
COMMUTATOR SUBGROUP OF SYLOW 2-SUBGROUPS OF ALTERNATING GROUP AND MILLER-MORENO GROUPS AS BASES OF NEW KEY EXCHANGE PROTOCOL

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ABSTRACT

The goal of this investigation is effective method of key exchange which based on non-commutative group G . The results of Ko et al. [6] is improved and generalized. The size of a minimal generating set for the commutator subgroup of Sylow 2-subgroups of alternating group is found. The structure of the commutator subgroup of Sylow 2-subgroups of the alternating group A_{2^k} is investigated and used in key exchange protocol which based on non-commutative group.

We consider non-commutative generalization of CDH problem [4, 3] on base of metacyclic group of Miller-Moreno type (minimal non-abelian group). We show that conjugacy problem in this group is intractable. Effectivity of computation is provided due to using groups of residues by modulo n . The algorithm of generating (designing) common key in non-commutative group with 2 mutually commuting subgroups is constructed by us.

Key words: the commutator subgroup of Sylow 2-subgroups, metacyclic group, conjugacy key exchange scheme, finite group, conjugacy problem.

2000 AMS subject classifications: 20B27, 20B22, 20F65, 20B07, 20E45.

1 Introduction

In this paper new conjugacy key exchange scheme is proposed. This protocol based on conjugacy problem in non-commutative group [2, 3, 4, 5, 10]. We slightly generalize Ko Lee's [6] protocol of key exchange. Public key cryptographic schemes based on the new systems are established. The conjugacy search problem in a group G is the problem of recovering an $(a \in G)$ from given $(w \in G)$ and $h = a^{-1}wa$. This problem is in the core of several recently suggested public key exchange protocols. One of them is most notably due to Anshel, Anshel, and Goldfeld [2] and another due to Ko et al. [6]. As we know if CCP problem is tractable in G then problem of finding w^{ab} by given w , $w^a = a^{-1}wa$, $w^b = b^{-1}wb$ for an arbitrary fixed $w \in G$ such that is not from center of G , w^{ab} is the common key that Alice and Bob have to generate.

Recently, a novel approach to public key encryption based on the algorithmic difficulty of solving the word and conjugacy problems for finitely presented groups has been proposed in [1, 2]. The method is based on having a canonical minimal length form for words in a given finitely presented group, which can be computed rather rapidly, and in which there is no corresponding fast solution for the conjugacy problem. A key example is the braid group.

We denote by w^x the conjugated element $u = x^{-1}wx$. We show that efficient algorithm that can distinguish between two probability distributions of (w^x, w^y, w^{xy}) and (w^g, w^h, w^{gh}) does not exist. Also, an efficient algorithm which recovers w^{xh} from w , w^x and w^y does not exist. This group has representation

$$G = \langle a, b \mid a^{p^m} = e, b^{p^n} = e, b^{-1}ab = a^{1+p^{m-1}}, m \geq 2, n \geq 1 \rangle.$$

As a generators a, b can be chosen two arbitrary commuting elements [8, 10, 7].

Consider non-metacyclic group of Millera Moreno. This group has representation

$$G = \langle a, b \mid |c| = p, |a| = p^m, |a| = p^n, m \geq 1, n \geq 1, b^{-1}ab = ac, b^{-1}cb = c \rangle.$$

To find a length of orbit of action by conjugation by b we consider the class of conjugacy of elements of form $a^j c^i$. This class has length p because of action $b^{-1}a^j c^i b = a^{j+1} c^i, \dots$, as well as $b^{-1}a^j c^{i+p-1} b = a^j c^{i+p} = a^j c^i$ increase the power of c on 1. Thus, the first repetition of initial power j in $a^j c^i$ occurs though n conjugations of this word by b , where $1 \leq j \leq p$. Therefore, the length of the orbit is p .

We need to have an effective algorithm for computation of conjugated elements, if we want to design a key exchange algorithm based on non-commutative DH problem [5]. Due to the relation in metacyclic group, which define the homomorphism $\varphi : \langle b \rangle \rightarrow \text{Aut}(\langle a \rangle)$ to the automorphism group of the $B = \langle b \rangle$, we obtain a formula for finding a conjugated element. Using this formula, we can efficiently calculate the conjugated to element by using the raising to the $1 + p^{m-1}$ -th power, where $m > 1$.

There is effective method of checking the equality of elements due to cyclic structure of group $A = \langle a \rangle$ and $B = \langle b \rangle$ in this group G .

We have an effective method of checking the equality of elements in the additive group Z_n because of reducing by finite modulo n .

2 Proof that conjugacy problem is \mathcal{NP} -hard in G . Size of a conjugacy class

The orbit of the given base element $w \in G$ must be long enough if we want to have problem of DL or equally problem of conjugacy in non-commutative group G like \mathcal{NP} -hard problem.

Let elements of G act by conjugation on $w \in G$, where $w \notin Z(G)$.

Theorem 1. *The length of conjugacy class of non-central element w is equal to p .*

Proof. Recall the inner automorphism in G is determined by the formula $b^{-1}ab = a^{1+p^{m-1}}$. Let us recall the structure of minimal non-abelian Metacyclic group, namely $G = B \rtimes_{\varphi} A$, where $A = \langle a \rangle$ and $B = \langle b \rangle$ are finite cyclic groups. Therefore, the formula $b^{-1}ab = a^{1+p^{m-1}}$ defines a homomorphism φ in the subgroup of inner automorphisms $\text{Aut}(\langle a \rangle)$. It is well-known that each finite cyclic group is isomorphic to the correspondent additive cyclic group modulo n residue Z_n . In this group equality of elements can be checked effectively due to reducing the elements of the module group.

Consider the orbit of element w under action by conjugation. The length of such orbit can be found from equality $w^{(1+p^{m-1})^s} = w$ as minimal power s for which this equality will be true. We apply Newton binomial formula to the expression $(1 + p^{m-1}) \equiv 1 \pmod{p^m}$ and taking into account the relation $a^{p^m} = e$. We obtain

$$1 + C_s^1 p^{m-1} + 1 + C_s^2 p^{2(m-1)} + \dots + p^{s(m-1)} \equiv 1 \pmod{p^m}$$

only if $s \equiv p^l \pmod{p^m}$ with $l < m$ because $1 + C_s^1 p^{m-1} = 1 + sp^{m-1} \not\equiv 1 \pmod{p^s}$ if $s < p$. It means that the minimal s when this congruence start to holds is equal to p . The prime number p can be chosen as big as we need [17] which completes the proof. □

Let us evaluate the size of subsets S_1, S_2 with mutually commutative elements. Each of this subset of generated by them subgroups H_1, H_2 can be chosen as the subgroups of center of group G . It is well-known that the semidirect product is closely related to wreath product. The center of the wreath product with non-faithful action were recently studied [11].

Proposition 1. *As it was proved by the author a center of the restricted wreath product with n non-trivial coordinates $(A, X) \wr B$ is direct product of normal closure of center of diagonal of $Z(B^n)$, i.e. $(E \times Z(\Delta(B^n)))$, trivial an element, and intersection of $(K) \times E$ with (A) . In other words,*

$$Z((A, X) \wr B) = \langle (1; \underbrace{h, h, \dots, h}_n), e(Z(A) \cap Z(K, X)) \wr E \rangle \simeq \langle Z(A) \cap K \rangle \times Z(\Delta(B^n))$$

where $h \in Z(B), |X| = n$.

Taking into consideration that a semidirect product is the partial case of wreath product the diagonal of B^n degenerates in B . Thus, we obtain such formula for the center of semidirect product:

$$Z((A, X) \rtimes B) = \langle Z(1; h), e, (Z(A) \cap K, X) \wr E \rangle \simeq \langle Z(A) \cap K \rangle \times Z(\Delta(B^n)).$$

This structure lead to constructive method of finding elements of the center. As it was noted above the elements x and y are parts of elements of secret key. Therefore as greater a size of center of a considered group as greater a size of a key space of this protocol.

Also commutator subgroup of Sylow 2-subgroup of alternating groups can be used as a support of CSP problem [12, 13, 15].

Definition 2.1. For an arbitrary $k \in \mathbb{N}$ we call a k -coordinate subgroup $U < G$ a subgroup, which is determined by k -coordinate sets $[U]_l, l \in \mathbb{N}$, if this subgroup consists of all Kaloujnine's tableaux $a \in I$ for which $[a]_l \in [U]_l$.

We denote by $G_k(l)$ a level subgroup of G_k , which consists of the tuples of v.p. from $X^l, l < k - 1$ of any $\alpha \in G_k$.

As a sets S_1 and S_2 consisting of mutually commutative elements we can use the set of elements of l -coordinate subgroup of G_k , where $l < k$, or the elements of $G_k(l)$ that is isomorphic to this subgroup. As it was proved by the author [12] the order of $Syl_2 A_{2^k}$ is $2^{2^k - k - 2}$. therefore the growth of mutually commutative sets of elements S_1 and S_2 is exponential function has.

According to [9] index of center of metacyclic group has index $|G : Z(G)| = p^2$, therefore the order of $Z(G) = p^{k-2}$. Thus, we have $p^2 - 1$ possibilities to choose an element w as an element of the open key, which is in the protocol of key exchange.

3 Key exchange protocol

Let S_1, S_2 be subsets from G consisting of mutually commutative elements. We make a generalisation of CDH by taking into consideration the subgroups $H_1 = \langle S_1 \rangle$ and $H_2 = \langle S_2 \rangle$ instead of using S_1, S_2 . We can do this because the groups H_1 and H_2 have generating sets S_1 and S_2 which commute. Because of these mutually commutative generating sets, we know that the subgroups are additionally mutually commutative.

4 Consideration of base steps of the protocol

Input: Elements w, w^x and w^y .

Alice selects a private x as the random element x from the subgroup H_1 and computes $w^x = x^{-1}wx$. The she sends it to Bob. Bob selects a private y as the random element y from the subgroup H_2 and computes w^y . Then he sends it to Alice. Bob computes $(w^x)^y = w^{xy}$ and Alice computes $(w^y)^x = w^{yx}$. Taking into consideration that H_1 and H_2 are mutually commutative groups we obtain that $xy = yx$. Therefore, we have that $w^{xy} = w^{yx}$.

Output: w^{xy} that is the common key of Alice and Bob.

Thus, the common key [3, 6, 2, 1] w^{xy} was successfully generated.

Resistance to a cryptanalysis. But if an analytic use for a cryptanalysis will use for cryptanalysis solving of conjugacy search problem the method of reduction to solving of decomposition problem [16], then it lead us to solving of discrete logarithm problem in the multiplicative cyclic group Z_p . This problem is NP-hard for big p .

5 Conclusion

We can choose mutually commutative H_1, H_2 as subgroups of $Z(G)$. As we said above, x, y are chosen from H_1, H_2 as components of key. According to [8] $Z(G) = p^{n+m-2}$ so size of key-space is $O(p^{n+m-2})$. It should be noted that the size of key-space can be chosen as arbitrary big number by choosing the parameters p, n, m . As an element for exponenting we can choose an arbitrary element $w \in A$ but $w \neq e$, because the size of orbit in result of action of inner automorphism φ is always not less than p .

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