

A Note On Clauser-Horne-Shimony-Holt Inequality

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Abstract. Clauser-Horne-Shimony-Holt inequality, an extension of Bell's inequality, is of great importance to modern quantum computation and quantum cryptography. So far, all experimental demonstrations of entanglement are designed to check Bell's inequality or Clauser-Horne-Shimony-Holt inequality. In this note, we specify the math assumptions needed in the argument for Clauser-Horne-Shimony-Holt inequality. We then show the math argument for this inequality is totally indispensable of any physical interpretation, including the hidden variable interpretation for EPR thought experiment and the Copenhagen interpretation for quantum mechanics.

Keywords. Bell's inequality, Clauser-Horne-Shimony-Holt inequality, quantum entanglement, quantum cryptography.

1 Introduction

The Copenhagen interpretation for quantum mechanics says that a measurement causes an instantaneous collapse of the wave function and the quantum system after the collapse is random. Einstein, et al. [9] insisted that quantum mechanics was incomplete and proposed a thought experiment (EPR paradox) to refute it. In 1964, J. Bell [2] formulated EPR paradox mathematically and constructed an inequality,

$$|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{c})| \leq 1 + E(\vec{b}, \vec{c})$$

which involves three vectors $\vec{a}, \vec{b}, \vec{c}$ in a Hilbert space. In 1969, Clauser, Horne, Shimony and Holt [7] extended Bell's inequality for the case involving four vectors $\vec{a}, \vec{b}, \vec{a}', \vec{b}'$, i.e.,

$$|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}') + E(\vec{a}', \vec{b}') + E(\vec{a}', \vec{b})| \leq 2.$$

In the past decades, many experiments have been designed to test the Bell's inequality or Clauser-Horne-Shimony-Holt inequality [1, 6–8, 10]. Today, the two inequalities are of great importance to quantum information and computation [3, 4, 11, 12].

Very recently, we [5] have shown the math contradictions in Bell's argument for his inequality. For the readers' request, we had been investigating the Clauser-Horne-Shimony-Holt inequality. In this note, we specify the math assumptions needed in the argument for Clauser-Horne-Shimony-Holt inequality. We then show the math argument for the inequality is totally indispensable of any physical interpretation.

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2 A purely mathematical argument

Theorem. *Suppose that \mathbb{H} is a Hilbert space and \mathbb{T} is a topological space. The random variable $\lambda \in \mathbb{T}$ satisfies $\int \rho(\lambda) d\lambda = 1$, where $\rho(\lambda)$ is the density function. Suppose $A : \mathbb{H} \times \mathbb{T} \rightarrow \{-1, 0, 1\}$, $B : \mathbb{H} \times \mathbb{T} \rightarrow \{-1, 0, 1\}$. The average values of A, B are denoted by \bar{A}, \bar{B} , respectively. Define the function*

$$E(\vec{a}, \vec{b}) = \int \bar{A}(\vec{a}, \lambda) \bar{B}(\vec{b}, \lambda) \rho(\lambda) d\lambda$$

Then for arbitrary $\vec{a}, \vec{a}', \vec{b}, \vec{b}' \in \mathbb{H}$, the inequality

$$|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}') + E(\vec{a}', \vec{b}') + E(\vec{a}', \vec{b})| \leq 2$$

always holds.

Proof. By the definition of the function E , we have

$$\begin{aligned} E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}') &= \int \bar{A}(\vec{a}, \lambda) \bar{B}(\vec{b}, \lambda) \rho(\lambda) d\lambda - \int \bar{A}(\vec{a}, \lambda) \bar{B}(\vec{b}', \lambda) \rho(\lambda) d\lambda \\ &= \int \bar{A}(\vec{a}, \lambda) \bar{B}(\vec{b}, \lambda) [1 \pm \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}', \lambda)] \rho(\lambda) d\lambda \\ &\quad - \int \bar{A}(\vec{a}, \lambda) \bar{B}(\vec{b}', \lambda) [1 \pm \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}, \lambda)] \rho(\lambda) d\lambda \end{aligned}$$

Hence, by triangle inequality, it gives

$$\begin{aligned} |E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}')| &\leq \left| \int \bar{A}(\vec{a}, \lambda) \bar{B}(\vec{b}, \lambda) [1 \pm \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}', \lambda)] \rho(\lambda) d\lambda \right| \\ &\quad + \left| \int \bar{A}(\vec{a}, \lambda) \bar{B}(\vec{b}', \lambda) [1 \pm \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}, \lambda)] \rho(\lambda) d\lambda \right| \end{aligned}$$

Since $|\bar{A}(\vec{a}, \lambda)| \leq 1$, $|\bar{B}(\vec{b}, \lambda)| \leq 1$, $|\bar{B}(\vec{b}', \lambda)| \leq 1$, we obtain

$$|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}')| \leq \left| \int [1 \pm \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}', \lambda)] \rho(\lambda) d\lambda \right| + \left| \int [1 \pm \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}, \lambda)] \rho(\lambda) d\lambda \right|$$

Since $\rho(\lambda) \geq 0$, $1 \pm \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}', \lambda) \geq 0$, $1 \pm \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}, \lambda) \geq 0$, we have

$$\begin{aligned} \left| \int [1 \pm \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}', \lambda)] \rho(\lambda) d\lambda \right| &= \int [1 \pm \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}', \lambda)] \rho(\lambda) d\lambda = 1 \pm \int \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}', \lambda) \rho(\lambda) d\lambda \\ \left| \int [1 \pm \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}, \lambda)] \rho(\lambda) d\lambda \right| &= \int [1 \pm \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}, \lambda)] \rho(\lambda) d\lambda = 1 \pm \int \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}, \lambda) \rho(\lambda) d\lambda \end{aligned}$$

Therefore,

$$|E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}')| \leq 2 \pm \left[\int \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}', \lambda) \rho(\lambda) d\lambda + \int \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}, \lambda) \rho(\lambda) d\lambda \right]$$

which means

$$\begin{aligned} |E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}')| &\leq 2 - \left| \int \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}', \lambda) \rho(\lambda) d\lambda + \int \bar{A}(\vec{a}', \lambda) \bar{B}(\vec{b}, \lambda) \rho(\lambda) d\lambda \right| \\ &= 2 - |E(\vec{a}', \vec{b}') + E(\vec{a}', \vec{b})| \end{aligned}$$

Thus,

$$2 \geq |E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}')| + |E(\vec{a}', \vec{b}') + E(\vec{a}', \vec{b})| \geq |E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}') + E(\vec{a}', \vec{b}') + E(\vec{a}', \vec{b})|.$$

This completes the proof. □

3 Conclusion

In this note, we remark that the argument for Clauser-Horne-Shimony-Holt inequality is purely mathematical and totally independent of any physical interpretations. Apparently, the inequality does not capture any essences of EPR thought experiment or the Copenhagen interpretation for quantum mechanics. It is just a math proposition.

References

- [1] A. Aspect et al., Phys. Rev. Lett. 47, 460 (1981), 49, 91 (1982); 49, 1804 (1982)
- [2] J. Bell, On the Einstein-Podolsky-Rosen paradox, Physics, 1, 195-200 (1964)
- [3] D. Bouwmeester, JW Pan, K. Mattle, M. Eibl, H. Weinfurter, A. Zeilinger: Experimental Quantum Teleportation, Nature vol. 390, 11, pp.575 (1997)
- [4] I. Bengtsson, K. Zyczkowski: Geometry of Quantum States, An Introduction to Quantum Entanglement, Cambridge University Press (2006)
- [5] Z. J. Cao, L. H. Liu: Remarks on Bell's Inequality, Cryptology ePrint Archive: Report 2018/231 (2018)
- [6] J. F. Clauser, M. A. Horne: Experimental consequences of objective local theories, Phys. Rev. D 10, 526-535 (1974)
- [7] J. F. Clauser, M. A. Horne, A. Shimony, R. A. Holt: Proposed experiment to test local hidden-variable theories, Phys.Rev.Lett. 23, 880-884 (1969)
- [8] J. F. Clauser, A. Shimony: Bell's theorem: experimental tests and implications, Reports on Progress in Physics 41, 1881 (1978)
- [9] A. Einstein, B. Podolsky, N. Rosen: Can quantum-mechanical description of physical reality be considered complete, Phys.Rev. 47(10), 777-780 (1935)
- [10] P. Grangier, G. Roger, A. Aspect: Experimental evidence for a photon anticorrelation effect on a beam splitter, a new light on single-photon interferences, Europhysics Letters 1, 173-179 (1986)
- [11] M. B. Plenio, S. Virmani: An introduction to entanglement measures, Quant. Inf. Comp. 7, 1 (2007)
- [12] E. G. Steward: Quantum Mechanics, Its Early Development and the Road to Entanglement, Imperial College Press (2008)