# FAST: Disk Encryption and Beyond

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**Abstract.** This work introduces FAST which is a new family of cryptographic primitives. Several instantiations of FAST are described. These are targeted towards two goals, the specific task of disk encryption and a more general scheme suitable for a wide variety of practical applications. A major contribution of this work is to present detailed and careful implementations of several instantiations of FAST in both software and hardware. For disk encryption, the results from the implementations show that FAST compares very favourably to the IEEE disk encryption standards XCB and EME2 as well as the more recent proposal AEZ. Formally, FAST is a new family of tweakable enciphering schemes. It is built using a fixed input length pseudo-random function and an appropriate hash function. FAST uses a single-block key, is parallelisable and can be instantiated using only the encryption function of a block cipher. The hash function can be instantiated using either the Horner's rule based usual polynomial hashing or hashing based on the more efficient Bernstein-Rabin-Winograd polynomials. Security of FAST has been rigorously analysed using the standard provable security approach and concrete security bounds have been derived. Based on our implementation results, we put forward FAST as a serious candidate for standardisation and deployment.
**Keywords: disk encryption, tweakable enciphering schemes, pseudo-random function, Horner, BRW.**

## 1 Introduction

There is a huge amount of data residing on various kinds of storage devices. For example, the Indian national repository of biometric data called Aadhaar runs into several petabytes[3]. In today's world, much of the data at rest are sensitive and require encryption to be protected from unwanted access or tampering. The solution is to use full disk encryption where the storage device holds the encryption of the data under a secret key. Reading from the disk requires decrypting the relevant portion of the disk, while writing to the disk requires encrypting the data and then storing it at an appropriate location on the disk. The tasks of encryption and decryption are performed using a disk encryption algorithm. To be useful in practice a disk encryption algorithm needs to be both secure and efficient. The goal of security is to ensure that unwanted access or tampering is indeed not feasible while the goal of efficiency is to ensure that there is no noticeable slowdown in the process of reading from or writing to the disk.

A logical level view of a hard disk and most other storage devices is as a collection of sectors where each sector can store a fixed number of bytes. For example, present day hard disks have 4096-byte sectors while some of the older disks had 512-byte sectors[4]. Each sector has a unique address. A read or write operation on a disk works at the granularity of sectors. A read operation will specify a bunch of sector addresses and the complete contents of those sectors will be returned.

[3] `https://www.cse.iitb.ac.in/~comad/2010/pdf/Industry Sessions/UID_Pramod_Varma.pdf`
[4] `https://en.wikipedia.org/wiki/Disk_sector`

Similarly, a write operation will specify the data and a bunch of sector addresses and the contents of the corresponding sectors will be overwritten with the new data.

A disk encryption algorithm proceeds sector by sector. The content of a sector is encrypted using the secret key and stored in-place, i.e., the content of the sector is overwritten using the encrypted content. The original unencrypted content is not stored anywhere. Just encryption is not sufficient for security as can be seen from the following simple attack. Suppose that the contents of two successive sectors $s_1$ and $s_2$ are $C_1$ and $C_2$ corresponding to plaintexts $P_1$ and $P_2$ respectively. An adversary may simply swap $C_1$ and $C_2$. Subsequent decryption will show $s_1$ containing $P_2$ and $s_2$ containing $P_1$ whereas decryption before the swap would have shown $s_1$ containing $P_1$ and $s_2$ containing $P_2$. If it turns out that $s_1$ containing $P_2$ and $s_2$ containing $P_1$ is meaningful data, then by a simple swap operation, the adversary has been able to alter the content of the disk to a meaningful data which was not originally stored on the disk.

To prevent the above possibility, the encryption of the content of a sector needs to be somehow tied to the sector address. Decryption of any adversarially modified content of a sector should result in a random looking string which is unlikely to be meaningful data.

Viewed in this manner, a disk encryption mechanism is an example of a length preserving encryption where the length of the ciphertext is equal to the length of the plaintext. Further, there is another quantity (which is the sector address in case of disk encryption) which determines the ciphertext but, is itself not encrypted. In the literature this quantity has been called a tweak. The functionality of a tweak-based length preserving encryption has been called a tweakable enciphering scheme (TES) [24].

While disk encryption is a very important application of a TES, the full functionality of a TES is much more broader than just disk encryption. For the specific case of disk encryption, messages are contents of a sector and so are fixed length strings. A TES can have a more general message space consisting of binary strings of different lengths. Similarly, in the case of disk encryption, the tweak is a sector address and can be encoded using a short fixed length string. More generally, the tweak space in a TES can also consist of strings of different lengths or even consist of vectors of strings.

### Our Contributions

This paper describes a new family of tweakable enciphering schemes called FAST which is built using a pseudo-random function (PRF) and a hash function with provably low collision and differential probabilities. The domain and the range of the pseudo-random function are both equal to the set of all $n$-bit binary strings for an appropriately chosen $n$. The hash function is built using arithmetic over the finite field $GF(2^n)$. Some of the salient aspects of FAST are described below.

**Wide range of applications:** FAST can be used in the following settings.

*Fixed length setting:* This setting is targeted towards disk encryption application. It supports an $n$-bit tweak and messages whose lengths are a fixed multiple of the block size $n$.

*General setting:* This setting is very general. Messages are allowed to have different lengths and tweaks are allowed to be vectors of binary strings where the numbers of components in the vectors can vary. The richness of the tweak space provides considerable flexibility in applications where there is a message and an associated set of attributes. The message is to be encrypted while the attributes are to be in the clear but the ciphertext needs to be bound to the attributes. We mention two possible applications for such a functionality.

1. The message is a data packet that is to be stored at a destination node while the vector of attributes encode the path taken by the data packet to reach the destination node with the components of the vector identifying the intermediate nodes.
2. The message consists of biometric information while the attributes are date-time, gender and other related information. A possible application would be to the Aadhaar database mentioned earlier.

We note that the idea of having associated data to be a vector of strings was earlier proposed [37] in the context of deterministic authenticated encryption. AEZ [26] provides a conceptual level description of how to handle a vector of strings as tweak using an almost XOR universal hash function to process the vector. A generic security bound is provided in terms of the collision probability of the hash function. No concrete proposal for the hash function is provided. Consequently, the efficiency of processing the tweak cannot be determined and neither it is possible to obtain a concrete security bound. In contrast, following our objective of practical implementation we put forward several concrete designs for hashing a vector of strings with associated concrete secuity bounds and detailed software and hardware implementations.

**Software and hardware implementations:** A major objective of the paper consists of both software and hardware implementations of FAST and the most important TES schemes in the literature. The goal of such implementations is to perform a comparative study of software and hardware performances of FAST with those of the previous schemes. To this end, we have carried out detailed software and hardware implementations of the IEEE standards XCB and EME2 as well as AEZ (instantiated with the encryption function of the AES block cipher) along with similar implementations of variants of FAST.

1. The software implementation is targeted towards modern Intel processors and is in Intel intrinsics using the specialised AES-NI instructions including the `pclmulqdq` instruction. The code for the software implementation of FAST is publicly available from `https://github.com/sebatighosh/FAST`. The software implementation of FAST covers both the fixed length and the general settings. We provide timing results for the Skylake and the Kabylake processors of Intel.
2. The hardware implementation is based on FPGA and is targeted towards the disk encryption application.

Results arising from the implementations show that the new proposal compares favourably to the most important previous constructions in both software and hardware. For the fixed length setting, the best speed achieved by FAST on the Intel Skylake platform is 1.24 cycles per byte (cpb). In comparison, XCB, EME2 and AEZ achieve speeds of 1.92 cpb, 2.07 cpb and 1.74 cpb respectively. The corresponding figures on Kabylake for FAST, XCB, EME2 and AEZ are 1.19, 1.85, 1.99 and 1.70 cpb respectively. Further timing details are provided later. On the Virtex 5 FPGA platform, the best throughput of FAST is 30.55 Gbps while XCB, EME2 and AEZ achieve throughputs of 28.05 Gbps, 24.77 Gbps and 22.70 Gbps respectively. The corresponding figures on the Virtex 7 FPGA for FAST, XCB, EME2 and AEZ are 39.73, 37.21, 33.90 and 30.45 Gbps respectively. Further details on timings and area (in terms of slices and block RAMs) are provided later.

Based on the detailed comparative results of software and hardware performances, we conclude that FAST is the TES of choice for both disk encryption and general purpose applications. This makes FAST the definitive candidate for standardisation and deployment.

**Dispensing with invertibility:** There are several concrete TES proposals in the literature. Most of these proposals including the ones that have been standardised are modes of operations of a block cipher and use both the encryption and the decryption functions of the underlying block cipher. FAST, on the other hand, uses a PRF and does not require the invertibility property of a block cipher. The PRF itself may be instantiated using the encryption function of a block cipher such as AES. This provides two distinct advantages.

1. From a practical point of view, the advantage is that the decryption function of the block cipher does not require to be implemented. This is an advantage in hardware implementation since it results in a smaller hardware. A software implementation also benefits by requiring a smaller size code.
2. From a theoretical point of view, a block cipher is modelled as a strong pseduo-random permutation (SPRP). A PRF assumption on the encryption function of a block cipher is a weaker assumption than an SPRP assumption on the block cipher. So security of FAST can be based on a weaker assumption on the underlying block cipher.

We note that a previous work [40] had pointed out the possibility of using only the encryption function of a block cipher to build a TES. The work was more at a conceptual level using generic components and some unnecessary operations. It did not provide any specific instantiation or implementation. Subsequent to [40], the constructions AEZ [26] and FMix [7] proposed single key TESs using only the encryption function of the block cipher. FMix is a sequential scheme while AEZ is parallelisable. Later we discuss in more details several issues regarding the comparison of FAST to previous schemes.

**Parallelisable:** At a top level, the construction applies a Feistel layer of encryption on the first two message blocks and sandwiches a counter type mode of operation in-between two layers of hashing for the rest of the message. The counter mode is fully parallelisable. This leads to efficient implementations in both hardware and software.

**Design of hash functions:** We provide instantiations using two kinds of hash functions both of which are based on arithmetic over the finite field $GF(2^n)$. The first kind of hash function is based on the usual polynomial based hashing using Horner's rule. The second kind is based on a class of polynomials [6] which was later called BRW polynomials [39]. For tackling variable length inputs, a combination of BRW and Horner based hashing called Hash2L [9] turns out to be advantageous. For the fixed length setting, we show instantiations using Horner and BRW while for the general setting, we use the vector version vecHorner of Horner and the vector version vecHash2L of Hash2L.

**Provable security treatment:** The security of the proposed scheme is analysed following the standard provable security methodology. The theoretical notion of security of a TES is shown to hold under the assumption that the encryption function of the underlying block cipher is a PRF. The proof requires the hash functions to satisfy certain properties. We show that the hash functions obtained from Horner, vecHorner, BRW and vecHash2L satisfy the required properties. Concrete security bounds are derived for the different instantiations. These bounds show that the security of FAST is adequate for practical purposes and is comparable to those achieved in previous designs.

### Previous Works on TES

The first proposal for the construction of a strong pseudorandom permutation using a hash-ECB-hash approach was by Naor and Reingold [34]. This work, though, did not consider tweaks since the paper predates the formal introduction of the notion tweaks and hence of a TES. The notion of a tweakable block cipher and its security was formalised by Liskov, Rivest and Wagner [29]. This was followed by a formalisation of the notion of a tweakable enciphering scheme by Halevi and Rogaway [24]. The paper also described a TES called CMC which is based on the CBC mode of operation. A subsequent work [25] by the same authors introduced a TES called EME which is a parallelisable mode of operation of a block cipher. EME was extended to handle arbitrary length messages by Halevi [21] and the resulting scheme was called EME$^*$. The EME family of TESs does not require finite field multiplication. The main cost of encryption is roughly two block cipher calls per block of the message.

Construction of a TES using a counter based mode of operation of a block cipher and a Horner type hash function was first proposed by McGrew and Fluhrer [30]. This scheme was called XCB. A later variant [31] of XCB was proposed to improve efficiency and reduce key size. Various security problems for XCB have been pointed out [10].

There have been a number of works proposing different constructions of TESs. Examples are PEP [15], ABL [32], HCTR [41], HCH [16], TET [22] and HEH [39]. An improved security analysis of HCTR has been done later [14]. A generalisation of EME using a general masking scheme has been proposed [38]. As mentioned earlier, the conceptual possibility of constructing a TES from a PRF (and hence using only the encryption function of a block cipher) has been suggested earlier [40]. The constructions AEZ [26] and FMix [7] are also TESs constructed from a PRF. The possibility of constructing TESs from stream ciphers has been considered [40]. Concrete proposals and detailed FPGA implementations of stream cipher based TESs have been described [13].

Another line of investigation has been the construction of ciphers that can securely encipher their own keys [23, 4]. A generic method is known [4] which converts a conventional TES to one which can be proved to be secure even under the possibility of encrypting its own key. This generic method has been applied to EME2 [4]. We note that the method can equally well be applied to the construction FAST proposed in the present work.

### Relation of FAST to Previous Works

As discussed above, there is a long line of constructions of the hash-encrypt-hash type TESs. FAST is also a construction of this type. The ideas behind the design of FAST have been gathered from existing works.

- FAST uses a counter type mode of operation for the encryption layer. XCB [30, 31] was the first to propose such a design. The actual counter type mode used in FAST was proposed in the HCTR [41] construction.
- FAST provides an option to use the Bernstein-Rabin-Winograd polynomials for the hash layers. This idea has been proposed earlier [39].
- The encryption layer of FAST is built from a PRF and this idea is based on an earlier suggestion [40].

So, FAST collects various good design ideas from the existing literature to come up with a definitive scheme.

The other approach to the construction of TES does not use hash functions. The known examples are CMC [24], EME2 [25, 21], AEZ [26] and FMix [7]. Of these, only AEZ is both parallelisable *and* is built from a PRF; CMC and FMix are sequential while EME2 requires to be instantiated using a strong pseudo-random permutation (SPRP).

**Related Primitives**

Authenticated encryption with associated data (AEAD) [28, 5] encrypts a message under a key and a nonce to produce a ciphertext. The ciphertext is longer than the message and contains a tag which serves the purpose of authentication. The notion of deterministic authenticated encryption with associated data (DAEAD) [37] has been proposed as a solution to the key-wrap problem and such a scheme does not use a nonce. A DAEAD is to be used when the message contains sufficient entropy. In terms of known constructions, AEAD schemes turn out to be the most efficient, followed by DAEAD schemes and then TESs. The difficulty of using AEAD schemes for disk encryption and the possibility of indeed using DAEAD schemes for this purpose have been discussed in details [12]. The notion of robust authenticated encryption (RAE) has been proposed [26] and it was shown that a TES is a special case of an RAE. Further, a simple and generic method of constructing an RAE from a TES is known [26]. This method can also be applied to $\mathsf{FAST}$ to construct an RAE.

**Standards and Patents**

IEEE [3] has standardised two tweakable enciphering schemes, namely EME2 and XCB. Essentially, EME2 is the variant EME* [21] while the standardised version of XCB is a variant [31] of the original scheme [30]. Both EME2 and XCB are patented algorithms. Till date there is no unpatented algorithm which has been standardised. Apart from offering superior performance guarantees with respect to previous schemes XCB, EME2 and AEZ, it is our hope that $\mathsf{FAST}$ will also fill the gap of providing an attractive solution which is unencumbered by intellectual property claims.

An earlier IEEE standard is XTS [2] which has also been standardised [18] by NIST of USA. This is based on the XEX construction of Rogaway [36]. XTS is not a TES and the security provided by XTS is not adequate for disk encryption application. Rogaway [1] himself mentioned that XTS only provides light security and should be preferred only when there is an overriding concern for speed.

## 2 Preliminaries

Throughout the paper, we fix a positive integer $n$ and a positive integer $\eta \geq 3$.

**Notation:** Let $X$ and $Y$ be binary strings.

- The length of $X$ will be denoted as $\mathsf{len}(X)$.
- The concatenation of $X$ and $Y$ will be denoted as $X||Y$.
- For an integer $i$ with $0 \leq i < 2^n$, $\mathsf{bin}_n(i)$ denotes the $n$-bit binary representation of $i$.

We define the following terminology.

$\mathsf{first}_i(X)$: For a binary string $X$, $0 < i \leq \mathsf{len}(X)$, $\mathsf{first}_i(X)$ will denote the first (or, the most significant) $i$ bits of $X$.

$\mathsf{pad}_n(X)$: For a binary string $X$ and $n > 0$, if $X$ is the empty string, then $\mathsf{pad}_n(X)$ will denote the string $0^n$; while if $X$ is non-empty, then $\mathsf{pad}_n(X)$ will denote $X||0^i$, where $i \geq 0$ is the minimum integer such that $n$ divides $\mathsf{len}(X||0^i)$.

$\mathsf{parse}_n(X)$: For a binary string $X$ such that $\mathsf{len}(X) \geq 2n$, $\mathsf{parse}_n(X)$ denotes $(X_1, X_2, X_3)$ where $\mathsf{len}(X_1) = \mathsf{len}(X_2) = n$ and $X = X_1||X_2||X_3$. In other words, $\mathsf{parse}_n(X)$ divides the string $X$ into three parts with the first two parts having length $n$ bits each with the remaining bits of $X$ (if any) forming the third part.

$\mathsf{format}_n(X)$: For a non-empty binary string $X$ and a positive integer $n$, $\mathsf{format}_n(X)$ denotes $(X_1, X_2, \ldots, X_m)$ where $X = X_1||X_2||\cdots||X_m$, $m = \lceil \mathsf{len}(X)/n \rceil$, $\mathsf{len}(X_i) = n$ for $1 \leq i \leq m-1$ and $1 \leq \mathsf{len}(X_m) \leq n$. In other words, $\mathsf{format}_n(X)$ divides the string $X$ into $m-1$ $n$-bit blocks $X_1, \ldots, X_{m-1}$ and a possibly partial last block $X_m$.

*Number of $n$-bit blocks:* Let $X$ be a binary string and suppose that $\mathsf{format}_n(\mathsf{pad}_n(X))$ returns $X_1||\cdots||X_m$. We will say that the number of $n$-bit blocks in $X$ is $m$. Note that if $X$ is the empty string, then $\mathsf{pad}(X)$ is $0^n$ and so $\mathsf{format}_n(\mathsf{pad}_n(X))$ is also $0^n$ whence $m = 1$, i.e., as per our formalism, the empty string has one $n$-bit block. The number of $n$-bit blocks in $X$ will be denoted by $\mathfrak{l}(X)$.

For a vector of binary strings $Y = (Y_1, \ldots, Y_k)$, by the number of $n$-bit blocks in $Y$ we will mean the sum of the numbers of $n$-bit blocks in the strings $Y_1, \ldots, Y_k$. The number of $n$-bit blocks in $Y$ will be denoted by $\mathfrak{t}(Y)$.

$\mathsf{superBlks}_{n,\eta}(Z)$: For a binary string $Z$, $\mathsf{superBlks}_{n,\eta}(Z)$ denotes the vector of strings $(Z_1, \ldots, Z_\ell)$ obtained as $(Z_1, \ldots, Z_\ell) \leftarrow \mathsf{format}_{n\eta}(\mathsf{pad}_n(Z))$. For $1 \leq i \leq \ell - 1$, $Z_i$ is an $n\eta$-bit string while $Z_\ell$ is a string whose length is at most $n\eta$ and is divisible by $n$. The strings $Z_1, \ldots, Z_\ell$ are called super-blocks. The first $\ell - 1$ of these super-blocks consist of exactly $\eta$ $n$-bit blocks while the last super-block consists of at most $\eta$ $n$-bit blocks. *We will say that the number of super-blocks in $Z$ is $\ell$.*

**Finite field:** Let $\mathbb{F} = GF(2^n)$ be the finite field of $2^n$ elements. Using a fixed irreducible polynomial of degree $n$ over $GF(2)$ to represent $\mathbb{F}$, the elements of $\mathbb{F}$ can be identified with the binary strings of length $n$. Viewed in this manner, an $n$-bit binary string will be considered to be an element of $\mathbb{F}$. The addition operation over $\mathbb{F}$ will be denoted by $\oplus$; for $X, Y \in \mathbb{F}$, the product will be denoted as $XY$. The additive identity of $\mathbb{F}$ will be denoted as $\mathbf{0}$ and will be represented as $0^n$; the multiplicative identity of $\mathbb{F}$ will be denoted as $\mathbf{1}$ and will be represented as $0^{n-1}1$.

For $n = 128$, let $\mathbb{F}$ be represented as $GF(2)[\alpha]/\psi(\alpha)$ where $\psi(\alpha) = \alpha^{128} \oplus \alpha^7 \oplus \alpha^2 \oplus \alpha \oplus 1$. The 128-bit string $X$ is considered to be a polynomial $X(\alpha) \in GF(2)[\alpha]$. Let $Y$ be a 128-bit string representing the polynomial $Y(\alpha) = \alpha X(\alpha) \bmod \psi(\alpha)$. The string $Y$ can be obtained from the string $X$ as $Y = (X \ll 1) \oplus (\mathsf{msb}(X) \cdot 135)$, where $\mathsf{msb}(X)$ denotes the most significant bit of $X$. Over $\mathbb{F}$, this operation corresponds to the 'multiply by $\alpha$' map and has been called a doubling operation [36].

**Pseudo-random function:** The construction requires a family of functions where each function in the family maps $n$-bit strings to $n$-bit strings. More precisely, let $\{\mathbf{F}_K\}_{K \in \mathcal{K}}$ be a family of functions, where for $K \in \mathcal{K}$, $\mathbf{F}_K : \{0,1\}^n \rightarrow \{0,1\}^n$. Here $\mathcal{K}$ is the key space of $\mathbf{F}$. The security requirement on $\{\mathbf{F}_K\}_{K \in \mathcal{K}}$ is that of a pseudo-random function family. Informally this means, for a randomly chosen $K$, on distinct inputs, the outputs of $\mathbf{F}_K(\cdot)$ appear independent and uniformly distributed to a computationally bounded adversary. We provide the formal definition later. It is possible to instantiate $\mathbf{F}$ using the encryption (or the decryption) function of a block cipher. In particular, one may use the encryption function of AES to instantiate $\mathbf{F}$. This, however, is an overkill, since the invertibility property of the block cipher is not required by the construction.

**Counter mode:** The PRF $\mathbf{F}$ can handle only $n$-bit strings. Longer strings are handled in the following manner. Let $X$ be a non-empty binary string. For $K \in \mathcal{K}$ and $S \in \{0,1\}^n$, we define $\mathsf{Ctr}_{K,S}(X)$ in the following manner.

$$\mathsf{Ctr}_{K,S}(X) = (S_1 \oplus X_1, \ldots, S_{m-1} \oplus X_{m-1}, \mathsf{first}_r(S_m) \oplus X_m) \tag{1}$$

where $(X_1, \ldots, X_m) \leftarrow \mathsf{format}_n(X)$, $\mathsf{len}(X_m) = r$ and $S_i = \mathbf{F}_K(S \oplus \mathsf{bin}_n(i))$. This variant of the counter mode was originally used in HCTR [41]. Note that the PRF $\mathbf{F}$ is used to define the counter mode, but, the counter mode itself as defined here is not a PRF.

## 3 Hash Functions

Let $\mathcal{D}$ and $\mathcal{G}$ be finite non-empty sets. Let $\{H_\tau\}_{\tau \in \mathsf{T}}$ be an indexed family of functions such that for each $\tau$, $H_\tau : \mathcal{D} \rightarrow \mathcal{G}$.

**Collision and differential probabilities:**

- For distinct $x, x' \in \mathcal{D}$, the *collision probability* of $\{H_\tau\}_{\tau \in \mathsf{T}}$ for the pair $(x, x')$ is defined to be $\Pr_\tau[H_\tau(x) = H_\tau(x')]$.
- Suppose $\mathcal{G}$ is an additively written group. For distinct $x, x' \in \mathcal{D}$ and any $y \in \mathcal{G}$, the *differential probability* of $\{H_\tau\}_{\tau \in \mathsf{T}}$ for the triplet $(x, x', y)$ is defined to be $\Pr_\tau[H_\tau(x) - H_\tau(x') = y]$.

The above probabilities are taken over uniform random choices of $\tau$ from $\mathsf{T}$.

**Almost universal function family:** The family $\{H_\tau\}$ is said to be $\epsilon$-almost universal ($\epsilon$-AU) if for all distinct $x$, $x'$ in $\mathcal{D}$, the collision probability for the pair $(x, x')$ is at most $\epsilon$.

**Almost XOR universal function family:** The family $\{H_\tau\}$ is said to be $\epsilon$-almost XOR universal ($\epsilon$-AXU) if for all distinct $x$, $x'$ in $\mathcal{D}$ and any $y \in \mathcal{G}$, the differential probability for the triplet $(x, x', y)$ is at most $\epsilon$.

We define two standard hash functions.

**Polynomials:** For $m \geq 0$, let $\mathsf{Horner} : \mathbb{F} \times \mathbb{F}^m \rightarrow \mathbb{F}$ be defined as follows.

$$\mathsf{Horner}(\tau, X_1, \ldots, X_m) = \begin{cases} \mathbf{0}, & \text{if } m = 0; \\ X_1\tau^{m-1} \oplus X_2\tau^{m-2} \oplus \cdots \oplus X_{m-1}\tau \oplus X_m, & \text{if } m > 0. \end{cases}$$

We write $\mathsf{Horner}_\tau(X_1, \ldots, X_m)$ to denote $\mathsf{Horner}(\tau, X_1, \ldots, X_m)$. The degree of $\mathsf{Horner}_\tau(X_1, \ldots, X_m)$ as a polynomial in $\tau$ is at most $m-1$. For $m > 0$, $\mathsf{Horner}_\tau(X_1, \ldots, X_m) = \tau \mathsf{Horner}_\tau(X_1, \ldots, X_{m-1}) \oplus X_m$ and so $\mathsf{Horner}_\tau(X_1, \ldots, X_m)$ can be evaluated using $m - 1$ field multiplications.

For a fixed value of $m$, $\{\mathsf{Horner}_\tau\}$ is $((m-1)/2^n)$-AU and the family $\{\tau \cdot \mathsf{Horner}_\tau\}$ is $(m/2^n)$-AXU.

**BRW polynomials:** Bernstein [6] defined a family of polynomials based on a previous work by Rabin and Winograd [35]. (These have been called the BRW polynomials [39]). For $m \geq 0$, let $\mathsf{BRW} : \mathbb{F} \times \mathbb{F}^m \rightarrow \mathbb{F}$ be defined as follows. We write $\mathsf{BRW}_\tau(\cdots)$ to denote $\mathsf{BRW}(\tau, \cdots)$.

- $\mathsf{BRW}_\tau() = \mathbf{0}$;
- $\mathsf{BRW}_\tau(X_1) = X_1$;
- $\mathsf{BRW}_\tau(X_1, X_2) = X_1\tau \oplus X_2$;
- $\mathsf{BRW}_\tau(X_1, X_2, X_3) = (\tau \oplus X_1)(\tau^2 \oplus X_2) \oplus X_3$;
- $\mathsf{BRW}_\tau(X_1, X_2, \cdots, X_m)$
  $= \mathsf{BRW}_\tau(X_1, \cdots, X_{t-1})(\tau^t \oplus X_t) \oplus \mathsf{BRW}_\tau(X_{t+1}, \cdots, X_m)$;
  if $t \in \{4, 8, 16, 32, \cdots\}$ and $t \leq m < 2t$.

From the definition it follows that for $m \geq 3$, $\mathsf{BRW}_\tau(X_1, X_2, \cdots, X_m)$ is a monic polynomial and for $m = 0, 1, 2$, $\mathsf{BRW}_\tau(X_1, \ldots, X_m) = \mathsf{Horner}_\tau(X_1, \ldots, X_m)$. We further note the following points about $\mathsf{BRW}$ polynomials [6].

1. For $m \geq 3$, $\mathsf{BRW}_\tau(X_1,\ldots,X_m)$ can be computed using $\lfloor m/2 \rfloor$ field multiplications and $\lfloor \lg m \rfloor$ additional field squarings to compute $\tau^2, \tau^4, \ldots$.
2. *Let* $\mathfrak{d}(m)$ *denote the degree of* $\mathsf{BRW}_\tau(X_1,\ldots,X_m)$. For $m \geq 3$, $\mathfrak{d}(m) = 2^{\lfloor \lg m \rfloor + 1} - 1$ and so $\mathfrak{d}(m) \leq 2m-1$; equality is achieved if and only if $m = 2^a$; and $\mathfrak{d}(m) = m$ if and only if $m = 2^a - 1$ for some integer $a \geq 2$.
3. The map from $\mathbb{F}^m$ to $\mathbb{F}[\tau]$ given by $(X_1,\ldots,X_m) \longmapsto \mathsf{BRW}_\tau(X_1,\ldots,X_m)$ is injective.

For a fixed value of $m$, $\{\mathsf{BRW}_\tau\}$ is $((2m-1)/2^n)$-AU and the family $\{\tau \cdot \mathsf{BRW}_\tau\}$ is $(2m/2^n)$-AXU.

### 3.1 Hash Function vecHorner

Let

$$\mathcal{VD} = \bigcup_{k=0}^{255} \{(M_1,\ldots,M_k) : M_i \in \{0,1\}^*,\ 0 \leq \mathsf{len}(M_i) \leq 2^{n-16}-1\}. \tag{2}$$

The upper bound of 255 on $k$ ensures that the value of $k$ fits in a byte and the upper bound of $2^{n-16}-1$ on the lengths of strings ensures that the lengths of such strings fit into an $(n-16)$-bit binary string. The definition of $\mathsf{vecHorner} : \mathbb{F} \times \mathcal{VD} \rightarrow \mathbb{F}$ is shown in Table 1 where we write $\mathsf{vecHorner}_\tau(\cdot)$ to denote $\mathsf{vecHorner}(\tau,\cdot)$. The degree of $\mathsf{vecHorner}_\tau(M_1,\ldots,M_k)$ is at most $k + \sum_{i=1}^{k} m_i$ and its constant term is $\mathbf{0}$. Here $m_i = \mathfrak{l}(M_i)$, $i = 1,\ldots,k$.

Table 1: Computations of vecHorner and vecHash2L. The string $1^n$ denotes the element of $\mathbb{F}$ whose binary representation consists of the all-one string.

| $\mathsf{vecHorner}_\tau(M_1,\ldots,M_k)$ | $\mathsf{vecHash2L}_\tau(M_1,\ldots,M_k)$ |
|---|---|
| if $k=0$ return $1^n\tau$; | if $k=0$ return $1^n\tau$; |
| $\mathsf{digest} \leftarrow \mathbf{0}$; | $\mathsf{digest} \leftarrow \mathbf{0}$; |
| for $i \leftarrow 1,\ldots,k-1$ do | for $i \leftarrow 1,\ldots,k-1$ do |
| $(M_{i,1},\ldots,M_{i,m_i}) \leftarrow \mathsf{format}_n(\mathsf{pad}_n(M_i))$; | $(M_{i,1},\ldots,M_{i,\ell_i}) \leftarrow \mathsf{superBlks}_{n,\eta}(M_i)$; |
| $L_i \leftarrow \mathsf{bin}_n(\mathsf{len}(M_i))$; | $L_i \leftarrow \mathsf{bin}_n(\mathsf{len}(M_i))$; |
| for $j \leftarrow 1,\ldots,m_i$ do | for $j \leftarrow 1,\ldots,\ell_i$ do |
| $\mathsf{digest} \leftarrow \tau\mathsf{digest} \oplus M_{i,j}$; | $\mathsf{digest} \leftarrow \tau^{\mathfrak{d}(\eta)+1}\mathsf{digest} \oplus \mathsf{BRW}_\tau(M_{i,j})$; |
| end for; | end for; |
| $\mathsf{digest} \leftarrow \tau\mathsf{digest} \oplus L_i$; | $\mathsf{digest} \leftarrow \tau\mathsf{digest} \oplus L_i$; |
| end for; | end for; |
| $(M_{k,1},\ldots,M_{k,m_k}) \leftarrow \mathsf{format}_n(\mathsf{pad}_n(M_k))$; | $(M_{k,1},\ldots,M_{k,\ell_k}) \leftarrow \mathsf{superBlks}_{n,\eta}(M_k)$; |
| $L_k \leftarrow \mathsf{bin}_8(k) \vert\vert 0^8 \vert\vert \mathsf{bin}_{n-16}(\mathsf{len}(M_k))$; | $L_k \leftarrow \mathsf{bin}_8(k) \vert\vert 0^8 \vert\vert \mathsf{bin}_{n-16}(\mathsf{len}(M_k))$; |
| for $j \leftarrow 1,\ldots,m_k$ do | for $j \leftarrow 1,\ldots,\ell_k$ do |
| $\mathsf{digest} \leftarrow \tau\mathsf{digest} \oplus M_{k,j}$; | $\mathsf{digest} \leftarrow \tau^{\mathfrak{d}(\eta)+1}\mathsf{digest} \oplus \mathsf{BRW}_\tau(M_{k,j})$; |
| end for; | end for; |
| $\mathsf{digest} \leftarrow \tau\mathsf{digest} \oplus L_k$; | $\mathsf{digest} \leftarrow \tau\mathsf{digest} \oplus L_k$; |
| $\mathsf{digest} \leftarrow \tau\mathsf{digest}$; | $\mathsf{digest} \leftarrow \tau\mathsf{digest}$; |
| return digest. | return digest. |

The following result shows that vecHorner is an AXU family.

**Proposition 1.** *Let* $k \geq k' \geq 0$*;* $\mathbf{M} = (M_1,\ldots,M_k)$ *and* $\mathbf{M}' = (M'_1,\ldots,M'_{k'})$ *be two distinct vectors in* $\mathcal{VD}$ *and* $\alpha \in \mathbb{F}$*. For a uniform random* $\tau \in \mathbb{F}$*,*

$$\Pr_\tau\left[\mathsf{vecHorner}_\tau(\mathbf{M}) \oplus \mathsf{vecHorner}_\tau(\mathbf{M}') = \alpha\right] \leq \frac{\max\left(k + \sum_{i=1}^{k} m_i, k' + \sum_{j=1}^{k'} m'_j\right)}{2^n} \tag{3}$$

*where $m_i$ (resp. $m'_j$) is the number of $n$-bit blocks in $\mathsf{pad}_n(M_i)$ (resp. $\mathsf{pad}_n(M'_j)$).*

*Proof.* Let $p(\tau) = \mathsf{vecHorner}_\tau(\mathbf{M}) \oplus \mathsf{vecHorner}_\tau(\mathbf{M'}) \oplus \alpha$. If $p(\tau)$ is a non-zero polynomial, then the degree of $p(\tau)$ is at most $\max\left(k + \sum_{i=1}^{k} m_i, k' + \sum_{j=1}^{k'} m'_j\right)$. The probability that a uniform random $\tau$ is a root of $p(\tau)$ is at most the stated bound. So, it is sufficient to argue that $p(\tau)$ is non-zero.

If $k' = 0$, then, as $\mathbf{M} \neq \mathbf{M'}$, $k > 0$. In this case, $\mathsf{vecHorner}_\tau(\mathbf{M'}) = 1^n\tau$ and the coefficient of $\tau$ in $\mathsf{vecHorner}_\tau(\mathbf{M})$ is $L_k \neq 1^n$. Hence, in this case $p(\tau)$ is a non-zero polynomial.

Let $M_{i_1,i_2}$ (resp. $M'_{j_1,j_2}$) be the $n$-bit blocks obtained from $\mathbf{M}$ (resp. $\mathbf{M'}$) using $\mathsf{format}$. If $k > k' > 0$, then the coefficient of $\tau$ in $p(\tau)$ is $L_k \oplus L'_{k'} \neq \mathbf{0}$ and so $p(\tau)$ is a non-zero polynomial. So, suppose $k = k'$. If there is an $i$ such that $L_i \neq L'_i$, let $i$ be the maximum such index. Using the maximality of $i$ it is possible to argue that $L_i \oplus L'_i$ occurs as a coefficient of some power of $\tau$ in $p(\tau)$ and again it follows that $p(\tau)$ is a non-zero polynomial. So, now suppose that $L_i = L'_i$ for all $1 \leq i \leq k = k'$. Since $\mathbf{M} \neq \mathbf{M'}$, there must be an $i$ and $j$ such that $M_{i,j} \neq M'_{i,j}$ again showing that $p(\tau)$ is a non-zero polynomial. □

### 3.2 Hash Function vecHash2L [9]

Two hash functions, namely Hash2L and vecHash2L, have been defined earlier [9]. Here we only recall the definition of vecHash2L since we will not be using the hash function Hash2L in this work. The definition of $\mathsf{vecHash2L} : \mathbb{F} \times \mathcal{VD} \to \mathbb{F}$ is given in Table 1 where we write $\mathsf{vecHash2L}_\tau(\cdot)$ to denote $\mathsf{vecHash2L}(\tau, \cdot)$. The degree of $\mathsf{vecHash2L}_\tau(M_1, \ldots, M_k)$ is at most $(\mathfrak{d}(\eta)+1)(\ell_1+\cdots+\ell_k)+k$, and its constant term is 0. The values of $\ell_1, \ldots, \ell_k$ are defined by the algorithm given in Table 1. Theorem 2 of [9] shows that vecHash2L is an AXU family. More precisely, the following is proved. Let $k \geq k' \geq 0$; $\mathbf{M} = (M_1, \ldots, M_k)$ and $\mathbf{M'} = (M'_1, \ldots, M'_{k'})$ be two distinct vectors in $\mathcal{VD}$. For a uniform random $\tau \in \mathbb{F}$ and for any $\alpha \in \mathbb{F}$,

$$\Pr_\tau\left[\mathsf{vecHash2L}_\tau(\mathbf{M}) \oplus \mathsf{vecHash2L}_\tau(\mathbf{M'}) = \alpha\right] \leq \frac{\max\left(k + (\mathfrak{d}(\eta)+1)\Lambda, k' + (\mathfrak{d}(\eta)+1)\Lambda'\right)}{2^n} \tag{4}$$

where $\Lambda = \sum_{i=1}^{k} \ell_i$ and $\Lambda' = \sum_{j=1}^{k'} \ell'_j$; $\ell_i$ (resp. $\ell'_j$) is the number of super-blocks in $M_i$ (resp. $M'_j$).

Note that the hash function vecHash2L is parameterised by the value of $\eta$. *In the rest of the paper, we will assume that $\eta + 1$ is a power of two so that the degree $\mathfrak{d}(\eta)$ of $\mathsf{BRW}_\tau(X_1, \ldots, X_\eta)$ is $\eta$.*

## 4 Construction

Formally, $\mathsf{FAST} = (\mathsf{FAST.Encrypt}, \mathsf{FAST.Decrypt})$ where

$$\mathsf{FAST.Encrypt}, \mathsf{FAST.Decrypt} : \mathcal{K} \times \mathcal{T} \times \mathcal{P} \to \mathcal{P}, \tag{5}$$

- $\mathcal{K}$ is a finite non-empty set called the key space,
- $\mathcal{T}$ is a finite non-empty set called the tweak space and
- $\mathcal{P}$ denotes both the message and the ciphertext spaces such that for any string $P \in \mathcal{P}$, $\mathsf{len}(P) > 2n$. So, for any $P \in \mathcal{P}$, the number of $n$-bit blocks in $\mathsf{pad}_n(P)$ is at least three. This requirement will be called the *length condition* on $\mathcal{P}$.

We emphasise that $\mathcal{P}$ does not necessarily contain all strings of lengths greater than $2n$. We provide the precise definitions of $\mathcal{P}$ for specific instantiations later.

For $K \in \mathcal{K}$, $T \in \mathcal{T}$ and $P \in \mathcal{P}$, we write $\mathsf{FAST.Encrypt}_K(T,P)$ to denote $\mathsf{FAST.Encrypt}(K,T,P)$; for $K \in \mathcal{K}$, $T \in \mathcal{T}$ and $C \in \mathcal{P}$, we write $\mathsf{FAST.Decrypt}_K(T,C)$ to denote $\mathsf{FAST.Decrypt}(K,T,C)$. The definitions of $\mathsf{FAST.Encrypt}_K(T,P)$ and $\mathsf{FAST.Decrypt}_K(T,C)$ are given in Table 2. These definitions use the functions $\mathbf{H}_\tau$, $\mathbf{G}_\tau$, $\mathbf{H}'_\tau$ and $\mathbf{G}'_\tau$ which themselves are defined using two hash functions $h$ and $h'$ in the following manner.

$$\begin{aligned}
\mathbf{H}_\tau(P_1,P_2,P_3,T) &= (P_1 \oplus h_\tau(T,P_3), P_2 \oplus \tau(P_1 \oplus h_\tau(T,P_3)));\\
\mathbf{G}_\tau(X_1,X_2,X_3,T) &= (X_1 \oplus h_\tau(T,X_3), X_2 \oplus \tau X_1);\\
\mathbf{H}'_\tau(C_1,C_2,C_3,T) &= (C_1 \oplus \tau(C_2 \oplus h'_\tau(T,C_3)), C_2 \oplus h'_\tau(T,C_3));\\
\mathbf{G}'_\tau(Y_1,Y_2,Y_3,T) &= (Y_1 \oplus \tau Y_2, Y_2 \oplus h'_\tau(T,Y_3)).
\end{aligned} \tag{6}$$

From the definitions of $\mathbf{H}_\tau, \mathbf{G}_\tau$ and $\mathbf{H}'_\tau, \mathbf{G}'_\tau$ it is easy to verify the following properties.

$$\begin{array}{lcl}
\mathbf{H}_\tau(P_1,P_2,P_3,T) = (A_1,F_1) & \text{implies} & \mathbf{G}_\tau(A_1,F_1,P_3,T) = (P_1,P_2);\\
\mathbf{H}'_\tau(C_1,C_2,C_3,T) = (F_2,B_2) & \text{implies} & \mathbf{G}'_\tau(F_2,B_2,C_3,T) = (C_1,C_2).
\end{array} \tag{7}$$

Note that for fixed $\tau$, $P_3$ and $T$, $\mathbf{H}_\tau(\cdot,\cdot,P_3,T)$ and $\mathbf{G}_\tau(\cdot,\cdot,P_3,T)$ are inverses of one another and similarly, $\mathbf{H}'_\tau(\cdot,\cdot,P_3,T)$ and $\mathbf{G}'_\tau(\cdot,\cdot,P_3,T)$ are inverses of one another.

The hash functions $h$ and $h'$ required in the definitions of $\mathbf{H}, \mathbf{G}$ and $\mathbf{H}', \mathbf{G}'$ respectively and the other components used in Table 2 are given below.

1. The two hash functions $h$ and $h'$ are defined in the following manner.

$$h, h' : \mathbb{F} \times \mathcal{T} \times \mathcal{M} \to \mathbb{F}, \tag{8}$$

   where

$$\mathcal{M} = \{x : w||x \in \mathcal{P} \text{ for some } w \in \{0,1\}^{2n}\}; \tag{9}$$

   $\mathbb{F}$ is the key space and also the digest space, $\mathcal{T}$ is the tweak space and $\mathcal{M}$ is the message space for the hash functions. For $\tau \in \mathbb{F}$, $T \in \mathcal{T}$ and $M \in \mathcal{M}$, we will write $h_\tau(T,M)$ (resp. $h'_\tau(T,M)$) to denote $h(\tau,T,M)$ (resp. $h'(\tau,T,M)$). Note that in $\mathsf{FAST}$, both $h$ and $h'$ share the same key $\tau$. Later we discuss the properties required of the pair of hash functions $(h,h')$ and how to construct such pairs using standard hash functions.
2. A PRF $\{\mathbf{F}_K\}_{K\in\mathcal{K}}$ where for $K \in \mathcal{K}$, $\mathbf{F}_K : \{0,1\}^n \to \{0,1\}^n$. The PRF is used in the $\mathsf{Ctr}$ mode as given in (1). Since strings in $\mathcal{P}$ are of length greater than $2n$, the $\mathsf{Ctr}$ mode is applied to non-empty strings.
3. A fixed $n$-bit string $\mathsf{fStr}$.
4. Sub-routines $\mathsf{Feistel}$ and $\mathsf{Feistel}^{-1}$ which are shown in Table 3.

From the descriptions of $\mathsf{FAST.Encrypt}_K(T,P)$ and $\mathsf{FAST.Decrypt}_K(T,C)$ in Table 2, the following two facts are easy to verify. For $K \in \mathcal{K}$, $T \in \mathcal{T}$ and $P \in \mathcal{P}$,

$$\mathsf{FAST.Decrypt}_K(T, \mathsf{FAST.Encrypt}_K(T,P)) = P; \tag{10}$$

$$\mathsf{len}\,(\mathsf{FAST.Encrypt}_K(T,P)) = \mathsf{len}(P). \tag{11}$$

From (10), it follows that the decryption function of $\mathsf{FAST}$ is the inverse of the encryption function, while (11) shows that the length of the ciphertext produced by the encryption function is equal to the length of the plaintext.

**Remarks:**

1. $\mathsf{FAST.Encrypt}_K(T, P)$ and $\mathsf{FAST.Decrypt}_K(T, C)$ can be seen as consisting of three distinct layers – hash-encrypt-hash. The quantity $\tau$ is the key for the hashing layers and is not used in the encryption layer while $K$ is used only in the encryption layer and not in the hashing layers. In an earlier version, we had used a different description, where the first two blocks were processed using a 4-round Feistel network where the first and the last round consisted of multiplication by $\tau$ so that one could consider the first two blocks to be processed by a strong pseudo-random permutation. The present description is equivalent to the previous description (except as discussed in the next point) and is perhaps more modular in the sense of separating the hashing and the encryption keys. The suggestion for providing such a modular description is due to a reviewer of the previous version.
2. The quantity $Z$ in $\mathsf{FAST.Encrypt}_K(T, P)$ and $\mathsf{FAST.Decrypt}_K(T, C)$ is defined to be equal to $F_1 \oplus F_2$. In an earlier version, we had defined $Z$ to be equal to $P_2 \oplus C_1$. The suggestion to define $Z$ as $F_1 \oplus F_2$ is due to Mridul Nandi. This saves a few cycles in a pipelined hardware implementation when $\mathbf{F}$ is instantiated with AES; it has no effect on the efficiency of software implementation.

Table 2: Encryption and decryption algorithms for $\mathsf{FAST}$.

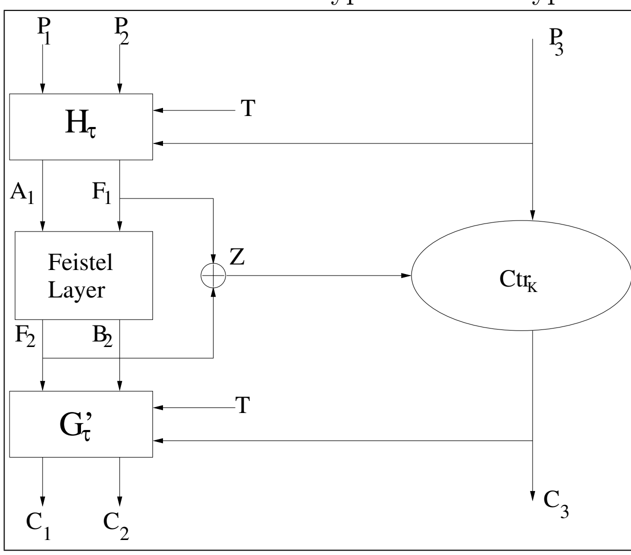
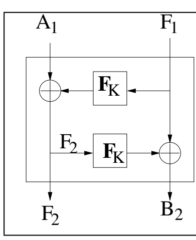
| | |
|---|---|
|  | **Algorithm** $\mathsf{FAST.Encrypt}_K(T, P)$<br>1. $\tau \leftarrow \mathbf{F}_K(\mathsf{fStr})$;<br>2. $(P_1, P_2, P_3) \leftarrow \mathsf{parse}_n(P)$;<br>3. $(A_1, F_1) \leftarrow \mathbf{H}_\tau(P_1, P_2, P_3, T)$;<br>4. $(F_2, B_2) \leftarrow \mathsf{Feistel}_K(A_1, F_1)$;<br>5. $Z \leftarrow F_1 \oplus F_2$;<br>6. $C_3 \leftarrow \mathsf{Ctr}_{K,Z}(P_3)$;<br>7. $(C_1, C_2) \leftarrow \mathbf{G}'_\tau(F_2, B_2, C_3, T)$;<br>8. return $(C_1 \vert\vert C_2 \vert\vert C_3)$.<br>**Algorithm** $\mathsf{FAST.Decrypt}_K(T, C)$<br>1. $\tau \leftarrow \mathbf{F}_K(\mathsf{fStr})$;<br>2. $(C_1, C_2, C_3) \leftarrow \mathsf{parse}_n(C)$;<br>3. $(F_2, B_2) \leftarrow \mathbf{H}'_\tau(C_1, C_2, C_3, T)$;<br>4. $(A_1, F_1) \leftarrow \mathsf{Feistel}^{-1}_K(F_2, B_2)$;<br>5. $Z \leftarrow F_1 \oplus F_2$;<br>6. $P_3 \leftarrow \mathsf{Ctr}_{K,Z}(C_3)$;<br>7. $(P_1, P_2) \leftarrow \mathbf{G}_\tau(A_1, F_1, P_3, T)$;<br>8. return $(P_1 \vert\vert P_2 \vert\vert P_3)$. |

Table 3: A two-round Feistel construction required in Table 2.

| | | |
|---|---|---|
|  | $\mathsf{Feistel}_K(A_1, F_1)$<br>1. $F_2 \leftarrow A_1 \oplus \mathbf{F}_K(F_1)$;<br>2. $B_2 \leftarrow F_1 \oplus \mathbf{F}_K(F_2)$;<br>return $(F_2, B_2)$. | $\mathsf{Feistel}^{-1}_K(F_2, B_2)$<br>1. $F_1 \leftarrow B_2 \oplus \mathbf{F}_K(F_2)$;<br>2. $A_1 \leftarrow F_2 \oplus \mathbf{F}_K(F_1)$;<br>return $(A_1, F_1)$. |

## 5 Instantiations of $\mathsf{FAST}$

Certain properties are required from the pair of hash functions $(h, h')$. These properties will be used in the security argument to show that in an information theoretic setting, the adversary's probability of breaking the security of $\mathsf{FAST}$ is low. The specific properties that will be required are formalised below.

**Definition 1.** *Let $(h, h')$ be a pair of hash functions where $h, h' : \mathbb{F} \times \mathcal{T} \times \mathcal{M} \to \mathbb{F}$ satisfy the following properties. For any $(T, M), (T', M') \in \mathcal{T} \times \mathcal{M}$, with $(T, M) \neq (T', M')$; any $\alpha, \beta \in \mathbb{F}$; and $\tau$ chosen uniformly at random from $\mathbb{F}$:*

$$\Pr[\tau(h_\tau(T, M) \oplus \alpha) = \beta] \leq \epsilon_1(\mathfrak{t}, \mathfrak{l}); \tag{12}$$

$$\Pr[\tau(h'_\tau(T, M) \oplus \alpha) = \beta] \leq \epsilon_1(\mathfrak{t}, \mathfrak{l}); \tag{13}$$

$$\Pr[\tau(h_\tau(T, M) \oplus h_\tau(T', M') \oplus \alpha) = \beta] \leq \epsilon_2(\mathfrak{t}, \mathfrak{l}, \mathfrak{t}', \mathfrak{l}'); \tag{14}$$

$$\Pr[\tau(h'_\tau(T, M) \oplus h'_\tau(T', M') \oplus \alpha) = \beta] \leq \epsilon_2(\mathfrak{t}, \mathfrak{l}, \mathfrak{t}', \mathfrak{l}'). \tag{15}$$

*For any $(T, M), (T', M') \in \mathcal{T} \times \mathcal{M}$; any $\alpha, \beta \in \mathbb{F}$; and $\tau$ chosen uniformly at random from $\mathbb{F}$:*

$$\Pr[\tau(h_\tau(T, M) \oplus h'_\tau(T', M') \oplus \alpha) = \beta] \leq \epsilon_2(\mathfrak{t}, \mathfrak{l}, \mathfrak{t}', \mathfrak{l}'). \tag{16}$$

*Here $\mathfrak{t} \equiv \mathfrak{t}(T)$, $\mathfrak{t}' \equiv \mathfrak{t}(T')$, $\mathfrak{l} \equiv \mathfrak{l}(M)$, $\mathfrak{l}' \equiv \mathfrak{l}(M')$; and $\epsilon_1$ and $\epsilon_2$ are functions of $\mathfrak{t}, \mathfrak{l}, \mathfrak{t}'$ and $\mathfrak{l}'$. Then $(h, h')$ is said to be an $(\epsilon_1, \epsilon_2)$-eligible pair of hash functions.*

We consider the following two scenarios for $\mathsf{FAST}$.

**Fixed length setting $\mathsf{Fx}_m$ for some positive integer $m > 2$:** For this setting, in (5), we define

$$\mathcal{T} = \{0,1\}^n, \quad \mathcal{P} = \{0,1\}^{mn} \text{ and so } \mathcal{M} = \{0,1\}^{n(m-2)}. \tag{17}$$

In other words, a tweak $T$ is an $n$-bit string while plaintexts and ciphertexts consist of $m$ $n$-bit strings. This particular setting is suited for disk encryption application, where for a fixed $n$, the number of blocks $m$ in a message is determined by the size of a disk sector. In this case, for $M \in \mathcal{M}$, $\mathfrak{l}(M) = m - 2$ and for $T \in \mathcal{T}$, $\mathfrak{t}(T) = 1$. By the length condition on $\mathcal{P}$, we must have $m \geq 3$.

Consider the encryption and decryption algorithms of $\mathsf{FAST}$. The number of $n$-bit blocks in $P$ (resp. $C$) is $m$ and so the number of $n$-bit blocks in $P_3$ (resp. $C_3$) is $m - 2$. The hash functions $h$ and $h'$ are invoked in $\mathsf{FAST}$ as $h(T, P_3)$ and $h'(T, C_3)$. So, in Definition 1, $\mathcal{T} = \mathbb{F}$ and $\mathcal{M} = \mathbb{F}^{m-2}$ and we have

$$h, h' : \mathbb{F} \times \mathbb{F} \times \mathbb{F}^{m-2} \to \mathbb{F}. \tag{18}$$

For the setting of $\mathsf{Fx}_m$, we describe two instantiations of $h$ and $h'$, one with $\mathsf{Horner}$ and the other with $\mathsf{BRW}$. The corresponding instantiations of $\mathsf{FAST}$ will be denoted as $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{Horner}]$ and $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}]$.

**General setting** $\mathsf{Gn}$**:** Let $\mathfrak{k}$ be a fixed integer in the range $\{0,\ldots,254\}$. For this setting, in (5), we define

$$\mathcal{T} = \bigcup_{k=0}^{\mathfrak{k}} \{(T_1,\ldots,T_k) : 0 \leq \mathsf{len}(T_i) \leq 2^{n-16}-1\}; \tag{19}$$

$$\mathcal{P} = \bigcup_{i>2n}^{2^{n-16}-1} \{0,1\}^i; \text{ and so} \tag{20}$$

$$\mathcal{M} = \bigcup_{i>0}^{2^{n-16}-2n-1} \{0,1\}^i. \tag{21}$$

A tweak $T$ is a vector $T=(T_1,\ldots,T_k)$ where $0 \leq k \leq \mathfrak{k}$ and each $T_i$ is a binary string. Since $\mathfrak{k} \leq 254$, $\mathfrak{k}+1 \leq 255$ and so the binary representation of $\mathfrak{k}+1$ will fit in a byte.

For $P \in \mathcal{P}$, suppose $M \in \mathcal{M}$ is such that $P = X||M$ for some binary string $X$ of length $2n$. Then $\mathfrak{l}(M) = m-2$, where $m$ is the number of blocks in $\mathsf{pad}_n(P)$. For a tweak $T=(T_1,\ldots,T_k)$, $\mathfrak{t}(T) = \sum_{i=1}^{k} m_i$, where $m_i$ is the number of blocks in $\mathsf{pad}_n(T_i)$.

The parameter $\mathfrak{k}$ controls the maximum number of components that can appear in a tweak. This does not imply that the number of components in all the tweaks is equal to $\mathfrak{k}$. Rather, the number of components in a tweak is between 0 and $\mathfrak{k}$. So, the above definition of the tweak space models tweaks as vectors having variable number of components. Since we put an upper bound of 254 on $\mathfrak{k}$, one possibility is to do away with the parameter $\mathfrak{k}$ and replace it with the value 254. The reason we do not do this is the following. The parameter $\mathfrak{k}$ enters the security bound. If we replace $\mathfrak{k}$ by 254, then this value would enter the security bound. If in practice, the actual value of $\mathfrak{k}$ is much less than 254 (as it is likely to be), then using 254 instead of $\mathfrak{k}$ will lead to a looser security bound than what it should actually be. It is to avoid this unnecessary looseness in the security bound that we introduce and work with the parameter $\mathfrak{k}$.

For the setting of $\mathsf{Gn}$, we describe two instantiations of $h$ and $h'$. One of these is based on $\mathsf{vecHorner}$ while the other is based on $\mathsf{vecHash2L}$. The parameter $\mathfrak{k}$ is required in both cases while the parameter $\eta$ is required only in the case of $\mathsf{vecHash2L}$. The instantiations of $\mathsf{FAST}$ in the general setting with $\mathsf{vecHorner}$ and $\mathsf{vecHash2L}$ will be denoted as $\mathsf{FAST}[\mathsf{Gn},\mathfrak{k},\mathsf{vecHorner}]$ and $\mathsf{FAST}[\mathsf{Gn},\mathfrak{k},\eta,\mathsf{vecHash2L}]$ respectively.

In the general setting, the lengths of the plaintexts can vary. Also, the tweak space has a rich structure which provides considerable flexibility in applications. Examples of such applications have been mentioned in the introduction. On the downside, the specific instantiations of the general setting are somewhat slower than the corresponding instantiations for the fixed length setting. So, for targeted applications such as disk encryption, it would be preferable to use the fixed length setting leaving out some of the extra overheads incurred in the general setting.

### 5.1 Hash Functions $h$ and $h'$ for $\mathsf{FAST}[\mathsf{Fx}_m,\mathsf{Horner}]$

Fix a positive integer $m \geq 3$ so that the length condition on $\mathcal{P}$ is satisfied. The hash functions $h, h'$ are defined using $\mathsf{Horner}$ as follows:

$$h_\tau(T, X_1||\cdots||X_{m-2}) = \tau\mathsf{Horner}_\tau(\mathbf{1}, X_1,\ldots,X_{m-2},T); \tag{22}$$

$$h'_\tau(T, X_1||\cdots||X_{m-2}) = \tau^2\mathsf{Horner}_\tau(\mathbf{1}, X_1,\ldots,X_{m-2},T). \tag{23}$$

Note that $\mathsf{Horner}_\tau(\mathbf{1}, X_1, \ldots, X_{m-2}, T)$ is a monic polynomial in $\tau$ of degree $m-1$. Consequently, $h$ and $h'$ are monic polynomials in $\tau$ of degrees $m$ and $m+1$ respectively whose constant terms are zero.

**Proposition 2.** *Let $m \geq 3$ be an integer. The pair $(h, h')$ of hash functions defined in (22) and (23) for the construction $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{Horner}]$ is an $(\epsilon_1, \epsilon_2)$-eligibile pair, where $\epsilon_1 = \epsilon_2 = (m+2)/2^n$.*

*Proof.* In this case, for $M \in \mathcal{M}$, $\mathfrak{l}(M) = m-2$ and for $T \in \mathcal{T}$, $\mathfrak{t}(T) = 1$. We write $\mathfrak{l}$ and $\mathfrak{t}$ instead of $\mathfrak{l}(M)$ and $\mathfrak{t}(T)$.

The polynomials $\tau(h_\tau(T, X_1||\cdots||X_{m-2}) \oplus \alpha) \oplus \beta$ and $\tau(h'_\tau(T, X_1||\cdots||X_{m-2}) \oplus \alpha) \oplus \beta$ are monic polynomials of degrees $\mathfrak{l} + \mathfrak{t} + 2 = m+1$ and $\mathfrak{l} + \mathfrak{t} + 3 = m+2$ in $\tau$ respectively. So, the probability that a uniform random $\tau$ in $\mathbb{F}$ is a root of $\tau(h_\tau(T, X_1||\cdots||X_{m-2}) \oplus \alpha) \oplus \beta$ (resp. $\tau(h'_\tau(T, X_1||\cdots||X_{m-2}) \oplus \alpha) \oplus \beta$) is $(\mathfrak{l}+\mathfrak{t}+2)/2^n = (m+1)/2^n$ (resp. $(\mathfrak{l}+\mathfrak{t}+3)/2^n = (m+2)/2^n$). This shows the value of $\epsilon_1$.

Let $X = X_1||\cdots||X_{m-2}$ and $X' = X'_1||\cdots||X'_{m-2}$ and $T, T'$ be such that $(T, X) \neq (T', X')$. Then $h_\tau(T, X) \oplus h_\tau(T', X')$ is a non-zero polynomial of degree at most $\mathfrak{l}+\mathfrak{t} = m-1$ whose constant term is zero. This is because the leading terms of $h_\tau(T, X)$ and $h_\tau(T', X')$ will cancel out in the sum $h_\tau(T, X) \oplus h_\tau(T', X')$ so that its degree will be at most $m-1$; $(T, X) \neq (T', X')$ ensures that $h_\tau(T, X) \oplus h_\tau(T', X')$ is a non-zero polynomial; and the constant terms of both $h_\tau(T, X)$ and $h_\tau(T', X')$ are zero. As a result, $\tau(h_\tau(T, X) \oplus h_\tau(T', X') \oplus \alpha) \oplus \beta$ is a non-zero polynomial in $\tau$ of degree at most $m$. So, the probability that a uniform random $\tau$ is a root of this polynomial is at most $(\mathfrak{l}+\mathfrak{t}+1)/2^n = m/2^n$. A similar reasoning shows that the probability that a uniform random $\tau$ is a root of $\tau(h'_\tau(T, X) \oplus h'_\tau(T', X') \oplus \alpha) \oplus \beta$ is at most $(\mathfrak{l}+\mathfrak{t}+2)/2^n = (m+1)/2^n$.

For any $(T, X)$ and $(T', X')$, the polynomial $h_\tau(T, X) \oplus h'_\tau(T', X')$ is a monic polynomial of degree $\mathfrak{l}+\mathfrak{t}+2 = m+1$ whose constant term is zero. Consequently, the polynomial $\tau(h_\tau(T, X) \oplus h'_\tau(T', X') \oplus \alpha) \oplus \beta$ is a monic polynomial of degree $m+2$ and so the probability that a uniform random $\tau$ is a root of this polynomial is $(\mathfrak{l}+\mathfrak{t}+3)/2^n = (m+2)/2^n$. This shows the value of $\epsilon_2$. $\square$

### 5.2 Hash Functions $h$ and $h'$ for $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}]$

Fix an integer $m \geq 4$. From the length condition on $\mathcal{P}$, we only need $m \geq 3$ and the condition $m \geq 4$ is a special requirement for $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}]$ as we explain below. In this case, the hash functions $h, h'$ are defined using $\mathsf{BRW}$ as follows:

$$h_\tau(T, X_1||\cdots||X_{m-2}) = \tau \mathsf{BRW}_\tau(X_1, \ldots, X_{m-2}, T); \tag{24}$$

$$h'_\tau(T, X_1||\cdots||X_{m-2}) = \tau^2 \mathsf{BRW}_\tau(X_1, \ldots, X_{m-2}, T). \tag{25}$$

Note that from the definition of $\mathsf{BRW}$ polynomials, for $m = 3$, $\mathsf{BRW}_\tau(X_1, \ldots, X_{m-2}, T)$ is not necessarily monic, while for $m \geq 4$, $\mathsf{BRW}_\tau(X_1, \ldots, X_{m-2}, T)$ is necessarily monic. It is to ensure the monic property that we enforce the condition $m \geq 4$ for $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}]$. An alternative would have been to prepend $\mathbf{1}$ as in the case of $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{Horner}]$. This though would create complications which do not seem to be necessary for the fixed length setting. Instead, we use this technique later in the context of the general setting.

Recall that the degree of $\mathsf{BRW}_\tau(X_1, \ldots, X_{m-2}, T)$ is denoted as $\mathfrak{d}(m-1)$. So, $h$ and $h'$ are also monic polynomials of degrees $1 + \mathfrak{d}(m-1)$ and $2 + \mathfrak{d}(m-1)$ respectively whose constant terms are zero.

**Proposition 3.** *Let $m \geq 4$ be an integer. The pair $(h, h')$ of hash functions defined in (24) and (25) for the construction $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}]$ is an $(\epsilon_1, \epsilon_2)$-eligibile pair, where $\epsilon_1 = \epsilon_2 = (3+\mathfrak{d}(m-1))/2^n$. Further, if $m$ is a power of two, then $(h, h')$ is an $((m+2)/2^n, (m+2)/2^n)$-eligibile pair.*

*Proof.* The proof is analogous to the proof of Proposition 2. It is required to use the expression $\mathfrak{d}(m-1)$ for the degree of $\mathsf{BRW}_\tau(X_1,\ldots,X_{m-2},T)$ and further the injectivity of the map $(X_1,\ldots,X_{m-2},T) \longmapsto \mathsf{BRW}_\tau(X_1,\ldots,X_{m-2},T)$ ensures that for $(T,X) \neq (T',X')$, the polynomial $\mathsf{BRW}_\tau(X_1,\ldots,X_{m-2},T) \oplus \mathsf{BRW}_\tau(X'_1,\ldots,X'_{m-2},T')$ is not zero.

The last statement follows from the previously mentioned fact that $\mathfrak{d}(m-1) = m-1$ if and only if $m \geq 4$ is a power of two. □

**Remark:** In the case where $m$ is a power of two, $(h,h')$ is an $((m+2)/2^n,(m+2)/2^n)$-eligible pair for both $\mathsf{FAST}[\mathsf{Fx}_m,\mathsf{Horner}]$ and $\mathsf{FAST}[\mathsf{Fx}_m,\mathsf{BRW}]$.

### 5.3 Hash Functions $h$ and $h'$ for $\mathsf{FAST}[\mathsf{Gn},\mathfrak{k},\mathsf{vecHorner}]$

In this setting, we define $h,h' : \mathbb{F} \times \mathcal{T} \times \mathcal{M} \rightarrow \mathbb{F}$ where $\mathcal{T}$ and $\mathcal{M}$ are given by (19) and (21) respectively. For $T=(T_1,\ldots,T_k) \in \mathcal{T}$ and $M \in \mathcal{M}$, let $d = \mathfrak{t}(T) + \mathfrak{l}(M) + k + 2$. We define

$$h_\tau(T,M) = \tau^d \oplus \mathsf{vecHorner}_\tau(T_1,\ldots,T_k,M); \tag{26}$$

$$h'_\tau(T,M) = \tau(\tau^d \oplus \mathsf{vecHorner}_\tau(T_1,\ldots,T_k,M)). \tag{27}$$

It is easy to see that $h$ and $h'$ are monic polynomials of degrees $d$ and $d+1$ respectively whose constant terms are zero. The computation of $\tau^d \oplus \mathsf{vecHorner}_\tau(T_1,\ldots,T_k,M)$ can be done by the following simple modification of the algorithm for computing $\mathsf{vecHorner}$ shown in Table 1. *The initialisation of* $\mathsf{digest}$ *using* $\mathsf{digest} = \mathbf{0}$ *is to be replaced with* $\mathsf{digest} = \mathbf{1}$.

**Proposition 4.** *The hash functions $h$ and $h'$ defined in (26) and (27) respectively for the construction* $\mathsf{FAST}[\mathsf{Gn},\mathfrak{k},\mathsf{vecHorner}]$ *form an $(\epsilon_1,\epsilon_2)$-eligibile pair, where*

$$\epsilon_1 = \frac{\mathfrak{t}+\mathfrak{l}+\mathfrak{k}+4}{2^n}; \qquad \epsilon_2 = \frac{\max(\mathfrak{t}+\mathfrak{l},\mathfrak{t}'+\mathfrak{l}')+\mathfrak{k}+4}{2^n}.$$

*Proof.* For $T=(T_1,\ldots,T_k)$, recall that $\mathfrak{t} = \mathfrak{t}(T) = \sum_{i=1}^k m_i$ where $m_i$ is the number of $n$-bit blocks in $\mathsf{pad}_n(T_i)$. Also, $\mathfrak{l} = \mathfrak{l}(M)$ is the number of $n$-bit blocks in $\mathsf{pad}_n(M)$.

The degree of $h_\tau(T,M)$ is $d = \mathfrak{t}+\mathfrak{l}+k+2 \leq \mathfrak{t}+\mathfrak{l}+\mathfrak{k}+2$ and the degree of $h'_\tau(T,M)$ is $d+1 = \mathfrak{t}+\mathfrak{l}+k+3 \leq \mathfrak{t}+\mathfrak{l}+\mathfrak{k}+3$. So, the polynomial $\tau(h_\tau(T,M)\oplus\alpha)\oplus\beta$ is a monic polynomial of degree at most $\mathfrak{t}+\mathfrak{l}+\mathfrak{k}+3$ and the polynomial $\tau(h'_\tau(T,M)\oplus\alpha)\oplus\beta$ is a monic polynomial of degree at most $\mathfrak{t}+\mathfrak{l}+\mathfrak{k}+4$. This shows the value of $\epsilon_1$.

Consider $(T',M') \neq (T,M)$ where $T' = (T'_1,\ldots,T'_{k'})$, $\mathfrak{t}' = \mathfrak{t}(T')$ and $\mathfrak{l}' = \mathfrak{l}(M')$. Without loss of generality assume that $k \geq k'$. Let $p(\tau) = \tau(h_\tau(T,M)\oplus h_\tau(T',M')\oplus\alpha)\oplus\beta$. If the degrees of $h_\tau(T,M)$ and $h_\tau(T',M')$ are not equal, then $p(\tau)$ is a polynomial of degree $\max(\mathfrak{t}+\mathfrak{l}+k+3,\mathfrak{t}'+\mathfrak{l}'+k'+3) \leq \max(\mathfrak{t}+\mathfrak{l}+\mathfrak{k}+3,\mathfrak{t}'+\mathfrak{l}'+\mathfrak{k}+3)$. So, suppose that the degrees of $h_\tau(T,M)$ and $h_\tau(T',M')$ are equal. The leading monic terms of the two polynomials cancel out. If $p(\tau)$ is a non-zero polynomial, then it has maximum degree $\max(\mathfrak{t}+\mathfrak{l}+\mathfrak{k}+2,\mathfrak{t}'+\mathfrak{l}'+\mathfrak{k}+2)$. So, it is sufficient to show that $p(\tau)$ is a non-zero polynomial. This argument is similar to that of Proposition 1. Further, a similar argument applies for $h'$ where the degree of $\tau(h'_\tau(T,M)\oplus h'_\tau(T',M')\oplus\alpha)\oplus\beta$ is at most $\max(\mathfrak{t}+\mathfrak{l}+\mathfrak{k}+4,\mathfrak{t}'+\mathfrak{l}'+\mathfrak{k}+4)$.

Now consider $(T,M)$ and $(T',M')$ which are not necessarily distinct and let $p(\tau) = \tau(h_\tau(T,M)\oplus h'_\tau(T',M')\oplus\alpha)\oplus\beta$. The coefficient of $\tau$ in $h_\tau(T,M)$ is $L = \mathsf{bin}_8(k+1)||0^8||\mathsf{bin}_{n-16}(\mathsf{len}(M)) \neq \mathbf{0}$ which is the coefficient of $\tau^2$ in $\tau h_\tau$. The coefficient of $\tau^2$ in $\tau h'_\tau(T',M')$ is $\mathbf{0}$ and so the coefficient of $\tau^2$ in $p(\tau)$ is $L \neq \mathbf{0}$. So, $p(\tau)$ is a non-zero polynomial. The degree of $p(\tau)$ is at most $\max(\mathfrak{t}+\mathfrak{l}+\mathfrak{k}+3,\mathfrak{t}'+\mathfrak{l}'+\mathfrak{k}+4)$.

This completes the proof. □

### 5.4 Hash Functions $h$ and $h'$ for $\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \eta, \mathsf{vecHash2L}]$

In this setting, we define $h, h' : \mathbb{F} \times \mathcal{T} \times \mathcal{M} \to \mathbb{F}$ where $\mathcal{T}$ and $\mathcal{M}$ are given by (19) and (21) respectively. For $T = (T_1, \ldots, T_k) \in \mathcal{T}$ and $M \in \mathcal{M}$, let the number of super-blocks in $\mathsf{pad}_n(T_i)$ be $\ell_i$ and the number of super-blocks in $\mathsf{pad}_n(M)$ be $\ell$. Let $d = (\mathfrak{d}(\eta) + 1)(\ell_1 + \cdots + \ell_k + \ell) + k + 2$. We define

$$h_\tau(T, M) = \tau^d \oplus \mathsf{vecHash2L}_\tau(T_1, \ldots, T_k, M); \tag{28}$$
$$h'_\tau(T, M) = \tau(\tau^d \oplus \mathsf{vecHash2L}_\tau(T_1, \ldots, T_k, M)). \tag{29}$$

The definition of $\mathsf{vecHash2L}$ requires choosing the value $\eta$. As mentioned earlier, we will assume that $\eta$ is chosen so that $\eta + 1$ is a power of two and so $\mathfrak{d}(\eta) = \eta$. The computation of $\tau^d \oplus \mathsf{vecHash2L}_\tau(T_1, \ldots, T_k, M)$ can be done by the following simple modification of the algorithm for computing $\mathsf{vecHash2L}$ shown in Table 1. *The initialisation of* $\mathsf{digest}$ *using* $\mathsf{digest} = \mathbf{0}$ *is to be replaced with* $\mathsf{digest} = \mathbf{1}$.

**Proposition 5.** *Let the parameter $\eta \geq 3$ required in the definition of* $\mathsf{vecHash2L}$ *be such that $\eta + 1$ is a power of two. The hash functions $h$ and $h'$ defined in (28) and (29) respectively for the construction* $\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \eta, \mathsf{vecHash2L}]$ *form an $(\epsilon_1, \epsilon_2)$-eligibile pair, where*

$$\epsilon_1 = \frac{((\eta + 1)/\eta)(\mathfrak{t} + \mathfrak{l}) + (\mathfrak{k} + 1)(\eta + 2) + 3}{2^n};$$
$$\epsilon_2 = \frac{((\eta + 1)/\eta)\max(\mathfrak{t} + \mathfrak{l}, \mathfrak{t}' + \mathfrak{l}') + (\mathfrak{k} + 1)(\eta + 2) + 3}{2^n}.$$

*Proof.* Since $\eta + 1 \geq 4$ is a power of two, $\mathfrak{d}(\eta) = \eta$.

For $i = 1, \ldots, k$, let the number of super-blocks in $\mathsf{pad}_n(T_i)$ be $\ell_i$ and the number of super-blocks in $\mathsf{pad}_n(M)$ be $\ell$. For $i = 1, \ldots, k$, let the number of $n$-bit blocks in $\mathsf{pad}_n(T_i)$ be $m_i$, so that $\mathfrak{t} = \mathfrak{t}(T) = \sum_{i=1}^{k} m_i$ and the number of $n$-bit blocks in $\mathsf{pad}_n(M)$ be $\mathfrak{l}$. In $\mathsf{pad}_n(T_i)$, each of the first $\ell_i - 1$ super-blocks contains exactly $\eta$ $n$-bit blocks and the last super-block contains at most $\eta$ blocks. Since the total number of $n$-bit blocks in $\mathsf{pad}_n(T_i)$ is $m_i$, we have $m_i > \eta(\ell_i - 1)$ from which we obtain $(\eta + 1)\ell_i < m_i((\eta + 1)/\eta) + \eta + 1$. Similarly, $(\eta + 1)\ell < \mathfrak{l}((\eta + 1)/\eta) + \eta + 1$. We have

$$\begin{aligned}
d &= (\mathfrak{d}(\eta) + 1)(\ell_1 + \cdots + \ell_k + \ell) + k + 2 \\
&\leq (\mathfrak{d}(\eta) + 1)(\ell_1 + \cdots + \ell_k + \ell) + \mathfrak{k} + 2 \\
&= (\eta + 1)(\ell_1 + \cdots + \ell_k + \ell) + \mathfrak{k} + 2 \quad (\text{since } \mathfrak{d}(\eta) = \eta) \\
&< ((\eta + 1)/\eta)(\mathfrak{t} + \mathfrak{l}) + (k + 1)(\eta + 1) + \mathfrak{k} + 2 \\
&\leq ((\eta + 1)/\eta)(\mathfrak{t} + \mathfrak{l}) + (\mathfrak{k} + 1)(\eta + 2) + 1.
\end{aligned} \tag{30}$$

So, the degree of $\tau h_\tau(T, M)$ is $d + 1$ which is at most $((\eta + 1)/\eta)(\mathfrak{t} + \mathfrak{l}) + (\mathfrak{k} + 1)(\eta + 2) + 2$ and the degree of $\tau h'_\tau(T, M)$ is $d + 2$ which is at most $((\eta + 1)/\eta)(\mathfrak{t} + \mathfrak{l}) + (\mathfrak{k} + 1)(\eta + 2) + 3$. This shows the value of $\epsilon_1$.

Consider $(T', M') \neq (T, M)$ where $T' = (T'_1, \ldots, T'_{k'})$, $\mathfrak{t}' = \mathfrak{t}(T')$ and $\mathfrak{l}' = \mathfrak{l}(M')$. Let $d'$ be the degree of $h_\tau(T', M')$. For any $\alpha, \beta \in \mathbb{F}$, we wish to bound the probability (over uniform random choice of $\tau$ in $\mathbb{F}$) that the polynomial $p_1(\tau) = \tau(h_\tau(T, M) \oplus h_\tau(T', M') \oplus \alpha) \oplus \beta$ is zero. If $d \neq d'$, then $p_1(\tau)$ is a monic polynomial of degree $\max(d + 1, d' + 1)$ and so the probability that it is zero is at most $\max(d + 1, d' + 1)/2^n$. If $d = d'$, then $p_1(\tau)$ is zero if and only if the polynomial

$$p_2(\tau) = \tau(\mathsf{vecHash2L}_\tau(T_1, \ldots, T_k, M) \oplus \mathsf{vecHash2L}_\tau(T'_1, \ldots, T'_{k'}, M') \oplus \alpha) \oplus \beta$$

is zero. Using Theorem 2 of [9] (see (4)), we have this probability to be at most $\max(d,d')/2^n$. So, the probability that $p_1(\tau)$ is zero is at most $\max(d+1,d'+1)/2^n$. Similarly, the probability that $\tau(h'_\tau(T,M) \oplus h'_\tau(T',M') \oplus \alpha) \oplus \beta$ is zero is at most $\max(d+2,d'+2)/2^n$.

Consider $(T,M)$ and $(T',M')$ which are not necessarily distinct. Fix $\alpha,\beta \in \mathbb{F}$ and consider $p(\tau) = \tau(h_\tau(T,M) \oplus h'_\tau(T',M') \oplus \alpha) \oplus \beta$. The coefficient of $\tau$ in

$$h_\tau(T,M) = \tau^d \oplus \mathsf{vecHash2L}_\tau(T_1,\ldots,T_k,M)$$

is

$$L = \mathsf{bin}_8(k+1)||0^8||\mathsf{bin}_{n-16}(\mathsf{len}(M)) \neq \mathbf{0},$$

which is the coefficient of $\tau^2$ in $\tau h_\tau(T,M)$. The coefficient of $\tau^2$ in $\tau h'_\tau(T',M')$ is $\mathbf{0}$ and so the coefficient of $\tau^2$ in $p(\tau)$ is $L \neq \mathbf{0}$. So, $p(\tau)$ is a non-zero polynomial. The degree of $p(\tau)$ is at most $\max(d+1,d'+2) \leq \max(d+2,d'+2)$. This shows the value of $\epsilon_2$. □

## 6 Security

In this section, we provide the formal definitions and the formal security statement for $\mathsf{FAST}$. The detailed security proof is provided in the Appendix.

An adversary $\mathcal{A}$ is a possibly probabilistic algorithm with access to one or more oracles. The output of an adversary is a single bit. The notation $\mathcal{A}^{\mathcal{O}_1,\mathcal{O}_2,\ldots} \Rightarrow 1$ denotes the fact that $\mathcal{A}$ outputs the bit 1 after interacting with the oracles $\mathcal{O}_1,\mathcal{O}_2,\ldots$. The interaction of $\mathcal{A}$ with its oracles is allowed to be adaptive, i.e., the adversary is allowed to choose an oracle and a query to be made to this oracle based on the responses it has received to its previous queries.

The important parameters of an adversary are its running time $\mathfrak{T}$, the number of queries $q$ that it makes to all its oracles and its query complexity $\sigma$. The query complexity is defined later.

The bulk of the actual security analysis will be in the information theoretic sense which in particular means that there is no restriction on the resources of the adversary. For such analysis, it is sufficient to consider the adversary to be a deterministic algorithm.

### 6.1 Pseudo-Random Function

Let $\{\mathbf{F}_K\}_{K\in\mathcal{K}}$ be a family of functions where for each $K \in \mathcal{K}$, $\mathbf{F}_K : \mathcal{D} \to \mathcal{R}$. Here $\mathcal{K}$ is the keyspace, $\mathcal{D}$ is the domain and $\mathcal{R}$ is the range. We require that all three of $\mathcal{K}$, $\mathcal{D}$ and $\mathcal{R}$ are finite non-empty sets.

Let $\rho$ be a function chosen uniformly at random from the set of all functions from $\mathcal{D}$ to $\mathcal{R}$. An equivalent and more convenient view of $\rho$ is the following. For distinct elements $X_1,\ldots,X_q$ from $\mathcal{D}$, the elements $\rho(X_1),\ldots,\rho(X_q)$ are independent and uniformly distributed elements of $\mathcal{R}$.

Roughly speaking, the function family $\{\mathbf{F}_K\}_{K\in\mathcal{K}}$ is said to be a PRF if for a uniform random $K \in \mathcal{K}$, the outputs of $\mathbf{F}_K(\cdot)$ on distinct elements of $\mathcal{D}$ are computationally indistinguishable from the outputs of $\rho$. This is formalised in the following manner. An adversary $\mathcal{A}$ has access to an oracle and can make queries to its oracle in an adaptive manner. We will assume that $\mathcal{A}$ does not repeat a query. The advantage of the adversary $\mathcal{A}$ in breaking the PRF-property of $\{\mathbf{F}_K\}_{K\in\mathcal{K}}$ is defined as follows.

$$\mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathcal{A}) = \Pr\left[K \xleftarrow{\$} \mathcal{K} : \mathcal{A}^{\mathbf{F}_K(\cdot)} \Rightarrow 1\right] - \Pr\left[\mathcal{A}^{\rho(\cdot)} \Rightarrow 1\right]. \tag{31}$$

Suppose $\mathcal{A}$ makes a total of $q$ queries which are $X^{(1)},\ldots,X^{(q)}$. The query complexity of $\mathcal{A}$ is the sum total of the number of $n$-bit blocks in $\mathsf{pad}_n(X^{(s)})$, $s=1,\ldots,q$.

By $\mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathfrak{T}, q, \sigma)$ we denote the maximum of $\mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathcal{A})$ over all adversaries $\mathcal{A}$ which run in time $\mathfrak{T}$, make $q$ queries and have query complexity $\sigma$. The function family $\{\mathbf{F}_K\}_{K\in\mathcal{K}}$ (or, more simply $\mathbf{F}$) is said to be a $(\mathfrak{T}, q, \sigma, \varepsilon)$-PRF if $\mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathfrak{T}, q, \sigma) \leq \varepsilon$.

### 6.2 Tweakable Enciphering Scheme

A tweakable enciphering scheme is a pair $\mathsf{TES} = (\mathsf{TES.Encrypt}, \mathsf{TES.Decrypt})$ where

$$\mathsf{TES.Encrypt}, \mathsf{TES.Decrypt} : \mathcal{K} \times \mathcal{T} \times \mathcal{P} \to \mathcal{P}$$

for finite non-empty sets $\mathcal{K}, \mathcal{T}$ and $\mathcal{P}$. The set $\mathcal{K}$ is called the key space, $\mathcal{T}$ is called the tweak space and $\mathcal{P}$ is called the message/ciphertext space. We write $\mathsf{TES.Encrypt}_K(\cdot,\cdot)$ ($\mathsf{TES.Decrypt}_K(\cdot,\cdot)$) to denote $\mathsf{TES.Encrypt}(K,\cdot,\cdot)$ (resp. $\mathsf{TES.Decrypt}(K,\cdot,\cdot)$). The functions $\mathsf{TES.Encrypt}$ and $\mathsf{TES.Decrypt}$ satisfy the following two properties. For $K \in \mathcal{K}$, $T \in \mathcal{T}$ and $P \in \mathcal{P}$,

1. $\mathsf{TES.Decrypt}_K(T, \mathsf{TES.Encrypt}_K(T, P)) = P$;
2. $\mathsf{len}\left(\mathsf{TES.Encrypt}_K(T, P)\right) = \mathsf{len}(P)$.

The first property states that the encrypt and the decrypt functions are inverses while the second property states that the length of the ciphertext is equal to the length of the plaintext. In other words, $\mathsf{TES.Encrypt}_K(T,\cdot)$ is a length preserving permutation of $\mathcal{P}$.

The notion of security for a TES that we consider is that of indistinguishability from uniform random strings. This implies other notions of security (see [24]). Let $\mathfrak{F}$ be the set of all functions $f$ from $\mathcal{T} \times \mathcal{P}$ to $\mathcal{P}$ such that for any $T \in \mathcal{T}$ and $P \in \mathcal{P}$, $\mathsf{len}(f(T, P)) = \mathsf{len}(P)$. Let $\rho_1$ and $\rho_2$ be two functions chosen independently and uniformly at random from $\mathfrak{F}$.

An adversary $\mathcal{A}$ attacking a TES has access to two oracles which we will call the left and the right oracles. Both the oracles are functions from $\mathfrak{F}$. An input to the left oracle is of the form $(T, P)$ and the response is $C$, while an input to the right oracle is of the form $(T, C)$ and the response is $P$. The adversary $\mathcal{A}$ adaptively queries its oracles possibly interweaving its queries to its left and right oracles. At the end, $\mathcal{A}$ outputs a bit.

We assume that the adversary does not make any pointless query. This means that $\mathcal{A}$ does not repeat a query to any of its oracles; does not query the right oracle with $(T, C)$ if it received $C$ in response to a query $(T, P)$ made to its left oracle; and does not query the left oracle with $(T, P)$ if it received $P$ in response to a query $(T, C)$ made to its right oracle. The advantage of $\mathcal{A}$ in breaking $\mathsf{TES}$ is defined as follows:

$$\begin{aligned}
&\mathbf{Adv}^{\pm\mathrm{rnd}}_{\mathsf{TES}}(\mathcal{A}) \\
&= \Pr\left[K \xleftarrow{\$} \mathcal{K} : \mathcal{A}^{\mathsf{TES.Encrypt}_K(\cdot,\cdot),\mathsf{TES.Decrypt}_K(\cdot,\cdot)} \Rightarrow 1\right] - \Pr\left[\mathcal{A}^{\rho_1(\cdot,\cdot),\rho_2(\cdot,\cdot)} \Rightarrow 1\right].
\end{aligned} \tag{32}$$

Suppose $\mathcal{A}$ makes $q$ queries $(T^{(1)}, X^{(1)}), \ldots, (T^{(q)}, X^{(q)})$ where $T^{(s)} = (T^{(s)}_1, \ldots, T^{(s)}_{k^{(s)}})$ and $X^{(s)}$ is either $P^{(s)}$ or $C^{(s)}$. For $j = 1, \ldots, k^{(s)}$, let $m^{(s)}_j$ be the number of $n$-bit blocks in $\mathsf{pad}_n(T^{(s)}_j)$ and let $m^{(t)}$ be the number of $n$-bit blocks in $\mathsf{pad}_n(X^{(t)})$. We will write $\mathfrak{t}^{(s)}$ to denote $\mathfrak{t}(T^{(s)})$. Let $X^{(s)} = X^{(s)}_1 || X^{(s)}_2 || X^{(s)}_3$ with $\mathsf{len}(X^{(s)}_1) = \mathsf{len}(X^{(s)}_2) = n$ and $\mathsf{len}(X^{(s)}_3) \geq 1$. Then $X^{(s)}_3 \in \mathcal{M}$ and $\mathfrak{l}(X^{(s)}_3) = m^{(s)} - 2$. We will write $\mathfrak{l}^{(s)}$ to denote $\mathfrak{l}(X^{(s)}_3)$.

The tweak query complexity $\theta$, the message query complexity $\omega$ and the total query complexity $\sigma$ are defined as follows.

$$\theta = \sum_{s=1}^{q} \mathfrak{t}(T^{(s)}) = \sum_{s=1}^{q} \mathfrak{t}^{(s)}; \tag{33}$$

$$\omega = \sum_{t=1}^{q} m^{(t)}; \tag{34}$$

$$\sigma = \theta + \omega. \tag{35}$$

By $\mathbf{Adv}^{\pm\mathrm{rnd}}_{\mathsf{TES}}(\mathfrak{T}, q, \theta, \omega)$ we denote the maximum of $\mathbf{Adv}^{\pm\mathrm{rnd}}_{\mathsf{TES}}(\mathcal{A})$ over all adversaries $\mathcal{A}$ which run in time $\mathfrak{T}$, make $q$ queries and have tweak query complexity $\theta$ and message query complexity $\omega$. $\mathsf{TES}$ is said to be $(\mathfrak{T}, q, \theta, \omega, \varepsilon)$-secure if $\mathbf{Adv}^{\pm\mathrm{rnd}}_{\mathsf{TES}}(\mathcal{A}) \leq \varepsilon$ for all $\mathcal{A}$ running in time $\mathfrak{T}$, making a total of $q$ queries with tweak query complexity $\theta$ and message query complexity $\omega$.

### 6.3 Security of FAST

The security statement for FAST is the following.

**Theorem 1.** *Let $n$ be a positive integer; $\mathbb{F} = GF(2^n)$ is represented using some fixed irreducible polynomial of degree $n$ over $GF(2)$; $\{\mathbf{F}_K\}_{K \in \mathcal{K}}$ where for $K \in \mathcal{K}$, $\mathbf{F}_K : \{0,1\}^n \to \{0,1\}^n$; $(h, h')$ is an $(\epsilon_1, \epsilon_2)$-eligible pair of hash functions, where $h, h' : \mathbb{F} \times \mathcal{T} \times \mathcal{M} \to \mathbb{F}$; and $\mathsf{fStr}$ is a fixed $n$-bit string used to build the TES*

$$\mathsf{FAST} = (\mathsf{FAST.Encrypt}, \mathsf{FAST.Decrypt})$$

*given in Table 2. Fix $q$, $\omega \geq q$ to be positive integers and $\theta$ to be a non-negative integer. For all $\mathfrak{T} > 0$,*

$$\mathbf{Adv}^{\pm\mathrm{rnd}}_{\mathsf{FAST}}(\mathfrak{T}, q, \theta, \omega) \leq \mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathfrak{T} + \mathfrak{T}', \omega + 1, \omega + 1) + \Delta(\mathsf{FAST}), \quad \textit{where}$$
$$\Delta(\mathsf{FAST}) = 2\omega \left( \sum_{s=1}^{q} \epsilon_1^{(s)} \right) + 3 \sum_{1 \leq s < t \leq q} \epsilon_2^{(s,t)} + \sum_{s=1}^{q} \epsilon_2^{(s,s)} + \frac{3\omega^2}{2^n}; \tag{36}$$

*$\mathfrak{T}'$ is the time required to answer $q$ queries with tweak query complexity $\theta$ and message query complexity $\omega$ plus some bookkeeping time; and for $1 \leq s, t \leq q$, $\epsilon_1^{(s)} = \epsilon_1(\mathfrak{t}^{(s)}, \mathfrak{l}^{(s)})$, $\epsilon_2^{(s,t)} = \epsilon_2(\mathfrak{t}^{(s)}, \mathfrak{l}^{(s)}, \mathfrak{t}^{(t)}, \mathfrak{l}^{(t)})$.*

We have the following consequences of Theorem 1 for the specific instantiations of FAST.

**Corollary 1.** *Let $m \geq 3$ be a fixed positive integer. Let $q$ and $\sigma \geq q$ be positive integers and $\omega = \sigma - q$. Consider the instantiations $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{Horner}]$ and $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}]$ of FAST. Then for all $\mathfrak{T} > 0$,*

$$\begin{aligned}
&\mathbf{Adv}^{\pm\mathrm{rnd}}_{\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{Horner}]}(\mathfrak{T}, q, q, \omega) \\
&\leq \mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathfrak{T} + \mathfrak{T}', \omega + 1, \omega + 1) + \frac{1}{2^n} \left( 5\omega^2 + \omega + \frac{11\omega q}{2} + 3q^2 + 2q \right).
\end{aligned} \tag{37}$$

*If $m \geq 4$, then for all $\mathfrak{T} > 0$,*

$$\begin{aligned}
&\mathbf{Adv}^{\pm\mathrm{rnd}}_{\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}]}(\mathfrak{T}, q, q, \omega) \\
&\leq \mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathfrak{T} + \mathfrak{T}', \omega + 1, \omega + 1) + \frac{1}{2^n} \left( 6\omega q + \frac{9q^2}{2} + 3q + 3\omega^2 + \mathfrak{d}(m-1) \left( 2\omega q + \frac{3q^2}{2} + q \right) \right) \\
&\leq \mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathfrak{T} + \mathfrak{T}', \omega + 1, \omega + 1) + \frac{1}{2^n} \left( 7\omega^2 + 3\omega q + 2\omega \right).
\end{aligned} \tag{38}$$

*Further, if $m \geq 4$ is a power of two, then for all $\mathfrak{T} > 0$,*

$$\begin{aligned}
&\mathbf{Adv}^{\pm\mathrm{rnd}}_{\mathsf{FAST}[\mathsf{Fx}_m,\mathsf{BRW}]}(\mathfrak{T}, q, q, \omega) \\
&\leq \mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathfrak{T} + \mathfrak{T}', \omega + 1, \omega + 1) + \frac{1}{2^n}\left(5\omega^2 + \omega + \frac{11\omega q}{2} + 3q^2 + 2q\right). \qquad (39)
\end{aligned}$$

*Proof.* Let $\Delta(\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{Horner}])$ and $\Delta(\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}])$ denote the expressions for $\Delta$ in (36) when $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{Horner}]$ and $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}]$ are respectively used.

In the setting of $\mathsf{Fx}_m$, each query consists of a single $n$-bit block for the tweak and $m$ $n$-bit blocks for the plaintext or the ciphertext, i.e., $m^{(s)} = m$ for all $1 \leq s \leq q$. So, $\theta = q$, $\omega = qm$ and $\sigma = \theta + \omega = q(m + 1)$. Also, $m \geq 3$ and $q \geq 1$ imply that $q < \omega$ and $m \leq \omega$.

From Proposition 2, we have $\epsilon_1^{(s)} = \epsilon_2^{(s,t)} = (m + 2)/2^n$ for $1 \leq s,t \leq q$. Using these, $\Delta(\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{Horner}])$ can be upper bounded as given in (37).

From Proposition 3, we have $\epsilon_1^{(s)} = \epsilon_2^{(s,t)} = (3 + \mathfrak{d}(m - 1))/2^n$ for $1 \leq s,t \leq q$. Using these, $\Delta(\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}])$ achieves the first bound and using $\mathfrak{d}(m - 1) \leq 2(m - 1) - 1$ yields the second bound.

In the case where $m \geq 4$ is a power of two, then $\mathfrak{d}(m-1) = m-1$. Also, from Proposition 3, we have $\epsilon_1^{(s)} = \epsilon_2^{(s,t)} = (m + 2)/2^n$ and so $\Delta(\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}]) = \Delta(\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{Horner}])$ which shows the required statement. □

**Remarks:**

1. As mentioned above, the query complexity $\sigma$ is equal to the sum of the tweak query complexity $\theta$ and the message query complexity $\omega$, i.e., $\sigma = \theta + \omega$. For the setting of $\mathsf{Fx}$, $\theta = q$ so that $\sigma = q + \omega$. Using this, the bounds in (37), (38) and (39) are all of the form $\mathfrak{c}\sigma^2$ for some small constant $\mathfrak{c}$. This is the typical form of the bound that is obtained for other constructions in the literature. So, FAST does not suffer any security loss compared to previous constructions.
2. Consider the application of FAST to disk encryption of 4096 byte sectors. Then $m = 2^8$. Using $\omega = mq$ the bounds for $\Delta(\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{Horner}])$ $\Delta(\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}])$ given by (37) and (39) respectively both reduce to the following form.

$$\frac{1}{2^n}\left(q^2(5 \cdot 2^{16} + 1441) + 258q\right) < \frac{1}{2^n}\left(5\sigma^2 + 2\sigma\right).$$

**Corollary 2.** *Let $q$, $\omega \geq q$ be positive integers and $\theta$ be a non-negative integer. Consider the instantiations* $\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \mathsf{vecHorner}]$ *and* $\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \eta, \mathsf{vecHash2L}]$ *of* FAST*. Then for all $\mathfrak{T} > 0$,*

$$\begin{aligned}
\mathbf{Adv}^{\pm\mathrm{rnd}}_{\mathsf{FAST}[\mathsf{Gn},\mathfrak{k},\mathsf{vecHorner}]}(\mathfrak{T}, q, \theta, \omega) &\leq \mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathfrak{T} + \mathfrak{T}', \omega + 1, \omega + 1) \\
&+ \frac{5\omega^2 + 2\omega\theta + \omega + \theta}{2^n} + \frac{3q(\theta + \omega) + (\mathfrak{k} + 2)((2\omega + 1)q + 3q^2) + 6q^2}{2^n}; \qquad (40) \\
\mathbf{Adv}^{\pm\mathrm{rnd}}_{\mathsf{FAST}[\mathsf{Gn},\mathfrak{k},\eta,\mathsf{vecHash2L}]}(\mathfrak{T}, q, \theta, \omega) &\leq \mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathfrak{T} + \mathfrak{T}', \omega + 1, \omega + 1) \\
&+ \frac{(3 + 2(\eta + 1)/\eta)\omega^2 + (2(\eta + 1)/\eta)\omega\theta + ((\eta + 1)/\eta)(\omega + \theta)}{2^n} \\
&+ \frac{3q((\eta + 1)/\eta)(\theta + \omega) + ((2\omega + 1)q + 3q^2)(\mathfrak{k} + 1)(\eta + 2) + 3q(2\omega + 1) + 9q^2}{2^n}. \qquad (41)
\end{aligned}$$

*Proof.* Let $\Delta(\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \mathsf{vecHorner}])$ and $\Delta(\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \eta, \mathsf{vecHash2L}])$ denote the expressions for $\Delta$ in (36) when $\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \mathsf{vecHorner}]$ and $\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \eta, \mathsf{vecHash2L}]$ are respectively used.

First consider $\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \mathsf{vecHorner}]$. From Proposition 4, $\epsilon_1^{(s)} = (\mathfrak{t}^{(s)} + \mathfrak{l}^{(s)} + \mathfrak{k} + 4)/2^n$ and $\epsilon_2^{(s,t)} = (\max(\mathfrak{t}^{(s)} + \mathfrak{l}^{(s)}, \mathfrak{t}^{(t)} + \mathfrak{l}^{(t)}) + \mathfrak{k} + 4)/2^n$. So, $\epsilon_2^{(s,s)} = \epsilon_1^{(s)}$. We have

$$\sum_{s=1}^{q} \epsilon_1^{(s)} = \sum_{s=1}^{q} \frac{\mathfrak{t}^{(s)} + \mathfrak{l}^{(s)} + \mathfrak{k} + 4}{2^n} = \frac{\theta + \omega}{2^n} + \frac{q(\mathfrak{k} + 2)}{2^n};$$

$$\sum_{1 \le s < t \le q} \epsilon_2^{(s,t)} \le \sum_{1 \le s < t \le q} \frac{\max(\mathfrak{t}^{(s)} + \mathfrak{l}^{(s)}, \mathfrak{t}^{(t)} + \mathfrak{l}^{(t)}) + \mathfrak{k} + 4}{2^n} \le \frac{q(\theta + \omega)}{2^n} + \frac{q^2(\mathfrak{k} + 4)}{2^n}.$$

Using these in Theorem 1, we obtain

$$\Delta(\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \mathsf{vecHorner}]) \le \frac{5\omega^2 + 2\omega\theta + \omega + \theta}{2^n} + \frac{3q(\theta + \omega) + (\mathfrak{k} + 2)((2\omega + 1)q + 3q^2) + 6q^2}{2^n}.$$

Now consider $\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \eta, \mathsf{vecHash2L}]$. From Proposition 5,

$$\epsilon_1^{(s)} = \frac{((\eta + 1)/\eta)(\mathfrak{t}^{(s)} + \mathfrak{l}^{(s)}) + (\mathfrak{k} + 1)(\eta + 2) + 3}{2^n};$$

$$\epsilon_2^{(s,t)} = \frac{((\eta + 1)/\eta) \max(\mathfrak{t}^{(s)} + \mathfrak{l}^{(s)}, \mathfrak{t}^{(t)} + \mathfrak{l}^{(t)}) + (\mathfrak{k} + 1)(\eta + 2) + 3}{2^n}.$$

So, $\epsilon_2^{(s,s)} = \epsilon_1^{(s)}$. We have

$$\sum_{s=1}^{q} \epsilon_1^{(s)} \le \frac{((\eta + 1)/\eta)(\theta + \omega) + q(\mathfrak{k} + 1)(\eta + 2) + 3q}{2^n};$$

$$\sum_{1 \le s < t \le q} \epsilon_2^{(s,t)} \le \frac{q((\eta + 1)/\eta)(\theta + \omega) + q^2(\mathfrak{k} + 1)(\eta + 2) + 3q^2}{2^n};$$

Using these in Theorem 1, we obtain

$$\begin{aligned} &\Delta(\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \eta, \mathsf{vecHash2L}]) \\ &\le \frac{(3 + 2(\eta + 1)/\eta)\omega^2 + (2(\eta + 1)/\eta)\omega\theta + ((\eta + 1)/\eta)(\omega + \theta)}{2^n} \\ &\quad + \frac{3q((\eta + 1)/\eta)(\theta + \omega) + ((2\omega + 1)q + 3q^2)(\mathfrak{k} + 1)(\eta + 2) + 3q(2\omega + 1) + 9q^2}{2^n}. \end{aligned}$$

□

**Remarks:**

1. Recall that $\mathfrak{k}$ is the number of components in the tweak while $\eta$ is the number of blocks in a BRW super-block. So, $\mathfrak{k}$ and $\eta$ are constants, i.e., they are independent of $n$.
2. The query complexity $\sigma$ is equal to the sum of the tweak query complexity $\theta$ and the message query complexity $\omega$, i.e., $\sigma = \theta + \omega$. Using this, it can be seen that the bounds on $\Delta(\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \mathsf{vecHorner}])$ and $\Delta(\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \eta, \mathsf{vecHash2L}])$ given by (40) and (41) respectively are of the form $\mathfrak{c}\sigma^2$ for some constant $\mathfrak{c}$. We provide illustrations below. As mentioned earlier, this is the typical form of the security bound for earlier constructions and so FAST does not have any security loss compared to the known constructions.

3. In our implementations, we take $\eta = 31$ and we use the bounds $(\eta+1)/\eta < 2$, $2(\eta+1)/\eta < 3$ and $3(\eta+1)/\eta < 4$. The value of $\mathfrak{k}$ and the lengths of the tweaks depend on the application. We provide two illustrations.
*Case* $\mathfrak{k} = 1$ *and each tweak is an* $n$*-bit string:* The bounds for $\Delta(\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \mathsf{vecHorner}])$ and $\Delta(\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \eta, \mathsf{vecHash2L}])$ can be shown to be respectively less than
$$\frac{5\sigma^2 + \sigma q + 12q^2 + \sigma + 3q}{2^n} \text{ and } \frac{6\sigma^2 + 135\sigma q + 72q^2 + 2\sigma + 69q}{2^n}.$$
*Case* $\mathfrak{k} = 8$ *where the components of a tweak can have variable lengths:*
The bounds for $\Delta(\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \mathsf{vecHorner}])$ and $\Delta(\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \eta, \mathsf{vecHash2L}])$ can be shown to be respectively less than
$$\frac{1}{2^n}\left(5\omega^2 + 2\omega\theta + \omega + \theta\right) + \frac{1}{2^n}\left(3q(\omega+\theta) + 10((2\omega+1)q + 3q^2) + 6q^2\right) \quad (42)$$
$$\text{and } \frac{1}{2^n}\left(6\omega^2 + 3\omega\theta + 2(\omega+\theta) + 6q(\theta+\omega) + 300q(2\omega+1) + 900q^2\right). \quad (43)$$
The expressions in (42) and (43) are determined by the values of $q$, $\omega$ and $\theta$. These quantities are in turn determined by the manner in which the adversary makes its queries. Since the adversary can make queries of varying lengths, it is not possible to obtain further simplifications of the expressions given in (42) and (43).

## 7 Comparison

This section provides a comparison of the design features of FAST with previously proposed TESs.

Several block cipher based TES constructions essentially use a layer of encryption using a mode of operation of the block cipher sandwiched between two layers of hashing. Differences arise in the choice of the mode of operation, the choice of the hash functions and other details.

1. For the mode of operation, the electronic codebook mode (ECB) has been suggested in TET [22] and HEH [39] while some form of the counter mode of operation has been used in XCB [30, 31], HCTR [41] and HCH [16]. In this paper, we use the counter mode of operation as described in the scheme HCTR [41].
2. For the hash functions, XCB, HCTR and HCH essentially use polynomial hashing based on Horner's rule. The cost of hashing in TET is higher. BRW based hashing has been suggested for HEH and implemented in hardware for fixed length messages [11].

All of the above mentioned TESs require both the encryption and the decryption functions of the block cipher. The possibility of using only the encryption function of a block cipher to build a TES has been reported [40] and for the convenience of description let us denote this scheme by TESX. The present work is based upon the idea behind TESX. In terms of similarity, both TESX and the present work use a Feistel layer on the first two blocks and a counter mode of operation on the rest of the blocks. There are several differences in the two constructions.

1. For TESX, inside the Feistel layer, the hash function $h$ is used to process $A_1$ and $B_2$. The key for this hash function is $\tau'$ which is independent of the key $\tau$ which is used for the hash function outside the Feistel layer. In contrast, FAST is organised such that the Feistel layer consists of only two encryption rounds and the entire hashing using a single key $\tau$ takes place outside the Feistel layer. In summary, TESX uses two hash keys while FAST uses a single hash key.

2. For $\mathsf{TESX}$, the hash of $P_3$ is masked with $\beta_1$ and XORed to both $P_1$ and $P_2$ and the hash of $C_3$ is masked with $\beta_2$ and XORed to both $C_1$ and $C_2$. $\mathsf{FAST}$ does away with the maskings with $\beta_1$ and $\beta_2$; the hash of $P_3$ is XORed to only $P_1$; and the hash of $C_3$ is XORed to only $C_2$.
3. For $\mathsf{TESX}$, the seed to the counter mode is generated as $A_1 \oplus P_2 \oplus C_1 \oplus B_2$. $\mathsf{FAST}$ generates the seed to the counter mode as $F_1 \oplus F_2$.
4. The counter mode suggested in $\mathsf{TESX}$ requires a doubling operation for each block. The counter mode used in $\mathsf{FAST}$ is given by (1) and is based on HCTR [41]; this counter mode does not require the doubling operation.

Some of the above differences such as reducing the hash key size and avoiding doubling operations are important from a practical point of view while the others are simplifications obtained by removing unnecessary operations. Keeping the similarities and the differences in mind, it would be proper to view the present work as a fine-tuned version of $\mathsf{TESX}$. This fine tuning is required from an engineering point of view where the goal is to obtain an efficient and clean design. More importantly, this work presents detailed implementations in both software and hardware and thereby actually demonstrates the advantages of the new proposal in comparison to the previous works.

Another class of block cipher based TESs such as CMC [24], EME [25, 21] (the scheme EME2 is essentially EME* [21]), AEZ [26] and FMix [7] essentially uses several layers of encryption using a mode of operation of a block cipher. CMC, EME and FMix use two layers of encryption whereas AEZ uses three layers of encryption. These TESs do not use any hash function. Out of these CMC and FMix are sequential while EME and AEZ are parallelisable. AEZ and FMix require only the encryption function of the underlying block cipher whereas CMC and EME require both the encryption and the decryption functions of the block cipher. The costs of encryption for CMC, EME and FMix are roughly two block cipher calls per block of the message and for AEZ the cost is roughly two-and-half block cipher calls per block of the message. CMC and FMix do not use any doubling operation while both EME and AEZ use doubling operations. Since AEZ and EME are parallelisable while CMC and FMix are not, any reasonable implementations of AEZ and EME will be faster than both CMC and FMix. Later we provide implemetation results which show the superiority of $\mathsf{FAST}$ in comparison to both AEZ and EME in both software and hardware and hence also imply the superiority of $\mathsf{FAST}$ over CMC and FMix.

Table 4 compares $\mathsf{FAST}$ to previously proposed schemes. From the viewpoint of efficiency, it is preferable to have schemes which are parallelisable. This would eliminate CMC and FMix from the comparison. Further, again from an efficiency point of view it would be preferable to have schemes which use only the encryption module of a block cipher. This restricts the comparison to AEZ and $\mathsf{TESX}$. As explained above, the current construction is a fine-tuned version of $\mathsf{TESX}$ and Table 4 shows the comparative advantage in terms of operation counts and key size. The inherent simplification of the design of $\mathsf{FAST}$ over that of $\mathsf{TESX}$ is not captured by these parameters. Since the design approaches of $\mathsf{FAST}$ and AEZ are different, the comparison between $\mathsf{FAST}$ and AEZ cannot be determined only from the operation counts.

Among the various schemes that have been proposed, only XCB and EME2 (which is essentially EME* [21]) have been standardised. Further, AEZ is a more recent proposal and has received a fair amount of attention as part of the CAESAR[5] competition. So, it is important to provide more detailed comparison to XCB, EME2 and AEZ. Below we provide details of the implementations of $\mathsf{FAST}$ in both software and hardware and the performance results of these implementations in comparison to those of XCB, EME2 and AEZ. For the purpose of such comparison, we have made efficient implementations of XCB, EME2 and AEZ in both software and hardware. In Appendix B

[5] `https://competitions.cr.yp.to/caesar.html`

Table 4: Comparison of different tweakable enciphering schemes. The block size is $n$ bits, the tweak is a single $n$-bit block and the number of blocks $m \geq 3$ in the message is fixed. [BC] denotes the number of block cipher calls; [M] denotes the number of field multiplications; [D] denotes the number of doubling ('multiplication by $\alpha$') operations; [BCK] denotes the number of block cipher keys; and [HK] denotes the number of blocks in the hash key.

| type | scheme | [BC] | [M] | [D] | [BCK] | [HK] | dec module | parallel |
|---|---|---|---|---|---|---|---|---|
| enc-mix-enc | CMC [24] | $2m+1$ | – | – | 1 | – | reqd | no |
| | EME2* [21] | $2m+1+m/n$ | – | 2 | 1 | 2 | reqd | yes |
| | AEZ [26] | $(5m+4)/2$ | – | $(m-2)/4$ | 1 | 2 | not reqd | yes |
| | FMix [7] | $2m+1$ | – | – | 1 | – | not reqd | no |
| hash-enc-hash | XCB [30] | $m+1$ | $2(m+3)$ | – | 3 | 2 | reqd | yes |
| | HCTR [41] | $m$ | $2(m+1)$ | – | 1 | 1 | reqd | yes |
| | HCHfp [16] | $m+2$ | $2(m-1)$ | – | 1 | 1 | reqd | yes |
| | TET [22] | $m+1$ | $2m$ | $2(m-1)$ | 2 | 3 | reqd | yes |
| | HEH-BRW[39] | $m+1$ | $2+2\lfloor(m-1)/2\rfloor$ | $2(m-1)$ | 1 | 1 | reqd | yes |
| | TESX with BRW [40] | $m+1$ | $4+2\lfloor(m-1)/2\rfloor$ | $2(m-1)$ | 1 | 2 | not reqd | yes |
| | FAST[$\mathsf{Fx}_m$, Horner] | $m+1$ | $2m+1$ | – | 1 | – | not reqd | yes |
| | FAST[$\mathsf{Fx}_m$, BRW] | $m+1$ | $2+2\lfloor(m-1)/2\rfloor$ | – | 1 | – | not reqd | yes |

we provide a brief overview of the software and hardware implementation of AEZ. This is of some independent interest.

## 8 Software Implementation

In this section, we describe our implementations of the various instantiations of FAST in software. For the implementation, we set $n = 128$ and so $\mathbb{F} = GF(2^{128})$. The implementation of the PRF $\mathbf{F}$ was done using the encryption function of AES.

Our target platforms for software implementation were modern Intel processors which support the AES-NI instructions which includes the 64-bit binary polynomial multiplication. The relevant Intel instructions for the two main tasks are the following.

**Computation of AES:** The relevant instructions are `aeskeygenassist` (for round key generation); `aesenc` (for one round of AES encryption) and `aesenclast` (for the last round of AES encryption). There are additional instructions for AES decryption. We do not mention these, since we do not require the AES decryption module.

**Computation of polynomial multiplication:** The relevant instruction is `pclmulqdq`. This instruction takes as input two 64-bit unsigned integers representing two polynomials each of maximum degree 63 over $GF(2)$ and returns a 128-bit quantity which represents the product of these two polynomials over $GF(2)$.

For software implementation, the two relevant parameters are latency and throughput. These are defined[6] as follows: "Latency is the number of processor clocks it takes for an instruction to have its data available for use by another instruction. Throughput is the number of processor clocks it takes for an instruction to execute or perform its calculations." On Skylake the latency/throughput figures of `aesenc`, `aesenclast` and `pclmulqdq` are 4/1.

[6] `https://software.intel.com/en-us/articles/measuring-instruction-latency-and-throughput`

### 8.1 Implementation of the Hash Functions

The several variants of the hash functions used in this work are all based on finite field computations over $\mathbb{F} = GF(2^n)$. For the implementation, we choose $n = 128$. Addition over this field is a simple XOR operation of 128-bit data types. Multiplication, on the other hand, is more involved.

We consider the field $\mathbb{F}$ to be represented using the irreducible polynomial $\psi(x) = x^{128} \oplus x^7 \oplus x^2 \oplus x \oplus 1$ over $GF(2)$. The elements of $\mathbb{F}$ are represented using polynomials over $GF(2)$ of degrees at most 127. Let $a(x)$ and $b(x)$ be two such polynomials. The multiplication of $a(x)$ and $b(x)$ in $\mathbb{F}$ consists of the following two operations. Compute the polynomial multiplication of $a(x)$ and $b(x)$ over $GF(2)$ and let $c(x)$ be the result. Then $c(x)$ is a polynomial over $GF(2)$ of degree at most 254. The product of $a(x)$ and $b(x)$ over $\mathbb{F}$ is $c(x) \bmod \psi(x)$. The above computation consists of two distinct steps, namely polynomial multiplication followed by reduction.

**Polynomial multiplication:** The instruction `pclmulqdq` multiplies two polynomials over $GF(2)$ of degrees at most 63 each and returns a polynomial of degree at most 126. This is a 64-bit polynomial multiplication over $GF(2)$. Our requirement is a 128-bit polynomial multiplication over $GF(2)$. Using the direct schoolbook method, a 128-bit polynomial multiplication can be computed using four 64-bit polynomial multiplications and hence using four `pclmulqdq` calls. Karatsuba's algorithm, on the other hand, allows the computation of a 128-bit polynomial multiplication using three `pclmulqdq` calls at the cost of a few extra XOR operations. Due to the low latency of `pclmulqdq` on Skylake processors, it turns out that schoolbook is faster than Karatsuba. This has been reported by Gueron[7] and we also observed this in our experiments. So, we opted to implement 128-bit polynomial multiplication using the schoolbook method.

**Reduction:** Efficient computation of $c(x) \bmod \psi(x)$ has been discussed earlier [19]. It was shown that this operation can be efficiently computed using two `pclmulqdq` instructions along with a few other shifts and xors. A more detailed description of this procedure can also be found elsewhere [9]. We implemented reduction using this method.

Horner: As mentioned earlier, $\mathsf{Horner}_\tau(X_1, \ldots, X_m)$ can be computed using $m-1$ multiplications in $\mathbb{F}$. Each multiplication consists of a polynomial multiplication followed by a reduction. Doing this directly, would lead to a count of $m-1$ polynomial multiplications and $m-1$ reductions.

The efficiency can be improved by using a delayed (or lazy) reduction strategy. Let $i > 1$ be a positive integer and suppose the powers $1, \tau, \tau^2, \ldots, \tau^{i-1}, \tau^i$ are available (i.e., the powers $\tau^2, \ldots, \tau^{i-1}, \tau^i$ have been pre-computed and stored). The expression $X_1\tau^{i-1} + \cdots + X_{i-1}\tau + X_i$ over $\mathbb{F}$ can be computed using $i-1$ polynomial multiplications followed by a single reduction. Extension to handle arbitrary number of blocks is easy. For simplicity, assume that $i|m$ and $\lambda = m/i$. The $m$ blocks are divided into $\lambda$ groups of $i$ blocks each. Each group of $i$ blocks is processed and suppose the outputs are $Y_1, Y_2, \ldots, Y_\lambda$. Then $\mathsf{Horner}_\tau(X_1, \ldots, X_m) = \tau^i(\cdots \tau^i(\tau^i Y_1 \oplus Y_2) \oplus \cdots \oplus Y_{\lambda-1}) \oplus Y_\lambda$. Processing of a single such group of $i$ blocks requires $i-1$ polynomial multiplications and a single reduction plus a multiplication by $\tau^i$. Note that the computation of $\tau^i Y_1 \oplus Y_2$ is done by performing the polynomial multiplication of $\tau^i$ and $Y_1$, computing $Y_2$ without the final reduction, adding the two results and then performing a reduction. Further, this strategy is also carried out for the intermediate computations. So, processing a group of $i$ blocks requires $i$ polynomial multiplications and a single reduction except for the last group. In the case where $i$ does not divide $m$, it is easy to modify this strategy to handle this case. We have implemented this strategy for $i = 8$ (for use

[7] `https://github.com/Shay-Gueron/AES-GCM-SIV/`

in Horner and vecHorner) and $i = 9$ (for use in vecHash2L). This strategy of delayed reduction has been earlier used [20] in the context of evaluation of POLYVAL which is a variant of Horner.

There is another technique which can result in efficiency improvement. The sequence $X_1, \ldots, X_m$ is decimated into $j$ subsequences $X_1, X_{j+1}, \ldots$; $X_2, X_{j+2}, \ldots$; $\ldots$; $X_j, X_{2j}, \ldots$; each subsequence is computed as a polynomial in $\tau^j$ and then the results are combined together to obtain the final result. This is a well known technique and we provide further details as part of the discussion on hardware implementation. The advantage of this technique is that the $j$ sub-sequences can be computed in parallel. This is a distinct advantage in hardware while in software the ability to batch $j$ independent multiplications allows the processor to efficiently pre-fetch and pipeline the corresponding operations. We have experimented with $j = 1$, 2 and 3 and later we report timing results for $j = 3$. There are cases, however, where $j = 1$ provides slightly better performance than $j = 3$.

**vecHorner:** The computation of vecHorner essentially boils down to Horner computation on several different blocks. The implementation of Horner is extended to implement the computation of vecHorner.

**BRW:** The implementation of BRW arises as part of the implementation of $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}]$. For this implementation, we chose $m = 256$ and $n = 128$ (corresponding to a 4096-byte disk sector to be encrypted using AES). With $m = 256$, BRW is invoked on $m-2+1 = 255$ (the first two message blocks are not hashed while the single tweak block is hashed) blocks. The implementation of BRW on 255 blocks has been done in the following manner. Write

$$\mathsf{BRW}_\tau(X_1, \ldots, X_{255}) = \mathsf{BRW}_\tau(X_1, \ldots, X_{127})(X_{128} \oplus \tau^{128}) \oplus \mathsf{BRW}_\tau(X_{129}, \ldots, X_{255}).$$

This shows that the 255-block BRW computation can be broken down into 2 127-block BRW computations. Continuing, we break up the 255-block BRW computation into 8 31-block BRW computations. A completely loop unrolled 31-block BRW computation can be implemented using 15 polynomial multiplications and 8 reductions [9]. We use this implementation of 31-block BRW computation to build the 255-block BRW computation. This strategy requires 127 polynomial multiplications and 71 reductions. Following the delayed reduction strategy for BRW computation [9], it is possible to have a completely loop unrolled 255-block BRW computation requiring 127 polynomial multiplications and 64 reductions. The code for such a loop unrolled implementation would be quite complex and could lead to a substantial performance penalty. This is the reason why we have chosen to build the 255-block BRW computation from the (loop unrolled) implementation of the 31-block BRW computation.

**vecHash2L:** The hash function vecHash2L is parameterised by two quantities, namely the block size $n$ and the super-block size $\eta$. The use of AES fixes $n$ to be 128. We have taken $\eta = 31$. This requires the implementations of 31-block BRW and also $i$-block BRW for $i = 1, \ldots, 30$ to tackle the last super-block which can possibly have less than 31 blocks. As mentioned earlier, an implementation of Hash2L for $n = 128$ and $\eta = 31$ was reported earlier [9], but, the implementation of vecHash2L was not considered in that work. The computation of vecHash2L can be conceptually seen as 31-block BRW computations whose outputs are combined using Horner. Additionally, after each component, the length block is processed. As discussed above, the computation of Horner can be improved by using the delayed reduction strategy and the decimation technique. We have experimented with various combinations and later we report the results for 3-decimated implementations with and without the delayed reduction strategy.

### 8.2 Implementation of FAST

We have described several variants of FAST. Software implementations of these variants are built from the implementation of AES and the implementations of the various hash functions. The AES based parts consist of the Ctr mode and the Feistel layer while the hash functions are built from either Horner or BRW in case of the fixed length setting and are built from either vecHorner or vecHash2L in the general setting. We describe these aspects below.

**Key schedule generation:** All versions of FAST use a single key $K$ which is the key to the underlying PRF $\mathbf{F}_K$. Instantiating $\mathbf{F}_K$ with the encryption function of AES requires generating the round keys. This is a one-time activity and is done using the instruction `aeskeygenassist`. The generated round keys are stored and used in both the Ctr mode and the Feistel layer.

**Ctr:** The Ctr mode defined in (1) requires a PRF $\mathbf{F}$ which is implemented using the encryption function of AES. Each invocation of the encryption function can be implemented using `aesenc` followed by `aesenclast`. The invocations of `aesenc` can be speeded up using an interleaving of multiple AES invocations. The AES encryption calls in the Ctr mode are fully parallelisable. Let $i \geq 1$ be a positive integer. The computations of the AES calls in the Ctr mode are done in batches of $i$ calls each. The inputs to one batch of $i$ encryptions are prepared; then the first rounds of AES encryptions of this batch of $i$ encryptions are computed using $i$ calls to `aesenc`; this is followed by the second rounds of this batch of $i$ encryptions again using `aesenc` and so on. This ensures that the second round of any AES encryption does not have to wait to obtain the output of the first round. This interleaved strategy leads to substantial speed-up over computing the complete AES encryptions one after another. In our implementation, we have used $i = 8$ which follows the earlier work by Gueron[8].

**Feistel:** The Feistel layer has two calls to AES encryptions. These calls are not parallelisable. So, these calls are implemented using a sequence of `aesenc` followed by a single call to `aesenclast` to perform the computation of a single AES encryption. The second encryption call in the Feistel layer can be executed in parallel with the first encryption call of the Ctr. We do not use this in the software implementation, though this aspect is exploited in the hardware implementation.

**Hash key generation:** The hash key $\tau$ is obtained by applying $\mathbf{F}_K$ to fStr. This is a one-time operation and the value of $\tau$ does not change during the life-time of $K$. So, it is possible to generate $\tau$ once and store it securely along with $K$. More generally, it is also possible to use a uniform random $\tau$ as the hash key instead of generating it by applying $\mathbf{F}_K$ to fStr. This will not affect the security analysis, but, will increase the key storage requirement. Alternatively, it is possible to generate $\tau$ once per session. The cost of generating $\tau$ is amortised over all the encryptions/decryptions per session and hence is negligible. Timing results provided later include the time for generating $\tau$.

**FAST[$\mathsf{Fx}_m$, Horner]:** In the setting of $\mathsf{Fx}_m$, tweaks consist of a single $n$-bit block while plaintexts and ciphertexts consist of $m$ $n$-bit blocks. In our implementation, we have taken $m = 256$ so that the total number of bytes in a plaintext/ciphertext is 4096. As mentioned earlier, this corresponds to the size of a modern disk sector. In this case, $P_3 = (P_{3,1}, \ldots, P_{3,m-2})$ consists of $(m-2)$ $n$-bit blocks and the hash function $\mathsf{Horner}_\tau(1, P_{3,1}, \ldots, P_{3,m-2}, T)$ needs to be computed. An implementation of Horner as mentioned above is used. This requires a total of 255 polynomial multiplications and a

[8] Interleaving of 8 AES encryptions has been called a sweat point in `https://crypto.stanford.edu/RealWorldCrypto/slides/gueron.pdf`

total of 32 reductions. Counting a single polynomial multiplication as 4 `pclmulqdq` and a reduction as 2 `pclumulqdq`, the total number of `pclmulqdq` calls required is 1084.

$\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}]$**:** As above, we take $m = 256$. The requirement is to compute

$$\mathsf{BRW}_\tau(P_{3,1}, \ldots, P_{3,m-2}, T).$$

This is done as described above which requires 127 polynomial multiplications and 71 reductions. The total number of `pclmulqdq` calls required is 650. This is 434 calls lesser than that required for computing $\mathsf{Horner}_\tau(1, P_{3,1}, \ldots, P_{3,m-2}, T)$. So, one would expect $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}]$ to be faster than $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{Horner}]$. Our implementation shows a speed-up, but, not as much as one might expect from the counts of the `pclmulqdq` calls. Indeed instruction cache and pipelining are rather complicated issues and precise information about these issues for Intel processors are not easily available[9]. So, it is possible that the code for BRW that we have developed can be tuned further to obtain speed improvements.

$\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \mathsf{vecHorner}]$**:** This requires the implementation of the hash function vecHorner which is an easy extension of the implementation of Horner.

$\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \eta, \mathsf{vecHash2L}]$**:** This requires the implementation of the hash function vecHash2L. The implementation of this hash function has been discussed above.

### 8.3 Timing Results

In this section, we provide timing results for the software implementations of all the four variants of FAST. The corresponding code is available at `https://github.com/sebatighosh/FAST`. The timing results for FAST are for both the settings of Fx and Gn.

In the setting of Fx, messages are 4096 bytes long, i.e., each message consists of 256 128-bit blocks and the tweak is a single 128-bit block. The timing results are shown in Table 5 along with the timing results for XCB, EME2 and AEZ.

In the setting of Gn, timing measurements are separately reported for messages of lengths 512, 1024, 4096 and 8192 bytes. For tweaks, the number of components has been considered to be 2, 3 and 4 and the sum of the lengths of the components of the tweaks has been taken to be 1024 bytes: For tweaks with 2 components, each component has length 512 bytes; for tweaks with 3 components, the 3 components have lengths 336, 336 and 352 bytes; whereas for tweaks with 4 components, each component has length 256 bytes. Two columns of measurements are shown for $\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \eta, \mathsf{vecHash2L}]$. The column with the heading 'delayed' reports measurements for the case where the Horner layer in vecHash2L has been implemented using the delayed reduction strategy while the column with the heading 'normal' reports measurements for the case where the Horner layer in vecHash2L has been implemented without using the delayed reduction strategy. The timing results are shown in Table 6.

The timing measurements were taken on two platforms.

- **Skylake:** The processor was Intel Core i7-6500U @ 2.5GHz. The operating system was 64-bit Ubuntu 14.04 LTS and the C codes were complied using GCC version 5.5.0.
- **Kabylake:** The processor was Intel Core i7-7700 @ 3.6GHz. The operating system was 64-bit Ubuntu 18.04 LTS and the C codes were complied using GCC version 7.3.0.

[9] `https://blog.cr.yp.to/20140517-insns.html`

For the setting of $\mathsf{Fx}$, we have carried out efficient implementations of XCB, EME2 and AEZ. In the implementation of AEZ, for the underlying block cipher, the full AES has been used unlike the reduced round versions considered earlier [26]. XCB uses hashing based on Horner's rule and we have used the same delayed reduction strategy in the implementation of this hashing as we did in the implementation of the hash function for $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{Horner}]$. The efficient software implementations of XCB, EME2 and AEZ are of independent interest. We provide a brief description of the software implementation of AEZ in Appendix B.1. Timing results from the setting of $\mathsf{Fx}$ show that all three of XCB, EME2 and AEZ are slower than $\mathsf{FAST}$. Consequently, we do not compare $\mathsf{FAST}$ to XCB, EME2 and AEZ in the setting of $\mathsf{Gn}$.

Based on Tables 5 and 6, we make the following observations.

1. In the setting of $\mathsf{Fx}$, $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{Horner}]$ and $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{BRW}]$ are faster than all three of XCB, EME2 and AEZ with $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{BRW}]$ being faster than $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{Horner}]$.
2. In the setting of $\mathsf{Gn}$ for $\mathsf{vecHorner}$, the speed decreases with increase in message length while for $\mathsf{vecHash2L}$ the speed increases with increase in message length. In both cases, for the same message length, the speed mostly does not vary much with increase in the number of components in the tweak. In the case of $\mathsf{vecHash2L}$, using the delayed reduction strategy for implementing the Horner layer results in improved speed than an implementation without using delayed reduction. Overall, on Kabylake $\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, 31, \mathsf{vecHash2L}]$ is faster than $\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \mathsf{vecHorner}]$ while on Skylake $\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, 31, \mathsf{vecHash2L}]$ is faster than $\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \mathsf{vecHorner}]$ for longer messages.

Table 5: Comparison of the cycles per byte measure of $\mathsf{FAST}$ with those of XCB, EME2 and AEZ in the setting of $\mathsf{Fx}_{256}$.

| scheme | Skylake | Kabylake |
|---|---|---|
| XCB | 1.92 | 1.85 |
| EME2 | 2.07 | 1.99 |
| AEZ | 1.74 | 1.70 |
| $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{Horner}]$ | 1.63 | 1.56 |
| $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{BRW}]$ | 1.24 | 1.19 |

Table 6: Report of cycles per byte measure for the setting of $\mathsf{Gn}$ for $\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, \mathsf{vecHorner}]$ and $\mathsf{FAST}[\mathsf{Gn}, \mathfrak{k}, 31, \mathsf{vecHash2L}]$.

| | | Skylake | | | Kabylake | | |
|---|---|---|---|---|---|---|---|
| msg len (bytes) | $\mathfrak{k}$ | vecHorner | vecHash2L (delayed) | vecHash2L (normal) | vecHorner | vecHash2L (delayed) | vecHash2L (normal) |
| 512 | 2 | 1.51 | 1.38 | 1.59 | 1.42 | 1.32 | 1.56 |
| | 3 | 1.40 | 1.38 | 1.39 | 1.32 | 1.31 | 1.35 |
| | 4 | 1.34 | 1.37 | 1.36 | 1.26 | 1.31 | 1.33 |
| 1024 | 2 | 1.53 | 1.34 | 1.48 | 1.42 | 1.27 | 1.42 |
| | 3 | 1.45 | 1.34 | 1.34 | 1.35 | 1.27 | 1.30 |
| | 4 | 1.40 | 1.33 | 1.32 | 1.29 | 1.27 | 1.30 |
| 4096 | 2 | 1.57 | 1.30 | 1.35 | 1.45 | 1.24 | 1.30 |
| | 3 | 1.54 | 1.29 | 1.31 | 1.43 | 1.24 | 1.27 |
| | 4 | 1.51 | 1.29 | 1.30 | 1.40 | 1.24 | 1.26 |
| 8192 | 2 | 1.57 | 1.27 | 1.32 | 1.45 | 1.22 | 1.27 |
| | 3 | 1.56 | 1.27 | 1.30 | 1.44 | 1.22 | 1.25 |
| | 4 | 1.54 | 1.27 | 1.30 | 1.43 | 1.22 | 1.25 |

**Remark:** From Table 5, it may be noted that AEZ is faster than EME2. From Table 4, it can be seen that the number of block cipher calls made by AEZ is more than that made by EME2. So, the fact that in practice AEZ turns out to be faster may be surprising. The explanation lies in the difference in the number of doubling operations made by EME2 and AEZ. From Table 4, EME2 requires roughly 2[BC]+2[D] operations per block whereas AEZ requires roughly 2.5[BC]+0.25[D] operations per block. Executing AES instructions in groups using pipelining results in very fast AES timings. Our experiment on the Skylake processor shows that AES requires about 0.65 cycles per byte. In contrast, while the doubling operation should in theory be much faster, there is no support for 128-bit shift and consequently doubling takes about 0.29 cycles per byte. (We refer to [17] for an elaborate discussion on various strategies for constant time doubling operation.) Using these figures, the operations count of 2[BC]+2[D] for EME2 translates to about 1.88 cycles per byte while the operations count of 2.5[BC]+0.25[D] for AEZ translates to about 1.70 cycles per byte. This provides an explanation of why AEZ is faster than EME2. Note that EME2 requires additional block cipher calls and so the actual observed time of 2.07 cycles per byte for EME2 is a bit higher than the estimated 1.88 cycles per byte whereas the observed and the estimated timings for AEZ are quite close.

## 9 Hardware Implementation

The fixed length variants of $\mathsf{FAST}$ (i.e., $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{Horner}]$ and $\mathsf{FAST}[\mathsf{Fx}_m, \mathsf{BRW}]$) have been implemented in reconfigurable hardware. The implementations were done keeping in mind the application of disk encryption. The design decisions that were made are as follows:

1. **PRF**: For the PRF $\mathbf{F}$, we have used the encryption function of AES. So, the block length $n = 128$.
2. **Message and tweak lengths**: We have assumed a message length of 4096 bytes. As mentioned earlier, this is the current sector size of commercially available hard disks. For $n = 128$, 4096 bytes correspond to 256 blocks, i.e., $m = 256$. For disk encryption application, the tweaks are sector addresses and we have assumed the tweak to be a single $n$-bit block. So, our implementations are those of $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{Horner}]$ and $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{BRW}]$.
3. **Choice of FPGA**: The basic design goal was speed and so the implementations were optimised for speed. Nevertheless, we tried to keep the area metric reasonable. The target devices were high end fast FPGAs. In particular, we have optimised our designs for the Xilinx Virtex 5 and Virtex 7 families.

With $m = 256$ and a single block tweak, the numbers of blocks in the inputs to the hash functions $h$ and $h'$ are both 255. The 255 blocks comprise of 254 blocks arising from $P_3$ or $C_3$ and one block from the tweak. Since 255 blocks are to be hashed, for $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{Horner}]$, the requirement is to implement 255-block $\mathsf{Horner}$ while for $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{BRW}]$, the requirement is to implement 255-block $\mathsf{BRW}$.

For both $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{Horner}]$ and $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{BRW}]$, we have implemented two variants, one with a single core of the AES encryption module and the other with two cores of the AES encryption module. We denote variants of $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{Horner}]$ and $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{BRW}]$ using a single AES core as $\mathsf{FAST}[\mathsf{AES},\mathsf{Horner}]$-1 and $\mathsf{FAST}[\mathsf{AES},\mathsf{BRW}]$-1 respectively. The variants of $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{Horner}]$ and $\mathsf{FAST}[\mathsf{Fx}_{256}, \mathsf{BRW}]$ using two AES cores are denoted as $\mathsf{FAST}[\mathsf{AES},\mathsf{Horner}]$-2 and $\mathsf{FAST}[\mathsf{AES},\mathsf{BRW}]$-2 respectively.

The two basic building blocks for all of these designs are the encryption function of the AES and a finite field multiplier.

In our implementations, we have used pipelined AES encryption cores. An AES encryption core requires a key generation module. For the two-core designs the same key generation module is shared by both the cores. The latency of each AES core is 11 cycles, i.e., the first block of ciphertext is produced after a delay of 11 cycles and thereafter one cipher block is obtained in each cycle. The design of the AES cores adopts some interesting ideas reported earlier [8]. The earlier design [8] was that of a sequential AES design tailored for the Virtex 5 family of devices. An important aspect of this design is that the S-boxes are implemented as $256 \times 8$ multiplexers and one S-box fits into 32 six-input LUTs which are available in Virtex 5 FPGAs. We have used the same idea to design the S-boxes of our pipelined AES core.

With $n = 128$, the requirement is to compute products in $GF(2^{128})$. For this, we have used a 4-stage pipelined Karatsuba multiplier. The number of stages was selected to match the maximum frequency of the AES encryption core, which is the only other significant component in the circuits. The multiplier design is the same as reported in a previous work [12].

To use the pipelined multiplier efficiently, it is important to schedule the multiplications in such a way that pipeline delays are minimised. The BRW computation is amenable to a very efficient pipelined implementation. This requires identifying an "optimal" order of the multiplications so that both pipeline delays and the necessity to store intermediate results are minimised. A detailed study of such an optimal ordering is available in the literature [11]. A circuit for computing BRW polynomials on 31 blocks of inputs using a 3-stage pipelined Karatsuba multiplier is known [11]. In the present work, the requirement is to compute BRW polynomials on 255 blocks using a 4-stage pipelined multiplier. We scale up the earlier design [11] suitably for our purpose.

For computing Horner using a pipelined multiplier the idea of decimation is used. This has been briefly mentioned in the context of software implementation. We provide some more details here. Let $(X_1, X_2, \ldots, X_m)$ and a positive integer $d$ be given. Let $\chi_i = m - i \pmod{d}$. The $d$-decimated Horner computation [9] is based on the following observation.

$$\begin{aligned}
&\mathsf{Horner}_\tau(X_1, X_2, \ldots, X_m) \\
&= \tau^{\chi_1}\mathsf{Horner}_{\tau^d}(X_1, X_{1+d}, X_{1+2d}, \ldots) \oplus \cdots \oplus \tau^{\chi_d}\mathsf{Horner}_{\tau^d}(X_d, X_{2d}, X_{3d}, \ldots).
\end{aligned}$$

So, $\mathsf{Horner}_\tau(X_1, X_2, \ldots, X_m)$ can be computed by evaluating $d$ *independent* polynomials at $\tau^d$ and then combining the results. This representation allows efficient use of a $d$-stage pipelined multiplier, as in each clock, $d$ independent multiplications can be scheduled.

In what follows, we give a detailed description of the architecture of FAST[AES,BRW]-2 followed by a short description of the architecture of FAST[AES,Horner]-2.

### 9.1 Architecture for FAST[AES,BRW]-2

FAST[AES,BRW]-2 uses two pipelined AES encryption cores and a 4-stage pipelined multiplier. An overview of the architecture is shown in Figure 1. We briefly describe its components and functioning.

The basic components of the architecture are the two AES encryption cores which are denoted as **AESodd** and **AESeven**. The module for the BRW polynomial evaluation using a 4-stage Karatsuba multiplier is shown as **BRWPoly_eval**.

The two AES cores, two multiplexers **M1** and **M2** and a counter named **Counter** are enclosed inside a dashed rectangle. This constitutes a module which implements the counter mode. The module can also perform AES encryption of a single block. The **AESeven** core is used only in counter mode whereas the **AESodd** core is used for both encryption in the counter mode and to encrypt single blocks. According to the algorithms in Tables 2 and 3, encryption of a single block is required for the blocks $F_1$ and $F_2$ in the Feistel function and for fStr in the main function.

The counter has two outputs, one for odd values and the other for even values. The even values are fed directly to the **AESeven** core and the odd values are fed to the **AESodd** core through the multiplexer **M1**. The block **BRWPoly_eval** performs the 255-block BRW computation. Additionally, this block also computes the single multiplications by $\tau$ required for the computation of $\mathbf{H}_\tau$ and $\mathbf{G}'_\tau$ (see Table 2).

The registers **Z**, **A1**, **F1**, **F2** and **B2** are used to store the intermediate values and these correspond to the variables $Z, A_1, F_1, F_2$ and $B_2$ respectively of the algorithms described in Tables 2 and 3.

The input ports **Podd** and **Peven** are used to feed in the odd numbered message blocks and even numbered message blocks respectively. The tweak is also fed in through **Podd**. The hash key is fed in through a separate port. The output ports **Ceven** and **Codd** output the even and odd numbered cipher blocks respectively.

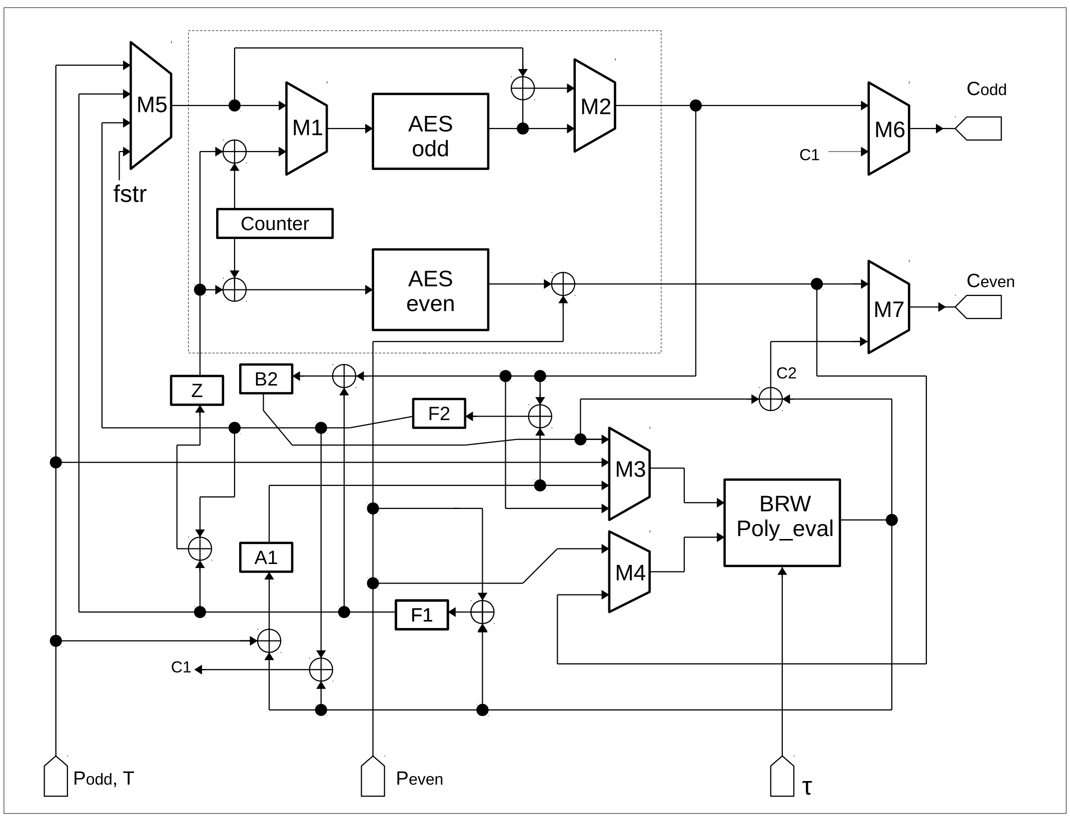



Fig. 1: Architecture for FAST[AES,BRW]-2.

The multiplexer **M5** selects the input to **AESodd** from one of the four possible inputs, namely, **Podd**, **F1**, **F2** or the string fStr. The multiplexer **M1** selects either the output of **M5** or $Z \oplus i$, where $i$ is the output from the odd port of **Counter**. This input design to the **AESodd** core through the multiplexers **M1** and **M5** allows **AESodd** to encrypt in the counter mode and also to encrypt the required single blocks.

The BRW computation module **BRWPoly_eval** is required to be fed two blocks of plaintext or ciphertext in each cycle. The multiplexer **M3** provides the first input to **BRWPoly_eval**. This input is selected by **M3** to be one of **Podd**, **Codd**, **A1** or **B2**. The inputs **Podd** and **Codd** are relevant for BRW while the inputs **A1** and **B2** are relevant when a single-block multiplication is required. The second input to **BRWPoly_eval** is the output of the multiplexer **M4** and can be either **Peven** or **Ceven**.

The final outputs of the circuit are selected using multiplexers **M6** and **M7**. Control signals are generated using a finite state machine which follows the algorithm of FAST.

**Timing analysis:** Figure 2 shows the timing diagram for the architecture for FAST[AES,BRW]-2. The first 11 clock cycles are required to compute the hash key $\tau$ by applying the AES encryption module to fStr. The computation of the hash function $\mathbf{H}_\tau$ (see (6)) requires a 255-block BRW computation and two subsequent field multiplications by $\tau$. The 255-block BRW computation requires 127 field multiplications. The 4-stage multiplier has a latency of 4 cycles. So, the BRW computation requires 131 cycles. The two subsequent multiplications require 4 cycles each. The computation of $\mathbf{H}_\tau$ is completed after 141 cyles which includes two additional synchronisation cycles. The Feistel network has two encryptions. The first encryption requires 11 cycles. After the first encryption, both $F_1$ and $F_2$ are available and so the input $Z = F_1 \oplus F_2$ to the counter can be obtained. Let $J_i = Z \oplus \mathsf{bin}_n(i)$, $i = 1, \ldots, 254$. **AESodd** performs the encryptions of $F_1, F_2, J_1, J_3, \ldots, J_{253}$ while **AESeven** performs the encryptions of $J_2, J_4, \ldots, J_{254}$. **AESodd** and **AESeven** are synchronised such that the encryptions of $J_{2j-1}$ and $J_{2j}$, $j = 1, \ldots, 127$, are obtained simultaneously. This allows the computation of $\mathbf{G}'_\tau$ to start after the encryptions of $J_1$ and $J_2$ are completed and be executed in parallel with the rest of the encryptions of the counter. The total computation requires 319 cycles which includes a few synchronisation cycles.

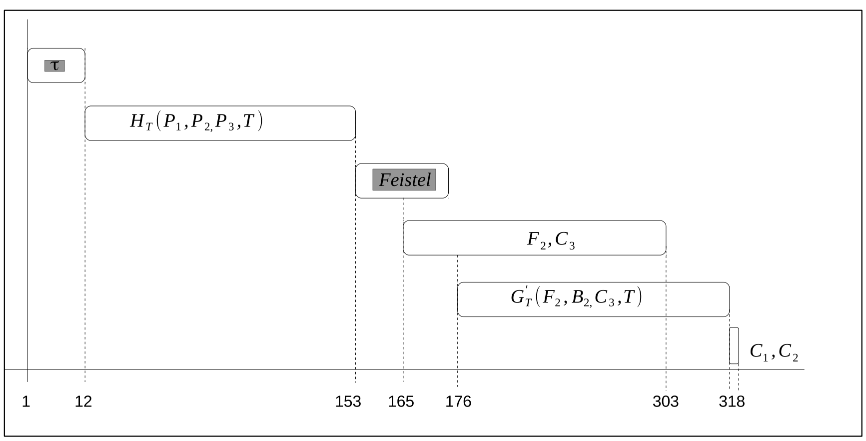



Fig. 2: Time diagram for encryption using FAST[AES,BRW]-2.

## 9.2 Architecture for FAST[AES,Horner]-2

To take the advantage of two AES cores in the design of FAST[AES,Horner]-2 it becomes necessary to use two multipliers. The reason is the following. The crucial parallelisation is in computing the second hash layer where the hash of the ciphertexts produced by the counter mode is computed.

Since two pipelined AES cores are used to implement the counter mode, after an initial delay, in each clock cycle two blocks of ciphertexts are produced. So, the hash module has to be capable of processing two ciphertext blocks in each cycle. For BRW based hashing, each multiplication involves two ciphertext blocks. On the other hand, in the case of Horner, each multiplication involves a single block. So, to process two ciphertext blocks in each cycle it is required to use two multipliers. Each multiplier operates in a 4-stage pipeline. For proper scheduling using the two multipliers, it is required to use a 8-decimated version of Horner. This allows the scheduling of four independent multiplications to each multiplier in every clock cycle.

### 9.3 Experimental Results

We present performance data for four implementations of FAST. The results are compared with the implementations of XCB, EME2 and AEZ. The implementations of XCB and EME2 are as reported in an earlier work [12]. Two architectures for each of EME2 and XCB are reported. These are named EME2-1, EME2-2 and XCB-1, XCB-2 respectively. For AEZ, we have carried out an efficient implementation which is described in Appendix B.2. This implementation uses two cores and we name the architecture AEZ-2. The hardware resources utilized in these architectures along with those used in the different architectures for FAST are summarized in Table 7.

Table 7: Summary of the main hardware resources in the architectures of FAST, EME2, XCB and AEZ.

| Scheme | Pipelined AES encryption core | Pipelined AES decryption core | Sequential AES decryption core | Pipelined multiplier |
|---|---|---|---|---|
| FAST[AES,BRW]-1 | 1 | 0 | 0 | 1 |
| FAST[AES,Horner]-1 | 1 | 0 | 0 | 1 |
| EME2-1 | 1 | 1 | 0 | 0 |
| XCB-1 | 1 | 0 | 1 | 1 |
| FAST[AES,BRW]-2 | 2 | 0 | 0 | 1 |
| FAST[AES,Horner]-2 | 2 | 0 | 0 | 2 |
| EME2-2 | 2 | 2 | 0 | 0 |
| XCB-2 | 2 | 0 | 1 | 2 |
| AEZ-2 | 2 | 0 | 0 | 0 |

Some important aspects of the architectures of XCB, EME2 and AEZ are as follows:

1. The encryption cores utilised in FAST are the same as those utilised in XCB, EME2 and AEZ. Further, the multiplier utilised in FAST is also utilised in XCB. The sequential decryption core required in XCB was optimised for speed. To match the critical path of the AES encryption core the sequential decryption core was implemented using T-boxes.
2. EME2 is an encrypt-mask-encrypt type construction which consists of two ECB layers with an intermediate masking. The ECB layers can be implemented with pipelined AES cores. For decryption, ECB in decryption mode is required; hence for efficient decryption functionality pipelined AES decryption cores are required to be used. The second layer of ECB in EME2 can only be computed once the first layer has been completed and so the intermediate results of the first layer of ECB encryption are required to be stored. Block RAMs are used for this purpose.

3. XCB is a hash-counter-hash type mode which involves a counter mode of operation sandwiched between two polynomial hash layers. The main encryption/decryption in XCB takes place through a variant of the counter mode (which is different from the counter mode used in FAST). The counter mode can be implemented using only the encryption module of AES. One call to the decryption module of AES is required in XCB for both encryption and decryption. For this, a sequential AES decryption core is utilised. Thus, XCB-2 uses two pipelined AES encryption cores which does the bulk encryption and in addition uses a sequential AES decryption core.
4. The polynomial hash layers in XCB consist of Horner computations. The second Horner computation in XCB can be computed in parallel with the counter mode. As in case of FAST[AES, Horner]-2 the counter mode in XCB-2 is implemented using two AES cores. So, in each clock cycle, two blocks of ciphertexts are obtained and to utilise this parallelisation two multipliers are required.
5. For AEZ, we do not consider an architecture consisting of a single AES core. The number of cycles required by such an architecture will be too high compared to the other schemes.
6. There are two architectures for AEZ, namely, AEZ-2-pre and AEZ-2-otf. In AEZ-2-pre, the required masks are precomputed whereas in AEZ-2-otf, the required masks are computed on the fly. A total of 145 cycles are required to precompute the masks in AEZ-2-pre.
7. The architecture for EME2 needs to store intermediate results of lengths equal to the message length. For doing this, EME2 requires 4 block RAMs. In contrast to EME2, both AEZ-2-pre and AEZ-2-otf require to store more intermediate results requiring 8 block RAMs. Further, AEZ-2-pre stores the precomputed masks which requires an additional 4 block RAMs. So, overall AEZ-2-otf requires 8 block RAMs while AEZ-2-pre requires 12 block RAMs.

The performance results presented in Table 8 are obtained after place and route process in ISE 14.7. The target device was `xc5vlx330t-2ff1738`. We tried many timing restrictions and the best case is reported.

Table 8: Implementation results for Virtex 5.

| Architecture | Area | | Frequency | Clock | Throughput |
|---|---|---|---|---|---|
| | slices | blk RAMs | (MHz) | cycles | (Gbps) |
| AES-PEC | 2859 | 0 | 300.56 | 1 | 38.47 |
| AES-PDC | 3110 | 0 | 239.34 | 1 | 30.72 |
| AES-SDC | 1800 | 0 | 292.48 | 11 | 3.40 |
| 128-bit mult | 1650 | 0 | 298.43 | 1 | 38.20 |
| FAST[AES,BRW]-2 | 7175 | 0 | 289.56 | 319 | 29.74 |
| FAST[AES,Horner]-2 | 8983 | 0 | 289.98 | 311 | **30.55** |
| XCB-2 | 9752 | 0 | 270.52 | 316 | 28.05 |
| EME2-2 | 10970 | 4 | 230.56 | 305 | 24.77 |
| AEZ-2-pre | 5646 | 12 | 269.56 | 389 (+145) | 22.70† |
| AEZ-2-otf | 5854 | 8 | 272.32 | 404 | 22.08 |
| FAST[AES, BRW]-1 | 5064 | 0 | 290.57 | 455 | 20.92 |
| FAST[AES, Horner]-1 | **4781** | 0 | 291.05 | 565 | 16.88 |
| XCB-1 | 6070 | 0 | 272.75 | 569 | 15.70 |
| EME2-1 | 6500 | 4 | 233.58 | 561 | 13.64 |

†: ignores the 145 cycles required for pre-computation.

The first part of Table 8 shows the performance of the basic modules, i.e., the pipelined encryption core (PEC), the pipelined decryption core (PDC), the sequential decryption core (SDC) and the 128-bit pipelined Karatsuba multiplier. The decryption cores are not required in FAST and AEZ. The pipelined decryption core is required for EME2 and the sequential decryption core is required for XCB. The results for individual AES cores in Table 8 include the area required for the key schedule module. For the implementations of modes of operation we have implemented only one key schedule, and it is shared between all the AES cores presented in the architecture.

From the results in Table 8 we observe the following:

1. Comparison of area.
   (a) AEZ requires two cores but no multiplier and so the number of slices is lesser than those required for 2 core architectures for FAST. On the other hand, the number of slices for AEZ is more than the single core architectures for FAST which use a single AES core and a multiplier.
   (b) Of all the two-core architectures, AEZ-2-pre requires the smallest number of slices and the highest number of block RAMs. FAST[AES, BRW]-2, on the other hand, requires more slices than AEZ, but no block RAM. Among the single-core architectures, FAST[AES, Horner]-1 is the smallest which is also the smallest design overall.
   (c) In comparison to Horner, the module for implementing BRW requires more registers and also circuits for squaring. As a result, FAST[AES, BRW]-1 requires 283 slices more than FAST[AES, Horner]-1.
   (d) For the two-core architectures, FAST[AES, Horner]-2 requires more area than FAST[AES, BRW]-2 since the implementation of FAST[AES, Horner]-2 requires two multipliers while the implementation of FAST[AES, BRW]-2 requires a single multiplier.
   (e) EME2 is the costliest in terms of area in both categories of single core and double core architectures. This is because it requires AES decryption cores. Further, both EME2-1 and EME2-2 require four block RAMs in addition to the slices.
   (f) The overall architecture of XCB is similar to that of FAST[AES, Horner]. The main difference is that XCB requires an additional sequential AES decryption core and this results in XCB being costlier than FAST[AES, Horner] in terms of area.
2. Comparison of throughput.
   (a) Among the two-core architectures, FAST[AES,Horner]-2 has the highest throughput while among the single-core architectures, FAST[AES,BRW]-1 has the highest throughput.
   (b) As computing BRW requires about half the number of multiplications required for computing Horner, in comparison to FAST[AES,Horner]-1, a significant number of clocks can be saved in computing the first hash in case of FAST[AES,BRW]-1. As a result, the total number of clocks required by FAST[AES,BRW]-1 is smaller than that required by FAST[AES,Horner]-1 and this leads to a better throughput for FAST[AES,BRW]-1.
   (c) FAST[AES,Horner]-2 is marginally better than FAST[AES,BRW]-2 in terms of throughput. This is due to the following reason. FAST[AES,Horner]-2 uses two multipliers which compensates for the gain from the use of BRW polynomials. Overall, FAST[AES,Horner]-2 requires slightly lesser number of clocks and utilises slightly higher frequency.
   (d) Both versions of XCB operate at a lower frequency than the corresponding versions of FAST. This leads to lower throughput of XCB compared to FAST. The lower frequency of XCB is essentially due to the use of the sequential AES decryption core which is not present in the architectures for FAST.

(e) Among the 2-core architectures, AEZ has the lowest throughput while EME2-1 has the lowest throughput overall. EME2 has the lowest frequency due to the use of the pipelined decryption core, which is absent in all other architectures.
(f) The frequency of AEZ is lower than FAST. This is due to the use of block RAMs.

To confirm the comparative performance of the different designs, we have also obtained results for the high end Virtex 7 FPGA. The target device was `xc7vx690t-3fgg1930`. The results are presented in Table 9. Based on Table 9, we make the following observations.

1. The frequency grows significantly in comparison with Virtex 5 results. This is basically a direct effect of the difference of the fabrication technology between the two families. While Virtex 5 family is built with 65 nm technology, Virtex 7 is built with 28 nm technology.
2. The number of slices for the AES cores is significantly lesser than the corresponding implementations in Virtex 5. This is due to the fact that slices in Virtex 7 include 8 Flip-Flops which is 4 more than that in Virtex 5.
3. In some cases, the number of slices grows in comparison with the Virtex 5. Examples are the 128-bit multiplier and FAST[AES,Horner]-1. This behaviour can be attributed to the optimisation performed by the tool.

Table 9: Implementation results for Virtex 7.

| Architecture | Area | | Frequency | Clock | Throughput |
|---|---|---|---|---|---|
| | slices | blk RAMs | (MHz) | cycles | (Gbps) |
| AES-PEC | 2093 | 0 | 405.02 | 1 | 51.84 |
| AES-PDC | 2352 | 0 | 352.19 | 1 | 45.08 |
| AES-SDC | 1575 | 0 | 390.056 | 11 | 4.54 |
| 128-bit mult | 1884 | 0 | 404.86 | 1 | 51.82 |
| FAST[AES,BRW]-2 | 7202 | 0 | 375.43 | 319 | 38.56 |
| FAST[AES,Horner]-2 | 8906 | 0 | 377.03 | 311 | **39.73** |
| XCB-2 | 9330 | 0 | 358.84 | 316 | 37.21 |
| EME2-2 | 11800 | 4 | 315.58 | 305 | 33.90 |
| AEZ-2-pre | 6072 | 12 | 361.52 | 389 (+ 145) | 30.45† |
| AEZ-2-otf | 5202 | 8 | 362.78 | 404 | 29.42 |
| FAST[AES, BRW]-1 | 5024 | 0 | 377.87 | 455 | 27.21 |
| FAST[AES, Horner]-1 | **4783** | 0 | 379.25 | 565 | 21.99 |
| XCB-1 | 5875 | 0 | 360.67 | 569 | 20.77 |
| EME2-1 | 6350 | 4 | 319.74 | 561 | 18.67 |

†: ignores the 145 cycles required for pre-computation.

## 10 Conclusion

In this paper we have presented a tweakable enciphering scheme called FAST. Instantiations of the scheme for both fixed length messages with single block tweaks and variable length messages with very general tweaks have been described. A detailed security analysis in the style of reductionist security proof has been provided. Software implementations of both kinds of instantiations have been made. The instantiation for fixed length messages with single block tweaks is appropriate for low-level disk encryption. An FPGA based hardware implementation has been done for this application. Both the software and the hardware implementations show that the new scheme outperforms previous schemes which makes the new scheme an attractive option for designers and standardisation bodies.

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## A Proof of Theorem 1

This section provides the proof of Theorem 1.

*Proof.* Let $\mathcal{A}$ be an adversary attacking $\mathsf{FAST}$. We use $\mathcal{A}$ to build an adversary $\mathcal{B}$ attacking the PRF-property of $\mathbf{F}$. $\mathcal{B}$ has access to an oracle which is either $\mathbf{F}_K(\cdot)$ for a uniform random $K$ in $\mathcal{K}$, or, the oracle is $\rho$ which is a uniform random function from $\{0,1\}^n$ to $\{0,1\}^n$. Adversary $\mathcal{B}$ uses the $(\epsilon_1, \epsilon_2)$-eligible pair of hash functions $(h, h')$ to set up an instance of $\mathsf{FAST}$ and invokes $\mathcal{A}$ to attack this instance. $\mathcal{A}$ makes a number of oracle queries to the encryption and decryption oracles of $\mathsf{FAST}$. $\mathcal{B}$ uses its own oracle and the hash functions $h$ and $h'$ to compute the responses which it provides to $\mathcal{A}$. At the end, $\mathcal{A}$ outputs a bit and $\mathcal{B}$ outputs the same bit. Note that both encryption and decryption queries by $\mathcal{A}$ can be answered using the oracle of $\mathcal{B}$ and the hash functions $h, h'$.

The running time of $\mathcal{B}$ is the running time of $\mathcal{A}$ along with the time required to compute the responses to the queries made by $\mathcal{A}$ using $\mathcal{B}$'s oracle plus some bookkeeping time which includes the time for set-up. So, the total running time of $\mathcal{B}$ is $\mathfrak{T} + \mathfrak{T}'$ as desired. Further, to answer $\mathcal{A}$'s queries, $\mathcal{B}$ needs to make a query to its oracle on $\mathsf{fStr}$ and to answer the $s$-th query, it needs to make $m^{(s)}$ queries to its oracle. So, the total number of times $\mathcal{B}$ queries its oracle is $1 + \sum_{s=1}^{q} m^{(s)} = \omega + 1$. Since each query of $\mathcal{B}$ consists of a single $n$-bit block, the query complexity is also $\omega + 1$.

If the oracle to $\mathcal{B}$ is the real oracle, i.e., the oracle is $\mathbf{F}_K$, then $\mathcal{A}$ gets to interact with the real encryption and decryption oracles of $\mathsf{FAST}$. So,

$$\Pr\left[K \stackrel{\$}{\leftarrow} \mathcal{K} : \mathcal{B}^{\mathbf{F}_K(\cdot)} \Rightarrow 1\right] = \Pr\left[K \stackrel{\$}{\leftarrow} \mathcal{K} : \mathcal{A}^{\mathsf{FAST.Encrypt}_K(\cdot,\cdot),\mathsf{FAST.Decrypt}_K(\cdot,\cdot)} \Rightarrow 1\right]. \tag{44}$$

Denote by $\mathsf{FAST}_\rho$ the instance of $\mathsf{FAST}$ where $\mathbf{F}_K$ is replaced by $\rho$. If the oracle to $\mathcal{B}$ is the random oracle, i.e., the oracle is $\rho$, then

$$\Pr\left[\mathcal{B}^{\rho(\cdot)} \Rightarrow 1\right] = \Pr\left[\mathcal{A}^{\mathsf{FAST}_\rho.\mathsf{Encrypt}(\cdot,\cdot),\mathsf{FAST}_\rho.\mathsf{Decrypt}(\cdot,\cdot)} \Rightarrow 1\right]. \tag{45}$$

So,

$$\begin{aligned}\mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathcal{B}) &= \Pr\left[K \stackrel{\$}{\leftarrow} \mathcal{K} : \mathcal{B}^{\mathbf{F}_K(\cdot)} \Rightarrow 1\right] - \Pr\left[\mathcal{B}^{\rho(\cdot)} \Rightarrow 1\right] \\ &= \Pr\left[K \stackrel{\$}{\leftarrow} \mathcal{K} : \mathcal{A}^{\mathsf{FAST.Encrypt}_K(\cdot,\cdot),\mathsf{FAST.Decrypt}_K(\cdot,\cdot)} \Rightarrow 1\right] \\ &\quad - \Pr\left[\mathcal{A}^{\mathsf{FAST}_\rho.\mathsf{Encrypt}(\cdot,\cdot),\mathsf{FAST}_\rho.\mathsf{Decrypt}(\cdot,\cdot)} \Rightarrow 1\right].\end{aligned} \tag{46}$$

The advantages of $\mathcal{A}$ and $\mathcal{B}$ are related as follows. Recall that $\mathfrak{F}$ is the set of all functions $f$ from $\mathcal{T} \times \mathcal{P}$ to $\mathcal{P}$ such that for any $T \in \mathcal{T}$ and $P \in \mathcal{P}$, $\mathsf{len}(f(T,P)) = \mathsf{len}(P)$. Let $\rho_1(\cdot,\cdot)$ and $\rho_2(\cdot,\cdot)$ be two independent and uniform random functions from $\mathfrak{F}$.

$$\begin{aligned}&\mathbf{Adv}^{\pm\mathrm{rnd}}_{\mathsf{FAST}}(\mathcal{A}) \\ &= \Pr\left[K \stackrel{\$}{\leftarrow} \mathcal{K} : \mathcal{A}^{\mathsf{FAST.Encrypt}_K(\cdot,\cdot),\mathsf{FAST.Decrypt}_K(\cdot,\cdot)} \Rightarrow 1\right] - \Pr\left[\mathcal{A}^{\rho_1(\cdot,\cdot),\rho_2(\cdot,\cdot)} \Rightarrow 1\right] \\ &= \Pr\left[K \stackrel{\$}{\leftarrow} \mathcal{K} : \mathcal{A}^{\mathsf{FAST.Encrypt}_K(\cdot,\cdot),\mathsf{FAST.Decrypt}_K(\cdot,\cdot)} \Rightarrow 1\right] - \Pr\left[\mathcal{A}^{\mathsf{FAST}_\rho.\mathsf{Encrypt}(\cdot,\cdot),\mathsf{FAST}_\rho.\mathsf{Decrypt}(\cdot,\cdot)} \Rightarrow 1\right] \\ &\quad + \Pr\left[\mathcal{A}^{\mathsf{FAST}_\rho.\mathsf{Encrypt}(\cdot,\cdot),\mathsf{FAST}_\rho.\mathsf{Decrypt}(\cdot,\cdot)} \Rightarrow 1\right] - \Pr\left[\mathcal{A}^{\rho_1(\cdot,\cdot),\rho_2(\cdot,\cdot)} \Rightarrow 1\right] \\ &= \Pr\left[\mathcal{A}^{\mathsf{FAST}_\rho.\mathsf{Encrypt}(\cdot,\cdot),\mathsf{FAST}_\rho.\mathsf{Decrypt}(\cdot,\cdot)} \Rightarrow 1\right] - \Pr\left[\mathcal{A}^{\rho_1(\cdot,\cdot),\rho_2(\cdot,\cdot)} \Rightarrow 1\right] + \mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathcal{B}).\end{aligned} \tag{47}$$

There are two events to consider, namely,

$$\mathcal{A}^{\mathsf{FAST}_\rho.\mathsf{Encrypt}(\cdot,\cdot),\mathsf{FAST}_\rho.\mathsf{Decrypt}(\cdot,\cdot)} \Rightarrow 1 \text{ and } \mathcal{A}^{\rho_1(\cdot,\cdot),\rho_2(\cdot,\cdot)} \Rightarrow 1.$$

Consider the event $\mathcal{A}^{\mathsf{FAST}_\rho.\mathsf{Encrypt}(\cdot,\cdot),\mathsf{FAST}_\rho.\mathsf{Decrypt}(\cdot,\cdot)} \Rightarrow 1$. Suppose $\mathcal{A}$ makes a total of $q$ queries with tweak query complexity $\theta$ and message query complexity $\omega$. For $1 \leq s \leq q$, let $\mathsf{ty}^{(s)} = \mathsf{enc}$ if the $s$-th query is an encryption query and $\mathsf{ty}^{(s)} = \mathsf{dec}$ if the $s$-th query is a decryption query. Denote the tweak, the plaintext and the ciphertext associated with the $s$-th query by $T^{(s)}$, $P^{(s)} = P_1^{(s)}||P_2^{(s)}||P_3^{(s)}$ and $C^{(s)} = C_1^{(s)}||C_2^{(s)}||C_3^{(s)}$ respectively. We have $\mathfrak{t}^{(s)} = \mathfrak{t}(T^{(s)})$ and $m^{(s)}$ is the number of $n$-bit blocks in $\mathsf{pad}_n(P^{(s)})$ and $\mathsf{pad}_n(C^{(s)})$. Also, $\mathfrak{l}^{(s)} = \mathfrak{l}(P_3^{(s)}) = \mathfrak{l}(C_3^{(s)}) = m^{(s)} - 2$.

The interaction of $\mathcal{A}$ with the oracle in this setting is given by the game $G_{\mathsf{real}}$ which is shown in Table 10. In this game, the random function $\rho$ is built incrementally. Whenever a "new" input to $\rho$ is received, the output is chosen independently and uniformly at random. The variable $\mathsf{bad}$ is set to $\mathsf{true}$ if it turns out that two inputs to $\rho$ collide. Let $\mathsf{Bad}_{\mathsf{real}}(\mathcal{A})$ be the event that $\mathsf{bad}$ is set to $\mathsf{true}$ in the game $G_{\mathsf{real}}$. Also, by $\mathcal{A}^{G_{\mathsf{real}}} \Rightarrow 1$ we denote the event that $\mathcal{A}$ outputs 1 in the game $G_{\mathsf{real}}$. Note that $\mathcal{A}^{G_{\mathsf{real}}} \Rightarrow 1$ is exactly the event $\mathcal{A}^{\mathsf{FAST}_\rho.\mathsf{Encrypt}(\cdot,\cdot),\mathsf{FAST}_\rho.\mathsf{Decrypt}(\cdot,\cdot)} \Rightarrow 1$.

If $\mathsf{Bad}_{\mathsf{real}}(\mathcal{A})$ does not occur, then the boxed instruction in game $G_{\mathsf{real}}$ is not executed. The absence of the boxed instruction does not affect the probability of $\mathsf{Bad}_{\mathsf{real}}(\mathcal{A})$. We consider the distributions of the plaintexts and ciphertexts when $\mathsf{Bad}_{\mathsf{real}}(\mathcal{A})$ does not occur. Let $Y_1^{(s)}$ denote the output of $\mathsf{Ch}\text{-}\rho(F_1^{(s)})$ and $Y_2^{(s)}$ denote the output of $\mathsf{Ch}\text{-}\rho(F_2^{(s)})$. Suppose $\mathsf{ty}^{(s)} = \mathsf{enc}$, i.e., the $s$-th query is an encryption query. Then from game $G_{\mathsf{real}}$, we can write

$$\begin{aligned}
C_1^{(s)} &= Y_1^{(s)} \oplus P_1^{(s)} \oplus \tau C_2^{(s)} \oplus \tau h'_\tau(T^{(s)}, C_3^{(s)}) \oplus h_\tau(T^{(s)}, P_3^{(s)});\\
C_2^{(s)} &= Y_2^{(s)} \oplus P_2^{(s)} \oplus \tau P_1^{(s)} \oplus h'_\tau(T^{(s)}, C_3^{(s)}) \oplus \tau h_\tau(T^{(s)}, P_3^{(s)});\\
C_{3,i}^{(s)} &= S_i^{(s)} \oplus P_{3,i}^{(s)} \text{ for } i = 1, \ldots, m^{(s)} - 3;\\
C_{3,m^{(s)}-2}^{(s)} &= \mathsf{first}_{r^{(s)}}(D^{(s)}) \oplus P_{3,m^{(s)}-2}^{(s)};
\end{aligned}$$

When $\mathsf{Bad}_{\mathsf{real}}(\mathcal{A})$ does not occur, $Y_1^{(s)}, Y_2^{(s)}, S_i^{(s)}, (i = 1, \ldots, m^{(s)} - 3), D^{(s)}$ are independent and uniform random strings. From the above relations, it is easy to argue that the ciphertext $C^{(s)}$ is also independent and uniform random. A similar argument shows that when $\mathsf{ty}^{(s)} = \mathsf{dec}$, i.e., the query is a decryption query, then $P^{(s)}$ is an independent and uniform random string. So, if $\mathsf{Bad}_{\mathsf{real}}(\mathcal{A})$ does not occur, then the adversary obtains independent and uniform random strings as responses to all its queries.

In the next step, the game $G_{\mathsf{real}}$ is modified to the game $G_{\mathsf{int}}$. This game is shown in Table 11. In this game, the outputs of $\rho$ are not chosen directly. Instead, these are defined from the plaintexts and the ciphertexts. For a enciphering query, the ciphertext is chosen independently and uniformly at random while for a deciphering query, the plaintext is chosen independently and uniformly at random. The outputs of $\rho$ are defined from these in the following manner.

$$\begin{aligned}
\rho(F_1^{(s)}) &= Y_1^{(s)} \leftarrow C_1^{(s)} \oplus P_1^{(s)} \oplus \tau C_2^{(s)} \oplus \tau h'_\tau(T^{(s)}, C_3^{(s)}) \oplus h_\tau(T^{(s)}, P_3^{(s)});\\
\rho(F_2^{(s)}) &= Y_2^{(s)} \leftarrow C_2^{(s)} \oplus P_2^{(s)} \oplus \tau P_1^{(s)} \oplus h'_\tau(T^{(s)}, C_3^{(s)}) \oplus \tau h_\tau(T^{(s)}, P_3^{(s)});\\
\rho(J_i^{(s)}) &= C_{3,i}^{(s)} \oplus P_{3,i}^{(s)} \text{ for } i = 1, \ldots, m^{(s)} - 3;\\
\rho(J_{m^{(s)}-2}^{(s)}) &= \begin{cases} D^{(s)} \oplus (P_{3,m^{(s)}-2}^{(s)}||0^{n-r^{(s)}}) \text{ if } \mathsf{ty}^{(s)} = \mathsf{enc};\\ E^{(s)} \oplus (C_{3,m^{(s)}-2}^{(s)}||0^{n-r^{(s)}}) \text{ if } \mathsf{ty}^{(s)} = \mathsf{dec};\end{cases}
\end{aligned} \tag{48}$$

As for an encryption query, $C_1^{(s)}, C_2^{(s)}, C_{3,1}^{(s)}, \ldots, C_{3,m^{(s)}-3}^{(s)}, D^{(s)}$ are chosen independently and uniformly at random, from (48) it follows that the outputs of $\rho$ are also independent and uniformly distributed. For a decryption query, $P_1^{(s)}, P_2^{(s)}, P_{3,1}^{(s)}, \ldots, P_{3,m^{(s)}-3}^{(s)}, E^{(s)}$ are chosen independently and uniformly at random. Again, from (48) it follows that the outputs of $\rho$ are also independent and uniformly distributed. So, as in game $G_{\mathsf{real}}$, in game $G_{\mathsf{int}}$ also the outputs of $\rho$ are independent and uniformly distributed.

Let $\mathsf{Bad}_{\mathsf{int}}(\mathcal{A})$ be the event that the variable $\mathsf{bad}$ is set to $\mathsf{true}$ in the game $G_{\mathsf{int}}$. Let $\mathcal{A}^{G_{\mathsf{int}}} \Rightarrow 1$ denote the event that $\mathcal{A}$ outputs 1 in the game $G_{\mathsf{int}}$. From the description of the games, it follows that if bad does not occur, then $\mathcal{A}$'s views in both $G_{\mathsf{real}}$ and $G_{\mathsf{int}}$ are the same. Also, the probabilities that bad occurs in the two games are equal. This gives the following.

**Claim 1.**

$$\Pr\left[\left(\mathcal{A}^{G_{\mathsf{real}}} \Rightarrow 1\right) \wedge \overline{\mathsf{Bad}_{\mathsf{real}}(\mathcal{A})}\right] = \Pr\left[\left(\mathcal{A}^{G_{\mathsf{int}}} \Rightarrow 1\right) \wedge \overline{\mathsf{Bad}_{\mathsf{int}}(\mathcal{A})}\right];$$
$$\Pr\left[\mathsf{Bad}_{\mathsf{real}}(\mathcal{A})\right] = \Pr\left[\mathsf{Bad}_{\mathsf{int}}(\mathcal{A})\right].$$

Next, the game $G_{\mathsf{int}}$ is changed to the game $G_{\mathsf{rnd}}$ which is shown in Table 12. In this game, there is no $\rho$. For an enciphering query, the ciphertext is chosen independently and uniformly at random and for a deciphering query, the plaintext is chosen independently and uniformly at random. These are returned to $\mathcal{A}$. After the interaction is over, in the finalisation step, the internal random variables are included in $\mathcal{D}$ and $\mathsf{bad}$ is set to $\mathsf{true}$ if there is a collision in $\mathcal{D}$. Let $\mathsf{Bad}_{\mathsf{rnd}}(\mathcal{A})$ be the event that the variable $\mathsf{bad}$ is set to $\mathsf{true}$ in the game $G_{\mathsf{rnd}}$. Let $\mathcal{A}^{G_{\mathsf{rnd}}} \Rightarrow 1$ denote the event that $\mathcal{A}$ outputs 1 in the game $G_{\mathsf{rnd}}$. If $\mathsf{bad}$ does not occur, then in both the games $G_{\mathsf{int}}$ and $G_{\mathsf{rnd}}$, $\mathcal{A}$ obtains independent and uniform random strings as responses to all its queries. Also, the probabilities that $\mathsf{bad}$ occurs in the two games are equal. So, we have the following.

**Claim 2.**

$$\Pr\left[\left(\mathcal{A}^{G_{\mathsf{int}}} \Rightarrow 1\right) \wedge \overline{\mathsf{Bad}_{\mathsf{int}}(\mathcal{A})}\right] = \Pr\left[\left(\mathcal{A}^{G_{\mathsf{rnd}}} \Rightarrow 1\right) \wedge \overline{\mathsf{Bad}_{\mathsf{rnd}}(\mathcal{A})}\right];$$
$$\Pr\left[\mathsf{Bad}_{\mathsf{int}}(\mathcal{A})\right] = \Pr\left[\mathsf{Bad}_{\mathsf{rnd}}(\mathcal{A})\right].$$

Note that the event $\mathcal{A}^{G_{\mathsf{rnd}}} \Rightarrow 1$ is exactly the event $\mathcal{A}^{\rho_1(\cdot,\cdot),\rho_2(\cdot,\cdot)} \Rightarrow 1$.

Using (47) along with Claims 1 and 2, we have the following.

$$\begin{aligned}
&\mathbf{Adv}^{\pm\mathrm{rnd}}_{\mathsf{FAST}}(\mathcal{A}) \\
&= \Pr\left[\mathcal{A}^{\mathsf{FAST}_\rho.\mathsf{Encrypt}(\cdot,\cdot),\mathsf{FAST}_\rho.\mathsf{Decrypt}(\cdot,\cdot)} \Rightarrow 1\right] - \Pr\left[\mathcal{A}^{\rho_1(\cdot,\cdot),\rho_2(\cdot,\cdot)} \Rightarrow 1\right] + \mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathcal{B}) \\
&= \mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathcal{B}) + \Pr\left[\mathcal{A}^{G_{\mathsf{real}}} \Rightarrow 1\right] - \Pr\left[\mathcal{A}^{G_{\mathsf{rnd}}} \Rightarrow 1\right] \\
&\leq \mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathcal{B}) + \Pr\left[\left(\mathcal{A}^{G_{\mathsf{real}}} \Rightarrow 1\right) \wedge \overline{\mathsf{Bad}_{\mathsf{real}}(\mathcal{A})}\right] + \Pr\left[\mathsf{Bad}_{\mathsf{real}}(\mathcal{A})\right] - \Pr\left[\left(\mathcal{A}^{G_{\mathsf{rnd}}} \Rightarrow 1\right) \wedge \overline{\mathsf{Bad}_{\mathsf{rnd}}(\mathcal{A})}\right] \\
&= \mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathcal{B}) + \Pr\left[\left(\mathcal{A}^{G_{\mathsf{int}}} \Rightarrow 1\right) \wedge \overline{\mathsf{Bad}_{\mathsf{int}}(\mathcal{A})}\right] + \Pr\left[\mathsf{Bad}_{\mathsf{int}}(\mathcal{A})\right] - \Pr\left[\left(\mathcal{A}^{G_{\mathsf{rnd}}} \Rightarrow 1\right) \wedge \overline{\mathsf{Bad}_{\mathsf{rnd}}(\mathcal{A})}\right] \\
&= \mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathcal{B}) + \Pr\left[\mathsf{Bad}_{\mathsf{rnd}}(\mathcal{A})\right]. \qquad (49)
\end{aligned}$$

Adversary $\mathcal{A}$ runs in time $\mathfrak{T}$. We instead consider an adversary $\mathcal{C}$ which is allowed unbounded runtime and also unbounded memory. Consider the interaction of $\mathcal{C}$ with the oracle in the game $G_{\mathsf{rnd}}$ and define the event $\Pr\left[\mathsf{Bad}_{\mathsf{rnd}}(\mathcal{C})\right]$ in a manner analogous to $\Pr\left[\mathsf{Bad}_{\mathsf{rnd}}(\mathcal{A})\right]$. Clearly, we have

$$\Pr\left[\mathsf{Bad}_{\mathsf{rnd}}(\mathcal{A})\right] \leq \Pr\left[\mathsf{Bad}_{\mathsf{rnd}}(\mathcal{C})\right]. \qquad (50)$$

So, it is sufficient to upper bound $\Pr\left[\mathsf{Bad}_{\mathsf{rnd}}(\mathcal{C})\right]$. Since $\mathcal{C}$ has unbounded computational power, without loss of generality, we may assume that $\mathcal{C}$ is deterministic.

**Upper bound on** $\Pr[\mathsf{Bad}_{\mathsf{rnd}}(\mathcal{C})]$**:** An upper bound on $\Pr[\mathsf{Bad}_{\mathsf{rnd}}(\mathcal{C})]$ is obtained by showing that in the game $G_{\mathsf{rnd}}$ the event that two random variables in $\mathcal{D}$ are equal occurs with low probability. The main crux of the whole proof is to argue this in the various cases that arise in considering the different pairs of random variables from $\mathcal{D}$. The claims below tackle all the different cases that can arise.

**Claim 3.** *For* $1 \leq s \leq q$, $\Pr[F_1^{(s)} = \mathsf{fStr}] \leq \epsilon_1^{(s)}$.

*Proof.*

$$\begin{aligned}
\Pr[F_1^{(s)} = \mathsf{fStr}] &= \Pr[\tau P_1^{(s)} \oplus \tau h_\tau(T^{(s)}, P_3^{(s)}) \oplus P_2^{(s)} = \mathsf{fStr}] \\
&= \Pr[\tau(h_\tau(T^{(s)}, P_3^{(s)}) \oplus P_1^{(s)}) = P_2^{(s)} \oplus \mathsf{fStr}] \\
&\leq \epsilon_1^{(s)}.
\end{aligned}$$

The last inequality follows from (12). □

**Claim 4.** *For* $1 \leq s \leq q$, $\Pr[F_2^{(s)} = \mathsf{fStr}] \leq \epsilon_1^{(s)}$.

*Proof.*

$$\begin{aligned}
\Pr[F_2^{(s)} = \mathsf{fStr}] &= \Pr[\tau C_2^{(s)} \oplus \tau h'_\tau(T^{(s)}, C_3^{(s)}) \oplus C_1^{(s)} = \mathsf{fStr}] \\
&= \Pr[\tau(h'_\tau(T^{(s)}, C_3^{(s)}) \oplus C_2^{(s)}) = C_1^{(s)} \oplus \mathsf{fStr}] \\
&\leq \epsilon_1^{(s)}.
\end{aligned}$$

The last inequality follows from (13). □

**Claim 5.** *For* $1 \leq s \leq q$, $1 \leq i \leq m^{(s)} - 2$, $\Pr[J_i^{(s)} = \mathsf{fStr}] = 1/2^n$.

*Proof.*

$$\begin{aligned}
J_i^{(s)} \oplus \mathsf{fStr} &= Z^{(s)} \oplus \mathsf{bin}_n(i) \oplus \mathsf{fStr} \\
&= F_1^{(s)} \oplus F_2^{(s)} \oplus \mathsf{bin}_n(i) \oplus \mathsf{fStr} \\
&= \tau(P_1^{(s)} \oplus h_\tau(T^{(s)}, P_3^{(s)})) \oplus P_2^{(s)} \oplus C_1^{(s)} \oplus \tau(C_2^{(s)} \oplus h'_\tau(T^{(s)}, C_3^{(s)})) \oplus \mathsf{bin}_n(i) \oplus \mathsf{fStr}.
\end{aligned}$$

When $\mathsf{ty}^{(s)} = \mathsf{enc}$, then $C_1^{(s)}$ is an $n$-bit uniform random string which is independent of the other quantities and when $\mathsf{ty}^{(s)} = \mathsf{dec}$, then $P_2^{(s)}$ is an $n$-bit uniform random string which is independent of the other quantities. So in both cases we have the required probability. □

**Claim 6.** *For* $s \neq t$, $\Pr[F_1^{(s)} = F_1^{(t)}] \leq \max\{\epsilon_2^{(s,t)}, 1/2^n\}$.

*Proof.*

$$F_1^{(s)} \oplus F_1^{(t)} = \tau(P_1^{(s)} \oplus P_1^{(t)}) \oplus \tau(h_\tau(T^{(s)}, P_3^{(s)}) \oplus h_\tau(T^{(t)}, P_3^{(t)})) \oplus P_2^{(s)} \oplus P_2^{(t)}.$$

There are four cases to consider.

**Case 1:** $\mathsf{ty}^{(s)} = \mathsf{ty}^{(t)} = \mathsf{enc}$. There are two sub-cases.

(a) **Case 1a:** $(T^{(s)}, P_1^{(s)}, P_3^{(s)}) = (T^{(t)}, P_1^{(t)}, P_3^{(t)})$.
As the adversary is not allowed to repeat a query, hence $(T^{(s)}, P_1^{(s)}, P_3^{(s)}) = (T^{(t)}, P_1^{(t)}, P_3^{(t)})$ implies $P_2^{(s)} \neq P_2^{(t)}$ and so $\Pr[F_1^{(s)} = F_1^{(t)}] = 0$.

(b) **Case 1b:** $(T^{(s)}, P_1^{(s)}, P_3^{(s)}) \neq (T^{(t)}, P_1^{(t)}, P_3^{(t)})$.
If $(T^{(s)}, P_3^{(s)}) = (T^{(t)}, P_3^{(t)})$, then $P_1^{(s)} \neq P_1^{(t)}$ and so $F_1^{(s)} \oplus F_1^{(t)} = \tau(P_1^{(s)} \oplus P_1^{(t)}) \oplus P_2^{(s)} \oplus P_2^{(t)}$ is a non-zero polynomial in $\tau$ of degree 1. Thus, $\Pr[F_1^{(s)} = F_1^{(t)}] = 1/2^n$.
If $(T^{(s)}, P_3^{(s)}) \neq (T^{(t)}, P_3^{(t)})$, then

$$\begin{aligned}\Pr[F_1^{(s)} = F_1^{(t)}] &= \Pr[\tau(h_\tau(T^{(s)}, P_3^{(s)}) \oplus h_\tau(T^{(t)}, P_3^{(t)}) \oplus P_1^{(s)} \oplus P_1^{(t)}) = P_2^{(s)} \oplus P_2^{(t)}] \\ &\leq \epsilon_2^{(s,t)}.\end{aligned}$$

The last inequality follows from (14).
**Case 2:** $\mathsf{ty}^{(s)} = \mathsf{ty}^{(t)} = \mathsf{dec}$. In this case all of $P_1^{(s)}, P_1^{(t)}, P_2^{(s)}, P_2^{(t)}$ are independent and uniformly distributed $n$-bit strings and so $\Pr[F_1^{(s)} = F_1^{(t)}] = 1/2^n$.
**Case 3:** $\mathsf{ty}^{(s)} = \mathsf{enc}$ and $\mathsf{ty}^{(t)} = \mathsf{dec}$. In this case $P_1^{(t)}$ and $P_2^{(t)}$ are independent and uniformly distributed $n$-bit strings and so $\Pr[F_1^{(s)} = F_1^{(t)}] = 1/2^n$.
**Case 4:** $\mathsf{ty}^{(s)} = \mathsf{dec}$ and $\mathsf{ty}^{(t)} = \mathsf{enc}$. In this case $P_1^{(s)}$ and $P_2^{(s)}$ are independent and uniformly distributed $n$-bit strings and so $\Pr[F_1^{(s)} = F_1^{(t)}] = 1/2^n$.

□

**Claim 7.** *For $s \neq t$, $\Pr[F_2^{(s)} = F_2^{(t)}] \leq \max\{\epsilon_2^{(s,t)}, 1/2^n\}$.*

The proof is almost the same as the proof of Claim 6.

**Claim 8.** *For $1 \leq s, t \leq q$, $\Pr[F_1^{(s)} = F_2^{(t)}] \leq \max\{\epsilon_2^{(s,t)}, 1/2^n\}$.*

*Proof.*

$$F_1^{(s)} \oplus F_2^{(t)} = \tau(P_1^{(s)} \oplus C_2^{(t)}) \oplus \tau(h_\tau(T^{(s)}, P_3^{(s)}) \oplus h'_\tau(T^{(t)}, C_3^{(t)})) \oplus (P_2^{(s)} \oplus C_1^{(t)}).$$

There are four cases.

**Case 1:** $\mathsf{ty}^{(s)} = \mathsf{ty}^{(t)} = \mathsf{enc}$. In this case, $C_1^{(t)}$ is an independent and uniform random $n$-bit string and so $\Pr[F_1^{(s)} = F_2^{(t)}] = 1/2^n$.
**Case 2:** $\mathsf{ty}^{(s)} = \mathsf{enc}$ and $\mathsf{ty}^{(t)} = \mathsf{dec}$. We have

$$\begin{aligned}\Pr[F_1^{(s)} = F_2^{(t)}] &= \Pr[\tau(P_1^{(s)} \oplus C_2^{(t)}) \oplus \tau(h_\tau(T^{(s)}, P_3^{(s)}) \oplus h'_\tau(T^{(t)}, C_3^{(t)})) = P_2^{(s)} \oplus C_1^{(t)}] \\ &= \Pr[\tau(h_\tau(T^{(s)}, P_3^{(s)}) \oplus h'_\tau(T^{(t)}, C_3^{(t)}) \oplus P_1^{(s)} \oplus C_2^{(t)}) = P_2^{(s)} \oplus C_1^{(t)}] \\ &\leq \epsilon_2^{(s,t)}.\end{aligned}$$

The last inequality follows from (16).
**Case 3:** $\mathsf{ty}^{(s)} = \mathsf{dec}$ and $\mathsf{ty}^{(t)} = \mathsf{enc}$. In this case, $P_2^{(s)}$ is an independent and uniform random $n$-bit string and so $\Pr[F_1^{(s)} = F_2^{(t)}] = 1/2^n$.
**Case 4:** $\mathsf{ty}^{(s)} = \mathsf{ty}^{(t)} = \mathsf{dec}$. In this case also, $P_2^{(s)}$ is an independent and uniform random $n$-bit string and so $\Pr[F_1^{(s)} = F_2^{(t)}] = 1/2^n$.

$\square$

**Claim 9.** *For $1 \leq s, t \leq q$ and $1 \leq i \leq m^{(t)} - 2$, $\Pr[F_1^{(s)} = J_i^{(t)}] \leq \epsilon_1^{(s)}$.*

*Proof.*

$$\begin{aligned}
\Pr[F_1^{(s)} = J_i^{(t)}] &= \Pr[F_1^{(s)} = F_1^{(t)} \oplus F_2^{(t)} \oplus \mathsf{bin}_n(i)] \\
&= \Pr[\tau(h_\tau(T^{(s)}, P_3^{(s)}) \oplus P_1^{(s)}) \oplus P_2^{(s)} \\
&\qquad = \tau(P_1^{(t)} \oplus h_\tau(T^{(t)}, P_3^{(t)})) \oplus P_2^{(t)} \\
&\qquad\quad \oplus C_1^{(t)} \oplus \tau(C_2^{(t)} \oplus h'_\tau(T^{(t)}, C_3^{(t)})) \oplus \mathsf{bin}_n(i)].
\end{aligned}$$

First suppose that $s \neq t$. If $\mathsf{ty}^{(t)} = \mathsf{enc}$, then $C_1^{(t)}$ is a uniform $n$-bit string which is independent of the other quantities and if $\mathsf{ty}^{(t)} = \mathsf{dec}$, then $P_2^{(t)}$ is a uniform $n$-bit string which is independent of the other quantities. In both cases, the above probability is $1/2^n$.

So, suppose that $s = t$. Then the required probability reduces to

$$\begin{aligned}
\Pr[F_1^{(s)} = J_i^{(s)}] &= \Pr[F_2^{(s)} = \mathsf{bin}_n(i)] \\
&= \Pr[\tau(C_2^{(s)} \oplus h'_\tau(T^{(s)}, C_3^{(s)})) = C_1^{(s)} \oplus \mathsf{bin}_n(i)] \\
&\leq \epsilon_1^{(s)}.
\end{aligned}$$

If $\mathsf{ty}^{(s)} = \mathsf{enc}$, then $C_1^{(s)}$ is a uniform $n$-bit string which is independent of the other quantities and so the probability is equal to $1/2^n$; on the other hand, if $\mathsf{ty}^{(s)} = \mathsf{dec}$, then the last inequality follows from (13). $\square$

**Claim 10.** *For $1 \leq s, t \leq q$ and $1 \leq i \leq m^{(t)} - 2$, $\Pr[F_2^{(s)} = J_i^{(t)}] \leq \epsilon_1^{(s)}$.*

The proof is almost the same as the proof of Claim 9.

**Claim 11.** *For $1 \leq s, t \leq q$, $1 \leq i \leq m^{(s)} - 2$, $1 \leq j \leq m^{(t)} - 2$ and $(s, i) \neq (t, j)$, $\Pr[J_i^{(s)} = J_j^{(t)}] \leq 1/2^n$.*

*Proof.*

$$\begin{aligned}
J_i^{(s)} \oplus J_j^{(t)} &= F_1^{(s)} \oplus F_2^{(s)} \oplus F_1^{(t)} \oplus F_2^{(t)} \oplus \mathsf{bin}_n(i) \oplus \mathsf{bin}_n(j) \\
&= \tau(P_1^{(s)} \oplus h_\tau(T^{(s)}, P_3^{(s)})) \oplus P_2^{(s)} \oplus C_1^{(s)} \oplus \tau(C_2^{(s)} \oplus h'_\tau(T^{(s)}, C_3^{(s)})) \\
&\quad \oplus \tau(P_1^{(t)} \oplus h_\tau(T^{(t)}, P_3^{(t)})) \oplus P_2^{(t)} \oplus C_1^{(t)} \oplus \tau(C_2^{(t)} \oplus h'_\tau(T^{(t)}, C_3^{(t)})) \\
&\quad \oplus \mathsf{bin}_n(i) \oplus \mathsf{bin}_n(j).
\end{aligned}$$

If $s = t$, then $i \neq j$ and so $\Pr[J_i^{(s)} = J_j^{(t)}] = \Pr[\mathsf{bin}_n(i) = \mathsf{bin}_n(j)] = 0$. Suppose that $s \neq t$. There are four cases to consider.

- If $\mathsf{ty}^{(s)} = \mathsf{ty}^{(t)} = \mathsf{enc}$, then both $C_1^{(s)}$ and $C_1^{(t)}$ are independent and uniform random strings which are independent of the other quantities.
- If $\mathsf{ty}^{(s)} = \mathsf{ty}^{(t)} = \mathsf{dec}$, then both $P_2^{(s)}$ and $P_2^{(t)}$ are independent and uniform random strings which are independent of the other quantities.

– If $\mathsf{ty}^{(s)} = \mathsf{enc}$ and $\mathsf{ty}^{(t)} = \mathsf{dec}$, then both $C_1^{(s)}$ and $P_2^{(t)}$ are independent and uniform random strings which are independent of the other quantities.
– If $\mathsf{ty}^{(s)} = \mathsf{dec}$ and $\mathsf{ty}^{(t)} = \mathsf{enc}$, then both $P_2^{(s)}$ and $C_1^{(t)}$ are independent and uniform random strings which are independent of the other quantities.

From the above it follows that if $s \neq t$, then in all cases the probability is equal to $1/2^n$. Thus, the claim follows. □

By Claims 3 to 11 and the union bound we have

$$\begin{aligned}
\Pr\left[\mathsf{Bad}_{\mathsf{rnd}}(\mathcal{C})\right] &\leq 2\sum_{s=1}^{q}\epsilon_1^{(s)} + \sum_{s=1}^{q}\left(\frac{m^{(s)}-2}{2^n}\right) + \sum_{1\leq s<t\leq q} 2\left(\epsilon_2^{(s,t)} + \frac{1}{2^n}\right) \\
&\quad + \sum_{1\leq s\leq t\leq q}\left(\epsilon_2^{(s,t)} + \frac{1}{2^n}\right) + 2\left(\sum_{s=1}^{q}\epsilon_1^{(s)}\right)\left(\sum_{t=1}^{q}(m^{(t)}-2)\right) + \frac{1}{2^n}\binom{\sum_{s=1}^{q}(m^{(s)}-2)}{2} \\
&= 2\left(\sum_{s=1}^{q}\epsilon_1^{(s)}\right)\left(1+\sum_{t=1}^{q}(m^{(t)}-2)\right) + \frac{3}{2^n}\frac{q(q-1)}{2} + \frac{q}{2^n} + 3\sum_{1\leq s<t\leq q}\epsilon_2^{(s,t)} + \sum_{s=1}^{q}\epsilon_2^{(s,s)} \\
&\quad + \sum_{s=1}^{q}\left(\frac{m^{(s)}-2}{2^n}\right) + \frac{1}{2^n}\binom{\sum_{s=1}^{q}(m^{(s)}-2)}{2} \\
&\leq 2\left(\sum_{s=1}^{q}\epsilon_1^{(s)}\right)(1+\omega-2q) + \frac{2q^2}{2^n} + 3\sum_{1\leq s<t\leq q}\epsilon_2^{(s,t)} + \sum_{s=1}^{q}\epsilon_2^{(s,s)} \\
&\quad + \frac{\omega-2q}{2^n} + \frac{1}{2^n}\frac{\omega(\omega-1)}{2} \\
&\leq 2\omega\left(\sum_{s=1}^{q}\epsilon_1^{(s)}\right) + 3\sum_{1\leq s<t\leq q}\epsilon_2^{(s,t)} + \sum_{s=1}^{q}\epsilon_2^{(s,s)} + \frac{3\omega^2}{2^n}. \qquad (51)
\end{aligned}$$

Putting together (49), (50) and (51) we obtain

$$\mathbf{Adv}^{\pm\mathrm{rnd}}_{\mathsf{FAST}}(\mathcal{A}) \leq \mathbf{Adv}^{\mathrm{prf}}_{\mathbf{F}}(\mathcal{B}) + 2\omega\left(\sum_{s=1}^{q}\epsilon_1^{(s)}\right) + 3\sum_{1\leq s<t\leq q}\epsilon_2^{(s,t)} + \sum_{s=1}^{q}\epsilon_2^{(s,s)} + \frac{3\omega^2}{2^n}.$$

The relations between the resources of $\mathcal{A}$ and $\mathcal{B}$ have been stated earlier. Maximising the left hand side on the resources shows the required result and completes the proof of Theorem 1. □

Table 10: Game $G_{\mathsf{real}}$.

| | |
|---|---|
| Subroutine $\mathsf{Ch}\text{-}\rho(X)$<br>$Y \xleftarrow{\$} \{0,1\}^n$;<br>if $X \in \mathcal{D}$ then $\mathsf{bad} \leftarrow \mathsf{true}$; $\boxed{Y \leftarrow \rho(X);}$ endif;<br>$\rho(X) \leftarrow Y$; $\mathcal{D} \leftarrow \mathcal{D} \cup \{X\}$; **return**$(Y)$;<br>Initialization:<br>$\tau \xleftarrow{\$} \{0,1\}^n$; $\mathcal{D} \leftarrow \{\mathsf{fStr}\}$; $\mathsf{bad} \leftarrow \mathsf{false}$. | |
| $\mathsf{ty}^{(s)} = \mathsf{enc}$: input $(T^{(s)}, P^{(s)})$ | $\mathsf{ty}^{(s)} = \mathsf{dec}$: input $(T^{(s)}, C^{(s)})$ |
| $(P_1^{(s)}, P_2^{(s)}, P_3^{(s)}) \leftarrow \mathsf{parse}_n(P^{(s)})$; | $(C_1^{(s)}, C_2^{(s)}, C_3^{(s)}) \leftarrow \mathsf{parse}_n(C^{(s)})$; |
| $A_1^{(s)} \leftarrow P_1^{(s)} \oplus h_\tau(T^{(s)}, P_3^{(s)})$; | $B_2^{(s)} \leftarrow C_2^{(s)} \oplus h'_\tau(T^{(s)}, C_3^{(s)})$; |
| $F_1^{(s)} \leftarrow \tau A_1^{(s)} \oplus P_2^{(s)}$; | $F_2^{(s)} \leftarrow C_1^{(s)} \oplus \tau B_2^{(s)}$; |
| $F_2^{(s)} \leftarrow A_1^{(s)} \oplus \mathsf{Ch}\text{-}\rho(F_1^{(s)})$; | $F_1^{(s)} \leftarrow B_2^{(s)} \oplus \mathsf{Ch}\text{-}\rho(F_2^{(s)})$; |
| $B_2^{(s)} \leftarrow F_1^{(s)} \oplus \mathsf{Ch}\text{-}\rho(F_2^{(s)})$; | $A_1^{(s)} \leftarrow F_2^{(s)} \oplus \mathsf{Ch}\text{-}\rho(F_1^{(s)})$; |
| $C_1^{(s)} \leftarrow \tau B_2^{(s)} \oplus F_2^{(s)}$; | $P_2^{(s)} \leftarrow \tau A_1^{(s)} \oplus F_1^{(s)}$; |
| $Z^{(s)} \leftarrow F_1^{(s)} \oplus F_2^{(s)}$; | $Z^{(s)} \leftarrow F_1^{(s)} \oplus F_2^{(s)}$; |
| for $i = 1$ to $m^{(s)} - 3$ do | for $i = 1$ to $m^{(s)} - 3$ do |
| $J_i^{(s)} \leftarrow Z^{(s)} \oplus \mathsf{bin}_n(i)$; | $J_i^{(s)} \leftarrow Z^{(s)} \oplus \mathsf{bin}_n(i)$; |
| $S_i^{(s)} \leftarrow \mathsf{Ch}\text{-}\rho(J_i^{(s)})$; | $S_i^{(s)} \leftarrow \mathsf{Ch}\text{-}\rho(J_i^{(s)})$; |
| $C_{3,i}^{(s)} \leftarrow P_{3,i}^{(s)} \oplus S_i^{(s)}$; | $P_{3,i}^{(s)} \leftarrow C_{3,i}^{(s)} \oplus S_i^{(s)}$; |
| end for; | end for; |
| $J_{m^{(s)}-2}^{(s)} \leftarrow Z^{(s)} \oplus \mathsf{bin}_n(m^{(s)} - 2)$; | $J_{m^{(s)}-2}^{(s)} \leftarrow Z^{(s)} \oplus \mathsf{bin}_n(m^{(s)} - 2)$; |
| $D^{(s)} \leftarrow \mathsf{Ch}\text{-}\rho(J_{m^{(s)}-2}^{(s)})$; | $E^{(s)} \leftarrow \mathsf{Ch}\text{-}\rho(J_{m^{(s)}-2}^{(s)})$; |
| $C_{3,m^{(s)}-2}^{(s)} \leftarrow P_{3,m^{(s)}-2}^{(s)} \oplus \mathsf{first}_{r^{(s)}}(D^{(s)})$; | $P_{3,m^{(s)}-2}^{(s)} \leftarrow C_{3,m^{(s)}-2}^{(s)} \oplus \mathsf{first}_{r^{(s)}}(E^{(s)})$; |
| $C_2^{(s)} \leftarrow B_2^{(s)} \oplus h'_\tau(T^{(s)}, C_3^{(s)})$; | $P_1^{(s)} \leftarrow A_1^{(s)} \oplus h_\tau(T^{(s)}, P_3^{(s)})$; |
| return $(C_1^{(s)} \| C_2^{(s)} \| C_3^{(s)})$. | return $(P_1^{(s)} \| P_2^{(s)} \| P_3^{(s)})$. |

Table 11: Game $G_{\mathsf{int}}$.

<table>
<tr><td colspan="2">Subroutine $\mathsf{ChkDom}(X)$<br>    if $X \in \mathcal{D}$ then $\mathsf{bad} \leftarrow \mathsf{true}$; endif;<br>    $\mathcal{D} \leftarrow \mathcal{D} \cup \{X\}$;<br><u>Initialization</u>:<br>    $\tau \xleftarrow{\$} \{0,1\}^n$; $\mathcal{D} \leftarrow \{\mathsf{fStr}\}$; $\mathsf{bad} \leftarrow \mathsf{false}$.</td></tr>
<tr><td><u>$\mathsf{ty}^{(s)} = \mathsf{enc}$: input $(T^{(s)}, P^{(s)})$</u><br>
$(P_1^{(s)}, P_2^{(s)}, P_3^{(s)}) \leftarrow \mathsf{parse}_n(P^{(s)})$;<br>
$C_1^{(s)} \xleftarrow{\$} \{0,1\}^n$; $C_2^{(s)} \xleftarrow{\$} \{0,1\}^n$;<br>
for $i = 1, \ldots, m^{(s)} - 3$ do $C_{3,i}^{(s)} \xleftarrow{\$} \{0,1\}^n$;<br>
$D^{(s)} \xleftarrow{\$} \{0,1\}^n$; $C_{3,m^{(s)}-2} \leftarrow \mathsf{first}_{r^{(s)}}(D^{(s)})$;<br>
$A_1^{(s)} \leftarrow P_1^{(s)} \oplus h_\tau(T^{(s)}, P_3^{(s)})$;<br>
$F_1^{(s)} \leftarrow \tau A_1^{(s)} \oplus P_2^{(s)}$; $\mathsf{ChkDom}(F_1^{(s)})$;<br>
$Y_1^{(s)} \leftarrow C_1^{(s)} \oplus P_1^{(s)} \oplus \tau(C_2^{(s)} \oplus h'_\tau(T^{(s)}, C_3^{(s)}))$<br>
        $\oplus h_\tau(T^{(s)}, P_3^{(s)})$;<br>
$\rho(F_1^{(s)}) \leftarrow Y_1^{(s)}$;<br>
$F_2^{(s)} \leftarrow A_1^{(s)} \oplus Y_1^{(s)}$; $\mathsf{ChkDom}(F_2^{(s)})$;<br>
$Y_2^{(s)} \leftarrow C_2^{(s)} \oplus P_2^{(s)} \oplus \tau P_1^{(s)} \oplus h'_\tau(T^{(s)}, C_3^{(s)})$<br>
        $\oplus \tau h_\tau(T^{(s)}, P_3^{(s)})$;<br>
$\rho(F_2^{(s)}) \leftarrow Y_2^{(s)}$;<br>
$B_2^{(s)} \leftarrow F_1^{(s)} \oplus Y_2^{(s)}$;<br>
$C_1^{(s)} \leftarrow \tau B_2^{(s)} \oplus F_2^{(s)}$;<br>
$Z^{(s)} \leftarrow F_1^{(s)} \oplus F_2^{(s)}$;<br>
for $i = 1$ to $m^{(s)} - 3$ do<br>
  $J_i^{(s)} \leftarrow Z^{(s)} \oplus \mathsf{bin}_n(i)$; $\mathsf{ChkDom}(J_i^{(s)})$;<br>
  $\rho(J_i^{(s)}) \leftarrow C_{3,i}^{(s)} \oplus P_{3,i}^{(s)}$;<br>
end for;<br>
$J_{m^{(s)}-2}^{(s)} \leftarrow Z^{(s)} \oplus \mathsf{bin}_n(m^{(s)} - 2)$; $\mathsf{ChkDom}(J_{m^{(s)}-2}^{(s)})$;<br>
$\rho(J_{m^{(s)}-2}^{(s)}) \leftarrow D^{(s)} \oplus (P_{3,m^{(s)}-2}^{(s)} \| 0^{n-r^{(s)}})$;<br>
return $(C_1^{(s)} \| C_2^{(s)} \| C_3^{(s)})$.</td>
<td><u>$\mathsf{ty}^{(s)} = \mathsf{dec}$: input $(T^{(s)}, C^{(s)})$</u><br>
$(C_1^{(s)}, C_2^{(s)}, C_3^{(s)}) \leftarrow \mathsf{parse}_n(C^{(s)})$;<br>
$P_1^{(s)} \xleftarrow{\$} \{0,1\}^n$; $P_2^{(s)} \xleftarrow{\$} \{0,1\}^n$;<br>
for $i = 1, \ldots, m^{(s)} - 3$ do $P_{3,i}^{(s)} \xleftarrow{\$} \{0,1\}^n$;<br>
$E^{(s)} \xleftarrow{\$} \{0,1\}^n$; $P_{3,m^{(s)}-2} \leftarrow \mathsf{first}_{r^{(s)}}(E^{(s)})$;<br>
$B_2^{(s)} \leftarrow C_2^{(s)} \oplus h'_\tau(T^{(s)}, C_3^{(s)})$;<br>
$F_2^{(s)} \leftarrow C_1^{(s)} \oplus \tau B_2^{(s)}$; $\mathsf{ChkDom}(F_2^{(s)})$;<br>
$Y_2^{(s)} \leftarrow C_2^{(s)} \oplus P_2^{(s)} \oplus \tau P_1^{(s)} \oplus h'_\tau(T^{(s)}, C_3^{(s)})$<br>
        $\oplus \tau h_\tau(T^{(s)}, P_3^{(s)})$;<br>
$\rho(F_2^{(s)}) \leftarrow Y_2^{(s)}$;<br>
$F_1^{(s)} \leftarrow B_2^{(s)} \oplus Y_2^{(s)}$; $\mathsf{ChkDom}(F_1^{(s)})$;<br>
$Y_1^{(s)} \leftarrow C_1^{(s)} \oplus P_1^{(s)} \oplus \tau(C_2^{(s)} \oplus h'_\tau(T^{(s)}, C_3^{(s)}))$<br>
        $\oplus h_\tau(T^{(s)}, P_3^{(s)})$;<br>
$\rho(F_1^{(s)}) \leftarrow Y_1^{(s)}$;<br>
$A_1^{(s)} \leftarrow F_2^{(s)} \oplus Y_1^{(s)}$;<br>
$P_2^{(s)} \leftarrow \tau A_1^{(s)} \oplus F_1^{(s)}$;<br>
$Z^{(s)} \leftarrow F_1^{(s)} \oplus F_2^{(s)}$;<br>
for $i = 1$ to $m^{(s)} - 3$ do<br>
  $J_i^{(s)} \leftarrow Z^{(s)} \oplus \mathsf{bin}_n(i)$; $\mathsf{ChkDom}(J_i^{(s)})$;<br>
  $\rho(J_i^{(s)}) \leftarrow C_{3,i}^{(s)} \oplus P_{3,i}^{(s)}$;<br>
end for;<br>
$J_{m^{(s)}-2}^{(s)} \leftarrow Z^{(s)} \oplus \mathsf{bin}_n(m^{(s)} - 2)$; $\mathsf{ChkDom}(J_{m^{(s)}-2}^{(s)})$;<br>
$\rho(J_{m^{(s)}-2}^{(s)}) \leftarrow E^{(s)} \oplus (C_{3,m^{(s)}-2}^{(s)} \| 0^{n-r^{(s)}})$;<br>
return $(P_1^{(s)} \| P_2^{(s)} \| P_3^{(s)})$.</td></tr>
</table>

Table 12: Game $G_{\mathsf{rnd}}$

Respond to the $s^{th}$ adversary query as follows:

$$\text{if } \mathsf{ty}^{(s)} = \mathsf{enc};\ C_1^{(s)}||C_2^{(s)}||C_{3,1}^{(s)}||\ldots||C_{3,m^{(s)}-3}^{(s)}||D^{(s)} \xleftarrow{\$} \{0,1\}^{nm^{(s)}};$$
$$C_{3,m^{(s)}-2}^{(s)} \leftarrow \mathsf{first}_{r^{(s)}}(D^{(s)}) \ \mathbf{return}\ C^{(s)} = C_1^{(s)}||C_2^{(s)}||C_{3,1}^{(s)}||\ldots||C_{3,m^{(s)}-2}^{(s)};$$
$$\text{if } \mathsf{ty}^{(s)} = \mathsf{dec};\ P_1^{(s)}||P_2^{(s)}||P_{3,1}^{(s)}||\ldots||P_{3,m^{(s)}-3}^{(s)}||E^{(s)} \xleftarrow{\$} \{0,1\}^{nm^{(s)}};$$
$$P_{3,m^{(s)}-2}^{(s)} \leftarrow \mathsf{first}_{r^{(s)}}(E^{(s)}) \ \mathbf{return}\ P^{(s)} = P_1^{(s)}||P_2^{(s)}||P_{3,1}^{(s)}||\ldots||P_{3,m^{(s)}-2}^{(s)};$$

Finalisation:

$$
\begin{aligned}
&\mathcal{D} \leftarrow \{\mathsf{fStr}\};\ \mathsf{bad} \leftarrow \mathsf{false};\ \tau \xleftarrow{\$} \{0,1\}^n;\\
&\text{for } s = 1 \text{ to } q \text{ do}\\
&\quad A_1^{(s)} \leftarrow P_1^{(s)} \oplus h_\tau(T^{(s)}, P_3^{(s)});\\
&\quad B_2^{(s)} \leftarrow C_2^{(s)} \oplus h'_\tau(T^{(s)}, C_3^{(s)});\\
&\quad F_1^{(s)} \leftarrow \tau A_1^{(s)} \oplus P_2^{(s)} = \tau(P_1^{(s)} \oplus h_\tau(T^{(s)}, P_3^{(s)})) \oplus P_2^{(s)};\\
&\quad \mathcal{D} \leftarrow \mathcal{D} \cup \{F_1^{(s)}\};\\
&\quad F_2^{(s)} \leftarrow C_1^{(s)} \oplus \tau B_2^{(s)} = C_1^{(s)} \oplus \tau(C_2^{(s)} \oplus h'_\tau(T^{(s)}, C_3^{(s)}));\\
&\quad \mathcal{D} \leftarrow \mathcal{D} \cup \{F_2^{(s)}\};\\
&\quad Z^{(s)} \leftarrow F_1^{(s)} \oplus F_2^{(s)};\\
&\quad \text{for } i = 1 \text{ to } m^{(s)} - 3;\\
&\quad\quad J_i^{(s)} \leftarrow Z^{(s)} \oplus \mathsf{bin}_n(i) = F_1^{(s)} \oplus F_2^{(s)} \oplus \mathsf{bin}_n(i);\\
&\quad\quad \mathcal{D} \leftarrow \mathcal{D} \cup \{J_i^{(s)}\};\\
&\quad \text{end for;}\\
&\quad \text{if } \mathsf{ty}^{(s)} = \mathsf{enc} \text{ then } J_{m^{(s)}-2}^{(s)} \leftarrow D^{(s)} \oplus (P_{3,m^{(s)}-2}^{(s)}||0^{n-r^{(s)}});\\
&\quad \text{if } \mathsf{ty}^{(s)} = \mathsf{dec} \text{ then } J_{m^{(s)}-2}^{(s)} \leftarrow E^{(s)} \oplus (C_{3,m^{(s)}-2}^{(s)}||0^{n-r^{(s)}});\\
&\quad \mathcal{D} \leftarrow \mathcal{D} \cup \{J_{m^{(s)}-2}^{(s)}\};\\
&\text{end for;}\\
&\text{if (some value occurs more than once in } \mathcal{D}) \text{ then } \mathsf{bad} \leftarrow \mathsf{true} \text{ endif;}
\end{aligned}
$$

## B Implementation of AEZ

For $\alpha \in \{0,1\}^{128}$ and $i \in \mathbb{N}$, the following operation has been defined [26] in the context of AEZ:

$$i \cdot \alpha = \begin{cases} \mathbf{0} & \text{if } i = 0; \\ \alpha & \text{if } i = 1; \\ (\alpha \ll 1) \oplus (\mathsf{msb}(\alpha) \cdot 135) & \text{if } i = 2; \\ 2 \cdot (j \cdot \alpha) & \text{if } i = 2j > 2; \\ (2j \cdot \alpha) \oplus \alpha & \text{if } i = 2j + 1 > 2. \end{cases} \tag{52}$$

The operation in (52) corresponding to $i = 2$ is the doubling operation. (See Section 2).

There are different versions of AEZ [26] built from different variants of AES. The version which is relevant to our work is the one where the proper AES algorithm is used. Messages of lengths at least $2n$ bits are handled differently from messages of lengths less than $2n$ bits. For our purpose, we will be considering the portion which can handle messages of lengths at least $2n$ bits. This portion has been called AEZ-Core [26]. By AEZ, we will denote AEZ-Core[AES].

The length of the message is written as $2nk + \mu$ bits with $0 \leq \mu < 2n$ and $k \geq 1$. Below we provide an overview of the encryption algorithm of AEZ where $\mu = 0$. Let $m = 2\mathfrak{m} + 2$ and consider a message having $m$ blocks with total length $n(2\mathfrak{m} + 2)$ bits. The message is partitioned into two parts: The first part consists of $2\mathfrak{m}n$ bits organised as $2\mathfrak{m}$ $n$-bit blocks $M_1, M'_1, \ldots, M_\mathfrak{m}, M'_\mathfrak{m}$ and the second part consists of $2n$ bits organised as 2 $n$-bit blocks $M_\mathsf{x}$ and $M_\mathsf{y}$. The ciphertext blocks are $C_i, C'_i$, $i = 1, \ldots, \mathfrak{m}$ and $C_\mathsf{x}, C_\mathsf{y}$.

At a conceptual level, this encryption consists of three layers. The first and the third layers consist of a sequence of 2-round Feistel networks where each Feistel network requires 2 block cipher calls. The second layer is a mixing layer and requires one block cipher call for each $i$. Let $E$ denote the encryption function of AES and for $\alpha \in \{0,1\}^n$, define $\widetilde{E}_K^{i,j}(\alpha) = E_K(\alpha \oplus (i+1) \cdot I \oplus j \cdot J)$ where $I = E_K(\mathbf{0})$ and $J = E_K(\mathbf{1})$ [26]. The encryption proceeds as follows:

*First layer:* for $i = 1, \ldots, \mathfrak{m}$, $W_i = M_i \oplus \widetilde{E}_K^{1,i}(M'_i)$; $X_i = M'_i \oplus \widetilde{E}_K^{0,0}(W_i)$;
$S_\mathsf{x} = \widetilde{E}_K^{0,1}(M_\mathsf{y}) \oplus M_\mathsf{x} \oplus X \oplus \mathit{\Delta}$; $S_\mathsf{y} = \widetilde{E}_K^{-1,1}(S_\mathsf{x}) \oplus M_\mathsf{y}$;
*Second layer:* for $i = 1, \ldots, \mathfrak{m}$, $S'_i = \widetilde{E}_K^{2,i}(S)$; $Y_i = S'_i \oplus W_i$; $Z_i = S'_i \oplus X_i$;
*Third layer:* for $i = 1, \ldots, \mathfrak{m}$, $C'_i = Y_i \oplus \widetilde{E}_K^{0,0}(Z_i)$; $C_i = Z_i \oplus \widetilde{E}_K^{1,i}(C'_i)$;
$C_\mathsf{y} = S_\mathsf{x} \oplus \widetilde{E}_K^{-1,2}(S_\mathsf{y})$; $C_\mathsf{x} = S_\mathsf{y} \oplus \widetilde{E}_K^{0,2}(C_\mathsf{y}) \oplus \mathit{\Delta} \oplus Y$;

Here $X = X_1 \oplus \cdots \oplus X_\mathfrak{m}$, $Y = Y_1 \oplus \cdots \oplus Y_\mathfrak{m}$, $S = S_\mathsf{x} \oplus S_\mathsf{y}$ and $\mathit{\Delta}$ is obtained by processing the tweak.

**Remarks:**

1. For both the software and hardware implementations, we have considered $n = 128$ and $m = 256$, i.e., messages of lengths equal to 4096 bytes. So, writing $m = 2\mathfrak{m} + 2$ we have $\mathfrak{m} = 127$.
2. In our software and hardware implementations, we have taken $\mathit{\Delta} = \mathbf{0}$, i.e., we have ignored the processing of the tweak. Since the resulting implementations of AEZ turn out to be less efficient than $\mathsf{FAST}$, considering the processing of the tweak for AEZ will result in further slowdown compared to $\mathsf{FAST}$.

### B.1 Software Implementation

The design of AEZ is based on OTR [33] and the parallelism in AEZ is the same as that in OTR. The encryptions in the first layer can be divided into two classes – one class consisting of the encryptions of $W_1, W_2, \ldots$ and the other class consisting of the encryptions of $M'_1, M'_2, \ldots$. Following the pipelining strategy for AES described in Section 8.2, the encryptions in each class have to be bunched into groups of eight. The encryptions proceed as follows. One bunch of $M'_i$'s is encrypted followed by the corresponding bunch of $W_i$'s; then the next bunch of $M'_i$'s is encrypted followed by the corresponding bunch of $W_i$'s and so on. After all the encryptions in the first layer are over, the encryptions in the second layer are to be computed. These encryptions are independent and can be executed in groups of eight. The strategy for executing the encryptions in the third layer is similar to that of the first layer. Though somewhat complicated, the above mentioned strategy can be used to obtain the benefits of pipelined AES execution.

One important efficiency issue is that of computing the values $j \cdot J$. As briefly mentioned in the paper introducing AEZ [26], by storing some of the previously generated values, it is possible to efficiently generate the required values. We provide some details. A queue of values $2 \cdot J, 3 \cdot J, \ldots$ is maintained. When $j \cdot J$ is computed, it is added to the tail of the queue. If the head of the queue contains $\jmath \cdot J$, then this value is used to generate $2\jmath \cdot J$ and $(2\jmath + 1) \cdot J$ and then the entry $\jmath \cdot J$ is deleted from the queue. Using this strategy, the computation of $2\jmath \cdot J$ from $\jmath \cdot J$ requires one doubling and the computation of $(2\jmath+1) \cdot J$ from $\jmath \cdot J$ requires one XOR. Overall, the computations of $2\jmath \cdot J$ and $(2\jmath+1) \cdot J$ require one doubling and one XOR. For the $2\mathfrak{m}$ blocks in the first part, the operation (52) will be required to be applied $\mathfrak{m}$ times. This will require $\lfloor \mathfrak{m}/2 \rfloor$ doubling operations and $\lfloor (\mathfrak{m} - 1)/2 \rfloor$ XOR operations. Since $m = 2\mathfrak{m} + 2$, the computations of all the masks require $\lfloor (m-2)/4 \rfloor$ doubling operations.

### B.2 Hardware Implementation

Hardware implementation of AEZ has been reported in the literature [27]. This work considered a reduced round version of AES, i.e., AEZ instantiated with 4-round AES. In contrast, we consider AEZ with full AES. Consequently, the pipelining issues considered in the earlier work [27] are different from our work. Below we provide a brief description of the architecture that we have designed to implement AEZ.

The architecture in Figure 3 allows the computation of AEZ encryption/decryption. The two AES cores are labelled as **AES** and **AES′** and the inputs to these cores are selected by **mux1** and **mux2** respectively. For computing the masking values $I$ and $J$ we need the encryptions of $\mathbf{0}$ and $\mathbf{1}$ respectively. This is enabled using **mux3**. **AES** and **AES′** work in parallel to compute $X_i$ and $W_i$ respectively. Since $X_i$ depends on $W_i$, **AES** waits for the first value of $W_i$ to be produced to start the computation. The computed values of $W_i$ and $X_i$ have to be stored and for that we use the single-port-block-RAMs **memWi** and **memXi** respectively. At a later stage, the values $S_{\mathsf{x}}$ and $S_{\mathsf{y}}$ are also stored in **memWi** and **memXi** respectively. The computation of $X = X_1 \oplus \cdots \oplus X_{127}$ is performed by **ACCX**. The values of $S'_i$'s are computed with the two AES cores. Due to data dependencies, these values need to be stored and for this purpose the dual-port-block-RAM **memSi** is used. In **memSi**, the last value is initialised to $\mathbf{0}$ and so $Z_{128} = S_{\mathsf{y}} \oplus \mathbf{0} = S_{\mathsf{y}}$ and $Y_{128} = S_{\mathsf{x}} \oplus \mathbf{0} = S_{\mathsf{x}}$. The computation of $Y = Y_1 \oplus \cdots \oplus Y_{127}$ is performed by **ACCY**. The input line marked $C'_i$ to **mux1** carries $C_{\mathsf{y}}$ at the end.

The masking values $j \cdot J$ can be computed in two different ways.

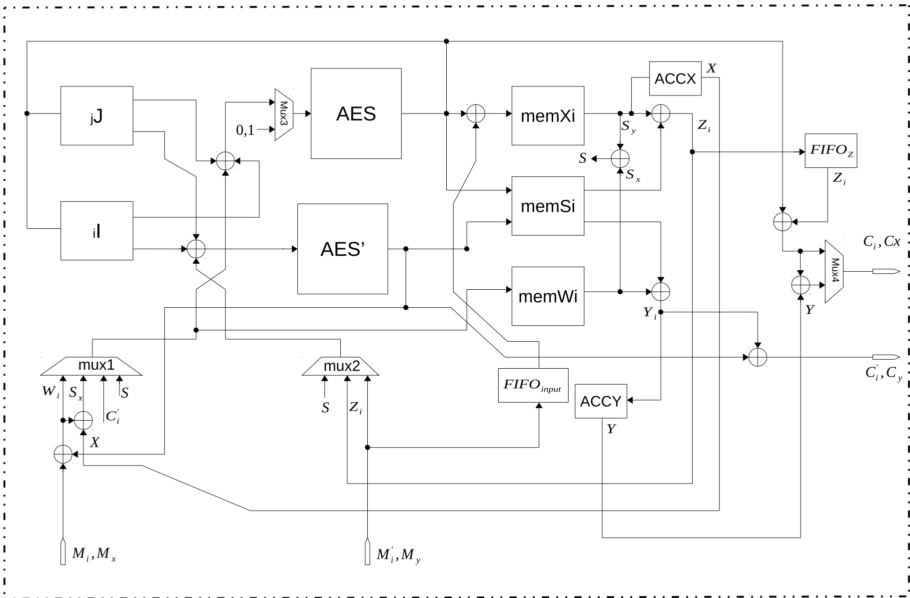
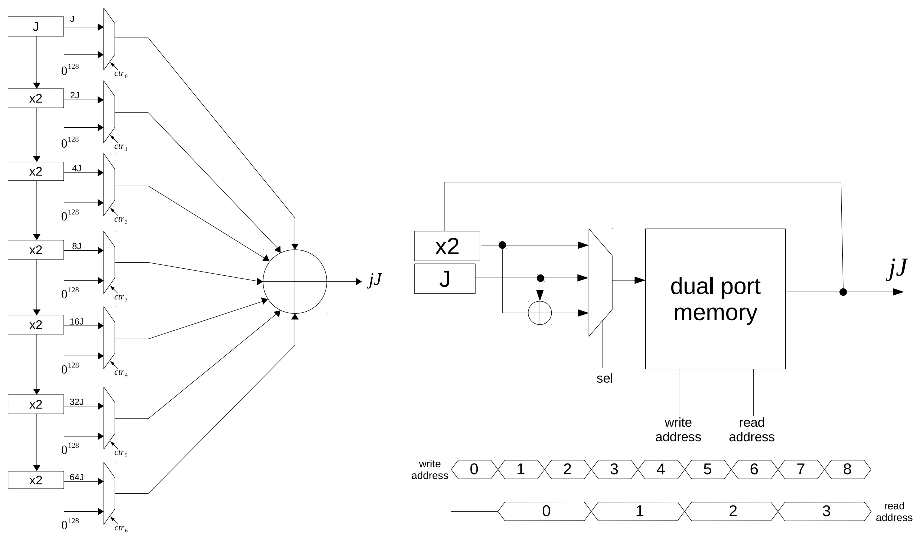



Fig. 3: Pipelined architecture for AEZ using two AES-encryption cores.

**Pre-computation:** The values of $I$, $J$ and all the necessary values $i \cdot I$, $j \cdot J$ can be precomputed. Computing $I$ and $J$ take 13 clock cycles while 127 clock cycles are necessary to compute all $j \cdot J$; $i \cdot I$ are only 4 values and are computed in parallel with $j \cdot J$. So the precomputation takes 145 clock cycles, taking into account the reset time and some clock cycles for synchronisation of the memory. In Figure 4b, we show the architecture to compute all the necessary values of $j \cdot J$ using double and add method. The values are stored in block RAMs. For the values $i \cdot I$, only the values $\{0, I, 2 \cdot I, 3 \cdot I\}$ are required and so they are computed and stored in registers.

**On the fly:** The values $i \cdot I$ are computed as above. For $j \cdot J$ we used the circuit in Figure 4a. It consists of the computations of $J, 2 \cdot J, 4 \cdot J, 8 \cdot J, 16 \cdot J, 32 \cdot J, 64 \cdot J$. Subsequently, depending on the binary representation of $j$ some of these are selected and XORed to obtain the required value $j \cdot J$. For example, $86 \cdot J = 64 \cdot J \oplus 16 \cdot J \oplus 4 \cdot J \oplus 2 \cdot J$.

The timing diagram for the architecture in Figure 3 is shown in Figure 5. Apart from the pre-computation, a total of 389 cycles is required to complete the encryption.




(a) Masks generated on the fly (b) Precomputing masks using double and add

Fig. 4: Two ways to compute the masks $j \cdot J$. In the figure $\times 2$ denotes doubling.

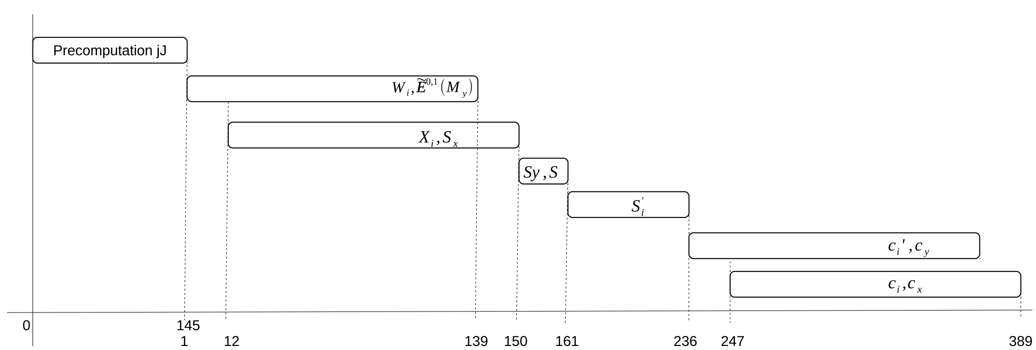



Fig. 5: Timing diagram for encryption using AEZ.