

Uncloneable Quantum Encryption via Random Oracles

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Abstract. Quantum information is well-known to achieve cryptographic feats that are unattainable using classical information alone. Here, we add to this repertoire by introducing a new cryptographic functionality called *uncloneable encryption*. This functionality allows the encryption of a classical message such that two collaborating but isolated adversaries are prevented from simultaneously recovering the message, even when the encryption key is revealed. Clearly, such functionality is unattainable using classical information alone.

We formally define uncloneable encryption, and show how to achieve it using Wiesner’s conjugate coding, combined with a quantum-secure pseudorandom function (qPRF). Modelling the qPRF as a quantum random oracle, we show security by adapting techniques from the quantum one-way-to-hiding lemma, as well as using bounds from quantum monogamy-of-entanglement games.

1 Introduction

One of the key distinctions between classical and quantum information is given by the *no-cloning principle*: unlike bits, arbitrary qubits cannot be perfectly copied [11, 18, 25]. This principle is the basis of many of the feats of quantum cryptography, including quantum money [24] and quantum key distribution (QKD) [6] (for a survey on quantum cryptography, see [9]).

In QKD, two parties establish a shared secret key, using public quantum communication combined with an authentic classical channel. The quantum communication allows to *detect* eavesdropping: when the parties detect only a small amount of eavesdropping, they can produce a shared string that is essentially guaranteed to be private. Gottesman [15] studied *quantum tamper-detection* in the case of *encryption schemes*: in this work, a classical message is encrypted into a quantum ciphertext such that, at decryption time, the receiver will *detect* if an adversary could have information about the plaintext when the key is revealed. We note that classical information alone cannot produce such encryption schemes, since it is always possible to perfectly *copy* ciphertexts.

Notably, Gottesman left open the question of an encryption scheme that would *prevent* the *splitting* of a ciphertext. In other words, would it be possible to encrypt a classical message into a quantum ciphertext, such that no attack at

the ciphertext level would be significantly successful in producing *two* quantum registers, each of which, when combined with the encryption key, could be used to reconstruct the plaintext?

In this work, we define, construct and prove security for a scheme that answers Gottesman’s question in the positive. We call this *uncloneable encryption*. The core technical aspects of this work were first presented in one of the author’s M.Sc. thesis [16].

1.1 Summary of Contributions

We consider encryption schemes that encode classical plaintexts into quantum ciphertexts, which we formalize in [Definition 2](#). Next, we define uncloneable encryption ([Definition 6](#)). Informally, this can be thought of as a game, played between the sender (Alice) and two recipient (Bob and Charlie). First, Alice picks a message $m \in \{0, 1\}^n$ and key $k \in \{0, 1\}^{\kappa(\lambda)}$ (κ is a polynomial in some security parameter, λ). She encrypts her message into a quantum ciphertext register R . Initially, Bob and Charlie are physically together, and they receive R . They apply a quantum map to produce two registers: Bob keeps register B and Charlie keeps register C . Bob and Charlie are then isolated. In the next phase, Alice reveals k to both parties. Using k and their quantum register, Bob and Charlie produce m_B and m_C respectively. Bob and Charlie *win* if and only if $m_B = m_C = m$.

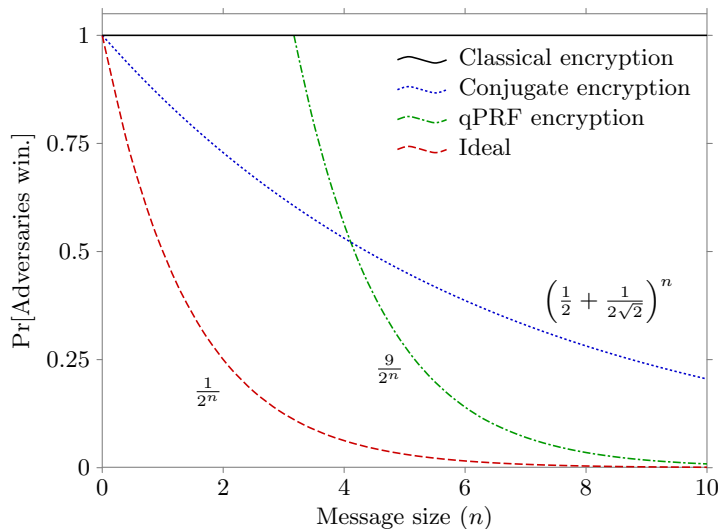


Fig. 1. Upper-bounds on winning probabilities for various types of encodings (up to negligible functions of λ) for messages sampled uniformly at random.

Assuming that Alice picks her message uniformly at random, our results are summarized in Fig. 1, where we plot upper bounds for the winning probability of Bob and Charlie against various types of encodings, according to the length of m . First of all, if the encoding is classical, then Bob and Charlie can each keep a copy of the ciphertext. Combined with the key k , each party decrypts to obtain m . This gives the horizontal line at $\Pr[\text{Adversaries win}] = 1$. Next, a lower bound on the winning probability for *any* encryption scheme is $\frac{1}{2^n}$ (corresponding to the parties coordinating a random guess). This is the *ideal* curve. Our goal is therefore to produce an encryption scheme that matches the ideal curve as close as possible.

It may seem that asking that Alice sample her message uniformly at random would be particularly restrictive, but this is not the case — we show in Theorem 2 that security in the case of uniformly sampled messages implies security in the case of non-uniformly sampled messages. Specifically, if Bob and Charlie can win with probability at most $2^{-n+t} + \eta(\lambda)$ when the message is sampled uniformly at random, for some t and some negligible function η , then they can win with probability at most $2^{-h+t} + \eta'(\lambda)$ if the message m is sampled from a distribution with a min-entropy of h . Note that η' is still a negligible function which, in general, is larger than η . Key to this reduction is the fact that the message length for the encryption schemes that we consider is fixed.

Our first attempt at realizing uncloneable encryption (Section 4.1) shows that the well-known Wiesner conjugate coding already achieves a security bound that is better than classical. For $r, \theta \in \{0, 1\}^n$, define the conjugate coding state $|x^\theta\rangle = H^{\theta_1}|x_1\rangle \otimes \dots \otimes H^{\theta_n}|x_n\rangle$. The encryption maps m into $\rho = |(m \oplus r)^\theta\rangle\langle(m \oplus r)^\theta|$. We sketch a proof that this satisfies a more usual definition of security for encryption schemes. The question of uncloneability then boils down to: “How well can an adversary *split* ρ into *two* registers, each of which, combined with (θ, r) can reconstruct m ?” This question is answered in prior work on *monogamy-of-entanglement games* [20]: a optimal strategy wins with probability $\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)^n$. This is again illustrated in Fig. 1.

In order to improve this bound, we use a quantum-secure pseudorandom function (qPRF) $f_\lambda : \{0, 1\}^\lambda \times \{0, 1\}^\lambda \rightarrow \{0, 1\}^n$ (see Definition 1). The encryption (see Section 4.2) consists in a quantum state $\rho = |r^\theta\rangle\langle r^\theta|$ for random $r, \theta \in \{0, 1\}^\lambda$, together with a classical string $c = m \oplus f_\lambda(s, r)$ for a random s . The key k consists in θ and s . Once again, it can be shown that this is an encryption scheme in a more usual sense and we sketch this argument in Section 4.2. Intuitively, the use of f_λ , affords us a gain in uncloneable security, because an adversary who wants to output m would need to know the pre-image of m under $f_\lambda(s, \cdot)$. Reaching a formal proof along these lines, however, is tricky. First, we model the qPRF using a quantum random oracle [8]; this limits the adversaries’ interaction with the qPRF to be black-box quantum queries. Next, the quantum random oracle model is notoriously tricky to use; in particular it is not possible to reason in terms of an adversary’s sequence of oracle calls, hence many of the techniques in the classical literature are not applicable. Fortunately, we can adapt techniques from Unruh’s quantum one-way-to-hiding lemma [21] to

the two-player setting, which enables us to recover a precise statement along the lines of the intuition above. We thus complete the proof of our main [Theorem 5](#), obtaining the bound $9 \cdot \frac{1}{2^n} + \text{negl}(\lambda)$. This is the fourth and final curve in [Fig. 1](#).

In addition to the above, we formally define a different type of uncloneable security: inspired by more standard security definitions of *indistinguishability*, we define *uncloneable-indistinguishability* ([Definition 8](#)). This security definition bounds the advantage that the adversaries have at *simultaneously* distinguishing between an encryption of 0^n and an encryption of a plaintext of length n , as prepared by the adversaries. In a series of results ([Theorems 3 and 6](#) and [Corollary 2](#)), we show that our main protocol achieves this security notion, but only against adversaries that use *unentangled strategies*.

We note that our protocols (both [Definition 9](#) and [Definition 10](#)) have the desirable property of being *prepare-and-measure* schemes. This means that the quantum technology for the honest users is limited to the preparation of single-qubit pure states, as well as to single-qubit measurements; these quantum technologies are mature and commercially available. (Note, however, that quantum storage remains a major challenge at the implementation level).

1.2 Application

As an application of uncloneable encryption, consider an untrusted cloud storage service. According to our encryption scheme (and as long as the cloud is a *quantum* cloud), Alice can store her data remotely in encrypted form, and have an *a priori* guarantee that, in the event that her private key is *completely* leaked, the damage caused by a malicious cloud would be contained. This is due to the security guarantee, according to which at most a *single* party could reconstruct the plaintext, given the key. Note that this application is a true prepare-and-measure scheme, since all storage is delegated to the cloud.

1.3 More on Related Work

Tamper-Evident Encryption. We referred above to what we called *tamper-evident encryption* [[15](#)]. However, we emphasize that the author originally called this contribution *uncloneable encryption*. We justify this choice of re-labelling in quoting the conclusion of the work:

One difficulty with such generalizations is that it is unclear to what extent the name “uncloneable encryption” is really deserved. I have not shown that a message protected by uncloneable encryption cannot be copied — only that Eve cannot copy it without being detected. Is it possible for Eve to create two states, (...), which can each be used (in conjunction with the secret key) to extract a good deal of information about the message? Or can one instead prove bounds, for instance, on the sum of the information content of the various purported copies? [[15](#)]

Since our work addresses this question, we have appropriately re-labeled prior work according to a seemingly more accurate name.

Quantum Copy-Protection. Further related work includes *quantum copy-protection*, as initiated by Aaronson [1]. Informally, this is a means to encode a function (from a given family of functions) into a quantum program state, such that an honest party can evaluate the function given the program state, but it would be impossible to somehow *split* the quantum program state so as to enable *two* parties to simultaneously evaluate the function. Aaronson gave protocols for quantum copy-protection in an oracle model, but left wide open the question of quantum copy-protection in the plain model. In a way, our uncloneable encryption is a first step towards quantum copy-protection, since it prevents copying of *data*, which can be seen as a unit of information that is even simpler than a function.

Quantum Key Recycling. The concept of quantum key recycling is a precursor to the QKD protocol, developed by Bennett and Brassard and Breidbart [7] (the manuscript was prepared in 1982 but only published recently). According to this protocol, it is possible to encrypt a classical message into a quantum state, such that information-theoretic security is assured, but in addition, a tamper detection mechanism would allow the one-time pad key to be re-used in the case that no eavesdropping is detected. Quantum key recycling has been the object of recent related work [10, 13]. To the best of our knowledge, the relationship between quantum key-recycling, tamper-evident encryption, and uncloneable encryption is unknown (see Section 1.4).

1.4 Outlook and Future Work

In this work, we show that, thanks to quantum information, one of the basic tacit assumptions of encryption, namely that an adversary can copy ciphertexts, is challenged. We believe that this has the potential to significantly change the landscape of cryptography, for instance in terms of techniques for *key management* [4]. Furthermore, our techniques could become building blocks for a theory of uncloneable cryptography.

Our work leads to many follow-up questions, broadly classified according to the following themes:

Improvements. There are many possible improvements to the current work. For instance: Could our scheme be made resilient to errors? Can we remove the reliance on the random oracle, and/or on the qPRF? Could an encryption scheme simultaneously be uncloneable *and* provide *tamper detection*? Would achieving uncloneable-indistinguishable security be possible, without any restrictions on the adversary’s strategy?

Links with related work. What are the links, if any, between uncloneable encryption, tamper-evident encryption [15], and quantum encryption with key recycling [7, 10, 13]? We note that both uncloneable encryption and quantum encryption with key recycling [13] make use of theorems developed in the context of one-sided device-independent QKD [20]. Can we make more formal links between these primitives?

More uncloneability. Finally, our work paves the way for the study more complex uncloneable primitives. Could this lead to uncloneable programs [1]? What about in complexity theory, could we define and realize uncloneable *proofs* [1]?

1.5 Outline

The remainder of the paper is structured as follows. In [Section 2](#), we introduce some basic notation and useful results from the literature. In [Section 3](#), we formally define uncloneable encryption schemes and their security. Our two protocols are described and proved secure in [Section 4](#).

2 Preliminaries

In this section, we present basic notation, together with techniques from prior work that are used in the remainder of the paper.

2.1 Notation and Basics of Quantum Information

We denote the set of all functions of the form $f : \{0, 1\}^n \rightarrow \{0, 1\}^m$ by $\text{Bool}(n, m)$.

We overload the expectation symbol \mathbb{E} in the following way. If X is a finite set and $f : X \rightarrow \mathbb{R}$ is some function, we define

$$\mathbb{E}_x f(x) = \frac{1}{|X|} \sum_{x \in X} f(x). \quad (1)$$

In other words, $\mathbb{E}_x f(x) = \mathbb{E} f(\mathcal{X})$ if \mathcal{X} is the random variable which is uniformly distributed over X . Our overloaded notation avoids having to formally define this random variable.

A comprehensive introduction to quantum information and quantum computing may be found in [17, 23]. We fix some notation in the following paragraphs.

Let $\mathcal{Q} = \mathbb{C}^2$ be the state space of a single qubit. In particular, \mathcal{Q} is a two-dimensional complex Hilbert space spanned by the orthonormal set $\{|0\rangle, |1\rangle\}$. For any $n \in \mathbb{N}^+$, we write $\mathcal{Q}(n) = \mathcal{Q}^{\otimes n}$ and note that

$$\{|s\rangle = |s_1\rangle \otimes |s_2\rangle \otimes \dots \otimes |s_n\rangle\}_{s \in \{0,1\}^n} \quad (2)$$

forms an orthonormal basis of $\mathcal{Q}(n)$.

Let \mathcal{H} be a Hilbert space. The set of all unitary and density operators on \mathcal{H} are denoted by $\mathcal{U}(\mathcal{H})$ and $\mathcal{D}(\mathcal{H})$, respectively. We recall that the operator norm of any linear operator $A : \mathcal{H} \rightarrow \mathcal{H}$ between finite dimensional Hilbert spaces is given by

$$\|A\| = \max_{\substack{v \in \mathcal{H} \\ \|v\|=1}} \|Av\| \quad (3)$$

and satisfies the property that $\|Av\| \leq \|A\| \cdot \|v\|$. If A is either a projector or a unitary operator, then $\|A\| = 1$.

We use the term “quantum state” to refer to both unit vectors $|\psi\rangle \in \mathcal{H}$ and to density operators $\rho \in \mathcal{D}(\mathcal{H})$ on some Hilbert space.

If $H \in \mathcal{U}(\mathcal{Q})$ is the Hadamard operator defined by

$$|0\rangle \mapsto \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \text{and} \quad |1\rangle \mapsto \frac{|0\rangle - |1\rangle}{\sqrt{2}} \quad (4)$$

then, for any strings $x, \theta \in \{0, 1\}^n$, we define

$$|x^\theta\rangle = H^{\theta_1} |x_1\rangle \otimes H^{\theta_2} |x_2\rangle \otimes \dots \otimes H^{\theta_n} |x_n\rangle \quad (5)$$

and note that $\{|s^\theta\rangle\}_{s \in \{0,1\}^n}$ forms an orthonormal basis of $\mathcal{Q}(n)$. Following their prominent use in [24], we call states of the form $|x^\theta\rangle$ *Wiesner states* and, for any fixed $\theta \in \{0, 1\}^n$, we call $\{|s^\theta\rangle\}_{s \in \{0,1\}^n}$ a *Wiesner basis*.

For any $n \in \mathbb{N}^+$, we define the Einstein-Podolski-Rosen [12] (EPR) state by

$$|\text{EPR}_n\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle |x\rangle \quad (6)$$

and note that it is an element of $\mathcal{Q}(2n)$.

We also recall that physically permissible transformation of a quantum system precisely coincide with the set of completely positive trace preserving (CPTP) maps. In particular, CPTP map will map density operators to density operators.

An efficient quantum circuit $\mathcal{C} = \{\mathcal{C}_\lambda\}_{\lambda \in \mathbb{N}^+}$ is a collection of quantum circuits indexed by \mathbb{N}^+ such that there exists a polynomial-time deterministic Turing machine T which, on input 1^λ , produces a description of \mathcal{C}_λ . Each circuit \mathcal{C}_λ defines and implements a certain CPTP map $C_\lambda : \mathcal{D}(\mathcal{H}_{\text{In},\lambda}) \rightarrow \mathcal{D}(\mathcal{H}_{\text{Out},\lambda})$, where the Hilbert spaces $\mathcal{H}_{\text{In},\lambda}$ and $\mathcal{H}_{\text{Out},\lambda}$ are implicitly defined by the circuit. Note that we consider general, which is to say possibly non-unitary, circuits. These were introduced in [2]. It is worth noting that a universal gate set for general quantum circuits exists which is composed of only unitary gates, implementing maps of the form $\rho \mapsto U\rho U^\dagger$ for some unitary operator U , and two non-unitary maps which are

- the single qubit partial trace map $\text{Tr} : \mathcal{D}(\mathcal{Q}) \rightarrow \mathcal{D}(\mathbb{C})$ and
- the state preparation map $\text{Aux} : \mathcal{D}(\mathbb{C}) \rightarrow \mathcal{D}(\mathcal{Q})$ defined by $1 \mapsto |0\rangle\langle 0|$.

Further information on the circuit model of quantum information may be found in [22].

2.2 Monogamy of Entanglement Games

Monogamy-of-entanglement games were introduced and studied in [20]. In short, a monogamy-of-entanglement game is played by Alice against cooperating Bob and Charlie. Alice describes to Bob and Charlie a collection of different POVMs which she could use to measure a quantum state on a Hilbert space \mathcal{H}_A . These POVMs are indexed by a finite set Θ and each reports a measurement result

taken from a finite set X . Bob and Charlie then produce a state $\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$, giving the A register to Alice, the B register to Bob and the C register to Charlie. Alice then picks a $\theta \in \Theta$, measures her subsystem with the corresponding POVM and obtains some result $x \in X$. She then announces θ to Bob and Charlie who are now isolated. Bob and Charlie win if and only if they can both simultaneously guess the result x .

Upper bounds on the winning probability of Bob and Charlie in such games was the primary subject of study in [20]. One of their main result, corresponding to a game where Alice measures in a random Wiesner basis, is as follows.

Theorem 1 ([20]). *Let $\lambda \in \mathbb{N}^+$ be an integer. For any Hilbert spaces \mathcal{H}_B and \mathcal{H}_C , any collections of POVMs*

$$\left\{ \{B_x^\theta\}_{x \in \{0,1\}^\lambda} \right\}_{\theta \in \{0,1\}^n} \quad \text{and} \quad \left\{ \{C_x^\theta\}_{x \in \{0,1\}^\lambda} \right\}_{\theta \in \{0,1\}^n} \quad (7)$$

on these Hilbert spaces, and any state $\rho \in \mathcal{D}(\mathcal{Q}(\lambda) \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$, we have that

$$\mathbb{E}_\theta \sum_{x \in \{0,1\}^\lambda} \text{Tr} [(|x^\theta\rangle\langle x^\theta| \otimes B_x^\theta \otimes C_x^\theta) \rho] \leq \left(\frac{1}{2} + \frac{1}{2\sqrt{2}} \right)^\lambda. \quad (8)$$

Using standard techniques, we can recast this theorem in a context where Alice sends to Bob and Charlie a random Wiesner state and Bob and Charlie split this state among themselves via some CPTP map Φ .

Corollary 1. *Let $\lambda \in \mathbb{N}^+$ be an integer. For any Hilbert spaces \mathcal{H}_B and \mathcal{H}_C , any collections of POVMs*

$$\left\{ \{B_x^\theta\}_{x \in \{0,1\}^\lambda} \right\}_{\theta \in \{0,1\}^n} \quad \text{and} \quad \left\{ \{C_x^\theta\}_{x \in \{0,1\}^\lambda} \right\}_{\theta \in \{0,1\}^n} \quad (9)$$

on these Hilbert spaces, and any CPTP map $\Phi : \mathcal{D}(\mathcal{Q}(\lambda)) \rightarrow \mathcal{D}(\mathcal{H}_B \otimes \mathcal{H}_C)$, we have that

$$\mathbb{E}_\theta \mathbb{E}_x \text{Tr} [(B_x^\theta \otimes C_x^\theta) \Phi (|x^\theta\rangle\langle x^\theta|)] \leq \left(\frac{1}{2} + \frac{1}{2\sqrt{2}} \right)^\lambda. \quad (10)$$

The proof is relegated to [Appendix A](#), but conceptually follows from a two-step argument. First, we only consider states of the form $(\mathbb{1} \otimes \Phi) |\text{EPR}_\lambda\rangle\langle \text{EPR}_\lambda|$ for some CPTP map Φ and where Alice keeps the intact subsystems from the EPR pairs. Then, we apply the correspondence between Alice measuring her half of an EPR pair in a random Wiesner basis and her sending a random Wiesner state. This correspondence is similar to the one used in the Shor-Prekill proof of security for the BB84 QKD protocol [19].

[Corollary 1](#) can be seen as the source of “unclonability” for our upcoming protocols. When Alice sends a state $|x^\theta\rangle\langle x^\theta|$, picked uniformly at random, to Bob and Charlie, she has a guarantee that it is unlikely for both of them to learn x even if she later divulges θ .

It is worth noting that [Theorem 1](#) and [Corollary 1](#) have no computational or hardness assumptions. This makes them an ideal corner stone on which to build uncloneable encryption.

2.3 Oracles and Quantum-Secure Pseudorandom Functions

A quantum-secure pseudorandom function is a keyed function which appears random to an efficient quantum adversary who only sees its input/output behaviour and is ignorant of the particular key being used. We formally define this notion with the help of oracles. Quantum accessible oracles have been previously studied in the literature, for example in [5, 8, 21].

Given a function $H \in \text{Bool}(n, m)$, a quantum circuit \mathbb{C} is said to have oracle access to H , denoted \mathbb{C}^H , if we add to its gate set a gate implementing the unitary operator $O^H \in \mathcal{U}(\mathcal{Q}(n)_Q \otimes \mathcal{Q}(m)_R)$ defined on computational basis states by

$$|x\rangle_Q \otimes |y\rangle_R \mapsto |x\rangle_Q \otimes |y \oplus H(x)\rangle_R . \quad (11)$$

Colloquially, we are giving \mathbb{C} a “block box” which essentially computes the function H . Note that if $H, H' \in \text{Bool}(n, m)$ are two functions, we can obtain the circuit $\mathbb{C}^{H'}$ from \mathbb{C}^H by replacing every instance of the O^H gate by the $O^{H'}$ gate.

We can now give a definition, inspired by the one in [26], of a quantum-secure pseudorandom function.

Definition 1 (Quantum-Secure Pseudorandom Function). *A quantum-secure pseudorandom function \mathcal{F} is a collection of functions*

$$\mathcal{F} = \left\{ f_\lambda : \{0, 1\}^\lambda \times \{0, 1\}^{\ell_{In}(\lambda)} \rightarrow \{0, 1\}^{\ell_{Out}(\lambda)} \right\}_{\lambda \in \mathbb{N}^+} \quad (12)$$

where $\ell_{In}, \ell_{Out} : \mathbb{N}^+ \rightarrow \mathbb{N}^+$ and such that:

1. *There is an efficient quantum circuit $F = \{F_\lambda\}_{\lambda \in \mathbb{N}^+}$ such that F_λ implements the CPTP map $F_\lambda(\rho) = U_\lambda \rho U_\lambda^\dagger$ where $U_\lambda \in \mathcal{U}(\mathcal{Q}(\lambda) \otimes \mathcal{Q}(\ell_{In}(\lambda)) \otimes \mathcal{Q}(\ell_{Out}(\lambda)))$ is defined by*

$$U_\lambda(|k\rangle|a\rangle|b\rangle) = |k\rangle|a\rangle|b \oplus f_\lambda(k, a)\rangle . \quad (13)$$

2. *For all efficient quantum circuits $D = \{D_\lambda^H\}_{\lambda \in \mathbb{N}^+}$ having oracle access to a function of the form $H \in \text{Bool}(\ell_{In}(\lambda), \ell_{Out}(\lambda))$, each implementing a CPTP map of the form $D_\lambda^H : \mathcal{D}(\mathbb{C}) \rightarrow \mathcal{D}(\mathcal{Q})$, there is a negligible function η such that:*

$$\left| \Pr_{k \leftarrow \{0, 1\}^\lambda} \text{Tr} [|0\rangle\langle 0| D_\lambda^{f_\lambda(k, \cdot)}(1)] - \Pr_{H \leftarrow \text{Bool}(\ell_{In}(\lambda), \ell_{Out}(\lambda))} \text{Tr} [|0\rangle\langle 0| D_\lambda^H(1)] \right| \leq \eta(\lambda) . \quad (14)$$

We should think of D as a circuit which attempts to distinguish two different cases: is it given oracle access to an instance of the pseudorandom function, which is to say $f(k, \cdot) : \{0, 1\}^{\ell_{In}(\lambda)} \rightarrow \{0, 1\}^{\ell_{Out}(\lambda)}$ for a randomly sampled $k \in \{0, 1\}^\lambda$? Or to a function that was sampled truly at random, $H \in \text{Bool}(\ell_{In}(\lambda), \ell_{Out}(\lambda))$?

The circuit takes no input and produces a single bit of output, via measuring a single qubit in the computational basis. The bound given in the definition

ensures that the probability distribution of the output does not change by much in both scenarios.

In his work on quantum-secure pseudorandom functions [26], Zhandry showed that certain pseudorandom functions that are secure against classical adversaries are insecure against quantum adversaries. Fortunately, Zhandry also showed that some common constructions of pseudorandom functions remain secure against quantum adversaries.

3 Uncloneable Encryption

The encryption of classical plaintexts into classical ciphertexts has been extensively studied. The study of encrypting quantum plaintexts into quantum ciphertexts has also received some attention, for example in [3]. Uncloneable encryption is a security notion for classical plaintexts which is impossible to achieve in any meaningful way with classical ciphertexts. Thus, we need to formally define a notion of quantum encryptions for classical messages.

3.1 Quantum Encryptions of Classical Messages

A quantum encryption of classical messages scheme is a procedure which takes as input a plaintext and a key, in the form of classical bit strings, and produces a ciphertext in the form of a quantum state. We model these schemes as quantum circuits and CPTP maps where classical bit strings are identified with computational basis states: $s \leftrightarrow |s\rangle\langle s|$. Our schemes are defined for fixed plaintext lengths and indexed by a security parameter λ . The size of the key and of the ciphertext may depend on λ . This is formalized in Definition 2.

Definition 2 (Quantum Encryption of Classical Messages). *Let n be an integer. A n -quantum encryption of classical messages (n -QECM) scheme is a triplet of uniform efficient quantum circuits $\mathcal{S} = (\text{Key}, \text{Enc}, \text{Dec})$ implementing CPTP maps of the form*

- $\text{Key}_\lambda : \mathcal{D}(\mathbb{C}) \rightarrow \mathcal{D}(\mathcal{H}_{K,\lambda})$,
- $\text{Enc}_\lambda : \mathcal{D}(\mathcal{H}_{K,\lambda} \otimes \mathcal{H}_{M,\lambda}) \rightarrow \mathcal{D}(\mathcal{H}_{T,\lambda})$, and
- $\text{Dec}_\lambda : \mathcal{D}(\mathcal{H}_{K,\lambda} \otimes \mathcal{H}_{T,\lambda}) \rightarrow \mathcal{D}(\mathcal{H}_{M,\lambda})$

where $\mathcal{H}_{M,\lambda} = \mathcal{Q}(n)$ is the plaintext space, $\mathcal{H}_{T,\lambda} = \mathcal{Q}(\ell(\lambda))$ is the ciphertext space, and $\mathcal{H}_{K,\lambda} = \mathcal{Q}(\kappa(\lambda))$ is the key space for functions $\ell, \kappa : \mathbb{N}^+ \rightarrow \mathbb{N}^+$.

For all $\lambda \in \mathbb{N}^+$, $k \in \{0, 1\}^{\kappa(\lambda)}$, and $m \in \{0, 1\}^n$, the maps must satisfy

$$\text{Tr}[|k\rangle\langle k| \text{Key}(1)] > 0 \implies \text{Tr}[|m\rangle\langle m| \text{Dec}_k \circ \text{Enc}_k |m\rangle\langle m|] = 1 \quad (15)$$

where λ is implicit, Enc_k is the CPTP map defined by $\rho \mapsto \text{Enc}(|k\rangle\langle k| \otimes \rho)$, and we define Dec_k analogously.

A short discussion on the key generation circuit, Key , is in order. First, note that Key takes no input. Indeed, the domain of Key_λ is $\mathcal{D}(\mathbb{C})$ and \mathbb{C} is the state space of zero qubits. In particular, there is a single valid quantum state on \mathbb{C} : $\mathcal{D}(\mathbb{C}) = \{1\}$. To generate a classical key to be used by the encryption and decryption circuits Enc_λ and Dec_λ , a party runs the circuit Key_λ and obtains the quantum state $\text{Key}_\lambda(1)$. This quantum state is then measured in the computational basis and the result of this measurement is used as the key. We then see that Eq. (15) is a correctness condition which imposes that, for all keys that may be generated, a valid ciphertext is always correctly decrypted.

3.2 Security Notions

Now that we have formal definition for QECMs, we can define security notions for these schemes. We define three such notions:

1. Indistinguishable security. Conceptually inspired by the original security notion of indistinguishable encryptions [14] and similar in details to an analogue definition in [3] which accounts for quantum ciphertexts, this security notions is essentially the weakest that should be satisfied by a QECM to provide some level of encryption. It is formally stated in Definition 4.
2. Uncloneable security. This security notion is novel to this work and captures, in the broadest sense, what we mean by an “uncloneable encryption scheme”. This security notion is defined in Definition 6 and is parametrized by a real value $0 \leq t \leq n$, where n is the message size. The case where $t = 0$ is ideal and $t = n$ is trivial. In particular, no encryption scheme with classical ciphertexts may achieve t -uncloneable security for $t < n$.
3. Uncloneable-indistinguishable security. This security notion is also novel to this work. It can be seen as a combination of indistinguishable and uncloneable security. It is formally defined in Definition 8.

Each of these security notions is defined in two steps. First, we define a type of attack (Definitions 3, 5 and 7). Then, we say that the QECM achieves the given security notion if all admissible attacks have their winning probability appropriately bounded (Definitions 4, 6 and 8).

We first define our notion of indistinguishable security.

Definition 3 (Indistinguishable Attack). *Let \mathcal{S} be an n -QECM scheme. An indistinguishable attack is a pair of efficient quantum circuits $\mathcal{A} = (\mathcal{G}, \mathcal{A})$ implementing CPTP maps of the form*

$$\begin{aligned} & - G_\lambda : \mathcal{D}(\mathbb{C}) \rightarrow \mathcal{D}(\mathcal{H}_{M,\lambda} \otimes \mathcal{H}_{S,\lambda}) \text{ and} \\ & - A_\lambda : \mathcal{D}(\mathcal{H}_{T,\lambda} \otimes \mathcal{H}_{S,\lambda}) \rightarrow \mathcal{D}(\mathcal{Q}) \end{aligned}$$

where $\mathcal{H}_{S,\lambda} = \mathcal{Q}(s(\lambda))$ for a function $s : \mathbb{N}^+ \rightarrow \mathbb{N}^+$ and $\mathcal{H}_{M,\lambda}$ and $\mathcal{H}_{T,\lambda}$ are as defined in \mathcal{S} .

Definition 4 (Indistinguishable Security). Let \mathcal{S} be an n -QECM. For any fixed and implicit λ , we define the CPTP map $Enc_k^1 : \mathcal{D}(\mathcal{H}_{M,\lambda}) \rightarrow \mathcal{D}(\mathcal{H}_{T,\lambda})$ by

$$\rho \mapsto \sum_{m \in \{0,1\}^n} \text{Tr}[|m\rangle\langle m| \rho] \cdot Enc_k(|m\rangle\langle m|) \quad (16)$$

and the CPTP map $Enc_k^0 : \mathcal{D}(\mathcal{H}_{M,\lambda}) \rightarrow \mathcal{D}(\mathcal{H}_{T,\lambda})$ by

$$\rho \mapsto Enc_k(|\mathbf{0}\rangle\langle\mathbf{0}|) \quad (17)$$

where $\mathbf{0} \in \{0,1\}^n$ is the all zero bit string.

Then, we say that \mathcal{S} is indistinguishable secure if for all indistinguishable attacks \mathcal{A} there exists a negligible function η such that

$$\mathbb{E}_b \mathbb{E}_{k \leftarrow \mathcal{K}} \text{Tr}[|b\rangle\langle b| A_\lambda \circ (\mathbb{1}_S \otimes Enc_k^b) \circ G(1)] \leq \frac{1}{2} + \eta(\lambda) \quad (18)$$

where λ is implicit on the left-hand side, $b \in \{0,1\}$, and \mathcal{K}_λ is the random variable distributed on $\{0,1\}^{\kappa(\lambda)}$ such that

$$\Pr[\mathcal{K}_\lambda = k] = \text{Tr}[|k\rangle\langle k| Key_\lambda(1)] \quad (19)$$

In Definition 4, the map Enc_k^0 should be seen as discarding whatever plaintext was given and producing the encryption of the all zero bit string. On the other hand, Enc_k^1 is the map which first measures the state given in the computational basis, to ensure that the plaintext is indeed a classical message, and then encrypts this message. We say that QECM scheme has indistinguishable security if no efficient adversary can distinguish between both of these scenarios with more than a negligible probability. Of course, this is one of the weakest security notion for encryption schemes, but it will be sufficient for our needs to show that the schemes we define do offer some level of security.

Next, we formalize the intuitive definition for uncloneable security as given in Section 1.1.

Definition 5 (Uncloneable Attack). Let \mathcal{S} be a n -QECM scheme. An uncloneable attack against the scheme \mathcal{S} is a triplet of uniform efficient quantum circuits $\mathcal{A} = (A, B, C)$ implementing CPTP maps of the form

- $A_\lambda : \mathcal{D}(\mathcal{H}_{T,\lambda}) \rightarrow \mathcal{D}(\mathcal{H}_{B,\lambda} \otimes \mathcal{H}_{C,\lambda})$,
- $B_\lambda : \mathcal{D}(\mathcal{H}_{K,\lambda} \otimes \mathcal{H}_{B,\lambda}) \rightarrow \mathcal{D}(\mathcal{H}_{M,\lambda})$, and
- $C_\lambda : \mathcal{D}(\mathcal{H}_{K,\lambda} \otimes \mathcal{H}_{C,\lambda}) \rightarrow \mathcal{D}(\mathcal{H}_{M,\lambda})$

where $\mathcal{H}_{B,\lambda} = \mathcal{Q}(\beta(\lambda))$ and $\mathcal{H}_{C,\lambda} = \mathcal{Q}(\gamma(\lambda))$ for some functions $\beta, \gamma : \mathbb{N}^+ \rightarrow \mathbb{N}^+$ and $\mathcal{H}_{K,\lambda}$, $\mathcal{H}_{M,\lambda}$, and $\mathcal{H}_{T,\lambda}$ are as defined by \mathcal{S} .

We sketch out the relation between the various CPTP maps and the underlying Hilbert spaces considered in Definition 5 in Fig. 2.

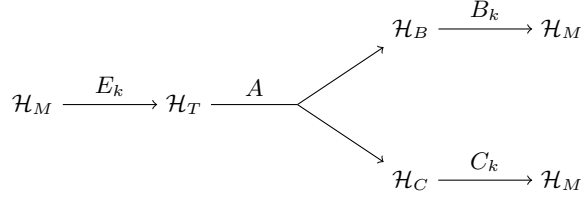


Fig. 2. Schematic representation of the maps considered in an uncloneable attack as given in [Definition 5](#).

Definition 6 (Uncloneable Security). An n -QECM scheme \mathcal{S} is t -uncloneable secure if for all random variables \mathcal{M} on $\{0, 1\}^n$ with min-entropy h and all uncloneable attacks \mathcal{A} against \mathcal{S} there exists a negligible function η such that

$$\mathbb{E}_{m \leftarrow \mathcal{M}} \mathbb{E}_{k \leftarrow \mathcal{K}} \text{Tr} [(|m\rangle\langle m| \otimes |m\rangle\langle m|) (B_k \otimes C_k) \circ A (|m\rangle\langle m|)] \leq 2^{-h+t} + \eta(\lambda) \quad (20)$$

where λ is implicit on the left-hand side, \mathcal{K}_λ is a random variable distributed on $\{0, 1\}^{\kappa(\lambda)}$ such that

$$\Pr [\mathcal{K}_\lambda = k] = \text{Tr} [|k\rangle\langle k| \text{Key}_\lambda(1)], \quad (21)$$

and B_k the CPTP map defined by $\rho \mapsto B(|k\rangle\langle k| \otimes \rho)$ and similarly for C_k .

We note that any encryption which produces classical ciphertexts cannot be t -uncloneable secure for any $t < n$. Indeed, an attack \mathcal{A} where \mathbf{A} duplicates the classical ciphertext and where $\mathbf{B} = \mathbf{C} = \text{Dec}$ succeeds with probability 1.

Our definition of uncloneable security is with respect to any distribution over the message space. However, it suffices to consider the uniform distribution.

Theorem 2. Let \mathcal{S} be an n -QECM scheme. Suppose that for any uncloneable attack \mathcal{A} on \mathcal{S} there exists a negligible function η such that

$$\mathbb{E}_{k \leftarrow \mathcal{K}} \mathbb{E}_m \| (|m\rangle\langle m| \otimes |m\rangle\langle m|) (B_k \otimes C_k) \circ A \circ \text{Enc}_k |m\rangle\langle m| \|^2 \leq 2^{-n+t} + \eta(\lambda). \quad (22)$$

Then, \mathcal{S} is t -uncloneable secure.

Proof. For all $k \in \{0, 1\}^{\kappa(\lambda)}$ and $m \in \{0, 1\}^n$, define

$$p(k, m) = \| (|m\rangle\langle m| \otimes |m\rangle\langle m|) (B_k \otimes C_k) \circ A \circ \text{Enc}_k (|m\rangle\langle m|) \|^2 \quad (23)$$

and let \mathcal{M} be a random variable on $\{0, 1\}^n$ with min-entropy h . By hypothesis, we have that

$$\mathbb{E}_{k \leftarrow \mathcal{K}} \mathbb{E}_{m \leftarrow \mathcal{M}} \|(|m\rangle\langle m| \otimes |m\rangle\langle m|) (B_k \otimes C_k) \circ A \circ Enc_k (|m\rangle\langle m|)\|^2 \quad (24)$$

$$= \sum_{m \in \{0,1\}^n} \Pr[\mathcal{M} = m] \mathbb{E}_{k \leftarrow \mathcal{K}} p(k, m) \quad (25)$$

$$\leq 2^{-h} \sum_{m \in \{0,1\}^n} \mathbb{E}_{k \leftarrow \mathcal{K}} p(k, m) \quad (26)$$

$$\leq 2^{-h} (2^t + 2^n \eta(\lambda)) \quad (27)$$

$$= 2^{-h+t} + 2^{-h+n} \eta(\lambda). \quad (28)$$

Finally, we note that $\lambda \mapsto 2^{-h+n} \eta(\lambda)$ is a negligible function of λ .

Finally, we define the notion of uncloneable-indistinguishable security.

Definition 7 (Uncloneable-Indistinguishable Attack). *Let \mathcal{S} be a n -QECM scheme. An uncloneable-indistinguishable-attack against the scheme \mathcal{S} is a tuple $\mathcal{A} = (G, A, B, C)$ of efficient quantum circuits implementing CPTP maps of the form*

- $G_\lambda : \mathcal{D}(\mathbb{C}) \rightarrow \mathcal{D}(\mathcal{H}_{S,\lambda} \otimes \mathcal{H}_{M,\lambda})$,
- $A_\lambda : \mathcal{D}(\mathcal{H}_{S,\lambda} \otimes \mathcal{H}_{T,\lambda}) \rightarrow \mathcal{D}(\mathcal{H}_{B,\lambda} \otimes \mathcal{H}_{C,\lambda})$,
- $B_\lambda : \mathcal{D}(\mathcal{H}_{K,\lambda} \otimes \mathcal{H}_{B,\lambda}) \rightarrow \mathcal{D}(\mathcal{Q})$, and
- $C_\lambda : \mathcal{D}(\mathcal{H}_{K,\lambda} \otimes \mathcal{H}_{C,\lambda}) \rightarrow \mathcal{D}(\mathcal{Q})$

where $\mathcal{H}_{S,\lambda} = \mathcal{Q}(s(\lambda))$, $\mathcal{H}_{B,\lambda} = \mathcal{Q}(\beta(\lambda))$, and $\mathcal{H}_{C,\lambda} = \mathcal{Q}(\alpha(\lambda))$ for some functions $s, \alpha, \beta : \mathbb{N}^+ \rightarrow \mathbb{N}^+$ and all other Hilbert spaces are as defined by \mathcal{S} .

We sketch out the relation between the various CPTP maps and the underlying Hilbert spaces considered in Definition 7 in Fig. 3.

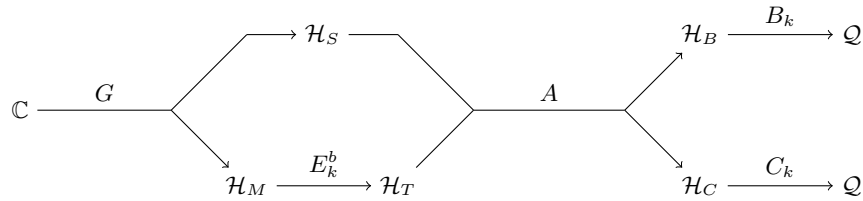


Fig. 3. Relation between the CPTP maps and Hilbert spaces considered in an uncloneable-indistinguishable attack as described in Definition 7.

Definition 8 (Uncloneable-Indistinguishable Security). Let \mathcal{S} be an n -QECM and define Enc_k^0 and Enc_k^1 as in Definition 4.

We say that \mathcal{S} is uncloneable-indistinguishable secure if for all uncloneable-indistinguishable attacks \mathcal{A} , there exists a negligible function η such that

$$\mathbb{E}_b \mathbb{E}_{k \leftarrow \mathcal{K}} \text{Tr} \left[\left(|b\rangle\langle b| \otimes |b\rangle\langle b| \right) \left(B_k \otimes C_k \right) A \left(\mathbb{1}_S \otimes E_k^b \right) G(1) \right] \leq \frac{1}{2} + \eta(\lambda) \quad (29)$$

where λ is implicit on the left-hand side, \mathcal{K}_λ is distributed on $\{0, 1\}^{\kappa\lambda}$ such that

$$\Pr[\mathcal{K} = k] = \text{Tr}[|k\rangle\langle k| K(1)], \quad (30)$$

and B_k is the CPTP map defined by $\rho \mapsto B(|k\rangle\langle k| \otimes \rho)$ and similarly for C_k .

It is trivial to see, but worth noting, that uncloneable-indistinguishable security implies indistinguishable security. We now briefly sketch the proof.

Let \mathcal{S} be an n -QECM and $\mathcal{A} = (\mathbf{G}, \mathbf{A})$ be an indistinguishable attack which shows that \mathcal{S} is not indistinguishable secure. Then, we can construct an uncloneable-indistinguishable attack $\mathcal{A}' = (\mathbf{G}', \mathbf{A}', \mathbf{B}', \mathbf{C}')$ which implies that \mathcal{S} is not uncloneable-indistinguishable secure.

Indeed, let $\mathbf{G}' = \mathbf{G}$ and \mathbf{B} and \mathbf{C} be the circuits which do nothing on a single qubit input. Then, we define \mathbf{A}' to first run \mathbf{A} and measure the output in the computational basis state. The result is a single classical bit which may then be copied and given to both \mathbf{B} and \mathbf{C} .

It is then simple to see that the winning probability of \mathbf{A} in the indistinguishable scenario is the same as the winning probability of \mathbf{A}' in the uncloneable-indistinguishable scenario.

Finally, it can also be shown that any 0-uncloneable secure QECM \mathcal{S} is uncloneable-indistinguishable secure.

Theorem 3. Let \mathcal{S} be a n -QECM. If \mathcal{S} is 0-uncloneable secure, then it is also uncloneable-indistinguishable secure.

Proof (Sketch). Let $\mathcal{A} = (\mathbf{G}, \mathbf{A}, \mathbf{B}, \mathbf{C})$ be an uncloneable-indistinguishable attack on \mathcal{S} . From \mathcal{A} , we can define an uncloneable attack. Let $\mathcal{A}' = (\mathbf{A}', \mathbf{B}', \mathbf{C}')$ be the uncloneable attack defined by

- \mathbf{A}' : Run \mathbf{G} from \mathcal{A} and obtain a state $G(1) \in \mathcal{D}(\mathcal{H}_S \otimes \mathcal{H}_M)$. Measure the M register and call the result m . Discard the register M and keep the register S . Then, run \mathbf{A} from \mathcal{A} on the state ρ received and the state that was kept in the register S . In addition, give a copy of m to both \mathbf{B}' and \mathbf{C}' .
- \mathbf{B}' Run \mathbf{B} from \mathcal{A} on the state obtained from the simulation of \mathbf{A} . Measure the output and if the result is 0, output $\mathbf{0}$, the all zero bit string. If the result is 1, output the m which was given by \mathbf{A}' .
- \mathbf{C}' Analogous to \mathbf{B}' .

Comparing the winning probability of \mathcal{A}' in the uncloneable scenario, with uniform messages, against the winning probability of \mathcal{A} in the uncloneable-indistinguishable scenario, we can show that the winning probability of \mathcal{A} is

at most 2^{n-1} times the winning probability of \mathcal{A}' . But, by assumption, the winning probability of \mathcal{A}' is upper bounded by $2^{-n} + \eta(\lambda)$ for some negligible η . Thus, the winning probability of \mathcal{A} is upper bounded by $\frac{1}{2} + 2^{n-1}\eta(\lambda)$. Noting that $\lambda \mapsto 2^{n-1}\eta(\lambda)$ is a negligible function completes the proof.

4 Two Protocols

In this section, we first present a protocol for the encryption of classical messages into quantum ciphertexts based on Wiesner's conjugate encoding (Section 4.1). This will also include a simple proof of its uncloneable security. Then, in Section 4.2, we present a refinement of this first protocol which uses quantum secure pseudorandom functions. The proof of the uncloneable security of this protocol is a bit more involved and so we present some technical lemmas in Section 4.3 before we give our final main results in Section 4.4.

4.1 Conjugate Encryption

Our first QECM scheme is a one-time pad encoded into Wiesner states. We emphasize that this will not offer much in terms of uncloneable security but it remains an instructive example.

Definition 9 (Conjugate Encryption). *Let n be an integer. We define the conjugate encryption n -QECM scheme by the following circuits.*

Circuit 1: The key generation circuit **Key**.

Input : *None.*

Output: *A state $\rho \in \mathcal{D}(\mathcal{Q}(n+n))$.*

- 1 *Sample $r \leftarrow \{0, 1\}^n$ uniformly at random.*
 - 2 *Sample $\theta \leftarrow \{0, 1\}^n$ uniformly at random.*
 - 3 *Output $\rho = |r\rangle\langle r| \otimes |\theta\rangle\langle \theta|$.*
-

Circuit 2: The encryption circuit **Enc**.

Input : *A plaintext $m \in \{0, 1\}^n$ and a key $(r, \theta) \in \{0, 1\}^{n+n}$.*

Output: *A ciphertext $\rho \in \mathcal{D}(\mathcal{Q}(n))$.*

- 1 *Output $\rho = |(m \oplus r)^\theta\rangle\langle (m \oplus r)^\theta|$.*
-

Circuit 3: The decryption circuit **Dec**.

Input : *A ciphertext $\rho \in \mathcal{D}(\mathcal{Q}(n))$ and a key $(r, \theta) \in \{0, 1\}^{n+n}$.*

Output: *A plaintext $m \in \{0, 1\}^n$.*

- 1 *Compute $\rho' = H^\theta \rho H^\theta$.*
 - 2 *Measure ρ' in the computational basis. Call the result c . Output $c \oplus r$.*
-

The correctness of this scheme is trivial to verify and it is indistinguishable secure. The latter follows from the fact that for any $\rho \in \mathcal{D}(\mathcal{H}_S \otimes \mathcal{Q}(n))$ we have that

$$\mathbb{E}_r \mathbb{E}_\theta \left(\mathbb{1}_S \otimes \text{Enc}_{(r,\theta)}^1 \right) (\rho) = \mathbb{E}_r \mathbb{E}_\theta \left(\mathbb{1}_S \otimes \text{Enc}_{(r,\theta)}^0 \right) (\rho) \quad (31)$$

where $Enc_{r,\theta}^0$ and $Enc_{r,\theta}^1$ are as defined in [Definition 4](#).

We will need one small technical lemma before proceeding to the proof of uncloneable security for this scheme.

Lemma 1. *Let $n \in \mathbb{N}^+$ be an integer, $f : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \mathbb{R}$ be a function and $s \in \{0, 1\}^n$ be a string. Then,*

$$\mathbb{E}_x f(x, x \oplus s) = \mathbb{E}_x f(x \oplus s, x). \quad (32)$$

Proof. See [Appendix A](#).

We can now show the following.

Theorem 4. *The scheme given in [Definition 9](#) is $n \log_2 \left(1 + \frac{1}{\sqrt{2}}\right)$ -uncloneable secure.*

Proof. By [Theorem 2](#), it suffices to show that for any uncloneable attack \mathcal{A} the quantity

$$\mathbb{E}_m \mathbb{E}_r \mathbb{E}_\theta \text{Tr} [(|m\rangle\langle m| \otimes |m\rangle\langle m|) (B_{(r,\theta)} \otimes C_{(r,\theta)}) \circ A (|(m \oplus r)^\theta\rangle\langle (m \oplus r)^\theta|)] \quad (33)$$

is upper bounded by $\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)^n$. By applying [Lemma 1](#) with respect to the expectation over m , this quantity is the same as

$$\mathbb{E}_m \mathbb{E}_r \mathbb{E}_\theta \text{Tr} [(|m \oplus r\rangle\langle m \oplus r| \otimes |m \oplus r\rangle\langle m \oplus r|) (B_{(r,\theta)} \otimes C_{(r,\theta)}) \circ A (|m^\theta\rangle\langle m^\theta|)]. \quad (34)$$

We then see that for any fixed r , we can apply [Corollary 1](#) to bound the expectation of the trace over m and θ by $\left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right)^n$, completing the proof.

4.2 Our Protocol

The motivation for our second QECM scheme is to use quantum-secure pseudo-random functions to attempt to “distill” the uncloneability found in the Wiesner state.

Definition 10 (\mathcal{F} -Conjugate Encryption). *Let $n \in \mathbb{N}^+$ be an integer. Let*

$$\mathcal{F} = \{f_\lambda : \{0, 1\}^\lambda \times \{0, 1\}^\lambda \rightarrow \{0, 1\}^n\}_{\lambda \in \mathbb{N}^+} \quad (35)$$

be a quantum-secure pseudorandom function. We define the \mathcal{F} -conjugate encryption n -QECM scheme by the following circuits.

Circuit 4: The key generation circuit **Key**.

Input : None.

Output: A state $\rho \in \mathcal{D}(\mathcal{Q}(\kappa(\lambda)))$.

- 1 Sample $s \leftarrow \{0, 1\}^\lambda$ uniformly at random.
 - 2 Sample $\theta \leftarrow \{0, 1\}^\lambda$ uniformly at random.
 - 3 Output $\rho = |s\rangle\langle s| \otimes |\theta\rangle\langle \theta|$.
-

Circuit 5: The encryption circuit **Enc**.

Input : A plaintext $m \in \{0, 1\}^n$ and a key $(s, \theta) \in \{0, 1\}^{\lambda+\lambda}$.

Output: A ciphertext $\rho \in \mathcal{D}(\mathcal{Q}(\ell(\lambda)))$.

- 1 Sample $r \leftarrow \{0, 1\}^\lambda$ uniformly at random.
 - 2 Compute $c = m \oplus f_\lambda(s, r)$.
 - 3 Output $\rho = |c\rangle\langle c| \otimes |r^\theta\rangle\langle r^\theta|$.
-

Circuit 6: The decryption circuit **Dec**.

Input : A ciphertext $|c\rangle\langle c| \otimes \rho \in \mathcal{D}(\mathcal{Q}(n + \lambda))$ and a key $(s, \theta) \in \{0, 1\}^{\lambda+\lambda}$.

Output: A plaintext $m \in \{0, 1\}^n$.

- 1 Compute $\rho' = H^\theta \rho H^\theta$.
 - 2 Measure ρ' in the computational basis. Call the result r .
 - 3 Output $m = c \oplus f_\lambda(s, r)$.
-

It is trivial to see that this scheme is correct and we can also show that it is indistinguishable secure. The latter follows from the fact that if we use a truly random function instead of the qPRF, then

$$\mathbb{E}_r \mathbb{E}_{H \in \text{Bool}(\lambda, n)} \left(\mathbb{1}_S \otimes \text{Enc}_{(r, H)}^0 \right) (\rho) = \mathbb{E}_r \mathbb{E}_{H \in \text{Bool}(\lambda, n)} \left(\mathbb{1}_S \otimes \text{Enc}_{(r, H)}^1 \right) (\rho) \quad (36)$$

where $\text{Enc}_{(r, H)}^0$ and $\text{Enc}_{(r, H)}^1$ are as given in [Definition 4](#). Thus, any adversary has no advantage in distinguishing the cases. When the truly random functions are replaced by a qPRF, the adversary may have at most a negligible advantage in distinguishing the cases.

4.3 Technical Lemmas

We first present a few technical lemmas which will be used in our proof of security.

Lemma 2. *Let R be a ring with $a, b \in R$ and $c = a + b$. Then, for all $n \in \mathbb{N}^+$, we have that*

$$c^n = a^n + \sum_{k=0}^{n-1} a^{n-k-1} b c^k. \quad (37)$$

Proof. See [Appendix A](#).

Lemma 3. *Let \mathcal{H} be a Hilbert space, $n \in \mathbb{N}^+$ be an integer, and $\{v_0, v_1, \dots, v_n\}$ be $n+1$ vectors in \mathcal{H} such that $\|v_i\| \leq 1$ for all $i \in \{1, \dots, n\}$ and $\|\sum_{i=0}^n v_i\| \leq 1$. Then,*

$$\left\| \sum_{i=0}^n v_i \right\|^2 \leq \|v_0\|^2 + (3n+2) \sum_{i=1}^n \|v_i\|. \quad (38)$$

Proof. See [Appendix A](#).

The following implicitly appears in [\[21\]](#).

Lemma 4. *Let $f : \text{Bool}(n, m) \rightarrow \mathbb{R}$ be a function and $x \in \{0, 1\}^n$ be a string. For any $H \in \text{Bool}(n, m)$ and $y \in \{0, 1\}^m$, define $H_{x,y} \in \text{Bool}(n, m)$ by*

$$s \mapsto \begin{cases} H(s) & \text{if } s \neq x, \\ y & \text{if } s = x. \end{cases} \quad (39)$$

Then,

$$\mathbb{E}_H f(H) = \mathbb{E}_H \mathbb{E}_y f(H_{x,y}). \quad (40)$$

The following two lemmas form the core of upcoming proofs of uncloneable security and they may be interpreted as follows. We consider two adversaries who have oracle access to a function $H \in \text{Bool}(\lambda, n)$ which is chosen uniformly at random. Their goal is to simultaneously guess the value $H(x)$ for some value of x . The adversaries share some quantum state which we interpret as representing all the information they may initially have on x . The lemmas relates the probability of both parties simultaneously guessing $H(x)$ to their probability of being able to both simultaneously guess x .

The first of these lemmas, [Lemma 5](#), considers this problem in a setting where the adversaries do not share any entanglement. The second, [Lemma 6](#) imposes no such restriction.

We show that the probability that both adversaries correctly guess $H(x)$ is upper bounded by

$$\frac{1}{2^n} + Q \cdot G \quad \text{or} \quad 9 \cdot \frac{1}{2^n} + Q' \cdot G' \quad (41)$$

where Q and Q' are polynomial functions of the number of queries the adversaries make to the oracle and G and G' quantifies their probability of guessing x with a particular strategy. The factor of 9 is present only if we allow the adversaries to share entanglement.

We can interpret G and G' in a manner very similar to its analogous quantity in Unruh's one-way-to-hiding lemma [\[21\]](#). The adversaries, instead of continuing until the end of their computation, will stop immediately before a certain (randomly chosen) query to the oracle and measure their query register in the computational basis. Then, G is related to the probability that this procedure

succeeds at letting both adversaries simultaneously obtain x , averaged over the possible stopping points and possible functions implemented by the oracle.

The key idea in the proof of these lemmas is that we can decompose the unitary operator representing each of the adversaries computations into a two “parts”. Explicitly, this decomposition appears in Eqs. (47) and (54). One of these “parts” will never query the oracle on x and the other could query the oracle on x . We note that this idea was present in the proof of Unruh’s one-way-to-hiding lemma [21].

Lemma 5. *Let $\lambda, n \in \mathbb{N}^+$ be integers. For $L \in \{B, C\}$, we let*

- $s_L, q_L \in \mathbb{N}^+$ be integers,
- $\mathcal{H}_{L_Q} = \mathcal{Q}(\lambda)$, $\mathcal{H}_{L_R} = \mathcal{Q}(n)$, and $\mathcal{H}_{L_S} = \mathcal{Q}(s_L)$,
- $U_L \in \mathcal{U}(\mathcal{H}_{L_Q} \otimes \mathcal{H}_{L_R} \otimes \mathcal{H}_{L_S})$ be a unitary operator, and
- $\{\pi_L^y\}_{y \in \{0,1\}^n}$ be a POVM on $\mathcal{H}_{L_Q} \otimes \mathcal{H}_{L_R} \otimes \mathcal{H}_{L_S}$.

Finally, let $|\psi\rangle = |\psi_B\rangle \otimes |\psi_A\rangle$ be a separable unit vector with $|\psi_L\rangle \in \mathcal{Q}(n + \lambda + s_L)$ for $L \in \{B, C\}$ and $x \in \{0, 1\}^\lambda$ be a string. Then, we have

$$\mathbb{E}_H \left\| \Pi^{H(x)} \left((U_B O_B^H)^{q_B} \otimes (U_C O_C^H)^{q_C} \right) |\psi\rangle \right\|^2 \leq \frac{9}{2^n} + (3q + 2)q^4 \sqrt{M} \quad (42)$$

where $\Pi^{H(x)} = \pi_B^{H(x)} \otimes \pi_C^{H(x)}$, $q = q_B + q_C$ and

$$M = \mathbb{E}_k \mathbb{E}_\ell \mathbb{E}_H \mathbb{E}_{H'} \left\| \left(|x\rangle\langle x|_{B_Q} \otimes |x\rangle\langle x|_{C_Q} \right) \left((U_B O_B^H)^k \otimes (U_C O_C^{H'})^\ell \right) |\psi\rangle \right\|^2 \quad (43)$$

with $k \in \{0, \dots, q_B\}$, $\ell \in \{0, \dots, q_C\}$, and $H \in \text{Bool}(n, m)$.

Proof. Note that since $|\psi\rangle$ is separable, we have that

$$M = \underbrace{\left(\mathbb{E}_H \mathbb{E}_k \left\| |x\rangle\langle x|_{B_Q} \mathcal{O}^{U_B, H, k} |\psi_B\rangle \right\|^2 \right)}_{=M_B} \cdot \underbrace{\left(\mathbb{E}_{H'} \mathbb{E}_\ell \left\| |x\rangle\langle x|_{C_Q} \mathcal{O}^{U_C, H', \ell} |\psi_C\rangle \right\|^2 \right)}_{=M_C}. \quad (44)$$

For the remainder of the proof, we fix $L \in \{B, C\}$ such that $M_L = \min\{M_B, M_C\}$. Note that $\sqrt{M_L} \leq \sqrt[4]{M}$. Once again using the fact that $|\psi\rangle$ is separable, we have that

$$\mathbb{E}_H \left\| \Pi^{H(x)} \left((U_B O_B^H)^{q_B} \otimes (U_C O_C^H)^{q_C} \right) |\psi\rangle \right\|^2 \leq \mathbb{E}_H \left\| \pi_L^{H(x)} (U_B O_B^H)^{q_L} |\psi_L\rangle \right\|^2. \quad (45)$$

With all this, it suffices to show that

$$\mathbb{E}_H \left\| \pi_L^{H(x)} (U_B O_B^H)^{q_L} |\psi_L\rangle \right\|^2 \leq \frac{1}{2^n} + (3q_L + 2)q_L \sqrt{M_L} \quad (46)$$

to obtain our result.

Let $P_L = |x\rangle\langle x|_{L_Q}$. Using the fact that $U_L O_L^H = U_L O_L^H P_L + U_L O_L^H (\mathbb{1} - P_L)$ and [Lemma 2](#), we have that

$$(U_L O_L^H)^{q_L} = \overbrace{(U_L O_L^H (\mathbb{1} - P_L))^{q_L}}^{=V_L^H} + \sum_{k=0}^{q_L-1} \underbrace{(U_L O_L^H (\mathbb{1} - P_L))^{q_L-k-1} U_L O_L^H P_L (U_L O_L^H)^k}_{=W_L^{H,k}} \quad (47)$$

and we define $W_L^H = \sum_{k=0}^{q_L-1} W_L^{H,k}$.

Using [Lemma 3](#), the definition of the various W operators, and properties of the operator norm on projectors and unitary operators, we have that

$$\left\| \pi^{H(x)} (V_L^H + W_L^H) |\psi\rangle \right\|^2 \leq \left\| \pi^{H(x)} V_L |\psi_L\rangle \right\|^2 + (3q_L + 2)q_L \mathbb{E}_k \left\| P_L (U_L O_L^H)^k |\psi\rangle \right\|^2. \quad (48)$$

Using Jensen's inequality, we have that

$$E_H \left\| \pi_L^{H(x)} \left((U_L O_L^H)^{q_L} \right) |\psi_L\rangle \right\|^2 \leq \mathbb{E} \left\| \pi_L^{H(x)} V_L |\psi_L\rangle \right\|^2 + (3q_L + 2)q_L \sqrt{M_L} \quad (49)$$

and so it suffices to show that $\mathbb{E}_H \left\| \pi_L^{H(x)} V_L^H |\psi_L\rangle \right\|^2 \leq 2^{-n}$. By [Lemma 4](#), it is then sufficient to show that

$$\mathbb{E}_H \mathbb{E}_y \left\| \pi_L^y V_L^{H_{x,y}} |\psi_L\rangle \right\|^2 \leq 2^{-n} \quad (50)$$

where $H_{x,y} \in \text{Bool}(\lambda, n)$ is defined by $H_{x,y}(x) = y$ and $H_{x,y}(s) = H(s)$ for all $s \neq x$. Recall that V_L^H is independent of the value of $H(x)$, in the sense that $V_L^{H_{x,y}} = V_L^H$ for all $y \in \{0, 1\}^n$. Indeed, prior to every query to H in V_L^H , we project the state on a subspace which does not query H on x . So, using the fact that each π_L^y projects on mutually orthogonal subspaces and that $\|V_L^H\| \leq 1$, we have that

$$\mathbb{E}_y \left\| \pi_L^y V_L^{H_{x,y}} |\psi_L\rangle \right\|^2 = \frac{1}{2^n} \|V_L^H |\psi_L\rangle\|^2 \leq \frac{1}{2^n} \quad (51)$$

which completes the proof.

Lemma 6. *Let $\lambda, n \in \mathbb{N}^+$ be integers. For $L \in \{B, C\}$, we let*

- $s_L, q_L \in \mathbb{N}^+$ be integers,
- $\mathcal{H}_{L_Q} = \mathcal{Q}(\lambda)$, $\mathcal{H}_{L_R} = \mathcal{Q}(n)$, and $\mathcal{H}_{L_S} = \mathcal{Q}(s_L)$,
- $U_L \in \mathcal{U}(\mathcal{H}_{L_Q} \otimes \mathcal{H}_{L_R} \otimes \mathcal{H}_{L_S})$ be a unitary operator, and
- $\{\pi_L^y\}_{y \in \{0,1\}^n}$ be a POVM on $\mathcal{H}_{L_Q} \otimes \mathcal{H}_{L_R} \otimes \mathcal{H}_{L_S}$.

Finally, let $|\psi\rangle \in \mathcal{Q}(2(\lambda + n) + s_B + s_C)$ be a unit vector and $x \in \{0, 1\}^\lambda$ be a string. Then, we have

$$\mathbb{E}_H \left\| \Pi^{H(x)} \left((U_B O_B^H)^{q_B} \otimes (U_C O_C^H)^{q_C} \right) |\psi\rangle \right\|^2 \leq \frac{9}{2^n} + (3q_B q_C + 2)q_B q_C \sqrt{M} \quad (52)$$

where $\Pi^{H(x)} = \pi_B^{H(x)} \otimes \pi_C^{H(x)}$ and

$$M = \mathbb{E}_k \mathbb{E}_\ell \mathbb{E}_H \left\| \left(|x\rangle\langle x|_{B_Q} \otimes |x\rangle\langle x|_{C_Q} \right) \left((U_B O_B^H)^k \otimes (U_C O_C^H)^\ell \right) |\psi\rangle \right\|^2 \quad (53)$$

with $k \in \{0, \dots, q_B\}$, $\ell \in \{0, \dots, q_C\}$, and $H \in \text{Bool}(n, m)$.

Proof. For $L \in \{B, C\}$, we define $P_L = |x\rangle\langle x|_{L_Q}$. Using [Lemma 2](#) and the fact that $U_L O_L^H = U_L O_L^H P_L + U_L O_L^H (\mathbb{1} - P_L)$, we have that

$$\begin{aligned} (U_L O_L^H)^{q_L} &= \overbrace{(U_L O_L^H (\mathbb{1} - P_L))^{q_L}}^{=V_L^H} + \\ &\quad \sum_{k=0}^{q_L-1} \underbrace{(U_L O_L^H (\mathbb{1} - P_L))^{q_L-k-1} U_L O_L^H P_L (U_L O_L^H)^k}_{=W_L^{H,k}} \end{aligned} \quad (54)$$

and we define $W_L^H = \sum_{k=0}^{q_L-1} W_L^{H,k}$. This implies that

$$\begin{aligned} &\left\| \Pi^{H(x)} \left((U_B O_B^H)^{q_B} \otimes (U_C O_C^H)^{q_C} \right) |\psi\rangle \right\|^2 \\ &= \left\| \Pi^{H(x)} \left((O_B O_B^H)^{q_B} \otimes V_C^H + V_B^H \otimes W_C^H + W_B^H \otimes W_C^H \right) |\psi\rangle \right\|^2. \end{aligned} \quad (55)$$

We now claim that the $W_B^H \otimes W_C^H$ operator corresponds to the M in the upper bound provided in the statement. Indeed, using [Lemma 3](#), the definition of the various W operators, and properties of the operator norm on projectors and unitary operators, we have that

$$\begin{aligned} &\left\| \Pi^{H(x)} \left((O_B O_B^H)^{q_B} \otimes V_C^H + V_B^H \otimes W_C^H + W_B^H \otimes W_C^H \right) |\psi\rangle \right\|^2 \\ &\leq \left\| \Pi^{H(x)} \left((O_B O_B^H)^{q_B} \otimes V_C^H + V_B^H \otimes W_C^H \right) |\psi\rangle \right\|^2 \\ &\quad + (3q_B q_C + 2)q_B q_C \mathbb{E}_k \mathbb{E}_\ell \left\| (P_B \otimes P_C) \left((U_B O_B^H)^k \otimes (U_C O_C^H)^\ell \right) |\psi\rangle \right\|^2. \end{aligned} \quad (56)$$

Using Jensen's inequality, we then have that

$$\begin{aligned} &\mathbb{E}_H \left\| \Pi^{H(x)} \left((O_B O_B^H)^{q_B} \otimes V_C^H + V_B^H \otimes W_C^H + W_B^H \otimes W_C^H \right) |\psi\rangle \right\|^2 \\ &\leq \mathbb{E}_H \left\| \Pi^{H(x)} \left((O_B O_B^H)^{q_B} \otimes V_C^H + V_B^H \otimes W_C^H \right) |\psi\rangle \right\|^2 + (3q_B q_C + 2)q_B q_C \sqrt{M}. \end{aligned} \quad (57)$$

It now suffices to show that

$$\mathbb{E}_H \left\| \Pi^{H(x)} \left((O_B O_B^H)^{q_B} \otimes V_C^H + V_B^H \otimes W_C^H \right) |\psi\rangle \right\|^2 \leq \frac{9}{2^n}. \quad (58)$$

By [Lemma 4](#), this is equivalent to showing that

$$\mathbb{E}_H \mathbb{E}_y \left\| \Pi^y \left((U_B O_B^{H_{x,y}})^{q_B} \otimes V_C^{H_{x,y}} + V_B^{H_{x,y}} \otimes W_C^{H_{x,y}} \right) |\psi\rangle \right\|^2 \leq \frac{9}{2^n} \quad (59)$$

In fact, it will be sufficient to show that for any particular H , the expectation over y is bounded by $9 \cdot 2^{-n}$. If, for any H , we define

$$\alpha = \mathbb{E}_y \left\| \Pi^y \left(\left(U_B O_B^{H_{x,y}} \right) \otimes V_C^{H_{x,y}} \right) |\psi\rangle \right\|^2 \quad (60)$$

and

$$\beta = \mathbb{E}_y \left\| \Pi^y \left(V_B^{H_{x,y}} \otimes W_C^{H_{x,y}} \right) |\psi\rangle \right\|^2 \quad (61)$$

then, using the triangle inequality and the fact that the operators in $\{\Pi^y\}_{y \in \{0,1\}^n}$ project on mutually orthogonal subspaces, we have that

$$\mathbb{E}_y \left\| \Pi^y \left(\left(O_B O_B^{H_{x,y}} \right)^{q_B} \otimes V_C^{H_{x,y}} + V_B^{H_{x,y}} \otimes W_C^{H_{x,y}} \right) |\psi\rangle \right\|^2 \leq \alpha + \beta + 2\sqrt{\alpha\beta}. \quad (62)$$

Now, noting that $V_B^{H_{x,y}}$ and $V_C^{H_{x,y}}$ do not depend on the value of y , as they always project on a subspace which does not query the oracle H on x , and using properties of the operator norm, we have that

$$\alpha = \mathbb{E}_y \left\| \Pi^y \left(\left(U_B O_B^{H_{x,y}} \right) \otimes V_C^{H_{x,y}} \right) |\psi\rangle \right\|^2 \quad (63)$$

$$\leq \mathbb{E}_y \left\| (\mathbb{1}_B \otimes \pi_C^y) \left(\mathbb{1}_B \otimes V_C^{H_{x,y}} \right) |\psi\rangle \right\|^2 \quad (64)$$

$$= \frac{1}{2^n} \left\| (\mathbb{1}_B \otimes V_C^H) |\psi\rangle \right\|^2 \leq \frac{1}{2^n}. \quad (65)$$

A similar reasoning yields that $\beta \leq 4 \cdot 2^{-n}$, where the 4 is a result of squaring the upper bound

$$\left\| W_C^{H_{x,y}} \right\| \leq \left\| \left(U_C O_C^{H_{x,y}} \right)^{q_C} \right\| + \left\| V_C^{H_{x,y}} \right\| \leq 2. \quad (66)$$

Finally, noting that $\alpha + \beta + 2\sqrt{\alpha\beta} \leq 9 \cdot 2^{-n}$ finished the proof.

4.4 Main Results

We now have all the necessary tools to prove our main results.

Theorem 5. *Let \mathcal{S} be the n -QECM scheme defined in [Definition 10](#). If the qPRF is modeled by a random oracle, then \mathcal{S} is $\log_2(9)$ -uncloneable secure.*

When we say that we model a qPRF as a random oracle, we mean that instead of sampling some $s \in \{0,1\}^\lambda$ and giving it to the parties so that they may compute the function $f_\lambda(s, \cdot) : \{0,1\}^\lambda \rightarrow \{0,1\}^n$, we instead give them oracle access to a function $H \in \text{Bool}(\lambda, n)$. Then, instead of taking the expectation over $s \in \{0,1\}^\lambda$, we take the expectation over all possible $H \in \text{Bool}(\lambda, n)$.

Proof. Let $\mathcal{A} = \{A, B, C\}$ be an uncloneable attack against \mathcal{S} as described in Definition 5. By Theorem 2, it suffices to show that the winning probability of the adversaries is bounded by $9 \cdot 2^{-n} + \eta(\lambda)$ for uniformly distributed messages and some negligible function η .

Accounting for the randomness of the encryption and for a fixed and implicit λ , the quantity we wish to bound is then given by

$$\omega = \mathbb{E}_H \mathbb{E}_\theta \mathbb{E}_x \mathbb{E}_m \text{Tr} \left[P^m (B_\theta^H \otimes C_\theta^H) \circ A(|m \oplus H(x)\rangle\langle m \oplus H(x)| \otimes |x^\theta\rangle\langle x^\theta|) \right] \quad (67)$$

where $P^m = |m\rangle\langle m| \otimes |m\rangle\langle m|$. Then, by using Lemma 1 with respect to the expectation over m to move the dependence on the string $H(x)$ from the state to the projector, we have that

$$\omega = \mathbb{E}_H \mathbb{E}_\theta \mathbb{E}_x \mathbb{E}_m \text{Tr} \left[P^{m \oplus H(x)} (B_\theta^H \otimes C_\theta^H) \circ A(|m\rangle\langle m| \otimes |x^\theta\rangle\langle x^\theta|) \right]. \quad (68)$$

Using standard purification arguments, we can add auxiliary states $|\text{aux-B}\rangle\langle \text{aux-B}|$ and $|\text{aux-C}\rangle\langle \text{aux-C}|$ to the state $A(|m\rangle\langle m| \otimes |x^\theta\rangle\langle x^\theta|)$, replace the maps B_θ^H and C_θ^H by unitary operators on the resulting larger Hilbert spaces and similarly replace the projectors $|m\rangle\langle m|$ by projectors $\{\pi_B^m\}_{m \in \{0,1\}^n}$ and $\{\pi_C^m\}_{m \in \{0,1\}^n}$ on these larger Hilbert spaces.

Following [8], these purified unitary operators will be of the form $(U_L^\theta O_L^H)^{q_L}$, acting on a Hilbert space of the form $\mathcal{Q}(\lambda)_{L_Q} \otimes \mathcal{Q}(n)_{L_R} \otimes \mathcal{Q}(s_L)_{L_S}$ for some integers $q_L, s_L \in \mathbb{N}^+$. We also assume that

$$\begin{aligned} \rho^{m,x,\theta} &= A(|m\rangle\langle m| \otimes |x^\theta\rangle\langle x^\theta|) \otimes |\text{aux-B}\rangle\langle \text{aux-B}| \otimes |\text{aux-C}\rangle\langle \text{aux-C}| \\ &\in \mathcal{D}(\mathcal{Q}(\lambda)_{B_Q} \otimes \mathcal{Q}(n)_{B_R} \otimes \mathcal{Q}(s_B)_{B_S} \otimes \mathcal{Q}(\lambda)_{C_Q} \otimes \mathcal{Q}(n)_{C_R} \otimes \mathcal{Q}(s_C)_{C_S}). \end{aligned} \quad (69)$$

Finally, we can write $\rho^{m,x,\theta}$ as an ensemble of pure states, which is to say that

$$\rho^{m,x,\theta} = \sum_{i \in I^{m,x,\theta}} p_i \left| \psi_i^{m,x,\theta} \right\rangle \left\langle \psi_i^{m,x,\theta} \right| \quad (70)$$

for some index set $I^{m,x,\theta}$, some probabilities p_i and some unit vectors $|\psi_i^{m,x,\theta}\rangle$.

It then follows that ω can be expressed as

$$\mathbb{E}_m \mathbb{E}_\theta \mathbb{E}_x \mathbb{E}_H \sum_{i \in I^{m,x,\theta}} p_i \left\| \left(\pi_B^{m \oplus H(x)} \otimes \pi_C^{m \oplus H(x)} \right) \left((U_B^\theta O_B^H)^{q_B} \otimes (U_C^\theta O_C^H)^{q_C} \right) \left| \psi_i^{m,x,\theta} \right\rangle \right\|^2. \quad (71)$$

Noting that we can bring the expectation with respect to H into the summation, we can then use Lemma 6 to express ω as

$$\frac{9}{2^n} + q \mathbb{E}_m \mathbb{E}_\theta \mathbb{E}_x \sum_{i \in I^{m,x,\theta}} p_i \sqrt{\mathbb{E}_H \mathbb{E}_k \mathbb{E}_\ell \left\| Q_x \left((U_B O_B^H)^{q_B} \otimes (U_C O_C^H)^{q_C} \right) \left| \psi_i^{m,x,\theta} \right\rangle \right\|^2} \quad (72)$$

where $q = (3q_B q_C + 2)q_B q_C$ and $Q_x = |x\rangle\langle x|_{Q_B} \otimes |x\rangle\langle x|_{Q_C}$. Defining

$$\beta_x^{\theta,H,k} = \left((U_B^\theta O_B^H)^{q_B} \right)^\dagger |x\rangle\langle x|_{Q_B} \left((U_B^\theta O_B^H)^{q_B} \right), \quad (73)$$

and similarly for $\gamma_x^{\theta,H,\ell}$ by replacing every instance of B with C , we use Jensen's lemma to bring the remaining expectations and sums into the square root and obtain

$$\omega = \frac{9}{2^n} + q \sqrt{\mathbb{E}_m \mathbb{E}_\theta \mathbb{E}_x \mathbb{E}_H \mathbb{E}_k \mathbb{E}_\ell \text{Tr} \left[\left(\beta_x^{\theta,H,k} \otimes \gamma^{\theta,H,k} \right) \rho^{m,x,\theta} \right]}. \quad (74)$$

Letting Φ_m to be the CPTP map defined by

$$\rho \mapsto A(|m\rangle\langle m| \otimes \rho) \otimes |\text{aux-B}\rangle\langle \text{aux-B}| \otimes |\text{aux-C}\rangle\langle \text{aux-C}| \quad (75)$$

we see that, for any fixed H, k, ℓ , and m , [Corollary 1](#) implies that

$$\mathbb{E}_x \mathbb{E}_\theta \text{Tr} \left[\left(\beta_x^{\theta,H,k} \otimes \gamma^{\theta,H,k} \right) \rho^{m,x,\theta} \right] \leq \left(\frac{1}{2} + \frac{1}{2\sqrt{2}} \right)^\lambda \quad (76)$$

since $\rho^{m,x,\theta} = \Phi_m(|x^\theta\rangle\langle x^\theta|)$. Thus,

$$\omega \leq \frac{9}{2^n} + q \left(\sqrt{\frac{1}{2} + \frac{1}{2\sqrt{2}}} \right)^\lambda. \quad (77)$$

Finally, since B and C are efficient quantum circuits, they may query the oracle a number of time which grows at most polynomially in λ . Thus, $q \leq p(\lambda)$ for some polynomial p . Noting that $\lambda \mapsto p(\lambda) \cdot \left(\sqrt{\frac{1}{2} + \frac{1}{2\sqrt{2}}} \right)^\lambda$ is a negligible function completes the proof.

Theorem 6. *Let \mathcal{S} be the n -QECM scheme given in [Definition 10](#). If the qPRF is modeled by a random oracle and the adversaries cannot share any entanglement, then \mathcal{S} is 0-uncloneable secure.*

Proof (Sketch). We follow the proof of [Theorem 5](#) but we use the bound given from [Lemma 5](#) instead of the one given by [Lemma 6](#). This shows that adversaries who do not share any entanglement cannot win with probability larger than $2^{-n} + \eta(\lambda)$ for a negligible η which concludes the proof.

Corollary 2. *Let \mathcal{S} be the QECM scheme given in [Definition 10](#). If the qPRF is modeled by a random oracle and the adversaries cannot share any entanglement, then \mathcal{S} is indistinguishable-uncloneable secure.*

Proof (Sketch). Use [Theorem 6](#) with [Theorem 3](#).

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A Technical Proofs

Proof (Corollary 1). It suffices to apply [Theorem 1](#) with the same POVMs and the state

$$\rho = (\mathbb{1}_A \otimes \Phi_{A'}) |\text{EPR}_\lambda\rangle\langle\text{EPR}_\lambda|_{AA'} = \frac{1}{2^n} \sum_{r,s \in \{0,1\}^\lambda} |r\rangle\langle s| \otimes \Phi(|r\rangle\langle s|), \quad (78)$$

which is the result of applying the map Φ to the second half of λ EPR pairs. Note that for all $\theta \in \{0, 1\}$ we have that

$$\frac{1}{2^n} \sum_{r,s \in \{0,1\}^\lambda} |r\rangle\langle s| \otimes \Phi(|r\rangle\langle s|) = \frac{1}{2^n} \sum_{r,s \in \{0,1\}^\lambda} |r^\theta\rangle\langle s^\theta| \otimes \Phi(|r^\theta\rangle\langle s^\theta|). \quad (79)$$

We then have that

$$\begin{aligned} & \mathbb{E}_\theta \sum_{x \in \{0,1\}^\lambda} \text{Tr} [(|x^\theta\rangle\langle x^\theta| \otimes B_x^\theta \otimes C_x^\theta) \rho] \\ &= \mathbb{E}_\theta \frac{1}{2^n} \sum_{x,r,s \in \{0,1\}^\lambda} \text{Tr} [(|x^\theta\rangle\langle x^\theta| \otimes B_x^\theta \otimes C_x^\theta) (|r^\theta\rangle\langle s^\theta| \otimes \Phi(|r^\theta\rangle\langle s^\theta|))] \\ &= \mathbb{E}_\theta \frac{1}{2^n} \sum_{x,r,s \in \{0,1\}^\lambda} \text{Tr} [|x^\theta\rangle\langle x^\theta| |r^\theta\rangle\langle s^\theta|] \cdot \text{Tr} [(B_x^\theta \otimes C_x^\theta) \Phi(|r^\theta\rangle\langle s^\theta|)] \\ &= \mathbb{E}_\theta \frac{1}{2^n} \sum_{x \in \{0,1\}^\lambda} \text{Tr} [(B_x^\theta \otimes C_x^\theta) \Phi(|x^\theta\rangle\langle x^\theta|)]. \end{aligned} \quad (80)$$

The first equality holds by definition of ρ and the linearity of the operators. The second holds by property of the trace and the third since

$$\mathrm{Tr} [|x^\theta\rangle\langle x^\theta| |r^\theta\rangle\langle s^\theta|] = \begin{cases} 1 & \text{if } x = r = s \\ 0 & \text{else.} \end{cases} \quad (81)$$

Thus the bound given in [Theorem 1](#) is directly applicable.

Proof (Lemma 1). Recall that for any fixed string $s \in \{0, 1\}^n$, the map $x \mapsto x \oplus s$ is a permutation which is its own inverse. If we define $g : \{0, 1\}^n \rightarrow \mathbb{R}$ by $x \mapsto f(x, x \oplus s)$, we then have that

$$\mathbb{E}_x f(x, x \oplus s) = \mathbb{E}_x g(x) = \mathbb{E}_x g(x \oplus s) = \mathbb{E}_x f(x \oplus s, x) \quad (82)$$

which concludes the proof.

Proof (Lemma 2). Proceed by induction over n with a trivial base case when $n = 1$ and note that

$$\left(a^n + \sum_{k=0}^{n-1} a^{n-k-1} b c^k \right) c = a^{n+1} + a^n b + \sum_{k=0}^{n-1} a^{n-k-1} b c^{k+1} \quad (83)$$

$$= a^{n+1} + \sum_{k=0}^{(n+1)-1} a^{(n+1)-k-1} b c^k. \quad (84)$$

Proof (Lemma 3). We first note that $\|v_0\| \leq \|\sum_{i=0}^n v_i\| + \sum_{i=1}^n \|v_i\| \leq 1 + n$. Then, using the triangle inequality, we have that

$$\left\| \sum_{i=0}^n v_i \right\|^2 \leq \left(\sum_{i=0}^n \|v_i\| \right)^2 = \sum_{i=0}^n \sum_{j=0}^n \|v_i\| \cdot \|v_j\|. \quad (85)$$

We consider the summands in the right hand side differently depending on the value of i . If $i = 0$, we note that

$$\sum_{j=0}^n \|v_0\| \cdot \|v_j\| \leq \|v_0\|^2 + (n+1) \sum_{j=1}^n \|v_j\|. \quad (86)$$

If $i \neq 0$, we note that

$$\sum_{j=0}^n \|v_i\| \cdot \|v_j\| \leq \|v_i\| \sum_{j=0}^n \|v_j\| \leq ((n+1) + n) \|v_i\|. \quad (87)$$

We obtain the result by adding each of these bounds, which is to say that

$$\left\| \sum_{i=0}^n v_i \right\|^2 \leq \|v_0\|^2 + (3n+2) \sum_{i=1}^n \|v_i\|. \quad (88)$$