ON ISOGENY GRAPHS OF SUPERSINGULAR ELLIPTIC CURVES
OVER FINITE FIELDS

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Abstract. We study the isogeny graphs of supersingular elliptic curves over finite fields, with an emphasis on the vertices corresponding to elliptic curves of $j$-invariant 0 and 1728.

1. Introduction

Let $\mathbb{F}_q$ be the finite field of order $q$ and characteristic $p > 3$, and let $\overline{\mathbb{F}}_q$ denote its algebraic closure. Let $\ell$ be a prime different from $p$. The isogeny graph $\mathcal{H}_\ell(\mathbb{F}_q)$ is a directed graph whose vertices are the $\overline{\mathbb{F}}_q$-isomorphism classes of elliptic curves defined over $\mathbb{F}_q$, and whose directed arcs represent degree-$\ell$ $\mathbb{F}_q$-isogenies (up to a certain equivalence) between elliptic curves in the isomorphism classes. See [10] and [15] for summaries of the theory behind isogeny graphs and for applications in computational number theory.

Every supersingular elliptic curve defined over $\mathbb{F}_p$ is isomorphic to one defined over $\mathbb{F}_{p^2}$. Pizer [12] showed that the subgraph $\mathcal{G}_\ell(\mathbb{F}_{p^2})$ of $\mathcal{H}_\ell(\mathbb{F}_{p^2})$ induced by the vertices corresponding to isomorphism classes of supersingular elliptic curves over $\mathbb{F}_{p^2}$ is an expander graph (and consequently is connected). This property of $\mathcal{G}_\ell(\mathbb{F}_{p^2})$ was exploited by Charles, Goren and Lauter [3] who proposed a cryptographic hash function whose security is based on the intractability of computing directed paths of a certain length between two vertices in $\mathcal{G}_\ell(\mathbb{F}_{p^2})$. In 2011, Jao and De Feo [8] (see also [5]) presented a key agreement scheme whose security is also based on the intractability of this problem for small $\ell$ (typically $\ell = 2, 3$). There have also been proposals for related signature schemes [18, 7] and an undeniable signature scheme [9].

In this paper, we study the supersingular isogeny graph $\mathcal{G}_\ell(\mathbb{F}_{p^2})$ whose vertices are (canonical representatives of) the $\mathbb{F}_{p^2}$-isomorphism classes of supersingular elliptic curves defined over $\mathbb{F}_{p^2}$, and whose directed arcs represent degree-$\ell$ $\mathbb{F}_{p^2}$-isogenies between the elliptic curves. Observe that the difference between the definitions of $\mathcal{G}_\ell(\mathbb{F}_{p^2})$ and $\mathcal{G}_\ell(\mathbb{F}_p)$ is that the isomorphisms and isogenies in the former are defined over $\mathbb{F}_{p^2}$ itself. This difference necessitates a careful treatment of the vertices corresponding to supersingular elliptic curves having $j$-invariant equal to 0 and 1728. We note that the security of the aforementioned cryptographic schemes in fact relies on the difficulty of constructing directed paths in $\mathcal{G}_\ell(\mathbb{F}_{p^2})$ (and not in $\mathcal{G}_\ell(\mathbb{F}_p)$ as stated in [3] and [5]). Thus, it is worthwhile to study the differences between $\mathcal{G}_\ell(\mathbb{F}_{p^2})$ and $\mathcal{G}_\ell(\mathbb{F}_p)$. We also note that Delfs and Galbraith [4] studied supersingular isogeny graphs $\mathcal{G}_\ell(\mathbb{F}_p)$, and observed that they have similar ‘volcano’ structures as the ordinary subgraphs of $\mathcal{H}_\ell(\mathbb{F}_p)$ [6]. The remainder of the paper is organized as follows. In §2 we provide a concise summary of the relevant

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background on elliptic curves and isogenies between them. Standard references for the material in §2 are the books by Silverman [14] and Washington [17]. The supersingular isogeny graph $\Gamma_{\ell}(\mathbb{F}_{p^2})$ is defined in §3. In §4 and §5, we completely describe the three small subgraphs of $\Gamma_{\ell}(\mathbb{F}_{p^2})$ whose vertices correspond to supersingular elliptic curves $E$ over $\mathbb{F}_{p^2}$ with $t = p^2 + 1 - \#E(\mathbb{F}_{p^2}) \in \{0, -p, p\}$; see Figure 1. In §6, we study the two large subgraphs of $\Gamma_{\ell}(\mathbb{F}_{p^2})$ whose vertices correspond to supersingular elliptic curves $E$ over $\mathbb{F}_{p^2}$ with $t = p^2 + 1 - \#E(\mathbb{F}_{p^2}) \in \{-2p, 2p\}$, and make some observations about the number of loops at the vertices corresponding to elliptic curves with $j$-invariant equal to 0 or 1728.

2. Elliptic curves

Let $k = \mathbb{F}_q$ be the finite field of order $q$ and characteristic $p \neq 2, 3$, and let $\overline{k} = \cup_{n \geq 1} \mathbb{F}_{q^n}$ denote its algebraic closure. Let $\sigma : x \mapsto x^q$ denote the $q$-power Frobenius map. An elliptic curve $E$ over $k$ is defined by a Weierstrass equation $E/k : Y^2 = X^3 + aX + b$ where $a, b \in k$ and $4a^3 + 27b^2 \neq 0$. The $j$-invariant of $E$ is $j(E) = 1728 \cdot 4a^3/(4a^3 + 27b^2)$. One can easily check that $j(E) = 0$ if and only if $a = 0$, and $j(E) = 1728$ if and only if $b = 0$. For any extension $K$ of $k$, the set of $K$-rational points on $E$ is $E(K) = \{(x, y) \in K \times K : y^2 = x^3 + ax + b\} \cup \{\infty\}$, where $\infty$ is the point at infinity; we write $E = E(\overline{k})$. The chord-and-tangent addition law transforms $E(K)$ into an abelian group. For any $n \geq 2$ with $p \nmid n$, the group of $n$-torsion points on $E$ is isomorphic to $\mathbb{Z}_n \oplus \mathbb{Z}_n$. In particular, if $n$ is prime then $E$ has exactly $n + 1$ distinct order-$n$ subgroups.

2.1. Isomorphisms and automorphisms. Two elliptic curves $E/k : Y^2 = X^3 + aX + b$ and $E'/k : Y^2 = X^3 + a'X + b'$ are isomorphic over the extension field $K/k$ if there exists $u \in K^*$ such that $a' = u^4a$ and $b' = u^6b$. If such a $u$ exists, then the corresponding isomorphism $f : E \to E'$ is defined by $(x, y) \mapsto (u^2x, u^3y)$. If $E$ and $E'$ are isomorphic over $K$, then $j(E) = j(E')$. Conversely, if $j(E) = j(E')$, then $E$ and $E'$ are isomorphic over $\overline{k}$. Elliptic curves $E_1/k$, $E_2/k$ that are isomorphic over $\mathbb{F}_{q^d}$ for some $d > 1$, but are not isomorphic over any smaller extension of $\mathbb{F}_q$, are said to be degree-d twists of each other. In particular, a degree-2 (quadratic) twist of $E_1/k : Y^2 = X^3 + aX + b$ is $E_2/k : Y^2 = X^3 + c^2aX + c^3b$ where $c \in k^*$ is a non-square, and $\#E_1(k) + \#E_2(k) = 2q + 2$. If $j \in \overline{k} \setminus \{0, 1728\}$, then

$$E_j : Y^2 = X^3 + \frac{3j}{1728 - j}X + \frac{2j}{1728 - j}$$

is an elliptic curve with $j(E) = j$. Also, $E : Y^2 = X^3 + 1$ has $j(E) = 0$ and $Y^2 = X^3 + X$ has $j(E) = 1728$.

An automorphism of $E/k$ is an isomorphism from $E$ to itself. The group of all automorphism of $E$ that are defined over $K$ is denoted by $\text{Aut}_K(E)$. If $j(E) \neq 0, 1728$, then $\text{Aut}_K(E)$ has order 2 with generator $(x, y) \mapsto (x, -y)$. If $j(E) = 1728$, then $\text{Aut}_K(E)$ is a cyclic group of order 4 with generator $\psi : (x, y) \mapsto (-x, yi)$ where $i \in \overline{k}$ is a primitive fourth root of unity. If $j(E) = 0$, then $\text{Aut}_K(E)$ is cyclic of order 6 with generator $\rho : (x, y) \mapsto (jx, -y)$ where $j \in \overline{k}$ is a primitive third root of unity.
2.2. **Isogenies.** Let $E, E'$ be elliptic curves defined over $k = \mathbb{F}_q$. An isogeny $\phi : E \to E'$ is a non-constant rational map defined over $\overline{k}$ with $\phi(\infty) = \infty$. An endomorphism on $E$ is an isogeny from $E$ to itself; the zero map $P \mapsto \infty$ is also considered to be an endomorphism on $E$. If the field of definition of $\phi$ is the extension $K$ of $k$, then $\phi$ is called a $K$-isogeny. If such an isogeny exists, then $E$ and $E'$ are said to be $K$-isogenous. Tate’s theorem asserts that $E$ and $E'$ are $K$-isogenous if and only if $\# E(K) = \# E'(K)$.

The isogeny $\phi$ is a morphism, is surjective, is a group homomorphism, and has finite kernel. Every $K$-isogeny $\phi$ can be represented as $\phi = (r_1(X), r_2(X) \cdot Y)$ where $r_1, r_2 \in K(X)$. Let $r_1(X) = p_1(X)/q_1(X)$, where $p_1, q_1 \in K[X]$ with $\gcd(p_1, q_1) = 1$. Then the degree of $\phi$ is $\max(\deg p_1, \deg q_1)$. Also, if $\phi$ is said to be separable if $r_1'(X) \neq 0$; otherwise it is inseparable. In fact, $\phi$ is separable if and only if $\# \ker \phi = \deg \phi$. Note that all isogenies of prime degree $\ell \neq p$ are separable.

For every $m \geq 1$, the multiplication-by-$m$ map $[m] : E \to E$ is a $k$-isogeny of degree $m^2$. Every degree-$m$ isogeny $\phi : E \to E'$ has a unique dual isogeny $\hat{\phi} : E' \to E$ satisfying $\hat{\phi} \circ \phi = [m]$ and $\phi \circ \hat{\phi} = [m]$. If $\phi$ is a $K$-isogeny, then so is $\hat{\phi}$. We have $\deg \hat{\phi} = \deg \phi$ and $\hat{\phi} = \phi$. If $E''$ is an elliptic curve defined over $k$ and $\psi : E' \to E''$ is an isogeny, then $\overline{\psi} \circ \phi = \hat{\phi} \circ \psi$.

2.3. **Vélu’s formula.** Let $E$ be an elliptic curve defined over $k = \mathbb{F}_q$. Let $\ell \neq p$ be a prime, and let $G$ be an order-$\ell$ subgroup of $E$. Then there exists an elliptic curve $E'$ over $\overline{k}$ and a degree-$\ell$ isogeny $\phi : E \to E'$ with $\ker \phi = G$. The elliptic curve $E'$ and the isogeny $\phi$ are both defined over $K = \mathbb{F}_q^t$ where $t$ is the smallest positive integer such that $G$ is $\sigma^t$-invariant, i.e., $\{\sigma^t(P) : P \in G\} = G$ where $\sigma(P) = (x^q, y^q)$ if $P = (x, y)$ and $\sigma(\infty) = \infty$. Furthermore, $\phi$ is unique in the following sense: if $E''$ is an elliptic curve defined over $K$ and $\psi : E \to E''$ is a degree-$\ell$ $K$-isogeny with $\ker \psi = G$, then there exists an isomorphism $f : E' \to E''$ defined over $K$ such that $\psi = f \circ \phi$.

Given the Weierstrass equation $Y^2 = X^3 + aX + b$ for $E/k$ and an order-$\ell$ subgroup $G$ of $E$, Vélu’s formula yields an elliptic curve $E'$ defined over $K$ and a degree-$\ell$ $K$-isogeny $\phi : E \to E'$ with $\ker \phi = G$.

Suppose first that $\ell = 2$ and $G = \{\infty, (\alpha, 0)\}$. Then the Weierstrass equation for $E'$ is

$$E' : Y^2 = X^3 - (4a + 15\alpha^2)X + (8b - 14\alpha^3),$$

and the isogeny $\phi$ is given by

$$\phi = \left( X + \frac{3a^2 + a}{X - \alpha}, \ Y - \frac{(3a^2 + a)Y}{(X - \alpha)^2} \right).$$

Suppose now that $\ell$ is an odd prime. For $Q = (x_Q, y_Q) \in G^*$, define

$$t_Q = 3x_Q^2 + a, \quad u_Q = 2y_Q^2, \quad w_Q = u_Q + t_Qx_Q.$$

Furthermore, define

$$t = \sum_{Q \in G^*} t_Q, \quad w = \sum_{Q \in G^*} w_Q,$$

and

$$r(X) = X + \sum_{Q \in G^*} \left( \frac{t_Q}{X - x_Q} + \frac{u_Q}{(X - x_Q)^2} \right).$$
Then the Weierstrass equation for $E'$ is
\begin{equation}
E' : Y^2 = X^3 + (a - 5t)X + (b - 7w),
\end{equation}
and the isogeny $\phi$ is given by
\begin{equation}
\phi = (r(X), r'(X)Y).
\end{equation}

We will henceforth denote the Vélu-generated elliptic curve $E'$ by $E/G$.

2.4. Modular polynomials. Let $\ell$ be a prime. The modular polynomial $\Phi_\ell(X,Y) \in \mathbb{Z}[X,Y]$ is a symmetric polynomial of the form $\Phi_\ell(X,Y) = X^{\ell+1} + Y^{\ell+1} - X\ell Y^\ell + \sum c_{ij}X^iY^j$, where the sum is over pairs of integers $(i,j)$ with $0 \leq i, j \leq \ell$ and $i + j < 2\ell$. Modular polynomials have the following remarkable property that for any elliptic curve $E$ characterizes the $j$-invariants of those elliptic curves $E'$ for which a degree-$\ell$ separable isogeny $\phi : E \to E'$ exists.

**Theorem 1.** Suppose that the characteristic of $k = \mathbb{F}_q$ is different from $\ell$. Let $E/k$ be an elliptic curve with $j(E) = j$. Let $G_1, G_2, \ldots, G_{\ell+1}$ be the order-$\ell$ subgroups of $E$. Let $j_i = j(E/G_i)$. Then the roots of $\Phi_\ell(j,Y)$ in $\overline{F}$ are precisely $j_1, j_2, \ldots, j_{\ell+1}$.

2.5. Supersingular elliptic curves. Hasse’s theorem states that if $E$ is defined over $\mathbb{F}_q$, then $\#E(\mathbb{F}_q) = q + 1 - t$ where $|t| \leq 2\sqrt{q}$. The integer $t$ is called the trace of the $q$-power Frobenius map $\sigma$ since the characteristic polynomial of $\sigma$ acting on $E$ is $Z^2 - tZ + q$. If $p | t$, then $E$ is supersingular; otherwise it is ordinary. Every supersingular elliptic curve $E$ over $\mathbb{F}_q$ is isomorphic to one defined over $\mathbb{F}_p^2$; in particular, $j(E) \in \mathbb{F}_p^2$. Henceforth, we shall assume that $q = p^2$ (and $p > 3$).

Supersingularity of an elliptic curve depends only on its $j$-invariant. We say that $j \in \mathbb{F}_p^2$ is supersingular if there exists a supersingular elliptic curve $E/\mathbb{F}_p^2$ with $j(E) = j$; if this is the case, then all elliptic curves with $j$-invariant equal to $j$ are supersingular. Note that $j = 0$ is supersingular if and only if $p \equiv 2 \pmod{3}$, and $j = 1728$ is supersingular if and only if $p \equiv 3 \pmod{4}$.

Schoof [13] determined the number of isomorphism classes of elliptic curves over a finite field. In particular, the number of isomorphism classes of supersingular elliptic curves $E$ over $\mathbb{F}_p^2$ with $\#E(\mathbb{F}_p^2) = p^2 + 1 - t$ is
\begin{equation}
N(t) = \begin{cases}
\frac{1}{12} (p + 6(\frac{-3}{p}) - 3(\frac{-4}{p})) & \text{if } t = \pm 2p, \\
1 - (\frac{-3}{p}) & \text{if } t = \pm p, \\
1 - (\frac{-4}{p}) & \text{if } t = 0,
\end{cases}
\end{equation}
where $(\frac{a}{p})$ is the Legendre symbol. It follows that the total number of isomorphism classes of supersingular elliptic curves over $\mathbb{F}_p^2$ is $\lfloor p/12 \rfloor + \epsilon$, where $\epsilon = 0, 5, 3, 8$ if $p \equiv 1, 5, 7, 11 \pmod{12}$ respectively. Furthermore, if $t = 0, -p$ or $p$ then $E(\mathbb{F}_p^2)$ is cyclic.

3. Supersingular isogeny graphs

Let $k = \mathbb{F}_q$ where $q = p^2$, and let $\ell \neq p$ be a prime. Recall that $\sigma$ is the $q$-th power Frobenius map. The supersingular isogeny graph $G_\ell(k)$ is a directed graph whose vertex set $V_\ell(k)$ consists of canonical representatives of the $k$-isomorphism classes of supersingular elliptic curves defined over $k$. The (directed) arcs of $G_\ell(k)$ are defined as follows. Let $E_1 \in V_\ell(k)$, and let $G$ be a $\sigma$-invariant order-$\ell$ subgroup of $E_1$. Let $\phi : E_1 \to E_1/G$ be
the Vélu isogeny with kernel $G$ (recall that $E_1/G$ and $\phi$ are both defined over $k$), and let $E_2$ be the canonical representative of the $k$-isomorphism class of elliptic curves containing $E_1/G$. Then $(E_1, E_2)$ is an arc; we call $E_1$ the tail and $E_2$ the head of the arc. Note that $G_\ell(k)$ can have multiple arcs (more than one arc $(E_1, E_2)$) and loops (arcs of the form $(E_1, E_1)$).

**Remark 1.** The definition of arcs is independent of the choice of isogeny with kernel $G$. This is because, as noted in §2.3, if $\phi': E_1 \to E_2'$ is any degree-$\ell$ isogeny with kernel $G$ where both $E_2'$ and $\phi'$ are defined over $k$, then $E_2'$ and $E_1/G$ are isomorphic over $k$ and consequently $\phi$ and $\phi'$ yield the same arc $(E_1, E_2)$.

**Remark 2.** The definition of $G_\ell(k)$ is independent of the choice of canonical representatives. Indeed, let $f: E'_1 \to E_1$ be a $k$-isomorphism of elliptic curves, and suppose that $E'_1$ was chosen as a canonical representative instead of $E_1$. Let $\psi = \phi \circ f$. Then Ker $\psi = f^{-1}(G)$, and thus the $\sigma$-invariant order-$\ell$ subgroup $f^{-1}(G)$ of $E'_1$ yields the arc $(E'_1, E_2)$. The claim now follows since $f^{-1}$ yields a one-to-one correspondence between the $\sigma$-invariant order-$\ell$ subgroups of $E_1$ and $E'_1$.

A consequence of Tate's theorem is that the graph $G_\ell(k)$ can be partitioned into five subgraphs whose vertices are the $k$-isomorphism classes of supersingular elliptic curves $E/k$ with trace $t = p^2 + 1 - \#E(k) \in \{0, -p, p, -2p, 2p\}$; we denote these subgraphs by $G_\ell(k, t)$. There are two such subgraphs ($t = \pm 2p$) when $p \equiv 1 \pmod{12}$, four subgraphs ($t = \pm p, \pm 2p$) when $p \equiv 5 \pmod{12}$, three subgraphs ($t = 0, \pm 2p$) when $p \equiv 7 \pmod{12}$, and five subgraphs ($t = 0, \pm p, \pm 2p$) when $p \equiv 11 \pmod{12}$. These subgraphs are further studied in §§4–6. We first fix the canonical representatives of the $k$-isomorphism classes of supersingular elliptic curves over $k$.

Suppose that $p \equiv 3 \pmod{4}$, and let $w$ be a generator of $k^*$. Munuera and Tena [11] showed that the representatives of the four isomorphism classes of elliptic curves $E/k$ with $j(E) = 1728$ can be taken to be

$$E_{1728,w^i} : Y^2 = X^3 + w^iX \quad \text{for } i \in [0, 3].$$

Of these curves, $E_{1728,w}$ and $E_{1728,w^3}$ have $p^2 + 1 \mathbb{F}_{p^2}$-rational points, and so we choose them as canonical representatives of the vertices of $G_\ell(k, 0)$. Furthermore, $\#E_{1728,1}(\mathbb{F}_{p^2}) = p^2 + 1 + 2p$ and $\#E_{1728,w^3}(\mathbb{F}_{p^2}) = p^2 + 1 - 2p$; hence, we select $E_{1728,1}$ and $E_{1728,w^3}$ as representatives of vertices in $G_\ell(k, -2p)$ and $G_\ell(k, 2p)$, respectively.

Suppose that $p \equiv 2 \pmod{3}$, and let $w$ be a generator of $k^*$. Munuera and Tena [11] also showed that the representatives of the six isomorphism classes of elliptic curves $E/k$ with $j(E) = 0$ can be taken to be

$$E_{0,w^i} : Y^2 = X^3 + w^i \quad \text{for } i \in [0, 5].$$

Of these curves, $E_{0,w}$ and $E_{0,w^5}$ have $p^2 + 1 + p \mathbb{F}_{p^2}$-rational points, and so we choose them as canonical representatives of the vertices of $G_\ell(k, -p)$. Similarly, $E_{0,w^2}$ and $E_{0,w^4}$ have $p^2 + 1 - p \mathbb{F}_{p^2}$-rational points, and so we choose them as canonical representatives of the vertices of $G_\ell(k, p)$. Finally, $\#E_{0,1}(\mathbb{F}_{p^2}) = p^2 + 1 + 2p$ and $\#E_{0,w^5}(\mathbb{F}_{p^2}) = p^2 + 1 - 2p$; hence, we select $E_{0,1}$ and $E_{0,w^5}$ as representatives of vertices in $G_\ell(k, -2p)$ and $G_\ell(k, 2p)$, respectively.

If $j \neq 0, 1728$ is supersingular, then $E_j$ (defined in (1)) and a quadratic twist $\tilde{E}_j$ are representatives of the two isomorphism classes of elliptic curves with $j$-invariant equal to
Furthermore, \(#E_j(\mathbb{F}_{p^2}) \in \{p^2 + 1 - 2p, p^2 + 1 + 2p\}\) and \(#E_j(\mathbb{F}_{p^2}) + #\tilde{E}_j(\mathbb{F}_{p^2}) = 2p^2 + 2\). We select \(E_j\) as the representative of a vertex in either \(G_{\ell}(k, -2p)\) or \(G_{\ell}(k, 2p)\) depending on whether \(#E_j(\mathbb{F}_{p^2}) = p^2 + 1 + 2p\) or \(p^2 + 1 - 2p\), and \(\tilde{E}_j\) as the representative of a vertex in the other graph.

4. The subgraph \(G_{\ell}(\mathbb{F}_{p^2}, 0)\)

\[
\begin{align*}
\ell = 2 & \quad \ell \equiv 1 \pmod{4} & \quad \ell \equiv 3 \pmod{4} \\
G_{\ell}(\mathbb{F}_{p^2}, 0) & \quad E_{1728, w} \quad E_{1728, w^3} & \quad E_{1728, w} \quad E_{1728, w^3} & \quad E_{1728, w} \quad E_{1728, w^3} \\
\ell = 3 & \quad \ell \equiv 1 \pmod{3} & \quad \ell \equiv 2 \pmod{3} \\
G_{\ell}(\mathbb{F}_{p^2}, -p) & \quad E_{0, w} \quad E_{0, w^5} & \quad E_{0, w} \quad E_{0, w^5} & \quad E_{0, w} \quad E_{0, w^5} \\
\ell = 3 & \quad \ell \equiv 1 \pmod{3} & \quad \ell \equiv 2 \pmod{3} \\
G_{\ell}(\mathbb{F}_{p^2}, p) & \quad E_{0, w^2} \quad E_{0, w^4} & \quad E_{0, w^2} \quad E_{0, w^4} & \quad E_{0, w^2} \quad E_{0, w^4}
\end{align*}
\]

Figure 1. The small subgraphs of \(G_{\ell}(\mathbb{F}_{p^2}), p \equiv 11 \pmod{12}\).

Let \(q = p^2\) where \(p \equiv 3 \pmod{4}\), \(w\) is a generator of \(\mathbb{F}_q^*\), and \(\ell \neq p\) is a prime. The graph \(G_{\ell}(\mathbb{F}_{p^2}, 0)\) has two vertices, \(E_{1728, w}\) and \(E_{1728, w^3}\); to ease the notation we will call them \(E_w\) and \(E_{w^3}\) in this section. The map \(\psi : (x, y) \mapsto (-x, iy)\) where \(i \in \mathbb{F}_q\) satisfies \(i^2 = -1\) is an automorphism of \(E_w\) and \(E_{w^3}\).

**Theorem 2.** Let \(p\) and \(\ell\) be primes with \(p \equiv 3 \pmod{4}\) and \(\ell \neq p\).

(i) \(G_2(\mathbb{F}_{p^2}, 0)\) has exactly two arcs, one loop at each of its two vertices.

(ii) If \(\ell \equiv 3 \pmod{4}\), then \(G_{\ell}(\mathbb{F}_{p^2}, 0)\) has no arcs.

(iii) If \(\ell \equiv 1 \pmod{4}\), then \(G_{\ell}(\mathbb{F}_{p^2}, 0)\) has exactly four arcs, two loops at each of its two vertices.

**Proof.** We describe the arcs originating at \(E_w\); the \(E_{w^3}\) case is similar.

Since \(t = 0\), the characteristic polynomial of the \(q\)-power Frobenius map \(\sigma\) is \(Z^2 + p^2\).
(i) If \( \ell = 2 \), then 1 is the only eigenvalue of \( \sigma \) acting on \( E_w[2] \). Since \((0,0)\) is the only point of order 2 in \( E_w(F_q) \), there is only one \( \sigma \)-invariant order-2 subgroup of \( E_w[2] \), namely \( G = \{ \infty, (0,0) \} \). Vélu’s formula (2) yields the isogeny \( \phi : E_w \to E_w/G \), where the Weierstrass equation of \( E_w/G \) is \( Y^2 = X^3 - 4wX \). Now, 
\[
(-4)^{(q-1)/4} = (-4)^{(p-1)(p+1)/4} = 1.
\]
Thus, \( E_w/G \) and \( E_w \) are isomorphic over \( F_q \) (cf. §2.1), whence \( G \) yields a loop at \( E_w \).

(ii) If \( \ell \equiv 3 \pmod{4} \), then \( Z^2 + p^2 \) has no roots modulo \( \ell \), and consequently \( E_w \) has no \( \sigma \)-invariant order-\( \ell \) subgroups.

(iii) If \( \ell \equiv 1 \pmod{4} \), then \( Z^2 + p^2 \) has two distinct roots modulo \( \ell \), namely \( \pm cp \) mod \( \ell \) where \( c \) is a square root of \(-1\) modulo \( \ell \). Let \( P_0 \in E_w[\ell]^* \) with \( \sigma(P_0) = [cp]P_0 \), and let \( G = \langle P_0 \rangle \). Then \( G \) is a \( \sigma \)-invariant order-\( \ell \) subgroup of \( E_w[\ell] \).

Vélu’s formula (4) yields an isogeny \( \phi : E_w \to E_w/G \). Since \( E_w \) and \( G \) are defined over \( F_q \), so is \( E' = E_w/G \). Since \( E' \) is in the same isogeny class as \( E_w \), its defining equation is of the form \( Y^2 = X^3 + w'X \) for some \( w' \in F_q^* \). Furthermore, \( \phi = (r(X), r'(X) \cdot Y) \), where

\[
\begin{align*}
r(X) &= X + \sum_{P \in G^*} \left( \frac{3x_P^2 + w}{X - x_P} + \frac{2y_P^2}{(X - x_P)^2} \right) \\
&= X + \sum_{P \in G^*} \frac{(3x_P^2 + w)(X - x_P) + 2y_P^2}{(X - x_P)^2} \\
&= X + \sum_{P \in G^*} \frac{(3x_P^2 + w)(X - x_P)(X + x_P)^2 + 2y_P^2(X + x_P)^2}{(X^2 - x_P^2)^2}.
\end{align*}
\]

Now, for all \( P \in G^* \), we have
\[
\sigma(\psi(P)) = \psi(\sigma(P)) = \psi([cp]P) = [cp]\psi(P)
\]
and hence \( \psi(P) \in G \). Let \( H \subset G^* \) be such that \( P \in H \) if and only if \( \psi(P) \not\in H \). Then
\[
r(X) = X + \sum_{P \in H} \frac{w_P(X)}{(X^2 - x_P^2)^2},
\]
where
\[
w_P(X) = (3x_P^2 + w)(X - x_P)(X + x_P)^2 + 2y_P^2(X + x_P)^2 \\
+ (3x_P^2 + w)(X + x_P)(X - x_P)^2 - 2y_P^2(X - x_P)^2 \\
= (3x_P^2 + w)(X^2 - x_P^2)(X + x_P) + 2y_P^2(X + x_P)^2 \\
+ (3x_P^2 + w)(X^2 - x_P^2)(X - x_P) - 2y_P^2(X - x_P)^2 \\
= 2X(3x_P^2 + w)(X^2 - x_P^2) + 8y_P^2x_PX.
\]
Thus
\[
r(X) = X \left( 1 + 2 \sum_{P \in H} \frac{(3x_P^2 + w)(X^2 - x_P^2) + 4y_P^2x_P}{(X^2 - x_P^2)^2} \right) \triangleq XS(X^2),
\]
where \( S \in F_q(X) \). Also,
\[
r'(X) = S(X^2) + 2X^2S'(X^2) \triangleq T(X^2),
\]
where $T \in \mathbb{F}_q(X)$.

Since $(0,0) \in E_w(\mathbb{F}_q)$ and $\phi$ is defined over $\mathbb{F}_q$, we have $\phi((0,0)) \in E'(\mathbb{F}_q)$. Since $E'(\mathbb{F}_q)$ is cyclic, $(0,0)$ is its only point of order 2, and hence $\phi((0,0)) = (0,0)$. Now, let $Q_0 = (x_0, y_0) \in E_w$ with $2Q_0 = (0,0)$; note that $x_0 \neq 0$ and $y_0 \neq 0$. From the elliptic curve point doubling formula, we deduce that $x_0 = \pm \sqrt{w}$. If $x_0 = \sqrt{w}$ then $y_0 = \pm \sqrt{2w^{3/4}}$, whereas if $x_0 = -\sqrt{w}$ then $y_0 = \pm i \sqrt{2w^{3/4}}$. Without loss of generality, suppose that $Q_0 = (\sqrt{w}, \sqrt{2w^{3/4}})$. Let $Q_0' = \phi(Q_0)$. Then

$$[2]Q_0' = [2]\phi(Q_0) = \phi([2]Q_0) = \phi((0,0)) = (0,0).$$

Therefore $Q_0' \in \{([w], \pm \sqrt{2w^{3/4}}), ([w], \pm i \sqrt{2w^{3/4}})\}$. Without loss of generality, suppose that $\phi(Q_0) = ([\sqrt{w}], \sqrt{2w^{3/4}})$. Then $\sqrt{w} = S(w)\sqrt{w}$ and so $(w'/w)^{1/2} \in \mathbb{F}_q$. Moreover, $\sqrt{2w^{3/4}} = \sqrt{2w^{3/4}}T(w)$, so $(w'/w)^{3/4} \in \mathbb{F}_q^*$. Thus, $(w'/w)^{3/4}/(w'/w)^{1/2} = (w'/w)^{1/4} \in \mathbb{F}_q^*$. Hence $E_w$ and $E'$ are isomorphic over $\mathbb{F}_q$, so $G$ yields a loop at $E_w$.

Similarly, the eigenspace of $-cp \bmod \ell$ yields a second loop at $E_w$.

\[\square\]

5. The Subgraphs $G_\ell(\mathbb{F}_{p^2}, \pm p)$

Let $q = p^2$ where $p \equiv 2 \pmod{3}$, $w$ is a generator of $\mathbb{F}_q^*$, and $\ell \neq p$ is a prime. The graph $G_\ell(\mathbb{F}_{p^2}, -p)$ has two vertices, $E_{0,w}$ and $E_{0,w^5}$; to ease the notation we will call them $E_w$ and $E_{w^5}$ in this section. The map $\rho: (x,y) \mapsto (jx,-y)$ where $j \in \mathbb{F}_q$ satisfies $j^2 + j + 1 = 0$ is an automorphism of $E_w$ and $E_{w^5}$.

**Theorem 3.** Let $p$ and $\ell$ be primes with $p \equiv 2 \pmod{3}$ and $\ell \neq p$.

(i) $G_3(\mathbb{F}_{p^2}, -p)$ has exactly two arcs, one loop at each of its two vertices.

(ii) If $\ell \equiv 2 \pmod{3}$, then $G_\ell(\mathbb{F}_{p^2}, -p)$ has no arcs.

(iii) If $\ell \equiv 1 \pmod{3}$, then $G_\ell(\mathbb{F}_{p^2}, -p)$ has exactly four arcs, two loops at each of its two vertices.

**Proof.** We describe the arcs originating at $E_w$; the $E_{w^5}$ case is similar.

Since $t = -p$, the characteristic polynomial of the $q$-power Frobenius map $\sigma$ is $Z^2 + pZ + p^2$.

(i) If $\ell = 3$, then $-1$ is the only eigenvalue of $\sigma$ acting on $E_w[3]$ with eigenvectors $(0, \pm \sqrt{w})$. Indeed, the $x$-coordinate of any point in $E_w[3]$ is a root of the 3-division polynomial $3X(X^3 + 4w)$, and if $x^3 + 4w = 0$ then $x^{p^2} \neq x$ since $(-4w)^{1/3} \notin \mathbb{F}_{p^2}$. Thus, $G = \langle(0, \sqrt{w})\rangle$ is the unique $\sigma$-invariant order-3 subgroup of $E_w$. Vélu’s formula (4) yields the isogeny $\phi: E_w \to E_w/G$, where the Weierstrass equation of $E_w/G$ is $Y^2 = X^3 - 27w$. Now,

$$(-27)^{(q-1)/6} = ((-27)^{p-1})^{(p+1)/6} = 1.$$ 

Thus, $E_w/G$ and $E_w$ are isomorphic over $\mathbb{F}_q$ (cf. §2.1), whence $G$ yields a loop at $E_w$.

(ii) If $\ell \equiv 2 \pmod{3}$, then $Z^2 + pZ + p^2$ has no roots modulo $\ell$, and consequently $E_w$ has no $\sigma$-invariant order-$\ell$ subgroups.

(iii) If $\ell \equiv 1 \pmod{3}$, then $Z^2 + pZ + p^2$ has two distinct roots modulo $\ell$, namely $\lambda_1 = (-1 + \sqrt{-3})p/2 \bmod \ell$ and $\lambda_2 = (-1 - \sqrt{-3})p/2 \bmod \ell$. The corresponding eigenspaces, $\langle P_0 \rangle$ and $\langle Q_0 \rangle$, are the only two $\sigma$-invariant order-$\ell$ subgroups of $E_w$. 


Let $G = \langle P_0 \rangle$, and let $\phi : E_w \to E'_w/G$ be the Vélu isogeny. Since $E_w$ and $G$ are defined over $\mathbb{F}_q$, so is $E' = E_w/G$. Since $E'$ is in the same isogeny class as $E_w$, its defining equation is of the form $Y^2 = X^3 + w'$ for some $w' \in \mathbb{F}_q^*$. Furthermore, $\phi = (r(X), r'(X) \cdot Y)$, where

$$r(X) = X + \sum_{P \in G^*} \left( \frac{3x_P^2}{X - x_P} + \frac{2y_P^2}{(X - x_P)^2} \right)$$

$$= X + \sum_{P \in G^*} \frac{3x_P^2(X^3 - x_P^3)(X^2 + x_P X + x_P^2) + 2y_P^2(X^2 + x_P X + x_P^2)^2}{(X^3 - x_P^3)^2}$$

$$= X + \sum_{P \in G^*} \frac{3x_P^2(X^3 - x_P^3)(X - jx_P)(X - j^2x_P) + 2y_P^2(X - jx_P)^2(X - j^2x_P)^2}{(X^3 - x_P^3)^2}.$$

Now, for all $P \in G^*$, we have

$$\sigma(\rho(P)) = \rho(\sigma(P)) = \rho([\lambda_1]P) = [\lambda_1]\rho(P)$$

and hence $\rho(P), \rho^2(P) \in G$. Let $H \subset G^*$ be such that $P \in H$ if and only if $\rho(P) \not\in H$ and $\rho^2(P) \not\in H$. Then

$$r(X) = X + \sum_{P \in H} \frac{w_1(X) + 2y_P^2 w_2(X)}{(X^3 - x_P^3)^2},$$

where

$$w_1(X) = 3x_P^2(X^3 - x_P^3)(X - jx_P)(X - j^2x_P)\left(X - j^2x_P\right)^2 + 3j^2x_P^2(X^3 - x_P^3)(X - jx_P)(X - j^2x_P)$$

$$= 9x_P^3X(X^3 - x_P^3)$$

and

$$w_2(X) = (X - jx_P)^2(X - j^2x_P)^2 + (X - x_P)^2(X - jx_P)^2 + (X - x_P)^2(X - j^2x_P)^2$$

$$= 3X(X^3 + 2x_P^3).$$

Thus we can write $r(X) = XS(X^3)$ where $S \in \mathbb{F}_q(X)$.

The order-2 points in $E_w$ and $E'$ are $(-j^u w^{1/3}, 0)$ ($u = 0, 1, 2$) and $(-j^u w^{1/3}, 0)$ ($u = 0, 1, 2$), respectively. Thus $\phi((w^{1/3}, 0)) = (j^u w^{1/3}, 0)$ for some $u \in \{0, 1, 2\}$. Since $r(X) = XS(X^3)$, we have $j^u w^{1/3} = w^{1/3}S(w)$, from which we deduce that

$$(w'/w)^{1/3} \in \mathbb{F}_q^*.$$  

Now consider the $q$-power Frobenius map $\sigma'$ acting on $E'$. As with the case of $E_w$, the only order-3 points $P' \in E'$ satisfying $\sigma'(P') = -P'$ are $(0, \pm \sqrt{w'})$. On the other hand

$$\sigma'((0, \sqrt{w})) = \phi(\sigma((0, \sqrt{w}))) = \phi((0, -\sqrt{w})) = -\phi((0, \sqrt{w})),$$

and hence $\phi((0, \sqrt{w})) = (0, \pm \sqrt{w'})$. Thus, $\pm \sqrt{w'} = \sqrt{w'}r'(0)$, implying that

$$(w'/w)^{1/2} \in \mathbb{F}_q^*.$$  

Finally, (10) and (11) give $(w'/w)^{1/6} \in \mathbb{F}_q^*$, and thus $E_w$ and $E'$ are isomorphic over $\mathbb{F}_q$. Hence $G$ yields a loop at $E_w$. Similarly, $\langle Q_0 \rangle$ yields a second loop at $E_w$.

The proof of Theorem 4 is similar to that of Theorem 3.

**Theorem 4.** Let $p$ and $\ell$ be primes with $p \equiv 2 \pmod{3}$ and $\ell \neq p$. 

\( N \) denote the number of arcs \((j \in \text{outdegree of } E)\) from \( E \) arcs \((\text{by Aut}(E) \text{automorphisms of } G)\) a root of multiplicity \( c \) the map \( \Phi \).

**Proof.**

As noted in §3, the vertices in \( G_\ell(E,F_{p^2},-2p) \) have distinct \( j \)-invariants. Moreover, there is an one-to-one correspondence between the vertices in \( G_\ell(F_{p^2},-2p) \) and the vertices in \( G_\ell(F_{p^2},2p) \); namely, if \( E \) is a vertex in \( G_\ell(F_{p^2},-2p) \) then the chosen quadratic twist \( \bar{E} \) is a vertex in \( G_\ell(F_{p^2},2p) \). Now, the characteristic polynomial of the \( q \)-power Frobenius map \( \sigma \) acting on any vertex \( E \) in \( G_\ell(F_{p^2},-2p) \) is \( Z^2 + 2pZ + p^2 = (Z+p)^2 \), so \((\sigma + [p])^2 = 0 \). Since nonzero endomorphisms are surjective, we must have \( \sigma + [p] = 0 \). Hence \( \sigma = [-p] \) and all order-\( \ell \) subgroups of \( E \) are \( \sigma \)-invariant. It follows that every vertex in \( G_\ell(F_{p^2},-2p) \) has outdegree \( \ell + 1 \). Similarly, every vertex in \( G_\ell(F_{p^2},2p) \) has outdegree \( \ell + 1 \).

By Theorem 1, the \( j \)-invariants of the heads of arcs with tail \( E \) in \( G_\ell(F_{p^2},-2p) \) are precisely the roots of \( \Phi_j(j(E),Y) \) (all \( \ell + 1 \) of which lie in \( F_{p^2} \)). These roots are also the \( j \)-invariants of the heads of arcs with tail \( \bar{E} \) in \( G_\ell(F_{p^2},2p) \). Hence the directed graphs \( G_\ell(F_{p^2},-2p) \) and \( G_\ell(F_{p^2},2p) \) are isomorphic.

Sutherland [15] defines the isogeny graph \( H_\ell(F_{p^2}) \) to have vertex set \( F_{p^2} \) and arcs \((j_1,j_2)\) present with multiplicity equal to the multiplicity of \( j_2 \) as a root of \( \Phi_j(j,Y) \) in \( F_{p^2} \). The following shows that \( G_\ell(F_{p^2}) \), the supersingular component of \( H_\ell(F_{p^2}) \), is isomorphic to \( G_\ell(F_{p^2},-2p) \).

**Theorem 5.** \( G_\ell(F_{p^2},-2p) \) and \( G_\ell(F_{p^2}) \) are isomorphic.

**Proof.** Recall that all supersingular elliptic curves over \( F_{p^2} \) are defined over \( F_{p^2} \). Hence the map \( \beta : E \rightarrow j(E) \) is a bijection between the vertex sets of \( G_\ell(F_{p^2},-2p) \) and \( G_\ell(F_{p^2}) \). Now, let \((E_1,E_2)\) be an arc of multiplicity \( c \geq 0 \) in \( G_\ell(F_{p^2},-2p) \). By Theorem 1, \( j(E_2) \) is a root of multiplicity \( c \) of \( \Phi_j(j(E_1),Y) \). Hence \((j(E_1),j(E_2))\) is an arc of multiplicity \( c \) in \( G_\ell(F_{p^2}) \). Thus, \( \beta \) preserves arcs and \( G_\ell(F_{p^2},-2p) \cong G_\ell(F_{p^2}) \).

**6.1. Indegree.** Suppose that \( p \) is prime and let \( E \) be a vertex in \( G_\ell(F_{p^2},-2p) \). Then all automorphisms of \( E \) are defined over \( F_{p^2} \); we denote the group of all automorphisms of \( E \) by \( \text{Aut}(E) \). Recall from §2.1 that \#\( \text{Aut}(E) \) is 4, 6 or 2 depending on whether \( j(E) = 1728 \), \( j(E) = 0 \) or \( j(E) \not= 0, 1728 \).

Let \( \ell \not= p \) be a prime. Let \( E_1,E_2 \) be two vertices in \( G_\ell(F_{p^2},-2p) \), and let \( \phi_1,\phi_2 : E_1 \rightarrow E_2 \) be two degree-\( \ell \) \( F_{p^2} \)-isogenies. We say that \( \phi_1 \) and \( \phi_2 \) are equivalent if they have the same kernel, or, equivalently, if there exist \( \rho_2 \in \text{Aut}(E_2) \) such that \( \phi_2 = \rho_2 \circ \phi_1 \). Thus, the arcs \((E_1,E_2)\) in \( G_\ell(F_{p^2},-2p) \) can be seen as the classes of equivalent degree-\( \ell \) \( F_{p^2} \)-isogenies from \( E_1 \) to \( E_2 \). We define \( \phi_1 \) and \( \phi_2 \) to be automorphic if there exists \( \rho_1 \in \text{Aut}(E_1) \) such that \( \phi_2 = \rho_2 \circ \phi_1 \circ \rho_1 \). Hence, if \( \phi_1 \) and \( \phi_2 \) are automorphic then there exist \( \rho_1 \in \text{Aut}(E_2) \) and \( \rho_2 \in \text{Aut}(E_2) \) such that \( \phi_2 = \rho_2 \circ \phi_1 \circ \rho_1 \). Since \( \phi_2 = \rho_2 \circ \phi_1 \circ \rho_1 \), it follows that the duals of automorphic isogenies are automorphic.

**Theorem 6.** Let \( E \) be a vertex in \( G_\ell(F_{p^2},-2p) \) and let \( n = \#\text{Aut}(E)/2 \). Let \( a \) and \( b \) denote the number of arcs \((E,E_1)\) and arcs \((E,E_0)\) in \( G_\ell(F_{p^2},-2p) \), respectively. Then the indegree of \( E \) is \((\ell + a + 2b + 1)/n \).
Proof. Let $E_1$, $E_2$ be two vertices in $\mathcal{G}_t(\mathbb{F}_{p^2}, -2p)$, and let $\text{Aut}(E_i) = (\rho_i)$ and $n_i = \#\text{Aut}(E_i)/2$ for $i = 1, 2$. Let $\phi : E_1 \rightarrow E_2$ be a degree-$\ell$ $\mathbb{F}_{p^2}$-isogeny.

Suppose first that the kernel of $\phi$ is not an eigenspace of $\rho_1$. Consider the set

$$\mathcal{A} = \{\rho_2^j \circ \phi \circ \rho_1^i : 0 \leq i < 2n_1, 0 \leq j < 2n_2\}$$

of isogenies automorphic to $\phi$. Since $\rho_i^{n_i} = -1$ for $i \in \{1, 2\}$, we have

$$\mathcal{A} = \{\rho_2^j \circ \phi \circ \rho_1^i : 0 \leq i < n_1, 0 \leq j < 2n_2\}.$$ 

One can check that if $(i, j) \neq (i', j')$ where $0 \leq i, i' < n_1$ and $0 \leq j, j' < 2n_2$, then $\rho_2^j \circ \phi \circ \rho_1^i = \rho_2^{j'} \circ \phi \circ \rho_1^{i'}$ implies that the kernel of $\phi$ is an eigenspace of $\rho_1$. Hence the set $\mathcal{A}$ has size exactly $2n_1n_2$ and the isogenies in $\mathcal{A}$ can be partitioned into $n_1$ classes of equivalent isogenies, each class comprised of $2n_2$ isogenies. Similarly, the set

$$\hat{\mathcal{A}} = \{\rho_1^i \circ \phi \circ \rho_2^j : 0 \leq i < n_1, 0 \leq j < 2n_2\}$$

of dual isogenies can be partitioned into $n_2$ classes of equivalent isogenies, each class comprised of $2n_1$ isogenies. Consequently, $\phi$ generates $n_1$ different arcs $(E_1, E_2)$ and $\hat{\phi}$ generates $n_2$ different arcs $(E_2, E_1)$. Because duals of automorphic isogenies are automorphic, if there is another degree-$\ell$ $\mathbb{F}_{p^2}$-isogeny $\psi$ from $E_1$ to $E_2$ not automorphic to $\phi$, then $\psi$ (resp. $\hat{\psi}$) generates a set of $n_1$ (resp. $n_2$) arcs $(E_1, E_2)$ (resp. $(E_2, E_1)$) disjoint from those generated by $\phi$ (resp. $\hat{\phi}$). Therefore, the number $r_{\text{out}}$ of arcs $(E_1, E_2)$ generated by isogenies whose kernels are not eigenspaces of $\rho_1$ and the number $r_{\text{in}}$ of arcs $(E_2, E_1)$ generated by their duals are multiples of $n_1$ and $n_2$, respectively. Moreover, we have

$$r_{\text{in}} = \frac{n_2 \cdot r_{\text{out}}}{n_1}. \tag{12}$$

Suppose now that the kernel of $\phi$ is an eigenspace of $\rho_1$. This scenario occurs only if $E_1$ has $j$-invariant 1728 or 0. Suppose $E_1$ has $j$-invariant 1728, and let $\rho_1$ be the automorphism $(x, y) \mapsto (-x, iy)$ where $i \in \mathbb{F}_{p^2}$ satisfies $i^2 = -1$. Denote by $G$ the kernel of $\phi$, and let $\phi' : E_1 \rightarrow E_1/G$ denote the Vélu isogeny. By (5), $E_1/G$ has equation $Y^2 = X^3 + aX - 7w$ for some $a \in \mathbb{F}_{p^2}$ and $w = \sum_{Q \in G^*}(5x_Q^3 + 3x_Q)$. Since $\rho_1(G) = G$, if $(x, y) \in G$ then $(-x, iy) \in G$. Hence $w = 0$ and we conclude that $E_1/G$ is isomorphic to $E_1$ over $\mathbb{F}_{p^2}$, i.e., $E_2 = E_1$. A similar argument using the automorphism $(x, y) \mapsto (jx, -y)$ with $j \in \mathbb{F}_{p^2}$ satisfying $j^2 + j + 1 = 0$ shows that we also have $E_2 = E_1$ when the $j$-invariant of $E_1$ is 0. Thus, if the kernel of $\phi$ is an eigenspace of $\rho_1$, the arcs generated by $\phi$ are loops at $E_1$. Therefore, we can generalize (12) to the total number $t_{\text{out}}$ of arcs $(E_1, E_2)$ and the total number $t_{\text{in}}$ of arcs $(E_2, E_1)$ and obtain

$$t_{\text{in}} = \frac{n_2 \cdot t_{\text{out}}}{n_1}. \tag{13}$$

Now, let $E$ be a vertex in $\mathcal{G}_t(\mathbb{F}_{p^2}, -2p)$ and $n = \#\text{Aut}(E)/2$. Denote by $E_j$ the vertex in $\mathcal{G}_t(\mathbb{F}_{p^2}, -2p)$ having $j$-invariant $j \in \mathbb{F}_{p^2}$. Let $a$ be the number of arcs $(E, E_{1728})$ and $b$ the number of arcs $(E, E_0)$. Note that the number of arcs $(E, E_j)$, $j \notin \{0, 1728\}$, is $c = \ell - a - b + 1$. From (13) we have

$$\text{indegree}(E) = \frac{c}{n} + \frac{2a}{n} + \frac{3b}{n}.$$
whence \[ \text{indegree}(E) = \frac{\ell + a + 2b + 1}{n}. \]

6.2. Loops. Let \( E_{1728} \) and \( E_0 \) denote the vertices in \( G_\ell(\mathbb{F}_{p^2}, -2p) \) with \( j \)-invariants 1728 and 0. In §6.2.1 and §6.2.2 we investigate the number of loops at \( E_{1728} \) and \( E_0 \) in \( G_\ell(\mathbb{F}_{p^2}, -2p) \).

6.2.1. \( E_{1728} \) loops. We begin by noting that
\[
\Phi_2(X, 1728) = (X - 1728)(X - 287496)^2.
\]
Since \( 287496 - 1728 = 2^3 \cdot 3^6 \cdot 7^2 \), we see that 1728 is a triple root of \( \Phi_2(X, 1728) \) in \( \mathbb{Z}_p[X] \) if \( p = 7 \) and a single root if \( p > 7 \). Hence the number of loops at \( E_{1728} \) in \( G_2(\mathbb{F}_{p^2}, -2p) \) is three if \( p = 7 \) and one if \( p > 7 \) (and \( p \equiv 3 \pmod{4} \)).

Lemma 7. Let \( p \equiv 3 \pmod{4} \) be a prime, and let \( \ell \neq p \) be an odd prime. Then the number of loops at \( E_{1728} \) is even. Moreover, if \( \ell \equiv 1 \pmod{4} \) then there are at least two loops at \( E_{1728} \).

Proof. Let \( \rho \) denote the automorphism \((x, y) \mapsto (-x, iy)\) of \( E_{1728} \) where \( i \in \mathbb{F}_{p^2} \) satisfies \( i^2 = -1 \). Since \#Aut(\( E_{1728} \))/2 = 2 we have from the first part of the proof of Theorem 6 that the number of loops at \( E_{1728} \) generated by isogenies whose kernels are not eigenspaces of \( \rho \) is even. The characteristic polynomial \( Z^2 + 1 \) of \( \rho \) splits modulo \( \ell \) if and only if \( \ell \equiv 1 \pmod{4} \). Hence, if \( \ell \equiv 3 \pmod{4} \) then all the loops at \( E_{1728} \) are generated by isogenies whose kernels are not eigenspaces of \( \rho \) and thus the number of loops is even. Now suppose that \( \ell \equiv 1 \pmod{4} \). The eigenspaces of \( \rho \) modulo \( \ell \) are two different order-\( \ell \) subgroups of \( E_{1728} \). The second part of the proof of Theorem 6 shows that the edges generated by these subgroups are loops at \( E_{1728} \).

Theorem 8. Let \( \ell \equiv 3 \pmod{4} \) be a fixed prime. Let \( p \equiv 3 \pmod{4} \), \( p \neq \ell \), be a prime for which \( E_{1728} \) has at least one loop in \( G_\ell(\mathbb{F}_{p^2}, -2p) \). Then \( p \leq \Phi_\ell(1728, 1728) \).

Proof. Let \( r \equiv 1 \pmod{4} \) be a prime. The elliptic curve \( E/\mathbb{F}_r : Y^2 = X^3 + X \) is ordinary, has \( j \)-invariant 1728, and has endomorphism ring \( \mathbb{Z}[i] \subseteq \mathbb{Q}(i) \) with discriminant \( D = -1 \). Proposition 23 of [10] tells us that there are exactly \( 1 + (D/\ell) = 0 \) degree-\( \ell \) isogenies over \( \mathbb{F}_r \) to \( E \). Hence by Theorem 1, we must have \( \Phi_\ell(1728, 1728) \neq 0 \pmod{r} \) and so \( \Phi_\ell(1728, 1728) \neq 0 \).

Now, since \( E_{1728} \) has a loop in \( G_\ell(\mathbb{F}_{p^2}, -2p) \), we have \( \Phi_\ell(1728, 1728) \equiv 0 \pmod{p} \) and hence \( p \leq \Phi_\ell(1728, 1728) \).

Theorem 9. Let \( \ell \equiv 1 \pmod{4} \) be a fixed prime. Then \( \Phi_\ell(1728, Y) = (Y - 1728)^2 G_\ell(Y) \) where \( G_\ell \in \mathbb{Z}[Y] \). Moreover, if \( p \equiv 3 \pmod{4} \) is a prime for which \( E_{1728} \) has at least three loops in \( G_\ell(\mathbb{F}_{p^2}, -2p) \), then \( p \leq \Phi_\ell(1728) \).

Proof. Write \( \Phi_\ell(1728, Y) = (Y - 1728)^2 G_\ell(Y) + H(Y) \), where \( G_\ell, H \in \mathbb{Z}[Y] \) and \( \deg H \leq 1 \). For any prime \( p \equiv 3 \pmod{4} \), since there are at least two loops at \( E_{1728} \) in \( G_\ell(\mathbb{F}_{p^2}, -2p) \), we have \( H(Y) = 0 \pmod{p} \). Hence \( H(Y) = 0 \).
Let \( r \equiv 1 \pmod{4} \) be a prime \( \neq \ell \). Proposition 23 of [10] tells us that there are exactly \( 1 + \left(\frac{-1}{p}\right) \) degree-\( \ell \) isogenies over \( \overline{\mathbb{F}}_r \) to \( E/\overline{\mathbb{F}}_r : Y^2 = X^3 + X \). Hence there can be at most two loops at \( E \) in \( H_\ell(\overline{\mathbb{F}}_r) \), and so by Theorem 1 we must have \( G_\ell(1728) \neq 0 \). Now, there are more than two loops at \( E_{1728} \) in \( G_\ell(\mathbb{F}_p^2, -2p) \) if and only if \( G_\ell(1728) \equiv 0 \pmod{p} \). It follows that \( p \leq G_\ell(1728) \). \( \square \)

6.2.2. \( E_0 \) loops. We have

\[
\Phi_2(X, 0) = (X - 2^4 \cdot 3^3 \cdot 5^3)^3,
\]
whence 0 is a triple root of \( \Phi_2(X, 0) \) in \( \mathbb{Z}_p[X] \) if \( p = 5 \) and not a root if \( p > 5 \). Hence the number of loops at \( E_0 \) in \( G_2(\mathbb{F}_p^2, -2p) \) is three if \( p = 5 \) and zero if \( p > 5 \) (and \( p \equiv 2 \pmod{3} \)). Similarly, since

\[
\Phi_3(X, 0) = X(X - 2^{15} \cdot 3 \cdot 5^3)^3,
\]
we conclude that the number of loops at \( E_0 \) in \( G_3(\mathbb{F}_p^2, -2p) \) is four if \( p = 5 \) and one if \( p > 5 \) (and \( p \equiv 2 \pmod{3} \)).

**Lemma 10.** Let \( p \equiv 2 \pmod{3} \) be a prime, and let \( \ell \neq 3, p \) be an odd prime. If \( \ell \equiv 2 \pmod{3} \), then the number of loops at \( E_0 \) is \( \equiv 0 \pmod{3} \). If \( \ell \equiv 1 \pmod{3} \), then the number of loops at \( E_0 \) is \( \equiv 2 \pmod{3} \).

**Proof.** Similar to the proof of Lemma 7. \( \square \)

**Theorem 11.** Let \( \ell \equiv 2 \pmod{3} \) be a fixed odd prime. Let \( p \equiv 2 \pmod{3} \), \( p \neq \ell \), be a prime for which \( E_0 \) has at least one loop in \( G_\ell(\mathbb{F}_p^2, -2p) \). Then \( p \leq \Phi_\ell(0, 0) \).

**Proof.** Let \( r \equiv 1 \pmod{3} \) be a prime. The elliptic curve \( E/\overline{\mathbb{F}}_r : Y^2 = X^3 + 1 \) is ordinary, has \( j \)-invariant 0, and has endomorphism ring \( \mathbb{Z}[(1+\sqrt{-3})/2] \subseteq \mathbb{Q}(\sqrt{-3}) \) with discriminant \( D = -3 \). Proposition 23 of [10] tells us that there are exactly \( 1 + \left(\frac{D}{p}\right) = 0 \) degree-\( \ell \) isogenies over \( \overline{\mathbb{F}}_r \) to \( E \). Hence by Theorem 1, we must have \( \Phi_\ell(0, 0) \neq 0 \pmod{r} \) and so \( \Phi_\ell(0, 0) \neq 0 \).

Now, since \( E_0 \) has a loop in \( G_\ell(\mathbb{F}_p^2, -2p) \) we have \( \Phi_\ell(0, 0) \equiv 0 \pmod{p} \) and hence \( p \leq \Phi_\ell(0, 0) \). \( \square \)

**Theorem 12.** Let \( \ell \equiv 1 \pmod{3} \) be a fixed prime. Let \( p \equiv 2 \pmod{3} \) be a prime for which \( E_0 \) has at least three loops in \( G_\ell(\mathbb{F}_p^2, -2p) \). Write \( \Phi_\ell(0, Y) = Y^2G_\ell(Y) + H(Y) \), where \( G_\ell, H \in \mathbb{Z}[Y] \) and \( \deg H \leq 1 \). Then \( p \leq G_\ell(0) \).

**Proof.** Similar to the proof of Theorem 9. \( \square \)

**Remark 3.** It is worth mentioning that portions of Lemmas 7 and 10 can be obtained using standards properties of modular polynomials and the \( j \) modular function. Indeed, these properties lets one prove that when \( p \equiv 3 \pmod{4} \) and \( \ell \equiv 1 \pmod{3} \) then there are at least two loops at \( E_{1728} \); also, when \( p \equiv 2 \pmod{3} \) and \( \ell \equiv 1 \pmod{3} \) then there are at least two loops at \( E_0 \). The proofs we have given are elementary and self contained.

For primes \( \ell \equiv 1 \pmod{4} \) (resp. \( \ell \equiv 3 \pmod{4} \)), let \( p^1_{1728}(\ell) \) (resp. \( p^3_{1728}(\ell) \)) denote the largest prime \( p \equiv 3 \pmod{4} \), \( p \neq \ell \), for which \( E_{1728} \) has at least three loops (resp. at least one loop) in \( G_\ell(\mathbb{F}_p^2, -2p) \). Similarly, for odd primes \( \ell \equiv 1 \pmod{3} \) (resp. \( \ell \equiv 2 \pmod{3} \)), let \( p^1_0(\ell) \) (resp. \( p^0_0(\ell) \)) denote the largest prime \( p \equiv 2 \pmod{3} \), \( p \neq \ell \), for which \( E_0 \) has at least three loops (resp. at least one loop) in \( G_\ell(\mathbb{F}_p^2, -2p) \). Table 1 lists \( p^1_{1728}(\ell), p^3_{1728}(\ell), p^1_0(\ell), p^0_0(\ell) \) for all primes \( \ell \leq 283 \). These values were obtained by factoring the
relevant values of the modular polynomial $\Phi_\ell$; the modular polynomials were obtained from Sutherland’s database [1, 16]. For example, $p_{1728}^3(\ell)$ is the largest prime factor of $\Phi_{1728}(1728, 1728)$ that is congruent to 3 modulo 4.

Bröker and Sutherland [2] proved that $h(\Phi_\ell) \leq 6\ell \log \ell + 18\ell$ for all primes $\ell$ and conjectured that $\lim \inf_{\ell \to \infty} \frac{h(\Phi_\ell) - 6\ell \log \ell}{\ell} > 11.8$; here $h(\Phi_\ell)$ denotes the natural logarithm of the largest coefficient (up to sign) of $\Phi(X,Y)$. Thus the coefficients of $\Phi_\ell(X,Y)$ are very large. So, for example, one expects that $\Phi_{1728}(1728, 1728)$ is a very large integer with at least one relatively large prime factor. In light of this, the size of the numbers in Table 1 are surprisingly small. For example, $\Phi_{19}(1728, 1728)$ is a 822-decimal digit number with prime factorization

$\Phi_{19}(1728, 1728) = 2^{180} \cdot 3^{124} \cdot 7^{104} \cdot 11^{24} \cdot 19^2 \cdot 23^8 \cdot 31^8 \cdot 47^8 \cdot 59^8 \cdot 67^4 \cdot 71^8$,

whereby $\Phi_{19}(1728, 1728)$ is 71-smooth and $p_{1728}^3(19) = 71$.

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Table 1. The values $p_{1728}^1(\ell)$, $p_{1728}^3(\ell)$, $p_0^1(\ell)$, $p_0^2(\ell)$ for all odd primes $\ell \leq 283$.

References


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