 W_2^{-1} \). These ratios are enough to factor \( x_0 \). Indeed, letting \( \beta_{1i}/\beta_{2i} = x_i/y_i \) for coprime integers \( x_i, y_i \) and using \( \beta_a \equiv \beta_{ai}/z \pmod{p_i} \), we deduce \( x_i \beta_2 - y_i \beta_1 \equiv (x_i \beta_{2i} - y_i \beta_{1i})/z \equiv 0 \pmod{p_i} \) for \( i \in [n] \) and therefore \( \gcd(x_i \beta_2 - y_i \beta_1, x_0) = p_i \) with high probability.

In summary, the Cheon et al. attack recovers all secret primes \( \{ p_i \} \) in polynomial time given the set \( \mathcal{A} \) of level-one encodings of zero and the sets \( \mathcal{B} \) and \( \mathcal{C} \).

5.2.3. Attacking CLT13 with fewer encodings. We consider the following CLT13-based problem.

**Definition 5.3 (CLT13 Problem).** Let \( n \geq 2 \) be the dimension of CLT13 and \( x_0 = \prod_{i=1}^n p_i \). Let \( \mathcal{E} \) be a finite non-empty set of encodings at level 1 and \( \mathcal{E}_0 \subseteq \mathcal{E} \) a non-empty subset such that every element of \( \mathcal{E}_0 \) is an encoding of zero. The CLT13 Problem is as follows: Given the sets \( \mathcal{E} \) and \( \mathcal{E}_0 \), factor \( x_0 \).

We refer to \( \mathcal{E} \) and \( \mathcal{E}_0 \) as the sets of "available encodings" and "available encodings of zero", respectively. It is not a loss of generality to consider level-one encodings. As in [CHL+15], we write \( \mathcal{E} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \) with \( \mathcal{A} \subseteq \mathcal{E}_0 \). As recalled above, [CHL+15] requires \#(\mathcal{E}_0) \geq n to factor \( x_0 \) and a total number of public encodings \#(\mathcal{E}) = 2n + 2.

We aim at reducing the number of encodings needed for the factorization of \( x_0 \) and treat the following questions independently:

(i) Factor \( x_0 \) with fewer available encodings of zero, i.e. \#(\mathcal{E}_0) < n

(ii) Factor \( x_0 \) with fewer available encodings, i.e. \#(\mathcal{E}) < 2n + 2
A naive improvement. As for the CRT-ACD Problem, there is a naive improvement using fewer encodings, but assuming $\kappa = 4$. One can form product encodings $\alpha_j \beta_a \gamma_k \delta_j$ at level 4, where every encoding is at level 1. These can be partitioned into sets $A, B$ and $C$ such that $A$ corresponds to encodings of zero with $\#A = O(\sqrt{n})$. However, this approach has the inconvenience of using $\kappa = 4$ and our improved attack aims at lowering the number of public encodings while $\kappa = 3$.

Minimizing the number of encodings of zero. We explain how to use Algorithm $A_C$ to factor $x_0$ using only $\#E_0 = O(\sqrt{n})$ level-one encodings of zero.

We fix integers $2 \leq p < n$ and $3 \leq t < n$ and assume again $\kappa = 3$. As in [CHL+15], we write $E = A \cup B \cup C$ with $A \subseteq E_0$. We let $\#A = p$, $\#B = t$ and $\#C = n$; and claim $p = O(\sqrt{n})$.

Every product encoding $c = \alpha_j \beta_a \gamma_k$ with $(\alpha_j, \beta_a, \gamma_k) \in A \times B \times C$ is an encoding of zero and by correct zero-testing we obtain integer matrix relations

\[ W_a = P \cdot U_a \cdot Q, \quad a \in [t] \]

for $P \in \mathbb{Z}^p \times n, Q \in \mathbb{Z}^t \times n$ corresponding to encodings in $A$ and $C$, respectively, and diagonal matrices $\{U_a\}_a$ corresponding to $B$. Exactly as in [CHL+15], the matrices $\{U_a\}_a$ contain integers $\beta_{ai}$ such that $\beta_a \equiv \beta_{ai} \pmod{p_i}$ for $i \in [n]$. With high probability the ranks of $P$ and $Q$ are $p$ and $n$, respectively. Defining $W_0 = W_1$ and $W_a = W_{a-1}$ for $2 \leq a \leq t$ we obtain an instance similar to Problem $D$ of Def. [1.1] but without a "special input matrix" $PQ$ (see Sec. [2.2]). Using Algorithm $A_C$, we reveal eigenvalues (the diagonal entries) of the matrices $\{U_a U_{a-1}^{-1}\}_a$ as it is likely that $U_1$ be invertible. We finally deduce the prime factorization of $x_0$ by taking greatest common divisors, as in [CHL+15].

By the optimization in Sec. [3.3] we choose $t = \lceil \sqrt{2n} \rceil$ and $\#A = p = \lceil \sqrt{2n} \rceil$.

Minimizing the total number of encodings. We now explain how to use Algorithm $A_D$ to factor $x_0$ using $\#E = \frac{2}{3}n + O(\sqrt{n})$ instead of $\#E = 2n + 2$ as in [CHL+15].

Contrary to the previous case, we now use a set $C$ with $\#C = p$; so $\#E = 2p + t$. It is now direct to see that upon correct zero-testing we derive equations as in (5.12) but with $Q \in \mathbb{Z}^t \times p$ instead. Thus, if both $P$ and $Q$ have rank $p$, we obtain Problem D of Def. [1.1] without "special input matrix" $W_0$. Then Algorithm $A_D$ reveals ratios of diagonal entries of $\{U_a U_{a-1}^{-1}\}$ and we consequently factor $x_0$.

Following Sec. [4.3] we are led to minimize $\#E(n) = 2p + t$ as a function of $n$. We can let $p = \lceil \frac{2}{3}n + \frac{1}{3} \sqrt{2n} \rceil$ and $t = \lceil \frac{1}{3} \sqrt{2n} + \frac{5}{3} \rceil$ and obtain

\[ \#E(n) = 2 \left( \frac{2}{3}n + \frac{1}{3} \sqrt{2n} \right) + \left( \frac{1}{3} \sqrt{2n} + \frac{5}{3} \right) = \frac{4}{3} n + O(\sqrt{n}). \]

Cryptanalysis with independent slots. In [CN19], Coron and Notarnicola cryptanalyze CLT13 when no encodings of zero are available beforehand, but instead only "partial-zero" encodings. Messages are non-zero modulo a product of several primes $g_1 \cdots g_\theta$ for some integer $\theta \in [n]$. We can improve this cryptanalysis following the same techniques as above. Let $\ell$ the number of partial-zero encodings. Since [CN19] is based on the algorithm of Cheon et al. to factor $x_0$, we can now replace it by Algorithm $A_C$ once $\ell$ encodings of zero are created. This means that we can set
ℓ = \mathcal{O}(\sqrt{n})$, which brings a twofold improvement: first, lattice reduction (in the orthogonal lattice attack [CN19 Sec. 4]) is only run on a lattice of dimension \mathcal{O}(\sqrt{n}); and second, the number of partial-zero encodings is reduced to \mathcal{O}(\sqrt{n}).

6. Computational Aspects and Practical Results

We describe practical parameters for algorithms \texttt{AC} and \texttt{AD}. We have implemented our algorithms in SageMath [S^+17]; our source code is provided in https://pastebin.com/Yg6QgZTh. Our experiments are done on a standard 3.3 GHz Intel Core i7 processor.

6.1. Instance Generation of Problems \texttt{C} and \texttt{D}. As is the case for applications in cryptanalysis, we consider matrices with integer entries. To generate instances of Problems \texttt{C} and \texttt{D}, given fixed parameters \(n, t, p\), we uniformly at random generate matrices \(P, Q\) and \(\{U_a\}_a\) with entries in \([-k, k] \cap \mathbb{Z}\) for some \(k \in \mathbb{Z} \geq 1\) as in Def. 1.1. We set \(W_0 = PQ\) and \(W_a = PU_aQ\) for \(a \in [t]\) to obtain instances of Problems \texttt{C} or \texttt{D}.

We perform the linear algebra over \(\mathbb{Z}/\ell\mathbb{Z}\) for a large prime \(\ell\), instead of over \(\mathbb{Q}\). It suffices to choose \(\ell\) slightly larger than the diagonal entries of \(\{U_a\}_a\) (e.g. \(\ell = \mathcal{O}(\max_{a,i}|u_{ai}|)\), where \(u_{ai}\) for \(i \in [n]\) denote the diagonal entries of \(U_a\)). The running time depends on the entry size of the generated matrices. The overall computational cost of our algorithms is dominated by the cost of solving systems of linear equations and performing simultaneous diagonalization, which can be done by standard algorithms for non-sparse linear algebra.

6.2. Practical Experiments. We gather practical parameters for problems \texttt{C} and \texttt{D}, and for our applications of Sec. 5. We compare \(p, t\) with the theoretical values \(p_0(n), t_0(n)\) obtained in Sec. 3 and 4. For Table 1, \(p_0(n) = \lceil \sqrt{2n} \rceil\) and \(t_0(n) = \lceil \sqrt{2n} - 1 \rceil\). For Table 2, \(p_0(n) = \lceil \frac{2}{3}n + \frac{\sqrt{n}}{3\sqrt{2}} \rceil\) and \(t_0(n) = \lceil \frac{1}{3}(\sqrt{2n} + 5) \rceil\). Here "entry size" is an approximation of the bit-size of the max-norm of each input matrix.

In Table 3, we compare our work with [CP19] for the CRT-ACD Problem and with [CHL^+15] for the cryptanalysis of CLT13. We give parameters for obtaining a complete factorization of \(M\) (in CRT-ACD) and \(x_0\) (in CLT13) of approximate bit-size \(n\eta\). For CRT-ACD, the column "this work" equals \#\(S = p + t\) (Series 1). For CLT13, "this work" shows \#\(E = 2p + t\) (Series 2) and \#\(E_0 = p\) (Series 3). For example, for \(n = 50\), our algorithm factors \(M\) (in CRT-ACD) using only 19 public samples, whereas [CP19] requires 51 samples; and similarly breaks CLT13 with only 10 public encodings of zero, while [CHL^+15] uses 50.

In conclusion, these practical experiments overall confirm our theory, as well as the quadratic improvement over [CP19] and [CHL^+15].

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References

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Table 1. Experimental data for Algorithm $\mathcal{A}_C$

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Series 3

Table 3. Experimental data for the CRT-ACD Problem and the CLT13 Problem


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