ON ISOGENY GRAPHS OF SUPERSINGULAR ELLIPTIC CURVES
OVER FINITE FIELDS

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Abstract. We study the isogeny graphs of supersingular elliptic curves over finite fields, with an emphasis on the vertices corresponding to elliptic curves of $j$-invariant 0 and 1728.

1. Introduction

Let $\mathbb{F}_q$ be the finite field of order $q$ and characteristic $p > 3$, and let $\mathbb{F}_q$ denote its algebraic closure. Let $\ell$ be a prime different from $p$. The isogeny graph $\mathcal{H}_\ell(\mathbb{F}_q)$ is a directed graph whose vertices are the $\mathbb{F}_q$-isomorphism classes of elliptic curves defined over $\mathbb{F}_q$, and whose directed arcs represent degree-$\ell \mathbb{F}_q$-isogenies (up to a certain equivalence) between elliptic curves in the isomorphism classes. See [10] and [15] for summaries of the theory behind isogeny graphs and for applications in computational number theory.

Every supersingular elliptic curve defined over $\mathbb{F}_p$ is isomorphic to one defined over $\mathbb{F}_{p^2}$. Pizer [12] showed that the subgraph $\mathcal{G}_\ell(\mathbb{F}_{p^2})$ of $\mathcal{H}_\ell(\mathbb{F}_{p^2})$ induced by the vertices corresponding to isomorphism classes of supersingular elliptic curves over $\mathbb{F}_{p^2}$ is an expander graph (and consequently is connected). This property of $\mathcal{G}_\ell(\mathbb{F}_{p^2})$ was exploited by Charles, Goren and Lauter [2] who proposed a cryptographic hash function whose security is based on the intractability of computing directed paths of a certain length between two vertices in $\mathcal{G}_\ell(\mathbb{F}_{p^2})$. In 2011, Jao and De Feo [8] (see also [4]) presented a key agreement scheme whose security is also based on the intractability of this problem for small $\ell$ (typically $\ell = 2, 3$). There have also been proposals for related signature schemes [19, 6] and an undeniable signature scheme [9].

In this paper, we study the supersingular isogeny graph $\mathcal{G}_\ell(\mathbb{F}_{p^2})$ whose vertices are (representatives of) the $\mathbb{F}_{p^2}$-isomorphism classes of supersingular elliptic curves defined over $\mathbb{F}_{p^2}$, and whose directed arcs represent degree-$\ell \mathbb{F}_{p^2}$-isogenies between the elliptic curves. Observe that the difference between the definitions of $\mathcal{G}_\ell(\mathbb{F}_{p^2})$ and $\mathcal{G}_\ell(\mathbb{F}_{p^2})$ is that the isomorphisms and isogenies in the former are defined over $\mathbb{F}_{p^2}$ itself. This difference necessitates a careful treatment of the vertices corresponding to supersingular elliptic curves having $j$-invariant equal to 0 and 1728. We note that the security of the aforementioned cryptographic schemes relies on the difficulty of constructing certain directed paths in $\mathcal{G}_\ell(\mathbb{F}_{p^2})$. On the other hand, [2] and [4] state that security is based on the hardness of constructing certain directed paths in $\mathcal{G}_\ell(\mathbb{F}_{p^2})$. Thus, it is worthwhile to study the differences between $\mathcal{G}_\ell(\mathbb{F}_{p^2})$ and $\mathcal{G}_\ell(\mathbb{F}_{p^2})$. We also note that Delfs and Galbraith [3] studied supersingular isogeny graphs $\mathcal{G}_\ell(\mathbb{F}_p)$, where the vertices are $\mathbb{F}_p$-isomorphism classes of supersingular elliptic curves defined over $\mathbb{F}_p$ and the arcs are equivalence classes of degree-$\ell$
The supersingular isogeny graph $G_\ell(F_{p^2})$ is defined in §3. In §4, we completely describe the three small subgraphs of $G_\ell(F_{p^2})$ whose vertices correspond to supersingular elliptic curves $E$ over $F_{p^2}$ with $t = p^2 + 1 - \#E(F_{p^2}) \in \{0, -p, p\}$; see Figure 1. In §5, we study the two large subgraphs of $G_\ell(F_{p^2})$ whose vertices correspond to supersingular elliptic curves $E$ over $F_{p^2}$ with $t = p^2 + 1 - \#E(F_{p^2}) \in \{-2p, 2p\}$, and make some observations about the number of loops at the vertices corresponding to elliptic curves with $j$-invariant equal to 0 or 1728. We make some concluding remarks in §6.

2. Elliptic curves

In the remainder of this paper, $p$ will denote a prime greater than 3. Let $k = F_q$ be the finite field of order $q$ and characteristic $p$, and let $\overline{k} = \bigcup_{n \geq 1} F_{q^n}$ denote its algebraic closure. Let $\sigma : \alpha \mapsto \alpha^q$ denote the $q$-power Frobenius map. An elliptic curve $E$ over $k$ is defined by a Weierstrass equation $E/k : Y^2 = X^3 + aX + b$ where $a, b \in k$ and $4a^3 + 27b^2 \neq 0$. The $j$-invariant of $E$ is $j(E) = \frac{1728 \cdot 4a^3}{4a^3 + 27b^2}$. One can easily check that $j(E) = 0$ if and only if $a = 0$, and $j(E) = 1728$ if and only if $b = 0$. For any extension $K$ of $k$, the set of $K$-rational points on $E$ is $E(K) = \{(x, y) \in K \times K : y^2 = x^3 + ax + b\} \cup \{\infty\}$, where $\infty$ is the point at infinity; we write $E = E(\overline{k})$. The chord-and-tangent addition law transforms $E(K)$ into an abelian group. For any $n \geq 2$ with $p \nmid n$, the group of $n$-torsion points on $E$ is isomorphic to $\mathbb{Z}_n \oplus \mathbb{Z}_n$. In particular, if $n$ is prime then $E$ has exactly $n+1$ distinct order-$n$ subgroups.

2.1. Isomorphisms and automorphisms. Two elliptic curves $E/k : Y^2 = X^3 + aX + b$ and $E'/k : Y^2 = X^3 + a'X + b'$ are isomorphic over the extension field $K/k$ if there exists $u \in K^*$ such that $a' = u^4a$ and $b' = u^6b$. If such a $u$ exists, then the corresponding isomorphism $f : E \to E'$ is defined by $(x, y) \mapsto (u^2x, u^3y)$. If $E$ and $E'$ are isomorphic over $K$, then $j(E) = j(E')$. Conversely, if $j(E) = j(E')$, then $E$ and $E'$ are isomorphic over $\overline{k}$. Elliptic curves $E_1/k$, $E_2/k$ that are isomorphic over $\mathbb{F}_{q^d}$ for some $d > 1$, but are not isomorphic over any smaller extension of $\mathbb{F}_q$, are said to be degree-$d$ twists of each other. In particular, a degree-2 (quadratic) twist of $E_1/k : Y^2 = X^3 + aX + b$ is $E_2/k : Y^2 = X^3 + c^2aX + c^3b$ where $c \in k^*$ is a non-square, and $\#E_1(k) = \#E_2(k) = 2q + 2$. If $j \in \overline{k} \setminus \{0, 1728\}$, then

$$E_j : Y^2 = X^3 + \frac{3j}{1728-j}X + \frac{2j}{1728-j}$$

is an elliptic curve with $j(E_j) = j$. Also, $E : Y^2 = X^3 + 1$ has $j(E) = 0$ and $Y^2 = X^3 + X$ has $j(E) = 1728$.

An automorphism of $E/k$ is an isomorphism from $E$ to itself. The group of all automorphisms of $E$ that are defined over $K$ is denoted by $\text{Aut}_K(E)$. If $j(E) \neq 0, 1728$, then $\text{Aut}_\overline{k}(E)$ has order 2 with generator $(x, y) \mapsto (x, -y)$. If $j(E) = 1728$, then $\text{Aut}_\overline{k}$ is cyclic of order 4 with generator $\psi : (x, y) \mapsto (-x, iy)$ where $i \in \overline{k}$ is a primitive fourth root of
unity. If \( j(E) = 0 \), then \( \text{Aut}_E \) is cyclic of order 6 with generator \( \rho : (x, y) \mapsto (\eta x, -y) \) where \( \eta \in \overline{k} \) is a primitive third root of unity.

2.2. Isogenies. Let \( E, E' \) be elliptic curves defined over \( k = \mathbb{F}_q \). An isogeny \( \phi : E \to E' \) is a non-constant rational map defined over \( \overline{k} \) with \( \phi(\infty) = \infty \). An endomorphism on \( E \) is an isogeny from \( E \) to itself; the zero map \( P \mapsto \infty \) is also considered to be an endomorphism on \( E \). If the field of definition of \( \phi \) is the extension \( K \) of \( k \), then \( \phi \) is called a \( K \)-isogeny. If such an isogeny exists, then \( E \) and \( E' \) are said to be \( K \)-isogenous. Tate’s theorem asserts that for finite \( K \), \( E \) and \( E' \) are \( K \)-isogenous if and only if \( \#E(K) = \#E'(K) \).

An isogeny \( \phi \) is a morphism, is surjective, is a group homomorphism, and has finite kernel. Every \( K \)-isogeny \( \phi \) can be represented as \( \phi = (r_1(X), r_2(X)) \cdot Y \) where \( r_1, r_2 \in K(X) \) (see p. 51 of [17]). Let \( r_1(X) = p_1(X)/q_1(X) \), where \( p_1, q_1 \in K[X] \) with \( \gcd(p_1, q_1) = 1 \). Then the degree of \( \phi \) is \( \max(\deg p_1, \deg q_1) \). Also, \( \phi \) is said to be separable if \( r_1'(X) \neq 0 \); otherwise it is inseparable. In fact, \( \phi \) is separable if and only if \( \# \ker \phi = \deg \phi \). Note that all isogenies of prime degree \( \ell \neq p \) are separable.

For every \( m \geq 1 \), the multiplication-by-\( m \) map \([m] : E \to E\) is a \( k \)-isogeny of degree \( m^2 \). Every degree-\( m \) isogeny \( \phi : E \to E' \) has a unique dual isogeny \( \hat{\phi} : E' \to E \) satisfying \( \hat{\phi} \circ \phi = [m] \) and \( \phi \circ \hat{\phi} = [m] \). If \( \phi \) is a \( K \)-isogeny, then so is \( \hat{\phi} \). We have \( \deg \hat{\phi} = \deg \phi \) and \( \hat{\phi} = \phi \). If \( E'' \) is an elliptic curve defined over \( k \) and \( \psi : E' \to E'' \) is an isogeny, then \( \overline{\psi \circ \phi} = \hat{\phi} \circ \hat{\psi} \).

2.3. Vélu’s formula. Let \( E \) be an elliptic curve defined over \( k = \mathbb{F}_q \). Let \( \ell \neq p \) be a prime, and let \( G \) be an order-\( \ell \) subgroup of \( E \). Let \( G^* = G \setminus \{ \infty \} \). Then there exists an elliptic curve \( E' \) over \( \overline{k} \) and a degree-\( \ell \) isogeny \( \phi : E \to E' \) with \( \ker \phi = G \). The elliptic curve \( E' \) and the isogeny \( \phi \) are both defined over \( K = \mathbb{F}_q \), where \( t \) is the smallest positive integer such that \( G \) is \( \sigma^t \)-invariant, i.e., \( \{ \sigma^t(P) : P \in G \} = G \) where \( \sigma \) is the \( q \)-power Frobenius map (so \( \sigma(P) = (x^q, y^q) \) if \( P = (x, y) \) and \( \sigma(\infty) = \infty \)). Furthermore, \( \phi \) is unique in the following sense: if \( E'' \) is an elliptic curve defined over \( K \) and \( \psi : E \to E'' \) is a degree-\( \ell \) \( K \)-isogeny with \( \ker \psi = G \), then there exists an isomorphism \( f : E' \to E'' \) defined over \( K \) such that \( \psi = f \circ \phi \).

Given the Weierstrass equation \( Y^2 = X^3 + aX + b \) for \( E/k \) and an order-\( \ell \) subgroup \( G \) of \( E \), Vélu’s formula yields an elliptic curve \( E' \) defined over \( K \) and a degree-\( \ell \) \( K \)-isogeny \( \phi : E \to E' \) with \( \ker \phi = G \).

Suppose first that \( \ell = 2 \) and \( G = \{ \infty, (\alpha, 0) \} \). Then the Weierstrass equation for \( E' \) is

\[
E' : Y^2 = X^3 - (4a + 15\alpha^2)X + (8b - 14\alpha^3),
\]

and the isogeny \( \phi \) is given by

\[
\phi = \left(\frac{X + 3\alpha^2 + a}{X - \alpha}, \frac{Y - (3\alpha^2 + a)Y}{(X - \alpha)^2}\right).
\]

Suppose now that \( \ell \) is an odd prime. For \( Q = (x_Q, y_Q) \in G^* \), define

\[
t_Q = 3x_Q^2 + a, \quad u_Q = 2y_Q^2, \quad w_Q = u_Q + t_Qx_Q.
\]

Furthermore, define

\[
t = \sum_{Q \in G^*} t_Q, \quad w = \sum_{Q \in G^*} w_Q.
\]
and
\begin{equation}
    r(X) = X + \sum_{Q \in G^*} \left( \frac{tQ}{X-xQ} + \frac{uQ}{(X-xQ)^2} \right).
\end{equation}

Then the Weierstrass equation for \( E' \) is
\begin{equation}
    E' : Y^2 = X^3 + (a-5t)X + (b-7w),
\end{equation}
and the isogeny \( \phi \) is given by
\begin{equation}
    \phi = (r(X), r'(X)Y).
\end{equation}

We will henceforth denote the Vélu-generated elliptic curve \( E' \) by \( E^C \).

2.4. Modular polynomials. Let \( \ell \) be a prime. The modular polynomial \( \Phi_\ell(X,Y) \in \mathbb{Z}[X,Y] \) is a symmetric polynomial of the form \( \Phi_\ell(X,Y) = X^{\ell+1} + Y^{\ell+1} - X^\ell Y^\ell + \sum c_{ij} X^i Y^j \), where the sum is over pairs of integers \((i,j)\) with \( 0 \leq i, j \leq \ell \) and \( i + j < 2\ell \). Modular polynomials have the following remarkable property.

**Theorem 1.** Suppose that the characteristic of \( k = \mathbb{F}_q \) is different from \( \ell \). Let \( E/k \) be an elliptic curve with \( j(E) = j \). Let \( G_1, G_2, \ldots, G_{\ell+1} \) be the order-\( \ell \) subgroups of \( E \). Let \( j_i = j(E^{G_i}) \). Then the (possibly repeated) roots of \( \Phi_\ell(j, Y) \) in \( \overline{k} \) are precisely \( j_1, j_2, \ldots, j_{\ell+1} \).

2.5. Supersingular elliptic curves. Hasse’s theorem states that if \( E \) is defined over \( \mathbb{F}_q \), then \( #E(\mathbb{F}_q) = q + 1 - t \) where \( |t| \leq 2\sqrt{q} \). The integer \( t \) is called the trace of the \( q \)-power Frobenius map \( \sigma \) since the characteristic polynomial of \( \sigma \) acting on \( E \) is \( Z^2 - tZ + q \). If \( p \mid t \), then \( E \) is called supersingular; otherwise it is said to be ordinary. Every supersingular elliptic curve \( E \) over \( \mathbb{F}_q \) is isomorphic to one defined over \( \mathbb{F}_{p^2} \); in particular, \( j(E) \in \mathbb{F}_{p^2} \). Henceforth, we shall assume that \( q = p^2 \) (and \( p > 3 \)).

Supersingularity of an elliptic curve depends only on its \( j \)-invariant. We say that \( j \in \mathbb{F}_{p^2} \) is supersingular if there exists a supersingular elliptic curve \( E/\mathbb{F}_{p^2} \) with \( j(E) = j \); if this is the case, then all elliptic curves with \( j \)-invariant equal to \( j \) are supersingular. Note that \( j = 0 \) is supersingular if and only if \( p \equiv 2 \) (mod 3), and \( j = 1728 \) is supersingular if and only if \( p \equiv 3 \) (mod 4).

Schoof [13, Theorem 4.6] determined the number of isomorphism classes of elliptic curves over a finite field. In particular, the number of isomorphism classes of supersingular elliptic curves \( E \) over \( \mathbb{F}_{p^2} \) with \( #E(\mathbb{F}_{p^2}) = p^2 + 1 - t \) is
\begin{equation}
N(t) = \begin{cases} 
    \left( p + 6 - 4\left(\frac{-3}{p}\right) - 3\left(\frac{-1}{p}\right) \right)/12, & \text{if } t = \pm 2p, \\
    1 - \left(\frac{-3}{p}\right), & \text{if } t = \pm p, \\
    1 - \left(\frac{-1}{p}\right), & \text{if } t = 0,
\end{cases}
\end{equation}
where \( (\frac{\cdot}{p}) \) is the Legendre symbol. It follows that the total number of isomorphism classes of supersingular elliptic curves over \( \mathbb{F}_{p^2} \) is \( [p/6] + \epsilon \), where \( \epsilon = 0, 6, 3, 9 \) if \( p \equiv 1, 5, 7, 11 \) (mod 12) respectively. Furthermore, if \( t = 0 \), \( -p \) or \( p \) then \( E(\mathbb{F}_{p^2}) \) is cyclic [13, Lemma 4.8].
3. Supersingular isogeny graphs

Let \( k = \mathbb{F}_q \) where \( q = p^2 \), and let \( \ell \neq p \) be a prime. Recall that \( \sigma \) is the \( q \)-th power Frobenius map. The supersingular isogeny graph \( \mathcal{G}_\ell(k) \) is a directed graph whose vertex set \( V_\ell(k) \) consists of representatives (chosen below) of the \( k \)-isomorphism classes of supersingular elliptic curves defined over \( k \). The (directed) arcs of \( \mathcal{G}_\ell(k) \) are defined as follows. Let \( E_1 \in V_\ell(k) \), and let \( G \) be a \( \sigma \)-invariant order-\( \ell \) subgroup of \( E_1 \). Let \( \phi : E_1 \to E_1' \) be the \( \sigma \)-isogeny with kernel \( G \) (recall that \( E_1' \) and \( \phi \) are both defined over \( k \)), and let \( E_2 \) be the representative of the \( k \)-isomorphism class of elliptic curves containing \( E_1' \). Then \((E_1,E_2)\) is an arc; we call \( E_1 \) the tail and \( E_2 \) the head of the arc. Note that \( \mathcal{G}_\ell(k) \) can have multiple arcs (more than one arc \((E_1,E_2)\)) and loops (arcs of the form \((E_1,E_1)\)).

**Remark 1.** The definition of arcs is independent of the choice of isogeny with kernel \( G \). This is because, as noted in \( \S 2.3 \), if \( \phi' : E_1 \to E_1' \) is any degree-\( \ell \) isogeny with kernel \( G \) where both \( E_1' \) and \( \phi' \) are defined over \( k \), then \( E_1' \) and \( E_1'' \) are isomorphic over \( k \) and consequently \( \phi \) and \( \phi' \) yield the same arc \((E_1,E_2)\).

**Remark 2.** The definition of \( \mathcal{G}_\ell(k) \) is independent of the choice of representatives. Indeed, let \( f : E_1' \to E_1 \) be a \( k \)-isomorphism of elliptic curves, and suppose that \( E_1' \) was chosen as a representative instead of \( E_1 \). Let \( \psi = \phi \circ f \). Then \( \text{Ker } \psi = f^{-1}(G) \), and thus the \( \sigma \)-invariant order-\( \ell \) subgroup \( f^{-1}(G) \) of \( E_1' \) yields the arc \((E_1',E_2)\). The claim now follows since \( f^{-1} \) yields a one-to-one correspondence between the \( \sigma \)-invariant order-\( \ell \) subgroups of \( E_1 \) and \( E_1' \).

A consequence of Tate’s theorem is that the graph \( \mathcal{G}_\ell(k) \) can be partitioned into subgraphs whose vertices are the \( k \)-isomorphism classes of supersingular elliptic curves \( E/k \) with trace \( t = p^2 + 1 - \#E(k) \in \{0, -p, p, -2p, 2p\} \); we denote these subgraphs by \( \mathcal{G}_\ell(k,t) \). There are two such subgraphs \((t = \pm 2p)\) when \( p \equiv 1 \) (mod 12), four subgraphs \((t = \pm p, \pm 2p)\) when \( p \equiv 5 \) (mod 12), three subgraphs \((t = 0, \pm 2p)\) when \( p \equiv 7 \) (mod 12), and five subgraphs \((t = 0, \pm p, \pm 2p)\) when \( p \equiv 11 \) (mod 12). These subgraphs are further studied in \( \S 4 \) and \( \S 5 \). We first fix the representatives of the \( k \)-isomorphism classes of supersingular elliptic curves over \( k \).

Suppose that \( p \equiv 3 \) (mod 4), and let \( w \) be a generator of \( k^* \). Munuera and Tena [11] showed that the representatives of the four isomorphism classes of elliptic curves \( E/k \) with \( j(E) = 1728 \) can be taken to be

\[
E_{1728,w^i}: Y^2 = X^3 + w^i X \quad \text{for} \quad i \in \{0, 3\}.
\]

Of these curves, \( E_{1728,w} \) and \( E_{1728,w^3} \) have \( p^2 + 1 \mathbb{F}_p \)-rational points, and so we choose them as the vertices of \( \mathcal{G}_\ell(k,0) \). Furthermore, \( \#E_{1728,1}(\mathbb{F}_p^2) = p^2 + 1 + 2p \) and \( \#E_{1728,w^2}(\mathbb{F}_p^2) = p^2 + 1 - 2p \); hence, we select \( E_{1728,1} \) and \( E_{1728,w^2} \) as the vertices of \( \mathcal{G}_\ell(k,-2p) \) and \( \mathcal{G}_\ell(k,2p) \), respectively.

Suppose that \( p \equiv 2 \) (mod 3), and let \( w \) be a generator of \( k^* \). Munuera and Tena [11] also showed that the representatives of the six isomorphism classes of elliptic curves \( E/k \) with \( j(E) = 0 \) can be taken to be

\[
E_{0,w^i}: Y^2 = X^3 + w^i \quad \text{for} \quad i \in \{0, 5\}.
\]
Of these curves, \( E_{0,w} \) and \( E_{0,w^5} \) have \( p^2 + 1 + p \mathbb{F}_{p^2} \)-rational points, and so we choose them as the vertices of \( G_\ell(k, -p) \). Similarly, \( E_{0,w^2} \) and \( E_{0,w^4} \) have \( p^2 + 1 - p \mathbb{F}_{p^2} \)-rational points, and so we choose them as the vertices of \( G_\ell(k, p) \). Finally, \( \#E_{0,1} = p^2 + 1 + 2p \) and \( \#E_{0,3} = p^2 + 1 - 2p \); hence, we select \( E_{0,1} \) and \( E_{0,3} \) as the vertices of \( G_\ell(k, -2p) \) and \( G_\ell(k, 2p) \), respectively.

If \( j \neq 0, 1728 \) is supersingular, then \( E_j \) (defined in (1)) and a quadratic twist \( \tilde{E}_j \) are representatives of the two isomorphism classes of elliptic curves with \( j \)-invariant equal to \( j \). Furthermore, \( \#E_j \mathbb{F}_{p^2} = \{p^2 + 1 - 2p, p^2 + 1 + 2p\} \) and \( \#E_j \mathbb{F}_{p^2} = 2p^2 + 2 - \#E_j \mathbb{F}_{p^2} \). We select \( E_j \) as a vertex in either \( G_\ell(k, -2p) \) or \( G_\ell(k, 2p) \) depending on whether \( \#E_j \mathbb{F}_{p^2} = p^2 + 1 + 2p \) or \( p^2 + 1 - 2p \), and \( \tilde{E}_j \) as a vertex in the other graph.

4. The subgraphs \( G_\ell(\mathbb{F}_{p^2}, 0) \) and \( G_\ell(\mathbb{F}_{p^2} \pm p) \)

For a supersingular elliptic curve \( E \) defined over a finite field \( \mathbb{F}_q \) of characteristic \( > 3 \), we denote by \( \text{End}(E) \) the ring of endomorphisms of \( E \) defined over \( \mathbb{F}_q \) and by \( K = \text{End}(E) \otimes \mathbb{Z} \mathbb{Q} \) the corresponding endomorphism algebra. We will use the following classical result of Waterhouse [18] (see also Theorem 2.1 in [3]) to describe the arcs in the subgraphs \( G_\ell(\mathbb{F}_{p^2}, 0) \) and \( G_\ell(\mathbb{F}_{p^2} \pm p) \) as depicted in Figure 1.

![Figure 1. The small subgraphs of \( G_\ell(\mathbb{F}_{p^2}), p \equiv 11 \pmod{12} \).](image-url)
Theorem 2. Let $E$ be a supersingular elliptic curve defined over $\mathbb{F}_q = \mathbb{F}_{p^n}$ with $p > 3$, and let $t = q + 1 - \#E(\mathbb{F}_q)$. Then one of the following holds:

(i) $n$ is even and $t = \pm 2\sqrt{q}$;
(ii) $n$ is even, $p \equiv 2 \pmod{3}$ and $t = \pm \sqrt{q}$;
(iii) $n$ is even, $p \equiv 3 \pmod{4}$ and $t = 0$;
(iv) $n$ is odd and $t = 0$.

Let $\sigma$ be the $q$-power Frobenius endomorphism of $E$. In case (i), $K$ is a quaternion algebra over $\mathbb{Q}$, $\sigma$ is a rational integer, and $\text{End}(E)$ is a maximal order in $K$. In cases (ii), (iii) and (iv), $K = \mathbb{Q}(\sigma)$ is an imaginary quadratic number field and $\text{End}(E)$ is an order in $K$ with conductor coprime to $p$.

4.1. The subgraph $G_\ell(\mathbb{F}_{p^2}, 0)$. Let $q = p^2$ where $p \equiv 3 \pmod{4}$, $w$ is a generator of $\mathbb{F}_q^*$, and $\ell \not\equiv p$ is a prime. The graph $G_\ell(\mathbb{F}_{p^2}, 0)$ has two vertices, $E_{1728,w}$ and $E_{1728,w^3}$; to ease the notation we will call them $E_w$ and $E_{w^3}$ in this section.

Theorem 3. Let $p > 3$ and $\ell$ be primes with $p \equiv 3 \pmod{4}$ and $\ell \not\equiv p$.

(i) $G_2(\mathbb{F}_{p^2}, 0)$ has exactly two arcs, one loop at each of its two vertices.
(ii) If $\ell \equiv 3 \pmod{4}$, then $G_\ell(\mathbb{F}_{p^2}, 0)$ has no arcs.
(iii) If $\ell \equiv 1 \pmod{4}$, then $G_\ell(\mathbb{F}_{p^2}, 0)$ has exactly four arcs, two loops at each of its two vertices.

Proof. We describe the arcs originating at $E_w$. Notice that these arcs are exactly the degree-$\ell$ endomorphisms of $E_w$, i.e., the non-unit factors of $\ell$ in $\text{End}(E_w)$. The case $E_{w^3}$ case is similar.

Since $t = 0$, by Theorem 2, $\text{End}(E_w)$ is an order in $K = \mathbb{Q}(\sigma)$ with conductor $c$ coprime to $p$. The characteristic polynomial of the $p^2$-power Frobenius map $\sigma$ is $Z^2 + p^2$, and so we have $K = \mathbb{Q}(\sqrt{-p^2}) = \mathbb{Q}(i)$ whose maximal order is $\mathbb{Z}[i]$, the Gaussian integers. Since $\sigma$ and multiplication by integers are in $\text{End}(E_w)$, we have

$$\mathbb{Z}[\sigma] = \mathbb{Z}[ip] \subseteq \text{End}(E_w) \subseteq \mathbb{Z}[i].$$

Thus, the conductor $c$ of $\text{End}(E_w)$ divides the conductor $p$ of $\mathbb{Z}[\sigma]$, whence $c = 1$ and $\text{End}(E_w) = \mathbb{Z}[i]$. We have the following cases.

(i) If $\ell = 2$, then $\ell$ factors as $2 = i(i - 1)^2$. Hence, since $\mathbb{Z}[i]$ is a unique factorization domain, there is a unique degree-$\ell$ endomorphism of $E_w$.
(ii) If $\ell \equiv 3 \pmod{4}$, then $\ell$ is prime in $\mathbb{Z}[i]$. Thus, there are no degree-$\ell$ endomorphisms.
(iii) If $\ell \equiv 1 \pmod{4}$, then $\ell$ splits as $\ell = \alpha\overline{\alpha}$ for some Gaussian prime $\alpha$. Hence, there are exactly two degree-$\ell$ endomorphisms of $E_w$.

4.2. The subgraphs $G_\ell(\mathbb{F}_{p^2}, \pm p)$. Let $q = p^2$ where $p \equiv 2 \pmod{3}$, $w$ is a generator of $\mathbb{F}_q^*$, and $\ell \not\equiv p$ is a prime. The graph $G_\ell(\mathbb{F}_{p^2}, -p)$ has two vertices, $E_{0,w}$ and $E_{0,w^3}$; to ease the notation we will call them $E_w$ and $E_{w^3}$ in this section.

Theorem 4. Let $p > 3$ and $\ell$ be primes with $p \equiv 2 \pmod{3}$ and $\ell \not\equiv p$.

(i) $G_3(\mathbb{F}_{p^2}, -p)$ has exactly two arcs, one loop at each of its two vertices.
(ii) If $\ell \equiv 2 \pmod{3}$, then $G_\ell(\mathbb{F}_{p^2}, -p)$ has no arcs.
(iii) If \( \ell \equiv 1 \pmod{3} \), then \( G_\ell(\mathbb{F}_{p^2}, -p) \) has exactly four arcs, two loops at each of its two vertices.

**Proof.** We describe the arcs originating at \( E_w \). Notice that these arcs are exactly the degree-\( \ell \) endomorphisms of \( E_w \), i.e., the non-unit factors of \( \ell \) in \( \text{End}(E_w) \). The \( E_w \)'s case is similar.

Since \( t = -p \), by Theorem 2, \( \text{End}(E_w) \) is an order in \( K = \mathbb{Q}(\sigma) \) with conductor \( c \) coprime to \( p \). The characteristic polynomial of the \( p \)-power Frobenius map \( \sigma \) is \( Z^2 + pZ + p^2 \), and thus we have \( K = \mathbb{Q}(\sqrt{-3}) \). Hence, the maximal order of \( K \) is Eisenstein integers \( \mathbb{Z}[\lambda] \) where \( \lambda = (-1 + \sqrt{3})/2 \). Since \( \sigma \) and multiplication by integers are in \( \text{End}(E_w) \), we have

\[
\mathbb{Z}[\sigma] = \mathbb{Z}[\lambda p] \subseteq \text{End}(E_w) \subseteq \mathbb{Z}[\lambda].
\]

Thus, the conductor \( c \) of \( \text{End}(E_w) \) divides the conductor \( p \) of \( \mathbb{Z}[\sigma] \), whence \( c = 1 \) and \( \text{End}(E_w) = \mathbb{Z}[\lambda] \). We have the following cases.

(i) If \( \ell = 3 \), then \( \ell \) factors as \( 3 = -\lambda^2(1 - \lambda)^2 \). Hence, since \( \mathbb{Z}[\lambda] \) is a unique factorization domain, there is a unique degree-\( \ell \) endomorphism of \( E_w \).

(ii) If \( \ell \equiv 2 \pmod{3} \), then \( \ell \) is prime in \( \mathbb{Z}[\lambda] \). Thus there are no degree-\( \ell \) endomorphisms.

(iii) If \( \ell \equiv 1 \pmod{3} \), then \( \ell \) splits as \( \ell = \alpha \bar{\alpha} \) for some Eisenstein prime \( \alpha \). Hence, there are exactly two degree-\( \ell \) endomorphisms of \( E_w \).

\( \square \)

The proof of Theorem 5 is similar to that of Theorem 4.

**Theorem 5.** Let \( p > 3 \) and \( \ell \) be primes with \( p \equiv 2 \pmod{3} \) and \( \ell \neq p \).

(i) \( G_3(\mathbb{F}_{p^2}, p) \) has exactly two arcs, one loop at each of its two vertices.

(ii) If \( \ell \equiv 2 \pmod{3} \), then \( G_\ell(\mathbb{F}_{p^2}, p) \) has no arcs.

(iii) If \( \ell \equiv 1 \pmod{3} \), then \( G_\ell(\mathbb{F}_{p^2}, p) \) has exactly four arcs, two loops at each of its two vertices.

5. THE SUBGRAPHS \( G_\ell(\mathbb{F}_{p^2}, \pm 2p) \)

As noted in §3, the vertices in \( G_\ell(\mathbb{F}_{p^2}, -2p) \) have distinct \( j \)-invariants. Moreover, there is a one-to-one correspondence between the vertices in \( G_\ell(\mathbb{F}_{p^2}, -2p) \) and the vertices in \( G_\ell(\mathbb{F}_{p^2}, 2p) \); namely, if \( E \) is a vertex in \( G_\ell(\mathbb{F}_{p^2}, -2p) \) then the chosen quadratic twist \( \tilde{E} \) is a vertex in \( G_\ell(\mathbb{F}_{p^2}, 2p) \). Now, the characteristic polynomial of the \( q \)-power Frobenius map \( \sigma \) acting on any vertex \( E \) in \( G_\ell(\mathbb{F}_{p^2}, -2p) \) is \( Z^2 + 2pZ + p^2 = (Z + p)^2 \), so \( (\sigma + [p])^2 = 0 \). Since nonzero endomorphisms are surjective, we must have \( \sigma + [p] = 0 \). Hence \( \sigma = [-p] \) and all order-\( \ell \) subgroups of \( E \) are \( \sigma \)-invariant. It follows that every vertex in \( G_\ell(\mathbb{F}_{p^2}, -2p) \) has outdegree \( \ell + 1 \). Similarly, every vertex in \( G_\ell(\mathbb{F}_{p^2}, 2p) \) has outdegree \( \ell + 1 \).

By Theorem 1, the \( j \)-invariants of the heads of arcs with tail \( E \) in \( G_\ell(\mathbb{F}_{p^2}, -2p) \) are precisely the roots of \( \Phi_\ell(j(E), Y) \) (all \( \ell + 1 \) of which lie in \( \mathbb{F}_{p^2} \)). These roots are also the \( j \)-invariants of the heads of arcs with tail \( \tilde{E} \) in \( G_\ell(\mathbb{F}_{p^2}, 2p) \). Hence the directed graphs \( G_\ell(\mathbb{F}_{p^2}, -2p) \) and \( G_\ell(\mathbb{F}_{p^2}, 2p) \) are isomorphic.

Sutherland [15] defines the isogeny graph \( \mathcal{H}_\ell(\mathbb{F}_{p^2}) \) to have vertex set \( \mathbb{F}_{p^2} \) and arcs \((j_1, j_2)\) present with multiplicity equal to the multiplicity of \( j_2 \) as a root of \( \Phi_\ell(j_1, Y) \) in \( \mathbb{F}_{p^2} \). The
following folklore result shows that $\mathcal{G}_\ell(\mathbb{F}_{p^2})$, the supersingular component of $\mathcal{H}_\ell(\mathbb{F}_{p^2})$, is isomorphic to $\mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p)$.

**Theorem 6.** $\mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p)$ and $\mathcal{G}_\ell(\mathbb{F}_{p^2})$ are isomorphic.

**Proof.** Recall that every supersingular elliptic curves over $\mathbb{F}_{p^2}$ is isomorphic to one defined over $\mathbb{F}_{p^2}$. Hence the map $\beta : E \mapsto j(E)$ is a bijection between the vertex sets of $\mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p)$ and $\mathcal{G}_\ell(\mathbb{F}_{p^2})$. Now, let $(E_1, E_2)$ be an arc of multiplicity $c \geq 0$ in $\mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p)$. By Theorem 1, $j(E_2)$ is a root of multiplicity $c$ of $\Phi_\ell(j(E_1), Y)$. Hence $(j(E_1), j(E_2))$ is an arc of multiplicity $c$ in $\mathcal{G}_\ell(\mathbb{F}_{p^2})$. Thus, $\beta$ preserves arcs and $\mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p) \cong \mathcal{G}_\ell(\mathbb{F}_{p^2})$. □

5.1. **Indegree.** Suppose that $p$ is prime and let $E$ be a vertex in $\mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p)$. Then all automorphisms of $E$ are defined over $\mathbb{F}_{p^2}$; we denote the group of all automorphisms of $E$ by $\text{Aut}(E)$. Recall from §2.1 that $\#\text{Aut}(E) = 4, 6$ or 2 depending on whether $j(E) = 1728$, $j(E) = 0$ or $j(E) \neq 0, 1728$.

Let $\ell \neq p$ be a prime. Let $E_1, E_2$ be two vertices in $\mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p)$, and let $\phi_1, \phi_2 : E_1 \to E_2$ be two degree-$\ell$ $\mathbb{F}_{p^2}$-isogenies. We say that $\phi_1$ and $\phi_2$ are equivalent if they have the same kernel, or, equivalently, if there exists $\rho_2 \in \text{Aut}(E_2)$ such that $\phi_2 = \rho_2 \circ \phi_1$. Thus, the arcs $(E_1, E_2)$ in $\mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p)$ can be seen as the classes of equivalent degree-$\ell$ $\mathbb{F}_{p^2}$-isogenies from $E_1$ to $E_2$. We define $\phi_1$ and $\phi_2$ to be automorphic if there exists $\rho_1 \in \text{Aut}(E_1)$ such that $\phi_1$ and $\phi_2$ are automorphic. Hence, if $\phi_1$ and $\phi_2$ are automorphic then there exist $\rho_1 \in \text{Aut}(E_1)$ and $\rho_2 \in \text{Aut}(E_2)$ such that $\phi_2 = \rho_2 \circ \phi_1 \circ \rho_1$. Since $\hat{\phi}_2 = \rho_1^{-1} \circ \phi_1 \circ \rho_2^{-1}$, it follows that the duals of automorphic isogenies are automorphic.

**Theorem 7.** Let $E$ be a vertex in $\mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p)$ and let $n = \#\text{Aut}(E)/2$. Let $a$ and $b$ denote the number of arcs $(E, E_1728)$ and arcs $(E, E_0)$ in $\mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p)$, respectively. Then the indegree of $E$ is $(\ell + a + 2b + 1)/n$.

**Proof.** Let $E_1, E_2$ be two vertices in $\mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p)$, and let $\text{Aut}(E_1) = \langle \rho_i \rangle$ and $n_i = \#\text{Aut}(E_i)/2$ for $i = 1, 2$. Let $\phi : E_1 \to E_2$ be a degree-$\ell$ $\mathbb{F}_{p^2}$-isogeny.

Suppose first that the kernel of $\phi$ is not an eigenspace of $\rho_1$. Consider the set

$$\mathcal{A} = \{\rho_2^i \circ \phi \circ \rho_1^i : 0 \leq i < 2n_1, 0 \leq j < 2n_2\}$$

of isogenies automorphic to $\phi$. Since $\rho_1^{n_i} = -1$ for $i \in \{1, 2\}$, we have

$$\mathcal{A} = \{\rho_2^i \circ \phi \circ \rho_1^i : 0 \leq i < n_1, 0 \leq j < 2n_2\}.$$ 

One can check that if $(i, j) \neq (i', j')$ where $0 \leq i, i' < n_1$ and $0 \leq j, j' < 2n_2$, then $\rho_2^i \circ \phi \circ \rho_1^i = \rho_2^j \circ \phi \circ \rho_1^j$ implies that the kernel of $\phi$ is an eigenspace of $\rho_1$. Hence the set $\mathcal{A}$ has size exactly $2n_1n_2$ and the isogenies in $\mathcal{A}$ can be partitioned into $n_1$ classes of equivalent isogenies, each class comprised of $2n_2$ isogenies. Similarly, the set

$$\hat{\mathcal{A}} = \{\rho_1^i \circ \hat{\phi} \circ \rho_2^j : 0 \leq i < 2n_1, 0 \leq j < 2n_2\}$$

of dual isogenies can be partitioned into $n_2$ classes of equivalent isogenies, each class comprised of $2n_1$ isogenies. Consequently, $\phi$ generates $n_1$ different arcs $(E_1, E_2)$ and $\rho_1$ generates $n_2$ different arcs $(E_2, E_1)$. Because duals of automorphic isogenies are automorphic, if there is another degree-$\ell$ $\mathbb{F}_{p^2}$-isogeny $\psi$ from $E_1$ to $E_2$ not automorphic to $\phi$, then $\psi$ (resp. $\hat{\psi}$) generates a set of $n_1$ (resp. $n_2$) arcs $(E_1, E_2)$ (resp. $(E_2, E_1)$) disjoint from those generated by $\phi$ (resp. $\hat{\phi}$). Therefore, the number $r_{out}$ of arcs $(E_1, E_2)$ generated
by isogenies whose kernels are not eigenspaces of \( \rho_1 \) and the number \( r_{in} \) of arcs \((E_2, E_1)\) generated by their duals are multiples of \( n_1 \) and \( n_2 \), respectively. Moreover, we have

\[
(10) \quad r_{in} = \frac{n_2 \cdot r_{out}}{n_1}.
\]

Suppose now that the kernel of \( \phi \) is an eigenspace of \( \rho_1 \). This scenario occurs only if \( E_1 \) has \( j \)-invariant 1728 or 0. Suppose \( E_1 \) has \( j \)-invariant 1728, and let \( \rho_1 \) be the automorphism \((x, y) \mapsto (-x, iy)\) where \( i \in \mathbb{F}_{p^2} \) satisfies \( i^2 = -1 \). Denote by \( G \) the kernel of \( \phi \), and let \( \phi' : E_1 \to E_1^G \) denote the Vélu isogeny. By (5), \( E_1^G \) has equation \( Y^2 = X^3 + aX - 7w \) for some \( a \in \mathbb{F}_{p^2} \) and \( w = \sum_{Q \in \Gamma_s} (5x_Q^3 + 3x_Q) \). Since \( \rho_1(G) = G \), if \((x, y) \in G\) then \((-x, iy) \in G\). Hence \( w = 0 \) and we conclude that \( E_1^G \) is isomorphic to \( E_1 \) over \( \mathbb{F}_{p^2} \), i.e., \( E_2 = E_1 \). A similar argument using the automorphism \((x, y) \mapsto (\eta x, -y)\) with \( \eta \in \mathbb{F}_{p^2} \) satisfying \( \eta^2 + \eta + 1 = 0 \) shows that we also have \( E_2 = E_1 \) when the \( j \)-invariant of \( E_1 \) is 0. Thus, if the kernel of \( \phi \) is an eigenspace of \( \rho_1 \), the arcs generated by \( \phi \) are loops at \( E_1 \). Therefore, we can generalize (10) to the total number \( t_{out} \) of arcs \((E_1, E_2)\) and the total number \( t_{in} \) of arcs \((E_2, E_1)\) and obtain

\[
(11) \quad t_{in} = \frac{n_2 \cdot t_{out}}{n_1}.
\]

Now, let \( E \) be a vertex in \( \mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p) \) and \( n = \#\text{Aut}(E) / 2 \). Denote by \( E_j \) the vertex in \( \mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p) \) having \( j \)-invariant \( j \in \mathbb{F}_{p^2} \). Let \( a \) be the number of arcs \((E, E_1)\) and \( b \) the number of arcs \((E, E_0)\). Note that the number of arcs \((E, E_j), j \notin \{0, 1728\}\), is \( c = \ell - a - b + 1 \). From (11) we have

\[
\text{indegree}(E) = \frac{c}{n} + \frac{2a}{n} + \frac{3b}{n},
\]

whence

\[
\text{indegree}(E) = \frac{\ell + a + 2b + 1}{n}.
\]

\[
\square
\]

5.2. Loops. Let \( E_{1728} \) and \( E_0 \) denote the vertices in \( \mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p) \) with \( j \)-invariants 1728 and 0. In §5.2.1 and §5.2.2 we investigate the number of loops at \( E_{1728} \) and \( E_0 \) in \( \mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p) \). In particular, we determine upper bounds on \( p \) for which \( E_0 \) and \( E_{1728} \) have unexpected loops, i.e., loops not arising from eigenspaces of the primitive automorphisms of \( E_0 \) and \( E_{1728} \).

5.2.1. \( E_{1728} \) loops. We begin by noting that

\[
\Phi_2(X, 1728) = (X - 1728)(X - 287496)^2.
\]

Since \( 287496 - 1728 = 2^3 \cdot 3^6 \cdot 7^2 \), we see that 1728 is a triple root of \( \Phi_2(X, 1728) \) in \( \mathbb{Z}_p[X] \) if \( p = 7 \) and a single root if \( p > 7 \). Hence the number of loops at \( E_{1728} \) in \( \mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p) \) is three if \( p = 7 \) and one if \( p > 7 \) (and \( p \equiv 3 \pmod{4} \)).

Lemma 8. Let \( p \equiv 3 \pmod{4} \) be a prime, and let \( \ell \neq p \) be an odd prime. Then the number of loops at \( E_{1728} \) is even. Moreover, if \( \ell \equiv 1 \pmod{4} \) then there are at least two loops at \( E_{1728} \).
Proof. Let $\rho$ denote the automorphism $(x,y) \mapsto (-x,iy)$ of $E_{1728}$ where $i \in \mathbb{F}_p^2$ satisfies $i^2 = -1$. Since $\# \text{Aut}(E_{1728})/2 = 2$ we have from the first part of the proof of Theorem 7 that the number of loops at $E_{1728}$ generated by isogenies whose kernels are not eigenspaces of $\rho$ is even.

The characteristic polynomial $Z^2 + 1$ of $\rho$ splits modulo $\ell$ if and only if $\ell \equiv 1 \pmod{4}$. Hence, if $\ell \equiv 3 \pmod{4}$ then all the loops at $E_{1728}$ are generated by isogenies whose kernels are not eigenspaces of $\rho$ and thus the number of loops is even. Now suppose that $\ell \equiv 1 \pmod{4}$. The eigenspaces of $\rho$ modulo $\ell$ are two different order-$\ell$ subgroups of $E_{1728}$. The second part of the proof of Theorem 7 shows that the arcs generated by these subgroups are loops at $E_{1728}$. □

Let $p$ be a prime and let $B_{p,\infty}$ denote the quaternion algebra over $\mathbb{Q}$ ramified at $p$ and $\infty$ with trace $\text{Tr}$ and norm $N$. From [7, Lemma 2.1.1], we have the following result.

**Lemma 9.** Let $R$ be a maximal order of $B_{p,\infty}$, and let $K_1, K_2$ be distinct imaginary quadratic subfields of $B_{p,\infty}$. Furthermore, suppose that there exist $k_i \in R$, $i = 1,2$, such that $\{1,k_i\}$ is a $\mathbb{Q}$-basis for $K_i$. Then $p \leq 4N(k_1)N(k_2)$.

**Theorem 10.** Let $\ell$ be a fixed prime, and let $p \equiv 3 \pmod{4}$ be a prime distinct from $\ell$. Suppose that $E_{1728}$ has at least one loop in $G_\ell(\mathbb{F}_{p^2}, -2p)$ when $\ell \equiv 3 \pmod{4}$, and at least three loops when $\ell \equiv 1 \pmod{4}$. Then $p < 4\ell$.

**Proof.** Let $\text{End}(E_{1728})$ be the endomorphism ring of $E_{1728}$. It is known that $\text{End}(E_{1728})$ is a maximal order in $B_{p,\infty}$ [18]. Since $\text{End}(E_{1728})$ contains the order-$4$ automorphism $\rho : (x,y) \mapsto (-x,iy)$, where $i \in \mathbb{F}_p^2$ satisfies $i^2 = -1$, we have $\mathbb{Q}(\rho) = \mathbb{Q}(\sqrt{-1}) \subset \text{End}(E_{1728})$.

Suppose that $E_{1728}$ has a loop in $G_\ell(\mathbb{F}_{p^2}, -2p)$, whence there exists $\alpha \in \text{End}(E_{1728})$ such that $N(\alpha) = \ell$. If $\ell \equiv 3 \pmod{4}$, then $\alpha \notin \mathbb{Q}(\rho)$ since $\ell$ is prime in $\mathbb{Z}[\rho]$. On the other hand, if $\ell \equiv 1 \pmod{4}$, then $\ell$ splits uniquely in $\mathbb{Z}[\rho]$ up to multiplication by units as $\ell = \delta \delta'$. If $E_{1728}$ has at least three loops in $G_\ell(\mathbb{F}_{p^2}, -2p)$, then we can further assume that $\alpha \neq u\delta$ for all units $u \in \mathbb{Z}[\rho]$ and again we conclude that $\alpha \notin \mathbb{Q}(\rho)$.

Every element $b \in B_{p,\infty}$ satisfies $b^2 - \text{Tr}(b)b + N(b) = 0$. Now, let $\gamma = 2\alpha - \text{Tr}(\alpha)$. Since $\text{Tr}(\gamma) = 0$, we have $\gamma^2 = -N(\gamma) < 0$. Hence $\mathbb{Q}(\alpha) = \mathbb{Q}(\gamma)$ is an imaginary quadratic field different from $\mathbb{Q}(\rho)$. Considering the bases $\{1,\rho\}$, $\{1,\alpha\}$ for $\mathbb{Q}(\rho)$, $\mathbb{Q}(\alpha)$, respectively, Lemma 9 yields $p \leq 4\ell$, and as $p$ is a prime number, we conclude that $p < 4\ell$. □

**5.2.2. $E_0$ loops.** We have

$$\Phi_2(X,0) = (X - 2^4 \cdot 3^3 \cdot 5^3)^3,$$

whence 0 is a triple root of $\Phi_2(X,0)$ in $\mathbb{Z}_p[X]$ if $p = 5$ and not a root if $p > 5$. Hence the number of loops at $E_0$ in $G_2(\mathbb{F}_{p^2}, -2p)$ is three if $p = 5$ and zero if $p > 5$ (and $p \equiv 2 \pmod{3}$). Similarly, since

$$\Phi_3(X,0) = X(X - 2^5 \cdot 3 \cdot 5^3)^3,$$

we conclude that the number of loops at $E_0$ in $G_3(\mathbb{F}_{p^2}, -2p)$ is four if $p = 5$ and one if $p > 5$ (and $p \equiv 2 \pmod{3}$).

**Lemma 11.** Let $p \equiv 2 \pmod{3}$ be a prime, and let $\ell \neq 3, p$ be an odd prime. If $\ell \equiv 2 \pmod{3}$, then the number of loops at $E_0$ is $\equiv 0 \pmod{3}$. If $\ell \equiv 1 \pmod{3}$, then the number of loops at $E_0$ is $\equiv 2 \pmod{3}$. 


Proof. Similar to the proof of Lemma 8. □

Theorem 12. Let \( \ell \) be a fixed prime. Let \( p \equiv 2 \) (mod 3), \( p \neq \ell \), be a prime for which \( E_0 \) has at least one loop in \( \mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p) \) if \( \ell \equiv 2 \) (mod 3) or at least three loops if \( \ell \equiv 1 \) (mod 3). Then \( p < 4\ell \).

Proof. Similar to the proof of Theorem 10. □

For primes \( \ell \equiv 1 \) (mod 4) (resp. \( \ell \equiv 3 \) (mod 4)), let \( p_1^{1728}(\ell) \) (resp. \( p_3^{1728}(\ell) \)) denote the largest prime \( p \equiv 3 \) (mod 4), \( p \neq \ell \), for which \( E_0 \) has at least three loops (resp. at least one loop) in \( \mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p) \). Similarly, for odd primes \( \ell \equiv 1 \) (mod 3) (resp. \( \ell \equiv 2 \) (mod 3)), let \( p_1^0(\ell) \) (resp. \( p_2^0(\ell) \)) denote the largest prime \( p \equiv 2 \) (mod 3), \( p \neq \ell \), for which \( E_0 \) has at least three loops (resp. at least one loop) in \( \mathcal{G}_\ell(\mathbb{F}_{p^2}, -2p) \). Table 1 lists \( p_1^{1728}(\ell) \), \( p_3^{1728}(\ell) \), \( p_1^0(\ell) \), \( p_2^0(\ell) \) for all primes \( \ell \leq 283 \). These values were obtained by factoring the relevant values of the modular polynomial \( \Phi_\ell \); the modular polynomials were obtained from Sutherland’s database [1, 16]. For example, \( p_3^{1728}(\ell) \) is the largest prime factor of \( \Phi_\ell(1728, 1728) \) that is congruent to 3 modulo 4. Table 1 indicates that the bounds \( p_1^{1728}(\ell) < 4\ell \) and \( p_3^{1728}(\ell) < 4\ell \) are tight, and suggests a tighter upper bound of \( 3\ell \) for \( p_1^0(\ell) \) and \( p_2^0(\ell) \).

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Table 1. The values \( p_1^{1728}(\ell) \), \( p_3^{1728}(\ell) \), \( p_1^0(\ell) \), \( p_2^0(\ell) \) for all odd primes \( \ell \leq 283 \).
6. Concluding remarks

We defined the supersingular isogeny graph $G_\ell(F_{p^2})$, and described the arcs of its small subgraphs $G_\ell(F_{p^2},0)$ and $G_\ell(F_{p^2},\pm p)$. We also investigated the existence of loops at vertices $E_0$ and $E_{1728}$ in the large subgraph $G_\ell(F_{p^2},-2p)$, and determined upper bounds on primes $p$ for which $E_0$ and $E_{1728}$ have unexpected loops in $G_\ell(F_{p^2},-2p)$.

Acknowledgements

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References
