Violating Clauser-Horne-Shimony-Holt Inequality Represents Nothing

Zhengjun Cao[1], Zhenfu Cao[2], Lihua Liu[3]

[1] *Department of Mathematics, Shanghai University, Shangda Road 99, Shanghai, 200444, China. caozhj@shu.edu.cn*
[2] *Software Engineering Institute, East China Normal University, Shanghai, 200062, China. zfcao@sei.ecnu.edu.cn*
[3] *Department of Mathematics, Shanghai Maritime University, Shanghai, 201306, China. liulh@shmtu.edu.cn*

Abstract

The Clauser-Horne-Shimony-Holt (CHSH) inequality is of great importance to quantum entanglement and quantum computers. We present a purely mathematical argument for the famous inequality. The essential assumptions in the new argument are totally independent of any physical interpretation, including the hidden variable interpretation for EPR thought experiment and the Copenhagen interpretation for quantum mechanics. The findings are helpful to comprehend the inequality from a new point of view.

Keywords: Bell's inequality, Clauser-Horne-Shimony-Holt inequality, quantum entanglement, EPR thought experiment.

1. Introduction

The Copenhagen interpretation for quantum mechanics says that a measurement causes an instantaneous collapse of the wave function and the quantum system after the collapse is random. Einstein, et al. [1] insisted that quantum mechanics was incomplete and proposed a thought experiment to refute it. In 1964, J. Bell [2] mathematically formulated EPR thought experiment and constructed an inequality,

$$|E(\vec{a},\vec{b}) - E(\vec{a},\vec{c})| \leq 1 + E(\vec{b},\vec{c})$$

which involves three vectors $\vec{a},\vec{b},\vec{c}$ in a Hilbert space. In 1969, Clauser, Horne, Shimony and Holt [3] extended Bell's formulation for the case involving four vectors $\vec{a},\vec{b},\vec{a}',\vec{b}'$, i.e.,

$$|E(\vec{a},\vec{b}) - E(\vec{a},\vec{b}') + E(\vec{a}',\vec{b}') + E(\vec{a}',\vec{b})| \leq 2.$$

In the past decades, many experiments [4, 5, 6, 7, 8] have been designed to test the Bell's inequality or CHSH inequality [3, 9, 10]. Today, the inequalities are of great importance to quantum information and computation [8, 11, 12]. In this note, we specify the essential mathematical assumptions in the argument for CHSH inequality. We then present a purely mathematical argument for the inequality. We find the argument is totally independent of any physical interpretation. We also report the current state of manufacturing quantum computers.

2. A purely mathematical argument

Theorem 1. *Suppose that $\mathbb{H}$ is a Hilbert space and $\mathbb{T}$ is a topological space. The random variable $\lambda \in \mathbb{T}$ satisfies $\int \rho(\lambda)\, d\lambda = 1$, where $\rho(\lambda)$ is the density function. Suppose $A : \mathbb{H}\times\mathbb{T} \longrightarrow \{-1,0,1\}$, $B : \mathbb{H}\times\mathbb{T} \longrightarrow \{-1,0,1\}$. The average values of A, B are denoted by $\bar{A}, \bar{B}$, respectively. Define the function*

$$E(\vec{a},\vec{b}) \triangleq \int \bar{A}(\vec{a},\lambda)\bar{B}(\vec{b},\lambda)\rho(\lambda)\, d\lambda \tag{1}$$

Then for arbitrary $\vec{a}, \vec{a}', \vec{b}, \vec{b}' \in \mathbb{H}$, the inequality

$$|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}') + E(\vec{a}', \vec{b}') + E(\vec{a}', \vec{b})| \leq 2 \tag{2}$$

holds.

Proof. By the definition of the function E, we have

$$\begin{aligned}
E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}') &= \int \bar{A}(\vec{a}, \lambda)\bar{B}(\vec{b}, \lambda)\rho(\lambda)\,\mathrm{d}\lambda - \int \bar{A}(\vec{a}, \lambda)\bar{B}(\vec{b}', \lambda)\rho(\lambda)\,\mathrm{d}\lambda \\
&= \int \bar{A}(\vec{a}, \lambda)\bar{B}(\vec{b}, \lambda)[1 \pm \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}', \lambda)]\rho(\lambda)\,\mathrm{d}\lambda \\
&\quad - \int \bar{A}(\vec{a}, \lambda)\bar{B}(\vec{b}', \lambda)[1 \pm \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}, \lambda)]\rho(\lambda)\,\mathrm{d}\lambda
\end{aligned}$$

Hence, by triangle inequality it gives

$$\begin{aligned}
|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}')| \leq& \left|\int \bar{A}(\vec{a}, \lambda)\bar{B}(\vec{b}, \lambda)[1 \pm \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}', \lambda)]\rho(\lambda)\,\mathrm{d}\lambda\right| \\
&+ \left|\int \bar{A}(\vec{a}, \lambda)\bar{B}(\vec{b}', \lambda)[1 \pm \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}, \lambda)]\rho(\lambda)\,\mathrm{d}\lambda\right|
\end{aligned}$$

Since $|\bar{A}(\vec{a}, \lambda)| \leq 1, |\bar{B}(\vec{b}, \lambda)| \leq 1, |\bar{B}(\vec{b}', \lambda)| \leq 1$, we obtain

$$|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}')| \leq \left|\int [1 \pm \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}', \lambda)]\rho(\lambda)\,\mathrm{d}\lambda\right| + \left|\int [1 \pm \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}, \lambda)]\rho(\lambda)\,\mathrm{d}\lambda\right|$$

Since $\rho(\lambda) \geq 0, 1 \pm \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}', \lambda) \geq 0, 1 \pm \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}, \lambda) \geq 0$, we have

$$\left|\int [1 \pm \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}', \lambda)]\rho(\lambda)\,\mathrm{d}\lambda\right| = \int [1 \pm \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}', \lambda)]\rho(\lambda)\,\mathrm{d}\lambda = 1 \pm \int \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}', \lambda)\rho(\lambda)\,\mathrm{d}\lambda$$

$$\left|\int [1 \pm \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}, \lambda)]\rho(\lambda)\,\mathrm{d}\lambda\right| = \int [1 \pm \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}, \lambda)]\rho(\lambda)\,\mathrm{d}\lambda = 1 \pm \int \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}, \lambda)\rho(\lambda)\,\mathrm{d}\lambda$$

Therefore,

$$|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}')| \leq 2 \pm \left[\int \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}', \lambda)\rho(\lambda)\,\mathrm{d}\lambda + \int \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}, \lambda)\rho(\lambda)\,\mathrm{d}\lambda\right]$$

which means

$$\begin{aligned}
|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}')| &\leq 2 - \left|\int \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}', \lambda)\rho(\lambda)\,\mathrm{d}\lambda + \int \bar{A}(\vec{a}', \lambda)\bar{B}(\vec{b}, \lambda)\rho(\lambda)\,\mathrm{d}\lambda\right| \\
&= 2 - |E(\vec{a}', \vec{b}') + E(\vec{a}', \vec{b})|
\end{aligned}$$

Thus,

$$2 \geq |E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}')| + |E(\vec{a}', \vec{b}') + E(\vec{a}', \vec{b})| \geq |E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}') + E(\vec{a}', \vec{b}') + E(\vec{a}', \vec{b})|.$$

This completes the proof. □

3. An analogical inequality

The CHSH inequality is derived entirely from the special maps A and B, independent of any physical objects and measurements. It is too trivial to reveal any essence of the product of Hilbert space $\mathbb{H}$ and topological space $\mathbb{T}$, let alone the hypothetical hidden-variable theory. Besides, we find *the fictitious variable λ is not truly invoked in the mathematical argument*. To illustrate this independency, we present an analogical inequality and argument without the symbol λ.

Theorem 2. *Suppose that $\mathbb{H}$ is a Hilbert space, and $A : \mathbb{H} \longrightarrow \{-1, 0, 1\}$, $B : \mathbb{H} \longrightarrow \{-1, 0, 1\}$. The average values of A, B are denoted by $\bar{A}, \bar{B}$, respectively. Define the function*

$$P(\vec{a}, \vec{b}) \triangleq \bar{A}(\vec{a})\bar{B}(\vec{b}) \tag{1'}$$

Then for arbitrary $\vec{a}, \vec{a}', \vec{b}, \vec{b}' \in \mathbb{H}$, the inequality

$$|P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{b}') + P(\vec{a}', \vec{b}') + P(\vec{a}', \vec{b})| \leq 2 \tag{2'}$$

holds.

Proof. By the definition of the function P, we have

$$\begin{aligned} P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{b}') &= \bar{A}(\vec{a})\bar{B}(\vec{b}) - \bar{A}(\vec{a})\bar{B}(\vec{b}') \\ &= \bar{A}(\vec{a})\bar{B}(\vec{b})[1 \pm \bar{A}(\vec{a}')\bar{B}(\vec{b}')] - \bar{A}(\vec{a})\bar{B}(\vec{b}')[1 \pm \bar{A}(\vec{a}')\bar{B}(\vec{b})]. \end{aligned}$$

Hence, by triangle inequality it gives

$$|P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{b}')| \leq \left|\bar{A}(\vec{a})\bar{B}(\vec{b})[1 \pm \bar{A}(\vec{a}')\bar{B}(\vec{b}')]\right| + \left|\bar{A}(\vec{a})\bar{B}(\vec{b}')[1 \pm \bar{A}(\vec{a}')\bar{B}(\vec{b})]\right|.$$

Since $|\bar{A}(\vec{a})| \leq 1, |\bar{B}(\vec{b})| \leq 1, |\bar{B}(\vec{b}')| \leq 1$, we obtain

$$|P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{b}')| \leq \left|1 \pm \bar{A}(\vec{a}')\bar{B}(\vec{b}')\right| + \left|1 \pm \bar{A}(\vec{a}')\bar{B}(\vec{b})\right|.$$

Since $1 \pm \bar{A}(\vec{a}')\bar{B}(\vec{b}') \geq 0$, $1 \pm \bar{A}(\vec{a}')\bar{B}(\vec{b}) \geq 0$, we have

$$\left|1 \pm \bar{A}(\vec{a}')\bar{B}(\vec{b}')\right| = 1 \pm \bar{A}(\vec{a}')\bar{B}(\vec{b}'), \quad \left|1 \pm \bar{A}(\vec{a}')\bar{B}(\vec{b})\right| = 1 \pm \bar{A}(\vec{a}')\bar{B}(\vec{b}).$$

Therefore,

$$|P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{b}')| \leq 2 \pm \left[\bar{A}(\vec{a}')\bar{B}(\vec{b}') + \bar{A}(\vec{a}')\bar{B}(\vec{b})\right]$$

which means

$$|P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{b}')| \leq 2 - \left|\bar{A}(\vec{a}')\bar{B}(\vec{b}') + \bar{A}(\vec{a}')\bar{B}(\vec{b})\right| = 2 - |P(\vec{a}', \vec{b}') + P(\vec{a}', \vec{b})|$$

Thus,

$$2 \geq |P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{b}')| + |P(\vec{a}', \vec{b}') + P(\vec{a}', \vec{b})| \geq |P(\vec{a}, \vec{b}) - P(\vec{a}, \vec{b}') + P(\vec{a}', \vec{b}') + P(\vec{a}', \vec{b})|.$$

This completes the proof. □

4. The dependency or independency between measurements is not exploited

Professor M. Paris argued that (in a personal email):

> *The proof of CHSH inequality, actually encompasses the assumption of locality from the very beginning, in Eq.(1), $\bar{A}$ not depending on $\vec{b}$ and $\bar{B}$ not depending on $\vec{a}$.*

But we don't think the pure mathematical definition Eq.(1) suffices to describe the philosophic notion – LOCALITY. As we see, $\bar{A}(\vec{a},\lambda)$ and $\bar{B}(\vec{b},\lambda)$ are physically interpreted as the two measuring results on two quantum entangled objects. The measurement $\bar{A}(\vec{a},\lambda)$ independent of the vector $\vec{b}$ does not imply that $\bar{A}(\vec{a},\lambda)$ is independent of the measurement $\bar{B}(\vec{b},\lambda)$ and vice versa. The dependency or independency between measurements is not exploited to prove the CHSH inequality. The inequality is just derived from two special maps, independent of any physical interpretations.

5. Inexplicit correlation function

The correlation function between $\bar{A}(\vec{a},\lambda)$ and $\bar{B}(\vec{b},\lambda)$ is not explicitly represented. In Eq.(1), the function $E(\vec{a},\vec{b})$ depends on the vector $\vec{a}$ and vector $\vec{b}$, corresponding to two projection directions, not on two measuring results $\bar{A}(\vec{a},\lambda)$ and $\bar{B}(\vec{b},\lambda)$. In order to construct a true correlation function between two measuring results, it is better to define

$$E_{\sigma_1,\sigma_2}(\vec{a},\vec{b}) \triangleq \int \bar{A}_{\sigma_1}(\vec{a},\lambda)\bar{B}_{\sigma_2}(\vec{b},\lambda)\rho(\lambda)\,\mathrm{d}\lambda \tag{3}$$

where σ_1,σ_2 are two quantum entangled objects. $\bar{A}_{\sigma_1}(\vec{a},\lambda)$ denotes the result of measuring σ_1, and $\bar{B}_{\sigma_2}(\vec{b},\lambda)$ denotes the result of measuring σ_2. By the similar argument, we obtain

$$|E_{\sigma_1,\sigma_2}(\vec{a},\vec{b}) - E_{\sigma_1,\sigma_2}(\vec{a},\vec{b}') + E_{\sigma_1,\sigma_2}(\vec{a}',\vec{b}') + E_{\sigma_1,\sigma_2}(\vec{a}',\vec{b})| \leq 2 \tag{4}$$

on the condition that *the same quantum entangled object σ_1 can be measured twice* to get $\bar{A}_{\sigma_1}(\vec{a},\lambda)$ and $\bar{A}_{\sigma_1}(\vec{a}',\lambda)$. So does the quantum entangled object σ_2. But this contradicts the quantum collapse theory. So, the inequality (4) cannot be practically used to distinguish the hidden variable interpretation from Copenhagen interpretation for EPR thought experiment.

6. How many quantum objects are measured

In order to distinguish different quantum objects to be measured, the four values in Eq.(4) should be explicitly represented as

$$E_{\sigma_1^1,\sigma_2^1}(\vec{a},\vec{b}),\ E_{\sigma_1^2,\sigma_2^2}(\vec{a},\vec{b}'),\ E_{\sigma_1^3,\sigma_2^3}(\vec{a}',\vec{b}'),\ E_{\sigma_1^4,\sigma_2^4}(\vec{a}',\vec{b}).$$

Namely, at least four pairs of entangled spins are involved. But we find

$$\begin{aligned}
E_{\sigma_1^1,\sigma_2^1}(\vec{a},\vec{b}) - E_{\sigma_1^2,\sigma_2^2}(\vec{a},\vec{b}') &= \int \bar{A}_{\sigma_1^1}(\vec{a},\lambda)\bar{B}_{\sigma_2^1}(\vec{b},\lambda)\rho(\lambda)\,\mathrm{d}\lambda - \int \bar{A}_{\sigma_1^2}(\vec{a},\lambda)\bar{B}_{\sigma_2^2}(\vec{b}',\lambda)\rho(\lambda)\,\mathrm{d}\lambda \\
&\neq \int \bar{A}_{\sigma_1^1}(\vec{a},\lambda)\bar{B}_{\sigma_2^1}(\vec{b},\lambda)[1 \pm \bar{A}_{\sigma_1^3}(\vec{a}',\lambda)\bar{B}_{\sigma_2^3}(\vec{b}',\lambda)]\rho(\lambda)\,\mathrm{d}\lambda \\
&\quad - \int \bar{A}_{\sigma_1^2}(\vec{a},\lambda)\bar{B}_{\sigma_2^2}(\vec{b}',\lambda)[1 \pm \bar{A}_{\sigma_1^4}(\vec{a}',\lambda)\bar{B}_{\sigma_2^4}(\vec{b},\lambda)]\rho(\lambda)\,\mathrm{d}\lambda
\end{aligned}$$

Hence, the analogical inequality

$$|E_{\sigma_1^1,\sigma_2^1}(\vec{a},\vec{b}) - E_{\sigma_1^2,\sigma_2^2}(\vec{a},\vec{b}') + E_{\sigma_1^3,\sigma_2^3}(\vec{a}',\vec{b}') + E_{\sigma_1^4,\sigma_2^4}(\vec{a}',\vec{b})| \leq 2 \tag{4'}$$

does not hold. Thus, violating the inequality (4′) represents nothing. In the past decades, the so-called violations [4, 5, 6, 7, 8] are actually tested in accordance with Eq.(4′). At least four pairs of (hypothetically entangled) spins are measured.

7. Practices are speaking out

The research of quantum computer in the past three decades was mainly driven by quantum entanglement theory. But we have noticed that the advancement of quantum computing machines was discouraging. In 2019, Google announced the achievement of quantum supremacy [13]. In 2022, IBM unveiled the world's largest quantum computer at 433 qubits [14]. In 2024, Google's claim of quantum supremacy has been completely smashed [15]. Very recently, Google and XPRIZE have launched \$5m prize to find actual uses for quantum computers [16]. This prize could calm some quantum computer fanatics down. Should we believe those experiments conducted in laboratories in colleges and universities, or the quantum machines produced by Google and IBM? Only practices, we think, are speaking out.

8. Conclusion

We present a purely mathematical argument for the Clauser-Horne-Shimony-Holt inequality. We find the argument is totally independent of any physical interpretations. To illustrate this point, we present an analogical inequality. The trivial mathematical assumptions for this inequality do not distinguish it from other mathematical propositions. We show that the original correlation function is not explicitly represented. The dependency or independency between measurements is not exploited at all. We believe that violating the inequality represents nothing.

References

[1] A. Einstein, B. Podolsky, N. Rosen, Can quantum-mechanical description of physical reality be considered complete, Phys.Rev. 47 (10) (1935) 777–780.

[2] J. S. Bell, On the einstein podolsky rosen paradox, Physics Physique Fizika 1 (1964) 195–200.

[3] J. F. Clauser, M. A. Horne, A. Shimony, R. A. Holt, Proposed experiment to test local hidden-variable theories, Phys. Rev. Lett. 23 (1969) 880–884.

[4] A. Aspect, P. Grangier, G. Roger, Experimental tests of realistic local theories via bell's theorem, Phys. Rev. Lett. 47 (1981) 460–463.

[5] A. Aspect, P. Grangier, G. Roger, Experimental realization of einstein-podolsky-rosen-bohm gedankenexperiment: A new violation of bell's inequalities, Phys. Rev. Lett. 49 (1982) 91–94.

[6] A. Aspect, J. Dalibard, G. Roger, Experimental test of bell's inequalities using time-varying analyzers, Phys. Rev. Lett. 49 (1982) 1804–1807.

[7] P. Grangier, G. Roger, A. Aspect, Experimental evidence for a photon anticorrelation effect on a beam splitter, a new light on single-photon interferences, Europhysics Letters 1 (4) (1986) 173–179.

[8] D. Bouwmeester, J. Pan, K. Mattle, M. Eibl, H. Weinfurter, A. Zeilinger, Experimental quantum teleportation, Nature 390 (1997) 575.

[9] J. F. Clauser, M. A. Horne, Experimental consequences of objective local theories, Phys. Rev. D 10 (1974) 526–535.

[10] J. F. Clauser, A. Shimony, Bell's theorem, experimental tests and implications, Reports on Progress in Physics 41 (12) (1978) 1881.

[11] I. Bengtsson, K. Zyczkowski, Geometry of Quantum States, An Introduction to Quantum Entanglement, Cambridge University Press, 2006.

[12] M. B. Plenio, S. S. Virmani, An Introduction to Entanglement Theory, Springer International Publishing, Cham, 2014, pp. 173–209.
[13] D. Cossins, It's official: Google has achieved quantum supremacyhttps://www.newscientist.com/article/2220968-its-official-google-has-achieved-quantum-supremacy/ (2019).
[14] K. P. Callaghan, Ibm unveils world's largest quantum computer at 433 qubitshttps://www.newscientist.com/article/2346074-ibm-unveils-worlds-largest-quantum-computer-at-433-qubits/ (2022).
[15] M. Sparkes, Google's claim of quantum supremacy has been completely smashedhttps://www.newscientist.com/article/2437886-googles-claim-of-quantum-supremacy-has-been-completely-smashed/ (2024).
[16] A. Wilkins, Google launches $5m prize to find actual uses for quantum computershttps://www.newscientist.com/article/2420137-google-launches-5m-prize-to-find-actual-uses-for-quantum-computers/ (2024).