

The discrete logarithm problem over prime fields: the safe prime case. The Smart attack, non-canonical lifts and logarithmic derivatives

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Abstract

In this brief note we connect the discrete logarithm problem over prime fields in the safe prime case to the logarithmic derivative.

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1 Introduction and the Main Idea

Let a_0 be a primitive root of a prime number $p > 2$. We know that for every $b_0 \in \{1, 2, \dots, p-1\}$ there exists a unique integer n_p modulo $p-1$ satisfying

$$a_0^{n_p} \equiv b_0 \pmod{p}. \quad (1)$$

n_p is called the discrete logarithm or index of b_0 to the base a_0 modulo p . In [4] the authors got the Teichmüller expansion using Hensel lifting of the discrete logarithm problem (1). This is got by raising both sides to the power p :

$$a_0^{n_p p} \equiv b_0^p \pmod{p^2} \quad (2)$$

which can be written as

$$(a_0 + a_1 p)^{n_p} \equiv (b_0 + b_1 p) \pmod{p^2}. \quad (3)$$

The Iwasawa logarithm of a p -adic number x is defined as $\frac{1}{p-1} \log x^{p-1}$. As this vanishes for a Teichmüller character the solution n_p could not be found out, but a formula

$$n_p \equiv \frac{(b_1 - \beta_{n_p})/b_0}{a_1/a_0} \pmod{p} \quad (4)$$

was obtained where β_{n_p} is the carry

$$a_0^{n_p} \equiv b_0 + \beta_{n_p} p \pmod{p^2}. \quad (5)$$

Kontsevich [7] and Riesel [10] point out that the difficulty arises because the problem is stated modulo p and the solution is needed modulo $p-1$. Hence we go to the discrete logarithm problem modulo the composite modulus $p(p-1)$. In this connection, see Bach [1].

In this paper we consider primes p of the form $2q+1$ where q is a prime number. p is called a safe prime as it is believed that the discrete logarithm problem is computationally difficult in this case when p is ‘large’.

From (1) we can go to the discrete logarithm problem

$$a_0^n \equiv b_0 \pmod{pq}. \quad (6)$$

(See Lemma 1.) From the assumptions made in Lemma 1 a_0 generates a subgroup of order $q\phi(q)$ modulo pq . Hensel lifting the problem modulo p^2q^2 we get

$$(a_0 + a_1pq)^n \equiv (b_0 + b_1pq) \pmod{p^2q^2}. \quad (7)$$

The order of the group generated by $a_0 + a_1pq$ remains as $q\phi(q)$ modulo p^2q^2 . Also

$$(a_0 + a_1pq)^{q\phi(q)} \equiv 1 \pmod{p^2q^3}. \quad (8)$$

(See Lemma 2.) Expanding (7) using the binomial theorem, we get

$$a_0^n + na_0^{n-1}a_1pq \equiv b_0 + b_1pq \pmod{p^2q^2}. \quad (9)$$

Writing

$$a_0^n \equiv b_0 + \beta_n pq \pmod{p^2q^2}, \quad (10)$$

will give

$$\beta_n + n \frac{b_0}{a_0} a_1 \equiv b_1 \pmod{pq}. \quad (11)$$

Here β_n is the carry of a_0^n modulo p^2q^2 and note that n and β_n are the two unknowns in the above linear congruence.

The summary of what we have done so far is that there are three problems when we try to solve the discrete logarithm problem modulo p :

1. The problem is given modulo p and the solution is needed modulo $p - 1$.
2. The Iwasawa logarithm of the Teichmüller expansion modulo p^2 is 0
3. The binomial theorem on the Teichmüller expansion modulo p^2 gives 'carry'.

We overcome the first problem by going modulo pq . The fact that we cannot get n arises from two possibilities being blocked as in the modulo p case. The analogue of the Teichmüller expansion does not have a non-zero logarithm (see (8)) and if the binomial theorem is used, a carry occurs as in the case of mod p , see (11).

However if we can construct a non-canonical lift modulo p^2q^2 then the problems dissolve. Thus solving the discrete logarithm problem is equivalent to the construction of a non-canonical lift.

The non-canonical lifts exist and can be written in the form

$$(a_0 + (a_1 + k)pq)^n \equiv (b_0 + (b_1 + l)pq) \pmod{p^2q^2}. \quad (12)$$

When $k = k_1p$ for some $k_1 \not\equiv 0 \pmod{q}$, then $l = l_1p$ for some $l_1 \pmod{q}$. In this case the order of the group is $q\phi(q)$. For the other k and l modulo pq the order of the group will be $pq\phi(q)$. On expanding (12) using the binomial theorem, one gets

$$(a_0 + a_1pq)^n + n(a_0 + a_1pq)^{n-1} k pq \equiv (b_0 + b_1pq) + l pq \pmod{p^2q^2} \quad (13)$$

and using (7)

$$n \equiv \frac{l_1/b_0}{k_1/a_0} \pmod{q}. \quad (14)$$

in the first case and

$$n \equiv \frac{l/b_0}{k/a_0} \pmod{pq}. \quad (15)$$

in the second case.

If we use the notation da_0 for k_1 and db_0 for l_1 then

$$n \equiv \frac{db_0/b_0}{da_0/a_0} \pmod{q}. \quad (16)$$

and if we use the notation da_0 for k and db_0 for l then

$$n \equiv \frac{db_0/b_0}{da_0/a_0} \pmod{pq}. \quad (17)$$

Thus n can be thought of as the logarithmic derivative. The non-canonical extensions (modulo p^2q^2) of the subgroup generated by $a_0 \bmod pq$ are labeled by da_0 . As $p = 2q + 1$, once we get $n \bmod q$, $n \bmod p - 1$ would be either n or $n + q \bmod p - 1$.

Note that we can get (16) and (17) by raising (12) to the powers $q\phi(q)$ and $pq\phi(q)$ respectively. In the second case we get

$$\left((a_0 + (a_1 + k)pq)^{pq\phi(q)} \right)^n \equiv (b_0 + (b_1 + l)pq)^{pq\phi(q)} \pmod{p^3q^3}, \quad (18)$$

which on expanding and using the notation in Section 2 will give

$$1 + n\left(q(a_0) + \frac{(a_1 + k)}{a_0}\phi(q)\right)p^2q^2 \equiv 1 + \left(q(b_0) + \frac{(b_1 + l)}{b_0}\phi(q)\right)p^2q^2 \pmod{p^3q^3}. \quad (19)$$

Using the formula for a_1 and b_1 one gets (17). This way of getting n is analogous to the attack on anomalous elliptic curves by Smart [13], Semaev [12], Satoh and Araki [11].

We would like to comment that derivatives of numbers have been studied historically for a long time starting from Kummer [6], [15], A. Weil (expanded by Kawada) [8] and more recently by A. Buium [2]. Hence the problem which is standing in isolation studied only by cryptologists gets connected to mainstream algebra and number theory. This was a complete surprise to the authors which is why we have written this brief note to bring it to the attention of experts in these areas.

2 Lemmas

We need some definitions and notations before we prove our lemmas. In [9] Lerch defined the Fermat quotient for a composite modulus. Let x be such that $\gcd(x, n) = 1$. Then $q(x)$ defined by

$$x^{\phi(n)} \equiv 1 + q(x)n \pmod{n^2}. \quad (20)$$

is called the Fermat quotient of x modulo n . We do not use the Euler's ϕ -function but we use Carmichael's λ function. $\lambda(n)$ is defined as follows [3]. $\lambda(2) = 1$, $\lambda(4) = 2$ and

$$\lambda(n) = \begin{cases} \phi(p^r), & \text{if } n = p^r \\ 2^{r-2}, & \text{if } n = 2^r, r \geq 3 \\ \text{lcm}(\lambda(p_1^{r_1}), \lambda(p_2^{r_2}), \dots, \lambda(p_k^{r_k})), & \text{if } n = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k} \end{cases} \quad (21)$$

When $n = p^2q^2$ where $p = 2q + 1$ q is a prime, $\phi(p^2q^2) = 2pq^2\phi(q)$ and $\lambda(p^2q^2) = pq\phi(q)$. In other words the order of the group of units modulo p^2q^2 is $\phi(p^2q^2)$ whereas the order of the largest cyclic group modulo p^2q^2 is $\lambda(p^2q^2)$. Hence we define $g(x)$ by the congruence

$$x^{pq\phi(q)} \equiv 1 + q(x)p^2q^2 \pmod{p^3q^3}. \quad (22)$$

Lemma 1 *Let a_0 be a primitive root of p and q . Let $\gcd(b_0, q) = 1$. Then the congruence $a_0^n \equiv b_0 \pmod{p}$ can be extended to*

$$a_0^{n_p} \equiv b_0 \pmod{pq}. \quad (23)$$

if and only if the Legendre symbols

$$\left(\frac{b_0}{p}\right) = \left(\frac{b_0}{q}\right). \quad (24)$$

Proof. $a_0^n \equiv b_0 \pmod{pq}$ if and only if

$$\begin{aligned} a_0^n &\equiv a_0^{n_p} \equiv b_0 \pmod{p} & \text{and} \\ a_0^n &\equiv a_0^{n_q} \equiv b_0 \pmod{q} & . \end{aligned}$$

This happens if and only if

$$n \equiv n_p \pmod{p-1} \text{ and} \quad (25)$$

$$n \equiv n_q \pmod{q-1}. \quad (26)$$

This is possible if and only if

$$2 = \gcd(p-1, q-1)|(n_p - n_q). \quad (27)$$

by Chinese Remainder theorem. That is

$$n_p \equiv n_q \pmod{2}. \quad (28)$$

In other words b_0 is a quadratic residue or nonresidue modulo p and q simultaneously. That is $\left(\frac{b_0}{p}\right) = \left(\frac{b_0}{q}\right)$.

Lemma 2 *If $a_0^n \equiv b_0 \pmod{pq}$ holds then*

$$(a_0 + a_1pq)^n \equiv (b_0 + b_1pq) \pmod{p^2q^2}, \quad (29)$$

where

$$a_1 = -\frac{q(a_0)a_0}{\phi(q)} \text{ and } b_1 = -\frac{q(b_0)b_0}{\phi(q)} \pmod{pq}. \quad (30)$$

Proof We want a_1 and b_1 to satisfy (30). Using the carry notation

$$a_0^n \equiv b_0 + \beta_n pq \pmod{p^2 q^2}, \quad (31)$$

we get the equation

$$\beta_n + n \frac{b_0}{a_0} a_1 \equiv b_1 \pmod{pq}. \quad (32)$$

Taking the power $pq\phi(q)$ on both sides of (31)

$$a_0^{npq\phi(q)} \equiv (b_0 + \beta_n pq)^{pq\phi(q)} \pmod{p^3 q^3} \quad (33)$$

and using (22) we get

$$nq(a_0) \equiv q(b_0) + \frac{\beta_n}{b_0} \phi(q) \pmod{pq}. \quad (34)$$

Comparing (32) and (34) will give the desired values of a_1 and b_1 .

Remark 1. Note that a_1 and b_1 can be calculated in polynomial time and the order of $(a_0 + a_1 pq)$ is $q\phi(q)$ modulo $p^2 q^2$.

Remark 2. Note that the Legendre symbols in (24) can be calculated in polynomial time.

Remark 3. We are given $b_0 \pmod{p}$. If (24) fails for the given b_0 we can check the same for $b_0 + kp$ for $k = 1, 2, 3 \dots$ until the condition is satisfied or we can multiply b_0 by a_0^k for some k and check the condition. In the first case n_p does not change and in the second case n_p becomes $n_p + k$ modulo $p - 1$ or

Remark 4. We can take $b_0^2 \pmod{pq}$ and consider the new discrete logarithm problem

$$a_0^n \equiv b_0^2 \pmod{pq}, \quad (35)$$

or

Remark 5. We can even relax the conditions in Lemma 1 as in our earlier preprint [5] as follows. Let $\gcd(a_0, q) = 1$ and $\gcd(b_0, q) = 1$. Let a_0 be a primitive root of p and let a_0 and b_0 satisfy $a_0^n \equiv b_0 \pmod{p}$. Then

$$a_0^{n\phi(q)} \equiv b_0^{\phi(q)} \pmod{pq}. \quad (36)$$

In this case the formulae corresponding to (16) and (17) would be

$$n \equiv \frac{db_0}{da_0} \pmod{q}. \quad (37)$$

and

$$n \equiv \frac{db_0/b_0^{\phi(q)}}{da_0/a_0^{\phi(q)}} \pmod{pq}. \quad (38)$$

3 Conclusion

For the composites p^2q^2 the Euler function $\phi(p^2q^2) = 2pq^2\phi(q)$ and the Carmichael function $\lambda(p^2q^2) = pq\phi(q)$ are not equal. Also $\lambda(p^2q^2) \nmid \phi(p^2q^2)$ and hence many non-canonical lifts exist. As is well known this would involve a suitable choice of polynomial for lifting. Recall that the polynomials are $x^{p-1} - 1$ and $x^{pq\phi(q)} - 1$ in the cases of Teichmüller lifting modulo p^2 and p^2q^2 respectively. This attack can be generalized to elliptic curve discrete logarithm problem over prime fields where q will be connected to the order of the group. See [14] for various ways of lifting the elliptic curve discrete logarithm problem.

References

- [1] E. Bach, Discrete Logarithms and Factoring, University of California at Berkeley Computer Science Division, Report UCB/CSD/84/186, 1984.
- [2] A. Buium, Arithmetic analogues of derivations, J. Algebra, Vol. 198,(1997) 290-299.
- [3] Peter J. Cameron and D. A. Preece, Notes on primitive lambda-roots, <https://cameroncounts.files.wordpress.com/2014/01/plr1.pdf>
- [4] H. Gopalkrishna Gadiyar, K M Sangeeta Maini and R. Padma, Cryptography, Connections, Cocycles and Crystals: A p -adic Exploration of the Discrete Logarithm Problem, Progress in Cryptology - Indocrypt 2004 (LNCS 3348), 305-314.
- [5] H. Gopalakrishna Gadiyar and R. Padma, The Discrete Logarithm Problem over Prime Fields can be transformed to a Linear Multivariable Chinese Remainder Theorem, <https://eprint.iacr.org/2016/836.pdf>
- [6] D. Hilbert, The theory of algebraic number fields, Springer, 1998.
- [7] M. Kontsevich, M. The $1/2$ -logarithm (Appendix to Elbaz-Vincent, Ph. and Gangl, H.: On Poly(ana)logs I MATH. Compos.Math.,130,161-210). Compos.Math., 2002, Volume 130, 211-214
- [8] Kawada, Y, On the Derivations in Number Fields, Annals of Mathematics Second Series, Vol. 54, No. 2 (Sep., (1951), 302-314

- [9] M. Lerch, Zur Theorie des Fermatschen Quotienten $\frac{a^{p-1} - 1}{p} = q(a)$,
Math. Annalen, Vol. 60 (1905) 471-490.
- [10] H. Riesel, Some Soluble classes of the Discrete Logarithm Problem, BIT
Numerical Mathematics, Vol. 28, Number 4, (1988) 839-851.
- [11] T. Satoh and K. Araki, Fermat quotients and the polynomial time
discrete log algorithm for anomalous elliptic curves. Comm. Math. Univ.
Sancti Pauli **47** (1998) 81-92.
- [12] I. A. Semaev, Evaluation of discrete logarithms on some elliptic curves.
Math. Comp. **67** (1998) 353-356.
- [13] N. P. Smart, The discrete logarithm problem on elliptic curves of trace
one. J. Crypto. **12** (1999) 193-196.
- [14] Joseph H. Silverman, Lifting and elliptic curve discrete logarithms, Pro-
ceedings of International Workshop on Selected Areas in Cryptography,
Springer (2008) 82-102.
- [15] L. C. Washington, Introduction to cyclotomic fields, Springer, 1997.