

Quantum authentication with key recycling

Christopher Portmann*

Institute for Theoretical Physics, ETH Zurich, 8093 Zurich, Switzerland.

March 29, 2017

Abstract

We show that a family of quantum authentication protocols introduced in [Barnum et al., FOCS 2002] can be used to construct a secure quantum channel and additionally recycle all of the secret key if the message is successfully authenticated, and recycle part of the key if tampering is detected. We give a full security proof that constructs the secure channel given only insecure noisy channels and a shared secret key. We also prove that the number of recycled key bits is optimal for this family of protocols, i.e., there exists an adversarial strategy to obtain all non-recycled bits. Previous works recycled less key and only gave partial security proofs, since they did not consider all possible distinguishers (environments) that may be used to distinguish the real setting from the ideal secure quantum channel and secret key resource.

1 Introduction

1.1 Reusing a one-time pad

A one-time pad can famously be used only once [Sha49], i.e., a secret key as long as the message is needed to encrypt it with information-theoretic security. But this does not hold anymore if the honest players can use quantum technologies to communicate. A quantum key distribution (QKD) protocol [BB84, SBPC⁺09] allows players to expand an initial short secret key, and thus encrypt messages that are longer than the length of the original key. Instead of first expanding a key, and then using it for encryption, one can also swap the order if the initial key is long enough: one first encrypts a message, then recycles the key. This is possible due to the same physical principles as QKD: quantum states cannot be cloned, so if the receiver holds the exact cipher that was sent, the adversary cannot have a copy, and thus does not have any information about the key either, so it may be reused.

*chportma@ethz.ch

This requires the receiver to verify the authenticity of the message received, and if this process fails, a net key loss occurs—the same happens in QKD: if an adversary tampers with the communication, the players have to abort and also lose some of the initial secret key.

1.2 Quantum authentication and key recycling

Some ideas for recycling encryption keys using quantum ciphers were already proposed in 1982 [BBB82]. Many years later, Damgård et al. [DPS05] (see also [DPS14, FS17]) showed how to encrypt a classical message in a quantum state and recycle the key. At roughly the same time, the first protocol for authenticating quantum messages was proposed by Barnum et al. [BCG⁺02], who also proved that quantum authentication necessarily encrypts the message as well. Gottesman [Got03] then showed that after the message is successfully authenticated by the receiver, the key can be leaked to the adversary without compromising the confidentiality of the message. And Oppenheim and Horodecki [OH05] adapted the protocol of [BCG⁺02] to recycle key. But the security definitions in these initial works on quantum authentication have a major flaw: they do not consider the possibility that an adversary may hold a purification of the quantum message that is encrypted. This was corrected by Hayden, Leung and Mayers [HLM11], who give a composable security definition for quantum authentication with key recycling. They then show that the family of protocols from [BCG⁺02] are secure, and prove that one can recycle part of the key if the message is accepted.

The security proof from [HLM11] does however not consider all possible environments. Starting in works by Simmons in the 80's and then Stinson in the 90's (see, for example, [Sim85, Sim88, Sti90, Sti94]) the classical literature on authentication studies two types of attacks: *substitution attacks*—where the adversary obtains a valid pair of message and cipher¹ and attempts to substitute the cipher with one that will decode to a different message—and *impersonation attacks*—where the adversary directly sends a forged cipher to the receiver, without knowledge of a valid message-cipher pair. To the best of our knowledge, there is no proof showing that security against impersonation attacks follows from security against substitution attacks, hence the literature analyzes both attacks separately.² This is particularly important in the case of composable security, which aims to prove the security of

¹Here we use the term *cipher* to refer to the authenticated message, which is often a pair of the original message and a tag or message authentication code (MAC), but not necessarily.

²In fact, one can construct examples where the probability of a successful impersonation attack is higher than the probability of a successful substitution attack. This can occur, because any valid cipher generated by the adversary is considered a successful impersonation attack, whereas only a cipher that decrypts to a different message is considered a successful substitution attack.

the protocol when used in any arbitrary environment, therefore also in an environment that first sends a forged cipher to the receiver, learns whether it is accepted or rejected, then provides a message to the sender to be authenticated, and finally obtains the cipher for this message. This is all the more crucial when key recycling is involved, since the receiver will already recycle (part of) the key upon receiving the forged cipher, which is immediately given to the environment. The work of Hayden et al. [HLM11] only considers environments that perform substitution attacks—i.e., first provide the sender with a message, then change the cipher, and finally learn the outcome of the authentication as well as receive the recycled key. Hence they do not provide a complete composable security proof of quantum authentication, which prevents the protocol from being composed in an arbitrary environment.³

More recently, alternative security definitions for quantum authentication have been proposed, both without [DNS12, BW16] and with [GYZ16] key recycling (see also [AM16]). These still only consider substitution attacks, and furthermore, they are, strictly speaking, not composable. While it is possible to prove that these definitions imply security in a composable framework (if one restricts the environment to substitution attacks), the precise way in which the error ε carries over to the framework has not been worked out in any of these papers. If two protocols with composable errors ε and δ are run jointly (e.g., one is a subroutine of the other), the error of the composed protocol is bounded by the sum of the individual errors, $\varepsilon + \delta$. If a security definition does not provide a bound on the composable error, then one cannot evaluate the new error after composition.⁴ For example, quantum authentication with key recycling requires a backwards classical authentic channel, so that the receiver may tell the sender that the message was accepted, and allow her to recycle the key. The error of the complete protocol is thus the sum of errors of the quantum authentication and classical authentication protocols. Definitions such as those of [DNS12, BW16, GYZ16] are not sufficient to directly obtain a bound on the error of such a composed protocol.

In the other direction, it is immediate that if a protocol is ε -secure according to the composable definition used in this work, then it is secure according to [DNS12, BW16, GYZ16] with the same error ε . More precisely, proving that the quantum authentication scheme constructs a secure channel is sufficient to satisfy [DNS12, BW16]—i.e., the ideal functionality is

³For example, QKD can be broken if the underlying authentication scheme is vulnerable to impersonation attacks, because Eve could trick Alice into believing that the quantum states have been received by Bob so that she releases the basis information.

⁴In an asymptotic setting, one generally does not care about the exact error, as long as it is negligible. But for any (finite) implementation, the exact value is crucial, since without it, it is impossible to set the parameters accordingly, e.g., how many qubits should one send to get an error $\varepsilon \leq 10^{-18}$.

a secure channel which only allows the adversary to decide if the message is delivered, but does not leak any information about the message to the adversary except its length (confidentiality), nor does it allow the adversary to modify the message (authenticity). And proving that the scheme constructs a secure channel that additionally generates fresh secret key is sufficient to satisfy the definition of *total authentication* from [GYZ16]. Garg et al. [GYZ16] also propose a definition of *total authentication with key leakage*, which can be captured in a composable framework by a secure channel that generates fresh key and leaks some of it to the adversary. This is however a somewhat unnatural ideal functionality, since it requires a deterministic leakage function, which may be unknown or not exist, e.g., in concrete protocols the bits leaked can depend on the adversary’s behavior — this is the case for the *trap code* [BGS13, BW16], which we discuss further in Section 5. The next natural step for players in such a situation is to extract a secret key from the partially leaked key, and thus the more natural ideal functionality is what one obtains after this privacy amplification step [BBCM95, RK05]: a secure channel that generates fresh secret key, but where the key generated may be shorter than the key consumed. The ideal functionality used in the current work provides this flexibility: the amount of fresh key generated is a parameter which may be chosen so as to produce less key than consumed, the same amount, or even more.⁵ Hence, with one security definition, we encompass all these different cases — no key recycling, partial key recycling, total key recycling, and even a net gain of secret key. Furthermore, having all these notions captured by ideal functionalities makes for a particularly simple comparison between the quite technical definitions appearing in [DNS12, BW16, GYZ16].

1.3 Contributions

In this work we use the Abstract Cryptography (AC) framework [MR11] to model the composable security of quantum authentication with key recycling. AC views cryptography as a resource theory: a protocol constructs a (strong) resource given some (weak) resources. For example, the quantum authentication protocols that we analyze construct two resources: a secure quantum channel — a channel that provides both *confidentiality* and *authenticity* — and a secret key resource that shares a fresh key between both players. In order to construct these resources, we require shared secret key, an insecure (noiseless) quantum channel and a backwards authentic classical channel. These are all resources, that may in turn be constructed from weaker resources, e.g., the classical authentic channel can be constructed from a shared secret key and an insecure channel, and noise-

⁵One may obtain more key than consumed by using the constructed secure channel to share secret key between the players. We use this technique to compensate for key lost in a classical authentication subroutine, that cannot be recycled.

less channels are constructed from noisy channels. Due to this constructive aspect of the framework, it is also called *constructive cryptography* in the literature [Mau12, MR16].

Although this approach is quite different from the Universal Composability (UC) framework [Can01, Can13], in the setting considered in this work — with one dishonest player and where recipients are denoted by classical strings⁶ — the two frameworks are essentially equivalent and the same results could have been derived with a quantum version of UC [Unr10]. In UC, the constructed resource would be called *ideal functionality*, and the resources used in the construction are setup assumptions.

We thus first formally define the ideal resources constructed by the quantum authentication protocol with key recycling — the secure channel and key resource mentioned in this introduction — as well as the resources required by this construction. We then prove that a family of quantum authentication protocols proposed by Barnum et al. [BCG⁺02] satisfy this construction, i.e., no distinguisher (called environment in UC) can distinguish the real system from the ideal resources and simulator except with an advantage ε that is exponentially small in the security parameter. This proof considers all distinguishers allowed by quantum mechanics, including those that perform impersonation attacks.

We show that in the case where the message is accepted, every bit of key may be recycled. And if the message is rejected, one may recycle all the key except the bits used to one-time pad the cipher.⁷ We prove that this is optimal for the family of protocols considered, i.e., an adversary may obtain all non-recycled bits of key. This improves on previous results, which recycled less key and only considered a subset of possible environments. More specifically, Hayden et al. [HLM11], while also analyzing protocols from [BCG⁺02], only recycle part of the key in case of an accept, and lose all the key in case of a reject. Garg et al. [GYZ16] propose a new protocol, which they prove can recycle all of the key in the case of an accept, but do not consider key recycling in the case of a reject either. The protocols we analyze are also more key efficient than that of [GYZ16]. We give two instances which need $\Theta(m + \log 1/\varepsilon)$ bits of initial secret key, instead of the $\Theta((m + \log 1/\varepsilon)^2)$ required by [GYZ16], where m is the length of the message and ε is the error. Independently from this work, Alagic and Majenz [AM16] proved that one of the instances analyzed here satisfies the weaker security definition of [GYZ16].

Note that the family of protocols for which we provide a security proof

⁶In a more general setting, a message may be in a superposition of “sent” and “not sent” or a superposition of “sent to Alice” and “sent to Bob”, which cannot be modeled in UC, but is captured in AC [PMM⁺17].

⁷Key recycling in the case of a rejected message is not related to any quantum advantage. A protocol does not leak more information about the key than (twice) the length of the cipher, so the rest may be reused. The same holds for classical authentication [Por14].

is a subset of the (larger) family introduced in [BCG⁺02]. More precisely, Barnum et al. [BCG⁺02] define quantum authentication protocols by composing a quantum one-time pad and what they call a *purity testing code* — which, with high probability, will detect any noise that may modify the encoded message — whereas we require a stricter notion, a *strong purity testing code* — which, with high probability, will detect any noise. This restriction on the family of protocols is necessary to recycle all the key. In fact, there exists a quantum authentication scheme, the *trap code* [BGS13, BW16], which is a member of the larger class from [BCG⁺02] but not the stricter class analyzed here, and which leaks part of the key to the adversary, even upon a successful authentication of the message — this example is discussed in Section 5.

We then give two explicit instantiations of this family of quantum authentication protocols. The first is the construction used in [BCG⁺02], which requires an initial key of length $2m + 2n$, where m is the length of the message and n is the security parameter, and has error $\varepsilon \leq 2^{-n/2+1} \sqrt{2m/n + 2}$. The second is an explicit unitary 2-design [Dan05, DCEL09] discovered by Chau [Cha05], which requires $5m + 4n$ bits of initial key⁸ and has error $\varepsilon \leq 2^{-n/2+1}$. Both constructions have a net loss of $2m + n$ bits of key if the message fails authentication. Since several other explicit quantum authentication protocols proposed in the literature are instances of this family of schemes, our security proof is a proof for these protocols as well — this is discussed further in Section 5.

Finally, we show how to construct the resources used by the protocol from nothing but insecure noisy channels and shared secret key, and calculate the joint error of the composed protocols. We also show how to compensate for the bits of key lost in the construction of the backwards authentic channel, so that the composed protocol still has a zero net key consumption if no adversary jumbles the communication.

There is currently no work in the literature that analyzes the composable security of quantum authentication without key recycling. And although a security proof with key recycling is automatically a security proof without key recycling, the parameters are not optimal — recycling the key results in a larger error — and the proof given in this paper is only valid for strong purity testing codes. So for completeness, we provide a proof of security for quantum authentication without key recycling in Appendix D, which is valid for weak purity testing codes and achieves an optimal error.

1.4 Structure of this paper

In Section 2 we give a brief introduction to the main concepts of AC, which are necessary to understand the notion of cryptographic construction and

⁸The complete design would require $5m + 5n$ bits of key, but we show that some of the unitaries are redundant when used for quantum authentication and can be dropped.

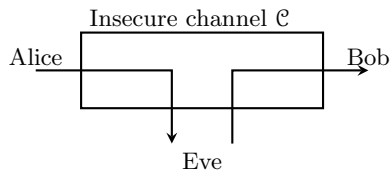
corresponding security definition. A more extended introduction to AC is provided in Appendix A. In Section 3 we then define the resources constructed and used by a quantum authentication scheme with key recycling. We introduce the family of protocols from [BCG⁺02] that we analyze in this work, and then prove that they construct the corresponding ideal resources. We also prove that the number of recycled bits is optimal. In Section 4 we show how to construct the various resources used by the quantum authentication protocol, and put the pieces together to get a security statement for the joint protocol that constructs the secure quantum channel and secret key resource from nothing but noisy insecure channels and shared secret key. Finally, in Section 5 we discuss the relation between some quantum authentication schemes that have appeared in the literature and those analyzed here, as well as some open problems. An overview of the appendices is given on page 39.

2 Constructive cryptography

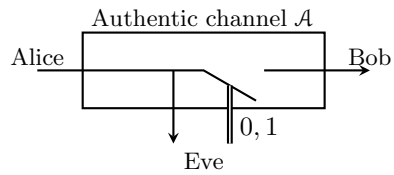
As already mentioned in Section 1.3, the AC framework [MR11] models cryptography as a resource theory. In this section we give a brief overview of how these constructive statements are formalized. We illustrate this with an example taken from [Por14], namely authentication of classical messages with message authentication codes (MAC). An expanded version of this introduction to AC is provided in Appendix A.

In an n player setting, a *resource* is an object with n interfaces, that allows every player to input messages and receive other messages at her interface. The objects depicted in Figure 1 are examples of resources. The insecure channel in Figure 1a allows Alice to input a message at her interface on the left and allows Bob to receive a message at his interface on the right. Eve can intercept Alice’s message and insert a message of her choosing at her interface. The authentic channel resource depicted in Figure 1b also allows Alice to send a message and Bob to receive a message, but Eve’s interface is more limited than for the insecure channel: she can only decide if Bob receives the message, an error symbol or nothing at all — by inputting 0, 1, or nothing, respectively, at her interface — but not tamper with the message being sent. The key resource drawn in Figure 1c provides each player with a secret key when requested. If two resources \mathcal{K} and \mathcal{C} are both available to the players, we write $\mathcal{K}||\mathcal{C}$ for the resource resulting from their parallel composition — this is to be understood as the resources being merged into one: the interfaces belonging to player i are simultaneously accessible to her as one new interface, which we depict in Figure 1d. In Appendix A we provide a more detailed description of the resources from Figure 1 along a discussion of how to model them mathematically.

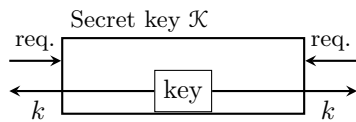
Converters capture operations that a player might perform locally at her



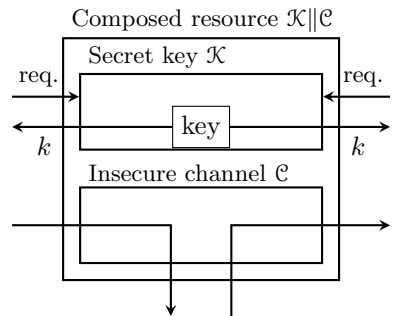
(a) An insecure channel from Alice (on the left) to Bob (on the right) allows Eve (below) to intercept the message and insert a message of her own.



(b) An authentic channel from Alice to Bob allows Eve (below) to receive a copy of the message and choose whether Bob receives it, an error symbol, or nothing at all.



(c) A secret key resource distributes a perfectly uniform key k to the players when they send a request **req.**



(d) If two resources \mathcal{K} and \mathcal{C} are available to the players, we denote the composition of the two as the new resource $\mathcal{K}||\mathcal{C}$.

Figure 1 – Some examples of resources. The insecure channel on the top left could transmit either classical or quantum messages. The authentic channel on the top right is necessarily classical, since it clones the message.

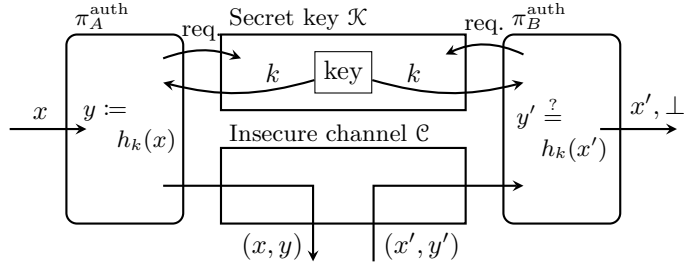


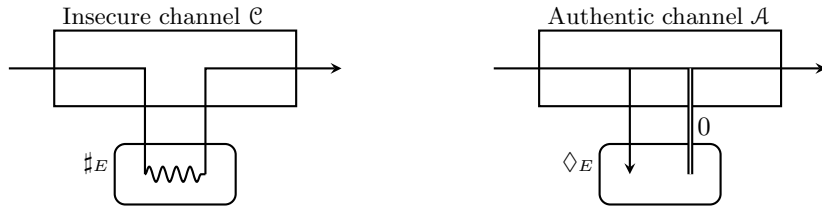
Figure 2 – The real system for a MAC protocol. Alice authenticates her message by appending a MAC to it. Bob checks if the MAC is correct and either accepts or rejects the message.

interface. For example, if the players share a key resource and an insecure channel, Alice might decide to append a MAC to her message. This is modeled as a converter π_A^{auth} that obtains the message x at the outside interface, obtains a key at the inside interface from a key resource \mathcal{K} and sends $(x, h_k(x))$ on the insecure channel \mathcal{C} , where h_k is taken from a family of strongly 2-universal hash functions [WC81, Sti94]. We illustrate this in Figure 2. Converters are always drawn with rounded corners. If a converter α_i is connected to the i interface of a resource \mathcal{R} , we write $\alpha_i\mathcal{R}$ or $\mathcal{R}\alpha_i$ for the new resource obtained by connecting the two.⁹

A protocol is then defined by a set of converters, one for every honest player. Another type of converter that we need is a *filter*. The resources illustrated in Figure 1 depict a setting with an adversary that has some control over these resources. For a cryptographic protocol to be useful it is not sufficient to provide guarantees on what happens when an adversary is present, one also has to provide a guarantee on what happens when no adversary is present, e.g., if no adversary tampers with the message on the insecure channel, then Bob will receive the message that Alice sent. We model this setting by covering the adversarial interface with a filter that emulates an honest behavior. In Figure 3 we draw an insecure and an authentic channel with filters \sharp_E and \diamond_E that transmit the message to Bob. In the case of the insecure channel, one may want to model an honest noisy channel when no adversary is present. This is done by having the filter \sharp_E add some noise to the message. A dishonest player removes this and has access to a noiseless channel as in Figure 1a.

We use the term *filtered resource* to refer to a pair of a resource \mathcal{R} and a filter \sharp_E , and often write $\mathcal{R}\sharp_E = (\mathcal{R}, \sharp_E)$. Such an object can be thought of as having two modes: it is characterized by the resource $\mathcal{R}\sharp_E$ when no adversary is present and by the resource \mathcal{R} when the adversary is present.

⁹In this work we adopt the convention of writing converters at the A and B interfaces on the left and converters at the E interface on the right, though there is no mathematical difference between $\alpha_i\mathcal{R}$ and $\mathcal{R}\alpha_i$.



(a) When no adversary is present, Alice’s message is delivered to Bob. In the case of a noisy channel, this noise is introduced by the filter $\#_E$.

(b) When no adversary is present, Bob receives the message sent by Alice.

Figure 3 – Channels with filters. The two channels from Figures 1a and 1b are represented with filters on Eve’s interface emulating an honest behavior, i.e., when no adversary is present.

The final object that is required by the AC framework to define the notion of construction and prove that it is composable, is a (pseudo-)metric defined on the space of resources that measures how close two resources are. In the following, we use a distinguisher based metric, i.e., the maximum advantage a distinguisher has in guessing whether it is interacting with resource \mathcal{R} or \mathcal{S} , which we write $d(\mathcal{R}, \mathcal{S})$. More specifically, let \mathcal{D} be a distinguisher, and let $\mathcal{D}[\mathcal{R}]$ and $\mathcal{D}[\mathcal{S}]$ be the binary random variables corresponding to \mathcal{D} ’s output when connected to \mathcal{R} and \mathcal{S} , respectively. Then the distinguishing advantage between \mathcal{R} and \mathcal{S} is defined as

$$d(\mathcal{R}, \mathcal{S}) := \sup_{\mathcal{D}} |\Pr[\mathcal{D}[\mathcal{R}] = 0] - \Pr[\mathcal{D}[\mathcal{S}] = 0]|.$$

Since we study information-theoretic security in this work, the supremum is taken over the set of all possible distinguishers allowed by quantum mechanics. This is discussed further in Appendix A.3.

We are now ready to define the security of a cryptographic protocol. We do so in the three player setting, for honest Alice and Bob, and dishonest Eve. Thus, in the following, all resources have three interfaces, denoted A , B and E , and a protocol is then given by a pair of converters (π_A, π_B) for the honest players. We refer to [MR11] for the general case, when arbitrary players can be dishonest.

Definition 2.1 (Cryptographic security [MR11]). Let $\pi_{AB} = (\pi_A, \pi_B)$ be a protocol and $\mathcal{R}_{\#} = (\mathcal{R}, \#)$ and $\mathcal{S}_{\diamond} = (\mathcal{S}, \diamond)$ denote two filtered resources. We say that π_{AB} constructs \mathcal{S}_{\diamond} from $\mathcal{R}_{\#}$ within ε , which we write $\mathcal{R}_{\#} \xrightarrow{\pi, \varepsilon} \mathcal{S}_{\diamond}$, if the two following conditions hold:

i) We have

$$d(\pi_{AB}\mathcal{R}_{\#E}, \mathcal{S}_{\diamond E}) \leq \varepsilon.$$

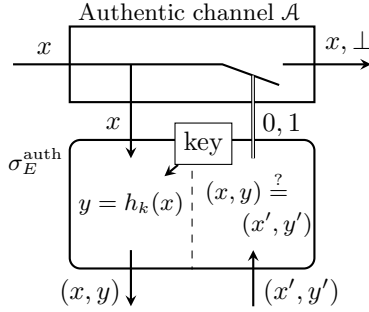


Figure 4 – The ideal system with simulator for a MAC protocol. The simulator σ_E^{auth} picks its own key and generates the MAC. If the value input by Eve is different from the output at her interface (or is input before an output is generated), the simulator prevents Bob from getting Alice’s message.

ii) There exists a converter¹⁰ σ_E — which we call simulator — such that

$$d(\pi_{AB}\mathcal{R}, \mathcal{S}\sigma_E) \leq \varepsilon.$$

If it is clear from the context what filtered resources $\mathcal{R}_{\#}$ and \mathcal{S}_{\diamond} are meant, we simply say that π_{AB} is ε -secure.

The first of these two conditions measures how close the constructed resource is to the ideal resource in the case where no malicious player is intervening, which is often called *correctness* in the literature. The second condition captures *security* in the presence of an adversary. For example, to prove that the MAC protocol π_{AB}^{auth} constructs an authentic channel \mathcal{A}_{\diamond} from a (noiseless) insecure channel \mathcal{C}_{\square} and a secret key \mathcal{K} within ε , we need to prove that the real system (with filters) $\pi_{AB}^{\text{auth}}(\mathcal{K}||\mathcal{C}_{\square})_E$ cannot be distinguished from the ideal system \mathcal{A}_{\diamond}_E with advantage greater than ε , and we need to find a converter σ_E^{auth} such that the real system (without filters) $\pi_{AB}^{\text{auth}}(\mathcal{K}||\mathcal{C})$ cannot be distinguished from the ideal system $\mathcal{A}_{\sigma_E^{\text{auth}}}$ with advantage greater than ε . For the MAC protocol, correctness is satisfied with error 0 and the simulator σ_E^{auth} drawn in Figure 4 satisfies the second requirement if the family of hash functions $\{h_k\}_k$ is ε -almost strongly 2-universal [Por14].

Remark 2.2. The protocols and simulators discussed in this work are all efficient. The protocols we consider are either trivially efficient or taken from other work, in which case we refer to these other works for proofs of efficiency. The efficiency of the simulator used to prove the security of quantum authentication has been analyzed in [BW16]. All other simulators used in the security proofs run the corresponding honest protocols, and

¹⁰For a protocol with information-theoretic security to be composable with a protocol that has computational security, one additionally requires the simulator to be efficient.

are thus efficient because the protocols are. We therefore do not discuss efficiency any further in this work.

3 Quantum authentication

We start with some technical preliminaries in Section 3.1, where we introduce (strong) purity testing codes, which are a key component of the family of quantum authentication protocols of [BCG⁺02]. In Section 3.2 we give a constructive view of quantum authentication with key recycling: we define the resources that such a protocol is expected to construct, as well as the resources that are required to achieve this. In Section 3.3 we describe the family of protocols that we analyze in this work, along with a variant in which the order of the encryption and encoding operations has been swapped, which we prove to be equivalent. In Section 3.4 we give a security proof for the family of quantum authentication protocols defined earlier. And in Section 3.5 we show that the number of recycled key bits is optimal. Finally, in Section 3.6 we give two explicit constructions of purity testing codes and get the exact parameters of the quantum authentication protocols with these codes.

3.1 Technical preliminaries

Pauli operators. To denote a Pauli operator on n qubits we write either $P_{x,z}$ or P_ℓ , where x and z are n -bit strings indicating in which positions bit and phase flips occur, and $\ell = (x, z)$ is the concatenation of x and z , which is used when we do not need to distinguish between x and z . Two Pauli operators P_j and P_ℓ with $j = (x, z)$ and $\ell = (x', z')$ commute (anti-commute) if the symplectic inner product

$$(j, \ell)_{\text{Sp}} := x \cdot z' - z \cdot x' \tag{1}$$

is 0 (is 1), where $x \cdot z$ is the scalar product of the vectors and the arithmetic is done modulo 2. Hence, for any P_j and P_ℓ

$$P_j P_\ell = (-1)^{(j, \ell)_{\text{Sp}}} P_\ell P_j.$$

We use several times the following equality

$$\sum_{j \in \{0,1\}^n} (-1)^{(j, \ell)_{\text{Sp}}} = \begin{cases} 2^n & \text{if } \ell = 0, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where $\ell = 0$ means that all bits of the string ℓ are 0.

Purity testing code. An error correcting code (ECC) that encodes an m qubit message in a $m + n$ qubit code word is generally defined by an isomorphism from \mathbb{C}^{2^m} to $\mathbb{C}^{2^{m+n}}$. In this work we define an ECC by a unitary $U : \mathbb{C}^{2^{m+n}} \rightarrow \mathbb{C}^{2^{m+n}}$. The code word for a state $|\psi\rangle$ is obtained by appending a n qubit state $|0\rangle$ to the message, and applying U , i.e., the encoding of $|\psi\rangle$ is $U(|\psi\rangle \otimes |0\rangle)$. We do not need to use the decoding properties of ECCs in this work, we only use them to detect errors, i.e., given a state $|\varphi\rangle \in \mathbb{C}^{2^{m+n}}$, we apply the inverse unitary U^\dagger and measure the last n qubits to see if they are $|0\rangle$ or not.

The first property we require of our codes, is that they map any Pauli error P_ℓ into another Pauli error $P_{\ell'}$, i.e.,

$$U^\dagger P_\ell U = e^{i\theta_\ell} P_{\ell'}, \quad (3)$$

for some global phase $e^{i\theta_\ell}$. This is always the case for any U that can be implemented with Clifford operators. In particular, all stabilizer codes have this property, which are used in [BCG⁺02] to define purity testing codes. Note that the mapping from ℓ to ℓ' defined by Eq. (3) is a permutation on the set of indices $\ell \in \{0, 1\}^{2m+2n}$ that depends only on the choice of code.

A code will detect an error P_ℓ if $P_{\ell'} = P_{x,z} \otimes P_{s,z'}$ for $s \neq 0$, where $P_{x,z}$ acts on the first m qubits and $P_{s,z'}$ on the last n . Measuring these last qubits would yield the syndrome s , since $P_{s,z'}$ flips the bits in the positions corresponding to the bits of s . And an error P_ℓ will act trivially on the message if $P_{\ell'} = P_{0,0} \otimes P_{s,z}$. In particular, if $P_{\ell'} = P_{0,0} \otimes P_{0,z}$, then this error will not be detected, but not change the message either.

For a code indexed by a key k , we denote by \mathcal{P}_k the set of Pauli errors that are not detected by this code, and by $\mathcal{Q}_k \subset \mathcal{P}_k$ we denote the undetected errors which act trivially on the message. A purity testing code is a set of codes $\{U_k\}_{k \in \mathcal{K}}$ such that when a code U_k is selected uniformly at random, it will detect with high probability all Pauli errors which act non-trivially on the message.

Definition 3.1 (Purity testing code [BCG⁺02]). A purity testing code with error ε is a set of codes $\{U_k\}_{k \in \mathcal{K}}$, such that for all Pauli operators P_ℓ ,

$$\frac{|\{k \in \mathcal{K} : P_\ell \in \mathcal{P}_k \setminus \mathcal{Q}_k\}|}{|\mathcal{K}|} \leq \varepsilon.$$

As mentioned in Section 1.3, we use a stricter definition of purity testing code in this work. We require that all non-identity Paulis get detected with high probability, even those that act trivially on the message. Intuitively, the reason for this is that, with the original definition of purity testing, if the adversary introduces some noise P_ℓ , by learning whether the message was accepted or not, she will learn whether that error acts trivially on the message or not, and thus learn something about the ECC used. This means

that the adversary learns something about the key used to choose the ECC, and hence it cannot be recycled in its entirety.¹¹

Definition 3.2 (Strong purity testing code.). A strong purity testing code with error ε is a set of codes $\{U_k\}_{k \in \mathcal{K}}$, such that for all non-identity Pauli operators P_ℓ ,

$$\frac{|\{k \in \mathcal{K} : P_\ell \in \mathcal{P}_k\}|}{|\mathcal{K}|} \leq \varepsilon.$$

In Section 3.6 we provide explicit constructions of strong purity testing codes.

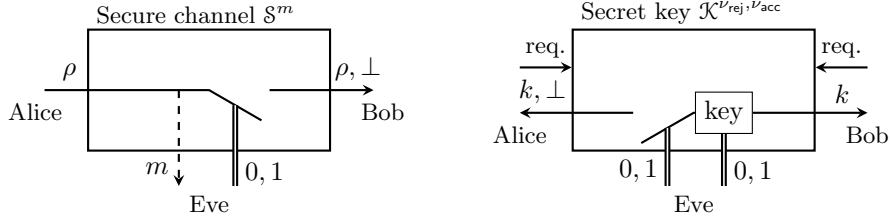
3.2 Secure channel & secret key resource

The main result in this paper is a proof that the family of quantum authentication protocols of Barnum et al. [BCG⁺02] restricted to strong purity testing codes can be used to construct a resource that corresponds to the parallel composition of a secure quantum channel \mathcal{S}^m and a secret key resource $\bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}}$, which are illustrated in Figure 5 and explained in more detail in the following paragraphs.

The secure quantum channel, \mathcal{S}^m , drawn in Figure 5a, allows an m -qubit message ρ to be transmitted from Alice to Bob, which Alice may input at her interface. Since in general the players cannot prevent Eve from learning that a message has been sent, Eve’s interface has one output denoted by a dashed arrow, which notifies her that Alice has sent an m -qubit message. But the players cannot prevent Eve from jumbling the communication lines either, which is captured in the resource \mathcal{S}^m by allowing Eve to input a bit that decides if Bob gets the message or an error symbol \perp — Eve may also decide not to provide this input (Eve cuts the communication lines), in which case the system is left waiting and Bob obtains neither the message nor an error. Note that the order in which messages are input to the resource \mathcal{S}^m is not fixed, Eve may well provide her bit before Alice inputs a message. In this case, Bob immediately receives an error \perp regardless of the value of Eve’s bit.

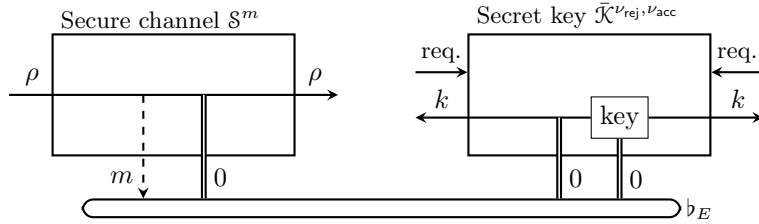
The secret key resource, $\bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}}$, depicted in Figure 5b distributes a uniformly random key to Alice and Bob. Unlike the simplified key resource from Figure 1c, here the adversary has some control over the length of the key produced. This is because in the real setting Eve can prevent the full key from being recycled by jumbling the message. This is reflected at Eve’s interface of $\bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}}$ allowing her to decide if the key generated is of length ν_{rej} or ν_{acc} . Furthermore, if in the real setting Alice were to recycle her key before Bob receives the cipher, Eve could use the information from the recycled key to modify the cipher without being detected. So Alice must wait

¹¹We conjecture that in this case only 1 bit of the key is leaked, see the discussion in Section 5.



(a) A secure channel \mathcal{S}^m is very similar to the authentic channel from Figure 1b. It allows Alice to send an m -qubit message, and Eve to decide if Bob gets it. But this time, Eve only receives the size of the message that has been sent (denoted by the dashed arrow), not a copy.

(b) A slightly weaker secret key resource than that from Figure 1c, $\bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}}$. It allows Eve to choose the length of the key generated, either $|k| = \nu_{\text{rej}}$ or $|k| = \nu_{\text{acc}}$. Furthermore, Eve can prevent Alice from getting the key at all.



(c) When no adversary is present, the filter b_E covers Eve's interface of the resource $\mathcal{S}^m \parallel \bar{\mathcal{K}}^{\nu_{\text{acc}}, \nu_{\text{rej}}}$. Once b_E is notified that a message has been sent, it allows the message through and notifies the secret key resource to prepare a key of length ν_{acc} .

Figure 5 – We depict here the filtered resource $(\mathcal{S}^m \parallel \bar{\mathcal{K}}^{\nu_{\text{acc}}, \nu_{\text{rej}}}, b_E)$ constructed by the quantum authentication protocols analyzed in this work. It can be seen as the composition of a secure channel \mathcal{S}^m (drawn in (a)) and a secret key resource $\bar{\mathcal{K}}^{\nu_{\text{acc}}, \nu_{\text{rej}}}$ (drawn in (b)). The filter b_E that emulates an honest behavior is drawn in (c).

for a confirmation of reception from Bob, which Eve can jumble, preventing Alice from ever recycling the key. This translates in the ideal setting to Eve having another control bit, deciding whether Alice receives the key or an error \perp . Note that if Eve provides her two bits in the wrong order, Alice always gets an error \perp . This key resource is modeled so that the honest players must request the key to obtain its value. If Bob does this before Eve has provided the bit deciding the key length, he gets an error instead of a key. If Alice makes the request before Eve has provided both her bits, she also gets an error. Otherwise they get the key k .

If no adversary is present, a filter \flat_E covers Eve's interface of the resources S^m and $\bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}}$, which is drawn in Figure 5c. This filter provides the inputs to the resources that allow Bob to get Alice's message and generate a key of length ν_{acc} that is made available to both players.

To construct the filtered resource $(S^m \parallel \bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}})_{\flat}$, the quantum authentication protocol will use a shared secret key to encrypt and authenticate the message. This means that the players must share a secret key resource. For simplicity we assume the players have access to a resource \mathcal{K}^{μ} as depicted in Figure 1c, that always provides them with a key of length μ .¹² Note that the security of the protocol is not affected if the players only have a weaker resource which might shorten the key or not deliver it to both players — such as the one constructed by the protocol, namely $\bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}}$ — because if either of the players does not have enough key, they simply abort, which is an outcome Eve could already achieve by cutting or jumbling the communication.

They also need to share an insecure quantum channel, which is used to send the message, and is illustrated in Figure 1a without a filter and in Figure 3a with a filter. The authentication protocol we consider is designed to catch any error, so if it is used over a noisy channel, it will always abort, even though no adversary is tampering with the message. We thus assume that the players share a noiseless channel, which we denote \mathcal{C}_{\square} , i.e., \mathcal{C} is controlled by the adversary as in Figure 1a. But if no adversary is present, the filter \square_E is noiseless. In Section 4.2 we explain how to compose the protocol with an error correcting code so as to run it over a noisy channel.

Finally, the players need a backwards authentic channel, that can send one bit of information from Bob to Alice. This is required so that Alice may learn whether the message was accepted and recycle the corresponding amount of key. The authentic channel and its filter \mathcal{A}_{\diamond} are drawn in Figures 1b and 3b. Putting all this together in the case of an active adversary, we get Figure 6, where the converters for Alice's and Bob's parts of the quantum authentication protocol are labeled $\pi_A^{\text{q-auth}}$ and $\pi_B^{\text{q-auth}}$, respectively.

According to Definition 2.1, a protocol $\pi_{AB}^{\text{q-auth}} = (\pi_A^{\text{q-auth}}, \pi_B^{\text{q-auth}})$ is then

¹²Since Eve's interface of \mathcal{K}^{μ} is empty, this resource has a trivial empty filter, which we do not write down.

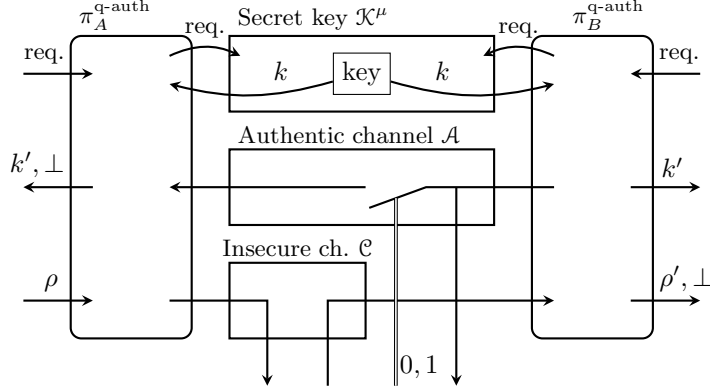


Figure 6 – The real system for quantum authentication with key recycling. Upon receiving a message ρ , $\pi_A^{\text{q-auth}}$ encrypts it with a key that it obtains from K^μ and sends it on the insecure channel. Upon receiving a quantum state on the insecure channel, $\pi_B^{\text{q-auth}}$ checks whether it is valid, and outputs the corresponding message ρ' or an error message \perp . It may then recycle (part of) the key, k' , and uses the authentic channel to notify $\pi_A^{\text{q-auth}}$ whether the message was accepted or not. $\pi_A^{\text{q-auth}}$ then recycles the key as well. Concrete protocols for this are given in Section 3.3.

a quantum authentication protocol (with key recycling) with error $\varepsilon^{\text{q-auth}}$ if it constructs $(S^m \parallel \bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}})_{\text{b}}$ from $\mathcal{C}_{\square} \parallel \mathcal{A}_{\diamond} \parallel \mathcal{K}^\mu$ within $\varepsilon^{\text{q-auth}}$, i.e.,

$$\mathcal{C}_{\square} \parallel \mathcal{A}_{\diamond} \parallel \mathcal{K}^\mu \xrightarrow{\pi_{AB}^{\text{q-auth}}, \varepsilon^{\text{q-auth}}} (S^m \parallel \bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}})_{\text{b}}. \quad (4)$$

In Section 3.3 we describe the protocol, and in Section 3.4 we prove that Eq. (4) is satisfied and provide the parameters $\mu, \nu_{\text{rej}}, \nu_{\text{acc}}, \varepsilon^{\text{q-auth}}$.

3.3 Generic protocol

The family of quantum authentication protocols from [BCG⁺02] consists in first encrypting the message to be sent with a quantum one-time pad, then encoding it with a purity testing code and a random syndrome. We do the same, but with a strong purity testing code. We also extend the protocol so that the players recycle all the key if the message is accepted, and the key used to select the strong purity testing code if the message is rejected. So that Alice may also recycle the key, Bob uses the backwards authentic classical channel to notify her of the outcome. We refer to this as the “encrypt-then-encode” protocol, the details of which are provided in Figure 7.

Alternatively, one may perform the encoding and encryption in the opposite order: Alice first encodes her message with the strong purity testing code with syndrome 0, then does a quantum one-time pad on the result-

Quantum authentication — encrypt-then-encode

1. Alice and Bob obtain uniform keys k , ℓ , and s from the key resource, where k is long enough to choose an element from a strong purity testing code that encodes m qubits in $m + n$ qubits, ℓ is $2m$ bits and s is n bits.
2. Alice encrypts the message ρ^A she receives with a quantum one-time pad using the key ℓ . She then appends an n qubit state $|s\rangle\langle s|^S$, and encodes the whole thing with a strong purity testing code, obtaining the cipher $\sigma^{AS} = U_k(P_\ell \rho^A P_\ell \otimes |s\rangle\langle s|^S) U_k^\dagger$.
3. Alice sends σ^{AS} to Bob on the insecure channel.
4. Bob receives a message $\tilde{\sigma}^{AS}$, he applies U_k^\dagger , decrypts the A part and measures the S part in the computational basis.
5. If the result of the measurement is s , he accepts the message and recycles k , ℓ and s . If the result is not s , he rejects the message, and recycles k .
6. Bob sends Alice a bit on the backwards authentic channel to tell her if he accepted or rejected the message.
7. When Alice receives Bob's bit, she either recycles all the keys or only k .

Figure 7 – This protocol is identical to the scheme from [BCG⁺02], except that the players use a strong purity testing code, recycle key, and have a backwards authentic channel so that Alice may learn the outcome.

ing $m + n$ qubit state. This “encode-then-encrypt” protocol is described in Figure 8.

The pseudo-code described in Figures 7 and 8 can easily be translated into converters as used in the AC formalism, i.e., the objects $\pi_A^{\text{q-auth}}$ and $\pi_B^{\text{q-auth}}$ from Figure 6. More precisely, if $\pi_A^{\text{q-auth}}$ receives a message at its outer interface, it requests a key from the key resource, encrypts the message as described and sends the cipher on the insecure channel. It may receive three symbols from the backwards authentic channel: an error \perp , in which case it does not recycle any key, a message 0 saying that $\pi_B^{\text{q-auth}}$ did not receive the correct state, in which case it recycles the part of the key used to choose the code, or a message 1 saying that $\pi_B^{\text{q-auth}}$ did receive the correct state, in which case it recycles all the key. If $\pi_A^{\text{q-auth}}$ first receives a message on the backwards authentic channel before receiving a message to send, it will not recycle any key. Similarly, when $\pi_B^{\text{q-auth}}$ receives a cipher on the insecure channel, it requests a key from the key resource, performs the decryption, outputs either the message or an error depending on the result of the decryption, and sends this result back to $\pi_A^{\text{q-auth}}$ on the authentic channel.

The encode-then-encrypt protocol uses n bits more key, and since these

Quantum authentication — encode-then-encrypt

1. Alice and Bob obtain uniform keys k and ℓ from the key resource, where k is long enough to choose an element from a strong purity testing code that encodes m qubits in $m + n$ qubits and ℓ is $2m + 2n$ bits long.
2. Alice appends a n qubit state $|0\rangle\langle 0|$ to the message ρ^A she receives, encodes it with a strong purity testing code chosen according to the key k , and encrypts the whole thing with a quantum one-time pad using the key ℓ . She thus obtains the cipher $\sigma^{AS} = P_\ell U_k(\rho^A \otimes |0\rangle\langle 0|^S) U_k^\dagger P_\ell$.
3. Alice sends σ^{AS} to Bob on the insecure channel.
4. Bob receives a message $\tilde{\sigma}^{AS}$, he applies P_ℓ , then U_k^\dagger , and measures the S part in the computational basis.
5. If the result of the measurement is 0, he accepts the message and recycles k and ℓ . Otherwise, he rejects the message, and recycles k .
6. Bob sends Alice a bit on the backwards authentic channel to tell her if he accepted or rejected the message.
7. When Alice receives Bob's bit, she either recycles all the keys or only k .

Figure 8 – This protocol is similar to the protocol from Figure 7, except that the order of the encryption and encoding have been reversed. To do this, the players need an extra n bits of key.

bits are not recycled in case of a reject, it is preferable to use the encrypt-then-encode protocol. These protocols are however identical: no external observer can detect which of the two is being run. This holds, because the encode-then-encrypt protocol performs phase flips on a syndrome that is known to be in a computational basis state $|s\rangle$. Thus, they have no effect and can be skipped. Likewise, Bob performs phase flips on S before measuring in the computational basis—he might as well skip these phase flips, since they have no effect either. We formalize this statement by proving (in Lemma 3.3) that the converters corresponding to the two different protocols are indistinguishable. This result is similar in spirit to proofs that some prepare-and-measure quantum key distribution (QKD) protocols are indistinguishable from entanglement-based QKD protocols, and thus security proofs for one are security proofs for the other [SP00].

Since these two protocols are indistinguishable, we provide a security proof in Section 3.4 for the encode-then-encrypt protocol. However, in Section 3.6, when we count the number of bits of key consumed, we count those of the encrypt-then-encode protocol.

Lemma 3.3. *Let $(\bar{\pi}_A^{q\text{-auth}}, \bar{\pi}_B^{q\text{-auth}})$ and $(\pi_A^{q\text{-auth}}, \pi_B^{q\text{-auth}})$ denote the pairs of converters modeling Alice's and Bob's behavior in the encrypt-then-encode*

and encode-then-encrypt protocols, respectively. Then

$$d(\bar{\pi}_A^{q\text{-auth}}, \pi_A^{q\text{-auth}}) = d(\bar{\pi}_B^{q\text{-auth}}, \pi_B^{q\text{-auth}}) = 0.$$

Proof. We start with Alice's part of the protocol. Let $\bar{\pi}_A^{q\text{-auth}}$ and $\pi_A^{q\text{-auth}}$ receive keys k, ℓ and s as in the protocol from Figure 7, as well as an extra key z of length n that is needed by $\pi_A^{q\text{-auth}}$, since it requires more key. The distinguisher prepares a state ρ^{RA} , and sends the A part to the system. $\bar{\pi}_A^{q\text{-auth}}$ outputs

$$\begin{aligned} U_k^{AS} P_\ell^A (\rho^{RA} \otimes |s\rangle\langle s|^S) P_\ell^A (U_k^{AS})^\dagger \\ &= U_k^{AS} (P_\ell^A \otimes P_{s,0}^S) (\rho^{RA} \otimes |0\rangle\langle 0|^S) (P_\ell^A \otimes P_{s,0}^S) (U_k^{AS})^\dagger \\ &= U_k^{AS} (P_\ell^A \otimes P_{s,z}^S) (\rho^{RA} \otimes |0\rangle\langle 0|^S) (P_\ell^A \otimes P_{s,z}^S) (U_k^{AS})^\dagger \\ &= P_{\ell'}^{AS} U_k^{AS} (\rho^{RA} \otimes |0\rangle\langle 0|^S) (U_k^{AS})^\dagger P_{\ell'}^{AS}, \end{aligned}$$

where in the last line we used Eq. (3). This is exactly the state output by $\pi_A^{q\text{-auth}}$ if when receiving the key k, ℓ, s, z , the protocol uses the Pauli $P_{\ell'}$ for the quantum one-time pad.

For Bob's part of the protocol, let the distinguisher prepare a state σ^{RAS} and send the AS part to the system. The subnormalized state held jointly by $\bar{\pi}_B^{q\text{-auth}}$ and the distinguisher after decoding and performing the measurement is given by

$$\begin{aligned} \langle s | P_\ell^A (U_k^{AS})^\dagger \sigma^{RAS} U_k^{AS} P_\ell^A | s \rangle \\ &= \langle 0 | (P_\ell^A \otimes P_{s,0}^S) (U_k^{AS})^\dagger \sigma^{RAS} U_k^{AS} (P_\ell^A \otimes P_{s,0}^S) | 0 \rangle \\ &= \langle 0 | (P_\ell^A \otimes P_{s,z}^S) (U_k^{AS})^\dagger \sigma^{RAS} U_k^{AS} (P_\ell^A \otimes P_{s,z}^S) | 0 \rangle \\ &= \langle 0 | (U_k^{AS})^\dagger P_{\ell'}^{AS} \sigma^{RAS} P_{\ell'}^{AS} U_k^{AS} | 0 \rangle. \end{aligned}$$

We again obtain the state that is jointly held by $\pi_B^{q\text{-auth}}$ and the distinguisher if when receiving the key k, ℓ, s, z , the protocol uses the Pauli $P_{\ell'}$ for the quantum one-time pad. \square

Remark 3.4. If part of the message is classical — i.e., it is diagonal in the computational basis and known not to have a purification held by the distinguisher — then running the same proof as Lemma 3.3, one can show that it is sufficient to perform bit flips on that part of the message, the phase flips are unnecessary. This is the case with the protocol from Section 4.3, that generates a key x locally and sends it to Bob using a quantum authentication scheme. We use this to save some bits of key in the bounds from Corollary 4.4 and Corollary 4.5.

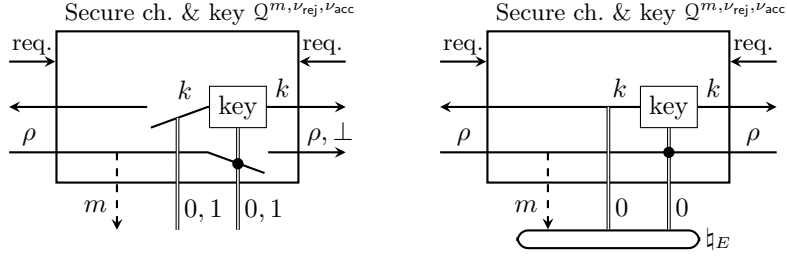


Figure 9 – The resource $Q^{m, \nu_{rej}, \nu_{acc}}$ is a restriction of the resource $S^m \parallel \mathcal{K}^{\nu_{rej}, \nu_{acc}}$ in which Eve’s interface only allows 1 bit to be input to decide both the length of the key and whether the message is received by Bob.

3.4 Security proof

As stated in Section 3.2 we wish to prove that the quantum authentication protocol considered constructs a secure channel S^m and secret key resource $\bar{\mathcal{K}}^{\nu_{rej}, \nu_{acc}}$. We will however prove this as a corollary of a theorem that makes a slightly stronger statement, namely that the protocol constructs a filtered resource $Q_{\natural}^{m, \nu_{rej}, \nu_{acc}}$ (depicted in Figure 9) that is equivalent to $(S^m \parallel \bar{\mathcal{K}}^{\nu_{rej}, \nu_{acc}})_b$, except for the fact that it provides one switch at the adversarial interface that decides both whether the message is delivered and the length of the recycled key. $\bar{\mathcal{K}}^{\nu_{rej}, \nu_{acc}}$ and S^m each provide Eve with an input bit to decide the length of the new key and whether the message is delivered, respectively, but in the protocol the two bits are correlated, since the players only recycle the full key if the message is successfully authenticated. One can thus make a slightly stronger statement, in which the ideal resource constructed only allows Eve to input one bit that decides both these things, which is what is achieved by $Q_{\natural}^{m, \nu_{rej}, \nu_{acc}}$.

The parameters of the construction are determined as follows. Let $\{U_k\}_{k \in \mathcal{K}}$ be a strong purity testing code of size $\log |\mathcal{K}| = \nu$ and with error ε that encodes an m qubit message in an $m + n$ qubit cipher. And let $\pi_{AB}^{q\text{-auth}} = (\pi_A^{q\text{-auth}}, \pi_B^{q\text{-auth}})$ denote Alice and Bob’s converters when running the encode-then-encrypt protocol from Figure 8. We are now ready to state the main theorem, namely that $\pi_{AB}^{q\text{-auth}}$ is a secure authentication scheme with key recycling.

Theorem 3.5. *Let $\pi_{AB}^{q\text{-auth}}$ denote converters corresponding to the protocol from Figure 8. Then $\pi_{AB}^{q\text{-auth}}$ constructs the secure channel and secret key filtered resource $Q_{\natural}^{m, \nu, \nu+2m+2n}$, given an insecure quantum channel \mathcal{C}_{\square} , a backwards authentic channel \mathcal{A}_{\diamond} and a secret key $\mathcal{K}^{\nu+2m+2n}$, i.e.,*

$$\mathcal{C}_{\square} \parallel \mathcal{A}_{\diamond} \parallel \mathcal{K}^{\nu+2m+2n} \xrightarrow{\pi_{AB}^{q\text{-auth}}, \varepsilon^{q\text{-auth}}} Q_{\natural}^{m, \nu, \nu+2m+2n},$$

with $\varepsilon^{q\text{-auth}} = \sqrt{\varepsilon} + \varepsilon/2$, where ε is the error of the strong purity testing code.

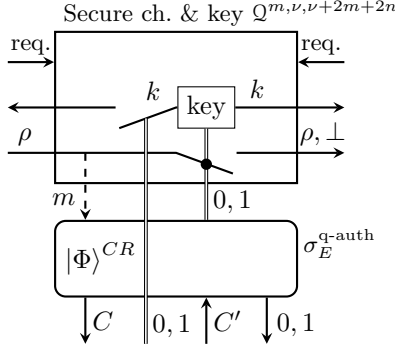


Figure 10 – The ideal quantum authentication system consisting of the constructed resource \mathcal{S}^m and $\mathcal{K}^{\nu, \nu+2m+2n}$, and the simulator $\sigma_E^{\text{q-auth}}$.

In order to prove this theorem, we need to find a simulator such that the real and ideal systems are indistinguishable except with advantage $\sqrt{\varepsilon} + \varepsilon/2$. The simulator that we use is illustrated in Figure 10, and works as follows. When it receives a notification from the ideal resource that a message is sent, it generates EPR pairs $|\Phi\rangle^{CR}$ and outputs half of each pair (the C register) at its outer interface. Once it receives a modified cipher (denoted C' in the picture), it measures this state and the half of the EPR pairs it kept in the Bell basis to decide if they were modified. It accordingly activates the switch on the resource $\mathcal{Q}^{m, \nu, \nu+2m+2n}$ controlling whether Bob gets the message and the length of the key generated, and outputs the bit of backward communication from Bob to Alice — which is always leaked to Eve. If it first receives the register C' before generating the EPR pairs, it always notifies the ideal resource to output an error and outputs 0 as the leak on the backwards authentic channel.

Proof. It is trivial to show that correctness holds with error 0, namely that

$$d\left(\pi_{AB}^{\text{q-auth}}(\mathcal{C} \square_E \| \mathcal{A} \diamond_E \| \mathcal{K}^{\nu+2m+2n}), \mathcal{Q}^{m, \nu, \nu+2m+2n} \natural_E\right) = 0. \quad (5)$$

We now prove the case of security, i.e.,

$$d\left(\pi_{AB}^{\text{q-auth}}(\mathcal{C} \| \mathcal{A} \| \mathcal{K}^{\nu+2m+2n}), \mathcal{Q}^{m, \nu, \nu+2m+2n} \sigma_E^{\text{q-auth}}\right) \leq \sqrt{\varepsilon} + \varepsilon/2. \quad (6)$$

The real and ideal systems, drawn in Figures 6 and 10 have 5 inputs. The distinguisher thus has the choice between $5!$ possible orders for providing inputs. However, most of these orders are redundant and do not need to be analyzed. Providing the requests for the secret keys before they are ready is pointless. So it is sufficient to look at the case where these requests are made as soon as the keys are available for recycling, i.e., after Bob has received the message from Alice and after Alice has received the confirmation from Bob.

What is more, neither sending Alice an error on the backwards authentic channel nor allowing her to get Bob's confirmation will help either way, since the distinguisher already knows what output Alice will produce, so we can completely ignore this input. That leaves only 2 in-ports, and thus 2 orders to analyze:

1. The distinguisher first inputs a message at Alice's interface, gets the cipher at Eve's interface, inputs a possibly modified cipher at Eve's interface, gets the output at Bob's interface, and requests the recycled key.
2. The distinguisher first inputs a fake cipher at Eve's interface, gets the output at Bob's interface, makes a request for his recycled key, then inputs a message at Alice's interface and receives the cipher for that message.

We start with the first case, the initial message is sent to Alice. The distinguisher prepares a message $|\psi\rangle^{ME}$ and inputs the M part at Alice's interface. The ideal channel then notifies the simulator that a message has been input. The simulator prepares a maximally entangled state $|\Phi\rangle^{CR}$ of dimension 2^{2m+2n} and outputs the C register at Eve's interface. The distinguisher now holds a bipartite state in CE , to which it applies a unitary U^{CE} . Without loss of generality, one may write the unitary as $U^{CE} = \sum_j P_j^C \otimes E_j^E$, where P_j^C are Paulis acting on the cipher register C and E_j^E act on the distinguisher's internal memory E . The resulting state in the C register is input back in the E interface. The simulator now measures CR in the Bell basis defined by the projectors $\{P_j \otimes I |\Phi\rangle\langle\Phi|^{CR} P_j \otimes I\}_j$. If the outcome is $j = 0$ — where $P_0 = I$ — it tells the resource $\mathcal{Q}^{m,\nu,\nu+2m+2n}$ that the cipher was not modified. In which case the contents of the register M is output at Bob's interface with an `acc` flag. Furthermore, $\mathcal{Q}^{m,\nu,\nu+2m+2n}$ generates a fresh uniform key (k, ℓ) , where $|k| = \nu$ and $|\ell| = 2m + 2n$. If the outcome is $j \neq 0$, then the simulator notifies the resource to delete the message and output a `rej` flag, as well as prepare only the shorter key k . The distinguisher then sends a request to obtain the fresh key. So the final state held by the distinguisher interacting with the ideal system is

$$\zeta = |\text{acc}\rangle\langle\text{acc}| \otimes \tau^K \otimes \tau^L \otimes \left[(I^M \otimes E_0^E) |\psi\rangle\langle\psi|^{ME} (I^M \otimes (E_0^E)^\dagger) \right] + \sum_{j \neq 0} |\text{rej}\rangle\langle\text{rej}| \otimes \tau^K \otimes E_j^E \rho^E (E_j^E)^\dagger, \quad (7)$$

where τ^K and τ^L are fully mixed states and $\rho^E = \text{tr}_M(|\psi\rangle\langle\psi|^{ME})$. One could append states \perp^L and \perp^M in the `rej` branch of Eq. (7) so that both terms have the same number of registers; we omit them for simplicity.

In the real system, for the secret key (k, ℓ) , the state before Bob's measurement of the syndrome is given by

$$\begin{aligned} |\varphi_{k,\ell}\rangle^{SME} &= \sum_j \left((U_k^{SM})^\dagger P_\ell^{SM} P_j^{SM} P_\ell^{SM} U_k^{SM} \otimes E_j^E \right) |0\rangle^S |\psi\rangle^{ME} \\ &= \sum_j (-1)^{(j,\ell)_{\text{Sp}}} \left((U_k^{SM})^\dagger P_j^{SM} U_k^{SM} \otimes E_j^E \right) |0\rangle^S |\psi\rangle^{ME}, \end{aligned}$$

where $(\cdot, \cdot)_{\text{Sp}}$ denotes the symplectic product defined in Eq. (1). Let \mathcal{J}_s^k be the set of indices j such that the error P_j^{SM} produces a syndrome s when code k is used, i.e., $(U_k^{SM})^\dagger P_j^{SM} U_k^{SM} = e^{i\theta_{k,j}} P_{s,z}^S \otimes P_{j'}^M$ for some $\theta_{k,j}$ (see Eq. (3) and discussion thereafter). For $j \in \mathcal{J}_s^k$, let

$$\begin{aligned} |s\rangle^S |\psi_{j,k}\rangle^{ME} &:= \left((U_k^{SM})^\dagger P_j^{SM} U_k^{SM} \otimes E_j^E \right) |0\rangle^S |\psi\rangle^{ME} \\ &= e^{i\theta_{k,j}} (P_{s,z}^S \otimes P_{j'}^M \otimes E_j^E) |0\rangle^S |\psi\rangle^{ME}. \end{aligned}$$

Then

$$\begin{aligned} |\varphi_{k,\ell}\rangle &= \sum_s \sum_{j \in \mathcal{J}_s^k} (-1)^{(j,\ell)_{\text{Sp}}} \left((U_k^{SM})^\dagger P_j^{SM} U_k^{SM} \otimes E_j^E \right) |0\rangle^S |\psi\rangle^{ME} \\ &= \sum_s \sum_{j \in \mathcal{J}_s^k} (-1)^{(j,\ell)_{\text{Sp}}} |s\rangle^S |\psi_{j,k}\rangle^{ME}. \end{aligned}$$

The next step in Bob's protocol consists in measuring the syndrome. If $s = 0$ is obtained, he outputs the message as well as the key (k, ℓ) and a flag `acc`. Otherwise he deletes the message, outputs k with the flag `rej`. The final state held be the distinguisher in this case is

$$\begin{aligned} \xi &= |\text{acc}\rangle\langle\text{acc}| \otimes \frac{1}{2^{\nu+2m+2n}} \sum_{k,\ell} |k, \ell\rangle\langle k, \ell| \\ &\quad \otimes \sum_{j_1, j_2 \in \mathcal{J}_0^k} (-1)^{(j_1 \oplus j_2, \ell)_{\text{Sp}}} |\psi_{j_1, k}\rangle\langle\psi_{j_2, k}|^{ME} \\ &+ |\text{rej}\rangle\langle\text{rej}| \otimes \frac{1}{2^{\nu+2m+2n}} \sum_{k,\ell} |k\rangle\langle k| \\ &\quad \otimes \sum_{s \neq 0} \sum_{j_1, j_2 \in \mathcal{J}_s^k} (-1)^{(j_1 \oplus j_2, \ell)_{\text{Sp}}} E_{j_1}^E \rho^E (E_{j_2}^E)^\dagger, \end{aligned}$$

where we have used $|\psi_{j,k}\rangle^{ME} = \left(V_{k,j}^M \otimes E_j^E \right) |\psi\rangle^{ME}$ for some unitary $V_{k,j}^M$.

Setting

$$\begin{aligned}
\zeta^{\text{acc}} &:= (I^M \otimes E_0^E) |\psi\rangle\langle\psi|^{ME} (I^M \otimes (E_0^E)^\dagger), \\
\zeta^{\text{rej}} &:= \sum_{j \neq 0} E_j^E \rho^E (E_j^E)^\dagger, \\
\xi_{k,\ell}^{\text{acc}} &:= \sum_{j_1, j_2 \in \mathcal{J}_0^k} (-1)^{(j_1 \oplus j_2, \ell)_{\text{Sp}}} |\psi_{j_1, k}\rangle\langle\psi_{j_2, k}|^{ME}, \\
\xi_k^{\text{rej}} &:= \frac{1}{2^{2m+2n}} \sum_{\ell, s \neq 0} \sum_{j_1, j_2 \in \mathcal{J}_s^k} (-1)^{(j_1 \oplus j_2, \ell)_{\text{Sp}}} E_{j_1}^E \rho^E (E_{j_2}^E)^\dagger,
\end{aligned}$$

the distance between real and ideal systems may be written as

$$\frac{1}{2} \|\zeta - \xi\|_{\text{tr}} = \frac{1}{2 \cdot 2^{\nu+2m+2n}} \sum_{k,\ell} \|\zeta^{\text{acc}} - \xi_{k,\ell}^{\text{acc}}\|_{\text{tr}} + \frac{1}{2 \cdot 2^\nu} \sum_k \|\zeta^{\text{rej}} - \xi_k^{\text{rej}}\|_{\text{tr}}.$$

ζ^{acc} and $\xi_{k,\ell}^{\text{acc}}$ are both pure states, so from Lemma C.1 we bound their distance as

$$\begin{aligned}
\frac{1}{2} \|\zeta^{\text{acc}} - \xi_{k,\ell}^{\text{acc}}\|_{\text{tr}} &\leq \left\| (I^M \otimes E_0^E) |\psi\rangle^{ME} - \sum_{j \in \mathcal{J}_0^k} (-1)^{(j,\ell)_{\text{Sp}}} |\psi_{j,k}\rangle^{ME} \right\| \\
&= \left\| \sum_{j \in \mathcal{J}_0^k \setminus \{0\}} (-1)^{(j,\ell)_{\text{Sp}}} |\psi_{j,k}\rangle^{ME} \right\| \\
&= \sqrt{\sum_{j_1, j_2 \in \mathcal{J}_0^k \setminus \{0\}} (-1)^{(j_1 \oplus j_2, \ell)_{\text{Sp}}} \langle\psi_{j_1, k}|\psi_{j_2, k}\rangle},
\end{aligned}$$

where $\| |a\rangle \| = \sqrt{\langle a|a\rangle}$ is the vector 2-norm and we used the fact that $|\psi_{0,k}\rangle^{ME} = (I^M \otimes E_0^E) |\psi\rangle^{ME}$. From Jensen's inequality and using Eq. (2) we obtain

$$\begin{aligned}
&\frac{1}{2 \cdot 2^{\nu+2m+2n}} \sum_{k,\ell} \|\zeta^{\text{acc}} - \xi_{k,\ell}^{\text{acc}}\|_{\text{tr}} \\
&\leq \sqrt{\frac{1}{2^{\nu+2m+2n}} \sum_{k,\ell} \sum_{j_1, j_2 \in \mathcal{J}_0^k \setminus \{0\}} (-1)^{(j_1 \oplus j_2, \ell)_{\text{Sp}}} \langle\psi_{j_1, k}|\psi_{j_2, k}\rangle} \\
&= \sqrt{\frac{1}{2^\nu} \sum_k \sum_{j \in \mathcal{J}_0^k \setminus \{0\}} \langle\psi_{j,k}|\psi_{j,k}\rangle}.
\end{aligned}$$

Finally, because the code is a strong purity testing code with error ε and

that $\langle \psi_{j,k} | \psi_{j,k} \rangle = \text{tr}(E_j^E \rho^E (E_j^E)^\dagger) =: p_j$ with $\sum_j p_j = 1$, we get

$$\begin{aligned} \frac{1}{2|\mathcal{K}||\mathcal{L}|} \sum_{k,\ell} \|\zeta^{\text{acc}} - \xi_{k,\ell}^{\text{acc}}\|_{\text{tr}} &\leq \sqrt{\frac{1}{|\mathcal{K}|} \sum_{j \neq 0} \sum_{k: j \in \mathcal{J}_0^k} \langle \psi_{j,k} | \psi_{j,k} \rangle} \\ &= \sqrt{\frac{1}{|\mathcal{K}|} \sum_{j \neq 0} \sum_{k: j \in \mathcal{J}_0^k} p_j} \\ &\leq \sqrt{\sum_{j \neq 0} \varepsilon p_j} \leq \sqrt{\varepsilon}. \end{aligned}$$

In the reject branch of the real system we have

$$\begin{aligned} \xi_k^{\text{rej}} &= \frac{1}{2^{2m+2n}} \sum_{\ell, s \neq 0} \sum_{j_1, j_2 \in \mathcal{J}_s^k} (-1)^{(j_1 \oplus j_2, \ell)_{\text{Sp}}} E_{j_1}^E \rho^E (E_{j_2}^E)^\dagger \\ &= \sum_{s \neq 0} \sum_{j \in \mathcal{J}_s^k} E_j^E \rho^E (E_j^E)^\dagger \\ &= \sum_{j \notin \mathcal{J}_0^k} E_j^E \rho^E (E_j^E)^\dagger, \end{aligned}$$

where we used again Eq. (2). Thus

$$\begin{aligned} \frac{1}{2 \cdot 2^\nu} \sum_k \|\zeta^{\text{rej}} - \xi_k^{\text{rej}}\|_{\text{tr}} &= \frac{1}{2 \cdot 2^\nu} \sum_k \left\| \sum_{j \in \mathcal{J}_0^k \setminus \{0\}} E_j^E \rho^E (E_j^E)^\dagger \right\|_{\text{tr}} \\ &\leq \frac{1}{2 \cdot 2^\nu} \sum_k \sum_{j \in \mathcal{J}_0^k \setminus \{0\}} p_j \leq \varepsilon/2. \end{aligned}$$

Putting all this together we get

$$\frac{1}{2} \|\zeta - \xi\|_{\text{tr}} \leq \sqrt{\varepsilon} + \varepsilon/2.$$

We now consider the second case: the distinguisher first prepares a state $|\psi\rangle^{CE}$ and inputs the C part at Eve's interface, then obtains the output at Bob's interface. Note that in the ideal case the channel always outputs a `rej` message at Bob's interface. Thus, if the cipher is accepted by Bob—who outputs a state ζ^{acc} —the distinguisher must be interacting with the real system and can already output this guess. In the case of a rejection, it now holds a bipartite system KE —the recycled key K and its purifying system E . It then applies an isometry $U : \mathcal{H}_{KE} \rightarrow \mathcal{H}_{KME}$ to this system and inputs the M part of the resulting state at Alice's interface. After which it obtains a cipher at Eve's interface and holds the tripartite system KCE —

the recycled key K , the cipher C and its internal memory E . We denote this state ζ in the ideal case and ξ^{rej} in the real case, and we need to bound

$$\frac{1}{2} \|\zeta - \xi^{\text{rej}}\|_{\text{tr}} + \frac{1}{2} \|\xi^{\text{acc}}\|_{\text{tr}}.$$

In a first step, we assume that the state $|\psi\rangle^{CE}$ prepared by the distinguisher is an antisymmetric fully entangled state, which we denote $|\Psi^-\rangle^{CE} = \sum_x (-1)^{w(x)} |x, \bar{x}\rangle^{CE}$, where $w(x)$ is the Hamming weight of $x \in \{0, 1\}^{m+n}$ and \bar{x} is the string x with all bits flipped. In the ideal case the simulator notifies the channel to reject the cipher, and the state $|\text{rej}\rangle\langle\text{rej}| \otimes \tau^K$ is output at Bob's interface. The distinguisher then holds $\zeta = \tau^K \otimes \tau^E$. In the real case, Bob applies the decoding algorithm, i.e., first a Pauli P_ℓ^C , then a unitary $(U_k^C)^\dagger$ and finally measures n bits of the syndrome in the computational basis. Since the antisymmetric state is invariant under $U \otimes U$, one could equivalently apply the inverse operation, $P_\ell U_k$, to the E system, i.e., the state after Bob's measurement is given by

$$\frac{1}{2^{\nu+3m+3n}} \sum_{k,\ell,s,x_1,x_2} (-1)^{w(x_1) \oplus w(x_2)} |k, \ell\rangle\langle k, \ell| \otimes (I^C \otimes P_\ell^E U_k^E) |s, x_1, \bar{s}, \bar{x}_1\rangle\langle s, x_2, \bar{s}, \bar{x}_1|^{CE} \left(I^C \otimes (U_k^E)^\dagger P_\ell^E \right).$$

If $s = 0$ Bob accepts the cipher as being valid, which happens with probability 2^{-n} , i.e., $\|\xi^{\text{acc}}\|_{\text{tr}} = 2^{-n}$. In the case where $s \neq 0$, he deletes the cipher, so the remaining state is given by

$$\frac{1}{2^{\nu+3m+3n}} \sum_{k,\ell,s \neq 0, x} |k, \ell\rangle\langle k, \ell| \otimes (I^C \otimes P_\ell^E U_k^E) |\bar{s}, \bar{x}\rangle\langle \bar{s}, \bar{x}|^{CE} \left(I^C \otimes (U_k^E)^\dagger P_\ell^E \right) = \tau^K \otimes \tau^L \otimes \tau^E - \rho^{KLE},$$

where

$$\rho^{KLE} = \frac{1}{2^{\nu+3m+3n}} \sum_{k,\ell,x} |k, \ell\rangle\langle k, \ell| \otimes P_\ell^E U_k^E |\bar{0}, \bar{x}\rangle\langle \bar{0}, \bar{x}|^E (U_k^E)^\dagger P_\ell^E,$$

K is made public and the L system is the part of the key kept secret by the players.

Let \mathcal{E} denote the completely positive, trace-preserving (CPTP) map consisting of the distinguisher's next step — the isometry $U : \mathcal{H}_{KE} \rightarrow \mathcal{H}_{KME}$ — and the final operation of the ideal system — deleting the message system M that is input at Alice's interface and outputting a fully mixed state τ^C . Let \mathcal{F} denote the CPTP map consisting of the distinguisher's next step and the final operation of the real system — encoding the message system

M according to the protocol and outputting the resulting cipher. We have $\zeta = \mathcal{E}(\tau^K \otimes \tau^E)$ and $\xi^{\text{rej}} = \mathcal{F}(\tau^K \otimes \tau^L \otimes \tau^E) - \mathcal{F}(\rho^{KLE})$. Thus,

$$\frac{1}{2} \|\zeta - \xi^{\text{rej}}\|_{\text{tr}} \leq \frac{1}{2} \|\mathcal{E}(\tau^K \otimes \tau^E) - \mathcal{F}(\tau^K \otimes \tau^L \otimes \tau^E)\|_{\text{tr}} + \frac{1}{2} 2^{-n},$$

since $\|\rho^{KLE}\|_{\text{tr}} = 2^{-n}$. Finally, note that we have

$$\mathcal{E}(\tau^K \otimes \tau^E) = \mathcal{F}(\tau^K \otimes \tau^L \otimes \tau^E) = \tau^C \otimes \sigma^{KE}$$

for $\sigma^{KE} = \text{tr}_M[U(\tau^K \otimes \tau^E)U^\dagger]$, since the random Pauli P_ℓ applied by the encryption algorithm completely decouples the cipher from KE . Putting this together, we get

$$\frac{1}{2} \|\zeta - \xi\|_{\text{tr}} \leq 2^{-n} \leq \sqrt{\varepsilon},$$

since a strong purity testing code will always have an error $\varepsilon \geq \frac{2^{2m+n}-1}{2^{2m+2n}-1} \geq 2^{-2n}$.

The final case that remains to consider is when the distinguisher prepares a state $|\psi\rangle^{CE}$ that is not the antisymmetric state. We will reduce this case to that of the entangled antisymmetric by using the entangled state $|\Psi^-\rangle^{CE}$ to teleport the C' part of any state $|\psi\rangle^{C'E'}$. Let the teleportation scheme be given by the projectors $\{M_a^{EC'}\}$ on EC' which incur a Pauli correction P_a^C on the C system, i.e.,

$$\text{tr}_{EC'} \left(\sum_a P_a^C \otimes M_a^{EC'} \left(|\Psi^-\rangle\langle\Psi^-|^{CE} \otimes |\psi\rangle\langle\psi|^{C'E'} \right) P_a^C \otimes M_a^{EC'} \right) = |\psi\rangle\langle\psi|_{CE'}.$$

So the distinguisher prepares an entangled state $|\Psi^-\rangle_{CE}$ and the state it wishes to send to Bob, $|\psi\rangle^{C'E'}$. It teleports the C' register to the C register, and sends this to Bob, who performs his decryption operation $(U_k^C)^\dagger P_\ell^C$. This results in the shared state

$$\begin{aligned} & \frac{1}{2^{\nu+2m+2n}} \sum_{k,\ell,a} |k,\ell\rangle\langle k,\ell| \otimes \left[\left((U_k^C)^\dagger P_\ell^C P_a^C \otimes M_a^{EC'} \right) \right. \\ & \quad \left. \left(|\Psi^-\rangle\langle\Psi^-|^{CE} \otimes |\psi\rangle\langle\psi|^{C'E'} \right) \left(P_a^C P_\ell^C U_k^C \otimes M_a^{EC'} \right) \right] \\ &= \frac{1}{2^{\nu+2m+2n}} \sum_{k,\ell,a} |k,\ell \oplus a\rangle\langle k,\ell \oplus a| \otimes \left[\left((U_k^C)^\dagger P_\ell^C \otimes M_a^{EC'} \right) \right. \\ & \quad \left. \left(|\Psi^-\rangle\langle\Psi^-|^{CE} \otimes |\psi\rangle\langle\psi|^{C'E'} \right) \left(P_\ell^C U_k^C \otimes M_a^{EC'} \right) \right] \\ &= \frac{1}{2^{\nu+2m+2n}} \sum_{k,\ell,a} X_a^L |k,\ell\rangle\langle k,\ell| X_a^L \otimes \left[\left(I^C \otimes M_a^{EC'} P_\ell^E U_k^E \right) \right. \\ & \quad \left. \left(|\Psi^-\rangle\langle\Psi^-|^{CE} \otimes |\psi\rangle\langle\psi|^{C'E'} \right) \left(I^C \otimes (U_k^E)^\dagger P_\ell^E M_a^{EC'} \right) \right], \end{aligned}$$

where X_a^L flips the bits of ℓ in the positions where $a_i = 1$. The teleportation of $|\psi\rangle_{C'E'}$ is thus equivalent to a measurement of the distinguisher's system followed by a correction of the secret key ℓ . This may however be performed after Bob measures the syndrome and accepts or rejects the cipher he received. The probability of accepting the cipher is thus unchanged, and plugging in the result from the case where the distinguisher sends Bob half of the anti-symmetric entangled state, we find that the state of the real system in the rejection branch after Bob's operations is

$$\begin{aligned} & \sum_a \left(X_a^L \otimes M_a^{EC'} \right) \tau^K \otimes \tau^L \otimes \tau^E \otimes |\psi\rangle\langle\psi|^{C'E'} \left(X_a^L \otimes M_a^{EC'} \right) \\ & \quad - \sum_a \left(X_a^L \otimes M_a^{EC'} \right) \rho^{KLE} \otimes |\psi\rangle\langle\psi|^{C'E'} \left(X_a^L \otimes M_a^{EC'} \right) \\ & = \tau^K \otimes \tau^L \otimes \sigma^{EC'E'} - \tilde{\rho}^{KLE C'E'} , \end{aligned}$$

where $\sigma^{EC'E'} = \sum_a M_a^{EC'} \left(\tau^E \otimes |\psi\rangle\langle\psi|^{C'E'} \right) M_a^{EC'}$. And in the ideal system the state after the ideal channel outputs a rejection is $\tau^K \otimes \sigma^{EC'E'}$. We thus obtain the same bound on the distance between real and ideal systems as in the previous case. \square

Corollary 3.6. *Let $\pi_{AB}^{q\text{-auth}}$ denote converteres corresponding to the protocol from Figure 8. Then $\pi_{AB}^{q\text{-auth}}$ constructs the secure channel and secret key filtered resource $(S^m \| \bar{\mathcal{K}}^{\nu, \nu+2m+2n})_b$, given an insecure quantum channel \mathcal{C}_\square , a backwards authentic channel \mathcal{A}_\diamond and a secret key $\mathcal{K}^{\nu+2m+2n}$, i.e.,*

$$\mathcal{C}_\square \| \mathcal{A}_\diamond \| \mathcal{K}^{\nu+2m+2n} \xrightarrow{\pi_{AB}^{q\text{-auth}}, \varepsilon^{q\text{-auth}}} (S^m \| \bar{\mathcal{K}}^{\nu, \nu+2m+2n})_b ,$$

with $\varepsilon^{q\text{-auth}} = \sqrt{\varepsilon} + \varepsilon/2$, where ε is the error of the strong purity testing code.

Proof. $\mathcal{Q}_\natural^{m, \nu_{\text{rej}}, \nu_{\text{acc}}}$ is a stronger resource than $(S^m \| \bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}})_b$, and one trivially has

$$\mathcal{Q}_\natural^{m, \nu_{\text{rej}}, \nu_{\text{acc}}} \xrightarrow{\text{id}, 0} (S^m \| \bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}})_b ,$$

with a simulator that forwards everything between the distinguisher and Eve's interface of $S^m \| \bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}}$ except for the bit deciding the secret key length and whether the message is accepted, which is copied and sent to both S^m and $\bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}}$. The corollary follows immediately from this and the composition theorem. \square

3.5 Optimality of the recycled key length

It follows from Lemma 3.3 that Theorem 3.5 and Corollary 3.6 are also proofs of security for the encrypt-then-encode protocol from Figure 7, i.e.,

$$\mathcal{C}_\square \| \mathcal{A}_\diamond \| \mathcal{K}^{\nu+2m+n} \xrightarrow{\pi_{AB}^{q\text{-auth}}, \varepsilon^{q\text{-auth}}} \mathcal{Q}_\natural^{m, \nu, \nu+2m+n} \xrightarrow{\text{id}, 0} (S^m \| \bar{\mathcal{K}}^{\nu, \nu+2m+n})_b ,$$

with $\varepsilon^{q\text{-auth}} = \sqrt{\varepsilon} + \varepsilon/2$. Thus, in the case where the message is not accepted by Bob, $2m + n$ bits of key are lost. We prove here that this is optimal: one cannot recycle any extra bit of key.

Lemma 3.7. *There exists an adversarial strategy to obtain all the secret bits that are not recycled in the encrypt-then-encode protocol.*

Proof. The distinguisher prepares EPR pairs $|\Phi\rangle^{ME}$ and provides the M part to Alice. It then receives the cipher and thus holds the state

$$U_k^{SM} P_\ell^M \left(|s\rangle^S \otimes |\Phi\rangle^{ME} \right),$$

which it keeps. It then sends a bogus cipher to Bob, and obtains the key k after Bob recycles it. It applies the decoding unitary $(U_k^{SM})^\dagger$, measures the S register to get the secret key s and measures the joint ME register in the Bell basis to get the secret key ℓ . \square

3.6 Explicit constructions

The protocols we have given in Section 3.3 use strong purity testing codes, and the parameters of the key used, key recycled and error depend on the parameters of these codes. In this section we give two constructions of purity testing codes. The first requires less initial secret key, the second has a better error parameter. Both have the same net consumption of secret key bits.

The first construction is from Barnum et al. [BCG⁺02]. They give an explicit strong purity testing code with $\nu = n$ and $\varepsilon = \frac{2m/n+2}{2^n}$.¹³ Plugging this in the parameters from Theorem 3.5 with the encrypt-then-encode protocol, we get the following.

Corollary 3.8. *The encrypt-then-encode protocol with the purity testing code of [BCG⁺02] requires an initial key of length $2m + 2n$. It recycles all bits if the message is accepted, and n bits if the message is rejected. The error is*

$$\varepsilon^{q\text{-auth}} = \sqrt{\frac{2m/n+2}{2^n}} + \frac{m/n+1}{2^n}.$$

The second construction we give is based on an explicit purity testing code by Chau [Cha05]—though he does not name it this way. Chau [Cha05] finds a set of unitaries $\mathcal{U} = \{U_k\}$ in dimension d such that, if k is chosen uniformly at random, any non-identity Pauli is mapped to every non-identity Pauli with equal frequency, i.e., $\forall P_j, P_\ell$ with $P_j \neq I$ and $P_\ell \neq I$,

$$\left| \left\{ U_k \in \mathcal{U} : U_k P_j U_k^\dagger = e^{i\theta_{j,k,\ell}} P_\ell \right\} \right| = \frac{|\mathcal{U}|}{d^2 - 1},$$

¹³In fact, [BCG⁺02] only prove that their construction is a purity testing code, not a strong one. But one can easily verify that it is strong with the same parameters. What is more, their construction has $\nu = \log(2^n + 1)$ and $\varepsilon = \frac{2m/n+2}{2^n+1}$. We remove one of the keys (and thus increase the error), so as to get simpler final expressions.

where $e^{i\theta_{j,k,\ell}}$ is some global phase.

We prove in Appendix B, Lemma B.2, that this is a strong purity testing code with $\varepsilon = 2^{-n}$ for $d = 2^{m+n}$. It also has $|\mathcal{U}| = 2^{m+n}(2^{2m+2n} - 1)$, hence $\nu = m + n + \log(2^{2m+2n} - 1) \leq 3m + 3n$. Note that when composed with Paulis as in the encode-then-encrypt protocol, $\{P_\ell U_k\}_{k,\ell}$ is a unitary 2-design [Dan05, DCEL09]. It follows that any (approximate) unitary t -design is a good quantum authentication scheme (see Appendix B for a formal proof).

Corollary 3.9. *The encrypt-then-encode protocol with the purity testing code of [Cha05] requires an initial key of length $5m + 4n$. It recycles all bits if the message is accepted, and $3m + 3n$ bits if the message is rejected. The error is $\varepsilon^{q\text{-auth}} = 2^{-n/2} + 2^{-n-1}$.*

4 Complete construction

We proved in Section 3 that the quantum authentication protocols from Figures 7 and 8 construct a secure channel and secret key filtered resource $(\mathcal{S}^m \parallel \bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}})_{\mathfrak{b}}$ from a shared secret key \mathcal{K}^μ , a noiseless insecure quantum channel \mathcal{C}_\square and a backwards authentic classical channel \mathcal{A}_\diamond , namely,

$$\mathcal{C}_\square \parallel \mathcal{A}_\diamond \parallel \mathcal{K}^\mu \xrightarrow{\pi_{AB}^{q\text{-auth}}, \varepsilon^{q\text{-auth}}} (\mathcal{S}^m \parallel \bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}})_{\mathfrak{b}}.$$

In this section we show how to construct the required resources from nothing but shared secret key and noisy channels, then put it all together to get the exact bounds of the composed protocols.

We discuss in Section 4.1 how to obtain the authentic channel: it can be constructed from a shared secret key and an insecure channel using any (classical) MAC-type authentication scheme [Por14]. Channels are however usually not noiseless. This is solved by using error correction: an error correction code constructs a noiseless channel given a noisy channel (with known noise), and is presented in Section 4.2. Were we to put things together at this point, we would construct the desired secure quantum channel from nothing but shared secret key and insecure noisy channels. But even in the case where no adversary is present, we would still consume some secret key, because we do not recycle the key from the backwards authentic channel. So in Section 4.3 we construct shared secret key given a secure channel — we simply share secret key using the channel. Combining all these pieces together, we obtain our secure quantum channel without any net consumption of key in the case where the adversary does not tamper with the messages. The security of the composed scheme follows immediately from the security of each component and the composition theorem of AC [MR11]. Finally, in Section 4.5 we discuss a setting in which the backwards authentic channel is not needed, thus allowing a more efficient use of the quantum channel —

since we do not need it to distribute key, and can thus use all of its capacity to send messages.

4.1 Authentic channel

We used classical authentication as an example in Section 2: an authentication protocol π_{AB}^{auth} constructs an authentic channel \mathcal{A}_\diamond given a (noiseless) insecure channel \mathcal{C}_\square and a secret key \mathcal{K}^η ,

$$\mathcal{C}_\square \parallel \mathcal{K}^\eta \xrightarrow{\pi_{AB}^{\text{auth}}, \varepsilon^{\text{auth}}} \mathcal{A}_\diamond. \quad (8)$$

The corresponding real and ideal systems were illustrated in Figures 2 and 4, respectively. Appending the MAC $h_k(x)$ to the message x is sufficient to construct the authentic channel with error ε if the family of functions $\{h_k\}_k$ is ε -almost strongly 2-universal [Por14]. In our case, Bob only needs to send a 1-bit message to Alice. If the key k has length η , a $2^{-\eta/2}$ -almost strongly 2-universal family of functions for 1-bit messages is given by $h_k(x) = k_x$, where k_0 are the first $\eta/2$ bits of k and k_1 are the last $\eta/2$ bits.

Lemma 4.1. *The authentication scheme described above satisfies Eq. (8) with $\varepsilon^{\text{auth}} = 2^{-\eta/2}$.*

Proof. Follows from [Por14, Lemma 9]. □

4.2 Noiseless channel

Both the classical authentication protocol discussed in Sections 2 and 4.1, as well as the quantum authentication protocol analyzed in Section 3 produce an error message as soon as there is any disturbance on the channel. Realistic channels are naturally noisy, so for such protocols to even make sense, one needs an extra layer of error correction that is designed to correct the specific noise on the channel. Here, we formalize this as a constructive statement. Let π_A^{ecc} encode a message with an error correcting code (ECC) given by the completely positive, trace-preserving (CPTP) map \mathcal{E} , and π_B^{ecc} decode the message with the CPTP map \mathcal{D} , as illustrated in Figure 11a. And let ε^{ecc} be the error of the ECC for noise given by a CPTP map \mathcal{F} , i.e.,

$$\frac{1}{2} \|\mathcal{D} \circ \mathcal{F} \circ \mathcal{E} - \text{id}\|_\diamond \leq \varepsilon^{\text{ecc}}. \quad (9)$$

Then $\pi_{AB}^{\text{ecc}} = (\pi_A^{\text{ecc}}, \pi_B^{\text{ecc}})$ constructs a noiseless (insecure) channel $\bar{\mathcal{C}}_\square$ from a noisy (insecure) channel \mathcal{C}_\sharp , if the filter \sharp_E introduces the noise \mathcal{F} , i.e.,

$$\bar{\mathcal{C}}_\square \xrightarrow{\pi_{AB}^{\text{ecc}}, \varepsilon^{\text{ecc}}} \mathcal{C}_\square. \quad (10)$$

Note that the resources \mathcal{C} and $\bar{\mathcal{C}}$ are of different dimension, since an error correcting code will map a quantum state to a new one of larger dimension.

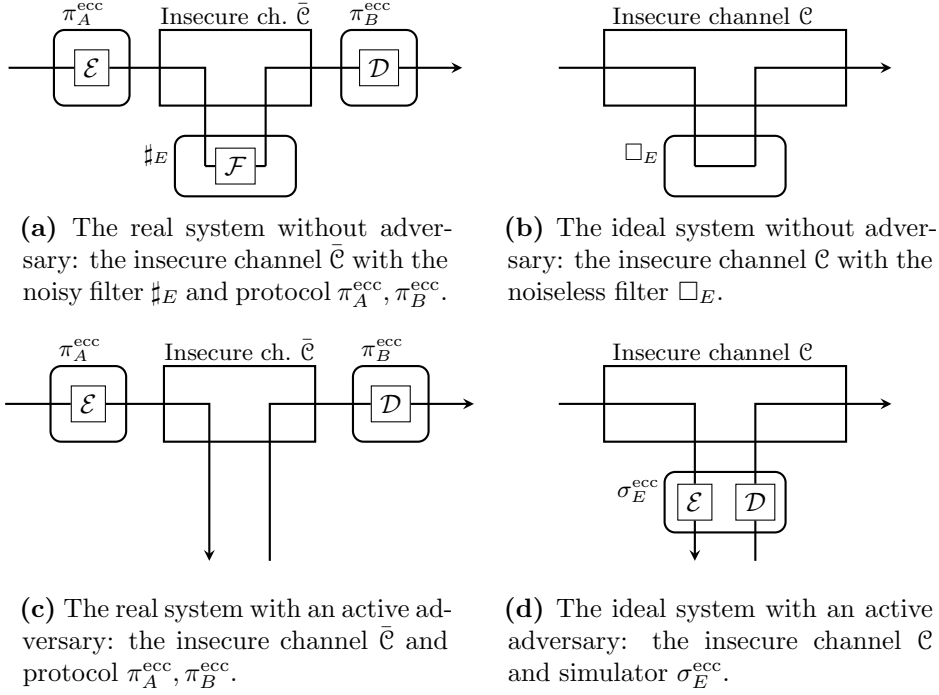


Figure 11 – An error correcting code constructs a noiseless channel \mathcal{C}_{\square} from a noisy channel $\bar{\mathcal{C}}_{\sharp}$.

In this work we generally we do not use different notation for channels of different dimensions, since the dimension is usually clear from the context, and we just write \mathcal{C} for an insecure channel. We distinguish a noiseless from a noisy channel by its filter, \square_E and \sharp_E , respectively.

Lemma 4.2. *If the filter \sharp_E introduces noise given by a CPTP map \mathcal{F} , and the encoding and decoding maps \mathcal{E} and \mathcal{D} satisfy Eq. (9), then the protocol π_{AB}^{ecc} that uses this ECC satisfies Eq. (10).*

Proof. We need to prove that $d(\pi_{AB}^{\text{ecc}}\bar{\mathcal{C}}_{\sharp_E}, \mathcal{C}_{\square_E}) \leq \varepsilon^{\text{ecc}}$ and $d(\pi_{AB}^{\text{ecc}}\bar{\mathcal{C}}, \mathcal{C}\sigma_E^{\text{ecc}}) \leq \varepsilon^{\text{ecc}}$ for some simulator σ_E^{ecc} to satisfy the two conditions from Definition 2.1. These systems are drawn in Figure 11. One can easily check from the figure that the first condition holds because the ECC was designed to achieve exactly this. $\pi_{AB}^{\text{ecc}}\bar{\mathcal{C}}$ and $\mathcal{C}\sigma_E^{\text{ecc}}$ are each a pair of channels, one performing the encoding operation \mathcal{E} , the other the decoding operation \mathcal{D} , so the second condition holds with distance 0. \square

Naturally, Lemma 4.2 only makes sense if there exists a code than can correct the errors introduced by \mathcal{F} , i.e., if there exist maps \mathcal{E}, \mathcal{D} that satisfy Eq. (9). In this work, when we talk about a noisy channel resource, we always mean such a channel that has non-zero capacity.

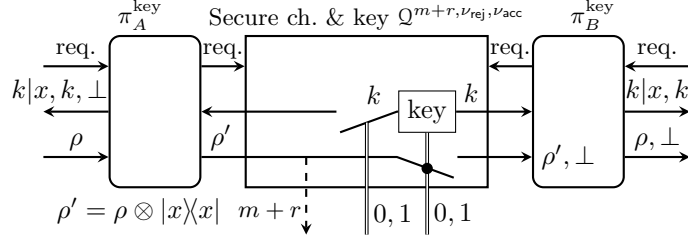


Figure 12 – Distributing a secret key using the channel $\mathcal{Q}^{m, \nu_{\text{rej}}, \nu_{\text{acc}}}$ constructed by a quantum authentication code. Alice generates a random string x of length r and appends it to an m qubit message ρ . If Bob successfully authenticates the message, he appends x to the recycled key. If Alice gets a confirmation from Bob that he received the correct message, she appends x to the recycled key.

4.3 Obtaining more key

It is trivial to share secret key using a secure channel: Alice generates a uniform string and sends it on the secure channel to Bob. If the channel can transmit an $m+r$ qubit state and we use it to share a r bit key, then m qubits can still be used to transmit a message. Writing this using the AC resource theory notation, we get

$$\mathcal{S}^{m+r} \rightarrow \mathcal{S}^m \parallel \tilde{\mathcal{K}}^r,$$

where $\tilde{\mathcal{K}}^r$ generates an r -bit key, but allows Eve to prevent Bob from receiving it.

Although this statement is correct, it is somewhat inconvenient, since $\tilde{\mathcal{K}}^r$ allows Eve to control whether Bob receives the key, but Alice does not know whether he received it; whereas in the real quantum authentication system analyzed, Alice actually learns whether Bob receives her message or not, since he sends her a confirmation bit on the backwards authentic channel. A stronger statement can be made if instead of using the final secure channel \mathcal{S}^m constructed in Corollary 3.6 to share secret key, we use the intermediary secure channel & key resource $\mathcal{Q}^{m, \nu_{\text{rej}}, \nu_{\text{acc}}}$ constructed in Theorem 3.5 (and illustrated in Figure 9), in which the secure channel \mathcal{S}^m and fresh key $\tilde{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}}$ are merged into one, with only one input bit at Eve's interface controlling both the length of the key and whether the message is delivered.

Using this secure channel to distribute key we get the system drawn in Figure 12, where Alice's converter π_A^{key} only outputs the r bits of key she inserted on the channel if she gets confirmation that Bob receives the message, i.e., if she obtains the longer key.

Lemma 4.3. *Let $\pi_{AB}^{\text{key}} = (\pi_A^{\text{key}}, \pi_B^{\text{key}})$ denote the protocol described above. It constructs a $\mathcal{Q}_{\text{q}}^{m, \nu_{\text{rej}}, \nu_{\text{acc}}+r}$ secure channel from a $\mathcal{Q}_{\text{q}}^{m+r, \nu_{\text{rej}}, \nu_{\text{acc}}}$ secure channel*

with no error, i.e.,

$$\mathcal{Q}_{\sharp}^{m+r, \nu_{\text{rej}}, \nu_{\text{acc}}} \xrightarrow{\pi_{AB}^{\text{key}, 0}} \mathcal{Q}_{\sharp}^{m, \nu_{\text{rej}}, \nu_{\text{acc}}+r}.$$

Proof. This lemma trivially holds with a simulator that changes the length of the message leaked at Eve's interface from $m+r$ to m and forwards the two control bits to the constructed $\mathcal{Q}^{m, \nu_{\text{rej}}, \nu_{\text{acc}}+r}$. \square

As in Corollary 3.6, it follows that

$$\mathcal{Q}_{\sharp}^{m+r, \nu_{\text{rej}}, \nu_{\text{acc}}} \xrightarrow{\pi_{AB}^{\text{key}, 0}} (\mathcal{S}^m \parallel \bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}+r})_{\flat}.$$

4.4 Putting it together

If we compose all the protocols described above, we obtain the system depicted in Figure 13. Let π_A denote the composition of Alice's converters and π_B denote the composition of Bob's converters. We then immediately get that the $\pi_{AB} = (\pi_A, \pi_B)$ constructs a secure channel and secret key filtered resource, $(\mathcal{S}^m \parallel \bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}+r})_{\flat}$, from secret keys \mathcal{K}^{μ} and \mathcal{K}^{η} , and two noisy channels $\bar{\mathcal{C}}_{\sharp}^{\rightarrow}$ and $\bar{\mathcal{C}}_{\sharp}^{\leftarrow}$ with error $\varepsilon = \max\{\varepsilon^{\text{q-auth}} + \varepsilon^{\text{auth}}, \bar{\varepsilon}^{\text{ecc}} + \leftarrow{\varepsilon}^{\text{ecc}}\}$, where the arrows are used to distinguish the forwards quantum channel and the backwards classical channel, i.e.,

$$\mathcal{K}^{\mu} \parallel \mathcal{K}^{\eta} \parallel \bar{\mathcal{C}}_{\sharp}^{\rightarrow} \parallel \bar{\mathcal{C}}_{\sharp}^{\leftarrow} \xrightarrow{\pi_{AB, \varepsilon}} (\mathcal{S}^m \parallel \bar{\mathcal{K}}^{\nu_{\text{rej}}, \nu_{\text{acc}}+r})_{\flat}.$$

In the following we assume for simplicity that $\bar{\varepsilon}^{\text{ecc}} + \leftarrow{\varepsilon}^{\text{ecc}} \leq \varepsilon^{\text{q-auth}} + \varepsilon^{\text{auth}}$ and take $\varepsilon = \varepsilon^{\text{q-auth}} + \varepsilon^{\text{auth}}$. Plugging in $r = \eta = n$ and the parameters from the two explicit quantum authentication protocols from Section 3.6, we get the following two corollaries. Corollary 4.4 uses the explicit construction proposed by Barnum et al. [BCG⁺02]. Corollary 4.5 uses the explicit unitary 2-design construction from Chau [Cha05] (see Section 3.6 for details).

Corollary 4.4. *For any m and n , there exist an explicit protocol that requires $2m+4n$ bits of secret key, a forwards noisy insecure quantum channel and a backwards noisy insecure classical channel, to construct a secure channel for an m qubit quantum message, which recycles all the key if the message is accepted and n bits if the message is rejected, and has error*

$$\varepsilon \leq 2^{-n/2}(1 + \sqrt{2m/n + 4}) + 2^{-n}(m/n + 2).$$

Proof. Follows from the composition theorem from AC [MR11] as well as Remark 3.4, Corollary 3.8, Lemma 4.1, Lemma 4.2, and Lemma 4.3. \square

Corollary 4.5. *For any m and n , there exists an explicit protocol that requires $5m+9n$ bits of secret key, a forwards noisy insecure quantum channel*

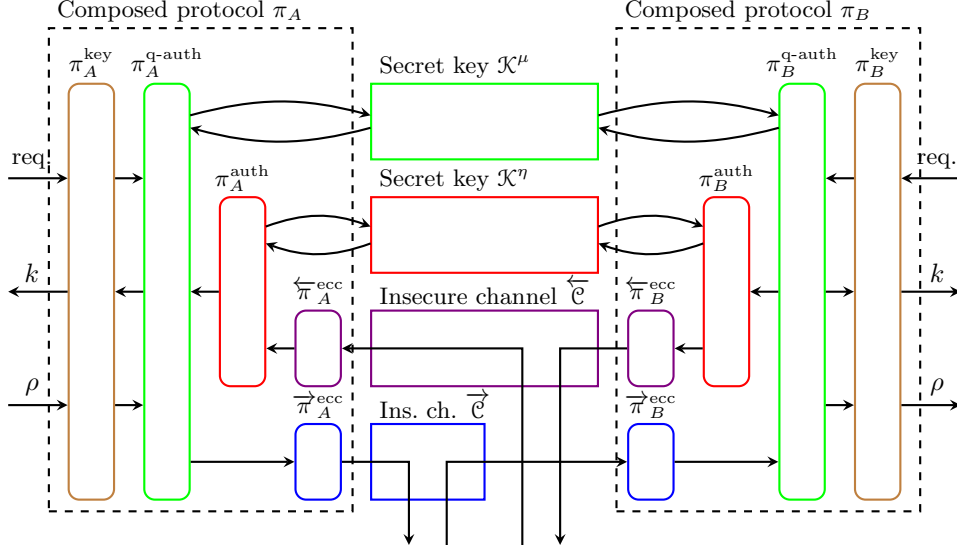


Figure 13 – Composing all the protocols discussed in this work results in the system depicted here. In blue we have drawn the systems constructing the noiseless insecure quantum channel, in violet the systems constructing the backwards noiseless classical channel, in red (along with the violet subprotocol) the systems constructing the backwards authentic channel, in green (along with the red and blue subprotocols) the systems constructing the secure quantum channel that recycles the key \mathcal{K}^μ , and in brown (along with the green subprotocol) the systems constructing the secure quantum channel that compensates for the lost key \mathcal{K}^η .

and a backwards noisy insecure classical channel, to construct a secure channel for an m qubit quantum message, which recycles all the key if the message is accepted and $3m + 6n$ bits if the message is rejected, and has error

$$\varepsilon \leq 2^{-n/2+1} + 2^{-n-1}.$$

Proof. Follows from the composition theorem from AC [MR11] as well as Remark 3.4, Corollary 3.9, Lemma 4.1, Lemma 4.2, and Lemma 4.3. \square

Note that both schemes have a loss of $2m + 3n$ key bits if the message is not successfully authenticated: $2m$ bits used to one-time pad the quantum message of length m , $2n$ bits to one-time pad the two classical strings of n bits—the n bits of key sent to replace those consumed by the backwards authentic channel and the n bits used as syndrome—and finally the n bits consumed by the backwards authentic channel cannot be replaced, so they are lost as well.

4.5 Removing the authentic channel

The backwards authentic channel is crucial in the construction of the secure channel with key recycling analyzed in this section, because without it, Alice would not know that her message arrived—let alone whether it was accepted or not—and thus not be able to recycle the key. One can however skip the authentic channel if the players share a stronger resource, e.g., another secure quantum channel that recycles key.

This is the case if the players construct secure quantum channels in both directions and alternate between the two: first Alice sends a quantum message to Bob, then Bob to Alice, then Alice to Bob, etc. Let them share two sets of keys, one set is used for the forward communication, the other is used for the backward communication. If at any point a message is not successfully authenticated, this means that an eavesdropper is disturbing the communication, and the players abort and stop communicating. Thus, Bob only sends his next message to Alice if he successfully received her message. So if Alice successfully authenticates Bob’s message, she knows that hers was received, and can recycle all of her key to send the next message. We thus avoid any explicit confirmation of reception, since sending the next message is in itself the confirmation.

One can easily show that the composition of n rounds of this protocol, each round sending an m qubit quantum message, constructs a secure channel for nm qubits, which we denote $\mathcal{R}^{m,n}$. Unlike the resource \mathcal{S}^m which gave the adversary a 1 bit input to decide if the message is delivered or not, $\mathcal{R}^{m,n}$ provides the adversary with a bit of input for every block of m qubits—but if one block is prevented from being delivered, none of the subsequent messages are delivered either. The error of this construction is n times the error of each round, $\varepsilon = n\varepsilon^{\text{q-auth}}$. This may continue arbitrary long if no adversary introduces noise on the channel, since the quantum authentication protocol recycles every bit of key.

5 Discussion and open questions

The family of quantum authentication protocols of Barnum et al. [BCG⁺02] as well as the subset analyzed in this work are large classes, which include many protocols appearing independently in the literature. The signed polynomial code [BCG⁺06, ABE10], the Clifford code [ABE10, DNS12, BW16] (which is a unitary 3-design [Zhu15, Web15]) and the unitary 8-design scheme from [GYZ16] are all instances which use a strong purity testing code. Our results apply directly to the Clifford and unitary 8-design schemes—which have in the same error as the unitary 2-design scheme from Corollary 3.9. But the signed polynomial code uses an ECC on qudits, not qubits, so our proof does not cover this case, and would have to be adapted to do so.

The trap code [BGS13, BW16] is an example of a quantum authentication

scheme that uses a purity testing code that is not a strong purity testing code, i.e., errors which do not modify the message do not necessarily provoke an abort. For example, if the adversary performs a simple bit flip in one position, this will provoke an abort with probability $2/3$ in the variant from [BGS13] and with probability $1/3$ in the variant from [BW16], but leaves the message unmodified if no abort occurs. If the adversary learns whether Bob accepted the message or not, she will learn whether the ECC used detects that specific bit flip or not, and thus learn something about the key used to select the ECC. Hence, the players cannot recycle the entire key, even in the case where the message is accepted. The restriction to strong purity testing codes is thus necessary to recycle every bit. It remains open how many bits of key can be recycled with the trap code, but we conjecture that this bit leaked due the decision to abort or not is the only part of the key leaked, and the rest can be recycled.

Another quantum authentication scheme, Auth-QFT-Auth, has been proposed in [GYZ16], where the authors prove that some of the key can be recycled as well. We do not know if this scheme fits in the family from [BCG⁺02] or not.

In the classical case, almost strongly 2-universal hash functions [WC81, Sti94] are used for authentication, and any new family of such functions immediately yields a new MAC. Likewise, any new purity testing code provides a new quantum authentication scheme. However, it is unknown whether all quantum authentication schemes can be modeled as a combination of a one-time pad and a purity testing code, or whether there exist interesting schemes following a different pattern.

We have proven that a loss of $2m + n$ bits of key is inevitable with these schemes if the adversary tampers with the channel. In the case of the unitary 2-design scheme, which has the smallest error, this is $2m + 2 \log 1/\varepsilon + 2$ bits of key which are consumed. A loss of $2m$ bits will always occur, since these are required to one-time pad the message. It remains open whether there exist other schemes — which do not fit the one-time pad + purity testing code model — which recycle more key.

The initial preprint of this work suggested that one should also investigate whether it is possible to find a prepare-and-measure scheme to encrypt and authenticate a classical message in a quantum state, so that all of the key may be recycled if it is successfully authenticated. At the time of writing, a possible solution had already been found by Fehr and Salvail [FS17]. Their protocol is however not known to be composable, and it remains open to prove that it achieves the desired result in such a setting.

Acknowledgments

CP would like to thank Anne Broadbent, Frédéric Dupuis and Debbie Leung for useful discussions.

CP is supported by the European Commission FP7 Project RAQUEL (grant No. 323970), US Air Force Office of Scientific Research (AFOSR) via grant FA9550-16-1-0245, the Swiss National Science Foundation (via the National Centre of Competence in Research ‘Quantum Science and Technology’) and the European Research Council – ERC (grant No. 258932).

Appendices

In Appendix A we provide a longer introduction to AC framework than already present in Section 2. In Appendix B we give an introduction to unitary 2-designs and prove that they are strong purity testing codes. Appendix C contains some technical lemmas used in the body of this work. And in Appendix D we provide a security proof for quantum authentication without key recycling.

A Abstract cryptography

As already mentioned in Section 1.3, the AC framework [MR11] models cryptography as a resource theory. The AC framework does however not explicitly define these resources. It follows a top-down paradigm and only specifies on each level of abstraction the properties of objects that are absolutely essential—the axioms these objects must satisfy. This simplifies the framework by removing unnecessary and cumbersome information—e.g., a model of computation—and results in more general framework that is not hard-coded with a specific communication or scheduling model. In this section we give a brief introduction to AC. We illustrate this with an example taken from [Por14], namely that appending a message authentication code (MAC) to a classical message is sufficient to construct a classical authentic channel given an insecure channel and a shared secret key. We refer the interested reader to the original AC paper [MR11] for a detailed treatment of the abstract layer and to [Mau12, PR14] for more gentle introductions to AC.

A.1 Resources

An \mathcal{I} -resource is an (abstract) system with interfaces specified by a set \mathcal{I} , e.g., $\mathcal{I} = \{A, B, E\}$. Each interface $i \in \mathcal{I}$ is accessible to a user i . The objects depicted in Figure 1 are examples of resources. The insecure channel in Figure 1a allows Alice to input a message at her interface on the left and

allows Bob to receive a message at his interface on the right. Eve can intercept Alice’s message and insert a message of her choosing at her interface. Mathematically, this can be captured by two identity channels, one from Alice to Eve and one from Eve to Bob, which may be used independently. The authentic channel resource depicted in Figure 1b also allows Alice to send a message and Bob to receive a message, but Eve’s interface is more limited than for the insecure channel: she can only decide if Bob receives the message, an error symbol or nothing at all—by inputting 0, 1, or nothing, respectively, at her interface—but not tamper with the message being sent. This can be modeled mathematically by two maps. The first goes from Alice to Eve and some internal memory of the system: upon receiving Alice’s message, it stores a copy in the memory sends the other to Eve. The second map goes from Eve and the internal memory to Bob: Bob either receives a copy of the memory or an error message depending on Eve’s input. Note that if the maps are activated in the other order, Bob will receive an error regardless of Eve’s input value. The key resource drawn in Figure 1c provides each player with a secret key when requested. This can be modeled by two maps, which each take the input `req.` from a player and return the key value (which is chosen uniformly at random). In Appendix A.3 we discuss how to model resources in general as mathematical objects.

Additionally, resources are equipped with a parallel composition operator, \parallel , that maps two resources to another resource. This is to be understood as both resources being merged into one, that can be accessed in any arbitrary order. For example, if players share a secret key resource \mathcal{K} and a channel resource \mathcal{C} , they have the resource $\mathcal{K}\parallel\mathcal{C}$ available, as depicted in Figure 1d. Given access to $\mathcal{K}\parallel\mathcal{C}$, the players could, e.g., get a key from \mathcal{K} and use it to encode a message that is sent on \mathcal{C} .

Converters capture operations that a player might perform locally at her interface. These are (abstract) systems with two interfaces, an *inside* interface and an *outside* interface. The inside interface connects to an interface of a resource, and the outside interface becomes the new interface of the resource resulting from the connection of this converter and resource. For example, in the setting described a paragraph higher, Alice might decide to append a MAC to her message. This is modeled as a converter π_A^{auth} that obtains the message x at the outside interface, obtains a key at the inside interface from a key resource \mathcal{K} and sends $(x, h_k(x))$ on the insecure channel \mathcal{C} , where h_k is taken from a family of strongly 2-universal hash functions [WC81, Sti94]. We illustrate this in Figure 2. Converters are always drawn with rounded corners.

If a converter α_i is connected to the i interface of a resource \mathcal{R} , we write $\alpha_i\mathcal{R}$ or $\mathcal{R}\alpha_i$ for the new resource obtained by connecting the two—in this work we adopt the convention of writing converters at the A and B interfaces on the left and converters at the E interface on the right, though there is no mathematical difference between $\alpha_i\mathcal{R}$ and $\mathcal{R}\alpha_i$. Serial and parallel

composition of converters is defined as follows:

$$(\alpha_i\beta_i)\mathcal{R} := \alpha_i(\beta_i\mathcal{R}) \quad \text{and} \quad (\alpha_i\|\beta_i)(\mathcal{R}\|\mathcal{S}) := (\alpha_i\mathcal{R})\|(\beta_i\mathcal{S}).$$

By definition, converters at different interfaces commute, i.e., $\alpha_i\beta_j\mathcal{R} = \beta_j\alpha_i\mathcal{R}$ if $i \neq j$. This allows us to draw systems as in Figure 2 without having to specify an order in which π_A^{auth} and π_B^{auth} are connected to the resource $\mathcal{K}\|\mathcal{C}$.

A protocol is then defined by a set of converters, one for every honest player. Another type of converter that we need is a *filter*. The resources illustrated in Figure 1 depict a setting with an adversary that has some control over these resources. For a cryptographic protocol to be useful it is not sufficient to provide guarantees on what happens when an adversary is present, one also has to provide a guarantee on what happens when no adversary is present, e.g., if no adversary tampers with the message on the insecure channel, then Bob will receive the message that Alice sent. We model this setting by covering the adversarial interface with a filter that emulates an honest behavior.¹⁴ In Figure 3 we draw an insecure and an authentic channel with filters \sharp_E and \diamond_E that transmit the message to Bob. In the case of the insecure channel, one may want to model an honest noisy channel when no adversary is present. This is done by having the filter \sharp_E add some noise to the message. A dishonest player removes this and has access to a noiseless channel as in Figure 1a.

We use the term *filtered resource* to refer to a pair of a resource \mathcal{R} and a filter \sharp_E , and often write $\mathcal{R}_\sharp = (\mathcal{R}, \sharp_E)$. Such an object can be thought of as having two modes: it is characterized by the resource \mathcal{R}_\sharp when no adversary is present and by the resource \mathcal{R} when the adversary is present. Parallel composition of filtered resources naturally follows from parallel composition of resources and converters:

$$\mathcal{R}_\sharp\|\mathcal{S}_\diamond := (\mathcal{R}\|\mathcal{S})_{\sharp\|\diamond}.$$

The final object that is required by the AC framework to define the notion of construction and prove that it is composable, is a (pseudo-)metric defined on the space of resources that measures how close two resources are. In the following, we use a distinguisher based metric, i.e., the maximum advantage a distinguisher has in guessing whether it is interacting with resource \mathcal{R} or \mathcal{S} , which we write $d(\mathcal{R}, \mathcal{S})$. This is discussed further in Appendix A.3.

¹⁴More generally, a filter covers the inputs and outputs that are only accessible to a dishonest player, but provides access to those that should be available to an honest player. The dishonest player can remove the filter to have more control over the resource. We however do not need such a feature in this work, since we only consider resources with E -interfaces that are blank if the adversary is not active.

A.2 Security definition

We are now ready to define the security of a cryptographic protocol. We do so in the three player setting, for honest Alice and Bob, and dishonest Eve. Thus, in the following, all resources have three interfaces, denoted A , B and E , and a protocol is then given by a pair of converters (π_A, π_B) for the honest players. We refer to [MR11] for the general case, when arbitrary players can be dishonest. For convenience, we reproduce here Definition 2.1.

Definition A.1 (Cryptographic security [MR11]). Let $\pi_{AB} = (\pi_A, \pi_B)$ be a protocol and $\mathcal{R}_\# = (\mathcal{R}, \#)$ and $\mathcal{S}_\diamond = (\mathcal{S}, \diamond)$ denote two filtered resources. We say that π_{AB} constructs \mathcal{S}_\diamond from $\mathcal{R}_\#$ within ε , which we write $\mathcal{R}_\# \xrightarrow{\pi, \varepsilon} \mathcal{S}_\diamond$, if the two following conditions hold:

i) We have

$$d(\pi_{AB}\mathcal{R}_\#_E, \mathcal{S}_\diamond_E) \leq \varepsilon.$$

ii) There exists a converter¹⁵ σ_E — which we call simulator — such that

$$d(\pi_{AB}\mathcal{R}, \mathcal{S}\sigma_E) \leq \varepsilon.$$

If it is clear from the context what filtered resources $\mathcal{R}_\#$ and \mathcal{S}_\diamond are meant, we simply say that π_{AB} is ε -secure.

The first of these two conditions measures how close the constructed resource is to the ideal resource in the case where no malicious player is intervening, which is often called *correctness* in the literature. The second condition captures *security* in the presence of an adversary. For example, to prove that the MAC protocol π_{AB}^{auth} constructs an authentic channel \mathcal{A}_\diamond from a (noiseless) insecure channel \mathcal{C}_\square and a secret key \mathcal{K} within ε , we need to prove that the real system (with filters) $\pi_{AB}^{\text{auth}}(\mathcal{K}||\mathcal{C}_\square_E)$ cannot be distinguished from the ideal system \mathcal{A}_\diamond_E with advantage greater than ε , and we need to find a converter σ_E^{auth} such that the real system (without filters) $\pi_{AB}^{\text{auth}}(\mathcal{K}||\mathcal{C})$ cannot be distinguished from the ideal system $\mathcal{A}\sigma_E^{\text{auth}}$ with advantage greater than ε . For the MAC protocol, correctness is satisfied with error 0 and the simulator σ_E^{auth} drawn in Figure 4 satisfies the second requirement if the family of hash functions $\{h_k\}_k$ is ε -almost strongly 2-universal [Por14].

It follows from the composition theorem of the AC framework [MR11] that if two protocols π and π' are ε - and ε' -secure, the composition of the two is $(\varepsilon + \varepsilon')$ -secure. More precisely, let protocols π and π' construct \mathcal{S}_\diamond from $\mathcal{R}_\#$ and \mathcal{T}_\square from \mathcal{S}_\diamond within ε and ε' , respectively, i.e.,

$$\mathcal{R}_\# \xrightarrow{\pi, \varepsilon} \mathcal{S}_\diamond \quad \text{and} \quad \mathcal{S}_\diamond \xrightarrow{\pi', \varepsilon'} \mathcal{T}_\square.$$

¹⁵For a protocol with information-theoretic security to be composable with a protocol that has computational security, one additionally requires the simulator to be efficient.

It then follows from the triangle inequality of the metric that $\pi'\pi$ constructs \mathcal{T}_\square from \mathcal{R}_\sharp within $\varepsilon + \varepsilon'$,

$$\mathcal{R}_\sharp \xrightarrow{\pi'\pi, \varepsilon + \varepsilon'} \mathcal{T}_\square.$$

A similarly statement holds for parallel composition. Let π and π' construct \mathcal{S}_\diamond and \mathcal{S}'_\square from \mathcal{R}_\sharp and \mathcal{R}'_\flat within ε and ε' , respectively, i.e.,

$$\mathcal{R}_\sharp \xrightarrow{\pi, \varepsilon} \mathcal{S}_\diamond \quad \text{and} \quad \mathcal{R}'_\flat \xrightarrow{\pi', \varepsilon'} \mathcal{S}'_\square.$$

If these resources and protocols are composed in parallel, we find that $\pi\|\pi'$ constructs $\mathcal{S}_\diamond\|\mathcal{S}'_\square$ from $\mathcal{R}_\sharp\|\mathcal{R}'_\flat$ within $\varepsilon + \varepsilon'$,

$$\mathcal{R}_\sharp\|\mathcal{R}'_\flat \xrightarrow{\pi\|\pi', \varepsilon + \varepsilon'} \mathcal{S}_\diamond\|\mathcal{S}'_\square.$$

Proofs of these statements can be found in [MR11, Mau12].

A.3 Instantiation

As stated at the beginning of this section, the AC framework [MR11] specifies only the necessary axioms that resources and converters must satisfy so that one can prove that the resulting notion of construction is composable. Modeling concrete systems such as those in Figures 1, 2, 3 or 4, requires an instantiation of the framework with mathematical objects that capture interactive quantum information-processing systems. Such an instantiation has been given in [PMM⁺17] and proven to satisfy the axioms of AC, where the interactive systems are called *causal boxes*.

Unlike the model of systems used in quantum UC [Unr10], in which the output of a system is given by a quantum message and a classical string denoting a recipient, causal boxes allow messages to be sent to a superposition of different players; they even allow superpositions of different numbers of messages to be generated in superpositions of different orders [PMM⁺17]. This generality is however not needed in the current work, because all converters and resources involved in the construction of secure channels have a very simple structure. For this work, a system \mathcal{S} can be modeled as having internal memory \mathcal{H}_{mem} , and sets of in-ports In and out-ports Out with message spaces $\{\mathcal{H}_i^{\text{in}}\}_{i \in \text{In}}$ and $\{\mathcal{H}_o^{\text{out}}\}_{o \in \text{Out}}$, respectively. Furthermore, upon receiving a message at $i \in \text{In}$, \mathcal{S} always produces outputs at a fixed set of out-ports $\mathcal{O}_i \subseteq \text{Out}$ such that $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset$ if $i \neq j$. A system \mathcal{S} is thus entirely described by a set of completely positive, trace-preserving (CPTP) maps

$$\{\mathcal{E}_i : \mathcal{L}(\mathcal{H}_i^{\text{in}} \otimes \mathcal{H}_{\text{mem}}) \rightarrow \mathcal{L}(\mathcal{H}_{\text{mem}} \otimes \mathcal{H}_{\mathcal{O}_i}^{\text{out}})\}_{i \in \text{In}}, \quad (11)$$

where $\mathcal{H}_{\mathcal{O}_i}^{\text{out}} = \bigotimes_{o \in \mathcal{O}_i} \mathcal{H}_o^{\text{out}}$ and $\mathcal{L}(\mathcal{H})$ is the space of linear operators on \mathcal{H} . Upon receiving a message at some port $i \in \text{In}$, the system \mathcal{S} then applies the map \mathcal{E}_i and outputs the messages in the registers \mathcal{O}_i .

For a fixed order of messages received, this specific type of system has been called a *quantum strategy* [GW07, Gut12], *quantum comb* [CDP09] or *operator tensor* [Har11, Har12, Har15]—here we use the terminology from [CDP09], namely *comb*. A comb can be represented more compactly as a single CPTP map $\mathcal{E} : \mathcal{L}(\mathcal{H}_{\text{In}}) \rightarrow \mathcal{L}(\mathcal{H}_{\text{Out}})$, and using the composition rules for combs [CDP09, Har12] or causal boxes [PMM⁺17], two such systems can be composed to obtain a new system of the same type. The exact formula for composing systems is not needed in the current work; in all special cases where we connect two systems, the resulting system can easily be worked out by hand. We refer the interested reader to [CDP09, Har12, PMM⁺17] for the generic cases.

As mentioned at the end of Appendix A.1 we use a distinguisher metric to define the distance between two systems \mathcal{R} and \mathcal{S} . This means that another system \mathcal{D} , a *distinguisher*, is given access to either \mathcal{R} or \mathcal{S} , and has to guess to which of the two it is connected. Let $\mathcal{D}[\mathcal{R}]$ and $\mathcal{D}[\mathcal{S}]$ be the binary random variables corresponding to \mathcal{D} 's guess, then the distinguishing advantage between \mathcal{R} and \mathcal{S} for this specific distinguisher is given by

$$d_{\mathcal{D}}(\mathcal{R}, \mathcal{S}) := |\Pr[\mathcal{D}[\mathcal{R}] = 0] - \Pr[\mathcal{D}[\mathcal{S}] = 0]|,$$

and the distance is given by

$$d(\mathcal{R}, \mathcal{S}) := \sup_{\mathcal{D}} d_{\mathcal{D}}(\mathcal{R}, \mathcal{S}),$$

where the supremum is taken over all distinguishers allowed by quantum mechanics¹⁶—it has been proven in [PMM⁺17] that $d(\cdot, \cdot)$ is indeed a metric.

Due to the simple structure of the systems considered in this work, the distinguishing metric can be reduced to the following strategy—for the general case of the distinguishing metric between causal boxes we refer the reader to [PMM⁺17]. Let the distinguisher have internal memory \mathcal{H}_R . It chooses an in-port i_1 , and prepares a state ρ_{RA_1} . The A_1 part is sent to the system at the i_1 port. It then receives the output on ports \mathcal{O}_{i_1} , which it appends to its internal memory. It measures its internal memory to decide on the next in-port i_2 , applies a map $\mathcal{F}_1 : \mathcal{L}(\mathcal{H}_R) \rightarrow \mathcal{L}(\mathcal{H}_{RA_2})$, and inputs the A_2 part at the corresponding port. This process is repeated until there are no more unused ports, after which it measures its internal memory and produces one bit of output, its final guess. In the case where the two systems being compared only have one in-port, this metric reduces to the diamond norm. And if the systems have no in-port (or one trivial in-port of dimension 1), this results in the trace-distance between the states output by the two systems.

¹⁶In the computational setting one would restrict the set of distinguishers to those that are efficient. In our information-theoretic setting the distance is defined over unbounded distinguishers as well.

B Unitary designs

The concept of a unitary 2-design was originally proposed in [Dan05,DCEL09]. The following (equivalent) definition is taken from [GAE07].

Definition B.1 (Unitary 2-design [Dan05, DCEL09, GAE07]). A unitary 2-design is a finite set of unitaries $\{V_j\}_{j \in \mathcal{J}}$ on $\mathcal{H} = \mathbb{C}^d$ such that for all $\rho \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$

$$\frac{1}{|\mathcal{J}|} \sum_{j \in \mathcal{J}} (V_j \otimes V_j) \rho (V_j^\dagger \otimes V_j^\dagger) = \int_{V(d)} (V \otimes V) \rho (V^\dagger \otimes V^\dagger) dV, \quad (12)$$

where dV is the Haar measure. Equivalently, $\{V_j\}_{j \in \mathcal{J}}$ is a unitary 2-design if for any quantum channel $\Lambda : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ and state $\rho \in \mathcal{L}(\mathcal{H})$,

$$\frac{1}{|\mathcal{J}|} \sum_{j \in \mathcal{J}} V_j^\dagger \Lambda(V_j \rho V_j^\dagger) V_j = \int_{V(d)} V^\dagger \Lambda(V \rho V^\dagger) V dV. \quad (13)$$

One way to construct a unitary 2-design is by finding a set of unitaries $\{U_k\}_{k \in \mathcal{K}}$ that map all non-identity Paulis to each other with equal frequency, i.e., $\forall P_\ell, P_{\ell'}$ such that $P_\ell \neq I$ and $P_{\ell'} \neq I$,

$$\left| \left\{ k \in \mathcal{K} : U_k P_\ell U_k^\dagger = e^{i\theta_{\ell, \ell', k}} P_{\ell'} \right\} \right| = \frac{|\mathcal{K}|}{d^2 - 1}, \quad (14)$$

where $e^{i\theta_{\ell, \ell', k}}$ is some global phase and d is the dimension of the Hilbert space. A unitary 2-design is then obtained by composing these unitaries with Paulis, i.e., the set $\{P_\ell U_k\}_{\ell, k}$ is a unitary 2-design. This has been used in [DCEL09] to show that the Clifford group is a unitary 2-design, and is further discussed in [GAE07].

Chau [Cha05] finds a set $\{U_k\}_{k \in \mathcal{K}}$ satisfying Eq. (14). To understand his construction, we must view the indices x and z of a Pauli operator $P_{x,z}$ as elements of a Galois field $x, z \in \text{GF}(d)$. Let $M = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix} \in \text{SL}(2, d)$ be any element of the special linear group of 2×2 matrices over the finite field $\text{GF}(d)$, i.e., matrices with determinant 1. Chau then finds unitaries U_M such that

$$U_M P_{x,z} U_M^\dagger = e^{i\theta} P_{\alpha x + \beta z, \delta x + \gamma z},$$

for some global phase $e^{i\theta}$ that may depend on M, x, z , where the arithmetic in the indices is done in $\text{GF}(d)$. By considering the entire set $\text{SL}(2, d)$ one can verify that Eq. (14) is satisfied. Since $|\text{SL}(2, d)| = d^3 - d$, we need $\log(d^3 - d) \leq 3 \log d$ bits of key to chose the unitary. We now show that this set is a strong purity testing code.

Lemma B.2. Any set $\{U_k\}_{k \in \mathcal{K}}$ satisfying Eq. (14) with $d = 2^{m+n}$ is a strong purity testing code with $\varepsilon = 2^{-n}$.

Proof. An error P_ℓ is not detected if it is mapped to $P_{\ell'} = P_{x,z} \otimes P_{0,z'}$. There are $2^{2m+n} - 1$ such Paulis $P_{\ell'}$ which are not identity. Since the unitaries $\{U_k\}_{k \in \mathcal{K}}$ are constructed to map P_ℓ to all non-identity $P_{\ell'}$ with equal frequency, then $\frac{2^{2m+n}-1}{2^{2m+2n}-1} \leq 2^{-n}$ of them will not detect P_ℓ . \square

A unitary t -design is defined similarly to a unitary 2-design, except that it has a t -fold tensor product instead of a 2-fold tensor product in Eq. (12). Intuitively, if a unitary 2-design is a good quantum authentication scheme, then so should any (approximate) unitary t -design for any $t \geq 2$. One can however not directly apply our proof to unitary t -designs. This is because we use Eq. (14) as an intermediary step, to show that a unitary 2-design is a strong purity testing code. Eq. (14) is also satisfied by unitary 3-designs (given by the Clifford group [Zhu15, Web15]), but not necessarily for $t \geq 4$, where the unitaries are not elements of the Clifford group anymore. One can however show directly from Eq. (13) that a unitary 2-design is a strong purity testing code, and since all (approximate) unitary t -designs (approximately) satisfy Eq. (13), they can all be used to construct quantum authentication schemes that have the same key recycling properties as unitary 2-designs.¹⁷

Lemma B.3. *Any δ -approximate t -design with $t \geq 2$ is a strong purity testing code with error $\delta + 2^{-n}$.*

Proof. To prove that a set of unitaries is a strong purity testing code, one has to show that it can be used to detect all non-identity Pauli errors with high probability. Setting $\Lambda(\rho) = P_\ell \rho P_\ell$ for a non-identity Pauli P_ℓ in Eq. (13), one can show that the RHS becomes (see, e.g., [DCEL09]),

$$\frac{d}{d^2-1} \mathbb{1} - \frac{1}{d^2-1} \rho.$$

If $d = 2^{m+n}$, $\rho = \rho' \otimes |0\rangle\langle 0|$ where ρ' is the first m qubits of ρ and the last n qubits are used as syndrome, then the probability of obtaining 0 when performing a measuring in the computational basis on the syndrome is

$$\frac{d^2 2^{-n}}{d^2-1} - \frac{1}{d^2-1} \leq 2^{-n}.$$

Thus, if the distance between the LHS and RHS of Eq. (13) is δ , then the probability of not detecting a Pauli error is less than $\delta + 2^{-n}$. \square

¹⁷The reason we used Eq. (14) and not Eq. (13) to prove the security of the unitary 2-design scheme, is that Chau's construction [Cha05] is a subset of a unitary 2-design that satisfies Eq. (14) but not Eq. (13). We still have to compose it with a random Pauli to obtain the unitary 2-design.

C Technical lemma

The following lemma is used in the proof of Theorem 3.5 to bound the trace distance between two states with the 2-norm.

Lemma C.1. *Let $|\psi\rangle$ and $|\varphi\rangle$ be two subnormalized states. Then*

$$\frac{1}{2} \|\psi\rangle\langle\psi| - |\varphi\rangle\langle\varphi|\|_{\text{tr}} \leq \| |\psi\rangle - |\varphi\rangle \|, \quad (15)$$

where $\| |a\rangle \| = \sqrt{\langle a|a\rangle}$ is the vector 2-norm.

Proof. It was shown in [Ren05, Lemma A.2.3] that if $\langle\psi|\varphi\rangle$ is real, then

$$\|\psi\rangle\langle\psi| - |\varphi\rangle\langle\varphi|\|_{\text{tr}} = \| |\psi\rangle - |\varphi\rangle \| \cdot \| |\psi\rangle + |\varphi\rangle \|.$$

For complex $\langle\psi|\varphi\rangle$ we define $|\varphi'\rangle := \frac{\langle\varphi|\psi\rangle}{|\langle\varphi|\psi\rangle|} |\varphi\rangle$. It then follows from [Ren05, Lemma A.2.3] that

$$\|\psi\rangle\langle\psi| - |\varphi\rangle\langle\varphi|\|_{\text{tr}} = \|\psi\rangle\langle\psi| - |\varphi'\rangle\langle\varphi'|\|_{\text{tr}} = \| |\psi\rangle - |\varphi'\rangle \| \cdot \| |\psi\rangle + |\varphi'\rangle \|.$$

To prove this lemma it remains to show that

$$\| |\psi\rangle - |\varphi'\rangle \| \cdot \| |\psi\rangle + |\varphi'\rangle \| \leq \| |\psi\rangle - |\varphi\rangle \| \cdot \| |\psi\rangle + |\varphi\rangle \|,$$

since combining this with $\| |\psi\rangle + |\varphi\rangle \| \leq 2$ we get Eq. (15).

Writing out the norms with the scalar product we obtain

$$\begin{aligned} & \| |\psi\rangle - |\varphi\rangle \|^2 \cdot \| |\psi\rangle + |\varphi\rangle \|^2 \\ &= (\langle\psi|\psi\rangle + \langle\varphi|\varphi\rangle - \langle\psi|\varphi\rangle - \langle\varphi|\psi\rangle)(\langle\psi|\psi\rangle + \langle\varphi|\varphi\rangle + \langle\psi|\varphi\rangle + \langle\varphi|\psi\rangle) \\ &= (\langle\psi|\psi\rangle + \langle\varphi|\varphi\rangle)^2 - (\langle\psi|\varphi\rangle + \langle\varphi|\psi\rangle)^2. \end{aligned}$$

Thus, using $|\langle\psi|\varphi\rangle| = \langle\psi|\varphi'\rangle$ we get

$$\begin{aligned} & \| |\psi\rangle - |\varphi\rangle \|^2 \cdot \| |\psi\rangle + |\varphi\rangle \|^2 - \| |\psi\rangle - |\varphi'\rangle \|^2 \cdot \| |\psi\rangle + |\varphi'\rangle \|^2 \\ &= (\langle\psi|\varphi'\rangle + \langle\varphi'|\psi\rangle)^2 - (\langle\psi|\varphi\rangle + \langle\varphi|\psi\rangle)^2 \\ &= \langle\psi|\varphi'\rangle^2 + \langle\varphi'|\psi\rangle^2 - \langle\psi|\varphi\rangle^2 - \langle\varphi|\psi\rangle^2 \\ &= 2|\langle\psi|\varphi\rangle|^2 - \langle\psi|\varphi\rangle^2 - \langle\varphi|\psi\rangle^2 \geq 0. \quad \square \end{aligned}$$

D Authentication without key recycling

The proof for authentication with key recycling provided in Section 3 is automatically a proof for authentication without key recycling, since the players do not have to reuse the key if they do not want to. But the parameters are not optimal in this case, because recycling the key causes the error to

change from ε to $\sqrt{\varepsilon}$. What is more, the proof given is only valid for strong purity testing codes, since these are essential to be able to recycle all the key. But if the users are not interested in recycling key, it is sufficient to use weak purity testing codes. For completeness, we provide a proof here that the entire family of quantum authentication protocols from [BCG⁺02] is secure, i.e., they construct the secure channel \mathcal{S}_b^m , where \mathcal{S}^m is the resource considered before — which is drawn in Figure 5a — and b_E is the obvious filter which lets the message through.

Like in the case of key recycling, we can consider an encrypt-then-encode and an encode-then-encrypt protocol, which are identical to Figures 7 and 8, except that the players use a weak purity testing code, do not recycle any key and do not use an backwards authentic channel. Let the weak purity testing code $\{U_k\}_{k \in \mathcal{K}}$ have size $\log |\mathcal{K}| = \nu$ and error ε , and encode an m qubit message in an $m+n$ qubit cipher. As previously, let $\pi_{AB}^{q\text{-auth}}$ denote the encode-then-encrypt converters and $\bar{\pi}_{AB}^{q\text{-auth}}$ denote the encrypt-then-encode version.

Lemma D.1. *The converters $\pi_{AB}^{q\text{-auth}}$ for the encode-then-encrypt protocol without key recycling construct the secure channel \mathcal{S}_b^m , given an insecure quantum channel \mathcal{C}_\square and a secret key $\mathcal{K}^{\nu+2m+2n}$, i.e.,*

$$\mathcal{C}_\square \parallel \mathcal{K}^{\nu+2m+2n} \xrightarrow{\pi_{AB}^{q\text{-auth}}, \varepsilon^{q\text{-auth}}} \mathcal{S}_b^m,$$

with $\varepsilon^{q\text{-auth}} = \max\{\varepsilon, 2^{-n}\}$.

Proof. Correctness of the protocol is trivial, so we only need to consider security. Just as in the proof of Theorem 3.5, the distinguisher has the choice between providing the inputs in two orders, first a message for Alice, then a (possibly modified) cipher at Eve’s interface, or first the cipher then the message. In the latter case, the simulator always tells the ideal channel to output an error; then when the ideal channel notifies the simulator that a message has been input at Alice’s interface, the simulator outputs a fully mixed state. This is exactly the same behavior as the simulator used in the proof of Theorem 3.5, except that no key is output at any point. We proved back then that the distinguisher has an advantage of at most 2^{-n} at distinguishing the real and ideal systems. The proof does not depend on the purity testing code, it follows directly from the random Paulis P_ℓ used for decrypting and encrypting. So in particular, it is also valid when the unitaries $\{U_k\}_{k \in \mathcal{K}}$ form a weak purity testing code. Hence in the case of weak purity testing codes without key recycling, the distinguishing advantage for this order of messages is also bounded by 2^{-n} .

The first case — when the distinguisher first provides a message at Alice’s interface, then modifies the cipher — requires a different simulator and different proof than that of Theorem 3.5 to go through with weak purity

testing codes. The simulator we use also prepares $n + m$ EPR pairs $|\Phi\rangle^{CR}$, but this time it picks a key k uniformly at random and applies U_k to the halves in the C system, which it outputs at Eve's interface. Upon receiving the (possibly modified) system C' back from the distinguisher, it applies the inverse U_k^\dagger , then measures in the Bell basis. Let the outcome of the measurement be j' . If the Pauli $P_{j'}$ acts trivially on the M' subsystem of $C' = S'M'$ and only flips phases on S' — i.e., $P_{j'} = P_{0,z} \otimes P_{0,0}$ — then the simulator tells the ideal resource to output the message, otherwise it should output an error.

Putting this together with the distinguisher that first prepares a state $|\psi\rangle^{ME}$, inputs the M part at Alice's interface, receives some cipher in the system C , applies a unitary $U^{CE} = \sum_j P_j^C \otimes E_j^E$ to the CE system, and inputs the modified C system back on the channel, we get the following final state in the ideal case:

$$\begin{aligned} \zeta = & |\text{acc}\rangle\langle\text{acc}| \otimes \frac{1}{2^\nu} \sum_k \sum_{j \in \mathcal{Q}_k} \left[(I^M \otimes E_j^E) |\psi\rangle\langle\psi|^{ME} (I^M \otimes (E_j^E)^\dagger) \right] \\ & + |\text{rej}\rangle\langle\text{rej}| \otimes \frac{1}{2^\nu} \sum_k \sum_{j \notin \mathcal{Q}_k} E_j^E \rho^E (E_j^E)^\dagger, \quad (16) \end{aligned}$$

where \mathcal{Q}_k is the set of Paulis that are not detected by the code and act trivially on M , i.e., the j for which $U_k^\dagger P_j^{SM} U_k = e^{i\theta_{k,j}} P_{0,z}^S \otimes P_{0,0}^M$.

In the real system, for the secret key (k, ℓ) , the state before Bob's measurement of the syndrome is given by

$$\begin{aligned} |\varphi_{k,\ell}\rangle^{SME} &= \sum_j \left((U_k^{SM})^\dagger P_\ell^{SM} P_j^{SM} P_\ell^{SM} U_k^{SM} \otimes E_j^E \right) |0\rangle^S |\psi\rangle^{ME} \\ &= \sum_j (-1)^{(j,\ell)_{\text{Sp}}} \left((U_k^{SM})^\dagger P_j^{SM} U_k^{SM} \otimes E_j^E \right) |0\rangle^S |\psi\rangle^{ME} \\ &= \sum_j (-1)^{(j,\ell)_{\text{Sp}}} e^{i\theta_{k,j}} \left(P_{k(j)}^{SM} \otimes E_j^E \right) |0\rangle^S |\psi\rangle^{ME}, \end{aligned}$$

where we denote by $k(j)$ the index of the Pauli operator such that $U_k^\dagger P_j U_k = e^{i\theta_{k,j}} P_{k(j)}$. Summing over k and ℓ , the state before Bob's measurement is given by

$$\begin{aligned} & \frac{1}{2^{\nu+2m+2n}} \sum_{k,\ell} |\varphi_{k,\ell}\rangle\langle\varphi_{k,\ell}|^{SME} \\ &= \frac{1}{2^{\nu+2m+2n}} \sum_{j_1, j_2, k, \ell} (-1)^{(j_1 \oplus j_2, \ell)_{\text{Sp}}} e^{i\theta_{k,j_1} - i\theta_{k,j_2}} \left(P_{k(j_1)}^{SM} \otimes E_{j_1}^E \right) \\ & \quad \left(|0\rangle\langle 0|^S \otimes |\psi\rangle\langle\psi|^{ME} \right) \left(P_{k(j_2)}^{SM} \otimes (E_{j_2}^E)^\dagger \right) \\ &= \frac{1}{2^\nu} \sum_{j,k} \left(P_{k(j)}^{SM} \otimes E_j^E \right) \left(|0\rangle\langle 0|^S \otimes |\psi\rangle\langle\psi|^{ME} \right) \left(P_{k(j)}^{SM} \otimes (E_j^E)^\dagger \right), \end{aligned}$$

where we used Eq. (2). The measurement of S yields $|\text{acc}\rangle$ if $P_{k(j)}^{SM} = P_{0,z'}^S \otimes P_{x,z}^M$, i.e., $j \in \mathcal{P}_k$, where \mathcal{P}_k denotes the set of Paulis that are not detected by the code k . Hence, the final state held by the distinguisher after Bob's measurement is given by

$$\begin{aligned} \xi = |\text{acc}\rangle\langle\text{acc}| \otimes \frac{1}{2^\nu} \sum_k \sum_{j \in \mathcal{P}_k} \left[\left(P_{k(j)}^M \otimes E_j^E \right) |\psi\rangle\langle\psi|^{ME} \left(P_{k(j)}^M \otimes (E_j^E)^\dagger \right) \right] \\ + |\text{rej}\rangle\langle\text{rej}| \otimes \frac{1}{2^\nu} \sum_k \sum_{j \notin \mathcal{P}_k} E_j^E \rho^E (E_j^E)^\dagger. \end{aligned} \quad (17)$$

The distinguishability between the real and ideal systems is given by the trace distance between Eqs. (16) and (17), namely

$$\begin{aligned} \frac{1}{2} \|\xi - \zeta\|_{\text{tr}} &= \frac{1}{2} \left\| \frac{1}{2^\nu} \sum_k \sum_{j \in \mathcal{P}_k \setminus \mathcal{Q}_k} \left[\left(P_{k(j)}^M \otimes E_j^E \right) |\psi\rangle\langle\psi|^{ME} \left(P_{k(j)}^M \otimes (E_j^E)^\dagger \right) \right] \right\|_{\text{tr}} \\ &\quad + \frac{1}{2} \left\| \frac{1}{2^\nu} \sum_k \sum_{j \in \mathcal{P}_k \setminus \mathcal{Q}_k} E_j^E \rho^E (E_j^E)^\dagger \right\|_{\text{tr}} \\ &\leq \frac{1}{2^\nu} \sum_k \sum_{j \in \mathcal{P}_k \setminus \mathcal{Q}_k} \text{tr} \left(E_j^E \rho^E (E_j^E)^\dagger \right) \\ &= \sum_j \text{tr} \left(E_j^E \rho^E (E_j^E)^\dagger \right) \frac{|\{k : j \in \mathcal{P}_k \setminus \mathcal{Q}_k\}|}{2^\nu} \leq \varepsilon. \quad \square \end{aligned}$$

It follows from this and Lemma 3.3 that the encrypt-then-encode protocol is also secure.

Corollary D.2. *The converters $\bar{\pi}_{AB}^{q\text{-auth}}$ for the encrypt-then-encode protocol without key recycling construct the secure channel \mathcal{S}_b^m , given an insecure quantum channel \mathcal{C}_\square and a secret key $\mathcal{K}^{\nu+2m+n}$, i.e.,*

$$\mathcal{C}_\square \parallel \mathcal{K}^{\nu+2m+n} \xrightarrow{\bar{\pi}_{AB}^{q\text{-auth}}, \varepsilon^{q\text{-auth}}} \mathcal{S}_b^m,$$

with $\varepsilon^{q\text{-auth}} = \max\{\varepsilon, 2^{-n}\}$.

References

- [ABE10] Dorit Aharonov, Michael Ben-Or, and Elad Eban. Interactive proofs for quantum computations. In *Proceedings of Innovations in Computer Science, ICS 2010*, pages 453–469. Tsinghua University Press, 2010. [arXiv:0810.5375].

- [AM16] Gorjan Alagic and Christian Majenz. Quantum non-malleability and authentication. Eprint, 2016. [arXiv:1610.04214].
- [BB84] Charles H. Bennett and Gilles Brassard. Quantum cryptography: Public key distribution and coin tossing. In *Proceedings of IEEE International Conference on Computers, Systems, and Signal Processing*, pages 175–179, 1984.
- [BBB82] Charles H. Bennett, Gilles Brassard, and Seth Breidbart. Quantum cryptography II: How to re-use a one-time pad safely even if $P=NP$. Original unpublished manuscript uploaded to arXiv in 2014, 1982. [arXiv:1407.0451].
- [BBCM95] Charles H. Bennett, Gilles Brassard, Claude Crépeau, and Ueli Maurer. Generalized privacy amplification. *IEEE Transaction on Information Theory*, 41(6):1915–1923, November 1995. [doi:10.1109/18.476316].
- [BCG⁺02] Howard Barnum, Claude Crépeau, Daniel Gottesman, Adam Smith, and Alain Tapp. Authentication of quantum messages. In *Proceedings of the 43rd Symposium on Foundations of Computer Science, FOCS '02*, pages 449–458. IEEE, 2002. [doi:10.1109/SFCS.2002.1181969, arXiv:quant-ph/0205128].
- [BCG⁺06] Michael Ben-Or, Claude Crépeau, Daniel Gottesman, Avinatan Hassidim, and Adam Smith. Secure multiparty quantum computation with (only) a strict honest majority. In *Proceedings of the 47th Symposium on Foundations of Computer Science, FOCS '06*, pages 249–260, 2006. [doi:10.1109/FOCS.2006.68, arXiv:0801.1544].
- [BGS13] Anne Broadbent, Gus Gutoski, and Douglas Stebila. Quantum one-time programs. In *Advances in Cryptology – CRYPTO 2013*, volume 8043 of *Lecture Notes in Computer Science*, pages 344–360. Springer, 2013. [doi:10.1007/978-3-642-40084-1_20, arXiv:1211.1080].
- [BW16] Anne Broadbent and Evelyn Wainewright. Efficient simulation for quantum message authentication. In *Proceedings of the 9th International Conference on Information Theoretic Security, ICITS 2016*, pages 72–91. Springer, 2016. [doi:10.1007/978-3-319-49175-2_4, arXiv:1607.03075].
- [Can01] Ran Canetti. Universally composable security: A new paradigm for cryptographic protocols. In *Proceedings of the 42nd Symposium on Foundations of Computer Science, FOCS '01*, pages 136–145. IEEE, 2001. [doi:10.1109/SFCS.2001.959888].

- [Can13] Ran Canetti. Universally composable security: A new paradigm for cryptographic protocols. Cryptology ePrint Archive, Report 2000/067, 2013. Updated version of [Can01]. [IACR e-print: 2000/067].
- [CDP09] Giulio Chiribella, Giacomo Mauro D’Ariano, and Paolo Perinotti. Theoretical framework for quantum networks. *Physical Review A*, 80:022339, August 2009. [doi:10.1103/PhysRevA.80.022339, arXiv:0904.4483].
- [Cha05] Hoi Fung Chau. Unconditionally secure key distribution in higher dimensions by depolarization. *IEEE Transactions on Information Theory*, 51(4):1451–1468, April 2005. [doi:10.1109/TIT.2005.844076, arXiv:quant-ph/0405016].
- [Dan05] Christoph Dankert. Efficient simulation of random quantum states and operators. Master’s thesis, University of Waterloo, 2005. [arXiv:quant-ph/0512217].
- [DCEL09] Christoph Dankert, Richard Cleve, Joseph Emerson, and Etera Livine. Exact and approximate unitary 2-designs and their application to fidelity estimation. *Physical Review A*, 80:012304, July 2009. [doi:10.1103/PhysRevA.80.012304, arXiv:quant-ph/0606161].
- [DNS12] Frédéric Dupuis, Jesper Buus Nielsen, and Louis Salvail. Actively secure two-party evaluation of any quantum operation. In *Advances in Cryptology – CRYPTO 2012*, volume 7417 of *Lecture Notes in Computer Science*, pages 794–811. Springer, 2012. [doi:10.1007/978-3-642-32009-5_46, IACR e-print: 2012/304].
- [DPS05] Ivan Damgård, Thomas Brochmann Pedersen, and Louis Salvail. A quantum cipher with near optimal key-recycling. In *Advances in Cryptology – CRYPTO 2005*, volume 3621 of *Lecture Notes in Computer Science*, pages 494–510. Springer, 2005. [doi:10.1007/11535218_30].
- [DPS14] Ivan Damgård, Thomas Brochmann Pedersen, and Louis Salvail. How to re-use a one-time pad safely and almost optimally even if $P = NP$. *Natural Computing*, 13(4):469–486, December 2014. [doi:10.1007/s11047-014-9454-5].
- [FS17] Serge Fehr and Louis Salvail. Quantum authentication and encryption with key recycling. To appear in the proceedings of EUROCRYPT, 2017. [arXiv:1610.05614].

- [GAE07] David Gross, Koenraad Audenaert, and Jens Eisert. Evenly distributed unitaries: on the structure of unitary designs. *Journal of Mathematical Physics*, 48:052104, 2007. [doi:10.1063/1.2716992, arXiv:quant-ph/0611002].
- [Got03] Daniel Gottesman. Uncloneable encryption. *Quantum Information and Computation*, 3:581, 2003. [arXiv:quant-ph/0210062].
- [Gut12] Gus Gutoski. On a measure of distance for quantum strategies. *Journal of Mathematical Physics*, 53(3):032202, 2012. [doi:10.1063/1.3693621, arXiv:1008.4636].
- [GW07] Gus Gutoski and John Watrous. Toward a general theory of quantum games. In *Proceedings of the 39th Symposium on Theory of Computing, STOC '07*, pages 565–574. ACM, 2007. [doi:10.1145/1250790.1250873, arXiv:quant-ph/0611234].
- [GYZ16] Sumegha Garg, Henry Yuen, and Mark Zhandry. New security notions and feasibility results for authentication of quantum data. Eprint, 2016. [arXiv:1607.07759].
- [Har11] Lucien Hardy. Reformulating and reconstructing quantum theory. Eprint, 2011. [arXiv:1104.2066].
- [Har12] Lucien Hardy. The operator tensor formulation of quantum theory. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 370(1971):3385–3417, 2012. [doi:10.1098/rsta.2011.0326, arXiv:1201.4390].
- [Har15] Lucien Hardy. Quantum theory with bold operator tensors. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 373(2047), 2015. [doi:10.1098/rsta.2014.0239].
- [HLM11] Patrick Hayden, Debbie Leung, and Dominic Mayers. The universal composable security of quantum message authentication with key recycling. Eprint, presented at QCrypt 2011, 2011. [arXiv:1610.09434].
- [Mau12] Ueli Maurer. Constructive cryptography—a new paradigm for security definitions and proofs. In *Proceedings of Theory of Security and Applications, TOSCA 2011*, volume 6993 of *Lecture Notes in Computer Science*, pages 33–56. Springer, 2012. [doi:10.1007/978-3-642-27375-9_3].

- [MR11] Ueli Maurer and Renato Renner. Abstract cryptography. In *Proceedings of Innovations in Computer Science, ICS 2011*, pages 1–21. Tsinghua University Press, 2011.
- [MR16] Ueli Maurer and Renato Renner. From indifferentiability to constructive cryptography (and back). In *Theory of Cryptography, Proceedings of TCC 2016-B, Part I*, volume 9985 of *Lecture Notes in Computer Science*, pages 3–24. Springer, 2016. [doi:10.1007/978-3-662-53641-4_1, IACR e-print: 2016/903].
- [OH05] Jonathan Oppenheim and Michał Horodecki. How to reuse a one-time pad and other notes on authentication, encryption, and protection of quantum information. *Physical Review A*, 72:042309, October 2005. [doi:10.1103/PhysRevA.72.042309, arXiv:quant-ph/0306161].
- [PMM⁺17] Christopher Portmann, Christian Matt, Ueli Maurer, Renato Renner, and Björn Tackmann. Causal boxes: Quantum information-processing systems closed under composition. To appear in *IEEE Trans. Inf. Theory*, 2017. [arXiv:1512.02240].
- [Por14] Christopher Portmann. Key recycling in authentication. *IEEE Transactions on Information Theory*, 60(7):4383–4396, July 2014. [doi:10.1109/TIT.2014.2317312, arXiv:1202.1229].
- [PR14] Christopher Portmann and Renato Renner. Cryptographic security of quantum key distribution. Eprint, 2014. [arXiv:1409.3525].
- [Ren05] Renato Renner. *Security of Quantum Key Distribution*. PhD thesis, Swiss Federal Institute of Technology (ETH) Zurich, September 2005. [arXiv:quant-ph/0512258].
- [RK05] Renato Renner and Robert König. Universally composable privacy amplification against quantum adversaries. In Joe Kilian, editor, *Theory of Cryptography, Proceedings of TCC 2005*, volume 3378 of *Lecture Notes in Computer Science*, pages 407–425. Springer, 2005. [doi:10.1007/978-3-540-30576-7_22, arXiv:quant-ph/0403133].
- [SBPC⁺09] Valerio Scarani, Helle Bechmann-Pasquinucci, Nicolas J. Cerf, Miloslav Dušek, Norbert Lütkenhaus, and Momtchil Peev. The security of practical quantum key distribution. *Reviews of Modern Physics*, 81:1301–1350, September 2009. [doi:10.1103/RevModPhys.81.1301, arXiv:0802.4155].
- [Sha49] Claude Shannon. Communication theory of secrecy systems. *Bell System Technical Journal*, 28(4):656–715, 1949.

- [Sim85] Gustavus J. Simmons. Authentication theory/coding theory. In *Advances in Cryptology – CRYPTO '84*, pages 411–431. Springer, 1985. [doi:10.1007/3-540-39568-7_32].
- [Sim88] Gustavus J. Simmons. A survey of information authentication. *Proceedings of the IEEE*, 76(5):603–620, May 1988. [doi:10.1109/5.4445].
- [SP00] Peter W. Shor and John Preskill. Simple proof of security of the BB84 quantum key distribution protocol. *Physical Review Letters*, 85:441–444, 2000. [doi:10.1103/PhysRevLett.85.441, arXiv:quant-ph/0003004].
- [Sti90] Douglas R. Stinson. The combinatorics of authentication and secrecy codes. *Journal of Cryptology*, 2(1):23–49, 1990. [doi:10.1007/BF02252868].
- [Sti94] Douglas R. Stinson. Universal hashing and authentication codes. *Designs, Codes and Cryptography*, 4(3):369–380, 1994. A preliminary version appeared at CRYPTO '91. [doi:10.1007/BF01388651].
- [Unr10] Dominique Unruh. Universally composable quantum multiparty computation. In *Advances in Cryptology – EUROCRYPT 2010*, volume 6110 of *Lecture Notes in Computer Science*, pages 486–505. Springer, 2010. [doi:10.1007/978-3-642-13190-5_25, arXiv:0910.2912].
- [WC81] Mark N. Wegman and Larry Carter. New hash functions and their use in authentication and set equality. *Journal of Computer and System Sciences*, 22(3):265–279, 1981.
- [Web15] Zak Webb. The Clifford group forms a unitary 3-design. *Quantum Information & Computation*, 16(15&16):1379–1400, 2015. [arXiv:1510.02769].
- [Zhu15] Huangjun Zhu. Multiqubit Clifford groups are unitary 3-designs. Eprint, 2015. [arXiv:1510.02619].