

# Improvements on the Individual Logarithm Step in Extended Tower Number Field Sieve

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**Abstract.** The hardness of discrete logarithm problem over finite fields is the foundation of many cryptographic protocols. When the characteristic of the finite field is medium or large, the state-of-art algorithms for solving the corresponding problem are the number field sieve and its variants. There are mainly three steps in such algorithms: polynomial selection, factor base logarithms computation, and individual logarithm computation. Note that the former two steps can be precomputed for fixed finite field, and the database containing factor base logarithms can be used by the last step for many times. In certain application circumstances, such as Logjam attack, speeding up the individual logarithm step is vital.

In this paper, we devise a method to improve the individual logarithm step by exploring subfield structures. Our method is based on the extended tower number field sieve algorithm, and achieves more significant improvement when the extension degree has a large proper factor. We also perform some experiments to illustrate our algorithm and confirm the result.

**Keywords:** Discrete logarithm problem, extended tower number field sieve, individual logarithm, smoothing phase.

## 1 Introduction

### 1.1 The discrete logarithm problem

The discrete logarithm problem (DLP) in finite fields has played an important role in public key cryptography, firstly used to construct Diffie-Hellman key exchange protocol [10], later used as an important ingredient to build torus-based [25] and pairing-based cryptographic schemes [17,9]. The Diffie-Hellman key exchange protocol makes use of a prime field  $\mathbb{F}_p$ , while the torus-based and pairing-based cryptosystem make use of finite fields  $\mathbb{F}_{p^n}$  and  $\mathbb{F}_{q^n}$  respectively.

It has long been realized that the characteristic of the underlying finite field affects the hardness of the corresponding discrete logarithm problem. Denote

$$L_Q(\alpha, c) = \exp((c + o(1))(\log Q)^\alpha (\log \log Q)^{1-\alpha}),$$

where  $Q$  is the cardinality of the field  $\mathbb{F}_{p^n}$ . For simplicity, we omit  $Q$  and  $c$  when there is no confusion.

Let  $p = L_Q(\alpha_p, c_p)$ . The state-of-art algorithms for solving the corresponding problem are number field sieve (NFS) [11,28,19,22], function field sieve (FFS) [1,2,18,5,12] and quasi-polynomial time algorithm (QPA) [5,12,13] when the characteristics of the finite fields are medium to large

( $\alpha_p > 1/3$ ), medium-small ( $0 < \alpha_p < 1/3$ ) and small ( $\alpha_p = 0$ ), respectively. While FFS and QPA enjoy heuristic  $L(\alpha_p)$  and quasi-polynomial time complexity, NFS still runs in  $L(1/3)$  time.

In 2016, Kim and Barbulescu [22] presented the extended tower number field sieve (exTNFS) and achieved a new complexity in the medium prime case. When the extension degree  $n$  can factor into two coprime integers and some other conditions are satisfied, the best complexity of exTNFS in the medium prime case is  $L_Q(1/3, \sqrt[3]{\frac{48}{9}})$ . Later, Jeong and Kim [16] removed the coprime restriction. Sarkar and Singh [26] combined the SS polynomial selection methods [27] and exTNFS to further loosen the conditions.

Briefly, NFS consist of three steps in general: polynomial selection, factor base logarithm computation, and individual logarithm computation. In polynomial selection step, two suitable polynomials are selected as a setup. The property of the selected polynomials affected the efficiency of the latter two steps. In recent years, some efficient polynomial selection methods have been proposed [23,4,27]. In factor base logarithm step, the logarithms among the factor base are computed and stored in a database. For several discrete logarithms computation, such as batch-DLP and delayed-target DLP, the polynomial selection step and factor base logarithm step can be computed only once. Then the efficiency of computing an individual logarithm will be more important. For instance, the Logjam attack [3] against the real-world Diffie-Hellman key exchange protocol highlights the necessity of faster individual DL method.

## 1.2 Related work

The individual logarithm step includes three phases: smoothing, descent and combination of logarithms. In smoothing phase, one randomizes the target element until it splits into several smooth elements. The complexity of this phase depends on the norm of the preimage of the target element. In Asiacrypt 2015, Guillevic [14] took advantage of the subfield of degree 1 or 2 to construct a preimage with small norm. It reduced the complexity of smoothing phase significantly when  $n$  is small.

## 1.3 Our contribution

In this paper, we aim at speeding up the smoothing phase further. Our method is a combination of exTNFS and generalization of Guillevic's idea. The main technique is to make full usage of the subfield structure.

Let the target finite field be  $\mathbb{F}_{p^n}$  with cardinality  $Q$ . Assume  $m$  is the largest factor of  $n$  and  $\ell$  is the largest prime factor of  $\#\mathbb{F}_{p^n}^\times$ . Let  $s$  be a random element in  $\mathbb{F}_{p^n}$  other than in a proper subfield of  $\mathbb{F}_{p^n}$  (otherwise, the DLP w.r.t  $s$  will be much easier). Let  $K_f$  be the corresponding number field where the smoothing phase will be done.

**Theorem 1.** *In the large characteristic case, i.e.  $\alpha_p > 2/3$ , there exists an element  $s'$  in  $K_f$  with norm bounded by  $O(Q^{1-\frac{m}{n}})$  such that  $\log s' \equiv \log s \pmod{\ell}$ .*

**Theorem 2.** *In the medium characteristic or boundary case, i.e.  $1/3 < \alpha_p \leq 2/3$ , there also exists an element  $s'$  in  $K_f$  with norm bounded by  $O(Q^{1-\frac{m}{n}})$  such that  $\log s' \equiv \log s \pmod{\ell}$ , if one of the following conditions holds:*

- (1)  $p^k \neq L_Q(2/3)$  for any factor  $k$  of  $n$  with  $k \neq m$ ;
- (2)  $K_f$  satisfies the conditions in Lemma 3.

For the remaining minor case, there exists an element  $s'$  with norm bounded by

$$\begin{cases} O(Q^{1-\frac{2k}{n}}), & \text{if } \mathbb{F}_{p^n} \text{ satisfies the conditions in Lemma 2} \\ O(Q^{1-\frac{k}{n}}), & \text{otherwise.} \end{cases}$$

When  $n$  is composite, the previous best result is  $1 - 2/n$ . Here, our result is  $1 - m/n$ , where  $m$  is the largest factor of  $n$ .

*Remark 1.* Very recently, Guillevic [15] has independently improved the individual discrete logarithm step by exploring the subfield structure. Our result is essentially the same as Guillevic's result when the characteristic is medium or large. However, there are some differences between the two methods:

- Since exTNFS performs better than traditional NFS when the extension degree is composite, we base our work on exTNFS. Guillevic's approach works also in the traditional NFS method.
- Although the basic idea of our work and Guillevic's work is to take usage of the largest subfield, the details differ. Particularly, in this method of our work, we construct the subfield explicitly according to the exTNFS method; while in Guillevic's method, a different approach is taken to construct a polynomial basis of such subfield.

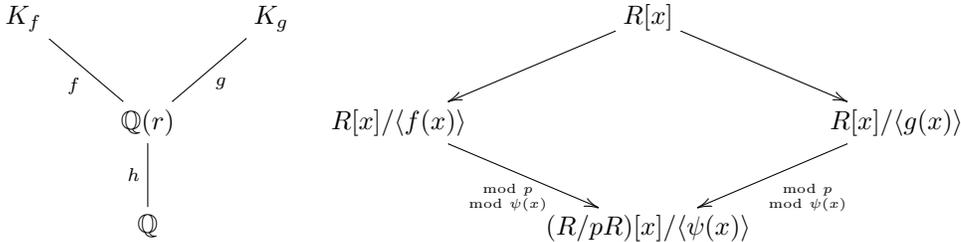
The rest of the paper is organized as follows. In Section 2, we introduce the extended tower number field sieve and Guillevic's work at Asiacrypt 15. In Section 3, we describe our improvement by taking advantage of the exTNFS and prove Theorem 1 and Theorem 2. In Section 4, we perform some numerical experiments to illustrate our two methods. In Section 5, we conclude the paper.

## 2 Preliminaries

### 2.1 The extended Tower Number Field Sieve

The tower number field sieve was first introduced by [29], and then rehabilitated by [6], and extended by [22]. Here, we briefly recall the exTNFS algorithm.

**Setup.** Let the target field be  $\mathbb{F}_Q$ , where  $Q = p^n$  and  $p = L_Q(\alpha_p, c_p)$  with  $\alpha_p > 1/3$ . Assume  $n = n_1 n_2$ . In (extended) TNFS, we consider two field extensions over a number field  $\mathbb{Q}(r)$ , which is defined by a monic irreducible polynomial  $h$  of degree  $n_1$ .  $K_f$  and  $K_g$  are two number fields above  $\mathbb{Q}(r)$  defined by irreducible polynomials  $f$  and  $g$  over a ring  $R$ , where  $R = \mathbb{Z}[r]/h(r)$ . Moreover, we need that  $h$  remains irreducible modulo  $p$ . Then  $p$  is inertia in  $R$  and  $R/pR \cong \mathbb{F}_{p^{n_1}}$ . Then we have the following commutative diagrams



where  $\psi(x)$  is the common factor of  $f$  and  $g$  over  $R/pR$ . To obtain the target finite field, the degree of  $\psi(x)$  should be  $n_2$ . Hence,  $(R/pR)[x]/\langle\psi(x)\rangle$  is isomorphic to  $\mathbb{F}_{p^n}$ .

**Polynomial selection.** The complexity of recent NFS algorithm and its variants highly rely on the size of the coefficients of the defining polynomials. To reduce the complexity, we have to select  $f, g$  and  $h$  with the coefficients as small as possible. To this end, we select  $h$  with coefficients of constant bound. Heuristically, we can find a suitable  $h$  with  $\|h\|_\infty = 1$ .

To select suitable  $f$  and  $g$ , which is similar to the classical case, there are several effective methods [19,23,4,27]. Table 1 summarises the results.

**Table 1.** The polynomial selection methods for NFS, where  $f$  and  $g$  are irreducible over  $\mathbb{Z}$  with a common factor modulo  $p$  of degree  $n$ .

Method	deg $f$	deg $g$	$\ f\ _\infty$	$\ g\ _\infty$
JLSV <sub>1</sub> [19]	$n$	$n$	$O(Q^{1/2n})$	$O(Q^{1/2n})$
JLSV <sub>2</sub> ( $D \geq n$ )[19]	$n$	$D$	$O(Q^{1/D+1})$	$O(Q^{1/D+1})$
Conj.[4]	$2n$	$n$	$O(\log p)$	$O(Q^{1/2n})$
GJL( $D \geq n$ )[23,4]	$D+1$	$D$	$O(\log p)$	$O(Q^{1/(D+1)})$
SS( $e n, d \geq n/e$ )[27]	$e(d+1)$	$ed$	$O(\log p)$	$O(Q^{1/e(d+1)})$

These results can be modified to adapt for exTNFS by replacing  $n$  by  $n_2$  and  $Q$  by  $p^{n_2}$ . Another difference is that the common factor of  $f$  and  $g$  is require to be irreducible over  $\mathbb{F}_{p^{n_1}}$  other than  $\mathbb{F}_p$ .

**Factor base logarithm.** In the exTNFS, we set  $n_1$ , the degree of  $h$ , such that  $p^{n_1} \geq L_Q(2/3)$ . Then we only need to sieve the polynomials of the form  $a(r) + b(r)x$ , where  $a(r)$  and  $b(r)$  are coprime polynomials in  $R = \mathbb{Z}[x]/h(x)$  of degree less than  $n_1$ .

After collecting enough relations among the factor base, we can form a sparse linear system. Using Wiedemann's algorithm [31], we solve the linear equations in time  $B^{2+o(1)}$  and obtain the virtual logarithms of the elements in the factor base.

**Individual logarithm.** To compute the logarithm of an element in  $\mathbb{F}_{p^n}^\times$ , in general it requires 2 phases. The first phase is smoothing phase, in which we randomize the target element  $s$  and test for  $L_Q(2/3)$ -smoothness with the ECM algorithm. We repeat this process until the principal ideal generated by  $s$  factors into prime ideals of small norm. Some of the prime ideals may not be in the factor base. So in the second phase, special- $q$  descent phase, we search for a relation between the prime ideal and other smaller ideals. We continue this process recursively until they all fall in the factor base.

**Complexity.** To achieve the optimal complexity, we usually balance the complexities of the relation collection step and the linear algebra step. The total complexity mainly depends on the sizes of the coefficients and degrees of  $f$  and  $g$ . Table 2 summarises the results.

In [22], in order to select  $f$  and  $g$  over  $\mathbb{Z}$  instead of  $R$ , the degrees  $n_1$  and  $n_2$  are required to be coprime. This restriction can be removed, see [16]. Of course, one can combine the exTNFS and SS polynomial selection method, which can loosen the above conditions in some sense, see [26]. When the characteristic of the field has a special form [30,20] or if we use multiple fields [7,24], we can achieve better performance.



**Lemma 1.** ([14]) Let  $\psi(X)$  be a monic irreducible polynomial of  $\mathbb{F}_p[X]$  of even degree  $n$  with a quadratic subfield  $\mathbb{F}_{p^2} = \mathbb{F}_p[Y]/(A(Y))$ . Moreover, assume that  $\psi$  splits over  $\mathbb{F}_p[Y]/(A(Y))$  as

$$\begin{aligned} \psi(X) &= (B(X) - Y)(B(X) - Y^p) \\ \text{or } \psi(X) &= (B(X) - YX)(B(X) - Y^pX) \end{aligned}$$

with  $B$  monic, of degree  $n/2$  and coefficients in  $\mathbb{F}_p$ . Let  $s \in \mathbb{F}_p[X]/(\psi(X))$  a random element,  $s = \sum_{i=0}^{n-1} s_i X^i$ .

Then there exists  $s' \in \mathbb{F}_{p^n}$ , monic and of degree  $n - 2$  in  $X$ , and  $u \in \mathbb{F}_{p^2}$ , such that  $s = u \cdot s'$  in  $\mathbb{F}_{p^n}$ .

According to the lemma, if the field contains a certain quadratic subfield, we can find two preimages  $s = \sum_{i=0}^{n-1} s_i x^i$  and  $s' = \sum_{i=0}^{n-2} s'_i x^i$ . Here, a preimage means its logarithm is congruent to the logarithm of  $s$  modulo  $\ell$ . Then we define the following lattice

$$\left( \begin{array}{cccc} p & & & \\ & \ddots & & \\ & & p & \\ s'_0 \dots s'_{n-3} & 1 & & \\ s_0 \dots s_{n-3} & s_{n-2} & 1 & \end{array} \right) \begin{array}{l} 0 \\ \vdots \\ n-3 \\ n-2 \\ n-1 \end{array} \left. \begin{array}{l} \\ \\ \\ \} \text{coef. of } s' \\ \} \text{coef. of } s \end{array} \right\} \begin{array}{l} n-2 \text{ rows} \\ \\ \end{array}$$

$n \times n$

Using it in place of the upper-left part of the lattice in the GJL and Conjugation cases, we can find a preimage with norm  $O(Q^{1-2/n})$ . This improvement is significant when  $n$  is small.

### 3 Using exTNFS to construct a preimage

#### 3.1 Main idea

Assume  $m$  is the largest proper factor of  $n$ , where  $n$  is the extension degree of the finite field. In this section, we use exTNFS to construct a preimage with norm  $O(Q^{1-m/n})$ . If  $n$  is even, the best result is to reduce the norm to  $O(Q^{1/2})$ .

Since  $m$  is the largest proper factor of  $n$ , the largest proper subfield of  $\mathbb{F}_{p^n}$  is  $\mathbb{F}_{p^m}$ . We set the degree of  $h$  in exTNFS to be  $m$  and the degree of  $\psi$  (the common factor of  $f$  and  $g$  over  $\mathbb{F}_{p^m}$ ) to be  $n' = n/m$ . Other settings are the same as section 2.1. Let  $d_f$  and  $d_g$  denote the degrees of  $f$  and  $g$  respectively.

For  $s \in \mathbb{F}_{p^n}^\times$ , each preimage of  $s$  in  $K_f$  is  $\sum_{i=0}^{n'-1} s_i(r)x^i$ , where  $s_i(r)$  is a polynomial in  $r$  of degree less than  $m$ . When  $s_{n'-1}(r) \neq 0$ , dividing each term by  $s_{n'-1}(r)$ , we obtain a preimage of  $s$  of the form  $\sum_{i=0}^{n'-2} s_i(r)x^i + x^{n'-1}$ . When  $s_{n'-1}(r) = 0$ , we can do the same thing to the highest nonzero term and obtain a shorter form, which is more advantageous for us to reduce the norm.

Next, we form the following lattice of dimension  $md_f$ :

$$\left( \begin{array}{cccccccc} p & & & & & & & \\ & \ddots & & & & & & \\ & & p & & & & & \\ & & & \ddots & & & & \\ & & & & p & & & \\ & & & & & \ddots & & \\ \mathbf{s}_0(\mathbf{r}) & \dots & \mathbf{s}_{n'-2}(\mathbf{r}) & & 1 & & & \\ \vdots & \vdots & \vdots & & \vdots & \ddots & & \\ \mathbf{r}^{m-1}\mathbf{s}_0(\mathbf{r}) & \dots & \mathbf{r}^{m-1}\mathbf{s}_{n'-2}(\mathbf{r}) & & 1 & & & \\ \psi_0(\mathbf{r}) & \psi_1(\mathbf{r}) & \dots & \psi_{n'-1}(\mathbf{r}) & & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\ \mathbf{r}^{m-1}\psi_0(\mathbf{r}) & \mathbf{r}^{m-1}\psi_1(\mathbf{r}) & \dots & \mathbf{r}^{m-1}\psi_{n'-1}(\mathbf{r}) & & 1 & & \\ & & & \vdots & & \vdots & \ddots & \\ & & & \psi_0(\mathbf{r}) & \psi_1(\mathbf{r}) & \dots & \psi_{n'-1}(\mathbf{r}) & 1 \\ & & & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & \mathbf{r}^{m-1}\psi_0(\mathbf{r}) & \mathbf{r}^{m-1}\psi_1(\mathbf{r}) & \dots & \mathbf{r}^{m-1}\psi_{n'-1}(\mathbf{r}) & 1 \end{array} \right)$$

where the algebraic numbers in bold stand for the row vectors of their coordinates. Applying the LLL algorithm to the lattice, we obtain a reduced element  $s' = \sum_{i=0}^{d_f-1} s'_i(r)x^i$  with

$$\log s' \equiv \log s \pmod{\ell}.$$

Since the determinant of the lattice is  $p^{m(n'-1)} = p^{n-m}$  and the dimension is  $md_f$ , we have

$$\|s'\|_\infty \leq Cp^{\frac{n-m}{md_f}},$$

where  $C$  is a small constant. According to [21,8], we have

$$N_{K_f/\mathbb{Q}}(s') = O(\|s'\|_\infty^{md_f} \|f\|_\infty^{md_{s'}}).$$

The value is

$$O(Q^{1-m/n})$$

in Conjugation, GJL or SS case, since the coefficients of  $f$  in these cases are small. In JLSV<sub>1</sub> and JLSV<sub>2</sub> cases, they are

$$O(Q^{3/2-m/n}) \text{ and } O(Q^{2-m/n})$$

respectively.

Thus, if there is no restriction on the degree of  $h$ , following the method above, we can construct a preimage of target element with norm  $O(Q^{1-m/n})$ , where  $m$  is the largest factor of  $n$ . Especially, when  $n$  is even, we can construct a preimage with norm  $O(Q^{1/2})$ . Then the complexity of the smoothing phase is reduced to  $L_Q(1/3, \sqrt[3]{\frac{3}{2}})$ .

However, in exTNFS, to achieve the optimal complexity, there are some restrictions on the choice of  $\deg(h)$ . Thus we first briefly recall the complexity analysis about exTNFS, then we can finish the proof of Theorem 1 and Theorem 2.

### 3.2 A brief analysis to recent results about exTNFS

There are lots of analysis to the complexity of the classical NFS whenever the characteristic is medium or large. Here we summarize the results of recent selection methods (GJL, Conj. and SS). One of their similarities is that they all set the coefficients of one polynomial to be small.

Let  $Q$  denote the cardinality of the target field and  $d_f$  (resp.  $d_g$ ) denote the degree of  $f$  (resp.  $g$ ) as before. Assume we sieve degree  $t-1$  polynomials of the form  $a_0 + a_1x + \dots + a_{t-1}x^{t-1}$ . Suppose the coefficients of  $f$  is small and the coefficients of  $g$  are bounded by  $O(Q^{1/n_g})$ . One can check  $n_g$  is compared with  $d_f$ .

Then the best complexity will have a uniform formula

$$L_Q(1/3, \sqrt[3]{\frac{c}{9}}),$$

where

$$c = 64 \frac{t-1}{t} \frac{d_f + d_g}{n_g}.$$

This result can be directly generalized to suit for exTNFS when we alter  $n_g$  to represent the coefficients of  $g$  are bounded by  $O(p^{n_2/n_g})$ . Moreover, it also suits for special NFS.

From the above formula, we can see that the optimal case is  $t = 2$ . This can be achieved only when  $p^{n_1} > L_Q(2/3)$ , where  $n_1$  is the degree of  $h$ . For the term  $\frac{d_f+d_g}{n_g}$ , it depends on the polynomial selection methods. When  $p^{n_1} = L_Q(2/3)$ , the best result is  $\frac{3}{2}$  achieved by Conjugation or SS method. Then the best complexity is  $L_Q(1/3, \sqrt[3]{\frac{48}{9}})$ . When  $p^{n_1} > L_Q(2/3)$  we cannot apply Conjugation method. The minimal value for  $\frac{d_f+d_g}{n_g}$  is 2 achieved by GJL or SS method. Then the total complexity is  $L_Q(1/3, \sqrt[3]{\frac{64}{9}})$ , which is larger than the former case.

Thus it is sufficient to consider the case that there is  $k|n$  such that  $p^k = L_Q(2/3)$ .

### 3.3 Reducing the norm in different cases

As before, let  $m$  be the largest proper factor of  $n$ .

If the characteristic is large, i.e.  $p > L_Q(2/3)$ , there is no problem. Thus we only need to consider the medium and boundary case. If  $p^k \neq L_Q(2/3)$  for any factor  $k$  of  $n$  with  $k \neq m$ , then we can set  $\deg(h)$  to be  $m$ .

For the remaining case,  $\deg(h)$  should be  $k < m$  and  $p^k$  is about  $L_Q(2/3)$ . Let  $q = p^k$  and  $n'' = n/k$ . In this case, we need to set  $f, g$  to have a common irreducible factor  $\psi$  of degree  $n''$  over  $\mathbb{F}_q$ . Note that, in this case, if we use the subfield  $\mathbb{F}_q$ , we can only reduce the norm to  $O(Q^{1-k/n})$  other than  $O(Q^{1-m/n})$ .

Firstly, we give a generalized version of the Lemma 1 to obtain a slightly better result.

**Lemma 2.** *Assume there is a proper subfield  $\mathbb{F}_{q^\lambda} = \mathbb{F}_q[Y]/A(Y)$  of  $\mathbb{F}_{q^{n''}}$  with  $\lambda > 1$  such that  $\psi$  splits over  $\mathbb{F}_{q^\lambda}$  as*

$$\psi(X) = \prod_{i=0}^{\lambda-1} (B(X) - Y^{q^i}),$$

where  $B(X)$  is a polynomial of degree  $n''/\lambda$  with coefficients in  $\mathbb{F}_q$ . Let  $s = \sum_{i=0}^{n''-1} s_i X^i$  be a random element in  $\mathbb{F}_q[X]/\psi(X)$ . We can find an element  $s'$  in  $\mathbb{F}_q[X]/\psi(X)$  of degree at most  $n''-2$  satisfying  $s = u \cdot s'$  with  $u \in \mathbb{F}_{q^\lambda}$ .

*Proof.* The proof is similar. We set the tower of fields as follows.

$$\begin{array}{c} \mathbb{F}_{q^{n''}} = \mathbb{F}_q[X]/\psi(X) = \mathbb{F}_q[X, Y]/(A(Y), B(X) - Y) \\ | \\ \mathbb{F}_{q^\lambda} = \mathbb{F}_q[Y]/A(Y) \\ | \\ \mathbb{F}_q \end{array}$$

We represent  $s$  as

$$s = \sum_{i=0}^{n''/\lambda-1} c_i(Y)X^i.$$

with  $c_i(Y)$  of degree in  $Y$  at most  $\lambda - 1$ . Dividing  $s$  by  $c_{n''/\lambda-1}(Y) (\in \mathbb{F}_{q^\lambda})$ , we obtain

$$\frac{s}{c_{n''/\lambda-1}(Y)} = \sum_{i=0}^{n''/\lambda-2} d_i(Y)X^i + X^{n''/\lambda-1},$$

with  $d_i(Y)$  of degree at most  $\lambda - 1$ . Substituting  $Y$  with  $B(X)$ , we obtain the right hand side is

$$\sum_{i=0}^{n''/\lambda-2} d_i(B(X))X^i + X^{n''/\lambda-1},$$

which is of degree at most  $\frac{n''}{\lambda}(\lambda - 1) + \frac{n''}{\lambda} - 2 = n'' - 2$ . □

Following the lemma, if the field has certain form, we can construct a preimage of degree at most  $n'' - 2$ . Then we can apply the LLL algorithm to obtain a preimage of norm  $O(Q^{1-2k/n})$ .

Next, we will show if some requirements for  $K_f$  can be met, we can construct a preimage with norm  $O(Q^{1-m/n})$ . Note that since  $k$ , the degree of  $h$ , satisfies  $p^k = L_Q(2/3)$ , we should use Conjugation method or SS method for polynomial selection. For simplicity, we consider the Conjugation method case while the other case is similar. In this case, the degree of  $f$  is  $2n/k = 2n''$ .

**Lemma 3.** *Let  $K_f = \mathbb{Q}(r)[X]/f(X) = \mathbb{Q}(r, x)$ . Assume there is a subfield  $\mathbb{Q}(r, y) \subseteq K_f$  of index  $2n'$  such that the coefficients of the minimal polynomials of  $y$  over  $\mathbb{Q}(r)$  and  $x$  over  $\mathbb{Q}(r, y)$  are both small, i.e. are bounded by  $O(\log p)$ . Let  $s$  be a random element in  $\mathbb{F}_{p^n}$ . We can construct a preimage of  $s$  in  $K_f$  with norm  $O(Q^{1-m/n})$ .*

*Proof.* Under this condition, we can view  $K_f$  as the extension field of  $\mathbb{Q}(r, y)$  by adding  $x$  and  $\mathbb{Q}(r, y)$  as the extension field of  $\mathbb{Q}(r)$  by adding  $y$ . Every element  $s$  in  $K_f$  can also be expressed as

$$\tilde{s} = \sum_{i=0}^{n'-1} \tilde{s}_i(r, y)x^i$$

where we use  $\tilde{s}$  to denote  $s$  in this expression. Note that, although  $\tilde{s}$  and  $s$  are the same element in  $K_f$ ,  $\|\tilde{s}\|_\infty$  and  $\|s\|_\infty$  are totally different.

Since the coefficients of the minimal polynomials of  $x$  and  $y$  are small, one can check the norm of  $s$  will be

$$N_{K_f/\mathbb{Q}}(s) = N_{K_f/\mathbb{Q}}(\tilde{s}) = O(\|\tilde{s}\|_\infty^{md_f}),$$

whose form is the same as before.

Now, let  $\tilde{s} \in K_f$  be a preimage of an element in  $\mathbb{F}_Q$ . Assume  $\tilde{s} = \sum_{i=0}^{n'-1} \tilde{s}_i(r, y)x^i$  with  $\tilde{s}_{n'-1}(r, y) \neq 0$ . We divide each term by  $\tilde{s}_{n'-1}(r, y)$ , and obtain

$$\tilde{s}' = \sum_{i=0}^{n'-1} \tilde{s}'_i(r, y)x^i + x^{n'-1}.$$

We can view it as a polynomial in  $x$  and  $y$  with coefficients in  $r$ . Then we can construct a vector whose components are the coefficients of  $y^i x^j$ . If we use the vector to replace the corresponding row of the lattice in section 3 and change the expression of  $\psi$ , then we can form a new lattice. Applying the LLL algorithm to the lattice, we can obtain a preimage  $\tilde{s}''$  with

$$\|\tilde{s}''\|_\infty \leq Cp^{\frac{n-m}{md_f}}.$$

Thus the norm of  $\tilde{s}''$  is bounded by  $O(Q^{1-m/n})$ . □

We give an example to illustrate the conditions in Lemma 3 can be satisfied.

**Example.** Let's consider the finite field  $\mathbb{F}_{p^{30}}$ , where  $p = 39614081257132168796771975177$ . The largest proper factor of 30 is 15. If we set  $\deg(h) = 5$ , we should set  $\deg(f) = 12$  in Conjugation method. Since 5 and 12 are coprime, it is sufficient to select  $f$  over  $\mathbb{Z}$ . Firstly, we choose two small coefficients polynomial  $x^6 - 1$  and  $x^3$ . Next, we choose the irreducible polynomial  $Y^2 + 1$  over  $\mathbb{Z}$  which has a root modulo  $p$ . Let  $f = \text{Res}_Y(Y^2 + 1, x^6 - 1 - x^3Y) = x^{12} - x^6 + 1$ . One can check  $f$  is irreducible over  $\mathbb{Z}$  and thus has a degree 6 irreducible factor modulo  $p$ . Let  $y$  be a root of the equation  $y^3 - 3y + 1$ . One can check  $f$  splits into 3 irreducible factor over  $\mathbb{Q}(y)$ . One of the factor is  $x^4 + yx^2 + 1$  with small coefficients. Hence in this example, the conditions in Lemma 3 are all satisfied.

Based the above discussion, we obtain the validity of Theorem 1 and Theorem 2.

## 4 Numerical Experiments

In this section, we give some numerical experiments to reduce the norm of the preimage.

**Example 1 ( $n = 6$  with GJL method).** We take a random prime number  $p$  of about 100-bit (30 decimal digit), and  $n = 6$ . The size of the field  $\mathbb{F}_{p^6}$  is about 180 decimal digits (dd). Since largest proper factor of  $n$  is 3, we set  $h$  to be a polynomial of degree 3 with small coefficients and irreducible modulo  $p$ . Let  $r$  be a root of  $h$ . We take  $f$  to be a degree 4 irreducible polynomial over  $\mathbb{Z}$  with small integer coefficients. Moreover we require that  $f$  has a degree 2 irreducible factor  $\psi$



Then  $f$  is irreducible over  $\mathbb{Z}$  and has an irreducible factor  $\psi(x) = x^2 - y$  over  $\mathbb{F}_p$ .

$$p = 7170914684772626399787694948453$$

$$h = r^3 + r + 1$$

$$f = x^4 + 1$$

$$\psi = x^2 + 294838925512229337898309576527$$

$$g = 966642759457218x^2 + 2497301836054577$$

$$s = (2767660019267865248076151275104r^2 + 357970798563045003813528394260r + 6311123219907587664235529021977)x \\ + 2234776892532612450942592349739r^2 + 5963032404036223471843728113344r + 2280716845436119385331170350884$$

with  $p$  a 31 dd prime number and  $p^6$  of 186 dd.

Taking  $s' = \frac{1}{s_1}s$ , we have

$$s' = x + 6829187035664634928218051355972r^2 + 2356513401811425063371831680423x + 7055298630876009777096508302136.$$

Since  $\psi$  doesn't have degree 1 term, we form a similar lattice in Example 1. Here we omit it. We use LLL algorithm to reduce the lattice and the returned short element  $s''$  is

$$(-243030r^2 - 1609858r - 14170476)x^3 + (17026360r^2 + 19611969r + 40385280)x^2 + \\ (-21368270r^2 - 25460768r + 45578231)x + 4785869r^2 - 5442349r - 3676839,$$

with coefficient at most 8 dd. Its norm is

$$N_{K_f/\mathbb{Q}}(s'') = 9408692079257501461183742234523910224598901984786177574687834188371565707188019033831132562049,$$

which is a 94 dd number. Its size is  $94/186 \approx 0.505$  of that of  $p^6$ .

**Example 3 ( $n = 12$  with GJL method).** In this example, we consider the case for  $n = 12$ . We want to take a 600-bit finite field. Then the characteristic  $p$  will be about 15 dd. The largest proper factor of  $n$  is 6, we set  $h$  to be a polynomial of degree 6 with small coefficients and irreducible modulo  $p$ . Let  $r$  be a root of  $h$  and  $R$  be the ring  $\mathbb{Z}[r]$ . We take  $f$  to be a degree 4 irreducible polynomial over  $R$  with small coefficients. Moreover we require that  $f$  has degree 2 irreducible factor  $\psi$  over  $\mathbb{F}_{p^6}$ . At last, we randomly pick an element  $s$  in  $\mathbb{F}_{p^6}$ .

$$p = 2251799813685269$$

$$h = r^6 + r - 1$$

$$f = x^4 + r$$

$$\psi = x^2 + 1993972645314362r^5 + 2014524994046034r^4 + 775349557393539r^3 + 2239410057339674r^2 + 1611508501046572r + 723760306664988$$

$$s = (664609958516367r^5 + 696970620962968r^4 + 772196105657867r^3 + 663786159251904r^2 + 1018587115350r + 871785303785789)x$$

$$+ 1254825522464853r^5 + 163391769589048r^4 + 1440697992754427r^3 + 833042729041497r^2 + 1146684997003032r + 2084950047673640$$

with  $p$  a 16 dd prime number and  $p^{12}$  of 185 dd. Here we omit the expression of  $g$ , since our computation doesn't involve  $g$ .





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