Network-Agnostic State Machine Replication

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Abstract. We study the problem of state machine replication (SMR)—the underlying problem addressed by blockchain protocols—in the presence of a malicious adversary who can corrupt some fraction of the parties running the protocol. Existing protocols for this task assume either a synchronous network (where all messages are delivered within some known time $\Delta$) or an asynchronous network (where messages can be delayed arbitrarily). Although protocols for the latter case give seemingly stronger guarantees, in fact they are incomparable since they (inherently) tolerate a lower fraction of corrupted parties.

We design an SMR protocol that is network-agnostic in the following sense: if it is run in a synchronous network, it tolerates $t_s$ corrupted parties; if the network happens to be asynchronous it is resilient to $t_a \leq t_s$ faults. Our protocol achieves optimal tradeoffs between $t_s$ and $t_a$.

1 Introduction

State machine replication (SMR) is a fundamental problem in distributed computing [18,19,31] that can be viewed as a generalization of Byzantine agreement (BA) [20, 30]. Roughly speaking, a BA protocol allows a set of $n$ parties to agree on a value once, whereas SMR allows those parties to agree on an infinitely long sequence of values with the additional guarantee that values input to honest parties are eventually included in the sequence. (See Section 3 for formal definitions. Note that SMR is not obtained by simply repeating a BA protocol multiple times; see further discussion in Section 1.1.) The desired properties should hold even in the presence of some fraction of corrupted parties who may behave arbitrarily. SMR protocols are deployed in real-world distributed data centers, and the problem has received renewed attention in the context of blockchain protocols used for cryptocurrencies and other applications.

Existing SMR protocols assume either a synchronous network, where all messages are delivered within some publicly known time bound $\Delta$, or an asynchronous network, where messages can be delayed arbitrarily. Although it may appear that protocols designed for the latter setting are strictly more secure, this is not the case because they also (inherently) tolerate a lower fraction of corrupted parties. Specifically, assuming a public-key infrastructure is available to the parties, SMR protocols tolerating up to $t_s < n/2$ adversarial corruptions are possible in a synchronous network, but in an asynchronous network SMR is achievable only for $t_a < n/3$ faults (see [8]).

We study here so-called network-agnostic SMR protocols that offer meaningful guarantees regardless of the network in which they are run. That is, fix thresholds $t_a, t_s$ with $0 \leq t_a < n/3 \leq t_s < n/2$. We seek to answer the following question: Is it possible to construct an SMR protocol that (1) tolerates $t_s$ (adaptive) corruptions if the network is synchronous, and moreover (2) tolerates $t_a$ (adaptive) corruptions even if the network is asynchronous? We show that the answer is positive iff $t_a + 2t_s < n$.

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Our work is directly inspired by recent results of Blum et al. [5], who study the same problem but for the simpler case of Byzantine agreement. We match their bounds on $t_a, t_s$ and, as in their work, show that these bounds are optimal in our setting.\footnote{One might expect that SMR implies BA, and so impossibility of BA when $t_a + 2t_s \geq n$ implies impossibility of SMR for the same thresholds. However, it is not clear that SMR implies BA (at least for the definitions we consider) when $t_a + 2t_s \geq n$.} While the high-level structure of our SMR protocol resembles the high-level structure of their BA protocol, in constructing our protocol we need to address several technical challenges (mainly due to the stronger liveness property required for SMR; see the following section) that do not arise in their work.

### 1.1 Related Work

There is extensive prior work on designing both Byzantine agreement and SMR/blockchain protocols; we do not provide an exhaustive survey, but instead focus only on the most relevant prior work.

As argued by Miller et al. [25], many well-known SMR protocols that tolerate malicious faults (e.g., [7, 16]) require at least partial synchrony in order to achieve liveness. Their HoneyBadger protocol [25] was designed specifically for asynchronous networks, but can only handle $t < n/3$ faults even if run in a synchronous network. Blockchain protocols are typically analyzed assuming synchrony [12, 26]; Nakamoto consensus, in particular, assumes that messages will be delivered much faster than the time required to solve proof-of-work puzzles.

We emphasize that SMR is not realized by simply repeating a (multi-valued) BA protocol multiple times. In particular, the validity property of BA only guarantees that if a value is input by all honest parties then that value will be output by all honest parties. In the context of SMR the parties each hold multiple inputs in a local buffer (where those inputs may arrive at arbitrary times), and there is no way to ensure that all honest parties will select the same value as input to some execution of an underlying BA protocol. Although generic techniques for compiling a BA protocol into an SMR protocol are known [8], those compilers are not network-agnostic and so do not suffice to solve our problem.

Our work focuses on protocols being run in a network that may be either synchronous or fully asynchronous. Other work looking at similar problems includes that of Malkhi et al. [24], who consider networks that may be either synchronous or partially synchronous; Liu et al. [21], who design a protocol that tolerates a minority of malicious faults in a synchronous network, and a minority of fail-stop faults in an asynchronous network; and Guo et al. [13] and Abraham et al. [2], who consider temporary disconnections between two synchronous network components.

A slightly different line of work [22,23,27,28] looks at designing protocols with good responsiveness. Roughly speaking, this means that the protocol still requires the network to be synchronous, but terminates more quickly if the actual message-delivery time is lower than the known upper bound $\Delta$. Kursawe [17] designed a protocol for an asynchronous network that terminates more quickly if the network is synchronous, but does not tolerate more faults in the latter case.

Finally, other work [3,9,10,29] considers a model where synchrony is available for some (known) limited period of time, but the network is asynchronous afterward.

### 1.2 Paper Organization

We define our model in Section 2, before giving definitions for the various tasks we consider in Section 3. In Section 4 we describe a network-agnostic protocol for the asynchronous common
subset (ACS) problem. The ACS protocol is used as a subprotocol of our main result, the network-agnostic SMR protocol, which is described and analyzed in Section 5.

In Section 6 we prove a lower bound showing that the thresholds we achieve are tight for network-agnostic SMR protocols. As discussed above, Blum et al. [5] show an analogous result for BA that does not directly apply to our setting.

2 Model

Setup assumptions and notation. We consider a network of \( n \) parties \( P_1, \ldots, P_n \) who communicate over point-to-point authenticated channels. We assume that the parties have established a public-key infrastructure prior to the protocol execution. That is, we assume that all parties hold the same vector \((pk_1, \ldots, pk_n)\) of public keys for a digital-signature scheme, and each honest party \( P_i \) holds the honestly generated secret key \( sk_i \) associated with \( pk_i \). A valid signature \( \sigma \) on \( m \) from \( P_i \) is one for which \( Vrfy_{pk_i}(m, \sigma) = 1 \). For readability, we use \( \langle m \rangle_i \) to denote a tuple \((i, m, \sigma)\) such that \( \sigma \) is a valid signature on message \( m \) signed using \( P_i \)'s secret key.

For simplicity, we treat signatures as ideal (i.e., perfectly unforgeable); we also implicitly assume that parties use some form of domain separation when signing (e.g., by using unique session IDs) to ensure that signatures are valid only in the context in which they are generated.

Where applicable, we use \( \kappa \) to denote a statistical security parameter.

Adversarial model. We consider the security of our protocols in the presence of an adversary who can adaptively corrupt some number of parties. The adversary may coordinate the behavior of corrupted parties, and cause them to deviate arbitrarily from the protocol. Note, however, that our claims about adaptive security are only with respect to the property-based definitions found in Section 3, not with respect to a simulation-based definition (cf. [11,14]).

Network model. We consider two possible settings for the network. In a synchronous network, all messages are delivered within some known time \( \Delta \) after they are sent, but the adversary can reorder and delay messages subject to this bound. (As a consequence, the adversary can potentially be rushing, i.e., it can wait to receive all incoming messages in a round before sending its own messages.) In this setting, we also assume all parties begin the protocol at the same time, and that parties’ clocks progress at the same rate. When we say the network is asynchronous, we mean that the adversary can delay messages for an arbitrarily long period of time, though messages must eventually be delivered. We do not make any assumptions on parties’ local clocks in the asynchronous case.

The network is either synchronous or asynchronous for the duration of the protocol (although we stress that the honest parties do not know which is the case).

3 Definitions

Although we are ultimately interested in state machine replication, our main protocol relies on various subprotocols for different tasks. We therefore provide relevant definitions here.

Throughout, when we say that a protocol achieves some property, we mean that it achieves that property with overwhelming probability.
3.1 Useful Subprotocols

Throughout this section we consider protocols where, in some cases, parties may not terminate (even upon generating output); for this reason, we mention termination explicitly in the definitions. Honest parties are those who are not corrupted by the end of the execution.

**Reliable broadcast.** A reliable broadcast protocol allows parties to agree on a value chosen by a designated sender. In contrast to the stronger notion of broadcast, here honest parties might not terminate (but, if so, then none of them terminate).

**Definition 1 (Reliable broadcast).** Let \( \Pi \) be a protocol executed by parties \( P_1, \ldots, P_n \), where a designated party \( P^* \in \{P_1, \ldots, P_n\} \) begins holding input \( v^* \) and parties terminate upon generating output.

- **Validity:** \( \Pi \) is \( t \)-valid if the following holds whenever at most \( t \) parties are corrupted: if \( P^* \) is honest, then every honest party outputs \( v^* \).
- **Consistency:** \( \Pi \) is \( t \)-consistent if the following holds whenever at most \( t \) parties are corrupted: either no honest party outputs, or else all honest parties output the same value \( v \in \{0, 1\} \).

If \( \Pi \) is \( t \)-valid and \( t \)-consistent, then we say it is \( t \)-secure.

**Byzantine agreement.** A Byzantine agreement protocol allows parties who each hold some initial value to agree on an output value. Note that for this definition we explicitly require termination.

**Definition 2 (Byzantine agreement).** Let \( \Pi \) be a protocol executed by parties \( P_1, \ldots, P_n \), where each party \( P_i \) begins holding input \( v_i \in \{0, 1\} \).

- **Validity:** \( \Pi \) is \( t \)-valid if the following holds whenever at most \( t \) of the parties are corrupted: if every honest party’s input is equal to the same value \( v \), then every honest party outputs \( v \).
- **Consistency:** \( \Pi \) is \( t \)-consistent if the following holds whenever at most \( t \) of the parties are corrupted: every honest party outputs the same value \( v \in \{0, 1\} \).
- **Termination:** \( \Pi \) is \( t \)-terminating if whenever at most \( t \) of the parties are corrupted, every honest party terminates with some output in \( \{0, 1\} \) (with overwhelming probability).

If \( \Pi \) is \( t \)-valid, \( t \)-consistent, and \( t \)-terminating, then we say it is \( t \)-secure.

**Asynchronous common subset (ACS).** Informally, a protocol for the asynchronous common subset (ACS) problem allows \( n \) parties, each with some input, to agree on a subset of those inputs. (The term “asynchronous” in the name is historical, and one can also consider protocols for this task in the synchronous setting.)

**Definition 3 (ACS).** Let \( \Pi \) be a protocol executed by parties \( P_1, \ldots, P_n \), where each \( P_i \) begins holding input \( v_i \in \{0, 1\}^* \), and parties output sets of size at most \( n \).

- **Validity:** \( \Pi \) is \( t \)-valid if the following holds whenever at most \( t \) parties are corrupted: if every honest party’s input is equal to the same value \( v \), then every honest party outputs \( \{v\} \).
- **Liveness:** \( \Pi \) is \( t \)-live if whenever at most \( t \) of the parties are corrupted, every honest party produces output.
- **Consistency:** \( \Pi \) is \( t \)-consistent if whenever at most \( t \) parties are corrupted, all honest parties output the same set \( S \).
- **Set quality:** \( \Pi \) has \( t \)-set quality if the following holds whenever at most \( t \) parties are corrupted: if an honest party outputs a set \( S \), then \( S \) contains the inputs of at least \( t + 1 \) honest parties.
3.2 State Machine Replication

Protocols for state machine replication (SMR) allow parties to maintain agreement on an ever-growing, ordered sequence of blocks, where a block is a set of values called transactions. An SMR protocol does not terminate but instead continues indefinitely. We model the sequence of blocks output by a party $P_i$ via a write-once array $\text{Blocks}_i[0], \text{Blocks}_i[1], \ldots$ maintained by $P_i$, each entry (or slot) of which is initially equal to $\perp$. We say that $P_i$ outputs a block in slot $j$ when $P_i$ writes a block to $\text{Blocks}_i[j]$; if $\text{Blocks}_i[j] \neq \perp$ then we refer to $\text{Blocks}_i[j]$ as the block output by $P_i$ in slot $j$. We do not require that honest parties output a block in slot $j - 1$ before outputting a block in slot $j$.

It is useful to define a notion of epochs for each party. (We stress that these are not global epochs; instead, each party maintains a local view of its current epoch.) Formally, we assume that each party $P_i$ maintains a write-once array $\text{Epochs}_i[0], \text{Epochs}_i[1], \ldots$, each entry of which is initialized to 0. We say $P_i$ enters epoch $j$ when it sets $\text{Epochs}_i[j] := 1$, and require:

- For $j > 0$, $P_i$ enters epoch $j - 1$ before entering epoch $j$.
- $P_i$ enters epoch $j$ before outputting a block in slot $j$.

An SMR protocol is run in a setting where parties asynchronously receive inputs (i.e., transactions) as the protocol is being executed; each party $P_i$ stores transactions it receives in a local buffer $\text{buf}_i$. We imagine these transactions as being provided to parties by some mechanism external to the protocol (which could involve a gossip protocol run among the parties themselves), and make no assumptions about the arrival times of these transactions at any of the parties.

We remark that according to these rules, $P_i$ may output a block for slot $j$ before it outputs a block for slot $j - 1$, but it enters epoch $j$ after entering epoch $j - 1$. If it is required for the application of the SMR protocol that parties output blocks in sequence, this is easily achievable by instructing parties to hold a block for slot $j$ without outputting until all blocks up to and including block $j - 1$ have been output.\(^2\)

**Definition 4 (State machine replication).** Let $\Pi$ be a protocol executed by parties $P_1, \ldots, P_n$ who are provided with transactions as input and locally maintain arrays $\text{Blocks}$ and $\text{Epochs}$ as described above.

- **Consistency:** $\Pi$ is $t$-consistent if the following holds whenever at most $t$ parties are corrupted: for all $j$, if an honest party outputs a block $B$ in slot $j$ then all parties that remain honest output $B$ in slot $j$.
- **Strong liveness:** $\Pi$ is $t$-live if the following holds whenever at most $t$ parties are corrupted: for any transaction $\text{tx}$ for which every honest party received $\text{tx}$ before entering epoch $j$, every party that remains honest outputs a block that contains $\text{tx}$ in some slot $j' \leq j$.
- **Completeness:** $\Pi$ is $t$-complete if the following holds whenever at most $t$ parties are corrupted: for all $j \geq 0$, every party that remains honest outputs some block in slot $j$.

If $\Pi$ is $t$-consistent, $t$-live, and $t$-complete, then we say it is $t$-secure.

\(^2\) More formally, let $\Pi$ denote the underlying SMR protocol that outputs blocks out of order. We can construct an SMR protocol $\Pi'$ that outputs blocks in order by simply buffering the blocks output by $\Pi$ and outputting the next block $i$ whenever all the blocks up to $i - 1$ have already been output. $\Pi'$ inherits strong liveness from strong liveness and completeness of $\Pi$. 

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Our liveness definition is stronger than usual, in that we require a transaction \( tx \) that appears in all honest parties’ buffers by epoch \( j \) to be included in a block output by each honest party in some slot \( j' \leq j \). (Typically, liveness only requires that each honest party eventually outputs a block containing \( tx \).) This stronger notion of liveness is useful for showing that SMR implies Byzantine agreement (see Section A.2) and is achieved by our protocol.

In our definition, a transaction \( tx \) is only guaranteed to be contained in a block output by an honest party if all honest parties receive \( tx \) as input. A stronger definition would be to require this to hold even if only a single honest party receives \( tx \) as input. It is easy to achieve the latter from the former, however, by simply having honest parties gossip all transactions they receive to the rest of the network.

### 4 An ACS Protocol with Higher Validity Threshold

Throughout this section, we assume an asynchronous network.

The asynchronous common subset (ACS) problem was originally introduced by Ben-Or et al. [4]. In this section, we construct an ACS protocol that is secure when the number of corrupted parties is below one threshold, and continues to provide validity even for some higher corruption threshold. That is, fix \( t_a \leq t_s \) with \( t_a + 2 \cdot t_s < n \). We show an ACS protocol that is \( t_a \)-secure, and achieves validity even for \( t_s \) corruptions. This protocol will be a key ingredient in our SMR protocol.

Our construction follows the high-level approach taken by Miller et al. [25], who devise an ACS protocol based on subprotocols for reliable broadcast and Byzantine agreement. In our case we need a reliable broadcast protocol that achieves validity for \( t_s \geq n/3 \) faults, and in Section 4.1 we show such a protocol. We then describe and analyze our ACS protocol in Section 4.2.

#### 4.1 Reliable Broadcast with Higher Validity

In Figure 1, we present a variant of Bracha’s (asynchronous) reliable broadcast protocol [6] that allows for a more general tradeoff between consistency and validity. Specifically, the protocol is parameterized by a threshold \( t_s \); for any \( t_a \leq t_s \) with \( t_a + 2 \cdot t_s < n \), the protocol achieves \( t_a \)-consistency and \( t_s \)-validity.

**Protocol** \( \Pi_{\text{BB}}^{t_s} \)

The sender \( P^* \) sends its input \( v^* \) to all parties. Then each party does:

- Upon receiving \( v^* \) from \( P^* \), send \( \langle \text{echo}, v^* \rangle \) to all parties.
- Upon receiving \( \langle \text{echo}, v^* \rangle \) messages on the same value \( v^* \) from \( n - t_s \) distinct parties, do: if \( \langle \text{ready}, v^* \rangle \) was not yet sent, then send \( \langle \text{ready}, v^* \rangle \) to all parties.
- Upon receiving \( \langle \text{ready}, v^* \rangle \) messages on the same value \( v^* \) from \( t_s + 1 \) distinct parties, do: if \( \langle \text{ready}, v^* \rangle \) was not yet sent, then send \( \langle \text{ready}, v^* \rangle \) to all parties.
- Upon receiving \( \langle \text{ready}, v^* \rangle \) messages on the same value \( v^* \) from \( n - t_s \) distinct parties, output \( v^* \) and terminate.

**Fig. 1.** Bracha’s reliable broadcast protocol, parameterized by \( t_s \).

**Lemma 1.** If \( t_s < n/2 \) then \( \Pi_{\text{BB}}^{t_s} \) is \( t_s \)-valid.
is asynchronous, it remains because \( C \)
bounded communication complexity in Lemma 8 below. Likewise, the state for
Lemma 3.
bounded, but since the state for \( \Pi \)
protocol that runs forever but has bounded communication complexity; we prove that
forever). Given that the SMR protocol itself runs indefinitely, it is reasonable to settle for an
\( \Pi \) BA
resp., \( \Pi \) ACS
BA
BB
BA
secure even when the network
\( t_a \)-live, not terminating (and in fact runs forever). Given that the SMR protocol itself runs indefinitely, it is reasonable to settle for an
\( \Pi \) ACS
protocol that runs forever but has bounded communication complexity; we prove that \( \Pi \) ACS
has bounded communication complexity in Lemma 8 below. Likewise, the state for \( \Pi \) ACS
may not be bounded, but since the state for \( \Pi \) SMR
is also unbounded, we consider this acceptable.

**Lemma 3.** Say \( t_s < n/2 \) and at most \( t_s \) parties are corrupted. Then if an honest \( P_i \) uses input \( v_i \)
in an execution of \( \Pi_{ACS}^{t_s,t_a} \), all honest parties receive output \( v_i \) from \( \text{Bcast}_i \).

**Proof.** The lemma follows from \( t_s \)-validity of \( \text{Bcast} \).

Before proceeding with the analysis, we note that because \( \text{BA} \) is \( t_a \)-secure even when the network
is asynchronous, it remains \( t_a \)-consistent and \( t_a \)-valid even once honest parties cease to participate
because \( C_1 \) is true. In proving the following lemmas, we implicitly rely on this fact.

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**Proof.** Assume there are at most \( t_s \) corrupted parties, and the sender is honest. All honest parties
receive the same value \( v^* \) from the sender, and consequently send (echo, \( v^* \)) to all other parties.
Since there are at least \( n - t_s \) honest parties, all honest parties receive (echo, \( v^* \)) from at least \( n - t_s \)
different parties, and as a result send (ready, \( v^* \)) to all other parties. By the same argument, all
honest parties receive (ready, \( v^* \)) from at least \( n - t_s \) parties, and so can output \( v^* \) (and terminate).
To complete the proof, we also argue that honest parties cannot output \( v \neq v^* \). Note first that
no honest party will send (echo, \( v \)) for any \( v \neq v^* \). Thus, any honest party will receive (echo, \( v \)) for
some \( v \neq v^* \) from at most \( t_s \) other parties. Since \( t_s < n - t_s \), no honest party will ever send (ready, \( v \))
for any \( v \neq v^* \). By the same argument, this shows that honest parties will receive (ready, \( v \)) for
some \( v \neq v^* \) from at most \( t_s \) other parties, and hence cannot output \( v \neq v^* \).

**Lemma 2.** Let \( t_a \) be such that \( t_a \leq t_s \) and \( t_a + 2 \cdot t_s < n \). Then \( \Pi_{BB}^{t_s} \) is \( t_a \)-consistent.

**Proof.** Suppose at most \( t_a \) parties are corrupted, and that an honest party \( P_i \) outputs \( v \). Then \( P_i \)
must have received (ready, \( v \)) messages from at least \( n - t_s \) distinct parties, at least \( n - t_s - t_a \geq t_s + 1 \)
of whom are honest. Thus, all honest parties receive (ready, \( v \)) messages from at least \( t_s + 1 \) distinct
parties, and so all honest parties send (ready, \( v \)) messages to everyone. It follows that all honest
parties receive (ready, \( v \)) messages from at least \( n - t_a \geq n - t_s \) parties, and so can output \( v \) as well.
To complete the proof, we argue that honest parties cannot output \( v' \neq v \). We argued above
that all honest parties send (ready, \( v \)) to everyone. Let \( P \) be the first honest party to do so. Since
\( t_a < t_s + 1 \), that party must have sent (ready, \( v \)) in response to receiving (echo, \( v \)) messages from
at least \( n - t_s \) distinct parties. If some honest \( P_j \) outputs \( v' \) then, arguing similarly, some honest
party \( P' \) must have received (echo, \( v' \)) messages from at least \( n - t_s \) distinct parties. But this is
a contradiction, since honest parties send only a single echo message but \( 2 \cdot (n - t_a) - t_a > n \).

### 4.2 An ACS Protocol

In Figure 2 we describe an ACS protocol \( \Pi_{ACS}^{t_s,t_a} \) that is parameterized by thresholds \( t_s, t_a \), where
\( t_s \leq t_a \) and \( t_a + 2 \cdot t_s < n \). Our protocol relies on two subprotocols: a reliable broadcast protocol
\( \text{Bcast} \) that is \( t_a \)-valid and \( t_a \)-consistent (such as the protocol \( \Pi_{BB}^{t_s} \) from the previous section), and
a Byzantine agreement protocol \( \text{BA} \) that is \( t_a \)-secure. (Since \( t_a < n/3 \), any asynchronous \( \text{BA} \) protocol secure for that threshold can be used.) Our ACS protocol runs several executions of these
protocols as sub-routines, and so to distinguish between them we denote the \( i \)th execution by \( \text{Bcast}_i \), resp., \( \text{BA}_i \). We say that the executions \( \text{Bcast}_i, \text{BA}_i \) correspond to party \( P_i \).

As we will see in the analysis below, \( \Pi_{ACS}^{t_s,t_a} \) is only \( t_a \)-live, not terminating (and in fact runs
forever). Given that the SMR protocol itself runs indefinitely, it is reasonable to settle for an
\( \Pi_{ACS}^{t_s,t_a} \) protocol that runs forever but has bounded communication complexity; we prove that \( \Pi_{ACS}^{t_s,t_a} \) has
bounded communication complexity in Lemma 8 below. Likewise, the state for \( \Pi_{ACS}^{t_s,t_a} \) may not be bounded, but since the state for \( \Pi_{SMR}^{t_s,t_a} \) is also unbounded, we consider this acceptable.

**Lemma 3.** Say \( t_s < n/2 \) and at most \( t_s \) parties are corrupted. Then if an honest \( P_i \) uses input \( v_i \)
in an execution of \( \Pi_{ACS}^{t_s,t_a} \), all honest parties receive output \( v_i \) from \( \text{Bcast}_i \).

**Proof.** The lemma follows from \( t_s \)-validity of \( \text{Bcast} \).
At any point during a party’s execution of the protocol, let $S^* \overset{\text{def}}{=} \{ i : \text{BA}_i \text{ output } 1 \}$ and let $s = |S^*|$. Define the following boolean conditions:

- $C_1(v)$: at least $n - t_a$ executions $\{ \text{Bcast}_i \}_{i \in [n]}$ have output $v$.
- $C_2(v)$: ready = true, all executions $\{ \text{BA}_i \}_{i \in [n]}$ have terminated, and a majority of the executions $\{ \text{Bcast}_i \}_{i \in S^*}$ have output $v$.
- $C_3$: ready = true, all executions $\{ \text{BA}_i \}_{i \in [n]}$ have terminated, and all executions $\{ \text{Bcast}_i \}_{i \in S^*}$ have terminated.

Each party initializes ready := false and then does:

- For all $i$: run Bcast, with $P_i$ as the sender, where $P_i$ uses input $v_i$.
- When Bcast terminates with output $v_i'$ do: if execution of BA, has not yet begun, run BA, using input 1.
- When $s \geq n - t_a$, set ready := true and run any executions $\{ \text{BA}_i \}_{i \in [n]}$ that have not yet begun, using input 0.
- (Exit 1:) If at any point $C_1(v)$ for some $v$, output $\{ v \}$.
- (Exit 2:) If at any point $\neg C_1 \land C_2(v)$ for some $v$, output $\{ v \}$.
- (Exit 3:) If at any point $\neg C_1 \land \neg C_2 \land C_3$, output $S := \{ v_i' \}_{i \in S^*}$.

After outputting:

- Continue to participate in any ongoing Bcast executions.
- Once $C_1 =$ true, stop participating in any ongoing BA executions.

**Lemma 4.** If $t_a + 2 \cdot t_s < n$, then $I^{t_s,t_a}_{\text{ACS}}$ is $t_s$-valid.

**Proof.** Note that $t_s < n/2$. Say at most $t_s$ parties are dishonest, and all honest parties have the same input $v$. Using Lemma 3, we see that at least $n - t_s$ executions of $\{ \text{Bcast}_i \}$ (namely, those for which $P_i$ is honest) will result in $v$ as output, and so all honest parties can take Exit 1 and output $\{ v \}$. It is not possible for an honest party to take Exit 1 and output something other than $\{ v \}$, since $t_s < n - t_s$. Thus, it only remains to show that if an honest party takes some other exit then it must also output $\{ v \}$. Consider the two possibilities:

**Exit 2:** Suppose some honest party $P$ takes Exit 2 and outputs $\{ v' \}$. Then, for that party, $C_2(v')$ is true, and so $P$ must have seen at least $\left\lceil \frac{n}{2} \right\rceil + 1$ of the $\{ \text{Bcast}_i \}_{i \in S^*}$ terminate with output $v'$. Moreover, $P$ must have ready = true and so $s \geq n - t_a$. Together, these imply that $P$ has seen at least $\left\lceil \frac{n - t_a}{2} \right\rceil + 1 > \left\lceil \frac{2t_s}{2} \right\rceil + 1 > t_s$ executions of $\{ \text{Bcast}_i \}$ terminate with output $v'$. At least one of those executions must correspond to an honest party. But then Lemma 3 implies that $v' = v$.

**Exit 3:** Assume an honest party $P$ takes Exit 3. Then $P$ must have ready = true (and so $s \geq n - t_a$), must have seen all executions $\{ \text{BA}_i \}_{i \in [n]}$ terminate, and must also have seen all executions $\{ \text{Bcast}_i \}_{i \in S^*}$ terminate. Because $|S^*| = s \geq n - t_a > 2t_s$, 

\[ 2t_s = 2 \cdot \left( \left\lceil \frac{n}{2} \right\rceil - \frac{t_a}{2} \right) + 1 > n - t_a, \]
a majority of the executions \( \{\text{Bcast}_i\}_{i \in S^*} \) that \( P \) has seen terminate must correspond to honest parties. Lemma 3 implies that all those executions must have resulted in output \( v \). But then \( C_2(v) \) must be true for \( P \), and it would not have taken Exit 3.

The following two lemmas prove that \( \Pi_{a}^{t_a,t_a} \) is \( t_a \)-consistent and \( t_a \)-live. First, we show that if two honest parties each output a set, then those sets are equal. Then we show that all honest parties do indeed output a set.

**Lemma 5.** Fix \( t_a, t_s \) with \( t_a \leq t_s \) and \( t_a + 2 \cdot t_s < n \), and assume at most \( t_a \) parties are corrupted. Then if honest parties \( P_1 \) and \( P_2 \) output sets \( S_1 \) and \( S_2 \), respectively, in an execution of \( \Pi_{a}^{t_a,t_a} \), it holds that \( S_1 = S_2 \).

**Proof.** We consider different cases based on the possible exits taken by the two honest parties, and show that in all cases their outputs agree.

**Case 1:** Either \( P_1 \) or \( P_2 \) takes Exit 1. Say \( P_1 \) takes Exit 1 and outputs \( \{v_1\} \). (The case where \( P_2 \) takes Exit 1 is symmetric.) We consider different sub-cases:

- **\( P_2 \) takes Exit 1:** Say \( P_2 \) outputs \( \{v_2\} \). Then \( P_1 \) and \( P_2 \) must have each seen at least \( n - t_s \) executions of \( \{\text{Bcast}_i\} \) output \( v_1 \) and \( v_2 \), respectively. Since \( t_s < n/2 \), at least one of those executions must be the same. But then \( t_a \)-consistency of \( \text{Bcast} \) implies that \( v_1 = v_2 \).

- **\( P_2 \) takes Exit 2:** Say \( P_2 \) outputs \( \{v_2\} \). For \( C_2(v_2) \) to be satisfied, \( P_2 \) must have \( s \geq n - t_a \), and must have seen at least

\[
\left\lfloor \frac{s}{2} \right\rfloor + 1 \geq \left\lfloor \frac{n - t_a}{2} \right\rfloor + 1
\]

executions of \( \{\text{Bcast}_i\} \) output \( v_2 \). As above, \( P_1 \) must have seen at least \( n - t_s \) executions of \( \{\text{Bcast}_i\} \) output \( v_1 \). But since

\[
(n - t_s) + \left\lfloor \frac{n - t_a}{2} \right\rfloor + 1 > n - t_s + \left\lfloor \frac{2t_s}{2} \right\rfloor + 1 > n,
\]

at least one of those executions must be the same and so \( t_a \)-consistency of \( \text{Bcast} \) implies that \( v_1 = v_2 \).

- **\( P_2 \) takes Exit 3:** We claim this cannot occur. Indeed, if \( P_2 \) takes Exit 3 then \( P_2 \) must have \( \text{ready} = \text{true} \) (and so \( s \geq n - t_a \)), and must have seen all executions \( \{\text{BA}_i\}_{i \in [n]} \) terminate and all executions \( \{\text{Bcast}_i\}_{i \in [S^*]} \) terminate. Because \( P_1 \) took Exit 1, \( P_1 \) must have seen at least \( n - t_s \) executions \( \{\text{Bcast}_i\}_{i \in [S^*]} \) output \( v_1 \), and therefore (by \( t_a \)-consistency of \( \text{Bcast} \)) there are at most \( t_s \) executions \( \{\text{Bcast}_i\}_{i \in [n]} \) that \( P_2 \) has seen terminate with a value other than \( v_1 \). The number of executions of \( \{\text{Bcast}_i\}_{i \in [S^*]} \) that \( P_2 \) has seen terminate with output \( v_1 \) is therefore at least \( (n - t_a) - t_s > t_s \), which is strictly greater than the number of executions \( \{\text{Bcast}_i\}_{i \in [S^*]} \) that \( P_2 \) has seen terminate with a value other than \( v_1 \). But then \( C_2(v_1) \) is true for \( P_2 \), and it would not take Exit 3.

**Case 2:** Neither \( P_1 \) nor \( P_2 \) takes Exit 1. We consider two sub-cases:

- **\( P_1 \) and \( P_2 \) both take Exit 2.** Say \( P_1 \) outputs \( \{v_1\} \) and \( P_2 \) outputs \( \{v_2\} \). Both \( P_1 \) and \( P_2 \) must have seen all \( \{\text{BA}_i\} \) terminate; by \( t_a \)-consistency of \( \text{BA} \) they must therefore hold the same \( S^* \). Since \( C_2(v_1) \) holds for \( P_1 \), it must have seen a majority of the executions \( \{\text{Bcast}_i\}_{i \in S^*} \) output \( v_1 \); similarly, \( P_2 \) must have seen a majority of the executions \( \{\text{Bcast}_i\}_{i \in S^*} \) output \( v_2 \). Then \( t_a \)-consistency of \( \text{Bcast} \) implies \( v_1 = v_2 \).
– Either \( P_1 \) or \( P_2 \) takes Exit 3. Say \( P_1 \) takes Exit 3. (The case where \( P_2 \) takes Exit 3 is symmetric.) As above, \( P_1 \) and \( P_2 \) agree on \( S^* \) (this holds regardless of whether \( P_2 \) takes Exit 2 or Exit 3). Since \( C_3 \) holds for \( P_1 \) but \( C_2 \) does not, \( P_1 \) must have seen all executions \( \{ \text{Bcast}_i \} \in S^* \) terminate but without any value being output by a majority of those executions. But then \( t_a \)-consistency of \( \text{Bcast} \) implies that \( P_2 \) also does not see any value being output by a majority of those executions, and so will not take Exit 2. Since \( P_2 \) instead must take Exit 3, it must have seen all executions \( \{ \text{Bcast}_i \} \in S^* \) terminate; \( t_a \)-consistency of \( \text{Bcast} \) then implies that \( P_2 \) outputs the same set as \( P_1 \).

**Lemma 6.** Fix \( t_a, t_s \) with \( t_a \leq t_s \) and \( t_a + 2 \cdot t_s < n \). Then \( \Pi^{t_s,t_a}_{\text{ACS}} \) has \( t_a \)-set quality.

**Proof.** Consider some honest party \( P \). We again consider the various possibilities. Say \( P \) takes Exit 1 and outputs \( S = \{ v \} \). Then \( P \) has seen at least \( n - t_s \) executions \( \{ \text{Bcast}_i \} \) terminate with output \( v \). Of these, at least \( n - t_s - t_a > t_s \geq t_a \) must correspond to honest parties. By Lemma 3, those honest parties all had input \( v \). This means that \( S \) contains the inputs of at least \( t_a + 1 \) honest parties.

Alternatively, say \( P \) takes Exit 2 or Exit 3 and outputs a set \( S \). Then \( P \) holds \( \text{ready} = \text{true} \), and so \( |S^*| \geq n - t_a \). At least
\[
\frac{n - 2 \cdot t_a}{t_a} > \frac{(n - t_a)}{2}, \quad t_a
\]
of the indices in \( S^* \) correspond to honest parties, and by Lemma 3 for each of those parties the corresponding output value \( v'_i \) that \( P \) holds is equal to that party’s input. Thus, regardless of whether \( P \) takes Exit 2 (and \( S \) contains the majority value output by \( \{ \text{Bcast}_i \} \in S^* \)) or Exit 3 (and \( S \) contains every value output by \( \{ \text{Bcast}_i \} \in S^* \)), the set \( S \) output by \( P \) contains the inputs of at least \( t_a + 1 \) honest parties.

**Lemma 7.** Fix \( t_a, t_s \) with \( t_a \leq t_s \) and \( t_a + 2 \cdot t_s < n \). Then \( \Pi^{t_s,t_a}_{\text{ACS}} \) is \( t_a \)-live.

**Proof.** Assume at most \( t_a \) parties are corrupted during an execution of \( \Pi^{t_s,t_a}_{\text{ACS}} \). We consider two cases: either some honest party takes Exit 1 during this execution, or no honest party ever takes Exit 1 during this execution. In the first case, the first honest party to take Exit 1 must have seen at least \( n - t_s \) executions \( \{ \text{Bcast}_i \} \in [n] \) output with the same value \( v \). Hence, \( t_a \)-consistency of \( \text{Bcast} \) implies that all other honest parties will eventually see at least those \( n - t_s \) executions output \( v \), and will output (if they have not already output via another exit).

In the second case, no honest party ever takes Exit 1. We argue that eventually all honest parties will set \( \text{ready} = \text{true} \) and output. If no honest party ever takes Exit 1, then all honest parties continue to participate in any still-running \( \text{BA} \) executions indefinitely. At all times \( t' \) such that no honest party has yet set \( \text{ready} = \text{true} \), for all \( \{ \text{BA}_i \} \in [n] \), each honest party has either input 1 or not yet provided input. Such executions are indistinguishable from an execution in which all honest parties have input 1, but some messages have been delayed, and therefore \( t_a \)-validity of \( \text{BA} \) implies that (so long as no honest parties input 0) these executions eventually output 1. There are now two possibilities: either it continues to be true that no honest parties input 0 to any \( \text{BA} \) execution, or some honest party inputs 0 to some \( \text{BA} \) execution. In the first case, \( t_a \)-validity of \( \text{BA} \) implies that at least \( n - t_a \) executions eventually output 1 for all honest parties, and therefore all honest parties set \( \text{ready} = \text{true} \). In the latter case, some honest party must have already set \( \text{ready} = \text{true} \) as a result of seeing at least \( n - t_a \) \( \text{BA} \) executions output 1. Therefore, by \( t_a \)-consistency of \( \text{BA} \), all honest parties will eventually see at least \( n - t_a \) \( \text{BA} \) executions output 1, and therefore all honest parties set \( \text{ready} = \text{true} \). Once \( \text{ready} = \text{true} \), each honest party will output as soon as all \( \{ \text{Bcast}_i \} \in S^* \)
output. Each Bcast\(_i\) such that \(i \in S^*\) is guaranteed to eventually output for the following reason: either the sender is honest, and Bcast\(_i\) terminates by Lemma 3, or the sender is dishonest, but by \(t_a\)-validity of BA, at least one honest party must have input 1 to BA\(_i\) as a result of seeing Bcast\(_i\) terminate, and therefore eventually everyone sees Bcast, terminate due to \(t_a\)-consistency of Bcast.

**Lemma 8.** Fix \(t_a, t_s\) with \(t_a \leq t_s\) and \(t_a + 2 \cdot t_s < n\). Then \(\Pi_{\text{ACS}}^{t_a,t_s}\) has bounded communication complexity under both of the following conditions:

1. At most \(t_a\) parties are corrupted.
2. At most \(t_s\) parties are corrupted and all honest parties input the same value \(v\).

**Proof.** Because Bcast has bounded communication complexity, it remains to show that all honest parties eventually stop participating in all BA executions, either because they terminate or because they set \(C_1 = \text{true}\) and stop participating in any still-running executions. We consider each condition separately.

**Case 1:** At most \(t_a\) parties are corrupted. We must show that either all BA executions will eventually terminate, or all honest parties eventually set \(C_1 = \text{true}\) and stop participating in any still-running BA executions.

Assume no honest parties take Exit 1 during an execution of BA. Then all honest parties continue to participate in all BA executions, and so bounded complexity follows from \(t_a\)-termination of BA.

Now assume some party takes Exit 1 during an execution of \(\Pi_{\text{ACS}}^{t_a,t_s}\). That party must have seen at least \(n - t_s\) executions \(\{\text{Bcast}\}_i\}_{i \in [n]}\) output with the same value. By \(t_a\)-consistency of Bcast, all honest parties eventually see those executions output with the same value, and thus can set \(C_1 = \text{true}\) and stop participating in any still-running BA executions.

**Case 2:** At most \(t_s\) parties are corrupted and all honest parties input the same value \(v\). Because all honest parties input the same value \(v\), \(t_s\)-validity of Bcast implies that all honest parties will eventually set \(C_1 = \text{true}\) and thus stop participating in any still-running BA executions.

**Theorem 1.** Fix \(t_a, t_s\) with \(t_a \leq t_s\) and \(t_a + 2 \cdot t_s < n\). Then \(\Pi_{\text{ACS}}^{t_a,t_s}\) is \(t_a\)-secure and \(t_s\)-valid.

**Proof.** Lemmas 5 and 7 together prove \(t_a\)-consistency. The theorem follows Lemmas 4 and 6.

## 5 A Network-Agnostic SMR Protocol

In this section, we show our main result: an SMR protocol that is \(t_s\)-secure in a synchronous network and \(t_a\)-secure in an asynchronous network. We begin in Section 5.1 by briefly introducing a useful subprotocol for what we call block agreement. (We defer a more detailed explanation of this protocol, along with relevant proofs, to the appendix.) We then use the block agreement protocol to construct an SMR protocol in Section 5.2.

### 5.1 Block Agreement

Recall that \((m)_i\) is shorthand for \((i, m, \sigma)\), where \(\sigma\) is a valid signature by \(P_i\) on message \(m\).

We define here a notion we call block agreement, and show a block-agreement protocol secure against any \(t < n/2\) corrupted parties. The structure of our protocol is inspired by the synod protocol of Abraham et al. [1]. Block agreement is a form of agreement where (1) in addition to an input, parties provide signatures (in a particular format) on those inputs, and (2) a stronger notion of validity is required. Specifically, consider pairs consisting of a block \(B\) along with a set \(\Sigma\) of signed buffers \((\text{buf}_j)_j\). We say a pair \((B, \Sigma)\) is valid if:
\(\Sigma\) contains signed buffers from strictly more than \(n/2\) distinct parties.

For each \(\langle \text{buf}_j \rangle \in \Sigma, \text{buf}_j \subseteq B\). (Note each buffer can be represented as a bit-vector of length \(|B|\).

**Definition 5 (Block agreement).** Let \(\Pi\) be a protocol executed by parties \(P_1, \ldots, P_n\), where each party \(P_i\) begins holding input \((B_i, \Sigma_i)\) and parties terminate upon generating output.

- **Validity:** \(\Pi\) is \(t\)-valid if whenever at most \(t\) of the parties are corrupted, then every honest party that outputs, outputs a \(t\)-valid pair.
- **Termination:** \(\Pi\) is \(t\)-terminating if the following holds when at most \(t\) of the parties are corrupted: every honest party outputs and terminates with probability \(1 - 2^{-\kappa}\).
- **Consistency:** \(\Pi\) is \(t\)-consistent if the following holds when at most \(t\) of the parties are corrupted: if every honest party inputs a \(t\)-valid pair, there is a \((B, \Sigma)\) such that every honest party outputs \((B, \Sigma)\).

If \(\Pi\) is \(t\)-valid, \(t\)-consistent, and \(t\)-terminating, then we say it is \(t\)-secure.

**Theorem 2.** There exists a block agreement protocol \(\Pi_{\text{BLA}}^{t_s}\) that is \(t\)-secure for any \(t < n/2\) when run in a synchronous network. Moreover, in a synchronous network, all honest parties terminate with probability \(1 - 2^{-\kappa}\) after time \(5\Delta\kappa\).

We now proceed directly to the SMR construction, deferring the block agreement construction and corresponding security proof until the Appendix.

### 5.2 State Machine Replication

At a high level, the protocol proceeds as follows. The parties attempt to achieve agreement on a block for each slot \(j\) using the block agreement protocol \(\Pi_{\text{BLA}}^{t_s}\). If that protocol succeeds in reaching agreement, parties use its output \(B\) as input to the ACS protocol \(\Pi_{\text{ACS}}^{t_a,t_s}\) from the previous section. Recall from Theorem 2 that \(\Pi_{\text{BLA}}^{t_s}\) terminates after time \(5\Delta\kappa\) with overwhelming probability when run in a synchronous network; this ensures that if the network is synchronous and at most \(t_s\) parties are corrupted, then all parties agree on their input \(B\) to \(\Pi_{\text{ACS}}^{t_a,t_s}\). Hence, \(t_a\)-validity of \(\Pi_{\text{ACS}}^{t_a,t_s}\) ensures that all parties output \(B\). On the other hand, if \(\Pi_{\text{BLA}}^{t_s}\) fails to output within time \(5\Delta\kappa\), parties abandon it and instead attempt to reach agreement using the ACS protocol directly. Now, \(t_a\)-consistency and set quality of the ACS protocol ensure agreement (for at most \(t_a\) corruptions) even if the network happens to be asynchronous.

In our protocol for state machine replication, parties agree on an ordered sequence of blocks. The data included in each block is determined by the output of the ACS subprotocol. Parties begin agreement on each new block at regular intervals (thanks to their synchronized clocks). All parties begin each iteration at the same time as long as the network is synchronous, though different parties may finish each block at different times, and indeed some parties may continue participating in an earlier iteration while at the same time beginning to participate in a new iteration. On the other hand, if the network is asynchronous, parties might not start iterations at the same time and the protocol must continue to ensure secure progress. In the following, we show how this can be achieved by relying on the network-agnostic properties of \(\Pi_{\text{ACS}}^{t_a,t_s}\).

**Theorem 3 (Consistency).** Fix \(t_a, t_s\) with \(t_a < n/3\) and \(t_a + 2 \cdot t_s < n\). The following guarantee holds for any execution of \(\Pi_{\text{SMR}}^{t_a,t_s}\) for which either (1) at most \(t_s\) parties are dishonest and the network
is synchronous, or (2) at most $t_a$ parties are dishonest and the network is asynchronous: If two honest parties $P_i$ and $P_j$ output $\text{Blocks}_i[k]$ and $\text{Blocks}_j[k]$, respectively, in slot $k$, then $\text{Blocks}_i[k] = \text{Blocks}_j[k]$.

Proof. We consider two cases. In the first, at most $t_s$ parties are dishonest and the network is synchronous, in the second, at most $t_a$ parties are dishonest and the network is asynchronous.

Case 1. In this case, by Theorem 2, all parties receive output $(B, \Sigma)$ from the block-agreement subprotocol after time at most $5\Delta \kappa$, and furthermore, $(B, \Sigma)$ must be $t_s$-valid. By $t_s$-validity of $\Pi^*_{\text{ACS}}$ (Lemma 4), any two honest parties $P_i$ and $P_j$ receive output $\text{BlockSet} = \{B\}$ from $\Pi^*_{\text{ACS}}$. Therefore, both parties set $\text{Blocks}_i[k] = \text{Blocks}_j[k] = B$ and terminate the subprotocol for slot $k$.

Case 2. In this case, by $t_a$-consistency of $\Pi^*_{\text{ACS}}$, we have that all honest parties agree on the same set $\text{BlockSet}$. Hence, everybody outputs the same block for slot $k$ and therefore in particular, $\text{Blocks}_i[k] = \text{Blocks}_j[k]$.

Theorem 4 (Strong liveness). Fix $t_s, t_a$ with $t_a < n/3$ and $t_a + 2 \cdot t_s < n$. The following guarantee holds for any execution of $\Pi^{t_s,t_a}_{\text{SMR}}$ for which either (1) at most $t_s$ parties are dishonest and the network is synchronous, or (2) at most $t_a$ parties are dishonest and the network is asynchronous: If a transaction $\text{tx}$ has been received by every honest party before entering epoch $k$, then for every honest party $P_i$, $\text{tx} \in \text{Blocks}_i[k']$ such that $k' \leq k$.

Proof. We consider two cases. In the first, at most $t_s$ parties are dishonest and the network is synchronous, in the second, at most $t_a$ parties are dishonest and the network is asynchronous. Note that in either case, all parties eventually hold a set $B$ of size $t_s + 1$, since there are at most $n - t_s + t_a + t_s > t_s$ corruptions in either case (since $t_s \leq t_s$).

Case 1. In this case, by consistency and validity of $\Pi^*_{\text{BLA}}$, all parties receive output $(B, \Sigma)$ from the block-agreement subprotocol after time at most $5\Delta \kappa$, and furthermore, $(B, \Sigma)$ must be $t_s$-valid. By $t_s$-validity of $\Pi^*_{\text{ACS}}$ (Lemma 4), all parties receive output $\{B\}$ from the ACS subprotocol and hence output $\{B\}$ as the block for slot $k$. Since $(B, \Sigma)$ is $t_s$-valid, it includes all the transactions held in some honest party $P_j$’s buffer at the point in time that it entered epoch $k$. Hence, either $\text{tx}$ was in $P_j$’s buffer (and so $B$ includes $\text{tx}$), or $\text{tx}$ was not in $P_j$’s buffer, which implies that $P_j$ has already output a block $\text{Blocks}_j[k']$ such that $k' < k$ and $\text{tx} \in \text{Blocks}_j[k']$ (and so by Theorem 3, all other honest parties eventually output that same block).
Case 2. In this case, note that honest parties only supply $B^*$ such that either $(B^*, \Sigma)$ or $(B^*, \Sigma')$ is $t_s$-valid as input to $\Pi_{ACS}^{t_s,t_a}$ (since they input either $B$ or $B'$). By $t_a$-consistency and $t_a$-set quality of $\Pi_{ACS}^{t_s,t_a}$, we have that all parties agree on a set $\text{BlockSet}$ containing at least $t_a + 1$ honest blocks. In particular, all of these honest blocks contain the buffer $\text{buf}$ of an honest party $P_j$ at the point it entered epoch $k$. Hence, either $\text{tx}$ was in $P_j$’s buffer (and so $\hat{B} \in \text{BlockSet}$ includes $\text{tx}$), or $\text{tx}$ was not in $P_j$’s buffer, which implies that $P_j$ has already output a block $\text{Blocks}_j[k']$ such that $k' < k$ and $\text{tx} \in \text{Blocks}_j[k]$ (and so by Theorem 3, all other honest parties eventually output that same block). In either case, there exists an epoch $k' \leq k$ such that all honest blocks in $\text{BlockSet}$ include $\text{tx}$, and it immediately follows that the union of blocks in $\text{BlockSet}$ also contains $\text{tx}$.

**Theorem 5 (Completeness).** Fix $t_a, t_s$ with $t_a < n/3$ and $t_a + 2 \cdot t_s < n$. The following guarantee holds for any execution of $\Pi_{SMR}^{t_s,t_a}$ for which either (1) at most $t_a$ parties are dishonest and the network is synchronous, or (2) at most $t_a$ parties are dishonest and the network is asynchronous: for all $k$, any honest party $P_i$ eventually outputs a block $\text{Blocks}_i[k]$ for slot $k$.

**Proof.** Regardless of whether the protocol is run in a synchronous network with at most $t_s$ corruptions, or in an asynchronous network with at most $t_a$ corruptions, the lemma follows by the liveness and termination properties of the subprotocols used.

### 6 Optimality of Our Thresholds

In this section we show that the parameters achieved by our protocol are optimal. This extends the analogous result by Blum et al. [5], who consider only the case of Byzantine agreement. We remark that, although SMR is generally thought of as a stronger form of consensus than BA, it is unclear whether a bound on the number of tolerable corruptions for BA implies one for SMR (under the same network conditions). For example, the folklore result that SMR implies BA only holds given less than $n/3$ corruptions. Because of this, impossibility of BA for $n/3$ corruptions does not imply impossibility of SMR under $n/3$ corruptions, and hence must be proven separately. Similarly to the above example, the bound of [5] does not imply ours.

**Lemma 9.** Fix $t_a, t_s, n$ with $t_a + 2t_s \geq n$. If an $n$-party SMR agreement protocol $\Pi$ is $t_s$-strongly live in a synchronous network, then it cannot also be $t_a$-consistent in an asynchronous network.

**Proof.** Assume $t_a + 2t_s = n$ and fix an SMR protocol $\Pi$. Partition the $n$ parties into sets $S_0, S_1, S_a$ where $|S_0| = |S_1| = t_s$ and $|S_a| = t_a$, and consider the following experiment:

- At global time 0, parties in $S_0$ begin running $\Pi$ with $m_0$ in their buffers, where $m_0, m_1$ are drawn uniformly and independently at random from the set $\{0, 1\}^\kappa$. All communication between parties in $S_0$ and parties in $S_1$ is blocked (but all other messages are delivered within time $\Delta$).

- Create virtual copies of each party in $S_a$, call them $S_a^0$ and $S_a^1$. Parties in $S_a^0$ begin running $\Pi$ (at global time 0) with $m_0$ in their buffers, and communicate only with each other and parties in $S_0$. Parties in $S_a^1$ begin running $\Pi$ with $m_1$ in their buffers, and communicate only with each other and parties in $S_1$.

Consider an execution of $\Pi$ in a synchronous network where parties in $S_1$ are corrupted and simply abort, and all remaining (honest) parties starting with $m_0$ in their buffers. The views of the honest parties in this execution are distributed identically to the views of $S_0 \cup S_a^0$ in the above experiment. In
particular, \( t_a \)-strong liveness of \( \Pi \) implies that all parties in \( S_0 \) include \( m_0 \) in Blocks\([0]\). Analogously, all parties in \( S_1 \) include \( m_1 \) in Blocks\([0]\).

Next consider an execution of \( \Pi \) in an asynchronous network where parties in \( S_a \) are corrupted, and run \( \Pi \) honestly with \( S_0 \) where all parties initially hold \( m_0 \) in their buffers at the start of the execution of \( \Pi \). Simultaneously, they run \( \Pi \) honestly with \( S_1 \) where all parties initially hold \( m_1 \) in their buffers at the start of the execution of \( \Pi \). Moreover, in this execution, all communication between the (honest) parties in \( S_0 \) and \( S_1 \) is delayed indefinitely. The views of the honest parties in this execution are distributed identically to the views of \( S_0 \cup S_1 \) in the above experiment, yet the conclusion of the preceding paragraph shows that \( t_a \)-consistency is violated with high probability, since a party in \( S_1 \) includes \( m_1 \) in Blocks\([0]\) with probability \( 1 - \frac{1}{2^\kappa} \).

References


A Appendix

A.1 Block Agreement

We now provide a construction and proofs for the block agreement protocol used in Section 5. Throughout this section, we assume a synchronous network.

We construct a block-agreement protocol in a modular fashion. We begin by defining a subprotocol $\Pi^{\text{Propose}}_P$ (see Figure 4) in which a designated party $P^*$ serves as a proposer. A tuple $(k, B, \Sigma, C)$ is called a $k$-vote on $(B, \Sigma)$ if $(B, \Sigma)$ is valid and either:

- $k = 0$, or
- $k > 0$ and $C$ is a set of valid signatures from a majority of the parties on messages of the form $(\text{Commit}, k', B, \Sigma)$ with $k' \geq k$ (where possibly different $k'$ can be used in different messages).

When the exact value of $k$ is unimportant, we simply refer to the tuple as a vote. A message of the form $\text{status} = (\text{Status}, k, B, \Sigma, C)_i$ is a correctly formed Status message (from party $P_i$) if $(k, B, \Sigma, C)$ is a vote. A message $(\text{Propose}, \text{status}_1, \ldots)_*$ is a correctly formed Propose message if it contains correctly formed Status messages from a majority of the parties.

We first show that any two honest parties who generate output in this protocol agree on their output.

**Lemma 10.** If honest parties $P_i$ and $P_j$ output $(B_i, \Sigma_i), (B_j, \Sigma_j) \neq \perp$, respectively, in an execution of $\Pi^{\text{Propose}}_P$, then $(B_i, \Sigma_i) = (B_j, \Sigma_j)$.
Protocol $\Pi_{\text{Propose}}^{P^*}$

We describe the protocol from the point of view of a party $P_i$ with input a vote $(k, B, \Sigma, C)$. Let $t = \lceil (n+1)/2 \rceil$.

1. At time 0, send status$_i := \langle \text{Status}, k, B, \Sigma, C \rangle_i$ to $P^*$.
2. At time $\Delta$, if $P^*$ has received at least $s \geq t$ correctly formed Status messages status$_1, \ldots, \text{status}_t$ (from distinct parties), then $P^*$ sets

   \[ m := (\text{Propose}, \text{status}_1, \ldots, \text{status}_t). \]

   and sends $\langle m \rangle_*$ to all parties.
3. At time $2\Delta$, if a correctly formed Propose message $\langle m \rangle_*$ has been received from $P^*$, then send $\langle m \rangle_*$ to all parties. Otherwise, output $\bot$.
4. At time $3\Delta$, let $\langle m \rangle'_j$ be the correctly formed Propose message received from $P_j$ (if any). If there exists $j$ such that $\langle m \rangle'_j \neq \langle m \rangle_*$, output $\bot$. Otherwise, let status$_{\text{max}} = \langle \text{Status}, k', B', \Sigma', C' \rangle$ be the status message in $\langle m \rangle_*$ with maximal $k'$ (picking the lowest index in case of ties). Output $(B', \Sigma')$. 

![Fig. 4. A protocol $\Pi_{\text{Propose}}^{P^*}$ with designated proposer $P^*$.

Proof. If $P_i$ outputs $(B_i, \Sigma_i) \neq \bot$, then $P_i$ must have received a correctly formed Propose message $\langle m \rangle_*$ by time $2\Delta$ that would cause it to output $(B_i, \Sigma_i)$. That message is forwarded by $P_i$ to $P_j$, and hence $P_j$ either outputs $\bot$ (if it detects an inconsistency) or the same value $(B_i, \Sigma_i)$.

Assume less than half the parties are corrupted. We show that if there is some $(B, \Sigma)$ such that the input of each honest party $P_i$ is a vote of the form $(k_i, B, \Sigma, C_i)$, and no honest party ever receives a vote $(k', B', \Sigma', C')$ with $k' \geq \min_i \{k_i\}$ and $(B', \Sigma') \neq (B, \Sigma)$, then the only value an honest party can output is $(B, \Sigma)$.

Lemma 11. Assume fewer than $n/2$ parties are corrupted, and that the input of each honest party $P_i$ to $\Pi_{\text{Propose}}^{P^*}$ is a $k_i$-vote on $(B, \Sigma)$. If no honest party ever receives a $k'$-vote on $(B', \Sigma') \neq (B, \Sigma)$ with $k' \geq \min_i \{k_i\}$, then every honest party outputs either $(B, \Sigma)$ or $\bot$.

Proof. Consider an honest party $P$ who does not output $\bot$. That party must have received a correctly formed Propose message $\langle m \rangle_*$ from $P^*$, which in turn must contain a correctly formed Status message from at least one honest party $P_i$. That Status message contains a vote $(k_i, B, \Sigma, C_i)$ and, under the assumptions of the lemma, any other vote $(k', B', \Sigma', C')$ contained in $\langle m \rangle_*$ with $k' \geq k_i$ has $(B', \Sigma') = (B, \Sigma)$. It follows that $P$ outputs $(B, \Sigma)$.

Finally, we show that when $P^*$ is honest then all honest parties do indeed generate output.

Lemma 12. Assume fewer than $n/2$ parties are corrupted. If every honest party’s input to $\Pi_{\text{Propose}}^{P^*}$ is a vote and $P^*$ is honest, then every honest party outputs the same valid $(B, \Sigma) \neq \bot$.

Proof. Since every honest party’s input is a vote, the honest $P^*$ will receive at least $\lceil (n+1)/2 \rceil$ correctly formed Status messages, and so sends a correctly formed Propose message to all honest parties. Since $P^*$ is honest, this is the only correctly formed Propose message the honest parties will receive, and so all honest parties will output the same valid $(B, \Sigma) \neq \bot$.

We now present a protocol $\Pi_{\text{GC}}^k$ that uses $\Pi_{\text{Propose}}^{P^*}$ to achieve a form of graded consensus on a valid pair $(B, \Sigma)$. (See Figure 5.) As in the protocol of Abraham et al. [1], we rely on an atomic
leader-election mechanism Leader with the following properties: On input $k$ from a majority of parties, Leader chooses a uniform leader $\ell \in \{1, \ldots, n\}$ and sends $(k, \ell)$ to all parties. This ensures that if less than half of all parties are corrupted, then at least one honest party must call Leader with input $k$ before the adversary can learn the identity of $\ell$. A leader-election mechanism tolerating any $t < n/2$ faults can be realized (in the synchronous model with a PKI) based on general assumptions [15]; it can also be realized more efficiently using a threshold unique signature scheme.

Below, we refer to a message $\langle \text{Commit}, k, B, \Sigma \rangle$ as a correctly formed Commit message (from $P_i$ on $(B, \Sigma)$) if $(B, \Sigma)$ is valid. We refer to a message $\langle \text{Notify}, k, B, \Sigma, C \rangle$ as a correctly formed Notify message on $(B, \Sigma)$ if $(B, \Sigma)$ is valid and $C$ is a set of valid signatures on $(\text{Commit}, k, B, \Sigma)$ from more than $n/2$ parties; in that case, $C$ is called a $k$-certificate for $(B, \Sigma)$.

![Graded block-consensus protocol](image)

**Protocol $\Pi^b_{GC}$**

We describe the protocol from the point of view of a party $P_i$ with input a vote $(k', B, \Sigma, C')$. Let $t = [(n + 1)/2]$.

1. At time 0, run parallel executions of $\Pi_{\text{Propose}}^{P_i}, \ldots, \Pi_{\text{Propose}}^{P_n}$, each using input $(k', B, \Sigma, C')$. Let $(B_j, \Sigma_j)$ be the output from the $j$th protocol.
2. At time $3\Delta$, call Leader($k$) to obtain the response $\ell$. If $(B_\ell, \Sigma_\ell) \neq \bot$, send $\langle \text{Commit}, k, B_\ell, \Sigma_\ell \rangle$ to every party.
3. At time $4\Delta$, if at least $t$ correctly formed Commit messages $\langle \text{Commit}, k, B_\ell, \Sigma_\ell \rangle$ from distinct parties have been received, then form a $k$-certificate $C$ for $(B_\ell, \Sigma_\ell)$, send $m := (\text{Notify}, k, B_\ell, \Sigma_\ell, C)$ to every party, output $((B_\ell, \Sigma_\ell, C), 2)$, and terminate.
4. At time $5\Delta$, if a correctly formed Notify message $\langle \text{Notify}, k, B, \Sigma, C \rangle$ has been received, output $((B, \Sigma, C), 1)$ and terminate. (If there is more than one such message, choose arbitrarily.) Otherwise, output $(\bot, 0)$ and terminate.

**Fig. 5.** A graded block-consensus protocol $\Pi^b_{GC}$, parameterized by $k$.

For an output $((B, \Sigma, C), g)$, we refer to $g$ as the grade and $(B, \Sigma, C)$ as the output. When a party’s output is $(B, \Sigma, C)$, we may also say that its output is a $k$-certificate for $(B, \Sigma)$.

**Lemma 13.** Assume fewer than $n/2$ parties are corrupted, and that the input of each honest party $P_i$ to $\Pi^b_{GC}$ is a $k_i$-vote on $(B, \Sigma)$. If no honest party ever receives a $k'$-vote on $(B', \Sigma') \neq (B, \Sigma)$ with $k' \geq \min_i \{k_i\}$ in step 1 of $\Pi^b_{GC}$, then (1) no honest party sends a Commit message on $(B', \Sigma') \neq (B, \Sigma)$ and (2) any honest party who outputs a nonzero grade outputs a $k$-certificate for $(B, \Sigma)$.

**Proof.** By Lemma 11, every honest party outputs either $(B, \Sigma)$ or $\bot$ in every execution of $\Pi_{\text{Propose}}$ in step 1. It follows that no honest party $P_i$ sends a Commit message on $(B', \Sigma') \neq (B, \Sigma)$, proving the first part of the lemma. Since less than half the parties are corrupted, this means an honest party will receive fewer than $[(n + 1)/2]$ correctly formed Commit messages on anything other than $(B, \Sigma)$; it follows that if an honest party outputs grade $g = 2$ then that party outputs $(B, \Sigma, C)$ with $C$ a $k$-certificate for $(B, \Sigma)$.

Arguing similarly, no honest party will receive a correctly formed Notify message on anything other than $(B, \Sigma)$. Hence any honest party that outputs grade 1 outputs $(B, \Sigma, C)$ with $C$ a $k$-certificate for $(B, \Sigma)$.

**Lemma 14.** Assume fewer than $n/2$ parties are corrupted. If an honest party outputs $(B, \Sigma, C)$ with a nonzero grade in an execution of $\Pi^b_{GC}$, then no honest party sends a Commit message on $(B', \Sigma') \neq (B, \Sigma)$.
Proof. Say an honest party outputs \((B, \Sigma, C)\) with a nonzero grade. That party must have received a correctly formed Notify message on \((B, \Sigma)\). Since that Notify message includes a \(k\)-certificate \(C\) with signatures from more than half the parties, at least one honest party \(P\) must have sent a Commit message on \((B, \Sigma)\). This means that \(P\) must have received \((B, \Sigma)\) as its output from \(\Pi_P^{\text{Propose}}\). By Lemma 10, this means the output of any other honest party from \(\Pi_P^{\text{Propose}}\) is either \((B, \Sigma)\) or \(\bot\). The lemma follows.

**Lemma 15.** Assume fewer than \(n/2\) parties are corrupted. If an honest party outputs \((B, \Sigma, C)\) with grade 2 in an execution of \(\Pi_k^\text{GC}\), then every honest party outputs a \(k\)-certificate on \((B, \Sigma)\) with a nonzero grade.

Proof. Say an honest party \(P\) outputs \((B, \Sigma, C)\) with a grade of 2. By Lemma 14, this means no honest party sent a correctly formed Commit message on \((B', \Sigma') \neq (B, \Sigma)\); it is thus impossible for any honest party to output \((B', \Sigma') \neq (B, \Sigma)\) with a nonzero grade. Since \(P\) sends a correctly formed Notify message on \((B, \Sigma)\) to all honest parties, every honest party will output \((B, \Sigma)\) with a nonzero grade.

**Lemma 16.** Assume fewer than \(n/2\) parties are corrupted. Then with probability at least \(1/2\) every honest party outputs a \(k\)-certificate on the same valid \((B, \Sigma)\) with a grade of 2.

Proof. The leader \(\ell\) chosen in step 2 was honest in step 1 with probability at least \(1/2\). We show that whenever this occurs, every honest party outputs grade 2. Agreement on a valid \((B, \Sigma)\) follows from Lemma 15.

Assume \(\ell\) was honest in step 1. Lemma 12 implies that every honest party holds the same valid \((B_\ell, \Sigma_\ell) \neq \bot\) in step 2, and so sends a correctly formed Commit message on \((B_\ell, \Sigma_\ell)\). Since there are at least \([(n + 1)/2]\) honest parties, the lemma follows.

In Figure 6 we describe our block-agreement protocol \(\Pi_t^\text{BLA}\).

**Protocol \(\Pi_t^\text{BLA}\)**

We describe the protocol from the point of view of a party \(P\) with input a valid pair \((B, \Sigma)\).
Initialize \((k^*, B^*, \Sigma^*, C^*) := (0, B, \Sigma, \emptyset)\) and \(k := 1\). While \(k \leq \kappa\) do:

1. At time \((5k - 5) \cdot \Delta\), run \(\Pi_k^\text{GC}\) using input \((k^*, B^*, \Sigma^*, C^*)\) to obtain output \(((B, \Sigma, C), g)\).
2. At time \(5k \cdot \Delta\) do: If \(g > 0\), set \((k^*, B^*, \Sigma^*, C^*) := (k, B, \Sigma, C)\). If \(g = 2\), output \((B, \Sigma)\). Increment \(k\).

**Fig. 6.** A block-agreement protocol \(\Pi_t^\text{BLA}\).

**Lemma 17.** If \(t < n/2\), then \(\Pi_t^\text{BLA}\) is \(t\)-secure.

Proof. Assume fewer than \(n/2\) parties are corrupted. Let \(k\) be the first iteration in which some honest party outputs \((B, \Sigma)\). We first show that in every subsequent iteration: (1) every honest party \(P_i\) uses as its input in step 1 a \(k_i\)-vote on \((B, \Sigma)\); and (2) corrupted parties cannot construct a \(k'\)-vote on \((B', \Sigma') \neq (B, \Sigma)\) for any \(k' \geq \min \{k_i\}\).

Say an honest party outputs \((B, \Sigma)\) in iteration \(k\). Then that party must have output a \(k\)-certificate for \((B, \Sigma)\) in the execution of \(\Pi_k^\text{GC}\) in iteration \(k\). By Lemma 15, this means every...
honest party output a $k$-certificate on $(B, \Sigma)$ in the same execution of $\Pi_{GC}^k$, and so (1) holds in iteration $k + 1$. Moreover, Lemma 14 implies that no honest party sent a Commit message on $(B', \Sigma') \neq (B, \Sigma)$ in the execution of $\Pi_{GC}^k$, and so (2) also holds in iteration $k + 1$. Lemma 13 implies, inductively, that the stated properties continue to hold in every subsequent iteration.

It follows from Lemma 13 that any other honest party $P$ who generates output in $\Pi_{BLA}^t$ also outputs $(B, \Sigma)$, regardless of whether they generate output in iteration $k$ or a subsequent iteration.

Lemma 16 shows that in each iteration of $\Pi_{BLA}^t$, with probability at least $\frac{1}{2}$ all honest parties output some (the same) valid $(B, \Sigma)$ in that iteration. Thus, after $\kappa$ iterations all honest parties have generated valid output with probability at least $1 - 2^{-\kappa}$ (note that all parties terminate after $\kappa$ iterations).

### A.2 SMR Implies BA with Liveness

For completeness, we give a brief discussion of how SMR relates to BA in the setting of network agnostic protocols. In Figure 7 we show how to use an SMR protocol $\Pi_{SMR}$ as a blackbox subroutine to achieve a slightly weakened form of Byzantine agreement, which we call “BA with liveness.” The key distinction between the BA definition given in Section 3 and the definition seen here is that the latter achieves liveness (not termination).

**Definition 6 (Byzantine agreement with liveness).** Let $\Pi$ be a protocol executed by parties $P_1, \ldots, P_n$, where each party $P_i$ begins holding input $v_i \in \{0, 1\}$.

- **Validity:** $\Pi$ is $t$-valid if the following holds whenever at most $t$ of the parties are corrupted: if every honest party’s input is equal to the same value $v$, then every honest party outputs $v$.
- **Consistency:** $\Pi$ is $t$-consistent if the following holds whenever at most $t$ of the parties are corrupted: every honest party outputs the same value $v \in \{0, 1\}$.
- **Liveness:** $\Pi$ is $t$-live if whenever at most $t$ of the parties are corrupted, every honest party outputs a value in $\{0, 1\}$ (with overwhelming probability).

If $\Pi$ is $t$-valid, $t$-consistent, and $t$-live, then we say it is $t$-secure.

In the following, we once again use $(v_i)_i$ as a shorthand for $(i, m, \sigma)$, where $\sigma_i \leftarrow \text{Sign}(sk_i, v_i)$.

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**Protocol $\Pi_{BA}$**

We describe the protocol from the point of view of a party $P_i$ with input $v_i$.

- Set $V_i := \emptyset$.
- Send $(v_i)_i$ to every party. Denote as $m_j$ the message $(v_j)_j$ received from party $P_j$.
- Upon receiving $m_j$, set $\text{buf}_i := \text{buf}_i \cup \{m_j\}$.
- Begin to run $\Pi_{SMR}$ at time $\Delta$.
- Upon outputting a block $\text{Blocks}_i[k]$ in $\Pi_{SMR}$ do: For each $m_j$ contained in $\text{Blocks}_i[k]$ for which $V_i$ does not already contain a value from party $P_j$, set $V_i := V_i \cup \{(v_j, j)\}$.
- If at any point during the execution $|V_i| \geq n - t_s$, then output the majority value among all values in $V_i$.

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Fig. 7. A protocol for Byzantine agreement.
Lemma 18 (Validity). Let \(2t_s + t_a < n\). If \(\Pi_{\text{SMR}}\) is \(t\)-live in a synchronous (resp., asynchronous) network, then \(\Pi_{\text{BA}}\) is \(t\)-valid in a synchronous (resp., asynchronous) network.

Proof. Suppose that all honest parties hold input \(v\). We consider two cases. In the first, the network is synchronous and there are at most \(t_s\) corrupted parties. In the second, the network is asynchronous and there are at most \(t_a\) corrupted parties.

**Case 1.** Because the network is synchronous, all parties begin to run the protocol at the same time 0 and all signed values \(\langle v \rangle\) sent by honest parties must be delivered to all honest parties within time \(\Delta\). Now, since all parties simultaneously start to run \(\Pi_{\text{SMR}}\) at time \(\Delta\), strong liveness of \(\Pi_{\text{SMR}}\) ensures that every honest party’s value-signature pair must appear in the first block \(\text{Blocks}_i[1]\) for any honest party \(P_i\). Hence, \(P_i\) holds \(V_i\) of size at least \(n - t_s\) immediately after outputting the block \(\text{Blocks}_i[1]\). Since all honest parties started with input \(v\), \(V_i\) must include this value at least \(n - t_s \geq t_s + 1\) times. Since at most \(t_s\) parties are corrupted, the honest majority among values in \(V_i\) can only be \(v\). Thus, all honest parties output \(v\).

**Case 2.** In this case, if any party holds \(V_i\) of size at least \(n - t_s\), it will always contain a (strict) majority of honest values, because \(\lceil \frac{n - t_s}{2} \rceil > t_a\) and there are at most \(t_a\) corrupted parties. Therefore, any honest party that produces output, will output \(v\) (since all honest parties’ input is \(v\)). Thus, it remains to show that all honest parties eventually hold \(V_i\) of size at least \(n - t_s\). We know that the \(n - t_s\) value-signature pairs sent by honest parties are eventually delivered to all honest parties; therefore, for each of these honest value-signature messages \(m_j = \langle v_j \rangle_j\), strong liveness of \(\Pi_{\text{SMR}}\) guarantees that all honest parties eventually output a block that includes \(m_j\). Thus, every honest party \(P_i\) eventually gathers \(V_i\) of size at least \(n - t_s\). \(\square\)

Lemma 19 (Consistency). Let \(2t_s + t_a < n\). If \(\Pi_{\text{SMR}}\) is \(t < t_s\)-consistent in a synchronous (resp., \(t < t_a\)-consistent in an asynchronous) network, then \(\Pi_{\text{BA}}\) is \(t\)-consistent in a synchronous (resp., asynchronous) network.

Proof. The lemma follows directly. \(\square\)

Lemma 20 (Liveness). Let \(2t_s + t_a < n\). If \(\Pi_{\text{SMR}}\) is \(t\)-live in a synchronous (resp., asynchronous) network, then \(\Pi_{\text{BA}}\) is \(t\)-live in a synchronous (resp., asynchronous) network.

Proof. As in the validity proof, we consider the two cases separately. In the first, the network is synchronous and there are at most \(t_s\) corrupted parties. In the second, the network is asynchronous and there are at most \(t_a\) corrupted parties.

**Case 1.** The proof proceeds similarly to Case 1 of the validity proof. All parties begin to run the protocol at the same time 0 and all signed values \(\langle v \rangle\) sent by honest parties must be delivered to all honest parties within time \(\Delta\). Strong liveness of \(\Pi_{\text{SMR}}\) ensures that every honest party’s value-signature pair must appear in the first block \(\text{Blocks}_i[1]\) for any honest party \(P_i\). Therefore, the set \(V_i\) held by \(P_i\) is of size at least \(n - t_s\) immediately after outputting the block \(\text{Blocks}_i[1]\), and so \(P_i\) can generate output.

**Case 2.** The proof proceeds similarly to Case 2 of the validity proof. It suffices to show that all honest parties eventually hold \(V_i\) of size at least \(n - t_s\). The \(n - t_s\) value-signature pairs sent by honest parties must eventually be delivered to all honest parties; therefore, strong liveness of \(\Pi_{\text{SMR}}\) guarantees that for each honest value-signature message \(m_j\), all honest parties eventually output a block that includes \(m_j\). Hence, every honest party \(P_i\) eventually holds a set \(V_i\) of size at least \(n - t_s\), and can output.