Faster Bootstrapping of FHE over the integers with large prime message space

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Abstract. Bootstrapping of FHE over the integer with large message is an open problem, which is to evaluate double modulo \((c \mod p) \mod Q\) arithmetic homomorphically for large \(Q\). In this paper, we express this double modulo reduction circuit as an arithmetic circuit of degree at most \(2 \log_2 \theta / 2\), with \(O(\theta \log_2 \theta)\) multiplication gates, where \(\theta = \frac{1}{\log p}\) and \(\lambda\) is the security parameter. The complexity of decryption circuit is independent of the message space size \(Q\) with a constraint \(Q > \theta \log_2 \theta / 2\).

Keywords: Fully homomorphic encryption, Bootstrapping, Restricted depth-3 circuit.

1 Introduction

Fully homomorphic encryption (FHE) enables computation of any circuits on the encryption data, which was first introduced by Rivest, Adleman and Dertouzos in 1978 [1]. Until 2009, Gentry presented a fully homomorphic encryption scheme based on ideal lattice [2]. Bootstrapping regard as Gentry’s breakthrough is a homomorphic evaluation of decryption which transforms the homomorphic encryption with limited homomorphic capability into the FHE scheme that can evaluate any circuits homomorphically.

The first FHE over the integers was proposed by van Dijk et al. [3] at Eurocrypt 2010, called DGHV. The security of DGHV relies on the hardness of the Approximate Greatest Common Divisor problem (AGCD) and the Sparse Subset Sum problem (SSSP). Several works have dramatically improved the efficiency and the hardness assumption needed to implement it, including [4–9]. Some of schemes above are leveled FHE scheme, but there essentially follow Gentry’s blueprint.

In DGHV scheme, the message space is binary space. It is easy to extend DGHV scheme with bigger message space such as the message \(m \in \mathbb{Z}_Q\), namely the ciphertext \(c = pq + Qe + m\) or \(c = p^2q + \lfloor \frac{Q}{p} \rfloor (m + Qr)\) according to the schemes [3, 7, 9, 11], where integer \(p\) is the secret key, and \(q, r, e\) are uniform randomly chosen integers from some prescribed intervals. Here the bootstrapping procedure is homomorphically evaluate the decryption

\[
m = c \mod p \mod Q = c - p[c/p] \mod Q = (c - p \left( \sum_{i=1}^{\epsilon} s_i z_i \right)) \mod Q
\]
where the complicated division by \( p \) is replace by using the hardness assumption of the SSSP: \( 1/p \approx \sum_{i=1}^{\Theta} s_i y_i \mod Q \), for secret key \( s = (s_1, \ldots, s_\Theta) \in \{0, 1\}^\Theta \) is \( \Theta \) length vector with hamming weight \( \theta = \lambda \), and \( z_i = (z_{i,0}, z_{i,-1}, \ldots, z_{i,-L})Q < Q \) is a real number with \( L = [\log Q\lambda] + 2 \) bits of precision after the \( Q \)-ary point, satisfying \( \sum_{i=1}^{\Theta} s_i z_i \approx c/p \).

At Eurocrypt 2015, Nuida and Kurosawa [8] proposed a polynomial of multiplicative degree of \( Q \) yielding the carry \( a \) in the procedure \( x + y = aQ + b \) for any \( x, y \in \mathbb{Z}_Q \), called \( Q \)-ary half adder. Then they expressed the decryption as \( s \mod Q \) arithmetic circuit of multiplicative degree \( Q^4 \).

In 2017, Cheon et al. [12] presented a faster bootstrapping method by using a homomorphic encryption scheme with various message spaces and homomorphic digit extraction technique. The degree of the decryption is \( O(1+\varepsilon) \), and the number of homomorphic multiplications is \( O(\log_2 \log Q) \), where \( \varepsilon \) is some small constant (be affected by the modulus \( Q \)).

However, Both of the above bootstrapping procedure introduced in [8, 12] only support the message space \( Q \) size of a constant number. For \( Q > 86^2 \), Cheon and Kim [13] expressed the decryption circuit as an \( \mathcal{L} \)-restricted depth-3 \( (\sum \prod \sum) \) circuit by the technique in [14]. The \( \mathcal{L} \)-degree is at most \( 86^2 \) and the number of product gates is at most \( 8\Theta^2 + \Theta + 1 \). As we know, \( \Theta = \omega(\lambda^6 \log \lambda) \) in [3] and is reduced to \( O(\lambda^3) \) in [4].

In 2018, Lian et al. [15] presented a bootstrapping algorithm of FHE over the integers with a large message space. In bootstrapping procedure, they using ciphertext of homomorphic encryption scheme with message space \( \mathbb{Z}_Q \) which is encryption for bit message. And they apply \( \text{mod-Q} \) arithmetic circuit to simulate the bit operations in its decryptionThey use Lagrange interpolating polynomial to reduce the number of simulating the bit XOR operation (it take \( \text{mod-Q} \) multiplier gates, which It will cause the multiplicative degree of decryption to grow rapidly). The decryption circuit is expressed as a polynomial of multiplicative degree \( 108 \theta \log^3 \theta \). It is clearly that the complexity of the decryption is independent of the message space size \( Q \) if \( Q > \theta \).

One problem in [15] is that the constant factor of the multiplicative degree of decryption \( 108 \) is bigger than the parameter \( \theta \). the parameter \( \theta \) is the hamming weight of the secret key \( sk \). It requires that \( \left( \frac{\Theta}{\theta/2} \right) \geq 2^\lambda \) to avoid an attack on the SSSP [16]. In [3, 15], \( \theta \) is set to be the security parameter \( \lambda \), namely \( \theta = \lambda \). While in [4], \( \theta \) can be chose smaller as long as the hardness assumption of SSSP hold, namely \( \theta = \frac{\log X}{\log \lambda} \). If we set the security parameter \( \lambda = 72 \), then \( \theta = \frac{\log X}{\log 72} = 15 \). It is easy to see that the constant factor in the multiplicative degree of decryption circuit is much bigger than \( \theta \). So, it requires to seek a method to reduce the measure of he constant factor.

In this paper, we modify the bootstrapping procedure presented in [15]. After using use Lagrange interpolating polynomial to obtain the hamming weight of some vectors in bits, we convert the result into a depth-3 circuit, and then, we apply the technique of [14] replace the 3-for-2 trick over \( \mathbb{Z}_Q \) in [15]. Finally, we express the decryption as a polynomial of degree \( \theta^2 \log^2 \theta/2 \). Compared to
Lian et al.’s bootstrapping algorithm, we get a quadratic function in security parameter, at first sight, it is worse complexity of decryption. In fact that, we improved it about more than 60 factor. For instance, if $\lambda = 72$, $\theta = 15$, our degree of decryption circuit is 1350, while [15] the one is 103680.

2 Preliminaries

2.1 Notations

For a real number $z$, we denote by $\lceil z \rceil$, $\lfloor z \rfloor$, $\lceil z \rceil$ the rounding of a up, down, or to the nearest integer. For integers $m$, $n$, we denote the integer sets $\{m, m+1, \ldots, n\}$ and $\{m, m+1, \ldots, n-1\}$ by $[m, n]$, and $[m, n]$, respectively. For a real number $r$, we use $r = (\ldots, r_{-2}, r_{-1}, r_0, r_1, \ldots)$ to denote the $Q$-ary representation of $r$ with $n$ bits of precision after the $Q$-ary point. When $Q = 2$, it denotes the binary representation of $r$. Given $x, p \in \mathbb{R}$, we let $[x]_p$ denote the unique number in $(-p/2, p/2]$ that is congruent to $x$ mod $p$. All logarithms in the text are base-2 unless stated otherwise.

2.2 Lagrange Interpolating Polynomial

The Lagrange interpolating polynomial is the polynomial $f(x)$ of degree $n-1$ that passes through the $n$ points $\{(x_0, y_0 = f(x_0)), \ldots, (x_{n-1}, y_{n-1} = f(x_{n-1}))\}$, and given by $f(x) = \sum_{j=0}^{n-1} f_j(x)$, where

$$f_j(x) = y_j \prod_{0 \leq k \leq n-1, k \neq j} \frac{x - x_k}{x_j - x_k} \mod Q.$$

Our goal of introducing the Lagrange interpolation polynomial is to obtain the mod-$Q$ arithmetic polynomial expression of computing any bit in the binary representation of the integer $x \in [0, \theta]$. For every integer $b \in [0, \theta]$, let $b = (b_{n-1}, \ldots, b_0)_2$, where $n = \lceil \log_\theta \rceil$. For each index $t \in [0, n-1]$, we construct a set consisting of integer $b$ and its $t$-th bit $b_t$, where $b = 0, 1, \ldots, \theta$, namely, denote the set as $\{(b, b_t)\}_{b \in [0, \theta]}$ for each $t$. So for each index $t \in [0, n-1]$, the $\theta + 1$ points set is

$$\{(x_b = b, y_b = b_t)\}_{b \in [0, \theta]} = \{(x_0 = 0, y_0 = 0_t), (x_1 = 1, y_1 = 1_t), \ldots, (x_\theta = \theta, y_\theta = \theta_t)\}.$$

If the variable $x$ equates to an integer $b \in [0, \theta]$, for the index $t \in [0, n-1]$, the output of the Lagrange interpolating polynomial

$$F_t(x) = \sum_{j=0}^{\theta} y_j \prod_{0 \leq k \leq \theta, k \neq j} \frac{x - x_k}{x_j - x_k} \mod Q$$

is $y_b$, which equates to the $t$-th bit in the binary representation of $x$. The multiplicative degree of the mod-$Q$ arithmetic circuit is $\theta$. 
2.3 Restricted Depth-3 Arithmetic Circuits

In [14], Gentry and Halevi show how to express the decryption function of Gentry’s FHE in [2] as a restricted depth-3 circuit over a large enough ring. Here we will show to express the decryption function of FHE over the integers with a large enough message space as a restricted depth-3 circuit. As mention in [14], By “restricted” means that the bottom sums in depth-3 circuit must come from a fixed (polynomial-size) set $L$ of polynomials, where $L$ is independent of the ciphertext.

**Definition 1.** *(Restricted Depth-3 Circuit)* Let $L = \{L_j(x_1, \cdots, x_n)\}$ be a set of polynomial, all in the same $n$ variables. An arithmetic circuit $C$ is a $L$-restricted depth-3 circuit over $(x_1, \cdots, x_n)$ if there exists multisets $S_1, \cdots, S_t \subseteq L$ and constants $\lambda_0, \cdots, \lambda_t$ such that

$$C(\vec{x}) = \lambda_0 + \sum_{i=1}^{t} \lambda_i \cdot \prod_{L_j \in S_i} L_j(x_1, \cdots, x_n)$$

The degree of $C$ with respect to $L$ is $d = \max_i |S_i|$ (we also call it the $L$-degree of $C$).

**Remark 1.** In all our instantiations of decryption circuits for FHE schemes over the integers, the $L_j$’s happen to be linear.

The following lemma 1 states Ben-or’s observation that multilinear symmetric polynomials can be computed by restricted depth-3 arithmetic circuits that perform interpolation. Here recall that a multilinear symmetric polynomials $M(\vec{x})$ is a symmetric polynomial where, for each $i$, every monomial is of degree at most 1 in $x_i$; there are no high powers of $x_i$.

**Lemma 1.** [14]: Let $Q > t^{2^n}$ be a prime, let a set $A \subset \mathbb{Z}_Q$ have cardinality $t^{2^n} + 1$, let $\vec{x} = (x_1, \ldots, x_{t^{2^n}+1})$ be variables, denote $\mathcal{L}_A = \{(a + x_i : a \in A, 1 \leq i \leq t^{2^n} + 1)\}$. For every multilinear symmetric polynomial $M(\vec{x})$ over $\mathbb{Z}_Q$, there is a circuit $C(\vec{x})$ such that:

- $C$ is a $\mathcal{L}_A$-restricted depth-3 circuit over $\mathbb{Z}_Q$ such that $C(\vec{x}) \equiv M(\vec{x}) \mod Q$.
- $C$ has $t^{2^n} + 1$ product gates of $\mathcal{L}_A$-degree $t^{2^n}$, one gate for each value $a_j \in A$, with the $j$-th gate compute the value $\lambda_j \prod_i (a_j + x_i)$ for some constant $\lambda_j$.
- A description of $C$ can be compute efficiently given the values $M(\vec{x})$ at all $\vec{x} = 1^{t^{2^n}+1}$.

The final bullet clarifies that Ben-or’s observation is constructive, we can compute the restricted depth-3 representation from any initial representation that lets us evaluate $M$.

3 Bootstrapping the Decryption

This section deals mainly with how to implement the decryption $m \leftarrow (c \mod p) \mod Q$ with a mod-$Q$ arithmetic circuit of a low degree.
3.1 Squashing the Decryption with SSSP Assumption

The decryption circuit is

\[ m \leftarrow (c \mod p) \mod Q = c - p\lfloor c/p \rfloor \mod Q. \]

Let \( y \) be a vector of \( \Theta \) rational number in \([0, Q]\) with \( \kappa \) bits of precision after the binary point, and let \( \mathbf{sk} = (s_1, s_2, \ldots, s_\Theta) \) be the secret key vector of \( \Theta \) bits with hamming weight \( \theta \) such that \( 1/p = (\mathbf{sk}, y) \mod Q \), where \(|c| < 2^{-\kappa}\).

We firstly compute \( z_i = [c \cdot y_i] \), keeping only \( n = \lceil \log(\theta + 1) \rceil \) bits of precision after the binary point for \( i = 1, 2, \ldots, \Theta \).

Therefore we have

\[ m \leftarrow (c \mod p) \mod Q = c - \left( \sum_{i=1}^{\Theta} s_i z_i \right) \mod Q \]

For \( i \in [1, \Theta] \), let \( z_i = z_i' + z_i'' \cdot 2^{-n} \mod Q \), where \( z_i' \in [0, Q) \) is the integer part of \( z_i \) and \( z_i'' < 2^n \) is the fractional part. Then we have

\[ m \leftarrow (c \mod p) \mod Q = c - \left( \sum_{i=1}^{\Theta} s_i (z_i' \mod Q) - [2^{-n} \sum_{i=1}^{\Theta} s_i z_i''] \right) \mod Q. \]

3.2 Bootstrapping

For the integer part, it is easy to compute \( (\sum_{i=1}^{\Theta} s_i z_i' \mod Q) \mod Q \) by using some mod-Q addition gates and multiplication-by-constant gates.

For the fractional part, in order to compute \( [2^{-n} \sum_{i=1}^{\Theta} s_i z_i''] \mod Q \), here we firstly denote the binary represent of \( z_i'' \) as \((z_{i,n-1}, \ldots, z_{i,1}, z_{i,0})_2\).

(1) \[ 2^{-n} \sum_{i=1}^{\Theta} s_i z_i'' = 2^{-n} \sum_{i=1}^{\Theta} s_i \sum_{j=0}^{n-1} z_{i,j}' 2^j = \sum_{j=0}^{n-1} 2^j \sum_{i=1}^{\Theta} s_i z_{i,j}' \]

Let \( n \) integer numbers \( \{W_j\}_{j \in [1, n]} \) such that \( W_j = \sum_{i=1}^{\Theta} s_i z_{i,j}' \), namely \( W_j \) is the hamming weight of the vector \((s_1 z_{1,j}', \ldots, s_\Theta z_{\Theta,j}')_2\). We can using mod-Q addition gates to sum-up directly the hamming weight.

(2) Since the hamming weight of the secret key vector \( \mathbf{sk} \) is \( \theta \), then \( W_j \) is not bigger than \( \theta \), i.e. \( W_j \leq \theta \). Let \( W_j = (w_{j,n}, \ldots, w_{j,1})_2 \). By the Lagrange interpolating polynomial, for \( 1 \leq t, j \leq n \), the bit values \( w_{j,t} = F_t(W_j) \), where the multiplicative degree of Lagrange interpolating polynomial \( F_t \) is \( \theta \).

(3) Now

\[  \sum_{j=0}^{n-1} 2^{j+n} W_j = \sum_{j=0}^{n-1} \sum_{t=0}^{n-1} (2^{j+t-n} \cdot w_{j,t}) = \sum_{j+t \geq n} 2^{j+t-n} w_{j,t} + 2^{-n} \sum_{j+t < n} 2^{j+t} w_{j,t}, \]

thus we have

\[ [2^{-n} \sum_{i=1}^{\Theta} s_i z_i''] \mod Q = \sum_{j+t \geq n} 2^{j+t-n} w_{j,t} + [2^{-n} \sum_{j+t < n} 2^{j+t} w_{j,t}] \mod Q \]
It is easy to compute $\sum_{j+t \geq n} 2^{j+t-n} w_{j,t}$ by using some mod-$Q$ addition gates and multiplication-by-constant gates. We now show how can compute $[2^{-n} \sum_{j+t < n} 2^{j+t} w_{j,t}] \mod Q$ in equation(2) using a $L_A$-restricted circuit.

Lemma 2. [14] Let $Q$ be a prime with $Q > 2^n$, there is a univariate polynomial $f(x)$ of degree $\log_2 \frac{n(n+1)}{2}$ such that $f(\sum_{i=0}^{t-1} w'_i u_i) = [2^{-n} \sum_{i=0}^{t-1} w'_i u_i] \mod Q$, where all $|u_i| < 2^n$.

Lemma 3. Let $t$ be positive integer and $f(x)$ a univariate polynomial over $\mathbb{Z}_Q$ (for $Q$ prime, $Q > 2^n$). Then there is a multilinear symmetric polynomial $M_f$ on $2^n$ variables such that

$$f(\sum_{i=0}^{t-1} w'_i u_i) = M_f(w'_0, \ldots, w'_{t-1}, 0, \ldots, 0)$$

for all $w'_i \in \{0, 1\}$, and $u_i \in [0, 2^n)$.

By Lemma 2, if $Q > \frac{n(n+1)}{2} \cdot 2^n$, there is a multilinear symmetric polynomial $M_f(\cdot)$ to compute the function $[2^{-n} \sum_{j+t < n} 2^{j+t} w_{j,t}] \mod Q$. Then by lemma 1, this multilinear symmetric polynomial $M_f(\cdot)$ can be expressed as $L_A$-restricted depth-3 circuit $C$ over $\mathbb{Z}_Q$ of degree at most $\frac{n(n+1)}{2} \cdot 2^n$, having at most $\frac{n(n+1)}{2} \cdot 2^n + 1$ product gates.

Thus we obtain the following results: If $Q$ be primes such that $Q > \frac{n(n+1)}{2} \cdot \theta$, the degree of the polynomial in the first step is 1, the degree of the polynomial in the second step is at most $\theta$, the degree of the polynomial in the third step is $\frac{n(n+1)}{2} \cdot \theta$.

Therefore the total degree of the decryption circuit over $\mathbb{Z}_Q$ is bounded by $\theta^2 \frac{n(n+1)}{2} \approx \theta^3 \log^2 \theta / 2$. The number of product gates is $\theta \cdot n + \left(\frac{n(n+1)}{2} \cdot \theta + 1\right)$. We notes that the complexity of decryption circuit is independent of $Q$ except a constraint $Q > \frac{n(n+1)}{2} \cdot \theta = \theta \log^2 \theta / 2$

4 Conclusion

We combine the techniques in [15] and [14], such as Lagrange interpolate and the trick to convert a rounding function into restricted depth-3 circuit, reduce the complexity of decryption circuit. The multiplicative degree of the decryption is $\theta^3 \log^2 \theta / 2$, and the multiplication gates is $O(\theta \log^2 \theta)$. Then we get a faster bootstrapping procedure than the one proposed in [15]. Table 1 shows that

Asymptotically, the parameter $\theta$, the hamming weight of the secret key vector, can be made as small as $\theta(\lambda/\log \lambda)(\text{see e.g. [7])}$. so we can set it to be $\lambda / \log \lambda$, rounded up to the next power of two minus one. For $\lambda = 72$, we have $\lambda / \log \lambda \approx 11.7$, so we set $\theta = 15$. Then the degree of the decryption circuit is $\theta^2 \frac{n(n+1)}{2} = 2250$, while the one in our last work is $108 \cdot \theta \log^3 \theta = 103680$.

The number of product gates is $\theta \cdot n + \left(\frac{n(n+1)}{2} \cdot \theta + 1\right) = 211$, while the one in last work is $\theta n' + 8n'^2 + 4n' + 9 = 534 (n' = \log \theta + 3)$. 

Table 1. Complexity of Decryption

<table>
<thead>
<tr>
<th>$Q$</th>
<th>multiplicative degree</th>
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<tbody>
<tr>
<td>$[15]$ $Q &gt; \theta$</td>
<td>$108\theta\log^3\theta$</td>
</tr>
<tr>
<td>$\lambda = 72, \theta = 15$ $Q &gt; 15$</td>
<td>$103680$</td>
</tr>
<tr>
<td>this paper $Q &gt; \theta\log^3\theta/2$</td>
<td>$\theta^2\log^3\theta/2$</td>
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<tr>
<td>$\lambda = 72, \theta = 15$ $Q &gt; 90$</td>
<td>$1350$</td>
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References