On relaxed security notions for secret sharing

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May 28, 2019

Abstract. Information ratio of an access structure is an important measure for efficiency of the best secret sharing scheme realizing it. The most common notion of secret sharing security is that of total (perfect) realization. Two well-known relaxations are the notions of *statistical* and *quasi-total* secret sharing. In this paper, we study the relation between different security notions. The most non-trivial and technical result of this paper is that quasi-total and total information ratios coincide for linear schemes. The proof is non-intuitive and uses tools from linear algebra in companion with a new relaxed security notion, called *partial* realization. We provide some intuition that why proving coincidence/separation between total and quasi-total information ratios for the class of *abelian* schemes is probably much more challenging.
We also present some additional results which shed further light on our understanding of different security notions. In particular, one of our results, in combination with a recent result, shows that statistical and total security notions coincide for the class of group-homomorphic schemes, or maybe even a larger class.

Keywords: Secret sharing · Access structure · Information ratio · Linear Algebra · Information theory.

1 Introduction

A *secret sharing scheme* [9, 42] is a powerful cryptographic tool that allows a dealer to share a secret among a set of participants such that only certain *qualified* subsets of participants are able to reconstruct the secret. The secret must remain information theoretically hidden from the remaining subsets, called *unqualified*. The collection of all qualified subsets is called an *access structure*, which is supposed to be monotone, i.e., closed under the superset operation. The original definition, known as *threshold* secret sharing, only dealt with access structures that include all subsets of size larger than a certain threshold and the general notion was later introduced in [26]. Another generalization is that of an *access function* [21]. This concept has been matured by building on a sequence of important works [10,32,43,46]. An access function is a monotone real function that specifies the percentage of the information on the secret that is obtained by each subset of participants. An access structure corresponds to a *total* access function which allows all-or-nothing recovery of the secret.

The *information ratio* [11, 13, 36] of a participant in a secret sharing scheme is defined as the ratio of the size of his share and the size of the secret. The most common interpretation of size of a random variable is its Shannon entropy. The information ratio of a secret sharing scheme is the maximum (also sometimes defined as the average) of all participants' information ratios. The information ratio of an access structure is defined as the infimum of the information ratios of all secret sharing schemes that realize it. Computation of information ratio of access structures is a challengingly difficult problem even when restricted to certain classes of schemes. In this paper, we are mainly interested in the information ratio for the class of *linear* and *abelian* schemes, and called the corresponding measure the linear (resp, abelian) information ratio. Some other classes will also show up in the paper.

Relaxed security notions. Most of the literature on secret sharing deals with total (perfect) realization of access structures by secret sharing schemes. In this notion, the security is considered for a single scheme, all the qualified sets recover the whole secret, and the secret remains information theoretically hidden from the unqualified sets. These requirements can be relaxed by loosening the reconstruction and privacy requirements. The qualified subsets may miss some information about the secret or may recover it with some error probability. The unqualified subsets are also allowed to gain some information on the secret. By considering a family of schemes, the information leak and incomplete reconstruction are required to be negligible. Two different approaches have been proposed in the literature. The first approach is a standard cryptographic relaxation, called *statistical secret sharing* (see [3] for probably the oldest modern definition and [8] for an old construction). The second one has been introduced in [30, 31], under the name of *quasi-perfect secret sharing.*

Our main motivation and main result. It is an open problem if the information ratio of an access structure is invariant with respect to different security notions. We look for subclasses of schemes for which this happens. It is an easy exercise to see that the statistical and total securities coincide for linear schemes [3]. But the situation for quasi-total security is unclear. We show that quasi-total and total information ratios are equal for the class of linear schemes. Additionally, we provide some intuition that why it might be more difficult to say something about any larger class, even the class of abelian schemes. On the other hand, as we will see, statistical security implies total security, for some non-trivial class of schemes which includes group-homomorphic[1] secret sharing schemes, and in particular, the abelian ones.

Our result on the equality of quasi-total and total linear information ratios is the most technical result of our paper, which is discussed is Section 1.1. Additional contributions will be mentioned in Section 1.2 and Section 1.3.

[1] A secret sharing scheme is called homomorphic if the product of the shares of two secrets produce a share for the product of the secrets. A homomorphic scheme is called group-homomorphic if the secret and share spaces are all groups.

1.1 Technicality of proving equality of total and quasi-total linear information ratios

We take the following steps for proving that the total and quasi-total information ratios coincide for the class of linear schemes.

Introducing the notion of partial secret sharing. We introduce an extremely relaxed security notion, called *partial* security, and a slightly more liberal one called *semi-partial*. These are the missing ingredients for proving coincidence of total and quasi-total information ratios for the class of linear schemes. We say that a secret sharing scheme partially realizes an access structure if the amount of information gained by any qualified set is strictly greater than that of any unqualified one. In other words, the qualified sets have a positive *advantage* δ over the unqualified ones with regard to the secret recovery. In the semi-partial realization, we additionally require that the secret remain perfectly hidden from the unqualified sets. See

Introducing the notion of partial information ratio. The total information ratio of a secret sharing scheme is defined on its own, i.e., regardless of what access structure it realizes, if any. However, we measure the efficiency of a (semi-) partial scheme, which we refer to as the *partial information ratio*, with respect to the access structure that it realizes. We define the partial information ratio as a scaled version of the total information ratio where the scale factor is $1/\delta$, where δ is the advantage mentioned above. The intuition behind this choice stems from decomposition constructions [24, 45, 46, 48].

Equality of partial and total linear information ratios. We prove that the (semi-) partial information ratio of an access structure is the same as its total information ratio for the class of all linear secret sharing schemes. The proof is somewhat technical and is handled via two linear algebraic lemmas. Our first lemma (Lemma 4.2) promises the existence of some linear maps that work for *any* subspace over a given finite field. The lemma does not hold if the space is not defined over a field that is not finite and its correctness is not trivial on finite fields a priori. On the other hand, our second lemma (Lemma 4.3) trivially holds for non-finite fields, but it is also true for finite fields that are sufficiently large.

Relation between partial, quasi-total and total information ratios. We prove that if the partial and total information ratios are equal for some class of schemes, the same holds true for the quasi-total and total information ratios. It remains open if the reverse holds true as well. Our main result on the equality of quasi-total and total linear information ratios then follows by the previous observation.

Challenges in extending our result. It remains open if the coincidence of partial and total information ratios holds for any class which is larger than the linear one. If they turn out to match for some class, so do the total and quasi-total information ratios. However, the converse might not necessarily hold true.

As we will discuss in next subsection, we even expect that the partial and total information ratios not to match for the class of abelian schemes. However, the coincidence/separation between total and quasi-total abelian information ratios is unclear.

1.2 More on partial information ratio

We continue to explore the relation between partial and total information ratios for the class of abelian schemes and with respect to the Shannon lower bound (i.e., by merely using the so-called Shannon-type information inequalities). We find both of the following observations non-trivial and somewhat surprising.

Abelian class. Even though we prove that the partial and total security notions coincide with respect to the linear upper-bound and the Shannon lower-bound (to be discussed next), it remains open if the two notions coincide for general schemes. Recently, an upper-bound on the abelian information ratio of the access structure $\mathcal{F}+\mathcal{N}$—the union of access structures induced by Fano and non-Fano matroids [4, 37]— has been computed in [28] (max$\leqslant 7/6$ and average$\leqslant 41/36$). Moreover, it has been conjectured that a non-trivial lower-bound (i.e., strictly greater than one) exists. We show that the partial abelian information ratio of this access structure is one. Therefore, we would not be surprised if total and partial information ratios turn out to become separate for abelian schemes. We remark that even if this turns out to be the case, it neither indicates general separation between partial and total information ratios nor abelian separation between quasi-total and total information ratios. See Section 6 for further details.

Shannon lower bound. It is easy to show that the lower-bound achieved for statistical and quasi-total information ratios, by merely considering Shannon-type inequalities, is the same as that of total information ratio [30]. We prove that the same thing happens for partial information ratio. Our proof is non-trivial and non-intuitive, since it is not a priori clear that the lower bound region (i.e., the polymatroidal set) is a polytope for the partial security, let alone being equal to that of the total security which is a polytope. See Section 7 for details.

It remains open if our result can be strengthened, e.g., by allowing certain additional non-Shannon type information inequalities [49], e.g., along the lines of [6, 38]). A corollary of our result is that Csirmaz sub-linear lower bound [17] also applies to partial security, which is not clear at a first glance.

1.3 Additional results

Some additional results mainly about various security notions are provided. The proofs are less technical as they mainly follow from standard techniques and known results in the literature.

On homomorphic statistical secret sharing. It has been observed in [3] that statistical and total security notions coincide for linear schemes. We show that statistical security notion implies total security for a sub-class of *group-characterizable* [15] schemes in which the secret group is *normal* in the main group. The proof is fairly easy and follows by properties of finite groups. What makes our result interesting is the following. In a recent work [29], it has been proved that *group-homomorphic* secret sharing schemes (those which are homomorphic and whose secret and share spaces are groups) are equivalent to group-characterizable schemes with normal subgroups. Combining both results, we conclude that statistical and total information ratios coincide for the class of group-homomorphic secret sharing schemes (or even maybe a larger class), which is non-trivial. It remains open if such a coincidence holds for homomorphic schemes whose secret and shares are weaker algebraic structures (such as monoid, magma, semi-group, etc). See Section 8.2 for details.

Relation between statistical and quasi-total notions. We prove that statistical security implies quasi-total security, using well-known results form information theory such as Fano's inequality and a more recent result [40], in a straightforward way. It is easy to argue that the other direction does not necessarily hold true. Nevertheless, it remains challengingly open if the statistical and quasi-total information ratios coincide. See Section 8.3 for details.

On length-based and entropy-based information ratios. Two different flavors of information ratio can be found in the literature [11, 13, 36], based on the interpretation of the "size" of a random variable, which we refer to as the "entropy-based" and "length-based" definitions. In the former, the information ratio is defined as the ratio between the share entropy and the secret entropy, which is adopted by us throughout this paper. In the latter one, it is defined as the ratio between the share length and secret length (where the length of a random variable is defined as the logarithm of the size of its support). We prove that the two definitions coincide for quasi-total security, due to a well-known result by Chan-Yeung [14], about characterizability of entropy region by group-characterizable random variables. Therefore, group-characterizable secret sharing schemes, which have uniform share and secret distributions, are "complete" for quasi-total security. It remains open if this is true for any other security notion, and in particular, the total one. See Section 9 for details.

On decomposition techniques. A common approach for finding upper bounds on the information ratio of access structures is the so-called *decomposition techniques*. These techniques have mainly been used to find upper-bounds on the information ratio of *concrete* access structures on a small number of participants [11,19,24,25,27,33,35,45,47]. They build on Stinson's λ-decomposition [45] by decomposing a given access structure into suitable sub-access structures [48] or sub-access functions [24,46]. In particular, the decomposition theorems in [24, 46] assume that in the linear partial sub-schemes, every subset of participants fully recovers a certain subset of secret elements and nothing more; that is, recovering a non-trivial linear combination of the secret elements is not allowed. Using the notion of partial information ratio and our result on the equality of partial and total linear information ratios, we show this strong requirement can be removed. See Section 10 for details.

1.4 Paper organization

In Section 2, we present the required preliminaries and introduce our notation. In Section 3, the partial and semi-partial security notions are introduced. Section 4 is devoted to the proof of the equality of (semi-) partial and total information ratios for the class of linear schemes. In Section 5, we study the quasi-total security notion and its relation with partial security. Section 6 includes some results on abelian secret sharing. In Section 7, we prove that the Shannon lower on the partial and total information ratios coincide. In Section 8, we study the statistical security notion and study its connection with total and quasi-total security. Section 9 discusses completeness of group-characterizable schemes for quasi-total security and equivalence of entropy-based and length-based information ratios. We conclude the paper in Section 11 by mentioning some problems which remain open in this paper.

Diagram 1 suggests how to read different sections of the paper after having read the preliminaries (Section 2).

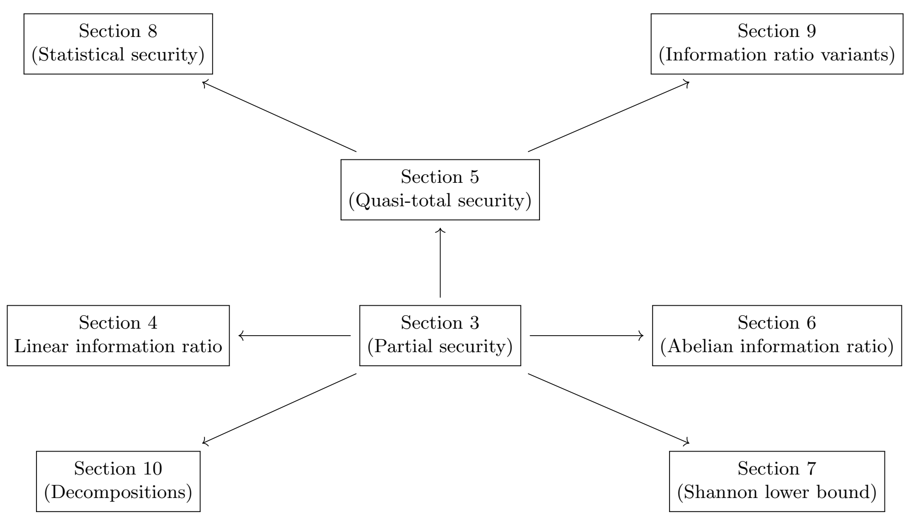

Fig. 1: Suggestion for reading the paper.

2 Secret sharing schemes

In this section, we provide the basic background along with some notations. We refer the reader to Beimel's survey [2] on secret sharing.

General notations. We denote the support of the random variable $\boldsymbol{X}$ by $\mathrm{supp}(\boldsymbol{X})$. All random variables are discrete in this paper. We assume that the reader is familiar with the Shannon entropy of a random variable $\boldsymbol{X}$, denoted by $\mathrm{H}(\boldsymbol{X})$, and the mutual information of random variables $\boldsymbol{X}, \boldsymbol{Y}$, denoted by $\mathrm{I}(\boldsymbol{X} : \boldsymbol{Y})$. For a positive integer m, we use $[m]$ to represent the set $\{1, \ldots, m\}$. Throughout the paper, $P = \{p_1, \ldots, p_n\}$ stands for a finite set of *participants*. A distinguished participant $p_0 \notin P$ is called *dealer* and we notate $Q = P \cup \{p_0\}$. Unless otherwise stated, we identify the participant p_i with its index i; i.e., $Q = \{0, 1, \ldots, n\}$. The set of positive integers and real numbers are respectively denoted by $\mathbb{N}$ and $\mathbb{R}$. All logarithms are to the base two. The closure of a topological set $\mathcal{X}$ is denoted by $\overline{\mathcal{X}}$, defined as the union of $\mathcal{X}$ with all its limit points.

Definition 2.1 (Access structure) *A non-empty subset $\Gamma \subseteq 2^P$, with $\varnothing \notin \Gamma$, is called an* access structure *on P if it is* monotone*; that is, $A \subseteq B \subseteq P$ and $A \in \Gamma$ imply that that $B \in \Gamma$.*

A subset $A \subseteq P$ is called *qualified* if $A \in \Gamma$; otherwise, it is called *unqualified*. A qualified subset is called *minimal* if none of its proper subsets is qualified.

Definition 2.2 (Access function [21]) *A mapping $\Phi : 2^P \to [0, 1]$ is called an* access function *if $\Phi(\varnothing) = 0$ and it is monotone; i.e., $A \subseteq B \subseteq P$ implies that $\Phi(A) \leqslant \Phi(B)$. An access function is called* rational *if $\Phi(A)$ is rational for every subset A and called* total *if $\Phi(A) \in \{0, 1\}$.*

Definition 2.3 (Secret sharing scheme) *A tuple $\Pi = \left(\boldsymbol{S}_i\right)_{i \in P \cup \{0\}}$ of jointly distributed random variables, with finite supports, is called a* secret sharing scheme *on participant set P when $\mathrm{H}(\boldsymbol{S}_0) > 0$. The random variable $\boldsymbol{S}_0$ is called the* secret *random variable and its support is called the* secret space*. The random variable $\boldsymbol{S}_i$, for any participant $i \in P$, is called the* share *random variable of the participant i and its support is called his* share space*.*

When we say that a secret $s \in \mathrm{supp}(\boldsymbol{S}_0)$ *is shared using Π*, we mean that a tuple $\left(s_i\right)_{i \in P \cup \{0\}}$ is sampled according to the distribution Π conditioned on the event $\{\boldsymbol{S}_0 = s\}$.

A secret sharing scheme Π is said to be *linear* if there exists a finite field $\mathbb{F}$ such that the support of every marginal random variable is an $\mathbb{F}$-vector space of finite dimension; additionally, we require that the joint distribution Π be uniform. When we want to emphasize the underlaid finite field, we call it an $\mathbb{F}$-*linear* scheme. When characteristic of $\mathbb{F}$ is p, a prime, we call it a p-linear scheme.

The most common definition of a linear scheme is based on linear maps. A secret sharing scheme $(\boldsymbol{S}_i)_{i \in Q}$ is said to be *linear* if there are finite dimensional vector spaces E and $(E_i)_{i\in Q}$, and linear maps $\mu_i : E \to E_i$, $i \in Q$ such that $\boldsymbol{S}_i = \mu_i(\boldsymbol{E})$, where $\boldsymbol{E}$ is the uniform distribution on E. In this paper, we use the following equivalent definition (see Appendix C or [28] for justification).

Definition 2.4 (Linear scheme) *A tuple $\Pi = (T; T_0, T_1, \ldots, T_n)$ is called an $\mathbb{F}$-linear (or simply a linear) secret sharing scheme if T is a finite dimensional $\mathbb{F}$-vector space, T_i is a subspace of T, for each $i \in [n]$, and $\dim T_0 \geqslant 1$. When there is no confusion, we omit T and simply write $\Pi = (T_i)_{i\in P\cup\{0\}}$. If the characteristic of $\mathbb{F}$ is p, we call the scheme p-linear.*

Definition 2.5 (Total realization) *We say that a secret sharing scheme $\Pi = \big(\boldsymbol{S}_i\big)_{i\in P\cup\{0\}}$ is a* (total) scheme *for Γ, or it* (totally) realizes *Γ, if the following two hold, where $\boldsymbol{S}_A = (\boldsymbol{S}_i)_{i\in A}$, for a subset $A \subseteq P$:*

(Correctness) $\mathrm{H}(\boldsymbol{S}_0|\boldsymbol{S}_A) = 0$ *for every qualified set $A \in \Gamma$ and,*
(Privacy) $\mathrm{I}(\boldsymbol{S}_0 : \boldsymbol{S}_B) = 0$ *for every unqualified set $B \in \Gamma^c$.*

Definition 2.6 (Access function/convec of a scheme) *The* access function *and the* (total) convec *of a secret sharing scheme $\Pi = \big(\boldsymbol{S}_i\big)_{i\in P\cup\{0\}}$ are respectively denoted by Φ_Π and $\mathrm{cv}(\Pi)$ and defined as follows:*

$$\Phi_\Pi(A) = \frac{\mathrm{I}(\boldsymbol{S}_0 : \boldsymbol{S}_A)}{\mathrm{H}(\boldsymbol{S}_0)} , \qquad \mathrm{cv}(\Pi) = \Big(\frac{\mathrm{H}(\boldsymbol{S}_i)}{\mathrm{H}(\boldsymbol{S}_0)}\Big)_{i\in P} .$$

For a linear scheme $\Pi = (T_i)_{i\in P\cup\{0\}}$, it is easy to verify that

$$\Phi_\Pi(A) = \frac{\dim(T_0 \cap T_A)}{\dim(T_0)} , \qquad \mathrm{cv}(\Pi) = \Big(\frac{\dim(T_i)}{\dim(T_0)}\Big)_{i\in P} .$$

Information ratio and convec set. Convec is short for contribution vector [27] and a norm on it can be used as a measure of efficiency of a secret sharing scheme. The convec set of an access structure can be defined with respect to a class of secret sharing schemes (e.g., linear, group-characterizable, abelian, etc).

Definition 2.7 (Total convec set) *The (total) convec set of an access structure Γ, denoted by $\Sigma_{\mathrm{t}}(\Gamma)$, is defined as the set of all convecs of all secret sharing schemes that (totally) realize Γ. When we restrict to the class* C *of secret sharing schemes, we use the notation $\Sigma_{\mathrm{t}}^{\mathrm{C}}(\Gamma)$.*

We use the notation $\Sigma_{\mathrm{t}}^{\mathrm{L}}$ (resp. Σ_{t}^{p}) when the convec set is restricted to the class of all linear (resp. p-linear) secret sharing schemes and call it the linear (resp. p-linear) convec set. The maximum and average information ratios of an access structure Γ on n participants, for the class C of secret sharing schemes, are respectively defined as:

$$\min\{\max(\boldsymbol{x}) : \boldsymbol{x} \in \overline{\Sigma_{\mathrm{t}}^{\mathrm{C}}(\Gamma)}\} \quad \text{and} \quad \tfrac{1}{n}\min\{\textstyle\sum_{i=1}^{n} x_i : (x_1, \ldots, x_n) \in \overline{\Sigma_{\mathrm{t}}^{\mathrm{C}}(\Gamma)}\} .$$

The polymatroidal set. Additionally, we introduce the K_{t}-set, called the total polymatroidal set, as a generalization of the κ-parameter [34]. The total polymatroidal set of an access structure Γ on n participants, denoted by $K_{\mathrm{t}}(\Gamma)$, is an n-dimensional polytope derived by taking into account all the Shannon inequalities as well as the correctness and privacy conditions. In Section 7, we present a precise definition; see Definition 7.5.

3 Partial and semi-partial secret sharing

In this section, we introduce two relaxed security notions for secret sharing schemes, referred to as *semi-partial* and *partial* realizations. A scheme is said to partially realize an access structure if the amount of information gained on the secret by every qualified set is strictly larger than that of any unqualified one. The semi-partial definition is less relaxed since it requires that the secret still remain information theoretically hidden from unqualified sets.

As it was mentioned in the introduction, our main motivation for introducing these security notions is to 1) prove that the quasi-total [30] and total convec sets coincide for linear schemes and 2) relax the requirements of the weighted-decompositions [24, 46].

3.1 Security definition

We begin by giving a formal definition of partial and semi-partial security notions.

Definition 3.1 (Partial and semi-partial realization) *We say that a secret sharing scheme Π is a* partial scheme *for Γ, or it* partially realizes *Γ, if:*

$$\delta = \min_{A \in \Gamma} \Phi_{\Pi}(A) - \max_{B \in \Gamma^c} \Phi_{\Pi}(B) > 0 \ . \tag{3.1}$$

We call it a semi-partial scheme*, if additionally $\Phi_{\Pi}(B) = 0$, for every unqualified set $B \in \Gamma^c$.*

The parameter δ is a *normalized* measure for quantifying the advantage of the qualified sets over the unqualified ones with respect to the amount of information that they gain on the secret. The intuition behind the choice of this factor and the following definition stems from decomposition constructions [24, 45, 46, 48], in which a similar scale factor appears. We will revisit decomposition methods in Section 10.

3.2 Partial convec

We measure the efficiency of a (semi-) partial scheme for an access structure via a scaled version of its usual (i.e., total) convec, that we call *partial convec.*

Definition 3.2 (Partial convec) *Let Π be a partial scheme for Γ. The* partial convec *of Π (with respect to Γ) is defined and denoted by*

$$\mathrm{pcv}(\Pi, \Gamma) = \frac{1}{\delta}\mathrm{cv}(\Pi),$$

where δ, the (normalized) advantage, is defined as in Equation (3.1). When there is no confusion, we simply use the notation $\mathrm{pcv}(\Pi)$.

The notions of partial and semi-partial realization give rise to two new convec sets.

Definition 3.3 (Partial and semi-partial convec sets) *The* partial convec set *of an access structure Γ, denoted by $\Sigma_{\mathrm{p}}(\Gamma)$, is defined as the set of all* partial convecs *of all secret sharing schemes that partially realize Γ. The* semi-partial convec set *is defined similarly and is denoted by $\Sigma_{\mathrm{sp}}(\Gamma)$. When we restrict to the class* C *of secret sharing schemes, we notate $\Sigma_{\mathrm{p}}^{\mathrm{C}}(\Gamma)$ and $\Sigma_{\mathrm{sp}}^{\mathrm{C}}(\Gamma)$.*

The K_{p}, $\Sigma_{\mathrm{p}}^{\mathrm{L}}$ and Σ_{p}^{p}-sets are defined similar to the case of total convec set. Similar notations are used for semi-partial security. The relation $\Sigma_{\mathrm{t}}^{\mathrm{C}}(\Gamma) \subseteq \Sigma_{\mathrm{sp}}^{\mathrm{C}}(\Gamma) \subseteq \Sigma_{\mathrm{p}}^{\mathrm{C}}(\Gamma)$ is immediate for any access structure Γ and any class C of secret sharing schemes. In Section 4 we prove that for the three security notions, the linear convec sets are the same (i.e., $\Sigma_{\mathrm{t}}^{\mathrm{L}}(\Gamma) = \Sigma_{\mathrm{sp}}^{\mathrm{L}}(\Gamma) = \Sigma_{\mathrm{p}}^{\mathrm{L}}(\Gamma)$). Also, in Section 7, we prove that the Shannon inequalities give the same lower-bound for the convec set (i.e., $K_{\mathrm{t}}(\Gamma) = K_{\mathrm{sp}}(\Gamma) = K_{\mathrm{p}}(\Gamma)$). In Section 6, we provide some evidence that for the class C of abelian schemes the inclusion $\Sigma_{\mathrm{t}}^{\mathrm{C}}(\Gamma) \subseteq \Sigma_{\mathrm{sp}}^{\mathrm{C}}(\Gamma)$ might be proper. The following proposition then follows.

Proposition 3.4 (Convec set relations) *For any access structure Γ, we have*

$$\Sigma_{\mathrm{t}}^{\mathrm{L}}(\Gamma) \subseteq \Sigma_{\mathrm{t}}(\Gamma) \subseteq \Sigma_{\mathrm{sp}}(\Gamma) \subseteq \Sigma_{\mathrm{p}}(\Gamma) \subseteq K_{\mathrm{t}}(\Gamma)\ .$$

Separation. Separation result between closures of $\Sigma_{\mathrm{t}}^{\mathrm{L}}$ and Σ_{t} has been proved in a recent work [28][2]. Separation between closure of Σ_{t} and K_{t} is also known [5] (based on an old result by Seymour [41]). It is easy to find examples that separate between Σ_{p} and K_{t}. Proving or disproving separations between Σ_{t} and Σ_{sp} and also between Σ_{sp} and Σ_{p} remains open.

Convexity. It is easy to show that the total convec set of any access structure is a set with *convex closure*. It remains open if this is also the case for the partial and semi-partial security notions.

[2] In this paper, we only focus on amortized definition of information ratio, i.e. the secret can be arbitrarily long. Refer to [1] for the role of amortization in secret sharing. In fact our definition of a linear scheme allows arbitrary secret dimension, whic is usually called *multi-linear* in the literature. In another variant, which we call *scaler-linear*, the secret is allowed to contain only one field element. Separation between scaler-linear and non-linear secret sharing was first proved by Beimel and Ishai in [3] under some plausible assumption. Later, such a separation was proved in [7] without relying on any assumption.

4 Equality of total and partial linear convec sets

In this section, we prove that the linear convec set is the same for the total, partial and semi-partial security notions. Two linear algebraic lemmas lie at the core of our proofs. The first one is used in Proposition 4.4 for transforming a semi-partial linear secret sharing scheme for a given access structure into a total one without changing its (partial) convec. But we also need the second lemma in Proposition 4.5 for proving a similar claim for partial schemes. The following theorem is then a direct corollary of both propositions.

Theorem 4.1 (Equality of partial and total linear convec sets) *Let p be a prime and Γ be an access structure. Then, $\Sigma_{\mathrm{p}}^{p}(\Gamma) = \Sigma_{\mathrm{sp}}^{p}(\Gamma) = \Sigma_{\mathrm{t}}^{p}(\Gamma)$, and in particular,*

$$\Sigma_{\mathrm{p}}^{\mathrm{L}}(\Gamma) = \Sigma_{\mathrm{sp}}^{\mathrm{L}}(\Gamma) = \Sigma_{\mathrm{t}}^{\mathrm{L}}(\Gamma) \ .$$

It remains open if the claim of Theorem 4.1 holds for other classes of schemes. In Section 6, we show that they probably become separate for the class of abelian schemes. However, their separation/coincidence for general secret sharing remains unclear.

4.1 Two linear algebraic lemmas

Our first lemma promises the existence of some linear maps that work for *any* subspace over a given finite field. The lemma does not hold if the space is not defined over a field that is not finite. So the claim is truly a property of finite fields.

Lemma 4.2 (Linear transformation lemma) *Let $1 \leqslant \lambda \leqslant m$ be integers. Let T_0 be a vector space over some finite field with dimension m. Then, there exist m linear maps $L_1, \ldots, L_m : T_0 \to T_0^{\lambda}$ such that for any subspace $E \subseteq T_0$ of dimension $\dim E \geqslant \lambda$, the following holds*

$$\sum_{i=1}^{m} L_i(E) = T_0^{\lambda} \ .$$

Proof. Without loss of generality we can assume that $T_0 = \mathbb{F}^m$, where $\mathbb{F}$ is the underlying finite field. We show that there exist m linear maps $L_1, \ldots, L_m :$ $\mathbb{F}^m \to \mathbb{F}^{m\lambda}$, such that for any λ linearly independent vectors $x_1, \ldots, x_\lambda \in \mathbb{F}^m$, the $m\lambda$ vectors $L_i(x_j) \in \mathbb{F}^{m\lambda}$, $i \in [m]$ and $j \in [\lambda]$, are linearly independent. The construction is explicit and is as follows.

Let $|\mathbb{F}| = q$ and identify $\mathbb{F}^m$ with a finite field $\mathbb{K}$ with q^m elements that is an extension of $\mathbb{F}$ with degree m. Choose a basis $w_1, ..., w_m$ for $\mathbb{K}$ over $\mathbb{F}$ and identify $\mathbb{F}^{m\lambda}$ with $\mathbb{K}^{\lambda}$.

Define L_i by sending $x \in \mathbb{K}$ to $(w_i x, w_i x^q, ..., w_i x^{q^{\lambda-1}}) \in \mathbb{K}^{\lambda}$. Note that the mappings $x \longmapsto x^q$ is an $\mathbb{F}$-linear map from $\mathbb{K}$ to $\mathbb{K}$ and $x \longmapsto x^{q^i}$ is the composition of this map with itself i times. Therefore, the mapping L_i is $\mathbb{F}$-linear too, for every $i \in [m]$. If there exist coefficients c_{ij}, $i \in [m]$ and $j \in [\lambda]$, such that $\Sigma_{j=1}^{\lambda}\Sigma_{i=1}^{m}c_{ij}L_i(x_j) = 0$, then $\sum_{j=1}^{\lambda}(\sum_{i=1}^{m} c_{ij}w_i)x_j^{q^{k-1}} = 0$ for every $k \in [\lambda]$. Since the $\lambda \times \lambda$ matrix $M = \left(x_i^{q^{k-1}}\right)_{i\in[\lambda],k\in[\lambda]}$ is invertible (to be proved at the end), we have $\sum_{i=1}^{m} c_{ij}w_i = 0$ for all $j \in [\lambda]$ and thus $c_{ij} = 0$, for every $i \in [m]$ and $j \in [\lambda]$, as the vectors $w_1, ..., w_m$ are linearly independent over $\mathbb{F}$. Therefore, the vectors $L_i(x_j)$, $i \in [m]$ and $j \in [\lambda]$, are linearly independent over $\mathbb{F}$.

We complete the proof by showing that the matrix M is invertible. Assume for a row vector $y = (y_1, \ldots, y_\lambda)$, we have $yM = 0$, hence $y_1x + y_2x^q + \ldots + y_\lambda x^{q^{\lambda-1}} = 0$ for every $x = x_1, \ldots, x_\lambda$. Since this polynomial is linear over the field $\mathbb{F}$, it vanishes on the span of these independent vectors over $\mathbb{F}$, a space with q^λ elements. However, as the polynomial is of degree $q^{\lambda-1}$, it is identically zero; i.e., $y = 0$. This shows that M is invertible. □

The following lemma is true for finite fields that are sufficiently large. In Appendix A we present an interesting probabilistic proof, proposed by one the Eurocrypt reviewers, but with a slightly stronger requirement on the field size.

Lemma 4.3 (Non-intersecting subspace lemma) *Let T_0 be a vector space of dimension m over a finite field with q elements and let $E_1, \ldots, E_N$ be subspaces of T_0 of dimension at most ω, $1 \leqslant \omega < m$. If $N < \frac{q^m-1}{q^{m-1}-1}$, then there is a subspace $S \subset T_0$ of dimension $m - \omega$ such that $S \cap E_i = 0$, for every $i \in [N]$.*

Proof. Without loss of generality we can assume that $\dim E_i = \omega$. Let $\mathbb{F}$ be the underlying finite field with q elements. We show that if $N < \frac{q^m-1}{q^{m-1}-1}$, then the required subspace S of dimension $m - w$ with zero intersection with E_i's exists. We prove this by induction on $m - w$. If $m - w = 1$, then each E_i has $q^{m-1} - 1$ non-zero elements so we have at most $N(q^{m-1} - 1)$ non-zero elements in their union. If $N < \frac{q^m-1}{q^{m-1}-1}$ then there is a non-zero element outside this union that generates the required subspace S. If E_i's are of dimension w, then since $N < \frac{q^m-1}{q^w-1}$ the above proof shows that there is a non-zero vector u outside their union. If we add this vector to each E_i we get subspace E_i' of dimension $w + 1$. Therefore, by induction, we have a subspace S' of dimension $m - w - 1$ that has zero intersection with each E_i'. Now the space generated by S and u is the required subspace of dimension $m - w$ and zero intersection with each E_i. □

4.2 A convec-preserving total linear scheme from a semi-partial linear one

The following proposition will be generalized in next section. However, we present it separately in this section since we will build on its proof in the course of the proof of Proposition 4.5.

Proposition 4.4 ($\boldsymbol{\Sigma_{\mathrm{sp}}^{p}} = \boldsymbol{\Sigma_{\mathrm{t}}^{p}}$) *Let Γ be an access structure and Π' be a semi-partial $\mathbb{F}$-linear secret sharing scheme for it. Then, there exists a total $\mathbb{F}$-linear secret sharing scheme Π for Γ such that $\mathrm{cv}(\Pi) = \mathrm{pcv}(\Pi')$.*

Proof. We first provide an informal proof by using duals of the linear maps introduced in Lemma 4.2. Identify the secret space of Π' by $\mathbb{F}^m$. Since Π' is a semi-partial scheme for Γ, there exists an integer λ, with $1 \leqslant \lambda \leqslant m$, such that every qualified participant set discovers at least λ independent linear relations on the secret. With a slight abuse of notation, let $L_1^\star, \ldots, L_m^\star : \mathbb{F}^{m\lambda} \to \mathbb{F}^m$ be the dual (transpose) of the linear maps of Lemma 4.2. We construct a total linear scheme Π for Γ with secret space $\mathbb{F}^{m\lambda}$ such that its convec is the same as the partial convec of Π'. To share a secret $s \in \mathbb{F}^{m\lambda}$, we share each of the m secrets $L_1^\star(s), \ldots, L_m^\star(s) \in \mathbb{F}^m$ using an independent instance of Π'. Each participant in Π receives a share from each instance of Π'. Hence, while the secret length has been multiplied by λ, the share of each participant has increased by a factor of at most m. By adding dummy shares, one can achieve an exact factor of m. Therefore, the total convec of Π and semi-partial convec of Π' are equal. Note that since the m different instances of Π' use independent randomnesses, any qualified set gains no information on the secret. By Lemma 4.2, each qualified set gets $m\lambda$ independent linear relations on s. We conclude that the scheme Π is total.

We now prove the lemma more formally by direct use of linear maps of Lemma 4.2. Let $\Pi' = (T'; T'_0, T'_1, \ldots, T'_n)$ be the $\mathbb{F}$-linear semi-partial scheme that satisfies $\lambda = \min_{A\in\Gamma}\{\dim(T'_A \cap T'_0)\} \geqslant 1$ and $\dim(T'_A \cap T'_0) = 0$ for all $A \in \Gamma^c$. Let $m = \dim(T_0) \geqslant 1$.

Our goal is to build a total $\mathbb{F}$-linear scheme $\Pi = (T; T_0, T_1, \ldots, T_n)$ such that $\dim(T_i) \leqslant m \dim(T'_i)$ for every $i \in [n]$ and $\dim(T_0) = m\lambda$.

Find an orthogonal complement R' for T'_0 inside T'; hence, $T' = T'_0 \oplus R'$. Let $T = {T'}_0^{\lambda} \oplus {R'}^m$.

Let $L_1, \ldots, L_m : T'_0 \to {T'}_0^{\lambda}$ be the linear maps of Lemma 4.2 and define $\phi : {T'}^m \to T$ by

$$\phi(s_1, \ldots, s_m, r_1, \ldots, r_m) = \Big(\sum_{i=1}^{m} L_i(s_i), r_1, \ldots, r_m \Big) ,$$

where $s_1, \ldots, s_m \in T'_0$ and $r_1, \ldots, r_m \in R'$.

We let $T_0 = {T'}_0^{\lambda}$ and $T_i = \phi({T'}_i^m)$. Then, the conditions on dimensions are clear and consequently $\mathrm{cv}(\Pi) \preceq \mathrm{pcv}(\Pi')$. It is straightforward to tweak the scheme such that the claimed vector equality holds. It remains to prove that Π totally realizes Γ.

For $A \subseteq [n]$, by linearity of ϕ, we have $T_A = \phi({T'}_A^m)$. Also, we have:

$$\begin{aligned} T_A \cap T_0 &= \phi({T'}_A^m) \cap {T'}_0^{\lambda} \\ &= \phi({T'}_A^m \cap {T'}_0^m) \\ &= \phi\big((T'_A \cap T'_0)^m\big) \\ &= \textstyle\sum_{i=1}^m L_i(T'_A \cap T'_0) , \end{aligned}$$

where the second equality follows from the following fact: $\phi(x) \in {T'}_0^{\lambda}$ if and only if $x \in {T'}_0^m$.

If $A \in \Gamma$, then $\dim(T'_A \cap T'_0) \geqslant \lambda$. Therefore, by Lemma 4.2, we have $T_A \cap T_0 = T_0$. Also, if $B \in \Gamma^c$, then $T'_B \cap T'_0 = 0$ and hence $T_B \cap T_0 = 0$. This shows that Π is a total scheme for Γ. □

4.3 A convec-preserving total linear scheme from a partial linear one

The following proposition is a generalization of Proposition 4.4. The proof expands on the proof of Proposition 4.4 by appropriately using Lemma 4.2.

Proposition 4.5 ($\boldsymbol{\Sigma}^p_{\mathrm{p}} = \boldsymbol{\Sigma}^p_{\mathrm{t}}$) *Let Γ be an access structure and Π' be a partial $\mathbb{F}$-linear secret sharing scheme for it. Then, there exists a finite extension $\mathbb{K}$ of $\mathbb{F}$ and a total $\mathbb{K}$-linear secret sharing scheme Π for Γ such that* $\mathrm{cv}(\Pi) = \mathrm{pcv}(\Pi')$. *Consequently, $\Sigma^p_{\mathrm{p}}(\Gamma) = \Sigma^p_{\mathrm{t}}(\Gamma)$, for every prime p.*

Proof. Let $\Pi' = (T'_0, \ldots, T'_n)$ and denote

$$\begin{aligned} \lambda &= \min\nolimits_{A \in \Gamma}\{\dim(T'_A \cap T'_0)\} \\ \omega &= \max\nolimits_{A \in \Gamma^c}\{\dim(T'_A \cap T'_0)\} \\ m &= \dim T'_0 \end{aligned}$$

where $1 \leqslant \lambda - \omega \leqslant m$.

Let N be the number of maximal unqualified subsets in Γ^c and $\mathbb{K}$ be an extension of $\mathbb{F}$ that satisfies $|\mathbb{K}| \geqslant N$. By the process of extending scalers, we can turn Π' into a $\mathbb{K}$-linear scheme with the same convec, access function and dimensions. For simplicity, we use the same notation for the new scheme; i.e., from now on Π' is considered to be a $\mathbb{K}$-linear scheme. In particular, the relations for λ, ω, m are still valid.

Construct $(T_0, \ldots, T_n)$ from Π' the same way as in the proof of Proposition 4.4 and recall that $\dim T_0 = m\lambda$ and $\dim T_i \leqslant m \dim T'_i$. The same argument, which was used in the proof of Proposition 4.4, shows that for any $A \in \Gamma$, we have $T_A \cap T_0 = T_0$. It is also trivial that for every $B \in \Gamma$, we have $\dim\left(T_B \cap T_0\right) \leqslant m\omega$.

By Lemma 4.3 (E_i is $T_B \cap T_0$ for some maximal unqualified set B, $\dim E_i \leqslant m\omega$ and $\dim T_0 = m\lambda$), one can choose $S \subseteq T_0$ of dimension $m(\lambda - \omega)$ such that $T_B \cap S = 0$, for every $B \in \Gamma^c$. Also, it is trivial that $T_A \cap S = S$, for every $A \in \Gamma$. Now, it is clear that $\Pi = (S, T_1, \ldots, T_n)$ is a total secret sharing scheme for Γ such that $\dim S = m(\lambda - \omega)$. Therefore, $\mathrm{cv}(\Pi) \preceq \mathrm{pcv}(\Pi')$. Again, it is straightforward to tweak the scheme such that the convec equality holds. □

5 On quasi-total security

In this section, we review the notion of quasi-total security, proposed in [30, 31]. What makes quasi-total security especial is that group-characterizable schemes are "complete" for it and, consequently, "length-based" and "entropy-based" definitions of information ratio coincide. This issue will be studied in Section 9.

In this section, we prove that if partial and total information ratios coincide for any class of secret sharing schemes, the same thing happens for the total and quasi-total information ratios. As a corollary of Theorem 4.1, the partial, quasi-total and total information ratios are all equal for the class of linear schemes.

5.1 Definition

We need the following definition before giving a formal definition of the quasi-total secret sharing and quasi-total convec set.

Definition 5.1 (Convec-converging family of schemes) *A sequence $\mathcal{F} = \{\Pi_k\}_{k\in\mathbb{N}}$ of secret sharing schemes on participants set P is called a* convec-converging family of schemes *if i) the entropy of secret does not vanish; i.e., $\mathrm{H}(\boldsymbol{S}_0^k) = \Omega(1)$ and, ii) the sequence $\{\mathrm{cv}(\Pi_k)\}_{k\in\mathbb{N}}$ is converging. The convec of the convec-converging family $\mathcal{F}$ is defined as $\mathrm{cv}(\mathcal{F}) = \lim_{k\to\infty} \mathrm{cv}(\Pi_k)$.*

Definition 5.2 (Quasi-total realization [30]) *Let Γ be an access structure on P and $\mathcal{F} = \{\Pi_k\}_{k\in\mathbb{N}}$ be a convec-converging family of secret sharing schemes. We say that $\mathcal{F}$ is a quasi-total family for Γ if $\lim_{k\to\infty} \Phi_{\Pi_k} = \Phi_\Gamma$, where $\Phi_\Gamma : 2^P \to \{0,1\}$ is a (monotone) mapping defined as $\Phi_\Gamma(A) = 1 \iff A \in \Gamma$.*

Definition 5.3 (Quasi-total convec set) *The quasi-total convec set of an access structure Γ, denoted by $\Sigma_{\mathrm{qt}}(\Gamma)$, is defined as the set of all convecs of all quasi-total families for Γ. When we restrict ourselves to the class C of secret sharing schemes, we use the notation $\Sigma_{\mathrm{qt}}^{\mathrm{C}}$.*

Notice that the quasi-total convec sets are closed. It is easy to prove that the Σ_{qt}-set (similar to the $\overline{\Sigma_{\mathrm{t}}}$-set) is convex, but recall that the closure convexity of the (semi-) partial convec set was left open.

5.2 Connections with partial and total security notions

We prove that if the partial and total convec sets are equal for some class of schemes, the same holds true for the quasi-total and total convec sets. It remains open if the reverse holds true as well.

Proposition 5.4 ($\overline{\boldsymbol{\Sigma}_{\mathbf{p}}^{\mathbf{C}}} = \overline{\boldsymbol{\Sigma}_{\mathbf{t}}^{\mathbf{C}}} \implies \boldsymbol{\Sigma}_{\mathbf{qt}}^{\mathbf{C}} = \overline{\boldsymbol{\Sigma}_{\mathbf{t}}^{\mathbf{C}}}$) *For any class C of schemes and any access structure Γ, if $\overline{\Sigma_{\mathrm{p}}^{\mathrm{C}}} = \overline{\Sigma_{\mathrm{t}}^{\mathrm{C}}}$ then $\Sigma_{\mathrm{qt}}^{\mathrm{C}} = \overline{\Sigma_{\mathrm{t}}^{\mathrm{C}}}$.*

Proof. It suffices to prove the inclusion $\Sigma_{\mathrm{qt}}^{\mathrm{C}}(\Gamma) \subseteq \overline{\Sigma_{\mathrm{t}}^{\mathrm{C}}(\Gamma)}$. Equivalently, we show that for every $\boldsymbol{\sigma} \in \Sigma_{\mathrm{qt}}^{\mathrm{C}}(\Gamma)$ we have $\boldsymbol{\sigma} \in \overline{\Sigma_{\mathrm{t}}^{\mathrm{C}}(\Gamma)}$. Let $\mathcal{F} = \{\Pi_k\}_{k\in\mathbb{N}}$ be a quasi-total family of class-C schemes for Γ with $\mathrm{cv}(\mathcal{F}) = \boldsymbol{\sigma}$. We construct a convec-converging family $\mathcal{F}' = \{\Pi'_k\}_{k\in\mathbb{N}}$ of class-C schemes such that: i) Π'_k is a total scheme for Γ for sufficiently large k and ii) $\mathrm{cv}(\mathcal{F}') = \boldsymbol{\sigma}$. This proves that $\boldsymbol{\sigma} \in \overline{\Sigma_{\mathrm{t}}^{\mathrm{C}}(\Gamma)}$.

Define $\lambda_k = \min_{A \in \Gamma}\{\Phi_{\Pi_k}(A)\}$ and $\omega_k = \max_{B \notin \Gamma}\{\Phi_{\Pi_k}(B)\}$. Since λ_k and ω_k respectively converge to 1 and 0, we have $\delta_k = \lambda_k - \omega_k > 0$ for sufficiently large k. This shows that Π_k is a partial class-C secret sharing scheme for Γ with partial convec $\mathrm{cv}(\Pi_k)/\delta_k$. By assumption, there exists a convec-converging family of class-C total schemes $\{\Pi'_{kj}\}_{j \in \mathbb{N}}$ for Γ with $\lim_{j\to\infty} \mathrm{cv}(\Pi'_{kj}) = \mathrm{cv}(\Pi_k)/\delta_k$. Let $\Pi'_k = \Pi'_{kk}$. Clearly, $\mathcal{F}' = \{\Pi'_k\}_{k \in \mathbb{N}}$ is a family of class-C total schemes for Γ with $\mathrm{cv}(\mathcal{F}') = \mathrm{cv}(\mathcal{F})$ since $\delta_k \to 1$, proving (i) and (ii). □

5.3 Main result

The main result of our paper, i.e., the equality of total and quasi-total information ratios for the class of linear schemes, is a corollary of Proposition 4.5 and Proposition 5.4.

Theorem 5.5 ($\boldsymbol{\Sigma^{\mathrm{L}}_{\mathrm{qt}} = \overline{\Sigma^{\mathrm{L}}_{\mathrm{t}}}}$) *For any access structure Γ and any prime p, we have $\Sigma^{p}_{\mathrm{qt}}(\Gamma) = \overline{\Sigma^{p}_{\mathrm{t}}(\Gamma)}$ and, consequently, $\Sigma^{\mathrm{L}}_{\mathrm{qt}}(\Gamma) = \overline{\Sigma^{\mathrm{L}}_{\mathrm{t}}(\Gamma)}$.*

It remains open if the claim of Theorem 5.5 holds for a class substantially larger than linear schemes. Even if it turns out that the partial and total convec sets do not coincide on some class larger than linear ones (e.g., the abelian ones which we guess to be the case and will discuss in Section 6), it does not provide sufficient evidence that this is also the case for total and quasi-total security notions. Therefore, we believe that proving coincidence/separation for larger classes demands innovative ideas and more advanced techniques.

6 On abelian information ratio

Equality of the linear information ratio for total and partial information ratios was proved in Section 5 and equality of Shannon lower bound will be proved in Section 7. Despite these lower bound and upper bound coincidences, in this section, we provide some evidence that the abelian information ratios probably do not match.

W study $\mathcal{F} + \mathcal{N}$, a well-known 12-participant access structure [4, 37] which has both Fano ($\mathcal{F}$) and non-Fano ($\mathcal{N}$) access structures as minors. The access structure $\mathcal{F}$ (resp. $\mathcal{N}$) is the port of Fano (resp. non-Fano) matroid and it is known [37] to be ideal only on finite fields with even (resp. odd) characteristic. As a result, their union (i.e., $\mathcal{F} + \mathcal{N}$) is *nearly ideal*. That is, its information ratio is one without admitting an ideal scheme. Very recently, in [28], the exact value of its linear information ratio has been determined (max= 4/3 and average= 41/36). Also, an upper-bound on its abelian information ratio has been provided (max⩽ 7/6 and average⩽ 41/36). Additionally, it has been conjectured in [28], that the exact value of its (total) abelian information ratio is strictly greater than one. Below, we show that the semi-partial abelian ramification ratio of this access structure is one.

Abelian schemes. An abelian scheme on a set P of participants is a collection $(G_i)_{i\in Q}$ of subgroups of a finite group G. An abelian scheme $\Pi = (G_i)_{i\in Q}$ realizes an access structure if 1) for every qualified set $A \subseteq P$ we have $G_0 \cap G_A = G_0$ and 2) for every unqualified set $A \subseteq P$ we have $G_0 \cap G_A = \{0\}$, where $G_A = \sum_{i\in A} G_i$.

The convec and access function of an abelian scheme $\Pi = (G_i)_{i\in Q}$ are computed as follows:

$$\Phi_\Pi(A) = \frac{\log |G_0 \cap G_A|}{\log |G_0|}\ , \qquad \mathrm{cv}(\Pi) = \Big(\frac{\log |G_i|}{\log |G_0|}\Big)_{i\in P}\ .$$

Every linear scheme is abelian. If $\Pi = (G_i)_{i\in Q}$ and $\Pi' = (G'_i)_{i\in Q}$ are abelian schemes for an access structure Γ, so is their direct sum $\Pi \oplus \Pi' = (G_i \oplus G'_i)_{i\in Q}$. In particular, if Π and Π' are linear schemes for Γ, then $\Pi \oplus \Pi'$ is an abelian scheme for Γ. The following corollary then becomes trivial. We refer to Appendix C or [28] for further discussion on abelian schemes.

Corollary 6.1 *For every even (resp. odd) number m, there exists an ideal abelian scheme for Fano (resp. non-Fano) access structure such that the order of all subgroups are m.*

A nearly ideal semi-partial abelian scheme for $\mathcal{F} + \mathcal{N}$. Let $k \in \mathbb{N}$ be an integer. Let $\Pi_k^{\mathcal{F}}$ (resp. $\Pi_k^{\mathcal{N}}$) be an ideal abelian scheme for $\mathcal{F}$ (resp. $\mathcal{N}$) whose subgroups all have order 2^k (resp. $2^k + 1$). We construct a nearly ideal semi-partial family of schemes $\{\Pi_k\}$ for $\mathcal{F} + \mathcal{N}$. Instead of describing the scheme Π_k using formal notation, we describe it informally. The secret space of Π_k is the direct sum of the secret spaces of $\Pi_k^{\mathcal{F}}$ and $\Pi_k^{\mathcal{N}}$, i.e., $G_0^{\mathcal{F}} \oplus G_0^{\mathcal{N}}$. To share a secret $(s^{\mathcal{F}}, s^{\mathcal{N}}) \in G_0^{\mathcal{F}} \oplus G_0^{\mathcal{N}}$, we share $s^{\mathcal{F}}$ via $\Pi_k^{\mathcal{F}}$ and share $s^{\mathcal{N}}$ via $\Pi_k^{\mathcal{N}}$, using independent randomnesses. It is easy see that Π_k is a semi-partial abelian scheme for $\mathcal{F} + \mathcal{N}$ and its information ratio converges to one as k goes to infinity.

Summary. Table 1 summarizes the known results on the $\mathcal{F} + \mathcal{N}$ access structure. We believe that, for the class of abelian schemes, computing the total (and consequently statistical by Theorem 8.7) information ratio of $\mathcal{F} + \mathcal{N}$ is reachable within known techniques (e.g., by manually using the *common information* method of [22] in a clever way), but as we discussed in Section 5.3, computing its quasi-total abelian information ratio probably demands substantially more advanced ideas and techniques.

7 Shannon lower-bound for partial information ratio

The main result of this section is to prove that the Shannon inequalities give the same lower-bound for the total and partial security notions. In other words, the polymatroidal sets of an access structure with respect to all security definitions are equal. It remains open if our result can be strengthened, e.g., by allowing certain additional non-Shannon type information inequalities, e.g., along the lines of [6, 38]). Our result shows that Csirmaz sub-linear lower bound [17] also applies to partial security.

<table>
<tr><th></th><th></th><th>total/statistical</th><th>quasi-total</th><th>(semi-)partial</th><th>reference</th></tr>
<tr><td rowspan="2">general</td><td>max</td><td rowspan="2" colspan="3">1</td><td rowspan="2">[4, 37]</td></tr>
<tr><td>average</td></tr>
<tr><td rowspan="2">abelian</td><td>max</td><td>$1 \leqslant \cdot \leqslant 7/6$</td><td>$1 \leqslant \cdot \leqslant 7/6$</td><td rowspan="2">1</td><td rowspan="2">Theorem 8.7
[28]</td></tr>
<tr><td>average</td><td>$1 \leqslant \cdot \leqslant 41/36$</td><td>$1 \leqslant \cdot \leqslant 41/36$</td></tr>
<tr><td rowspan="2">linear</td><td>max</td><td colspan="3">4/3</td><td rowspan="2">Theorems 4.1, 5.5
[28], [3]</td></tr>
<tr><td>average</td><td colspan="3">41/36</td></tr>
</table>

Table 1: Known results on the max/average information ratio of the access structure $\mathcal{F}+\mathcal{N}$ w.r.t. different security notions and different classes of schemes.

We define the polymatroidal sets precisely and then prove our claim. We use the following definition for a polymatroid, first introduced by Edmonds [20] in 1970. The relation between polymatroids and random variables was realized by Fujishige [23] in 1978. We refer the reader to Padro's lecture notes [39] for a leaner introduction to matroids, polymatroids and their connection to secret sharing.

Definition 7.1 (Polymatroid) *Let Q be a finite set. We say that $\mathcal{S} = (Q, r)$ is a polymatroid with ground set Q and rank function $r : 2^Q \to \mathbb{R}$, when:*

a) $r(\varnothing) = 0$,
b) $r(X) \leqslant r(Y)$, *for every subsets* $X \subseteq Y \subseteq Q$ *(monotonicity),*
c) $r(X) + r(Y) \geqslant r(X \cup Y) + r(X \cap Y)$, *for every subsets* $X, Y \subseteq Q$ *(submodularity).*

We simply denote the rank function of a singleton set $\{p\}$ by $r(p)$. We let $Q = P \cup \{p_0\}$ where $P = \{p_1, \cdots, p_n\}$ and assume that $r(p_0) > 0$. We borrow the following notation from [21].

Notation 7.2 *Let $\mathcal{S} = (Q, r)$ be a polymatroid and A and B be subsets of Q. We notate*

$$r(A|B) = r(AB) - r(B),$$

$$\varDelta_r(A : B) = r(A) + r(B) - r(AB).$$

7.1 Total polymatroidal set

Informally, the total polymatroidal set of an access structure $\varGamma$ on n participants, denoted by $K_{\mathrm{t}}(\varGamma)$, is the n-dimensional polytope derived by taking into account all the Shannon inequalities as well as the correctness and privacy conditions.

Definition 7.3 (Total polymatroid) *Let $\varGamma$ be an access structure on P and $\mathcal{S} = (Q, r)$ be a polymatroid. We say that $\mathcal{S}$ is a* total polymatroid *for $\varGamma$ when:*

a) $\varDelta_r(\{p_0\} : A) = r(p_0)$, *for every qualified set* $A \in \varGamma$ *and,*
b) $\varDelta_r(\{p_0\} : B) = 0$, *for every unqualified set* $B \in \varGamma^c$.

Definition 7.4 (Total convec of a polymatroid) *The* total convec *of a polymatroid* $\mathcal{S} = (Q, r)$ *is defined and denoted by* $\mathrm{cv}(\mathcal{S}) = \frac{1}{r(p_0)}(r(p))_{p \in P}$.

Definition 7.5 (Total polymatroidal set) *The* K_{t}*-set or* total polymatroidal set *of an access structure* Γ*, denoted by* $K_{\mathrm{t}}(\Gamma)$*, is defined as the set of all total convecs of all polymatroids for* Γ.

The following proposition is an extention of the inequality $\kappa(\Gamma) \leqslant \sigma(\Gamma)$ [34].

Proposition 7.6 ($\boldsymbol{\Sigma}_{\mathbf{t}}(\boldsymbol{\Gamma}) \subseteq \boldsymbol{K}_{\mathbf{t}}(\boldsymbol{\Gamma})$) *For any access structure* Γ*, it holds that* $\Sigma_{\mathrm{t}}(\Gamma) \subseteq K_{\mathrm{t}}(\Gamma)$.

7.2 Partial and semi-partial polymatroidal sets

Definition 7.7 (Partial and semi-partial polymatroid) *Let* Γ *be an access structure on* P *and* $\mathcal{S} = (Q, r)$ *be a polymatroid. We say that* $\mathcal{S}$ *is a* partial polymatroid *for* Γ *when:*

$$\delta = \min_{A \in \Gamma} \Delta_r(\{p_0\} : A) - \max_{B \in \Gamma^c} \Delta_r(\{p_0\} : B) > 0 \ . \tag{7.1}$$

If for every unqualified set $B \in \Gamma^c$ *it additionally holds that* $\Delta_r(\{p_0\} : B) = 0$*, we call it a semi-partial polymatroid for* Γ.

Definition 7.8 (Partial and semi-partial convec of a polymatroid) *Let* Γ *be an access structure on* P *and* $\mathcal{S} = (Q, r)$ *be a partial polymatroid for* Γ*. The* partial convec *of* $\mathcal{S}$ *(with respect to* Γ*) is defined and denoted by*

$$\mathrm{pcv}(\mathcal{S}, \Gamma) = \frac{1}{\delta}(r(p))_{p \in P} \ .$$

where δ*, the advantage, is defined as in Equation (7.1). When there is no confusion, we simply use the notation* $\mathrm{pcv}(\mathcal{S})$.

Definition 7.9 (Partial and semi-partial polymatroidal convec sets) *The* partial polymatroidal convec set *of an access structure* Γ*, denoted by* $K_{\mathrm{p}}(\Gamma)$*, is defined as the set of all* partial convecs *of all polymatroids that partially realize* Γ*. The* semi-partial polymatroidal convec set *is defined similarly and is denoted by* $K_{\mathrm{sp}}(\Gamma)$.

Proposition 7.6 also holds for the partial security; that is, for any access structure Γ, it holds that $\Sigma_{\mathrm{p}}(\Gamma) \subseteq K_{\mathrm{p}}(\Gamma)$ and $\Sigma_{\mathrm{sp}}(\Gamma) \subseteq K_{\mathrm{sp}}(\Gamma)$. The relation $K_{\mathrm{t}}(\Gamma) \subseteq K_{\mathrm{sp}}(\Gamma) \subseteq K_{\mathrm{p}}(\Gamma)$ is immediate for any access structure Γ. In next section we prove that these sets are indeed the same.

7.3 Main claim

Even though the K_{t}-set is trivially a polytope, it is not trivial that so are the other two sets, let alone being identical to the K_{t}-set.

Theorem 7.10 ($\boldsymbol{K}_{\mathbf{p}} = \boldsymbol{K}_{\mathbf{sp}} = \boldsymbol{K}_{\mathbf{t}}$) *For any access structure, the total, partial and semi-partial polymatroidal sets are identical.*

Proof. We know that $K_{\mathrm{t}}(\Gamma) \subseteq K_{\mathrm{sp}}(\Gamma) \subseteq K_{\mathrm{p}}(\Gamma)$ for any access structure Γ. It is sufficient to prove that $K_{\mathrm{p}}(\Gamma) \subseteq K_{\mathrm{t}}(\Gamma)$. Suppose that $\boldsymbol{a}' \in K_{\mathrm{p}}(\Gamma)$. Then, there exists a partial polymatroid $\mathcal{S}' = (P \cup \{p_0\}, r')$ for Γ and $\boldsymbol{a}' = \mathrm{pcv}(\mathcal{S}')$. We construct a total polymatroid $\mathcal{S} = (P \cup \{p_0\}, r)$ from $\mathcal{S}'$ for Γ such that $\mathrm{cv}(\mathcal{S}) = \mathrm{pcv}(\mathcal{S}')$. Let δ be as in Definition 7.1 and define α, β as follows,

$$\alpha = \min_{A \in \Gamma} \Delta_r(\{p_0\} : A)/r'(p_0) \quad , \quad \beta = \max_{B \in \Gamma^c} \Delta_r(\{p_0\} : B)/r'(p_0).$$

Define the function $r : 2^{P \cup \{p_0\}} \to [0, \infty)$ as follows:

$r(A) = r'(A)/\alpha$ for $A \in \Gamma^c$,
$r(A) = r'(A|\{p_0\})/\alpha + r'(p_0)$ for $A \in \Gamma$,
$r(A \cup \{p_0\}) = r(A)$ for $A \in \Gamma$,
$r(A \cup \{p_0\}) = r(A) + \frac{\alpha-\beta}{\alpha} r'(p_0)$ for $A \in \Gamma^c$;

Note that we have $r(\varnothing) = 0$ and $r(p_0) = \frac{\alpha-\beta}{\alpha} r'(p_0)$.

We claim that r is a rank function of a polymatroid with ground set $P \cup \{p_0\}$. First, we show that r has the monotonicity property. We check the monotonicity property only for the following nontrivial case: $A \cup \{p_0\} \subseteq B \cup \{p_0\}$ where A is a unqualified set and B is qualified. Checking the monotonicity property for the other cases is easier and left to the reader. Since $A \cup \{p_0\} \subseteq B \cup \{p_0\}$, the monotonicity of r' implies that $r'(A \cup \{p_0\}) \leqslant r'(B \cup \{p_0\})$. Therefore $r'(A|\{p_0\}) \leqslant r'(B|\{p_0\})$. Since A is unqualified we have $r'(A) \leqslant r'(A|\{p_0\}) + \beta r'(\{p_0\})$. Thus

$$\begin{aligned}
r(A \cup \{p_0\}) &= r(A) + \frac{\alpha - \beta}{\alpha} r'(\{p_0\}) \\
&= \frac{r'(A)}{\alpha} + r'(\{p_0\}) - \frac{\beta}{\alpha} r'(\{p_0\}) \\
&\leqslant \frac{r'(A|\{p_0\})}{\alpha} + \frac{\beta}{\alpha} r'(\{p_0\}) + r'(\{p_0\}) - \frac{\beta}{\alpha} r'(\{p_0\}) \\
&\leqslant \frac{r'(B|\{p_0\})}{\alpha} + r'(\{p_0\}) \\
&= r(B) \\
&= r(B \cup \{p_0\}).
\end{aligned}$$

For the sub-modularity property, we only check the sets $A, B \subseteq P$ where A, B and $A \cap B$ are unqualified and $A \cup B$ is qualified and other cases which are simpler are left to the reader. Since r' is sub-modular, we have $r'(A) + r'(B) \geqslant$

$r'(A \cup B) + r'(A \cap B)$. Since $A \cup B$ is qualified, by definition of α we have $\alpha \leqslant \Delta_{r'}(\{p_0\} : A \cup B)/r'(\{p_0\})$, or equivalently, $r'(A \cup B) \geqslant r'(A \cup B|\{p_0\}) + \alpha r(p_0)$. We observe that

$$\begin{aligned} r(A) + r(B) &= r'(A)/\alpha + r'(B)/\alpha \\ &= \frac{1}{\alpha}[r'(A) + r'(B)] \\ &\geqslant \frac{1}{\alpha}[r'(A \cup B) + r'(A \cap B)] \\ &\geqslant \frac{1}{\alpha}[r'(A \cup B|\{p_0\}) + \alpha r'(p_0) + r'(A \cap B)] \\ &= \big(r'(A \cup B|\{p_0\})/\alpha + r'(p_0)\big) + \big(r'(A \cap B)/\alpha\big) \\ &= r(A \cup B) + r(A \cap B). \end{aligned}$$

Now, we show that $\mathcal{S}$ is a total polymatroid for Γ. For every qualified set $A \in \Gamma$, we have $r(A \cup \{p_0\}) = r(A)$ by definition of r. Also, for every unqualified set $B \in \Gamma^c$, we have $r(B \cup \{p_0\}) = r(B) + r(p_0)$. Therefore $\mathcal{S} = (P \cup \{p_0\}, r)$ is total for Γ.

It remains to show that $\mathrm{cv}(\mathcal{S}) = \mathrm{pcv}(\mathcal{S}')$. Therefore, by definition of r, we have $r(p) = r'(p)/\alpha$ for any participant $p \in P$ (we have assumed that no singleton set is qualified, but it is easy to remove this assumption). Thus,

$$\begin{aligned} \mathrm{cv}(\mathcal{S}) &= \frac{1}{r(\{p_0\})}\big(r(\{p\})\big)_{p \in P} \\ &= \frac{1}{\frac{\alpha-\beta}{\alpha} r'(\{p_0\})}\big(r'(\{p\})/\alpha\big)_{p \in P} \\ &= \frac{1}{(\alpha - \beta) r'(p_0)}(r'(p))_{p \in P} \\ &= \frac{1}{\delta}(r'(p))_{p \in P} \\ &= \mathrm{pcv}(\mathcal{S}') \end{aligned}$$

Consequently, $K_{\mathrm{p}}(\Gamma) \subseteq K_{\mathrm{t}}(\Gamma)$. □

8 On statistical security

In this section, we study a standard cryptographic relaxation of secret sharing, called *statistical security*. See [3] for probably the oldest modern definition and [8] for an old construction.

We prove that 1) statistical security coincides with total security for a class of secret sharing schemes that includes group-homomorphic secret sharing schemes and 2) statistical security implies quasi-total security. The convexity of the statistical convec set is more technical than that of the total and quasi-total security notions and is given in Appendix B.

8.1 Definition

We provide some definitions that simplify the exposition of this section.

Definition 8.1 (Negligible and polynomial functions) *A function $f : \mathbb{N} \to \mathbb{R}$ is called negligible if $f(k) = k^{-\omega(1)}$. It is called polynomial if $f(k) = k^{\mathrm{O}(1)}$.*

Definition 8.2 (Statistical distance) *The* statistical distance *between two random variables $\boldsymbol{X}_0$ and $\boldsymbol{X}_1$ is defined by $\mathrm{SD}(\boldsymbol{X}_0, \boldsymbol{X}_1) = \frac{1}{2}\sum_x |\Pr[\boldsymbol{X}_0 = x] - \Pr[\boldsymbol{X}_1 = x]|$.*

Definition 8.3 (Maximum Reconstruction Error) *Let $(\boldsymbol{X}, \boldsymbol{Y})$ be jointly distributed random variables. The* Maximum Reconstruction Error (MRE) *of $\boldsymbol{X}$ from $\boldsymbol{Y}$ is defined as below, where* max *is taken over all $x \in \mathrm{supp}(\boldsymbol{X})$ and* min *is taken over all reconstruction functions $\textsc{Recon} : \mathrm{supp}(\boldsymbol{Y}) \to \mathrm{supp}(\boldsymbol{X})$,*

$$\mathrm{MRE}(\boldsymbol{X}|\boldsymbol{Y}) = \min_{\textsc{Recon}} \max_{x} \Pr[\textsc{Recon}(\boldsymbol{Y}) \neq x | \boldsymbol{X} = x] \ .$$

Definition 8.4 (Maximum Conditional Statistical Distance) *Let $(\boldsymbol{X}, \boldsymbol{Y})$ be jointly distributed random variables. The* Maximum Conditional Statistical Distance (MCSD) *of $\boldsymbol{Y}$ with respect to $\boldsymbol{X}$ is defined as follows, where* max *is taken over all pairs $x_0, x_1 \in \mathrm{supp}(\boldsymbol{X})$,*

$$\mathrm{MCSD}(\boldsymbol{Y}|\boldsymbol{X}) = \max_{x_0, x_1} \mathrm{SD}(\boldsymbol{Y}|\{\boldsymbol{X} = x_0\}, \boldsymbol{Y}|\{\boldsymbol{X} = x_1\}] \ .$$

The notion of statistical secret sharing and statistical convec set can be defined as follows. The reader may recall the definition of a convec-converging family of schemes (Definition 5.1).

Definition 8.5 (Statistical realization) *Let Γ be an access structure on P and $\mathcal{F} = \{\Pi^k\}_{k \in \mathbb{N}}$ be a convec-converging family of secret sharing schemes. Denote $\Pi^k = (\boldsymbol{S}_i^k)_{i \in P \cup \{0\}}$. We say that $\mathcal{F}$ is a statistical family for Γ if the following three hold:*

- **(Polynomial secret length)** $\log_2 |\mathrm{supp}(\boldsymbol{S}_0^k)|$ *is polynomial in k,*
- **(Statistical correctness)** *For every qualified set $A \in \Gamma$, $\mathrm{MRE}(\boldsymbol{S}_0^k|\boldsymbol{S}_A^k)$ is negligible,*
- **(Statistical privacy)** *For every unqualified set $A \in \Gamma$, $\mathrm{MCSD}(\boldsymbol{S}_A^k|\boldsymbol{S}_0^k)$ is negligible.*

We remark that the conditions on polynomial growth of the secret size and negligibility of the correctness and privacy errors are required for technical reasons. In particular, these conditions are critical in the proof of Proposition 8.9.

Definition 8.6 (Statistical convec set) *The statistical convec set of an access structure Γ, denoted by $\Sigma_{\mathrm{s}}(\Gamma)$, is defined as the set of all convecs of all statistical families for Γ. When we restrict ourselves to the class* C *of secret sharing schemes, we use the notation $\Sigma_{\mathrm{s}}^{\mathrm{C}}$.*

The statistical convec set is closed too and similar to the Σ_{qt} and $\overline{\Sigma_{\mathrm{t}}}$-sets, it is convex. However, because of our requirements, i.e., polynomial growth of secret length and negligibility of the correctness/privacy errors, the convexity proof becomes more technical than the case of total and quasi-total security notions. We refer the reader to Appendix B for a proof.

8.2 On group-homomorphic schemes

In [3], it has been mentioned that the notions of total and statistical secret sharing coincide in the case of linear schemes. Here we extend this observation to the sub-class N_0 of all group-characterizable schemes (recall Definition 9.1) whose secret subgroup is *normal* in the main group (i.e. $G_0 \trianglelefteq G$). In a recent work [29], it has been shown that group-characterizable secret sharing schemes with normal subgroups (i.e., $G_i \trianglelefteq G$ for every $i \in P \cup \{0\}$) are equivalent to *group-homomorphic* secret sharing schemes. A homomorphic scheme is called group-homomorphic if the secret and share spaces are all groups. It then follows that group-homomorphic statistical secret sharing schemes coincide with total security.

Theorem 8.7 (Statistical $\overset{\mathbf{N_0}}{\Longrightarrow}$ Total) *Let Γ be an access structure and $\mathcal{F} = \{\Pi_k\}_{k\in\mathbb{N}}$ be a statistical family of group-characterizable secret sharing schemes for Γ such that the secret group is normal in the main group (call this class N_0). Then, for every sufficiently large k, Π_k totally realizes Γ. Consequently, $\Sigma_{\mathrm{s}}^{\mathrm{N}_0}(\Gamma) = \overline{\Sigma_{\mathrm{t}}^{\mathrm{N}_0}(\Gamma)}$.*

Proof. The proof follows by the following observation. Let $\Pi = (\boldsymbol{S}_0, \boldsymbol{S}_1, \ldots, \boldsymbol{S}_n)$ be a group-characterizable secret sharing scheme (not necessarily form the class N_0) induced by groups $(G : G_0, G_1, \ldots, G_n)$. We know that for every non-empty subset $A \subseteq \{0, 1, \ldots, n\}$, we have $\mathrm{H}(\boldsymbol{S}_A) = \log \frac{|G|}{|G_A|}$, where $\boldsymbol{S}_A = (\boldsymbol{S}_i)_{i\in A}$ and $G_A = \bigcap_{i\in A} G_i$. It then follows that $\mathrm{H}(\boldsymbol{S}_A|\boldsymbol{S}_B) = \log \frac{|G_B|}{|G_{A\cap B}|}$ and $\mathrm{I}(\boldsymbol{S}_A : \boldsymbol{S}_B) = \log \frac{|G|}{|G_A * G_B|}$, for non-empty subsets $A, B \subseteq \{0, 1, \ldots, n\}$. This implies that if $\mathrm{H}(\boldsymbol{S}_A|\boldsymbol{S}_B) > 0$, then the quantity must be at least 1. The reason is that $G_{A\cap B}$ is a (proper) subgroup of G_B and, hence, its order divides $|G_B|$; i.e., the ratio $\frac{|G_B|}{|G_{A\cap B}|}$ is at least two. However, in general, the analogous statement is not true for the mutual information since $G_A * G_B$ is not necessarily a subgroup of G_B to ensure that its size divides $|G_B|$. Nevertheless, if one of these subgroups is a normal subgroup in G, then $G_A * G_B$ is a subgroup of G_B and, therefore, $\mathrm{I}(\boldsymbol{S}_A : \boldsymbol{S}_B)$ must be at least one if it is positive. We conclude that if an access structure is realizable by some group-characterizable family from the class N_0 statistically, then it is so totally. ▫

Technical discussion. As we discussed in the proof, our argument does not go through for the (general) class of group-characterizable schemes. In general, one can construct an example where the ratio between the order of G and the order (size) of $G_A * G_B$ is arbitrarily close to 1. Let G be the group of order

$p(p+1)$ generated by two elements a of order p and b of order $p+1$ where p is a given prime number. The only relation between a and b is $ab = b^p a$. Then it is easy to check that if we take G_A and G_B to be the subgroups of order p generated by a and bab^{-1}, respectively, then $G_A * G_B$ will be a subset of order p^2 and hence the ratio is $1 + 1/p$ which can be made arbitrarily close to 1. Nevertheless, this does not show that the theorem may not be true for general group-characterizable schemes. Therefore, it remains open if Theorem 8.7 holds for a class substantially larger than N_0. In particular, it remains open if any of the inclusions $\overline{\Sigma_{\mathrm{t}}^{\mathrm{G}}(\Gamma)} \subseteq \Sigma_{\mathrm{s}}^{\mathrm{G}}(\Gamma)$ or $\overline{\Sigma_{\mathrm{t}}(\Gamma)} \subseteq \Sigma_{\mathrm{s}}(\Gamma)$ is proper for some access structure.

The following then follows from [29].

Proposition 8.8 (Group-homomorphic schemes) *Let Γ be an access structure and $\mathcal{F} = \{\Pi_k\}_{k\in\mathbb{N}}$ be a statistical family of group-homomorphic secret sharing schemes for Γ. Then, for every sufficiently large k, Π_k totally realizes Γ.*

8.3 Relation with quasi-total security

Notice that by Theorem 5.5 and Theorem 8.7, the equality $\Sigma_{\mathrm{s}}^{\mathrm{L}}(\Gamma) = \Sigma_{\mathrm{qt}}^{\mathrm{L}}(\Gamma)$ holds. It remains open if the equality holds for any larger class. We close this section by proving the following proposition on the relation between the quasi-total and statistical convec sets.

Proposition 8.9 (Statistical $\Longrightarrow$ Quasi-total) *Let $\mathcal{F}$ be a statistical family of schemes for an access structure Γ. Then it is a quasi-total family for Γ too. Consequently, $\Sigma_{\mathrm{s}}^{\mathrm{C}}(\Gamma) \subseteq \Sigma_{\mathrm{qt}}^{\mathrm{C}}(\Gamma)$, for every class* C *of secret sharing schemes.*

Proof. It is easy to see that the proof follows from the following claim.

Claim. Let $\{(\boldsymbol{X}_k, \boldsymbol{Y}_k)\}_{k\in\mathbb{N}}$ be a family of jointly distributed random variables and assume that $\log|\mathrm{supp}(\boldsymbol{X}_k)|$ is polynomial in k. Then:

(i) If $\mathrm{MRE}(\boldsymbol{X}_k|\boldsymbol{Y}_k)$ is negligible in k, then $\lim_{k\to\infty} \mathrm{H}(\boldsymbol{X}_k|\boldsymbol{Y}_k) = 0$.
(ii) If $\mathrm{MCSD}(\boldsymbol{Y}_k|\boldsymbol{X}_k)$ is negligible in k, then $\lim_{k\to\infty} \mathrm{I}(\boldsymbol{X}_k : \boldsymbol{Y}_k) = 0$.

Part (i) follows by Fano's inequality [16] which is stated as follows. Suppose that we wish to estimate the random variable $\boldsymbol{X}$, with support $\mathcal{X}$, by an estimator $\boldsymbol{Y}$, and furthermore, assume that $\epsilon = \Pr[\boldsymbol{X} \neq \boldsymbol{Y}]$. Then, $\mathrm{H}(\boldsymbol{X}|\boldsymbol{Y}) \leqslant \mathrm{H}(\epsilon) + \epsilon \log(|\mathcal{X}| - 1)$, where $\mathrm{H}(\epsilon)$ is the entropy of a Bernoulli random variable with parameter ϵ. Denote $n(k) = |\mathrm{supp}(\boldsymbol{X}_k)|$ and $\epsilon(k) = \mathrm{MRE}(\boldsymbol{Y}_k|\boldsymbol{X}_k)$. By Fano's inequality and definition of MRE, we have $\mathrm{H}(\boldsymbol{X}_k \mid \boldsymbol{Y}_k) \leqslant \mathrm{H}(\epsilon(k)) + \epsilon(k)\log(n(k) - 1)$. This proves that $\lim_{k\to\infty} \mathrm{H}(\boldsymbol{X}_k \mid \boldsymbol{Y}_k) = 0$ since $\log n(k)$ is polynomial and $\epsilon(k)$ is negligible. Part (ii) has been implicitly proved in [40]. ▭

The reverse of the above proposition is not true, however.

Proposition 8.10 (Quasi-total $\not\Longrightarrow$ Statistical) *A quasi-total family for an access structure does not necessarily realize it statistically.*

Proof. Let $P = \{p_1\}$ and Γ be the trivial access structure on P. For every $k \in \mathbb{N}$, let T^k be a vector space on $\mathbb{F}_2$ with dimension k. Let $T_0^k = T^k$ and T_1^k be a subspace of T^k of dimension $k-1$. Then $\Pi_k = (T^k; T_0^k, T_1^k)$ quasi-totally realizes Γ but not statistically. The reason is that while the amount of information that the only qualified set gains on the secret converges to the entropy of the secret, the secret can only be guessed with probability at most 1/2. □

We remark that, nevertheless, the above proposition does not refute the coincidence of the quasi-total and statistical convec sets, i.e., if the inclusion $\Sigma_{\mathrm{qt}}^{\mathrm{C}}(\Gamma) \subseteq \Sigma_{\mathrm{s}}^{\mathrm{C}}(\Gamma)$ is proper for some access structure Γ and any class C of schemes.

9 On length-based and entropy-based efficiency measures

Two different flavors of information ratio can be found in the literature [11,13,36]. One is defined based on the ratio between the share entropy and the secret entropy, also adopted by us in the course of this paper. The other one is defined as the ratio between the share length (i.e., the logarithm of the share space size) and the secret length. Consequently, the information ratio of an access structure can be defined in two different ways, with respect to every security notion.

In this section, we show that the two definitions coincide for the quasi-total security notion. The key observation is that group-characterizable schemes are "complete" for this security notion. It remains an open problem if these claims are true for other security notion, and in particular, the total one[3].

Definition 9.1 (Group-characterizable scheme [15]) *A tuple $\Pi = (G : G_0, G_1, \ldots, G_n)$ is called a group-characterizable secret sharing scheme if G is a finite group, G_i is a subgroup of G, for each $i \in [n]$, and $|G|/|G_0| \geqslant 2$.*

A group-characterizable scheme $\Pi = (G : G_0, G_1, \ldots, G_n)$ induces a secret sharing scheme $(\boldsymbol{S}_0, \boldsymbol{S}_1, \ldots, \boldsymbol{S}_n)$ by letting $\boldsymbol{S}_i = \boldsymbol{X} G_i$, where $\boldsymbol{X}$ is a uniform random variable on G; hence, the support of $\boldsymbol{S}_i$ is the left cosets of G_i.

For a group-characterizable scheme $\Pi = (G : G_0, G_1, \ldots, G_n)$, it is easy to verify that

$$\Phi_\Pi(A) = \frac{\log\big(|G|/|G_A * G_0|\big)}{\log\big(|G|/|G_0|\big)}\ , \qquad \mathrm{cv}(\Pi) = \Big(\frac{\log\big(|G|/|G_i|\big)}{\log\big(|G|/|G_0|\big)}\Big)_{i\in[n]}\ ,$$

where $G_A = \bigcap_{i\in A} G_i$.

Let G denote the class of group-characterizable schemes. In the case of total security (as well as all other security notions in this paper), it remains open if the inclusion $\overline{\Sigma_{\mathrm{t}}^{\mathrm{G}}(\Gamma)} \subseteq \overline{\Sigma_{\mathrm{t}}(\Gamma)}$ is proper for some access structure Γ. The

[3] In the case of total security, it is known that the size-based and length-based definitions are the same for ideal access structures [12]. But it is open if group-characterizable schemes are complete for ideal access structures.

following theorem, whose proof follows by a well-known theorem by Chan and Yeung in [15], asserts that the answer is negative for quasi-total security; that is group-characterizable schemes are "complete" for this security notion.

Theorem 9.2 ($\boldsymbol{\Sigma}_{\mathbf{qt}}^{\mathbf{G}} = \boldsymbol{\Sigma}_{\mathbf{qt}}$) *The class of group-characterizable schemes,* G*, is complete for quasi-total security; that is, for every access structure Γ, it holds that $\Sigma_{\mathrm{qt}}^{\mathrm{G}}(\Gamma) = \Sigma_{\mathrm{qt}}(\Gamma)$.*

Proof. The Chan-Yeung's theorem [15, Theorem 4.1] is about random variables and can be stated for secret sharing schemes as follows: for every scheme $\Pi = (\boldsymbol{S}_i)_{i \in P \cup \{p_0\}}$, there exists a sequence $\{\Pi_k\}$ of group-characterizable schemes, with $\Pi_k = (\boldsymbol{S}_i^k)_{i \in P \cup \{p_0\}}$, such that for every $A \subseteq P \cup \{p_0\}$ it holds that $\lim_{k\to\infty} \frac{1}{k}\mathrm{H}(\boldsymbol{S}_A^k) = \mathrm{H}(\boldsymbol{S}_A)$. It then follows that $\lim_{k\to\infty} \mathrm{cv}(\Pi_k) = \mathrm{cv}(\Pi)$ and $\lim_{k\to\infty} \Phi(\Pi_k) = \Phi(\Pi)$.

Now we return to the proof of our theorem. Let Γ be an access structure and $\boldsymbol{\sigma} \in \Sigma_{\mathrm{qt}}(\Gamma)$. We need to show that $\boldsymbol{\sigma} \in \Sigma_{\mathrm{qt}}^{\mathrm{G}}(\Gamma)$. Let $\mathcal{F} = \{\Pi_m\}_{m\in\mathbb{N}}$ be a quasi-total family for Γ with $\mathrm{cv}(\mathcal{F}) = \boldsymbol{\sigma}$. Therefore, by Chan-Yeung's theorem, for each scheme Π_m, there exists a a sequence $\{\Pi_{k,m}\}$ of group-characterizable schemes such that $\lim_{k\to\infty} \mathrm{cv}(\Pi_{k,m}) = \mathrm{cv}(\Pi_m)$ and $\lim_{k\to\infty} \Phi(\Pi_{k,m}) = \Phi(\Pi_m)$. It is then easy to see that the family $\mathcal{F}' = \{\Pi_{j,j}\}$ of group-characterizable schemes satisfies $\mathrm{cv}(\mathcal{F}') = \lim_{j\to\infty} \mathrm{cv}(\Pi_{j,j}) = \mathrm{cv}(\mathcal{F})$ and $\lim_{k\to\infty} \Phi_{\Pi_{j,j}} = \Phi_\Gamma$; that is, $\boldsymbol{\sigma} \in \Sigma_{\mathrm{qt}}^{\mathrm{G}}(\Gamma)$. □

The following corollary is then immediate since the secret and shares of a group-characterizable scheme are all uniform.

Corollary 9.3 (Equivalence of entropy-based and length-based ratios) *Assume that in the definition of convec of a scheme (Definition 2.6), one replaces* $\mathrm{H}(\boldsymbol{S}_i)$ *with* $\log|\mathrm{supp}(\boldsymbol{S}_i)|$*. Then, the quasi-total convec set remains invariant.*

10 On decomposition theorems

The (λ, ω)-weighted-decomposition theorem of [24] (as well as its predecessor [46]) has the following limitation. They require that in the linear sub-schemes every subset of participants fully recovers a certain subset of the secret elements and nothing more; in other words, recovering a non-trivial linear combination of the secret elements is not allowed.

In Section 10.1, we show that the above strong requirement on the (λ, ω)-weighted-decomposition can be removed. The main tool that allows us to do this is the notion of partial secret sharing and the result of Section 4 on the equality of partial and total linear information ratios.

In Section 10.2, we present a unified decomposition theorem, that we refer to as the δ-decomposition, which captures the advantages of the (λ, ω)-decomposition [18, 48] and the (λ, ω)-weighted-decomposition [24] at one place. The theorem is essentially a restatement of known and folklore results. We introduce the notion of δ-decomposition, first, for the sake of completeness and, second, to provide the intuition behind the definition of partial security (Definition 3.1) and partial convec (Definition 3.2). The reader may compare those definitions with Definition 10.3.

Notation. In this section, we use the simplified notation Σ for total convec set and Λ (resp. Λ_p) for its restrictions to the class of all linear (resp. p-linear) schemes. We remark that the linear convec set of a (rational-valued) access function Φ is defined as the set of all convecs of all linear secret sharing schemes whose access function is Φ.

10.1 $(\boldsymbol{\lambda}, \boldsymbol{\omega})$-weighted-decomposition revisited

The following definition is a restatement of Definition 3.4 in [24].

Definition 10.1 ($(\boldsymbol{\lambda}, \boldsymbol{\omega})$–weighted decomposition) *Let $\lambda, \omega, N, m_1, \cdots, m_N$, be non-negative integers, with $0 \leqslant \omega < \lambda$. Let Γ be an access structure and $\Phi_1, \ldots, \Phi_N$ be (rational) access functions all defined on the same participants set and further assume that $m_j\Phi_j$ is an integer-valued function for every $j \in [N]$. We call $(m_1, \Phi_1), \ldots, (m_N, \Phi_N)$ a (λ, ω)-weighted-decomposition for Γ if the following two hold:*

- $\sum_{j=1}^{N} m_j\Phi_j(A) \geqslant \lambda$, *for every qualified set $A \in \Gamma$,*
- $\sum_{j=1}^{N} m_j\Phi_j(B) \leqslant \omega$, *for every unqualified set $B \in \Gamma^c$.*

The following decomposition theorem is an extension of Theorem 3.2 in [24], which was stated for a subclass of linear schemes. The proof essentially relies on Proposition 4.5

Theorem 10.2 ($(\boldsymbol{\lambda}, \boldsymbol{\omega})$–weighted decomposition) *Let p be a prime. Consider a (λ, ω)-weighted-decomposition $(m_1, \Phi_1), \ldots, (m_N, \Phi_N)$ for an access structure Γ and let $\boldsymbol{\sigma}_j \in \Lambda_p(\Phi_j)$, $j \in [N]$. Then, $\frac{1}{\lambda-\omega}\sum_{j=1}^{N} m_j\boldsymbol{\sigma}_j \in \Lambda_p(\Gamma)$.*

Proof. Let $\Pi_j = (T_{ij})_{i\in P}$ be a p-linear secret sharing scheme for Φ_j with convec $\boldsymbol{\sigma}_j$, for $j \in [N]$. Without loss of generality, we assume that all sub-schemes are $\mathbb{F}$-linear for a common finite field $\mathbb{F}$ with characteristic p. Let $T'_i = \oplus_{j\in[N]}T_{ij}$, for every $i \in P$. For every $i \in P$, we have $\dim T'_i = \sum_{j\in[N]} \dim T_{ij}$ which implies that

$$\left(\dim T'_i\right)_{i\in P} = \sum_{j=1}^{N} m_j\boldsymbol{\sigma}_j \ .$$

Also, for every subset A of participants, it holds that:

$$\begin{aligned}\dim(T'_A \cap T'_0) &= \sum_{j\in[N]} \dim(T_A \cap T_0)\\ &= \sum_{j\in[N]} m_j\Phi_{\Pi_j}(A)\\ &= \sum_{j\in[N]} m_j\Phi_j(A) \ .\end{aligned}$$

By definition of the (λ, ω)–weighted decomposition, we have

$$\Delta = \min_{A \in \Gamma} \dim(T'_A \cap T'_0) - \max_{B \in \Gamma^c} \dim(T'_B \cap T'_0) \geqslant \lambda - \omega \ .$$

Consequently, $\Pi' = (T'_i)_{i \in P}$ is an $\mathbb{F}$-linear partial secret sharing scheme for Γ with the following partial convec:

$$\mathrm{pcv}(\Pi') = \frac{1}{\Delta} \sum_{j=1}^{N} m_j \boldsymbol{\sigma}_j \ .$$

Then, by Proposition 4.5, there exists a finite extension $\mathbb{K}$ of $\mathbb{F}$, such that Γ has a total $\mathbb{K}$-linear scheme Π with the above convec. It is straightforward to modify the scheme to have a scheme with the convec $\frac{1}{\lambda-\omega} \sum_{j=1}^{N} m_j \boldsymbol{\sigma}_j$. □

10.2 δ-decomposition

We present the notion of δ-decomposition, which captures all the weighted [24,46] and non-weighted [18, 44] decompositions simultaneously, and even in a more general form. It also justifies the intuition behind the definition of partial security (Definition 3.1) and partial convec (Definition 3.2).

Definition 10.3 (δ-decomposition) *Let N be an integer and $\delta, h_1, \ldots, h_N$ be positive real numbers. Let Γ be an access structure and $\Phi_1, \ldots, \Phi_N$ be access functions all on participants set P. We say that $(h_1, \Phi_1), \ldots, (h_N, \Phi_N)$ is a δ–decomposition for Γ if*

$$\delta = \min_{A \in \Gamma} \sum_{j=1}^{N} h_j \Phi_j(A) - \max_{B \in \Gamma^c} \sum_{j=1}^{N} h_j \Phi_j(B) \ .$$

The proof of the following theorem is easy and we leave it to the reader.

Theorem 10.4 (δ-decomposition) *Let Γ be an access structure and consider a δ–decomposition $(h_1, \Phi_1), \ldots, (h_N, \Phi_N)$ for it. Then, the followings hold:*

(i) ***(Rational sub-access functions)*** *Let p be a prime, Φ_j be rational and $\boldsymbol{\sigma}_j \in \overline{\Lambda_p(\Phi_j)}$, for every $j \in [N]$. Then $\boldsymbol{\sigma} = \frac{1}{\delta} \sum_{j=1}^{N} h_j \boldsymbol{\sigma}_j \in \overline{\Lambda_p(\Gamma)}$.*

(ii) ***(Total sub-access functions)*** *Let Φ_j be total and $\boldsymbol{\sigma}_j \in \overline{\Sigma(\Phi_j)}$, for every $j \in [N]$. Then, $\boldsymbol{\sigma} = \frac{1}{\delta} \sum_{j=1}^{N} h_j \boldsymbol{\sigma}_j \in \overline{\Sigma(\Gamma)}$.*

11 Conclusion

We studied some questions about secret sharing schemes with respect to different security notions. Some questions were answered but several ones remained open, listed below.

1. The closures of Σ_{t}, Σ_{s} and Σ_{qt}-sets are all convex. What about the Σ_{p} and Σ_{sp}-sets?

2. Ignoring closures, the equalities $\Sigma_{\mathrm{t}}^{\mathrm{L}} = \Sigma_{\mathrm{sp}}^{\mathrm{L}} = \Sigma_{\mathrm{p}}^{\mathrm{L}} = \Sigma_{\mathrm{qt}}^{\mathrm{L}}$ hold. Do they hold for any class larger than the linear one, L?
3. Ignoring closures, the equality $\Sigma_{\mathrm{s}}^{\mathrm{N}_0} = \Sigma_{\mathrm{t}}^{\mathrm{N}_0}$ holds, where N_0 is the class of group-characterizable schemes whose secret group is normal in the main group? Does it hold for any larger class?
4. While statistical security implies the quasi-total security (and consequently $\Sigma_{\mathrm{s}} \subseteq \Sigma_{\mathrm{qt}}$), the reverse implication does not necessarily hold true. However, it remains open if their corresponding convec sets coincide, i.e., if $\Sigma_{\mathrm{qt}} = \Sigma_{\mathrm{s}}$.
5. The class of group-characterizable schemes, G, are complete for quasi-total security (i.e., $\Sigma_{\mathrm{qt}}^{\mathrm{G}} = \Sigma_{\mathrm{qt}}$). Is this also true for other security notions?
6. The length-based and entropy-based definitions of information ratio coincide for quasi-total security. What about other notions?
7. The Shannon lower bound on partial and total information ratio coincide. Do they coincide for a larger class of information inequalities (e.g., by allowing certain additional non-Shannon type information inequalities, along the lines of [6, 38])?
8. Compute the abelian information ratio of access structure $\mathcal{F}+\mathcal{N}$ with respect to the total and quasi-total information security notions.

Acknowledgment. We would like to thank Reza Kaboli for his proof of Theorem 7.10.

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A A probabilistic proof of Lemma 4.3 for $N < \frac{q-1}{q^{m-w}-1}\frac{q^m-1}{q^w-1}$

Without loss of generality we can assume that $\dim E_i = \omega$. Let $\mathbb{F}$ be the underlying finite field with q elements. Given a random subspace S of dimension $m-w$ there are exactly $\frac{q^{m-w}-1}{q-1}$ lines in S through the origin. For a given random line, the probability that it lies inside a subspace E_i of dimension w is $(q^w-1)/(q^m-1)$. Therefore, by the union bound, the probability that at least one of the lines in S is inside E_i is at most

$$\frac{q^{m-w}-1}{q-1}\frac{q^w-1}{q^m-1}\ .$$

Hence if $N < \frac{q-1}{q^{m-w}-1}\frac{q^m-1}{q^w-1}$, the probability that S has a non-trivial intersection with one of these subspaces is less than 1 and hence there exists a subspace of dimension $m-w$ that has zero intersection with all of the E_i's.

Notice that if $w=1$, this bound coincides with the bound $N < \frac{q^m-1}{q^{m-1}-1}$ of Lemma 4.3, proved using a non-probabilistic argument. Even though the bound of Lemma 4.3 is better in general, the difference is negligible (the maximum happens at $w = m/2$).

B Convexity of statistical convec sets

We need to take some care when proving the convexity of the statistical convec set. The subtleties are due to the requirements of statistical secret sharing on polynomial growth of the secret length and negligibility of the correctness and privacy errors. First, we present some preliminaries before getting to the main proof.

Definition B.1 ($(\boldsymbol{\epsilon}, \boldsymbol{\delta})$-statistical scheme/family) *Let Γ be an access structure on P and ϵ, δ be non-negative real numbers. We say that the scheme $\Pi = (\boldsymbol{S}_i)_{i \in P \cup \{p_0\}}$ is an (ϵ, δ)-statistical scheme for Γ if the following two hold:*

- **(Statistical correctness)** $\mathrm{MRE}(\boldsymbol{S}_0 | \boldsymbol{S}_A) \leqslant \epsilon$, *for every qualified set $A \in \Gamma$,*
- **(Statistical privacy)** $\mathrm{MCSD}(\boldsymbol{S}_A | \boldsymbol{S}_0) \leqslant \delta$, *for every unqualified set $B \in \Gamma$.*

Now let $\epsilon, \delta : \mathbb{N} \to \mathbb{R}$ be two functions. We say that a family $\mathcal{F} = \{\Pi_k\}_{k \in \mathbb{N}}$ is an (ϵ, δ)–statistical family for Γ, if for every integer k the scheme Π_k is an $\big(\epsilon(k), \delta(k)\big)$-statistical scheme for Γ.

The proof of the following lemma is an easy exercise and is left to the reader.

Notation B.2 ($\bigotimes_{\boldsymbol{j}} \boldsymbol{\Pi}_{\boldsymbol{j}}^{\boldsymbol{m_j}}$) *Let m be an integer. For secret sharing schemes $\Pi_1, \ldots, \Pi_m$, where $\Pi_j = (\boldsymbol{S}_{pj})_{p \in P \cup \{p_0\}}$, notate $\Pi_1 \otimes \ldots \otimes \Pi_m = (\boldsymbol{S}_p)_{p \in P \cup \{p_0\}}$, where $\boldsymbol{S}_p = (\boldsymbol{S}_{p1}, \ldots, \boldsymbol{S}_{pm})$ for every $p \in P \cup \{p_0\}$. We use the notation Π^m for the m-fold product $\Pi \otimes \ldots \otimes \Pi$.*

Lemma B.3 (Properties of $\bigotimes_{\boldsymbol{j}} \boldsymbol{\Pi}_{\boldsymbol{j}}^{\boldsymbol{m_j}}$) *Let $\Pi_1, \ldots, \Pi_m$ be secret sharing scheme, $m_1, \ldots, m_k$ be positive integers, Γ be an access structure, $\mathbb{F}$ be a finite field and p be a prime. Then:*

1. *If every Π_j realizes Γ, then so does $\bigotimes_{j=1}^{m} \Pi_j^{m_j}$.*
2. *If every Π_j is $\mathbb{F}$-linear (resp. p-linear, abelian, or group-characterizable), then so is $\bigotimes_{j=1}^{m} \Pi_j^{m_j}$.*
3. *If every Π_j is linear, then $\bigotimes_{i=j}^{m} \Pi_j^{m_j}$ is abelian.*

Lemma B.4 *Let α, β be a real numbers with $\alpha \in [0,1]$. Let $\alpha_k = \lfloor k\alpha \rfloor / k$ and $\beta_k = \lfloor k\beta \rfloor / k$ for every integer $k \in \mathbb{N}$. Then, for every integer $k \geqslant 1/\beta$,*

$$\Big| \frac{\alpha_k}{\alpha_k + (1-\alpha_k)\beta_k/\beta} - \alpha \Big| \leqslant (1 + \frac{1}{\beta}) \frac{1}{k} \ .$$

The inequality still holds if we replace β_k/β with β/β_k.

Lemma B.5 (Convex combination lemma) *Let $\alpha \in [0,1]$ be a real number and Π', Π'' be secret sharing schemes on participants set P. Denote the secret random variables of Π' and Π'' by $\boldsymbol{S}_0^{'}$ and $\boldsymbol{S}_0^{''}$, respectively, and let $\beta = \mathrm{H}(\boldsymbol{S}_0^{'})/\mathrm{H}(\boldsymbol{S}_0^{''})$.*

Then, there exists a sequence $\{\Pi_k\}_{k\in\mathbb{N}}$ of secret sharing schemes such that:

1. *$\Pi_k = \left(\Pi'\right)^{m'(k)} \otimes \left(\Pi''\right)^{m''(k)}$, where $m', m'' : \mathbb{N} \to \mathbb{N}$ are respectively $\mathrm{O}(k^2)$ and $\mathrm{O}(\beta k^2)$ functions,*
2. *For every integer $k \geqslant 1/\beta$, it holds that*

$$\|\mathrm{cv}(\Pi_k) - \alpha \mathrm{cv}(\Pi') - (1-\alpha)\mathrm{cv}(\Pi'')\| \leqslant \left(|\mathrm{cv}(\Pi')| + |\mathrm{cv}(\Pi'')|\right)(1+\frac{1}{\beta})\frac{1}{k} \ . \tag{B.1}$$

Proof. Let $\Pi' = \left(\boldsymbol{S}_0^{'}\right)_{p\in P\cup\{p_0\}}$ and $\Pi'' = \left(\boldsymbol{S}_p^{''}\right)_{p\in P\cup\{p_0\}}$. Denote $\beta = \mathrm{H}(\boldsymbol{S}_0^{'})/\mathrm{H}(\boldsymbol{S}_0^{''})$ and let $\{\alpha_k\}, \{\beta_k\}$ be two sequences of positive rational numbers respectively converging to α and β. Let $\alpha_k = c_k/d_k$ and $\beta_k = e_k/f_k$ where c_k, d_k, e_k, f_k are positive integers. Define

$$\Pi_k = \left(\boldsymbol{S}_p^k\right)_{p\in P\cup\{p_0\}} = \left(\Pi'\right)^{m'(k)} \otimes \left(\Pi''\right)^{m''(k)} \ ,$$

where $m'(k) = f_k c_k$ and $m''(k) = e_k(d_k - c_k)$. For proving the claimed quadratic orders of m', m'' in Part 1, it is sufficient to choose

$$c_k = \lfloor k\alpha \rfloor, e_k = \lfloor k\beta \rfloor, d_k = f_k = k \ ,$$

which proves Part (1).

For every $A \subseteq P$ and $p \in P \cup \{p_0\}$, by independence of the random variables, we have:

$$\mathrm{H}(\boldsymbol{S}_i^k) = f_k c_k \mathrm{H}(\boldsymbol{S}_i^{'}) + e_k(d_k - c_k)\mathrm{H}(\boldsymbol{S}_i^{''}) \ .$$

Therefore,

$$\begin{aligned}\frac{\mathrm{H}(\boldsymbol{S}_p^k)}{\mathrm{H}(\boldsymbol{S}_0^k)} &= \frac{f_k c_k \mathrm{H}(\boldsymbol{S}_p^{'}) + e_k(d_k - c_k)\mathrm{H}(\boldsymbol{S}_p^{''})}{f_k c_k \mathrm{H}(\boldsymbol{S}_0^{'}) + e_k(d_k - c_k)\mathrm{H}(\boldsymbol{S}_0^{''})} \\ &= \frac{\alpha_k}{\alpha_k + (1-\alpha_k)\beta_k/\beta}\frac{\mathrm{H}(\boldsymbol{S}_p^{'})}{\mathrm{H}(\boldsymbol{S}_0^{'})} + \frac{(1-\alpha_k)}{\alpha_k\beta/\beta_k + (1-\alpha_k)}\frac{\mathrm{H}(\boldsymbol{S}_p^{''})}{\mathrm{H}(\boldsymbol{S}_0^{''})} \ ,\end{aligned}$$

which can be compactly written as follows:

$$\mathrm{cv}(\Pi_k) = \frac{\alpha_k}{\alpha_k + (1-\alpha_k)\beta_k/\beta}\mathrm{cv}(\Pi') + \frac{(1-\alpha_k)}{\alpha_k\beta/\beta_k + (1-\alpha_k)}\mathrm{cv}(\Pi'') \ . \tag{B.2}$$

Part 2 then follows by Lemma B.4. □

Proposition B.6 (Convexity of statistical convec sets) *The statistical convec set of any access structure Γ (i.e., $\Sigma_{\mathrm{s}}(\Gamma)$) is a set with convex closure.*

Proof. Let $\mathcal{F}' = \{\Pi'_j\}_{j\in\mathbb{N}}$ and $\mathcal{F}'' = \{\Pi''_j\}_{j\in\mathbb{N}}$ be, respectively, (ϵ', δ') and (ϵ'', δ'')–statistical families for Γ, where $\epsilon', \delta', \epsilon'', \delta''$ are all negligible functions. We construct a statistical family $\mathcal{F}$ for Γ with:

$$\mathrm{cv}(\mathcal{F}) = \alpha\mathrm{cv}(\mathcal{F}') + (1-\alpha)\mathrm{cv}(\mathcal{F}'') \ .$$

Denote the secret random variables of Π'_j and Π''_j by $\boldsymbol{S}'_{0,j}$ and $\boldsymbol{S}''_{0,j}$, respectively, and let $\beta_j = \mathrm{H}(\boldsymbol{S}'_{0,j})/\mathrm{H}(\boldsymbol{S}''_{0,j})$. For each $j \in \mathbb{N}$, plug in $\Pi' = \Pi'_j$ and $\Pi'' = \Pi''_j$ in Lemma B.5. Let $\Pi_{j,k} = \left(\Pi'_j\right)^{m'_j(k)} \otimes \left(\Pi''_j\right)^{m''_j(k)}$ where $m'_j(k) = \mathrm{O}(k^2)$ and $m''_j(k) = \mathrm{O}(\beta_j k^2)$. Note that since the secret size of both Π'_j and Π''_j grows polynomially in j, so does β_j. Without loss of generality we may assume that $\beta_j = \Omega(1)$.

By Part 2, for every integer $k \geqslant \beta_j$, it holds that

$$|\mathrm{cv}(\Pi_{j,k}) - \alpha\mathrm{cv}(\Pi'_j) - (1-\alpha)\mathrm{cv}(\Pi''_j)| \leqslant \big(|\mathrm{cv}(\Pi'_j)| + |\mathrm{cv}(\Pi''_j)|\big)(1 + \frac{1}{\beta_j})\frac{1}{k} \ .$$

We let $\mathcal{F} = \{\Pi_{j,\beta_j}\}_{j\in\mathbb{N}}$ (we ignore using β_j instead of $\lceil\beta_j\rceil$). The claim on the convec of $\mathcal{F}$ is clear. Notice that the scheme Π_{j,β_j} is an $\big(\epsilon(j), \delta(j)\big)$-statistical scheme for Γ with the following parameters:

$$\big(\epsilon(j), \delta(j)\big) = \big(m'_j(\beta_j)\epsilon'(j) + m''_j(\beta_j)\epsilon''(j), m'_j(\beta_j)\delta'(j) + m''_j(\beta_j)\delta''(j)\big) \ .$$

Since $\beta_j, m'_j(\beta_j), m''_j(\beta_j)$ all grow polynomially in j and $\epsilon', \delta', \epsilon'', \delta''$ are all negligible functions, it follows that ϵ and δ are both negligible. It is also easy to see that the secret length of Π_{j,β_j} grows polynomially in j. We conclude that $\mathcal{F} = \{\Pi_{j,j}\}_{j\in\mathbb{N}}$ is a statistical family for Γ with the required convec. □

C Abelian and linear secret sharing

Recall the definition of a group-characterizable scheme (Definition 9.1). A group-characterizable scheme $\Pi = (G : G_0, G_1, \ldots, G_n)$ is called *abelian* if its main group G is abelian.

It is easy to show that (e.g., see [28]) every abelian scheme $\Pi = (G : G_0, G_1, \ldots, G_n)$, with respect to this definition induces an abelian scheme $\Pi' = (G'; G'_0, G_1, \ldots, G'_n)$, with respect to the following definition, and vice versa, with the same access function and convec.

Definition C.1 (Abelian scheme) *A tuple $\Pi = (G; G_0, G_1, \ldots, G_n)$ is called an abelian secret sharing scheme if G is a finite abelian group, G_i is a subgroup of G, for each $i \in [n]$, and $|G_0| \geqslant 2$. When there is no confusion, we simply write $\Pi = (G_i)_{i\in P\cup\{0\}}$.*

Definition C.2 (Linear scheme) *When T is a finite dimensional vector space on some finite field and $T_0, T_1, \dots, T_n$ are sup-spaces of T, the abelian secret sharing scheme $\Pi = (T; T_0, T_1, \dots, T_n)$ is called linear.*

Table 2 shows the simplified access functions and convecs for different types of schemes.

type	Π	$\Phi_\Pi(A)$	$\mathrm{cv}(\Pi)$	notation
group char.	$(G : G_0, G_1, \dots, G_n)$	$\dfrac{\log\big(\lvert G\rvert/\lvert G_A * G_0\rvert\big)}{\log\big(\lvert G\rvert/\lvert G_0\rvert\big)}$	$\left(\dfrac{\log\big(\lvert G\rvert/\lvert G_i\rvert\big)}{\log\big(\lvert G\rvert/\lvert G_0\rvert\big)}\right)_{i\in[n]}$	$G_A = \bigcap_{i\in A} G_i$
abelian	$(G; G_0, G_1, \dots, G_n)$	$\dfrac{\log\lvert G_0 \cap G_A\rvert}{\log\lvert G_0\rvert}$	$\left(\dfrac{\log\lvert G_i\rvert}{\log\lvert G_0\rvert}\right)_{i\in[n]}$	$G_A = \sum_{i\in A} G_i$
linear	$(T; T_0, T_1, \dots, T_n)$	$\dfrac{\dim(T_0 \cap T_A)}{\dim(T_0)}$	$\left(\dfrac{\dim(T_i)}{\dim(T_0)}\right)_{i\in[n]}$	$T_A = \sum_{i\in A} T_i$

Table 2: The access function and convec of different scheme types.