Finite field mapping to elliptic curves of j-invariant 1728

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Abstract. This article generalizes the simplified Shallue–van de Woestijne–Ulas (SWU) method of deterministic finite field mapping $\mathbb{F}_q \to E(\mathbb{F}_q)$ to the case of any elliptic $\mathbb{F}_q$-curve E of j-invariant 1728. More precisely, we obtain a rational $\mathbb{F}_q$-curve C (and its explicit quite simple proper $\mathbb{F}_q$-parametrization $par\colon \mathbb{P}^1 \to C$) on the Kummer surface K associated with the direct product $E \times E'$, where E' is the quadratic $\mathbb{F}_q$-twist of E. The SWU method consists in computing the direct image of par and a subsequent inverse image (P, Q) of the natural two-sheeted covering $\rho\colon E \times E' \to K$. Denoting by $\sigma\colon E' \xrightarrow{\sim} E$ the corresponding $\mathbb{F}_{q^2}$-isomorphism, it is easily seen that $P \in E(\mathbb{F}_q)$ or $\sigma(Q) \in E(\mathbb{F}_q)$.

Key words: finite fields, pairing-based cryptography, elliptic curves of j-invariant 1728, Kummer surfaces, rational curves, Weil restriction, isogenies.

Introduction

Since its invention in the early 2000s, *pairing-based cryptography* (on an elliptic curve $E\colon y^2 = f(x)$ over an finite field $\mathbb{F}_q$ of characteristic p) has become more and more popular every year, for example in cryptocurrencies. One of the latest reviews of standards, commercial products and libraries for this type of cryptography is given in [42, §5].

Many pairing (and other) protocols often use some (not necessarily injective or surjective) mapping $h\colon \mathbb{F}_q \to E(\mathbb{F}_q)$ (sometimes called *hashing* or *encoding*) that is efficiently computable. Reviews of this topic are represented in [13, Ch. 8], [33]. Of course, we can just randomly change few bits of a given element $a \in \mathbb{F}_q$ such that $\sqrt{f(a)} \in \mathbb{F}_q$, but this approach is vulnerable to timing attacks [13, §8.2]. Another deterministic method consists in the scalar multiplication $a \mapsto [a]P$ for some point $P \in E(\mathbb{F}_q)$. Unfortunately, it is also insecure [13, §8.1].

There are many safe (at least at first glance) constructions of the desired deterministic mapping such as Boneh–Franklin (bijective) mapping [7, §5.2] for supersingular curves of $j(E) = 0$, Icart mapping [23] for $q \equiv 2 \pmod 3$, or Elligator 2 [5, §5] provided that $2 \mid \#E(\mathbb{F}_q)$ and $j(E) \neq 1728$. However the unique method valid for arbitrary E and $\mathbb{F}_q$ was proposed in [37] (based on [31, Th. 14.1]) and improved in [35]. Now it is often called in honor of some its authors: *Shallue, van de Woestijne*, and sometimes *Ulas*.

The SWU method consists in parametrizing a (possibly singular) *rational* $\mathbb{F}_q$*-curve* C (see, e.g., [34, §4.1]) lying on some *Calabi–Yau* $\mathbb{F}_q$*-threefold* T (see, e.g., [40]). The latter is a minimal singularity resolution of some *generalized Kummer threefold* (studied in [1, §4.2], [10,

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§4], [12, §4.1.1]), namely the geometric quotient of E^3 under some action of $(\mathbb{Z}/2)^2$. Looking at the definition, we obtain the affine model

$$T\colon y^2 = f(x_1)f(x_2)f(x_3) \ \subset\ \mathbb{A}^4,$$

where (x_i, y_i) are three general points of E and $y := y_1y_2y_3$. Having a point $P \in C(\mathbb{F}_q)$, for at least one coordinate $a_i := x_i(P)$ the value $f(a_i)$ is a quadratic residue in $\mathbb{F}_q$. Therefore we get the points $\big(a_i, \pm\sqrt{f(a_i)}\big) \in E(\mathbb{F}_q)$.

According to [26, Th. 2] T is not uniruled threefold [11, Ch. 4], but can be represented in the form $(K \times E)/(\mathbb{Z}/2)$ [41], where K is the *Kummer surface* for E^2 (see, e.g., [6, §4]). By virtue of the latter and the Bogomolov–Tschinkel theorem [6, Th. 1.1] the surface K and hence threefold T are covered over $\overline{\mathbb{F}_q}$ by rational curves. We stress that over the field $\mathbb{C}$ (unlike a prime characteristic) this would lead to a contradiction. Besides, for $a \in \mathbb{F}_q$ such that $b := f(a)$ is a quadratic non-residue in $\mathbb{F}_q$ consider the hyperplane section

$$K'\colon y^2 = f(x_1)f(x_2)b \ \subset\ \mathbb{A}^3.$$

As you can see, this is the quadratic $\mathbb{F}_q$-twist of K, which in itself is the Kummer surface for $E \times E'$, where $E'\colon y^2 = f(x)b$ is the quadratic $\mathbb{F}_q$-twist of E.

If for SWU method we take a rational $\mathbb{F}_q$-curve on K' one obtains so-called *simplified SWU method* [8, §7]. In comparison with (classical) SWU method it allows to avoid 1 quadratic residuosity test in $\mathbb{F}_q$, whose computational complexity is approximately equal (by the quadratic reciprocity law [24, Th. 5.1]) to that of inversion in $\mathbb{F}_q$, which is a quite laborious operation. Nevertheless, despite Bogomolov–Tschinkel theorem finding a rational $\mathbb{F}_q$-curve C on K' (unlike T) is not a very simple task. For $j(E) \neq 0, 1728$ a desired curve (even for a larger class of Kummer surfaces) was first constructed in [29] (see also [32], [38, §2]). Interestingly, Articles [28, §1], [29] then use C to prove some arithmetic results over the field $\mathbb{Q}$.

However in pairing-based cryptography ordinary elliptic curves of $j(E) = 0, 1728$ are only interesting [13, Ch. 4]. This is due to the existence of high-degree twists for them, leading to faster pairing computation [13, §3.3]. In [43, §4.3] for some curve E with $j(E) = 0$ over the field $\mathbb{F}_p$ (resp. $\mathbb{F}_{p^2}$) it is proposed to use an ascending $\mathbb{F}_p$-isogeny (resp. $\mathbb{F}_{p^2}$-isogeny) $\mathcal{E} \to E$ of degree 11 (resp. 3) from certain auxiliary elliptic curve $\mathcal{E}$ with $j(\mathcal{E}) \neq 0, 1728$. Unfortunately, this approach highly depends on $\mathbb{F}_q$, that is in some cases there is no a desired $\mathbb{F}_q$-isogeny of small degree, which could be rapidly computed.

In this work we resolve the problem of constructing a rational $\mathbb{F}_q$-curve $C \subset K'$ for all elliptic $\mathbb{F}_q$-curves $E_a\colon y^2 = x^3 - ax$ with $j = 1728$. The most famous example of such pairing-friendly curves are Kachisa–Schaefer–Scott (KSS) curves of embedding degree 16 [25, Exam. 4.2], which have become (according to [3], [4], [17]) a popular alternative for those of $j = 0$. It is worth noting that to derive C we actively use (among other things) the theory of *Weil restriction* (*descent*) [18, §8.1] for elliptic curves with respect to the extension $\mathbb{F}_{q^2}/\mathbb{F}_q$. The cryptographic community knows this operation as an instrument of cryptanalysis [9, §22.3]. Interestingly, coefficients of our functions defining a proper parametrization of C are almost entirely some powers of 2 and 3. This allows to compute the corresponding mapping $h\colon \mathbb{F}_q \to E_a(\mathbb{F}_q)$ very quickly. Finally, let us remark that at worst h is 8:1 map (as in the classical SWU method), that is for almost every point from $E_a(\mathbb{F}_q)$ its inverse image (under h) contains at most 8 elements.

The article is organized as follows. In paragraphs 1 and 2 we recall basic facts about the Weil restriction of elliptic curves (with respect to $\mathbb{F}_{q^2}/\mathbb{F}_q$) and the Kummer surface for the direct product of two elliptic curves respectively. Next, §3 is dedicated to the new construction of a (singular) rational $\mathbb{F}_q$-curve on K' for elliptic curves E_a (of j-invariant 1728), providing explicit formulas for the mapping $h\colon \mathbb{F}_q \to E_a(\mathbb{F}_q)$. Finally, in §4 we make some remarks and conclusions, including the estimation of cardinality for the image of h.

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Contents

1 The Weil restriction of an elliptic $\mathbb{F}_{q^2}$-curve

In this paragraph we freely use some terms from the language of abelian varieties (for details see [30]). For a prime $p > 3$ and any its power q consider the finite field extension $\mathbb{F}_{q^2} = \mathbb{F}_q(\sqrt{\gamma})$, where $\gamma \in \mathbb{F}_q$, $\sqrt{\gamma} \notin \mathbb{F}_q$. Besides, for $i \in \{0, 1\}$ consider two elliptic $\mathbb{F}_{q^2}$-curves $\overline{E_i}$ given by the affine Weierstrass forms

$$E_i\colon y_i^2 = x_i^3 + a^{q^i} x_i + b^{q^i} \quad \subset \quad \mathbb{A}^2_{(x_i,y_i)}.$$

In other words, $\overline{E_i} = E_i \sqcup \{P_\infty\} \subset \mathbb{P}^2$, where $P_\infty := (0:1:0)$. These curves are obviously isogenous by means of the Frobenius maps $\mathrm{Fr}\colon E_0 \to E_1$, $\mathrm{Fr}\colon E_1 \to E_0$ over $\mathbb{F}_q$.

Consider the Weil restriction R_i (resp. $\overline{R_i}$) of E_i (resp. $\overline{E_i}$) with respect to $\mathbb{F}_{q^2}/\mathbb{F}_q$ (see, e.g., [18, §8.1]). We stress that $\overline{R_i} \not\simeq R_i \cup \{P_\infty\}$ even over $\overline{\mathbb{F}_q}$, however we will identify E_i (resp. R_i) with $\overline{E_i}$ (resp. $\overline{R_i}$) for simplicity of the notation. Let $A := E_0 \times E_1$ and

$$a := a_0 + a_1\sqrt{\gamma}, \qquad b := b_0 + b_1\sqrt{\gamma}, \qquad x_0 := u_0 + u_1\sqrt{\gamma}, \qquad y_0 := v_0 + v_1\sqrt{\gamma},$$

where $a_0, a_1, b_0, b_1 \in \mathbb{F}_q$. By definition $R_i(\mathbb{F}_q) = E_i(\mathbb{F}_{q^2})$ and

$$R_i\colon \begin{cases} v_0^2 + \gamma v_1^2 = u_0^3 + 3\gamma u_0 u_1^2 + a_0 u_0 + (-1)^i a_1 \gamma u_1 + b_0, \\ 2v_0 v_1 = \gamma u_1^3 + 3u_0^2 u_1 + a_0 u_1 + (-1)^i (a_1 u_0 + b_1) \end{cases} \quad \subset \quad \mathbb{A}^4_{(u_0,v_0,u_1,v_1)}.$$

Although j-invariants of the curves E_0, E_1 may be different, we always have the involution

$$s\colon \mathbb{A}^4 \xrightarrow{\sim} \mathbb{A}^4, \qquad (u_0, v_0, u_1, v_1) \mapsto (u_0, v_0, -u_1, -v_1)$$

such that $s\colon R_0 \xrightarrow{\sim} R_1$ and $s|_{R_0(\mathbb{F}_q)} = \mathrm{Fr}|_{E_0(\mathbb{F}_{q^2})}$. Thus we will also identify R_0 with R_1, omitting the index. Besides, there is an $\mathbb{F}_{q^2}$-isomorphism

$$\theta\colon \mathbb{A}^4_{(u_0,v_0,u_1,v_1)} \xrightarrow{\sim} \mathbb{A}^4_{(x_0,y_0,x_1,y_1)} \qquad \text{s.t.} \qquad \theta\colon R \xrightarrow{\sim} A$$

given by the matrix

$$\theta := \begin{pmatrix} 1 & 0 & \sqrt{\gamma} & 0 \\ 0 & 1 & 0 & \sqrt{\gamma} \\ 1 & 0 & -\sqrt{\gamma} & 0 \\ 0 & 1 & 0 & -\sqrt{\gamma} \end{pmatrix}, \qquad \text{where} \qquad \theta^{-1} = \frac{1}{2\sqrt{\gamma}} \begin{pmatrix} \sqrt{\gamma} & 0 & \sqrt{\gamma} & 0 \\ 0 & \sqrt{\gamma} & 0 & \sqrt{\gamma} \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

Consider the permutation

$$s' := \theta \circ s \circ \theta^{-1}\colon \mathbb{A}^4 \xrightarrow{\sim} \mathbb{A}^4, \qquad (x_0, y_0, x_1, y_1) \mapsto (x_1, y_1, x_0, y_0)$$

and the "twisted" Frobenius endomorphism

$$\pi\colon \mathbb{A}^4 \xrightarrow{\sim} \mathbb{A}^4, \qquad (x_0, y_0, x_1, y_1) \mapsto (x_1^q, y_1^q, x_0^q, y_0^q) \qquad \text{s.t.} \qquad \pi\colon A \xrightarrow{\sim} A.$$

It is easily checked that $\theta^{-1} \circ \pi \circ \theta$ is the (usual) Frobenius endomorphism. Thus π-invariant (hence $\mathbb{F}_{q^2}$-rational) curves $C \subset A$ and maps $\varphi\colon A \to \mathbb{A}^4_{(x_0,y_0,x_1,y_1)}$ correspond to $\mathbb{F}_q$-ones

$$\theta^{-1}(C) \subset R, \qquad \theta^{-1} \circ \varphi \circ \theta\colon R \to \mathbb{A}^4_{(u_0,v_0,u_1,v_1)}.$$

This means that

$$C = s'\big(C^{(1)}\big), \qquad \varphi = \big(\varphi_{x_0}, \varphi_{y_0}, \varphi_{x_0}^{(1)} \circ s', \varphi_{y_0}^{(1)} \circ s'\big),$$

where $C^{(1)}$ is the $\mathbb{F}_q$-conjugate curve to C and $\varphi_{x_0}^{(1)}, \varphi_{y_0}^{(1)}$ are the $\mathbb{F}_q$-conjugate functions to some $\varphi_{x_0}, \varphi_{y_0} \in \mathbb{F}_{q^2}(A)$.

It is also worth noting that on A there are natural involutions $[-1]$ and $[-1]^i \times [-1]^{i+1}$ (for $i \in \{0, 1\}$), which are transformed to R by θ as

$$\begin{aligned} &(u_0, v_0, u_1, v_1) \mapsto (u_0, -v_0, u_1, -v_1), \\ &(u_0, v_0, u_1, v_1) \mapsto \Big(u_0,\ (-1)^i \sqrt{\gamma} v_1,\ u_1,\ (-1)^i (\sqrt{\gamma})^{-1} v_0\Big) \end{aligned}$$

respectively.

Hereafter we assume that $a, b \in \mathbb{F}_q$ (i.e., $E := E_0 = E_1$). In this case $s'\colon E^2 \xrightarrow{\sim} E^2$. Let $\Delta, \Delta' \subset E^2$ be the diagonal and antidiagonal respectively. Then

$$\theta^{-1}(\Delta) = R \cap \{u_1 = v_1 = 0\} = E, \qquad \theta^{-1}(\Delta') = R \cap \{u_1 = v_0 = 0\} = E',$$

where the latter is the quadratic $\mathbb{F}_q$-twist of E:

$$E'\colon \gamma y^2 = x^3 + ax + b, \qquad \sigma\colon E' \xrightarrow{\sim} E, \qquad (x, y) \mapsto (x, \sqrt{\gamma} y).$$

Consider the exact sequences

$$0 \to E \hookrightarrow R \xrightarrow{\tau'} E' \to 0, \qquad 0 \to E' \hookrightarrow R \xrightarrow{\tau} E \to 0,$$

of $\mathbb{F}_q$-(homo)morphisms, where $\tau := [1] + s$, $\tau' := [1] - s$. Note that $\tau|_{R(\mathbb{F}_q)}$ is just the trace map on E with respect to $\mathbb{F}_{q^2}/\mathbb{F}_q$. As a result, we obtain the $\mathbb{F}_q$-rational $(2, 2)$-isogeny

$$\chi := \tau \times \tau'\colon R \to E \times E' \qquad \text{with} \qquad \ker(\chi) = E \cap E' = E[2] = E'[2].$$

Finally, the $(2, 2)$-isogenies

$$\psi := \chi \circ \theta^{-1}\colon E^2 \to E \times E', \qquad \psi = \begin{pmatrix} 1 & 1 \\ \sigma^{-1} & -\sigma^{-1} \end{pmatrix}$$

and $\widehat{\chi}\colon E \times E' \to R$ (dual to χ) have the kernels

$$\ker(\psi) = \Delta \cap \Delta' = \Delta[2] = \Delta'[2], \qquad \ker(\widehat{\chi}) = \Gamma \cap \Gamma' = \Gamma[2] = \Gamma'[2],$$

where Γ, Γ' are the graphs of σ and $-\sigma = \mathrm{Fr} \circ \sigma \circ \mathrm{Fr}^{-1}$ respectively.

2 Kummer surfaces

In this paragraph we handle some concepts of two-dimensional algebraic geometry, which can be found, for example, in [19, Ch. V]. For $i \in \{0, 1\}$ consider any two elliptic $\mathbb{F}_q$-curves $\overline{E_i} \subset \mathbb{P}^2$ given by the Weierstrass forms

$$E_i\colon y_i^2 = f_i(x_i) := x_i^3 + a_i x_i + b_i \quad \subset \quad \mathbb{A}^2_{(x_i, y_i)}$$

and their direct product

$$A := E_0 \times E_1 \subset \mathbb{A}^4_{(x_0, y_0, x_1, y_1)}, \qquad \overline{A} = \overline{E_0} \times \overline{E_1} \hookrightarrow \mathbb{P}^8,$$

where the second map is the Segre embedding. For $j \in \{0, 1, 2\}$ let r_j (resp. s_j) are roots of f_0 (resp. f_1) and $P_{r_j} := (r_j, 0)$ (resp. $P_{s_j} := (s_j, 0)$) are order 2 points on E_0 (resp. E_1). Also, let $\infty := (1 : 0)$ and $P_\infty := (0 : 1 : 0)$. Note that

$$\overline{A} = A \;\sqcup\; \overline{E_0} \times \{P_\infty\} \;\cup\; \{P_\infty\} \times \overline{E_1}.$$

Hereafter we will identify E_0, $\overline{E_0}$, $\overline{E_0} \times \{P_\infty\}$ (resp. E_1, $\overline{E_1}$, $\{P_\infty\} \times \overline{E_1}$), and A, $\overline{A}$.

By definition, the *Kummer surface* K_A of A (see, e.g., [6, §4]) is the minimal singularity resolution bl of the geometric quotient $A/[-1]$, which is sometimes called (*singular*) *Kummer surface*. In other words, bl is blowing up 16 nodes, which form the image of $A[2]$ to $A/[-1]$. If $E_0 \simeq E_1$, then at least over $\overline{\mathbb{F}_q}$ the Kummer surface K_A is birationally isomorphic to a quartic in $\mathbb{P}^3$ with 12 nodes. It is so-called *desmic surface*, which is related to the *desmic system* of three tetrahedrons (for more details see, e.g., [21, §B.5.2]).

There are also natural models

$$A/[-1]\colon y^2 = f_0(x_0)f_1(x_1) \qquad \subset \qquad \mathbb{A}^3_{(x_0,x_1,y)},$$
$$K_A\colon y_0^2 f_1(x_1) = y_1^2 f_0(x_0) \qquad \subset \qquad \mathbb{A}^2_{(x_0,x_1)} \times \mathbb{P}^1_{(y_0:y_1)}$$

and the two-sheeted maps

$$\rho\colon A \to A/[-1], \qquad (x_0, y_0, x_1, y_1) \mapsto (x_0, x_1, y_0 y_1),$$
$$\rho'\colon A \dashrightarrow K_A, \qquad (x_0, y_0, x_1, y_1) \mapsto \big((x_0, x_1), (y_0 : y_1)\big).$$

Therefore blowing up and blowing down maps have the form

$$bl = \rho \circ (\rho')^{-1}\colon K_A \to A/[-1], \qquad \big((x_0, x_1), (y_0 : y_1)\big) \mapsto \Big(x_0, x_1, f_1(x_1)\frac{y_0}{y_1}\Big) = \Big(x_0, x_1, f_0(x_0)\frac{y_1}{y_0}\Big),$$
$$bl^{-1} = \rho' \circ \rho^{-1}\colon A/[-1] \dashrightarrow K_A, \qquad (x_0, x_1, y) \mapsto \Big((x_0, x_1), \big(y : f_1(x_1)\big)\Big) = \Big((x_0, x_1), \big(f_0(x_0) : y\big)\Big)$$

respectively. Further, the involutions $[1]\times[-1]$, $[-1]\times[1]$ on A are induced to $A/[-1]$ as $(x_0, x_1, y) \mapsto (x_0, x_1, -y)$. The quotient of $A/[-1]$ under this new involution is $\mathbb{P}^1 \times \mathbb{P}^1$ and the corresponding natural map is denoted by *pr*. In simple words, it is the projection to the coordinates x_0, x_1.

For $r \in \{r_0, r_1, r_2, \infty\}$, $s \in \{s_0, s_1, s_2, \infty\}$ let

$$L_r := \rho(\{P_r\} \times E_1), \qquad M_s := \rho(E_0 \times \{P_s\})$$

and $E_{r,s}$ be the exceptional (-2)-curve on K_A corresponding to the point $\rho(P_r, P_s)$. For $r, s \neq \infty$ it is easily seen that

$$L_r = \{x_0 = r, y = 0\}, \qquad M_s = \{x_1 = s, y = 0\}, \qquad E_{r,s} = \{x_0 = r, x_1 = s\}.$$

Since $Ram := \bigsqcup_{r,s}(L_r \cup M_s)$ is exactly the ramification locus of *pr*, we will identify the lines L_r, $pr(L_r)$ and M_s, $pr(M_s)$. Note that

$$\overline{A}/[-1] = A/[-1] \sqcup L_\infty \cup M_\infty, \qquad \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{A}^2_{(x_0,x_1)} \sqcup L_\infty \cup M_\infty.$$

It is well known that K_A is a K3 surface [22], i.e., its canonical class and the first cohomology space $\mathrm{H}^1(K_A, \mathcal{O}_{K_A})$ of the structure sheaf $\mathcal{O}_{K_A}$ are zero. According to [39, §2.8.4], [22, Ch. 17] we have:

$$\mathrm{NS}(A) \simeq \mathbb{Z}[E_0, E_1] \oplus \mathrm{Hom}(E_0, E_1) \qquad \mathrm{NS}(K_A) \simeq \mathrm{Pic}(K_A) \simeq \mathrm{NS}(A) \oplus \mathbb{Z}[\{E_{r,s}\}]^{\mathrm{Fr}}$$

In particular, ranks of these free groups (i.e., Picard $\mathbb{F}_q$-numbers) satisfy the inequalities

$$2 \leqslant \rho(A) \leqslant 6, \qquad 8 \leqslant \rho(K_A) \leqslant 22$$

If $\rho(A) = 2$, then the curves E_0, E_1 are not isogenous over $\mathbb{F}_q$. At the same time, from $\rho(A) = 6$ follows that E_0, E_1 are supersingular and the surface K_A is geometrically unirational.

For an absolutely irreducible (possibly singular) $\mathbb{F}_q$-curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1 \setminus Ram$ we denote by r_C the count (over $\overline{\mathbb{F}_q}$) of branches B [20, §4.3] on C (i.e., discrete valuations of $\overline{\mathbb{F}_q}(C)$ [32, §1]) such that the intersection number (multiplicity) $B \cdot Ram$ is odd.

Theorem 1 ([32, Prop. 1.2.3]). *Suppose that C is a rational curve. Let $D := pr^{-1}(C)$ and k be one of the fields $\mathbb{F}_q, \mathbb{F}_{q^2}$.*

1. *If $r_C = 0$, then D consists of two absolutely irreducible rational curves D_0, D_1 defined at most over $\mathbb{F}_{q^2}$. Moreover, D is reducible over k if and only if $y = \sqrt{f_0(x_0)f_1(x_1)} \in k(C)$. In this case $pr\colon D_0 \to C$, $pr\colon D_1 \to C$ are birational k-morphisms.*

2. *If $r_C > 0$, then D is an absolutely irreducible (possibly singular) $\mathbb{F}_q$-curve of geometric genus $r_C/2 - 1$ (in particular, $2 \mid r_C$). Moreover, for $r_C > 2$ the curve D is hyperelliptic.*

Theorem 2 ([6, Lem. 4.1], [32, §2.1]).

1. *A curve $D \subset A/[-1]$ is rational if and only if $H := \rho^{-1}(D) \subset A$ is a hyperelliptic curve such that $H \cap A[2] \neq \emptyset$ and the hyperelliptic involution on H is the restriction of $[-1]$.*

2. *Moreover, if the image $C := pr(D)$ is of bidegree $(1,1)$ and $r_C = 2$, then geometric genus $g(H) = 2$ and H also has a non-hyperelliptic involution i such that $H/i = E_0$, $H/{-i} = E_1$ (such hyperelliptic curves were studied in detail, e.g., in [16]).*

Suppose that E_0, E_1 are $\mathbb{F}_q$-conjugate elliptic $\mathbb{F}_{q^2}$-curves as in §1 (hence we will use its notation). Let K_R be the Kummer surface of the Weil restriction R and

$$Q := K_R/\langle [1]\times[-1] \rangle \;=\; R/\langle [1]\times[-1], [-1]\times[1] \rangle.$$

It can be checked that Q is the Weil restriction of $\mathbb{P}^1$ with respect to $\mathbb{F}_{q^2}/\mathbb{F}_q$, which is also the unique (up to a change of variables) quadratic surface in $\mathbb{P}^3$ without $\mathbb{F}_q$-lines [18, Exer. 8.1.6.iii]. Looking at the transformation $\theta\colon R \xrightarrow{\sim} A$, we see that in affine coordinates the natural two-sheeted maps have the form

$$\rho\colon R \to R/[-1], \qquad (u_0, v_0, u_1, v_1) \mapsto (u_0, u_1, v_0^2 - \gamma v_1^2),$$
$$pr\colon R/[-1] \to Q, \qquad (u_0, u_1, v) \mapsto (u_0, u_1).$$

Finally, denote by $\overline{\theta}$, $\overline{\overline{\theta}}$ isomorphisms over $\mathbb{F}_{q^2}$ that are the restrictions of θ to $R/[-1]$ and Q respectively. Thus we obtain the commutative diagram

$$\begin{array}{ccc} R & \overset{\theta}{\xrightarrow{\sim}} & A \\ \rho\downarrow & & \downarrow\rho \\ R/[-1] & \overset{\overline{\theta}}{\xrightarrow{\sim}} & A/[-1] \\ pr\downarrow & & \downarrow pr \\ Q & \overset{\overline{\overline{\theta}}}{\xrightarrow{\sim}} & \mathbb{P}^1\times\mathbb{P}^1 \end{array}$$

3 Constructing a rational $\mathbb{F}_q$-curve on the Kummer surface

We will often use notation and results from §1-2. Consider a finite field $\mathbb{F}_q$ of characteristic $p > 3$ and $n := \log_p(q)$. We are interested in elliptic $\mathbb{F}_q$-curves $E_a \colon y^2 = x^3 - ax$ of j-invariant 1728. According to [36, Exam. V.4.5] they are ordinary if and only if $p \equiv 1 \pmod 4$, i.e., $\sqrt{-1} \in \mathbb{F}_p$. For definiteness we will suppose this condition, because for pairing-based cryptography supersingular curves are insecure at the moment.

Any two $\mathbb{F}_q$-curves of $j = 1728$ are isomorphic (at most over $\mathbb{F}_{q^4}$) by the map

$$E_a \xrightarrow{\sim} E_{a'}, \qquad (x, y) \mapsto (\sqrt{\alpha}x, \sqrt[4]{\alpha^3}y),$$

where $\alpha := a'/a$. Therefore provided that $\sqrt{a} \notin \mathbb{F}_q$ (hence $\sqrt[4]{a} \notin \mathbb{F}_{q^2}$) curves E_{a^i} (for $i \in \mathbb{Z}/4$) are unique ones (up to $\mathbb{F}_q$-isomorphism) of $j = 1728$. For E_1, E_a there are the quadratic $\mathbb{F}_q$-twists

$$E'_1 \colon ay^2 = x^3 - x, \qquad E'_a \colon ay^2 = x^3 - ax$$

and the corresponding $\mathbb{F}_{q^2}$-isomorphisms $\sigma \colon E'_1 \xrightarrow{\sim} E_1$, $\sigma \colon E'_a \xrightarrow{\sim} E_a$. It is obvious that

$$E'_1 \xrightarrow{\sim} E_{a^2}, \qquad E'_a \xrightarrow{\sim} E_{a^3}, \qquad (x, y) \mapsto (ax, a^2y).$$

The curves E_{a^i} are pairwise non-isogenous over $\mathbb{F}_q$ [13, Prop. 2.5]. Hence, in particular, Picard $\mathbb{F}_q$-numbers are equal to

$$\rho(K_{E_1 \times E'_1}) = 18, \qquad \rho(K_{E_a \times E'_a}) = 12.$$

In this paragraph we focus on constructing a rational $\mathbb{F}_q$-curve only on the Kummer surface $K_{E_a \times E'_a}$, because this is more difficult than analogous task for $K_{E_1 \times E'_1}$.

Obviously,

$$E_a[2] = E'_a[2] = \{P_0, P_\pm, P_\infty\}, \qquad P_0 := (0, 0), \qquad P_\pm := (\pm\sqrt{a}, 0).$$

According to the Vélu formulas [15, §25.1.1] we obtain:

$$E_a/P_0 \simeq_{\mathbb{F}_q} E_a, \qquad E_\pm := E_a/P_\pm \colon\ y^2 = x^3 - 11ax \mp 14a\sqrt{a},$$

where $j(E_\pm) = 287496$, and the corresponding vertical dual to each other 2-isogenies

$$\widehat{\varphi_\pm} \colon E_a \to E_\pm, \qquad \varphi_\pm \colon E_\pm \to E_a$$

have the form

$$\widehat{\varphi_\pm} = \begin{cases} x := x + \dfrac{2a}{x \mp \sqrt{a}}, \\ y := \Big(1 - \dfrac{2a}{(x \mp \sqrt{a})^2}\Big)y, \end{cases} \qquad \varphi_\pm = \begin{cases} x := \Big(x + \dfrac{a}{x \pm 2\sqrt{a}}\Big)/4, \\ y := \Big(1 - \dfrac{a}{(x \pm 2\sqrt{a})^2}\Big)y/8. \end{cases}$$

For compactness we will often use the value $\alpha_\pm := 1 \pm 2\sqrt{2}$. Note that

$$E_+[2] = \{Q_0^{(0)}, Q_\pm^{(0)}, P_\infty\}, \qquad E_-[2] = \{Q_0^{(1)}, Q_\pm^{(1)}, P_\infty\},$$

where

$$Q_0^{(i)} := \big((-1)^{(i+1)} 2\sqrt{a}, 0\big), \qquad Q_\pm^{(i)} := \big((-1)^i \alpha_\pm \sqrt{a}, 0\big).$$

Clearly,

$$\begin{aligned} \widehat{\varphi_+}(P_0) &= \widehat{\varphi_+}(P_-) = Q_0^{(0)}, & \varphi_+\big(Q_\pm^{(0)}\big) &= P_+, \\ \widehat{\varphi_-}(P_0) &= \widehat{\varphi_-}(P_+) = Q_0^{(1)}, & \varphi_-\big(Q_\pm^{(1)}\big) &= P_- \end{aligned}$$

and hence

$$E_a = E_+/Q_0^{(0)} = E_-/Q_0^{(1)}.$$

Finally, letting

$$A_\pm := E_+ \times E_-, \qquad A_a := E_a \times E_a, \qquad A'_a := E_a \times E'_a,$$

consider the dual to each other $(2,2)$-isogenies

$$\widehat{\varphi} := \widehat{\varphi_+} \times \widehat{\varphi_-} \colon\ A_a \to A_\pm, \qquad \varphi := \varphi_+ \times \varphi_- \colon\ A_\pm \to A_a,$$

which are π-invariant.

Next, let

$$\begin{aligned} \overline{\varphi} &:= \rho \circ \varphi \circ \rho^{-1} \colon\ A_\pm/[-1] \to A_a/[-1], & \overline{\psi} &:= \rho \circ \psi \circ \rho^{-1} \colon\ A_a/[-1] \to A'_a/[-1], \\ \overline{\overline{\varphi}} &:= pr \circ \overline{\varphi} \circ pr^{-1} \colon\ \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1, & \overline{\overline{\psi}} &:= pr \circ \overline{\psi} \colon\ A_a/[-1] \to \mathbb{P}^1 \times \mathbb{P}^1, \end{aligned}$$

where ψ is taken from §1. These maps form a commutative diagram represented in Figure 4. Note that $\overline{\psi}$ does not descend to any map $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. Looking at the formulas of the isogeny φ, we obtain:

$$\overline{\varphi} = \begin{cases} x_0 := \Big(x_0 + \dfrac{a}{x_0 + 2\sqrt{a}}\Big)/4, \\ x_1 := \Big(x_1 + \dfrac{a}{x_1 - 2\sqrt{a}}\Big)/4, \\ y := \Big(1 - \dfrac{a}{(x_0 + 2\sqrt{a})^2}\Big)\Big(1 - \dfrac{a}{(x_1 - 2\sqrt{a})^2}\Big) y/64. \end{cases}$$

At the same time, using the famous formulas of addiction and subtraction on elliptic curves (see, e.g., [15, §9.1]) yields:

$$\overline{\overline{\psi}} = \begin{cases} x_0 = \dfrac{x_0^2 x_1 + x_0 x_1^2 - a(x_0 + x_1) - 2y}{(x_0 - x_1)^2}, \\ x_1 = \dfrac{x_0^2 x_1 + x_0 x_1^2 - a(x_0 + x_1) + 2y}{(x_0 - x_1)^2}. \end{cases}$$

Below we will often use the computer algebra system Magma to produce equations or formulas and check theoretical facts (see the corresponding code in [27]). Consider on $\mathbb{A}^2_{(x_0,x_1)} \subset \mathbb{P}^1 \times \mathbb{P}^1$ the π-invariant conic

$$C_1\colon 6x_0x_1 - 11\sqrt{a}x_0 + 11\sqrt{a}x_1 - 20a,$$

which is unique bidegree $(1,1)$ curve (see Figure 1) passing through the points

$$\big(-2\sqrt{a},\ 2\sqrt{a}\big), \qquad \big(\alpha_+\sqrt{a},\ -\alpha_-\sqrt{a}\big), \qquad \big(\alpha_-\sqrt{a},\ -\alpha_+\sqrt{a}\big).$$

Using Magma, one can compute the defining polynomial of $C_2 := \overline{\overline{\varphi}}(C_1)$, namely

$$C_2\colon 24x_0^2x_1 + 25\sqrt{a}x_0^2 - 24x_0x_1^2 - 62\sqrt{a}x_0x_1 - 40ax_0 + 25\sqrt{a}x_1^2 + 40ax_1 + 16a\sqrt{a}.$$

This is a π-invariant cubic (of bidegree $(2,2)$) having the node $(\sqrt{a}, -\sqrt{a})$ (see Figure 2). Note that $r_{C_1} = r_{C_2} = 2$, hence by Theorem 1 the π-invariant curves

$$D_1 := pr^{-1}(C_1) \ \subset\ A/[-1], \qquad D_2 := pr^{-1}(C_2) = \overline{\varphi}(D_1) \ \subset\ A_a/[-1]$$

are rational. It turns out that the restriction $\overline{\overline{\varphi}}\colon C_1 \to C_2$ (and hence $\overline{\varphi}\colon D_1 \to D_2$) is invertible. Indeed,

$$(\overline{\overline{\varphi}})^{-1}\colon C_2 \dashrightarrow C_1, \qquad (\overline{\overline{\varphi}})^{-1} = \begin{cases} x_0 := \dfrac{24x_0^2 - 24x_0x_1 - 49\sqrt{a}x_0 + 25\sqrt{a}x_1 + 26a}{6(x_0 - \sqrt{a})}, \\[2ex] x_1 := \dfrac{11\sqrt{a}x_0 + \sqrt{a}x_1 - 10a}{6(x_0 - \sqrt{a})}. \end{cases}$$

Finally, denote by $C_2^{(1)}$ $\big($resp. $D_2^{(1)} = pr^{-1}\big(C_2^{(1)}\big)\big)$ the curve $\mathbb{F}_q$-conjugate to C_2 (resp. D_2).

Again, by means of Magma we get the image $C_8 := \overline{\overline{\psi}}(D_2) = \overline{\overline{\psi}}\big(D_2^{(1)}\big)$ (see Figure 3) given by the symmetric $\mathbb{F}_q$-polynomial

$$\begin{aligned} &C_8\colon 5764801a^3s_1^8 - 921984a^2s_1^6s_2^2 + 3471884416a^3s_1^6s_2 + 6914880000a^4s_1^6 + 36864as_1^4s_2^4 - \\ &64633336448a^2s_1^4s_2^3 + 216401113088a^3s_1^4s_2^2 - 1634869760000a^4s_1^4s_2 + 2073600000000a^5s_1^4 + \\ &966524928as_1^2s_2^5 - 3811311616a^2s_1^2s_2^4 - 941125009408a^3s_1^2s_2^3 + 10180198400000a^4s_1^2s_2^2 - \\ &14745600000000a^5s_1^2s_2 - 37748736s_2^7 + 1124073472as_2^6 - 56463720448a^2s_2^5 + \\ &757642297344a^3s_2^4 - 15920005120000a^4s_2^3 + 26214400000000a^5s_2^2, \end{aligned}$$

where $s_1 := x_0 + x_1$, $s_2 := x_0x_1$ are the elementary symmetric polynomials. Note that $\mathrm{bideg}(C_8) = (8,8)$. Since $r_{C_8} = 0$ it follows from Theorem 1 that the inverse image $pr^{-1}(C_8)$ consists of two different rational curves $D_8 := \overline{\psi}(D_2)$ and $D_8' := \overline{\psi}\big(D_2^{(1)}\big)$ such that the restrictions $pr\colon D_8 \to C_8$, $pr\colon D_8' \to C_8$ are birational. Moreover, D_8, D_8' are defined over the field $\mathbb{F}_q$, since $D_2, D_2^{(1)}$ are π-invariant (for better comprehension see §1). Further, according to the Magma computation we obtain

Lemma 1.

1. *The curve $C_8 \subset \mathbb{P}^1 \times \mathbb{P}^1$ has exactly* 42 *singular points, where* $(0,0)$, (∞,∞) *are unique ones from the ramification locus Ram.*

2. *The point* $(0,0)$ *is a non-ordinary singularity of multiplicity* 4 *with two different tangents (each one of multiplicity* 2*).*

3. *The point* (∞,∞) *is a node, whose tangents are the lines* L_∞, M_∞. *Moreover, this point is an inflexion one with respect to each of the two local branches.*

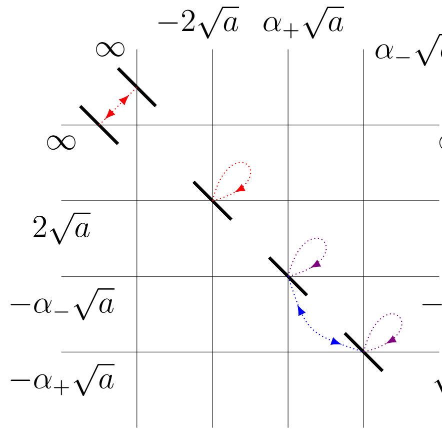

Figure 1: The curve C_1

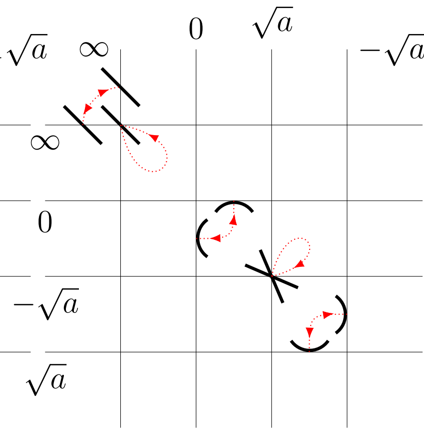

Figure 2: The curve C_2

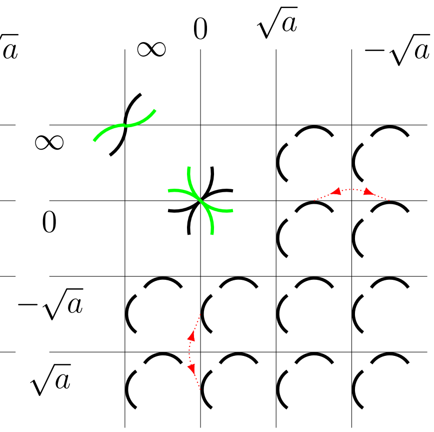

Figure 3: The curve C_8

Dotted arrows denote the action of the endomorphism π: blue ones if $\sqrt{2} \in \mathbb{F}_q$, violet ones if $\sqrt{2} \notin \mathbb{F}_q$, and red ones in both cases. Besides, we draw some branches in green for clarity.

Now we are going to parametrize the curve C_8. Note that C_1 has the π-invariant point $(-5/3\sqrt{a},\ 5/3\sqrt{a})$ and the projection from it gives:

$$pr_{C_1}\colon C_1 \dashrightarrow^{\simeq} \mathbb{A}^1_x, \quad x := \frac{3\sqrt{a}(x_0+x_1)}{3(x_0-x_1)+10\sqrt{a}} \qquad \text{s.t.} \qquad pr_{C_1}^{-1}\colon \mathbb{A}^1_x \dashrightarrow^{\simeq} C_1, \quad \begin{cases} x_0 := \dfrac{-5\sqrt{a}x+6a}{3(x-\sqrt{a})}, \\ x_1 := \dfrac{5\sqrt{a}x+6a}{3(x+\sqrt{a})}. \end{cases}$$

After substituting these formulas into D_1 we obtain the $\mathbb{F}_q$-curve

$$D_1'\colon 3^6x^6y^2 + 2^6a^3x^6 - 3^7ax^4y^2 - 2^4 3^2 a^4x^4 + 3^7a^2x^2y^2 + 3^4a^5x^2 - 3^6a^3y^2 \quad \subset \quad \mathbb{A}^2_{(x,y)}.$$

Thus there are birational isomorphisms

$$\chi := pr_{C_1} \times \mathrm{id}_y\colon D_1 \dashrightarrow^{\simeq} D_1', \qquad \chi^{-1} = pr_{C_1}^{-1} \times \mathrm{id}_y\colon D_1' \dashrightarrow^{\simeq} D_1$$

Further, Magma allows to compute the anticanonical map from D_1' to the $\mathbb{F}_q$-conic

$$Q\colon 2^6a^3u^2 + 3^6v^2 - 2^6a^4 \quad \subset \quad \mathbb{A}^2_{(u,v)}$$

given by the $\mathbb{F}_q$-formulas

$$\varphi_{-K}\colon D_1' \overset{\simeq}{\dashrightarrow} Q, \quad \begin{cases} u := x, \\ v := \dfrac{2^3a^3(3^2a - 2^3x^2)x}{3^6(x^2-a)y} \end{cases} \quad \text{s.t.} \quad \varphi_{-K}^{-1}\colon Q \overset{\simeq}{\dashrightarrow} D_1', \quad \begin{cases} x := u, \\ y := \dfrac{(2^3u^2 - 3^2a)uv}{2^3(u^2-a)^2}. \end{cases}$$

Finally, the projection from the point $(2^3a^2/3^3, 0) \in Q(\mathbb{F}_q)$ has the form

$$pr_Q\colon Q \overset{\simeq}{\dashrightarrow} \mathbb{A}^1_t, \quad t := \frac{3^3v - 2^3a^2}{3^3u} \quad \text{s.t.} \quad pr_Q^{-1}\colon \mathbb{A}^1_t \overset{\simeq}{\dashrightarrow} Q, \quad \begin{cases} u := \dfrac{-2^43^3a^2t}{2^6a^3 + 3^6t^2}, \\ v := \dfrac{2^3a^2(2^6a^3 - 3^6t^2)}{3^3(2^6a^3 + 3^6t^2)}. \end{cases}$$

Thus we obtain the $\mathbb{F}_q$-rational map

$$par := \overline{\overline{\psi}} \circ \overline{\varphi} \circ \chi^{-1} \circ \varphi_{-K}^{-1} \circ pr_Q^{-1}\colon\ \mathbb{A}^1_t \to C_8$$

given by the functions

$$par = \begin{cases} x_0 := \dfrac{(3^8t^2 + 2^6a^3)^2 g(t)}{2^{14}3^2a^4(3^7t^2 - 2^6a^3)^2 t}, \\ x_1 := \dfrac{(3^4t^2 + 2^6a^3)^2 g(t)}{2^83^6a(3^5t^2 - 2^6a^3)^2t^3}, \end{cases} \qquad \text{where} \qquad g(t) := 3^{12}t^4 - 2^73^47a^3t^2 + 2^{12}a^6.$$

It is easily seen that $g(t)$ has no multiple roots and the functions are in the reduced form, that is the numerators and denominators have no common roots. By [34, Th. 4.21] we get

Theorem 3. *The map par (or, equivalently,* $\overline{\overline{\psi}}|_{D_2}$*) is birational.*

Another proof consists in applying the projection formula [11, §1.2] with respect to $\overline{\overline{\psi}}$ and the fact that $\overline{\overline{\psi}}_*(D_2) = \deg\big(\overline{\overline{\psi}}|_{D_2}\big)C_8$. The inverse map par^{-1} can be also computed (by Magma), but we do not write out it here in the sake of compactness. Interestingly, according to [34, Cor. 6.14] the curve C_8 is not polynomial, i.e., it cannot be parametrized by two polynomials (even over $\overline{\mathbb{F}_q}$).

By Theorem 2 the inverse images

$$H_i := \rho^{-1}(D_i), \qquad H_2^{(1)} := \rho^{-1}\big(D_2^{(1)}\big), \qquad H_8' := \rho^{-1}(D_8'),$$

where $i \in \{1, 2, 8\}$, are hyperelliptic curves. Note that the maps

$$\varphi\colon H_1 \to H_2, \qquad \psi\colon H_2 \to H_8, \qquad \psi\colon H_2^{(1)} \to H_8'$$

are birational and hence they all have geometric genus 2 and a non-hyperelliptic involution.

Finally, we summarize the paragraph by the commutative diagrams

$$\begin{array}{ccccc} A_{\pm} & \overset{\varphi}{\to} & A_a & \overset{\psi}{\to} & A'_a \\ \rho\downarrow & & \rho\downarrow & & \downarrow\rho \\ A_{\pm}/[-1] & \overset{\overline{\varphi}}{\to} & A_a/[-1] & \overset{\overline{\psi}}{\to} & A'_a/[-1] \\ pr\downarrow & & pr\downarrow & \overset{\overline{\overline{\psi}}}{\searrow} & \downarrow pr \\ \mathbb{P}^1\times\mathbb{P}^1 & \overset{\overline{\overline{\varphi}}}{\to} & \mathbb{P}^1\times\mathbb{P}^1 & & \mathbb{P}^1\times\mathbb{P}^1 \end{array} \qquad \begin{array}{ccccc} H_1 & \overset{\varphi}{\to} & H_2 & \overset{\psi}{\to} & H_8 \\ \rho\downarrow & & \rho\downarrow & & \downarrow\rho \\ D_1 & \overset{\overline{\varphi}}{\to} & D_2 & \overset{\overline{\psi}}{\to} & D_8 \\ pr\downarrow & & pr\downarrow & \overset{\overline{\overline{\psi}}}{\searrow} & \downarrow pr \\ C_1 & \overset{\overline{\overline{\varphi}}}{\to} & C_2 & & C_8 \end{array}$$

Figure 4

4 Remarks and conclusions

Let us keep a notation of previous paragraphs. First of all, we would like to deal with the case $\sqrt{a} \in \mathbb{F}_q$ (in fact, it is sufficient to take $a = 1$). Let E'_-, E'_a be the quadratic $\mathbb{F}_q$-twists of E_-, E_a respectively (by the $\mathbb{F}_{q^2}$-isomorphism σ) and

$$A'_{\pm} := E_+ \times E'_-, \qquad A'_a := E_a \times E'_a.$$

By means of

$$[1]\times\sigma\colon A'_{\pm} \xrightarrow{\sim} A_{\pm}, \qquad [1]\times\sigma\colon A'_a \xrightarrow{\sim} A_a$$

the morphisms $\varphi, \overline{\varphi}$ are identically transformed to

$$A'_{\pm} \to A'_a, \qquad A'_{\pm}/[-1] \to A'_a/[-1]$$

respectively, hence we save the notation. Finally, for $i \in \{1, 2\}$ consider the $\mathbb{F}_q$-curves

$$H'_i := ([1]\times\sigma^{-1})(H_i), \qquad D'_i := \rho(H'_i).$$

Thus $D'_2 = \overline{\varphi}(D'_1)$ is a desired rational $\mathbb{F}_q$-curve on the Kummer surface of A'_a and we obtain the commutative diagrams

$$\begin{array}{ccc} A'_{\pm} & \overset{\varphi}{\to} & A'_a \\ \rho\downarrow & & \downarrow\rho \\ A'_{\pm}/[-1] & \overset{\overline{\varphi}}{\to} & A'_a/[-1] \\ pr\downarrow & & \downarrow pr \\ \mathbb{P}^1\times\mathbb{P}^1 & \overset{\overline{\overline{\varphi}}}{\to} & \mathbb{P}^1\times\mathbb{P}^1 \end{array} \qquad \begin{array}{ccc} H'_1 & \overset{\varphi}{\to} & H'_2 \\ \rho\downarrow & & \downarrow\rho \\ D'_1 & \overset{\overline{\varphi}}{\to} & D'_2 \\ pr\downarrow & & \downarrow pr \\ C_1 & \overset{\overline{\overline{\varphi}}}{\to} & C_2 \end{array}$$

Now we return to the more interesting case $\sqrt{a} \notin \mathbb{F}_q$. In particular, under the condition $q \equiv 5 \pmod 8$ it is sufficient to take $a \in \{2, 8\}$, because it is known that the Legendre symbol

$$\left(\frac{2}{p}\right) = 2^{\frac{p^2-1}{8}} = \begin{cases} 1 & \text{if} \quad p \equiv 1, 7 \pmod 8, \\ -1 & \text{if} \quad p \equiv 3, 5 \pmod 8, \end{cases} \qquad \left(\frac{2}{q}\right) = \begin{cases} 1 & \text{if} \quad 2 \mid n, \\ \left(\frac{2}{p}\right) & \text{if} \quad 2 \nmid n. \end{cases}$$

Fortunately, for $q \not\equiv 1 \pmod 8$ a square root in $\mathbb{F}_q$ can be computed by means of one exponentiation in $\mathbb{F}_q$ (see, e.g., [13, §5.1.7]), hence the simplified SWU method can be implemented quite efficiently.

It is time to clarify which sign of the square root $y = \sqrt{r}$ (for a quadratic residue $r \in \mathbb{F}_q^*$) should be chosen by default. Let $\mathbb{F}_q = \mathbb{F}_p(\gamma)$ and $y = \sum_{i=0}^{n-1} y_i\gamma^i \in \mathbb{F}_q^*$, where $0 \leqslant y_i < p$. If i_0 is the minimal index with $y_{i_0} \neq 0$, then we take y such that the value from $\{y_{i_0}, p - y_{i_0}\}$ is even (or odd). Another way is to compare what the value is greater than $(p-1)/2$. Let $R := \mathbb{P}^1 \setminus par^{-1}(Ram)$ and

$$h' \colon C_8(\mathbb{F}_q) \setminus Ram \to E_a(\mathbb{F}_q) \setminus E_a[2], \qquad (x_0, x_1) \mapsto \begin{cases} \big(x_0, \sqrt{f(x_0)}\big) & \text{if} \quad \sqrt{f(x_0)} \in \mathbb{F}_q, \\ \big(x_1, -\sqrt{f(x_1)}\big) & \text{if} \quad \sqrt{f(x_1)} \in \mathbb{F}_q. \end{cases}$$

Thus the parametrization $par \colon \mathbb{P}^1 \to C_8$ induces the mapping

$$h := h' \circ par \colon R(\mathbb{F}_q) \to E_a(\mathbb{F}_q).$$

Of course, we could extend h to all the field $\mathbb{F}_q$, but let us simplify the paragraph, not dealing with the exceptional cases. The defining polynomial of C_8 is symmetric, hence both points $\pm h(t)$ are in the image of h. More precisely, it can be checked that $h\big(2^6a^3/(3^6t)\big) = -h(t)$. Finally, since the curve C_8 is of bidegree $(8, 8)$, for most points $P \in E_a(\mathbb{F}_q)$ $(\neq P_0, P_\infty)$ it follows that $|h^{-1}(P)| \leqslant 8$.

Theorem 4. *We have the bounds*

$$\frac{q-54}{8} \leqslant \big|h\big(R(\mathbb{F}_q)\big)\big| \leqslant |E_a(\mathbb{F}_q)| - 2.$$

Proof. By the adjunction formula [19, Exer. V.1.3.a] arithmetic genus $p_a = 49$ for the curve $C_8 \subset \mathbb{P}^1 \times \mathbb{P}^1$, because a canonical divisor $K_{\mathbb{P}^1\times\mathbb{P}^1}$ is of bidegree $-(2, 2)$. Besides, for a point $P \in C_8(\mathbb{F}_q)$ consider the values

$$\alpha_P := \big|par^{-1}(P)(\mathbb{F}_q)\big|, \qquad \delta'_P := \begin{cases} 0 & \text{if} \quad \alpha_P = 0, \\ \alpha_P - 1, & \text{otherwise}, \end{cases}$$

and δ_P [19, Exam. V.3.9.3]. Using [2, Lem. 2.2] and [19, Exam. V.3.9.(2-3)], we obtain the inequalities

$$\big|par\big(\mathbb{P}^1(\mathbb{F}_q)\big)\big| \;=\; q + 1 - \sum_{P \in C_8(\mathbb{F}_q)} \delta'_P \;\geqslant\; q + 1 - \sum_{P \in C_8(\mathbb{F}_q)} \delta_P \;\geqslant\; q + 1 - p_a \;=\; q - 48.$$

Thus

$$\frac{q-54}{8} \leqslant \frac{\big|par\big(\mathbb{P}^1(\mathbb{F}_q)\big)\big| - \big|(C_8 \cap Ram)(\mathbb{F}_q)\big|}{8} \leqslant \big|h\big(R(\mathbb{F}_q)\big)\big|$$

and the upper bound is trivial. $\square$

To be more precise the formula for $|E_a(\mathbb{F}_q)|$ is given in [13, Prop. 2.5], [24, Th. 18.5]. The lower bound can be probably improved by the Chebotarev density theorem (in the function field case) as well as for another mappings (see [14, §3.2]).

In pairing-based cryptography non-supersingular (i.e, for $p \equiv 1 \pmod 3$) elliptic $\mathbb{F}_q$-curves $E_b\colon y^2 = x^3 - b$ of j-invariant 0 are only used in practice at the moment [42, Tab. 1]. Thus it is actual to generalize the simplified SWU method to them. More precisely, there is the following

Problem 1. *Let E_b be an elliptic $\mathbb{F}_q$-curve of $j = 0$ and E_b' be its quadratic $\mathbb{F}_q$-twist. How to explicitly construct a rational $\mathbb{F}_q$-curve D on the Kummer surface K_b' of the direct product $E_b \times E_b'$ such that bidegree of the image $C := pr(D) \subset \mathbb{P}^1 \times \mathbb{P}^1$ does not depend on $\mathbb{F}_q$?*

Unfortunately, the approach of this work does not allow to resolve this problem, because in the case $\sqrt[3]{b} \notin \mathbb{F}_q$ it seems that there is no a natural $\mathbb{F}_{q^2}$-isogeny from any elliptic curve of $j \neq 0$, that is an ascending $\mathbb{F}_{q^2}$-isogeny to E_b.

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