

Optimized quantization in Zero Leakage Helper Data Systems

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Abstract

Helper Data Systems are a cryptographic primitive that allows for the reproducible extraction of secrets from noisy measurements. Redundancy data called Helper Data makes it possible to do error correction while leaking little or nothing ('Zero Leakage') about the extracted secret string. We study the case of non-discrete measurement outcomes. In this case a quantization step is required. Recently de Groot et al. described a generic method to perform the quantization in a Zero Leakage manner. We extend their work and show how the quantization intervals should be set to maximize the amount of extracted secret key material when noise is taken into account.

1 Introduction

1.1 Helper Data Systems

Security with noisy data is the art of reproducibly extracting secret data from noisy measurements on a physical system. The two main applications are read-proof storage of cryptographic keys using Physical Unclonable Functions (PUFs) [16, 17, 10, 14, 2, 13] and privacy-preserving storage of biometric data. Power-off storage of keys in *digital* memory can often be considered insecure. (For instance, fuses can be optically inspected with a microscope; flash memory may be removed and then read digitally.) PUFs provide an alternative way to store keys, namely in analog form, which allows the designer to exploit the inscrutability of analog physical behavior. Keys stored in this way are referred to as Physically Obfuscated Keys (POKs) [9].

In both the biometrics and the PUF/POK case, one faces the problem that some form of error correction has to be performed, but under the constraint that the redundancy data (which is visible to attackers) does not endanger the secret. This problem has been addressed by the introduction of a special security primitive, the *Helper Data System* (HDS). A HDS in its most general form is shown in Fig. 1. The **Gen** procedure takes as input a measurement X . **Gen** outputs a secret S and (public) Helper Data W . The helper data is stored. In the reproduction phase, a fresh measurement Y is obtained. Typically Y is a noisy version of X , close to X (in terms of e.g. Euclidean distance or Hamming distance) but not necessarily identical. The **Rec** procedure takes Y and W as input. It outputs \hat{S} , an estimate of S . If Y is not too noisy then $\hat{S} = S$.

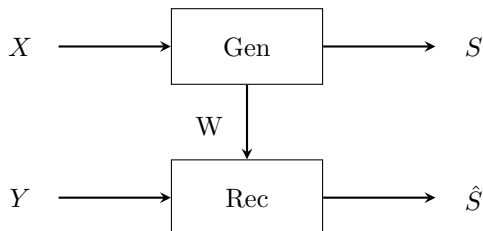


Figure 1: *Data flow in a generic Helper Data System.*

Two special cases of the general HDS are the Secure Sketch (SS) and the Fuzzy Extractor (FE) [8]. The Secure Sketch has $S = X$ (and $\hat{S} = \hat{X}$, an estimator for X). If X is not uniformly distributed,

then S is not uniform. The SS is suitable for privacy-preserving biometrics, where the stored biometric enrollment data is a cryptographic hash of X , just like hashed storage of passwords; high entropy of S (given W) is required, but not uniformity. The Fuzzy Extractor is required to have a (nearly) uniform S given W . The FE is typically used for extracting keys from PUFs and POKs. Note that there is a generic construction to obtain a FE from a SS: privacy amplification on X by applying a suitable information-theoretic hash function. This can be either a Universal Hash Function (UHF) [3, 15, 12] or, more sophisticatedly, a q -wise independent hash function. UHFs have the advantage of being simple to implement and providing information-theoretic security guarantees for all applications of the extracted key; however, they waste a lot of source entropy. Key derivation with q -wise independent hash functions can be done almost without any entropy loss [7] but gives information-theoretic guarantees only for ‘unpredictability applications’, which include signatures, Message Authentication Codes and keyed hashing.

In this paper we consider the general HDS case: $S \neq X$ and S is not necessarily uniform. The general HDS is of particular interest when X is a continuum variable: (i) The least significant digits of X are not interesting for key extraction and (ii) In view of the excellent performance of q -wise independent hashes [7] it is best to first extract from X a non-uniform high-entropy discrete secret and then compress it to make it more uniform.

1.2 Zero Leakage quantisation

In the biometrics case and in several PUF/POK scenarios the raw measurement data X is analog or nearly analog. A typical HDS then consists of two stages. The first stage is a HDS that maps the continuous X to a discrete space, i.e. it discretizes (quantizes) X . The second stage is a HDS acting on a discrete source, e.g. the Code Offset Method [1, 11, 8, 6, 19]. Both stages make use of helper data, and in both stages one has to worry about leakage.

In the first stage it is possible to make a construction such that W leaks nothing about S . Intuitively speaking, W contains the ‘least significant bits’ of X , which are noisy, while S contains the ‘most significant bits’. A HDS that achieves independence of S and W is called a Zero Leakage HDS (ZLHDS).

Verbitskiy et al. [18] introduced a Zero Leakage Fuzzy Extractor (ZLFE) for $X \in \mathbb{R}$.¹ They divided the space \mathbb{R} into N intervals A_0, \dots, A_{N-1} that are equiprobable in the sense that $\Pr[X \in A_j] = 1/N$ for all j . At enrollment, if X lies in interval A_j then S is set to j . For the helper data they introduced a further division of each interval A_j into m equiprobable subintervals $(A_{jk})_{k=0}^{m-1}$. If the enrollment measurement X lies in interval A_{jk} then the index k is stored as helper data. The fact that all these subintervals are equiprobable leads to independence between the helper data and the secret.

De Groot et al. [5] took the limit $m \rightarrow \infty$ and showed that the resulting scheme is not just a ZLFE but the generic best performing ZLFE for $X \in \mathbb{R}$; other ZLFEs for $X \in \mathbb{R}$ can be derived from the generic scheme. Furthermore, de Groot et al. generalized the scheme of [18] from ZLFEs to general ZLHDSs by allowing intervals A_0, \dots, A_{N-1} that are not equiprobable. Several questions were left open regarding the Rec procedure in general ZLHDSs and the performance of ZLHDSs compared to ZLFEs.

1.3 Contributions and outline

We investigate ZLHDSs for $X \in \mathbb{R}$.

- First we derive an optimal Rec procedure that minimises the probability of reconstruction errors. We obtain analytic formulas for Gaussian noise and Lorentz-distributed noise.
- Using this Rec procedure we study the performance of ZLHDSs compared to ZLFEs. We define performance as the mutual information between S and \hat{S} conditioned on the fact that the adversary knows W . This mutual information $I(S; \hat{S}|W)$ represents the maximum amount of secret key material that can be extracted from X using a ZLHDS. It turns out that the intricacies of the Rec procedure cause the mutual information to become a

¹A high-dimensional measurement is usually split into one-dimensional components, e.g. using Principal Component Analysis or similar methods. A HDS is then applied to each component individually. The results are combined and then serve as input for the 2nd stage.

very complicated function of the choice of quantisation intervals A_0, \dots, A_{N-1} . We have to resort to numerics. Our numerical results for Gaussian source and Gaussian noise show that optimisation of the quantisation intervals yields an improvement with respect to the ZLFE in terms of mutual information as well as reconstruction error probability. In most cases the gain in $I(S; \hat{S}|W)$ is modest (a few percent), but the reduction of the error rate can be substantial. *We conclude that in practice it is better to use a ZLHDS than a ZLFE.*

In Section 2 we introduce the notation used in this paper and give a rather long summary of the results of de Groot et al. [5]. In Section 3 we derive the optimal Rec procedure and provide analytic expressions (as far as possible) for the mutual information and the error rate. Section 4 presents the numerical results for Gaussian X and Gaussian noise.

2 Preliminaries

2.1 Notation and terminology

We use capital letters to represent random variables, and lowercase letters for their realizations. The input and output variables of the HDS are as depicted in Fig. 1. Sets are denoted by calligraphic font. The set \mathcal{S} is defined as $\mathcal{S} = \{0, \dots, N - 1\}$. For $\alpha \in \mathcal{S}$ we define $p_\alpha = \Pr[X \in A_\alpha]$. The expected value with respect to a random variable Z is denoted as \mathbb{E}_z . The mutual information (see e.g. [4]) between X and Y is denoted as $I(X; Y)$, and the mutual information conditioned on the third variable Z as $I(X; Y|Z)$. The probability density function (pdf) of the random variable $X \in \mathbb{R}$ is written as $f(x)$ and its cumulative distribution function (cdf) as $F(x)$.

2.2 Zero Leakage definition

For technical reasons, de Groot et al. used the following definition of the Zero Leakage property.

Definition 2.1 (Zero Leakage). *Let $W \in \mathcal{W}$. We call a HDS a Zero Leakage HDS if and only if*

$$\forall \mathcal{V} \subseteq \mathcal{W} \quad \Pr[S = s|W \in \mathcal{V}] = \Pr[S = s]. \quad (1)$$

The property (1) implies $I(W; S) = 0$.

2.3 Noise model

We adopt the noise model from [5]. The X and Y are considered to be noisy versions of an underlying ‘true’ value. Without loss of generality X is taken to have zero mean. The standard deviations of $X, Y \in \mathbb{R}$ are denoted as σ_X and σ_Y respectively. The verification sample Y is related to the enrollment measurement as $Y = \lambda X + R$, where $\lambda \in [0, 1]$ is the *attenuation parameter* and R is zero-mean additive noise, independent of X . We have $\sigma_Y^2 = \lambda^2 \sigma_X^2 + \sigma_R^2$. The correlation between X and Y is

$$\rho \stackrel{\text{def}}{=} \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sigma_X \sigma_Y} = \lambda \frac{\sigma_X}{\sigma_Y}, \quad (2)$$

with $\rho \in [-1, 1]$. The relation between λ, ρ, σ_X , and σ_R is given by $\lambda^2 = \frac{\rho^2}{1-\rho^2} \frac{\sigma_R^2}{\sigma_X^2}$. Two special cases are often considered:

Perfect enrollment. During enrollment there is no noise. The X equals the ‘true’ value. In this situation it holds that $\sigma_Y^2 = \sigma_X^2 + \sigma_R^2$ and $\lambda = 1$.

Identical conditions. The amount of noise is the same during enrollment and reconstruction. In this situation it holds that $\sigma_Y^2 = \sigma_X^2$ and $\lambda^2 = \rho^2 = 1 - \sigma_R^2/\sigma_X^2$.

The pdf of Y given $X = x$ is denoted as $\psi(y|x) = v(y - \lambda x)$. The noise is considered to be symmetric and fading, i.e. $v(-z) = v(z)$ and $v(z)$ is a decreasing function of $|z|$. The cdf corresponding to v is denoted as V .

2.4 The ZL scheme of [5]

The helper data is considered to be continuous, $W \in \mathcal{W} \subset \mathbb{R}$, and without loss of generality de Groot et al. set $\mathcal{W} = [0, 1)$. The left boundary of the quantisation region A_α is denoted as Ω_α , $\alpha \in \mathcal{S}$. (See Fig. 2.) It holds that

$$\Omega_\alpha = F^{\text{inv}} \left(\sum_{i=0}^{\alpha-1} p_i \right), \quad (3)$$

where F^{inv} stands for the inverse function of F . Note that $\Omega_0 = -\infty$. The **Gen** procedure is written as $s = Q(x)$, $w = g(x)$, where the Q and g functions are given by

$$Q(x) = \max\{\alpha \in \mathcal{S} : x \geq \Omega_\alpha\} \quad ; \quad g(x) = \frac{F(x) - F(\Omega_{Q(x)})}{p_{Q(x)}} = \frac{F(x) - \sum_{i=0}^{Q(x)-1} p_i}{p_{Q(x)}}. \quad (4)$$

The relation between x , s and w can be written in a more friendly form as

$$F(x) = F(\Omega_s) + wp_s = \sum_{i=0}^{s-1} p_i + wp_s. \quad (5)$$

The thus defined $w \in [0, 1)$ is called *quantile* helper data since it measures which quantile of the probability mass p_s is located between $F(\Omega_s)$ and x . It was shown that the random variable W , given S , has a uniform pdf. Consequently the scheme is a ZLHDS.

The mapping of x to (s, w) is a bijection. For the mapping of (s, w) to x the following notation is used,²

$$\xi_{s,w} \stackrel{\text{def}}{=} F^{\text{inv}} \left(\sum_{i=0}^{s-1} p_i + wp_s \right). \quad (6)$$

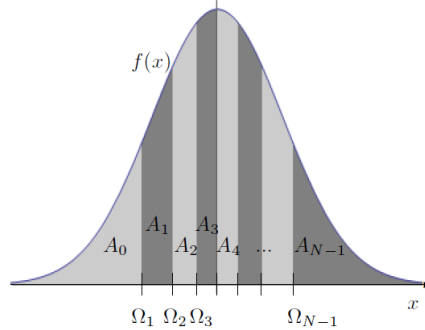


Figure 2: Illustration of the quantization boundaries Ω_α and regions A_α .

In the case of the Fuzzy Extractor ($p_\alpha = 1/N$ for all $\alpha \in \mathcal{S}$) the optimal reconstruction procedure was found to be the following maximum-likelihood ‘decoder’,

$$\hat{s} = \text{Rec}^{\text{FE}}(y, w) = \arg \max_{\alpha \in \mathcal{S}} \psi(y | \xi_{\alpha w}). \quad (7)$$

Eq. (7) can be conveniently implemented by defining decision boundaries $(\tau_{\alpha w})_{\alpha=0}^N$. If $y \in [\tau_{\alpha w}, \tau_{\alpha+1, w})$, then $\hat{s} = \alpha$. In the case of symmetric fading noise the location of the decision boundaries dictated by (7) was found to be

$$\tau_{\alpha w}^{\text{FE}} = \lambda \frac{\xi_{\alpha-1, w} + \xi_{\alpha w}}{2}. \quad (8)$$

Here one has to read $\xi_{-1, w} = -\infty$ and $\xi_{Nw} = \infty$, resulting in $\tau_{0w} = -\infty$, $\tau_{Nw} = \infty$. Fig. 3 shows how to intuitively understand (8). Each pdf $\psi(y | \xi_{\alpha w})$ in (7) is centered around $y = \lambda \xi_{\alpha w}$ and drops off symmetrically. The crossing point where one α -value becomes more likely than another lies exactly halfway between the centers of two neighbouring pdfs; such a crossing point is a decision boundary.

²We often omit the comma and write ξ_{sw} .

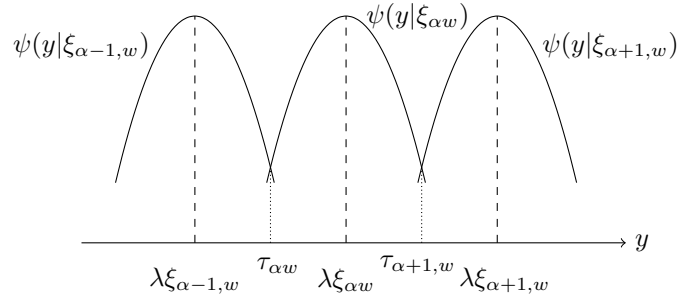


Figure 3: Visual representation of the decision boundaries for the reconstruction phase.

3 Optimization of the general ZLHDS

In this section we extend the results of de Groot et al. [5]. We generalize equations (7) and (8). Then we derive analytic expressions for $I(S; \hat{S}|W)$ and the reconstruction error probability P_{err} in terms of the scheme's parameters. We also discuss the relation between P_{err} and the bit error rate.

3.1 ZLHDS reconstruction

For the sake of completeness we explicitly show that W given $S = s$ is uniform. (This fact was implicit in [5] and was not separately stated.)

Lemma 3.1. *The probability density function of the helper data W given the secret S is uniform.*

Proof. For the pdf of W given $S = \alpha$ we write $\rho(w|\alpha)$. We start from $p_\alpha \rho(w|\alpha) dw = f(\xi_{\alpha w}) d\xi_{\alpha w}$. The validity of this equation is readily verified. Applying \int_0^1 to the left hand side yields p_α by definition; on the right hand side the equivalent operation is integration over $\xi_{\alpha w}$ on the interval A_α , which also yields p_α . Now we can write $\rho(w|\alpha) = \frac{f(\xi_{\alpha w})}{p_\alpha dw/d\xi_{\alpha w}} = \frac{f(\xi_{\alpha w})}{dF(\xi_{\alpha w})/d\xi_{\alpha w}} = \frac{f(\xi_{\alpha w})}{f(\xi_{\alpha w})} = 1$. In the second equality we used (5) with $s = \alpha$ kept constant while w varies. \square

Lemma 3.2. *For the general HDS the optimal reconstruction procedure is given by*

$$\hat{s} = \text{Rec}(y, w) = \arg \max_{\alpha \in \mathcal{S}} p_\alpha \psi(y|\xi_{\alpha w}). \quad (9)$$

Proof. This is a slight modification of Lemma 4.1 in [5], with the same starting point.

$$\text{Rec}(y, w) = \arg \max_{\alpha \in \mathcal{S}} \Pr[S = \alpha | Y = y, W = w] = \arg \max_{\alpha \in \mathcal{S}} \frac{\Pr[S = \alpha, Y = y, W = w]}{\Pr[Y = y, W = w]}. \quad (10)$$

Since the denominator does not depend on α , it can be eliminated.

$$\begin{aligned} \text{Rec}(y, w) &= \arg \max_{\alpha \in \mathcal{S}} \Pr[S = \alpha, Y = y, W = w] \\ &= \arg \max_{\alpha \in \mathcal{S}} \Pr[Y = y | S = \alpha, W = w] \rho(w|\alpha) p_\alpha. \end{aligned}$$

Using Lemma 3.1 we get

$$\hat{s} = \text{Rec}(y, w) = \arg \max_{\alpha \in \mathcal{S}} p_\alpha \Pr[Y = y | S = \alpha, W = w]. \quad (11)$$

Since (α, w) uniquely defines $\xi_{\alpha w}$, the probability $\Pr[Y = y | S = \alpha, W = w]$ equals $\Pr[Y = y | X = \xi_{\alpha w}]$, for which the notation $\psi(y|\xi_{\alpha w})$ is used. \square

From (9) we can derive an optimal placement of the boundaries $\tau_{\alpha w}$ for general noise and general HDS.

Lemma 3.3. For a ZLHDS the reconstruction boundary $\tau_{\alpha w}$ obtained using pdf intersections satisfies the following equation:

$$p_{\alpha-1}\psi(\tau_{\alpha w}|\xi_{\alpha-1,w}) = p_{\alpha}\psi(\tau_{\alpha w}|\xi_{\alpha w}). \quad (12)$$

Proof. From Lemma 3.2 we see that the decision boundary is the point y where the function $p_{\alpha}\psi(y|\xi_{\alpha w})$ intersects the function $p_{\alpha-1}\psi(y|\xi_{\alpha-1,w})$. \square

In the FE case, $p_{\alpha-1} = p_{\alpha}$ and (12) reduces to $\psi(\tau_{\alpha w}|\xi_{\alpha-1,w}) = \psi(\tau_{\alpha w}|\xi_{\alpha w})$, which directly yields (8). In the general HDS case, however, the difference between the p_{α} parameters changes the heights of the pdfs $\psi(y|\dots)$ in Fig. 3, which leads to a more complicated solution for the decision boundaries.

Theorem 3.4. Let the noise be Gaussian with zero mean and variance σ_R^2 . Then the intersection points as specified in (12) are given by

$$\tau_{\alpha w} = \lambda \frac{\xi_{\alpha-1,w} + \xi_{\alpha w}}{2} + \frac{\sigma_R^2 \ln \frac{p_{\alpha-1}}{p_{\alpha}}}{\lambda(\xi_{\alpha w} - \xi_{\alpha-1,w})}. \quad (13)$$

Proof. The Gaussian noise is given by $\psi(y|x) = \frac{1}{\sqrt{2\pi}\sigma_R} e^{-\frac{(y-\lambda x)^2}{2\sigma_R^2}}$. Eq. (12) then becomes

$$\frac{p_{\alpha-1}}{\sqrt{2\pi}\sigma_R} e^{-\frac{(\tau_{\alpha w} - \lambda \xi_{\alpha-1,w})^2}{2\sigma_R^2}} = \frac{p_{\alpha}}{\sqrt{2\pi}\sigma_R} e^{-\frac{(\tau_{\alpha w} - \lambda \xi_{\alpha w})^2}{2\sigma_R^2}}. \quad (14)$$

Taking the logarithm on both sides of the equation yields a linear equation in $\tau_{\alpha w}$, with solution (13). \square

Theorem 3.5. Let the noise be Lorentz-distributed, $\psi(y|x) = \frac{1/\sigma_R}{1 + \pi^2(y-\lambda x)^2/\sigma_R^2}$. Let $p_{\alpha} \neq p_{\alpha-1}$. If the following condition holds

$$p_{\alpha}p_{\alpha-1}(\lambda\xi_{\alpha,w} - \lambda\xi_{\alpha-1,w})^2 \geq \sigma_R^2 \frac{(p_{\alpha} - p_{\alpha-1})^2}{\pi^2}, \quad (15)$$

then the reconstruction boundary $\tau_{\alpha w}$ is given by

$$\tau_{\alpha w} = \frac{p_{\alpha-1}\lambda\xi_{\alpha w} - p_{\alpha}\lambda\xi_{\alpha-1,w}}{p_{\alpha-1} - p_{\alpha}} - \frac{1}{p_{\alpha-1} - p_{\alpha}} \sqrt{p_{\alpha}p_{\alpha-1}(\lambda\xi_{\alpha w} - \lambda\xi_{\alpha-1,w})^2 - \frac{\sigma_R^2}{\pi^2}(p_{\alpha-1} - p_{\alpha})^2}. \quad (16)$$

Proof. Substitution of the Lorentz distribution into (12) yields

$$\frac{p_{\alpha}}{1 + \pi^2\sigma_R^{-2}(\tau_{\alpha w} - \lambda\xi_{\alpha w})^2} = \frac{p_{\alpha-1}}{1 + \pi^2\sigma_R^{-2}(\tau_{\alpha w} - \lambda\xi_{\alpha-1,w})^2}. \quad (17)$$

Inversion of both sides of the equation gives a quadratic equation in $\tau_{\alpha w}$. (If $p_{\alpha} = p_{\alpha-1}$ then it reduces to a linear equation with (8) as the solution.) The quadratic equation has solutions only if the discriminant is nonnegative, which is equivalent to the condition (15). Finally we have to choose the correct sign preceding the square root of the determinant. We choose the sign in such a way that $\lambda\xi_{\alpha-1,w} < \tau_{\alpha w} < \lambda\xi_{\alpha w}$. We verify as follows that (16) indeed satisfies these inequalities. On the one hand, (16) can be written as

$$\tau_{\alpha w} = \lambda\xi_{\alpha w} + \frac{p_{\alpha}\lambda(\xi_{\alpha w} - \xi_{\alpha-1,w}) - \sqrt{\dots}}{p_{\alpha-1} - p_{\alpha}}. \quad (18)$$

Note that $\xi_{\alpha w} - \xi_{\alpha-1,w} > 0$. If $p_{\alpha-1} > p_{\alpha}$ then the $\sqrt{\dots}$ ‘wins’ and the numerator of the fraction is negative, as it should be. If $p_{\alpha-1} < p_{\alpha}$ then the denominator is negative and the $\sqrt{\dots}$ ‘loses’, making the numerator positive. On the other hand, (16) can also be written as

$$\tau_{\alpha w} = \lambda\xi_{\alpha-1,w} + \frac{p_{\alpha-1}\lambda(\xi_{\alpha w} - \xi_{\alpha-1,w}) - \sqrt{\dots}}{p_{\alpha-1} - p_{\alpha}}. \quad (19)$$

If $p_{\alpha-1} > p_{\alpha}$ then the $\sqrt{\dots}$ ‘loses’ and the fraction is positive. If $p_{\alpha-1} < p_{\alpha}$ then the $\sqrt{\dots}$ ‘wins’ and the fraction is again positive. \square

Remark. If one adopts (13) as decision boundaries, an incorrect reconstruction procedure may result under some pathological circumstances. This can happen, for example, if for some α it happens that $p_\alpha \ll p_{\alpha-1}$ and $p_\alpha \ll p_{\alpha+1}$; then in Fig. 3 the middle curve is located beneath the intersection of its neighbours, and \hat{s} cannot equal α even if $s = \alpha$. In practice we will never see this pathological case.

3.2 Optimization of the quantization intervals

As announced in Section 1.3, we want to maximize the amount of key material extracted from X by the ZLHDS. We have to take into account two effects: the noise, which limits how much of the entropy of X can be recovered in the reconstruction phase, and the fact that the adversary knows W . The quantity of interest is the mutual information between S and \hat{S} given W : $I(S; \hat{S}|W)$. This represents the ‘secrecy capacity’ or quality of the channel from S to \hat{S} created by the ZLHDS. If a perfect error correction mechanism is used as the second-stage HDS, i.e. one that achieves the Shannon bound, then $I(S; \hat{S}|W)$ is the achievable key length.

We note that even though $H(S|W) = H(S)$, we have $I(S; \hat{S}|W) \neq I(S; \hat{S})$ because \hat{S} is not independent of W .

Lemma 3.6. For a zero leakage helper data system the mutual information can be expressed as

$$I(S; \hat{S}|W) = H(S) - H(S|\hat{S}, W) = I(S; \hat{S}, W). \quad (20)$$

Proof. We write $I(S; \hat{S}|W) = H(S|W) - H(S|\hat{S}, W)$. Due to the ZL property it holds that $H(S|W) = H(S)$. \square

The mutual information $I(S; \hat{S}|W)$ can be seen as a function of the system parameters p_0, \dots, p_{N-1} . These parameters completely fix the **Gen** and **Rec** procedures. (The λ , σ_X and σ_R are given by nature and cannot be chosen). Hence we want to determine how to set vector $(p_\alpha)_{\alpha \in \mathcal{S}}$ as a function of λ , σ_X , σ_R so as to maximize our target function. Unfortunately, $I(S; \hat{S}|W)$ depends on the p_α parameters in a very complicated way. The **Gen** is simple enough, but the **Rec** procedure has decision boundaries $\tau_{\alpha w}$ (12) that depend on p_0, \dots, p_{N-1} not only directly but also via the $\xi_{\alpha w}$ points as specified in (6); this dependence is quite convoluted as the $\xi_{\alpha w}$ invoke the non-smooth stepwise function Q as well as the nonlinear F^{inv} . Analytic maximisation of $I(S; \hat{S}|W)$ is intractable. It is clear, however, that a maximum must exist. Consider the ZLFE at fixed $N \geq 3$. Not all intervals A_α have equal width, which leads to unequal probabilities for jumping from one interval to another due to noise. Making the narrowest intervals slightly broader reduces the reconstruction error probability (with a positive effect on our target function) and the entropy of S (with a negative effect). It is intuitively clear that at large σ_R the effect of reconstruction errors weighs more heavily than the $H(S)$ effect; then we expect a nontrivial maximum at a p_α setting different from the FE’s $p_\alpha = 1/N$. The numerics in Section 4 show that this is indeed the case.

For the efficiency of the numerical optimisation we now look for a simple form in which to represent $I(S; \hat{S}|W)$. We introduce the following notation,

$$\Upsilon_{\hat{s}|sw} \stackrel{\text{def}}{=} \Pr[\hat{S} = \hat{s}|S = s, W = w] = \int_{\tau_{\hat{s}w}}^{\tau_{\hat{s}+1,w}} \psi(y|\xi_{sw}) dy = V(\tau_{\hat{s}+1,w} - \lambda\xi_{sw}) - V(\tau_{\hat{s}w} - \lambda\xi_{sw}). \quad (21)$$

We can express the mutual information entirely in terms of the p_α and $\Upsilon_{\hat{s}|sw}$ parameters.

Lemma 3.7. For the ZLHDS the mutual information can be written as

$$I(S; \hat{S}|W) = \sum_{s=0}^{N-1} \sum_{\hat{s}=0}^{N-1} \int_0^1 dw p_s \Upsilon_{\hat{s}|sw} \log \frac{\Upsilon_{\hat{s}|sw}}{\sum_{\beta=0}^{N-1} p_\beta \Upsilon_{\hat{s}|\beta w}}. \quad (22)$$

Proof.

$$\begin{aligned} I(S; \hat{S}|W) &= \mathbb{E}_{s\hat{s}w} \log \frac{\Pr[S = s, \hat{S} = \hat{s}|W = w]}{\Pr[S = s|W = w]\Pr[\hat{S} = \hat{s}|W = w]} \\ &= \mathbb{E}_w \sum_{s, \hat{s}=0}^{N-1} \Pr[S = s|W = w] \Upsilon_{\hat{s}|sw} \log \frac{\Pr[S = s, \hat{S} = \hat{s}|W = w]}{\Pr[S = s|W = w]\Pr[\hat{S} = \hat{s}|W = w]}. \end{aligned} \quad (23)$$

In the last line we used the chain rule $\Pr[S = s, \hat{S} = \hat{s}, W = w] = \mathbb{E}_w \Pr[S = s | W = w] \Upsilon_{\hat{s}|sw}$. Next we use $\mathbb{E}_w(\cdots) = \int_0^1 dw(\cdots)$ as implied by Lemma 3.1, and $\Pr[S = s | W = w] = p_s$ by the ZL property. Finally we apply these rules, and $\Pr[\hat{S} = \hat{s} | W = w] = \sum_s p_s \Upsilon_{\hat{s}|sw}$, inside the logarithm. \square

3.3 Reconstruction errors

While we are mainly interested in the mutual information, we also care about the practical implementation aspects of the second-stage HDS. The second-stage HDS typically employs an Error-Correcting Code (ECC). If the output of the first-stage HDS has a high bit error rate, this causes problems for the ECC. In our numerics we keep track of the error rate.

We write $P_{\text{err}} = \Pr[\hat{S} \neq Q(X)]$ for the overall probability that \hat{S} is not equal to S . This is an averaged quantity, i.e. averaged over X . For fixed x we have

$$\Pr[\hat{S} = Q(X) | X = x] = \Upsilon_{Q(x)|Q(x),g(x)}. \quad (24)$$

Averaging over x gives

$$1 - P_{\text{err}} = \mathbb{E}_x \Pr[\hat{S} = Q(X) | X = x] = \mathbb{E}_x \Upsilon_{Q(x)|Q(x),g(x)} = \sum_{s \in \mathcal{S}} p_s \int_0^1 dw \Upsilon_{s|sw}. \quad (25)$$

In the last step we used that x uniquely maps to $(s, w) = (Q(x), g(x))$. Eq. (25) together with (21) is the most convenient way to analytically express the reconstruction error probability.

We consider the case where s is encoded as a Gray code. This is a well known technique to reduce the number of bit flips when a reconstruction error occurs. Table 1 lists the Gray code that we use. (Other, equivalent, encodings are possible.) We will look at $N \in \{3, 4, 5, 6\}$. The length of the Gray code is $\lceil \log N \rceil$ bits.

s	1st bit	2nd bit	3rd bit
0	0	0	0
1	0	0	1
2	0	1	1
3	0	1	0
4	1	1	0
5	1	1	1

Table 1: *Three-bit Gray code used for $N = 5$ and $N = 6$. The highlighted cell shows the two-bit Gray code that we use for $N = 3$ and $N = 4$.*

The Bit Error Rate (BER) is given by

$$\text{BER} = \frac{\mathbb{E}[\# \text{ bit errors}]}{\lceil \log N \rceil} = \frac{1}{\lceil \log N \rceil} \sum_{t=0}^{\lceil \log N \rceil} t \Pr[\# \text{ bit errors} = t]. \quad (26)$$

We introduce the following notation,

$$\Delta_{\hat{s}|s} \stackrel{\text{def}}{=} \Pr[\hat{S} = \hat{s} | S = s] = \mathbb{E}_w \Upsilon_{\hat{s}|sw}. \quad (27)$$

All the probabilities in (26) can be calculated in terms of the $\Delta_{\hat{s}|s}$ probabilities. The details are given in the Appendix.

4 Numerical results

We present numerical results for the optimization described in Section 3, for $N \in \{3, 4, 5, 6\}$. We consider a Gaussian source X and Gaussian noise. (This is already a rather accurate model for Coating PUFs [16]). Without loss of generality we set $\sigma_X = 1$. Only the ratio σ_R/σ_X matters. We consider the two cases defined in Section 2.3: *perfect enrollment* and *identical conditions*. We implemented (22) in Wolfram Mathematica 10.2 as a symbolic function. We used the built-in function `FindMaximum` to obtain optimum values for p_0, \dots, p_{N-1} . In order to reduce the dimension of the search space we imposed the symmetry $p_{N-1-\alpha} = p_\alpha$ by hand.

Fig. 4 shows $I(S; \hat{S}|W)$ versus P_{err} for various σ_R .

- When σ_R is small, the optimum setting of the HDS is close to the FE setting $p_\alpha = 1/N$, and it is clearly visible that increasing N has a very large benefit for the mutual information.
- For somewhat larger σ_R , there is a clear difference between the optimised HDS and the FE. For example, in the $\lambda = 1$ graph at $\sigma_R = 0.25$ we see that at $N = 6$ the transition from FE to HDS brings a modest improvement of the mutual information and a reduction of P_{err} from $\approx 23\%$ to $\approx 10\%$. The reduced P_{err} means that the ECC in the second stage is much easier to implement for the HDS than for the FE.
- At $\sigma_R > 0.5$ the noise is so bad that the HDS and the FE perform almost equally badly (though the HDS is always slightly better). Increasing N improves the mutual information only slightly, and at the cost of a large increase in P_{err} .

Fig. 5 shows the same data, but with the BER on the horizontal axis. The ‘zigzag’ at the transition from $N = 4$ to $N = 5$ occurs because the Gray code jumps from a 2-bit representation of s to a 3-bit representation, with little noise in the first of the three bits.

Fig. 6 shows the BER as a function of σ_R/σ_X . The curves for $N = 4$ and $N = 5$ cross each other; this causes the ‘zigzag’ in Fig. 5. The graphs of P_{err} as a function of σ_R/σ_X (Fig. 7) are much smoother. For completeness Fig. 8 plots the BER versus P_{err} . The relation is clearly nonlinear.

Fig. 9 shows the optimal values of p_0, \dots, p_{N-1} for the perfect enrollment case ($\lambda = 1$). At $\sigma_R = 0$ it holds that $p_\alpha = 1/N$ for all α , which is the FE configuration. When σ_R increases, the outer regions A_0, A_{N-1} shrink while the central region(s) become broader. Then at some point this trend reverses. At very large σ_R the p_α values stabilize, but not in the FE configuration.

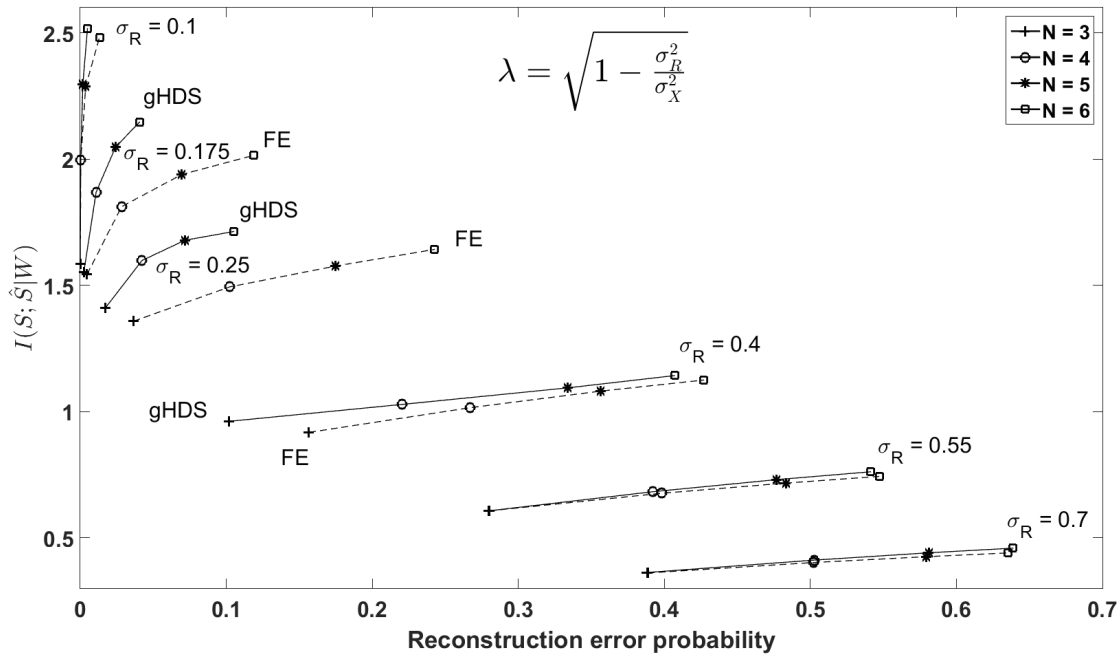
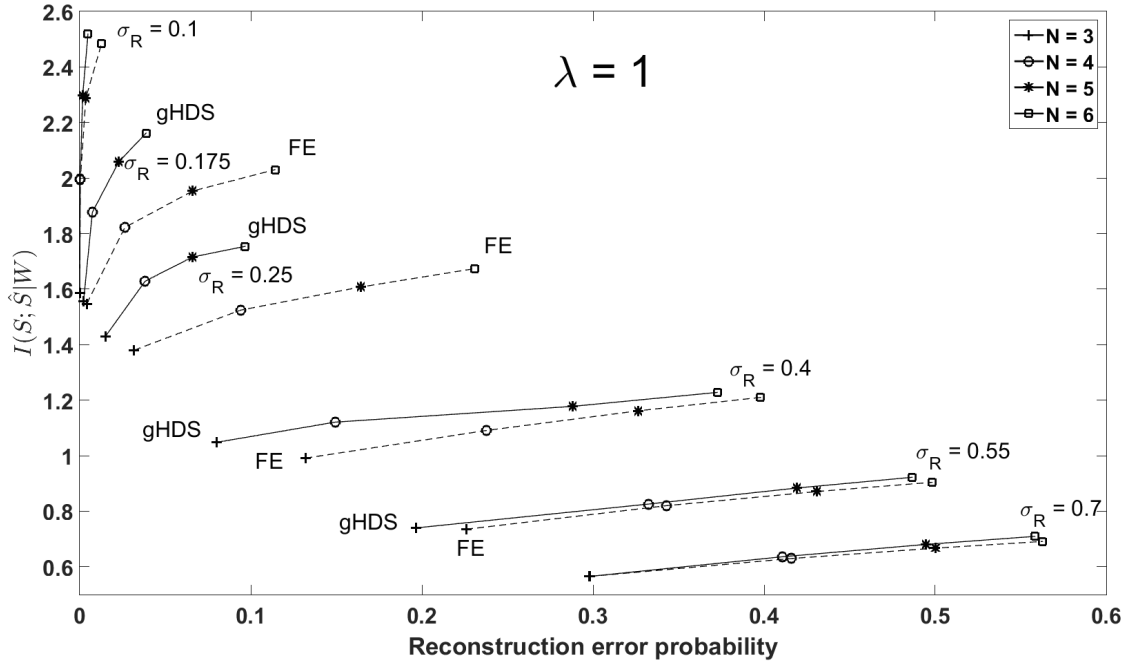


Figure 4: Mutual information versus P_{err} for perfect enrollment (upper figure) and identical conditions (lower figure). At fixed σ_R , data points for the general HDS are connected with a solid line, while a dashed line corresponds to the FE.

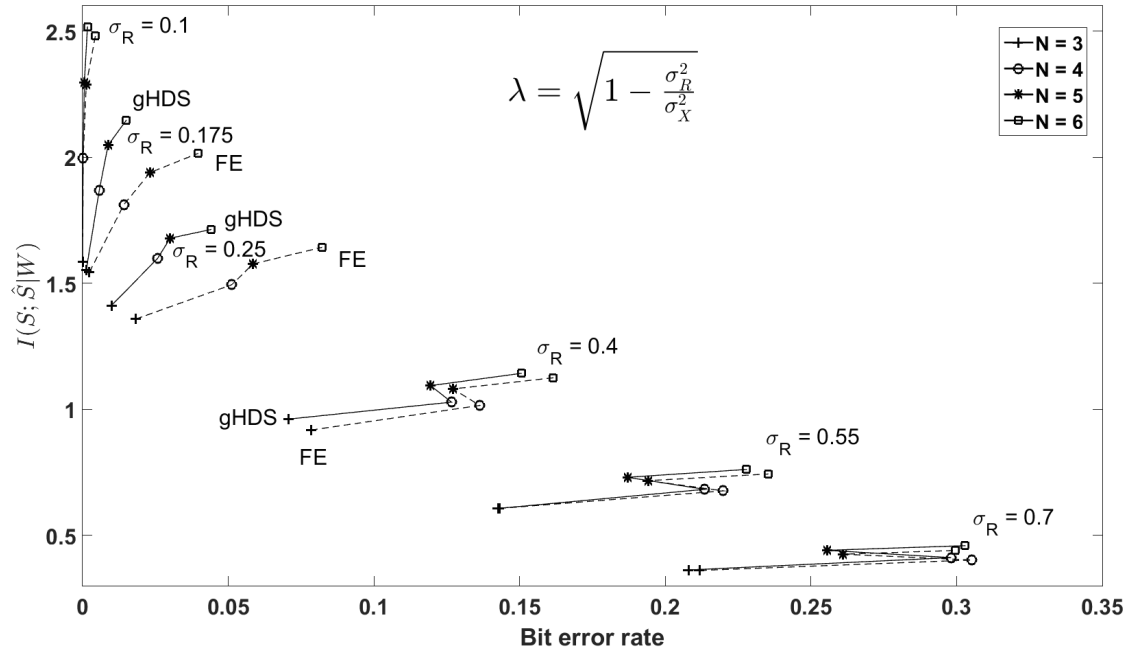
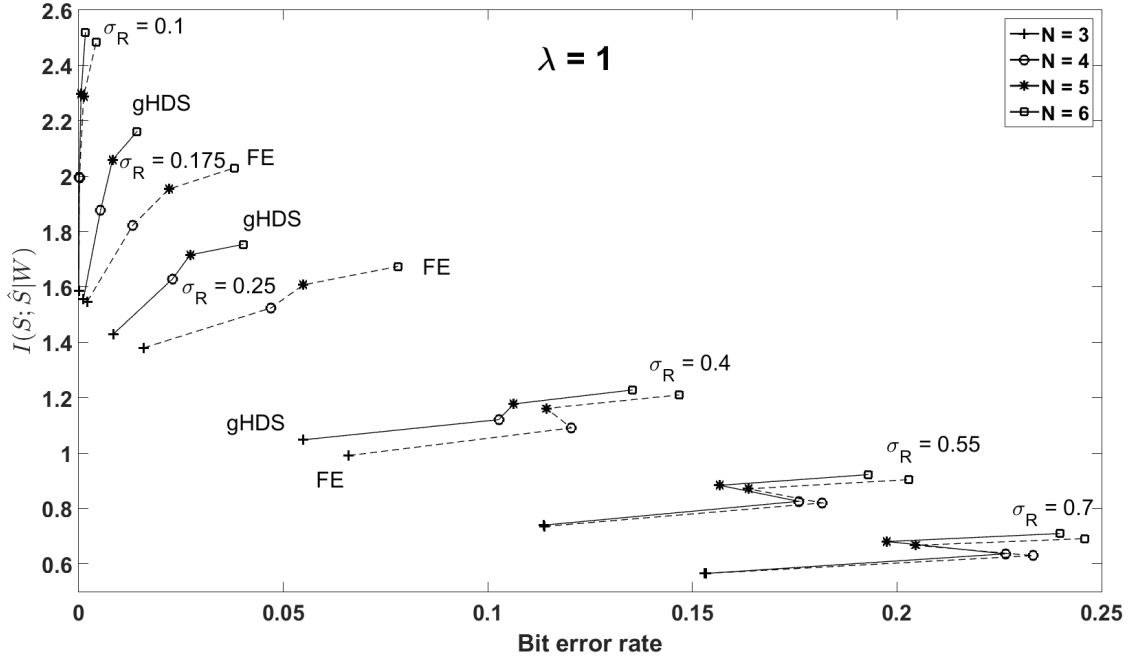


Figure 5: *Mutual information versus BER for perfect enrollment (upper figure) and identical conditions (lower figure). At fixed σ_R , data points for the general HDS are connected with a solid line, while a dashed line corresponds to the FE.*

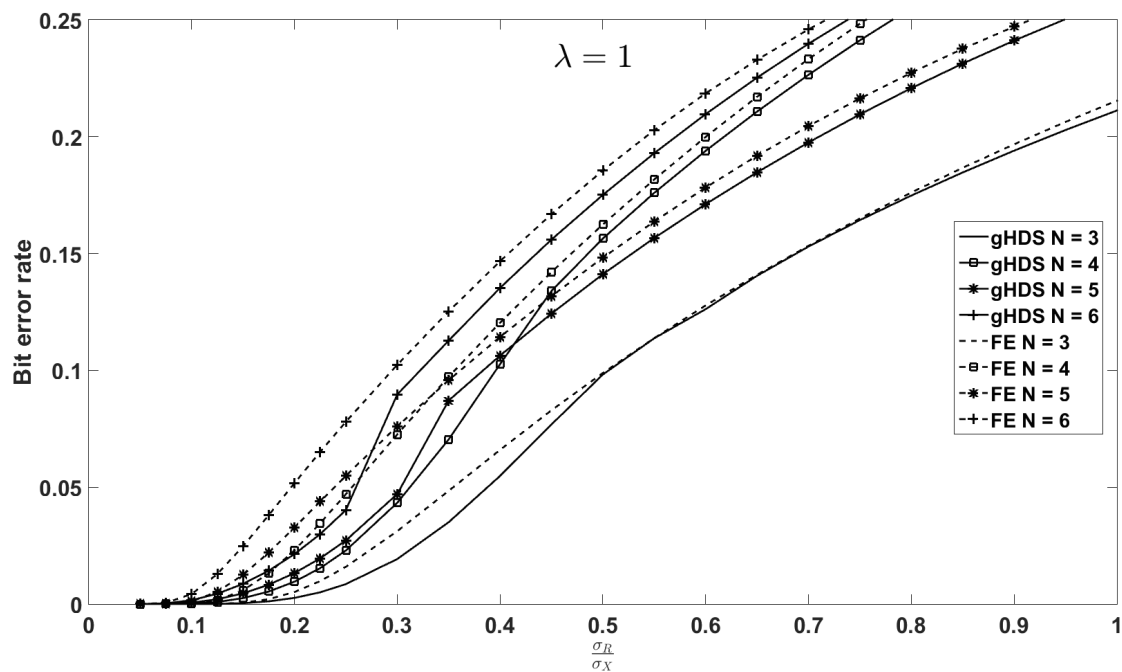


Figure 6: Bit Error Rate as a function of the noise parameter σ_R/σ_X . Perfect enrollment

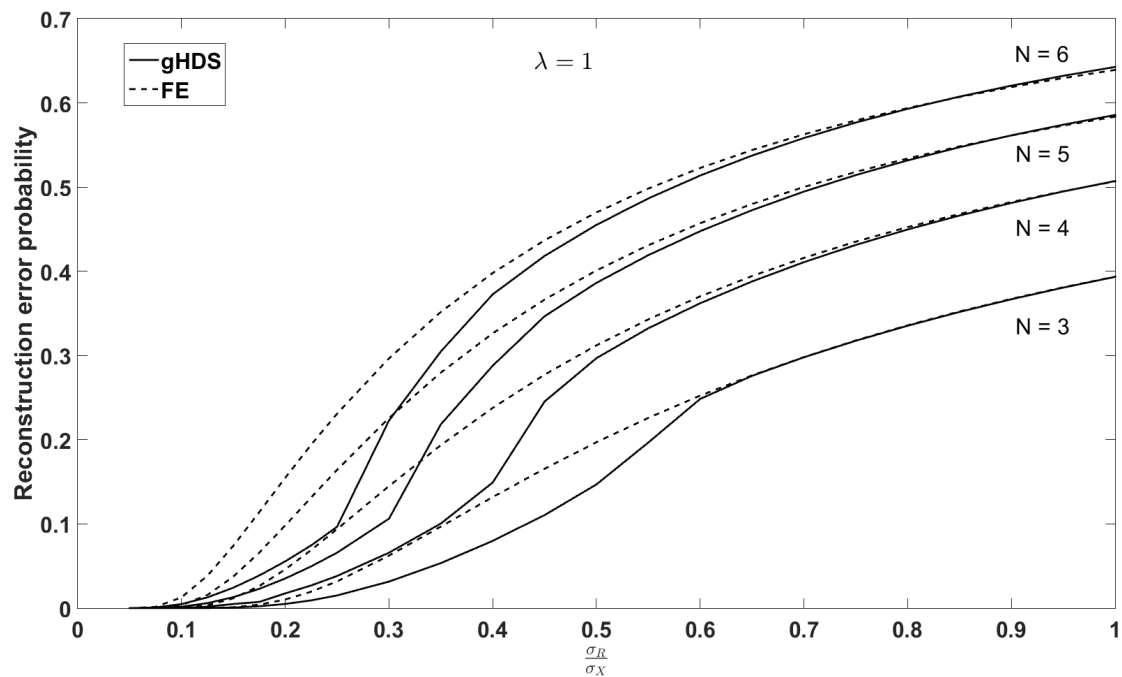


Figure 7: P_{err} as a function of the noise parameter σ_R/σ_x . Perfect enrollment.

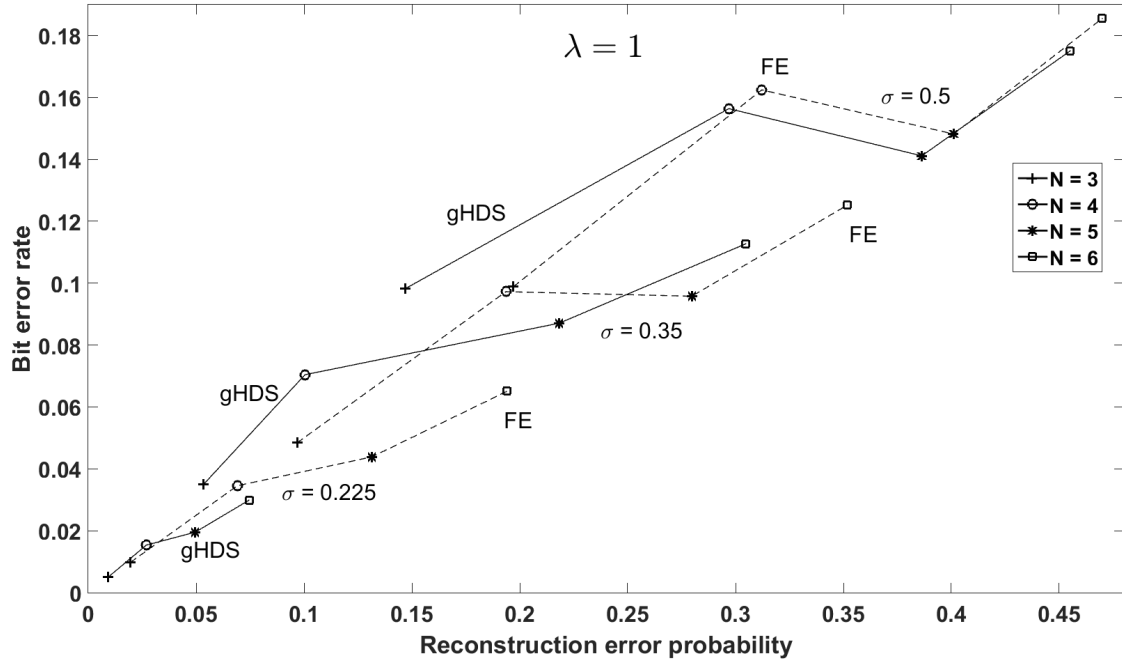


Figure 8: *BER versus reconstruction error probability P_{err} . Perfect enrollment. At given σ_R , data points for the HDS are connected with a solid line, while a dashed line corresponds to the FE.*

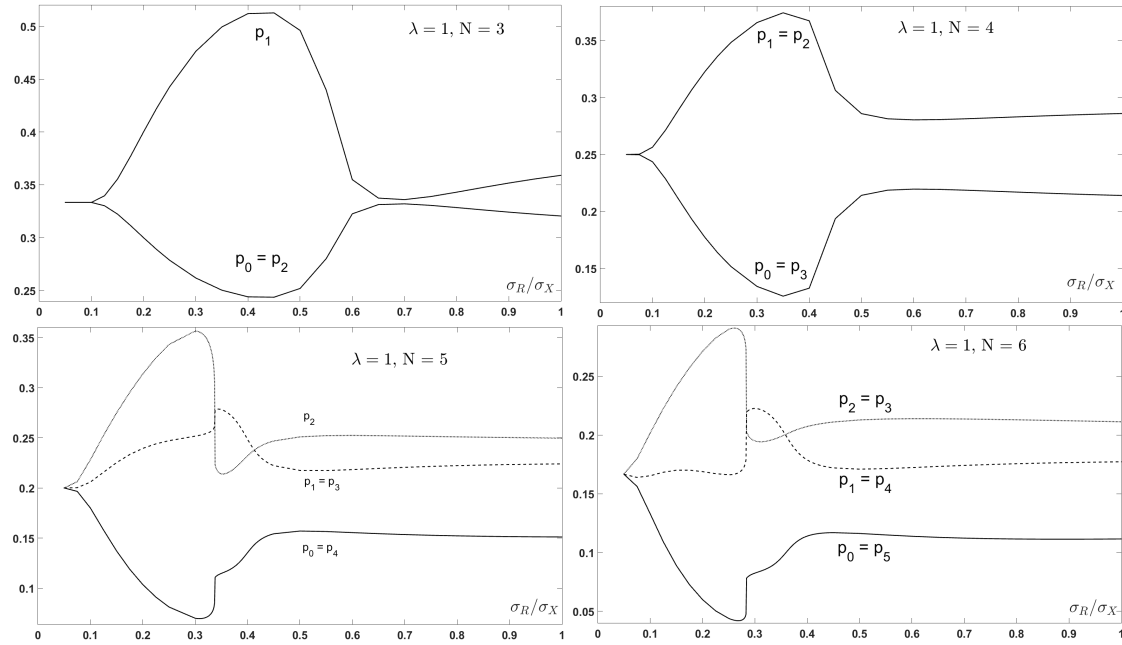


Figure 9: *The p_α values as a function of the noise parameter σ_R/σ_X , for $\lambda = 1$, $N = 3, 4, 5, 6$.*

5 Summary

We have extended the results of de Groot et al. [5] in the case of non-equiprobable quantisation intervals. Lemma 3.3 gives the recipe for finding the optimal decision boundaries used in **Rec**. The result for Gaussian and Lorentzian noise is given in Theorems 3.4 and 3.5.

We have studied the mutual information $I(S; \hat{S}|W)$, which is an upper bound on the amount of secret key material that can be robustly extracted from X . The mutual information is most conveniently expressed in terms of the p_s and $\Upsilon_{\hat{s}|sw}$ parameters (22). The dependence of the $\Upsilon_{\hat{s}|sw}$ on p_0, \dots, p_{N-1} is so complicated that optimisation of $I(S; \hat{S}|W)$ cannot be done analytically. The figures in Section 4 show the results of numerical optimisation in a simple model where the source and the noise are Gaussian. Such a model is reasonably accurate for Coating PUFs. For every combination $(N, \sigma_R/\sigma_X)$ the optimized ZLHDS clearly performs better than the ZLFE in terms of both mutual information and reconstruction error rate. The reduction in P_{err} is substantial. This makes the design of a second-stage HDS much more practical, since it makes is easier to implement an ECC that can cope with the bit errors introduced by reconstruction errors.

As future work we will apply the numerical optimisation to different source distributions, matching e.g. biometric data.

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Appendix: Bit error rates

We list expressions for the BER (26) in terms of the $\Delta_{\hat{s}|s}$ probabilities (27), when the Gray code of Table 1 is used. We assume a symmetric source pdf f and symmetric noise. As a result the optimal p_α values have the symmetry $p_{N-1-\alpha} = p_\alpha$, and there is a large number of symmetries between the Δ_{\dots} values, $\Delta_{N-1-\hat{s}|N-1-s} = \Delta_{\hat{s}|s}$.

N	$N \cdot \text{BER}$
3	$2p_0(\Delta_{1 0} + 2\Delta_{2 0}) + 2p_1\Delta_{2 1}$
4	$2p_0(\Delta_{1 0} + \Delta_{3 0} + 2\Delta_{2 0}) + 2p_1(\Delta_{0 1} + \Delta_{2 1} + 2\Delta_{3 1})$
5	$2p_0(\Delta_{1 0} + \Delta_{3 0} + 2\Delta_{2 0} + 2\Delta_{4 0}) + 2p_1(\Delta_{0 1} + \Delta_{2 1} + 2\Delta_{3 1} + 3\Delta_{4 1}) + 2p_2(\Delta_{1 2} + 2\Delta_{0 2})$
6	$2p_0(\Delta_{1 0} + \Delta_{3 0} + 2\Delta_{2 0} + 2\Delta_{4 0} + 3\Delta_{5 0}) + 2p_1(\Delta_{0 1} + \Delta_{2 1} + 2\Delta_{3 1} + 2\Delta_{5 1} + 3\Delta_{4 1}) + 2p_2(\Delta_{1 2} + \Delta_{3 2} + 2\Delta_{0 2} + 2\Delta_{4 2})$

The p -index in this table runs only to $\lceil N/2 \rceil - 1$ because of the $\alpha \leftrightarrow N - 1 - \alpha$ symmetry; this also gives rise to the factor 2 in front of each p_α . Inside the parentheses, the numerical factor in front of each Δ indicates the number of bit flips that occur due to that specific transition.

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