

Ideal Multilinear Maps based on Ideal Lattices

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Abstract. Cryptographic multilinear maps have many applications, such as multipartite key exchange and software obfuscation. However, the encodings of three current constructions are “noisy” and their multilinearity levels are fixed and bounded in advance. In this paper, we describe a candidate construction of ideal multilinear maps by using ideal lattices, which supports arbitrary multilinearity levels. The security of our construction depends on new hardness assumptions.

Keywords. Ideal multilinear maps, ideal lattices, multipartite Diffie–Hellman key exchange, witness encryption, zeroizing attack

1 Introduction

The construction of multilinear maps has been a long-standing open problem since 2003. Many studies on the applications of bilinear maps, such as [SOK00, Jou00, BF01, Sma03], have influenced research on cryptographic multilinear maps [BS03, RS09, PTT10, Rot13]. Boneh and Silverberg [BS03] first introduced the notion of multilinear maps, which are an extension of bilinear maps. However, they suspected that such maps come from the realm of algebraic geometry.

Garg, Gentry, and Halevi (GGH) recently described the first candidate construction of multilinear maps from ideal lattices [GGH13]. The GGH construction, whose encodings are randomized with noise and bounded with a fixed maximum degree, is different from the ideal multilinear maps envisioned by Boneh and Silverberg [BS03]. Construction security depends on the new hardness assumptions of GCDH/GDDH, which provided extensive cryptanalysis in [GGH13]. Langlois, Stehlé, and Steinfeld [LSS14] presented a variant of GGH by reanalyzing its re-randomization process to improve its efficiency. However, by using the zeroizing attack proposed by [GGH13], the application of multipartite key exchange (MPKE) based on GGH was broken by Hu and Jia [HJ15a].

One line of work focused on new constructions of multilinear maps. Following the GGH framework, the second candidate construction of multilinear maps was presented by Coron, Lepoint, and Tibouchi (CLT) [CLT13]. The CLT construction changes from working over ideal lattices to working over integers and is implemented by using many heuristic optimization techniques. However, by using the zeroizing attack, the CLT construction was broken by Cheon et al. [CHL+14]. Boneh, Wu, and Zimmerman [BWZ14] and Garg, Gentry, Halevi, and Zhandry [GGHZ14] proposed two independent approaches to fix the CLT construction [CLT13]. However, Coron, Lepoint, and Tibouchi [CTL14] showed that two fixes can be broken by using an extension of the attack proposed by Cheon et al. [CHL+14]. Recently, Coron, Lepoint, and Tibouchi [CTL15] presented a new variant of CLT by modifying the zero-testing parameter.

The third candidate construction of graph-induced multilinear maps from lattices was proposed by Gentry, Gorbunov, and Halevi [GGH15]. The security of their construction depends on new hardness assumptions and cannot be reduced to LWE or other classic hardness assumptions.

Another line of work focused on the new cryptographic applications of multilinear maps: witness encryption [GGS+13], general program obfuscation [GGH+13b, Zim15], function encryption [GGH+13b], and other applications [GGH+13a, BZ14].

However, all known constructions are noisy multilinear maps. These noisy encodings restrict the number of operations that can be performed and further restrict their applications. In this study, we propose a candidate construction of ideal multilinear map that supports any multilinearity degree.

1.1 Our Results

Our main contribution describes a candidate construction of ideal multilinear map by using ideal lattices. The security of our construction depends on new hardness assumptions. The starting point of our work is that, given $k+1$ ring elements $\mathbf{y}_i = \mathbf{a}_i \mathbf{g} + \mathbf{t}_i \mathbf{f}$, all products of the form $\mathbf{b}_j = \mathbf{a}_j \prod_{i \neq j} \mathbf{y}_i \bmod \mathbf{f}$ are identical to element $\mathbf{g}^k \prod_{i=1}^{k+1} \mathbf{a}_i \bmod \mathbf{f}$.

Our construction includes two layers, namely, the inner layer that works over polynomial rings $R = \mathbb{Z}[x]/(x^n + 1)$ and $R_q = R/qR$ and the outer layer that works over matrix ring $\mathbb{Z}^{2n \times 2n}$.

First, we select secret short ring elements $\mathbf{f}, \mathbf{g} \in R$. Given an element $\mathbf{a} \in R$, the level-1 encoding is $\mathbf{c} = (\mathbf{a}\mathbf{g}) \bmod \mathbf{f}$. However, one can compute $\mathbf{g} = (\mathbf{c}/\mathbf{a}) \bmod \mathbf{f}$ when \mathbf{f} , \mathbf{a} , and \mathbf{c} are known. To prevent this simple division attack, we replace \mathbf{f} with $\mathbf{q} = \mathbf{q}_0 \mathbf{f}$ and add noise $\mathbf{r}\mathbf{f}$ to the encoding \mathbf{c} , where \mathbf{q}_0 is a short ring element. Thereafter, we transform \mathbf{a} and \mathbf{c} into new encodings $\mathbf{x} = (\mathbf{a} + \mathbf{e}\mathbf{f}) \bmod \mathbf{q}$ and $\mathbf{y} = \mathbf{c} + \mathbf{t}\mathbf{f} = (\mathbf{a}\mathbf{g} + \mathbf{t}\mathbf{f}) \bmod \mathbf{q}$, where \mathbf{e} and \mathbf{t} are short random elements drawn from R . In our construction, we regard \mathbf{g}^k as level- k encoding and $\mathbf{r}\mathbf{f}$ as random encoding of zero.

The aforementioned encodings support the addition and multiplication operations. Given level-1 encodings $\mathbf{u}_i = \mathbf{c}_i + \mathbf{r}_i \mathbf{f} = (\mathbf{a}_i \mathbf{g} + \mathbf{r}_i \mathbf{f}) \bmod \mathbf{q}$, $i = 1, 2$, we derive the following expressions:

$$\begin{aligned} \mathbf{u}_1 + \mathbf{u}_2 &= (\mathbf{c}_1 + \mathbf{r}_1 \mathbf{f}) + (\mathbf{c}_2 + \mathbf{r}_2 \mathbf{f}) \\ &= ((\mathbf{a}_1 + \mathbf{a}_2)\mathbf{g} + (\mathbf{r}_1 + \mathbf{r}_2)\mathbf{f}) \bmod \mathbf{q}, \\ \mathbf{u}_1 \cdot \mathbf{u}_2 &= (\mathbf{c}_1 + \mathbf{r}_1 \cdot \mathbf{f}) \cdot (\mathbf{c}_2 + \mathbf{r}_2 \cdot \mathbf{f}) \\ &= ((\mathbf{c}_1 \cdot \mathbf{c}_2) + (\mathbf{c}_1 \mathbf{r}_2 + \mathbf{c}_2 \mathbf{r}_1 + \mathbf{r}_1 \mathbf{r}_2 \mathbf{f})\mathbf{f}) \bmod \mathbf{q} \\ &= ((\mathbf{a}_1 \mathbf{a}_2)\mathbf{g}^2 + (\mathbf{c}_1 \mathbf{r}_2 + \mathbf{c}_2 \mathbf{r}_1 + \mathbf{r}_1 \mathbf{r}_2 \mathbf{f})\mathbf{f}) \bmod \mathbf{q} \end{aligned}$$

One can be decided whether an encoding is 0. Given $\mathbf{q} = \mathbf{q}_0 \mathbf{f}$, all level- k encodings can have the form $(\mathbf{a}\mathbf{g}^k + \mathbf{r}\mathbf{f}) \bmod \mathbf{q}$. That is, the norm of any encoding is less than $\|\mathbf{q}\|_\infty$, where $\|\mathbf{q}\|_\infty$ is the maximum norm of \mathbf{q} . Thus, to decide whether \mathbf{u} is an encoding of zero, a zero-testing parameter $\mathbf{p}_{zt} = \mathbf{h} \cdot (\mathbf{f}^{-1} \bmod q) \bmod q$ is included in the public parameters, where $q \gg \|\mathbf{q}\|_\infty$. If the norm of $[\mathbf{p}_{zt} \cdot \mathbf{u}]_q$ is small, \mathbf{u} is an encoding of zero; otherwise, \mathbf{u} is an encoding of a nonzero element.

However, this construction can be broken when \mathbf{q} and τ pairs of encodings $\mathbf{x}_i = (\mathbf{a}_i + \mathbf{e}_i \mathbf{f}) \bmod \mathbf{q}$ and $\mathbf{y}_i = (\mathbf{a}_i \mathbf{g} + \mathbf{t}_i \mathbf{f}) \bmod \mathbf{q}$ are included in the public parameters. Because one can compute the basis of \mathbf{f} , the inverse of $Rot(\mathbf{a}_1)$, and the inverse of $Rot(\mathbf{g})$ by cross-multiplying $\mathbf{x}_i \mathbf{y}_{i_2} - \mathbf{x}_{i_2} \mathbf{y}_i = \mathbf{r}_{i_1, i_2} \mathbf{f}$ of $\mathbf{x}_i, \mathbf{y}_i$, where notation $Rot(\mathbf{v})$ denotes the anti-cyclic matrix of $\mathbf{v} \in R$.

To prevent the cross-multiplying attack, we introduce random matrices to enclose the ring elements in the aforementioned construction. We first select two short random

unimodular matrices \mathbf{T} , \mathbf{S} . Then we transform \mathbf{x}_i , \mathbf{y}_i into $\mathbf{X}_i = \mathbf{S}Rot(\mathbf{x}_i)\mathbf{S}^{-1}$, $\mathbf{Y}_i = \mathbf{T}^{-1}Rot(\mathbf{y}_i)\mathbf{T}$, and \mathbf{p}_{zt} into $\mathbf{P}_{zt} = [\mathbf{S}Rot(\mathbf{p}_{zt})\mathbf{T}]_q$. For modulo \mathbf{q} , we generate two list encodings of zero $\mathbf{M}_j = \mathbf{S}Rot(\mathbf{m}_j\mathbf{q})\mathbf{S}^{-1}$, $\mathbf{N}_j = \mathbf{T}^{-1}Rot(\mathbf{n}_j\mathbf{q})\mathbf{T}$, $j \in \llbracket n \rrbracket$, where \mathbf{m}_j and \mathbf{n}_j are short random elements drawn from R . We represent \mathbf{M} and \mathbf{N} as $n^2 \times n$ matrices whose column vectors are \mathbf{M}_j and \mathbf{N}_j , respectively. Thus, we can easily generate encodings by scalar product; that is, sampling τ small random integers d_i , a level-1 encoding is generated as $\mathbf{U} = \sum_{i=1}^{\tau} d_i \mathbf{Y}_i \bmod \mathbf{N}$, and the level-0 encoding corresponding to \mathbf{U} is $\mathbf{D} = \sum_{i=1}^{\tau} d_i \mathbf{X}_i \bmod \mathbf{M}$. Similarly, whether the encoding \mathbf{U} is an encoding of zero can be determined by computing $\mathbf{V} = [\mathbf{E} \cdot \mathbf{P}_{zt} \cdot \mathbf{U}]_q$ and checking the norm of \mathbf{V} , where $\mathbf{E} = \sum_{i=1}^{\tau} r_i \mathbf{X}_i \bmod \mathbf{M}$ and can be set identity matrix. By cross-multiplication, an adversary can obtain $\mathbf{V} = [\mathbf{E} \cdot \mathbf{P}_{zt} \cdot \mathbf{U}]_q = \mathbf{S}Rot(\mathbf{r})\mathbf{T}$, which is not reduced modulus q .

Very recently, Pellet-Mary and Stehlé [PS15] presented an efficient attack for our construction using random matrix. This is because one can compute $\mathbf{Y}_i^{-1} \bmod \mathbf{N}_j$ if $\det(\mathbf{Y}_i)$ and $\det(\mathbf{N}_j)$ are coprime. Without loss of generality, assume $\mathbf{C} = \mathbf{Y}_1^{-1} \bmod \mathbf{N}_1$. Given the public parameters par and $k+1$ level-1 encodings $\mathbf{U}_r, r=0, \dots, k$, then one computes $\mathbf{U} = (\prod_{r=0}^k \mathbf{U}_r) \bmod \mathbf{N}$, $\mathbf{U}' = (\mathbf{C} \prod_{r=0}^k \mathbf{U}_r) \bmod \mathbf{N}$, and $\mathbf{V} = [\mathbf{X}_1 \cdot \mathbf{P}_{zt} \cdot \mathbf{U}']_q$. Let \mathbf{D}_0 be the level-0 encoding corresponding to \mathbf{U}_0 . It is not difficult to verify that the significant bits of each entry in \mathbf{V} are same as that in $\mathbf{V}_0 = [\mathbf{D}_0 \cdot \mathbf{P}_{zt} \cdot \mathbf{U}]_q$. To avoid this attack, we multiply ring elements in the inner layer by a random short element $\mathbf{h} \in R$. In this case, one can no longer find an invertible encoding over modulo \mathbf{N}_j .

However, the encodings $\mathbf{X}_i = \mathbf{S}Rot(\mathbf{x}_i)\mathbf{S}^{-1}$ works over the integers, \mathbf{X}_i and $Rot(\mathbf{x}_i)$ are similar matrices. According to Stehlé [Ste15], one can obtain \mathbf{S} by computing the singular values of \mathbf{X}_i and recovering ring element \mathbf{x}_i . To thwart this attack, we extend encodings from single ring element to multiple ring element. That is, we generate new encodings and the zero-testing parameter as follows:

$$\mathbf{X}_i = \mathbf{S} \begin{pmatrix} Rot(\mathbf{x}_{1,1,i}) & Rot(\mathbf{x}_{1,2,i}) \\ Rot(\mathbf{x}_{2,1,i}) & Rot(\mathbf{x}_{2,2,i}) \end{pmatrix} \mathbf{S}^{-1}, \mathbf{Y}_i = \mathbf{T}^{-1} \begin{pmatrix} Rot(\mathbf{y}_{1,1,i}) & Rot(\mathbf{y}_{1,2,i}) \\ Rot(\mathbf{y}_{2,1,i}) & Rot(\mathbf{y}_{2,2,i}) \end{pmatrix} \mathbf{T},$$

$$\mathbf{P}_{zt} = \left[\mathbf{S} \begin{pmatrix} Rot(\mathbf{p}_{zt,1}) & \mathbf{r}_1 \\ \mathbf{r}_2 & Rot(\mathbf{p}_{zt,2}) \end{pmatrix} \mathbf{T} \right]_q \text{ with short ring elements } \mathbf{r}_1, \mathbf{r}_2.$$

For using as modulo the encodings $\mathbf{M}_j, \mathbf{N}_j$ of zero, we similarly transform them into the above form. This briefly describes our construction of ideal multilinear maps.

Our second contribution is to describe two applications using our ideal construction: multipartite Diffie–Hellman key exchange protocol (MPKE), which supports any number of participants, and witness encryption scheme (WE). In our construction, the size of modulus q does not depend on the multilinearity levels. Thus, the MPKE and WE using our ideal multilinear maps are practical.

1.2 Organization

The remainder of this paper is organized as follows. We recall several preliminaries in Section 2. We describe the construction of ideal multilinear maps that use ideal lattices in Section 3. We extend our construction to asymmetric and commutative variants in Section 4. Finally, we propose two applications by using our ideal construction in Section 5.

2 Preliminaries

2.1 Notations

We denote $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ as the integer ring, rational number field, and real number field, respectively. The vectors and matrices are denoted in bold. For a positive integer k , we let $\llbracket k \rrbracket = \{1, 2, \dots, k\}$. For n the power of 2, we let $R = \mathbb{Z}[x]/\langle x^n + 1 \rangle$, $R_q = R/qR$, and $\mathbb{k} = \mathbb{Q}[x]/\langle x^n + 1 \rangle$. We denote an element of the polynomial ring as a coefficient vector for simplicity. For the element $\mathbf{a} \in R$, we denote $\|\mathbf{a}\|_\infty$ ($\|\mathbf{a}\|$ for short) as the maximum norm of \mathbf{a} .

Throughout this study, we use the absolute minimum residual system, that is, $[a]_q = a \bmod q \in (-q/2, q/2]$. Similarly, notation $[\mathbf{a}]_q$ denotes each entry (or each coefficient) $a_i \in (-q/2, q/2]$ of \mathbf{a} .

2.2 Lattices and Ideal Lattices

The n -dimensional full-rank lattice $L \subset \mathbb{R}^n$ is the set of all integer linear combinations $\sum_{i=1}^n y_i \mathbf{b}_i$ of n linearly independent vector $\mathbf{b}_i \in \mathbb{R}^n$. If we arrange the vectors of \mathbf{b}_i as the columns of matrix $\mathbf{B} \in \mathbb{R}^{n \times n}$, then $L = \{\mathbf{B}\mathbf{y} : \mathbf{y} \in \mathbb{Z}^n\}$. We can state that \mathbf{B} spans L if \mathbf{B} is a basis for L . For basis \mathbf{B} of the lattice, we denote its parallelization cell as $P(\mathbf{B}) = \{\mathbf{B}\mathbf{z} \mid \mathbf{z} \in \mathbb{R}^n, \forall i : -1/2 \leq z_i < 1/2\}$. We let $\det(\mathbf{B})$ be the determinant of \mathbf{B} .

For elements $\mathbf{a}, \mathbf{g} \in R$, we let $I = \langle \mathbf{g} \rangle$ be the principal ideal lattice generated by \mathbf{g} , and $Rot(\mathbf{g}) = (\mathbf{g}, x \cdot \mathbf{g}, \dots, x^{n-1} \cdot \mathbf{g})$ the basis of R . We denote $[\mathbf{a}]_{\mathbf{g}}$ as the reduction of \mathbf{a} modulo $Rot(\mathbf{g})$, that is, $[\mathbf{a}]_{\mathbf{g}} \in P(Rot(\mathbf{g}))$ and $(\mathbf{a} - [\mathbf{a}]_{\mathbf{g}}) \in L(Rot(\mathbf{g}))$.

Given $\mathbf{c} \in \mathbb{R}^n$, $\sigma > 0$, we define $D_{L, \sigma, \mathbf{c}} = \rho_{\sigma, \mathbf{c}}(\mathbf{x}) / \rho_{\sigma, \mathbf{c}}(L)$ as the Gaussian distribution of lattice L , where $\mathbf{x} \in L$, $\rho_{\sigma, \mathbf{c}}(\mathbf{x}) = \exp(-\pi \|\mathbf{x} - \mathbf{c}\|^2 / \sigma^2)$, and $\rho_{\sigma, \mathbf{c}}(L) = \sum_{\mathbf{x} \in L} \rho_{\sigma, \mathbf{c}}(\mathbf{x})$. Thereafter, we write $D_{\mathbb{Z}^n, \sigma, 0}$ as $D_{\mathbb{Z}^n, \sigma}$ for simplicity. We denote a Gaussian sample as $\mathbf{x} \leftarrow D_{L, \sigma}$ (or $\mathbf{d} \leftarrow D_{I, \sigma}$) over the lattice L (or the ideal lattice I).

2.3 Multilinear Maps

Definition 2.1 (Multilinear Map [BS03]). For $k+1$ cyclic groups G_1, \dots, G_k, G_T of the same order p , a k -multilinear map $e : G_1 \times \dots \times G_k \rightarrow G_T$ has the following properties:

(1) Elements $\{g_j \in G_j\}_{j=1,\dots,k}$, index $j \in \llbracket k \rrbracket$, and integer $a \in \mathbb{Z}_p$ hold that

$$e(g_1, \dots, a \cdot g_j, \dots, g_k) = a \cdot e(g_1, \dots, g_k).$$

(2) Map e is a nondegenerate in the following sense: if elements $\{g_j \in G_j\}_{j=1,\dots,k}$

are generators of their respective groups, then $e(g_1, \dots, g_k)$ is a generator of G_T .

Definition 2.2 (k -Graded Encoding System [GGH13]). A k -graded encoding system over R is a set system of $S = \{S_j^{(\alpha)} \subset R : \alpha \in R, j \in \llbracket k \rrbracket\}$ with the following properties:

(1) For every index $j \in \llbracket k \rrbracket$, the sets $\{S_j^{(\alpha)} : \alpha \in R\}$ are disjoint.

(2) Binary operations “+” and “−” exist, such that every α_1, α_2 , every index $j \in \llbracket k \rrbracket$, and every $u_1 \in S_j^{(\alpha_1)}$ and $u_2 \in S_j^{(\alpha_2)}$ hold that $u_1 + u_2 \in S_j^{(\alpha_1 + \alpha_2)}$ and $u_1 - u_2 \in S_j^{(\alpha_1 - \alpha_2)}$, where $\alpha_1 + \alpha_2$ and $\alpha_1 - \alpha_2$ are the addition and subtraction operations in R , respectively.

(3) Binary operation “ \times ” exists, such that every α_1, α_2 , every index $j_1, j_2 \in \llbracket k \rrbracket$ with $j_1 + j_2 \leq k$, and every $u_1 \in S_{j_1}^{(\alpha_1)}$ and $u_2 \in S_{j_2}^{(\alpha_2)}$ hold that $u_1 \times u_2 \in S_{j_1 + j_2}^{(\alpha_1 \times \alpha_2)}$, where $\alpha_1 \times \alpha_2$ is the multiplication operation in R and $j_1 + j_2$ is the integer addition.

3 Ideal Multilinear Maps

In this section, we describe the construction of ideal multilinear maps applying unimodular matrices. For our construction, its inner layer works over the polynomial ring, and its outer layer works over the matrix ring. We use different unimodular matrices for elements of level-0 and level-1 encodings to prevent the cross-multiplication between level-0 and level-1 encodings. For simplicity, we only consider our construction as graded encoding.

3.1 Construction

Setting the parameters. We let λ be the security parameter and n be the dimension of polynomial ring R . Concrete parameters are set as $\sigma = \sqrt{\lambda n}$, $\sigma' = \lambda n^{1.5}$, $n = O(\lambda^2)$, $N = 4n$, $\xi = \lambda^{O(m)}$, $q \geq 20n^{16} \xi^2 2^\eta$, and $\tau = O(n^2)$.

Instance generation: $(\text{par}) \leftarrow \text{InstGen}(1^\lambda)$.

(1) Select a sufficiently large prime q .

(2) Generate parameters in the inner layer:

(2.1) Sample $\mathbf{f}_1, \mathbf{f}_2 \leftarrow D_{\mathbb{Z}^n, \sigma}$ such that $\mathbf{f}_1, \mathbf{f}_2$ are co-prime and $\mathbf{f}_t^{-1} \in \mathbb{k}$ with

$$\|\mathbf{f}_t^{-1}\| \leq O(n^2), \quad t \in \llbracket 2 \rrbracket, \text{ and set } \mathbf{f} = \mathbf{f}_1 \mathbf{f}_2.$$

(2.2) Sample $\mathbf{g}_t \leftarrow D_{\mathbb{Z}^n, \sigma}$, $\mathbf{a}_{t,s,i}, \mathbf{e}_{t,s,i}, \mathbf{d}_{t,s,i} \leftarrow D_{\mathbb{Z}^n, \sigma'}$, $t \in \llbracket 2 \rrbracket, s \in \llbracket 2 \rrbracket, i \in \llbracket \tau \rrbracket$.

(2.3) Sample $\mathbf{h}, \mathbf{q}_0 \leftarrow D_{\mathbb{Z}^n, \sigma'}$, and set $\mathbf{q} = \mathbf{h} \mathbf{q}_0 \mathbf{f}$ such that $\mathbf{q}^{-1} \in \mathbb{k}$.

(2.4) For $t \in \llbracket 2 \rrbracket, s \in \llbracket 2 \rrbracket$ and $t = s$, set

$$\mathbf{x}_{t,s,i} = \mathbf{h}(\mathbf{a}_{t,s,i} + \mathbf{e}_{t,s,i} \mathbf{f}_t) \bmod \mathbf{q}, \quad \mathbf{y}_{t,s,i} = \mathbf{h}(\mathbf{a}_{t,s,i} \mathbf{g}_t + \mathbf{d}_{t,s,i} \mathbf{f}_t) \bmod \mathbf{q}.$$

For $t \in \llbracket 2 \rrbracket, s \in \llbracket 2 \rrbracket$ and $t \neq s$, set

$$\mathbf{x}_{t,s,i} = \mathbf{h} \mathbf{e}_{t,s,i} \mathbf{f} \text{ and } \mathbf{y}_{t,s,i} = \mathbf{h} \mathbf{d}_{t,s,i} \mathbf{f}.$$

(2.5) Set $\mathbf{p}_{z,t} = \left[\mathbf{h}_t \mathbf{f}_t^{-1} \right]_q$, $t \in \llbracket 2 \rrbracket$, where $\mathbf{h}_t \leftarrow D_{\mathbb{Z}^n, \sigma}$.

(2.6) Sample $\mathbf{m}_{t,s,j}, \mathbf{n}_{t,s,j} \leftarrow D_{\mathbb{Z}^n, \sigma}$, $t \in \llbracket 2 \rrbracket$, $s \in \llbracket 2 \rrbracket$, $j \in \llbracket N \rrbracket$ such that \mathbf{M}', \mathbf{N}' are invertible over \mathbb{Q} , where

$$\mathbf{M}' = \begin{pmatrix} \mathbf{m}_{1,1,1} \mathbf{q} & \mathbf{m}_{1,1,2} \mathbf{q} & \cdots & \mathbf{m}_{1,1,N} \mathbf{q} \\ \mathbf{m}_{1,2,1} \mathbf{q} & \mathbf{m}_{1,2,2} \mathbf{q} & \cdots & \mathbf{m}_{1,2,N} \mathbf{q} \\ \mathbf{m}_{2,1,1} \mathbf{q} & \mathbf{m}_{2,1,2} \mathbf{q} & \cdots & \mathbf{m}_{2,1,N} \mathbf{q} \\ \mathbf{m}_{2,2,1} \mathbf{q} & \mathbf{m}_{2,2,2} \mathbf{q} & \cdots & \mathbf{m}_{2,2,N} \mathbf{q} \end{pmatrix},$$

$$\mathbf{N}' = \begin{pmatrix} \mathbf{n}_{1,1,1} \mathbf{q} & \mathbf{n}_{1,1,2} \mathbf{q} & \cdots & \mathbf{n}_{1,1,N} \mathbf{q} \\ \mathbf{n}_{1,2,1} \mathbf{q} & \mathbf{n}_{1,2,2} \mathbf{q} & \cdots & \mathbf{n}_{1,2,N} \mathbf{q} \\ \mathbf{n}_{2,1,1} \mathbf{q} & \mathbf{n}_{2,1,2} \mathbf{q} & \cdots & \mathbf{n}_{2,1,N} \mathbf{q} \\ \mathbf{n}_{2,2,1} \mathbf{q} & \mathbf{n}_{2,2,2} \mathbf{q} & \cdots & \mathbf{n}_{2,2,N} \mathbf{q} \end{pmatrix}.$$

(3) Generate parameters in the outer layer:

(3.1) Sample randomly $2n \times 2n$ -dimensional unimodular matrices \mathbf{S} and \mathbf{T} such that $\|\mathbf{S}\| = \|\mathbf{S}^{-1}\| = \|\mathbf{T}\| = \|\mathbf{T}^{-1}\| \leq O(n^2)$.

(3.2) For $i \in \llbracket \tau \rrbracket$, set

$$\mathbf{X}_i = \mathbf{S} \begin{pmatrix} \text{Rot}(\mathbf{x}_{1,1,i}) & \text{Rot}(\mathbf{x}_{1,2,i}) \\ \text{Rot}(\mathbf{x}_{2,1,i}) & \text{Rot}(\mathbf{x}_{2,2,i}) \end{pmatrix} \mathbf{S}^{-1},$$

$$\mathbf{Y}_i = \mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\mathbf{y}_{1,1,i}) & \text{Rot}(\mathbf{y}_{1,2,i}) \\ \text{Rot}(\mathbf{y}_{2,1,i}) & \text{Rot}(\mathbf{y}_{2,2,i}) \end{pmatrix} \mathbf{T}.$$

(3.3) For $j \in \llbracket N \rrbracket$, set

$$\mathbf{M}_j = \mathbf{S} \begin{pmatrix} \text{Rot}(\mathbf{m}_{1,1,j} \mathbf{q}) & \text{Rot}(\mathbf{m}_{1,2,j} \mathbf{q}) \\ \text{Rot}(\mathbf{m}_{2,1,j} \mathbf{q}) & \text{Rot}(\mathbf{m}_{2,2,j} \mathbf{q}) \end{pmatrix} \mathbf{S}^{-1},$$

$$\mathbf{N}_j = \mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\mathbf{n}_{1,1,j} \mathbf{q}) & \text{Rot}(\mathbf{n}_{1,2,j} \mathbf{q}) \\ \text{Rot}(\mathbf{n}_{2,1,j} \mathbf{q}) & \text{Rot}(\mathbf{n}_{2,2,j} \mathbf{q}) \end{pmatrix} \mathbf{T}.$$

We denote by $\mathbf{M}, \mathbf{N} \in \mathbb{Z}^{(4n^2) \times (4n)}$ the matrix of column vectors \mathbf{M}_j and \mathbf{N}_j , where \mathbf{M}_j and \mathbf{N}_j are considered as $4n^2$ -dimensional column vectors. Assume $\|\mathbf{M}\| = \|\mathbf{N}\| \leq \xi$.

(4) Generate the parameters of zero-testing and extraction:

(4.1) Sample $\mathbf{s}, \mathbf{t} \leftarrow D_{\mathbb{Z}^{2n}, \sigma}$ and $\mathbf{r}_1, \mathbf{r}_2 \leftarrow D_{\mathbb{Z}^n, \sigma'}$.

(4.2) Randomly select $\mathbf{z}_s, \mathbf{z}_t \in R_q$ such that $\mathbf{z}_s^{-1}, \mathbf{z}_t^{-1} \in R_q$.

(4.3) Set

$$\mathbf{s}^* = \left[\mathbf{s}^T \begin{pmatrix} \text{Rot}(\mathbf{z}_s^{-1}) & \\ & \text{Rot}(\mathbf{z}_s^{-1}) \end{pmatrix} \mathbf{S}^{-1} \right]_q,$$

$$\mathbf{t}^* = \left[\mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\mathbf{z}_t^{-1}) & 0 \\ 0 & \text{Rot}(\mathbf{z}_t^{-1}) \end{pmatrix} \mathbf{t} \right]_q.$$

$$(4.4) \text{ Set } \mathbf{P}_{zt} = \left[\mathbf{S} \begin{pmatrix} \text{Rot}(\mathbf{z}_s \mathbf{z}_i \mathbf{p}_{zt,1}) & \text{Rot}(\mathbf{z}_s \mathbf{z}_i \mathbf{r}_1) \\ \text{Rot}(\mathbf{z}_s \mathbf{z}_i \mathbf{r}_2) & \text{Rot}(\mathbf{z}_s \mathbf{z}_i \mathbf{p}_{zt,2}) \end{pmatrix} \mathbf{T} \right]_q.$$

(5) Output the public parameters $\text{par} = \{q, \mathbf{M}, \mathbf{N}, \{\mathbf{X}_i, \mathbf{Y}_i\}_{i \in [\tau]}, \mathbf{P}_{zt}, \mathbf{s}^*, \mathbf{t}^*\}$.

Generating level- k random encodings: $\mathbf{U} \leftarrow \text{Enc}(\text{par}, k, \{w_i\}_{i \in [\tau]}).$

Select $w_i \leftarrow D_{\mathbb{Z}, \sigma}$, $i \in [\tau]$ and generate a level-0 encoding $\mathbf{D} = \sum_{i=1}^{\tau} w_i \cdot (\mathbf{X}_i)^k \bmod \mathbf{M}$ and a level- k encoding $\mathbf{U} = \sum_{i=1}^{\tau} w_i \cdot (\mathbf{Y}_i)^k \bmod \mathbf{N}$.

Notice that here $\mathbf{R} = \sum_{i=1}^{\tau} w_i \cdot (\mathbf{X}_i)^k$ is considered as a $4n^2$ -dimensional column vector, and $\mathbf{R} \bmod \mathbf{M}$ means to map \mathbf{R} into parallelepiped of lattice generated by \mathbf{M} . Similarly, $\sum_{i=1}^{\tau} w_i \cdot (\mathbf{Y}_i)^k \bmod \mathbf{N}$.

Adding encodings: $\mathbf{U} \leftarrow \text{Add}(\text{par}, k, \mathbf{U}_1, \dots, \mathbf{U}_s).$

Given s level- k encodings $\mathbf{U}_r, r \in [s]$, their sum $\mathbf{U} = \sum_{r=1}^s \mathbf{U}_r \bmod \mathbf{N}$ is a level- k encoding.

Multiplying encodings: $\mathbf{U} \leftarrow \text{Mul}(\text{par}, 1, \mathbf{U}_1, \dots, \mathbf{U}_k).$

Given k level-1 encodings $\mathbf{U}_r, r \in [k]$, their product $\mathbf{U} = \prod_{r=1}^k \mathbf{U}_r \bmod \mathbf{N}$ is a level- k encoding.

Zero-testing: $\text{isZero}(\text{par}, \mathbf{D}, \mathbf{U}).$

Given a level- k encoding \mathbf{U} and a level-0 encoding \mathbf{D} , we determine whether \mathbf{U} is an encoding of zero. We compute $v = [\mathbf{s}^* \cdot \mathbf{D} \cdot \mathbf{P}_{zt} \cdot \mathbf{U} \cdot \mathbf{t}^*]_q$ in \mathbb{Z}_q and check whether v is small, as follows:

$$\text{isZero}(\text{par}, \mathbf{D}, \mathbf{U}) = \begin{cases} 1 & \text{if } |v| < q/2^n \\ 0 & \text{otherwise} \end{cases}.$$

Extract: $sk \leftarrow \text{Ext}(\text{par}, \mathbf{D}, \mathbf{U}).$

Given a level- k encoding \mathbf{U} and a level-0 encoding \mathbf{D} , $\text{Ext}(\text{par}, \mathbf{D}, \mathbf{U}) = \text{Extract}_s(\text{msbs}_{\gamma}([\mathbf{s}^* \cdot \mathbf{D} \cdot \mathbf{P}_{zt} \cdot \mathbf{U} \cdot \mathbf{t}^*]_q))$, where msbs_{γ} extracts the $\gamma = \eta - \lambda$ most significant bits from the result. Extract_s is a strong randomness extractor using the seed s .

Remark 3.1 (1) In our construction, we can replace vectors $\mathbf{s}, \mathbf{t} \leftarrow D_{\mathbb{Z}^{2n}, \sigma}$ with matrices

$$\mathbf{S}_1 \leftarrow D_{\mathbb{Z}^{k_1 \times 2n}, \sigma}, \mathbf{T}_1 \leftarrow D_{\mathbb{Z}^{2n \times k_2}, \sigma}, \text{ and } \mathbf{s}^*, \mathbf{t}^* \text{ with } \mathbf{S}^* = \left[\mathbf{S}_1 \begin{pmatrix} \text{Rot}(\mathbf{z}_s^{-1}) \\ \text{Rot}(\mathbf{z}_s^{-1}) \end{pmatrix} \mathbf{S}_1^{-1} \right]_q,$$

$$\mathbf{T}^* = \left[\mathbf{T}_1^{-1} \begin{pmatrix} \text{Rot}(\mathbf{z}_t^{-1}) \\ \text{Rot}(\mathbf{z}_t^{-1}) \end{pmatrix} \mathbf{T}_1 \right]_q. \text{ In this case, we compute a } k_1 \times k_2 \text{-matrix}$$

$[\mathbf{S}^* \cdot \mathbf{D} \cdot \mathbf{P}_{zt} \cdot \mathbf{U} \cdot \mathbf{T}^*]_q$ as the final result, where $k_1 \times k_2 < n$ to guarantee the security of

construction. (2) We can take $\tau = \lambda n$ or $\tau = 8n$ according to an optimization in [HJ15c].

(3) The matrix \mathbf{D} can be taken the identity matrix. Our aim is to demonstrate how to use level-0 encodings when constructing the MPKE protocol. (4) When our construction is only applied to multipartite Diffie–Hellman key exchange, $\mathbf{P}_{zt}, \mathbf{X}_i$ in the public parameters can

be replaced by $\mathbf{P}_{zt,i} = [\mathbf{X}_i \mathbf{P}_{zt}]_q$ and \mathbf{S} does not require a unimodular matrix. Moreover, the matrix $\mathbf{P}_{zt,i} = [\mathbf{X}_i \mathbf{P}_{zt}]_q$ may be further modified into a vector $\mathbf{p}_{zt,i} = [\mathbf{s}^* \mathbf{X}_i \mathbf{P}_{zt}]_q$.

3.2 Correctness

Lemma 3.2 $\text{InstGen}(1^\lambda)$ is a probabilistic polynomial time algorithm.

Proof. (1) The unimodular matrices \mathbf{S} and \mathbf{T} can be generated using the method of [GGH15].

(2) If $\det(\text{Rot}(\mathbf{q})) \neq 0$, then $\mathbf{q}^{-1} \in \mathbb{k}$. So, one can efficiently generate \mathbf{q} .

(3) For the matrices \mathbf{M}' , we have

$$\mathbf{M}' = \begin{pmatrix} \text{Rot}(\mathbf{q}) & & & \\ & \text{Rot}(\mathbf{q}) & & \\ & & \text{Rot}(\mathbf{q}) & \\ & & & \text{Rot}(\mathbf{q}) \end{pmatrix} \begin{pmatrix} \mathbf{m}_{1,1,1} & \mathbf{m}_{1,1,2} & \cdots & \mathbf{m}_{1,1,N} \\ \mathbf{m}_{1,2,1} & \mathbf{m}_{1,2,2} & \cdots & \mathbf{m}_{1,2,N} \\ \mathbf{m}_{2,1,1} & \mathbf{m}_{2,1,2} & \cdots & \mathbf{m}_{2,1,N} \\ \mathbf{m}_{2,2,1} & \mathbf{m}_{2,2,2} & \cdots & \mathbf{m}_{2,2,N} \end{pmatrix} \\ = \mathbf{Q} \times \mathbf{M}''$$

Since $\text{Rot}(\mathbf{q})$ is invertible, the diagonal matrix \mathbf{Q} is invertible over \mathbb{Q} . So, with high probability, \mathbf{M}' is invertible over \mathbb{Q} since a random sampling matrix \mathbf{M}'' is invertible over \mathbb{Q} with high probability.

Similarly, it is easy to show that \mathbf{N}' is invertible over \mathbb{Q} .

All other elements in $\text{InstGen}(1^\lambda)$ can be generated in polynomial time. \blacksquare

Lemma 3.3 The ranks of \mathbf{M} and \mathbf{N} in the public parameter par are $4n$.

Proof. We prove the result by contradiction and assume that the rank of \mathbf{M} is less than $4n$. Without loss of generality, assume that there exist $4n$ non-all-zero real numbers k_j such that $\sum_{j=1}^{4n} k_j \mathbf{M}_j = \mathbf{0}^{2n \times 2n}$, namely, $\mathbf{k} \neq \mathbf{0}$. Thus, we derive the following expression:

$$\sum_{j=1}^{4n} k_j \mathbf{M}_j = \mathbf{S} \left(\sum_{j=1}^{4n} k_j \begin{pmatrix} \text{Rot}(\mathbf{m}_{1,1,j} \mathbf{q}) & \text{Rot}(\mathbf{m}_{1,2,j} \mathbf{q}) \\ \text{Rot}(\mathbf{m}_{2,1,j} \mathbf{q}) & \text{Rot}(\mathbf{m}_{2,2,j} \mathbf{q}) \end{pmatrix} \right) \mathbf{S}^{-1} = \mathbf{0}^{2n \times 2n}.$$

Given that \mathbf{S} is invertible over \mathbb{R} , we derive

$$\sum_{j=1}^{4n} k_j \begin{pmatrix} \text{Rot}(\mathbf{m}_{1,1,j} \mathbf{q}) & \text{Rot}(\mathbf{m}_{1,2,j} \mathbf{q}) \\ \text{Rot}(\mathbf{m}_{2,1,j} \mathbf{q}) & \text{Rot}(\mathbf{m}_{2,2,j} \mathbf{q}) \end{pmatrix} = \mathbf{0}^{2n \times 2n}.$$

That is, $\sum_{j=1}^{4n} k_j \begin{pmatrix} \mathbf{m}_{1,1,j} \mathbf{q} & \mathbf{m}_{1,2,j} \mathbf{q} \\ \mathbf{m}_{2,1,j} \mathbf{q} & \mathbf{m}_{2,2,j} \mathbf{q} \end{pmatrix} = \mathbf{0}^{2 \times 2n}$. So, we have

$$\sum_{j=1}^{4n} k_j \begin{pmatrix} \mathbf{m}_{1,1,j} \mathbf{q} \\ \mathbf{m}_{1,2,j} \mathbf{q} \\ \mathbf{m}_{2,1,j} \mathbf{q} \\ \mathbf{m}_{2,2,j} \mathbf{q} \end{pmatrix} = \mathbf{0}^{4n \times 1}.$$

Given that \mathbf{M}' is invertible, we have $\mathbf{M}' \cdot \mathbf{k} = \mathbf{0}$ and $\mathbf{k} \neq \mathbf{0}$. A contradiction is generated. Similarly, we can prove that the rank of \mathbf{N} is $4n$. \blacksquare

Lemma 3.4 Suppose that $\text{Space}(\mathbf{M})$ and $\text{Space}(\mathbf{N})$ are linear spaces spanned by \mathbf{M} and \mathbf{N} , respectively. Then for $\{\mathbf{X}_i\}_{i \in [\tau]}$, $\{\mathbf{Y}_i\}_{i \in [\tau]}$ in the public parameter par ,

$\mathbf{X}_i \in \text{Space}(\mathbf{M})$ and $\mathbf{Y}_i \in \text{Space}(\mathbf{N})$.

Proof. Given that \mathbf{M}' is invertible, for $\mathbf{X}_i = \mathbf{S} \begin{pmatrix} \text{Rot}(\mathbf{x}_{1,1,i}) & \text{Rot}(\mathbf{x}_{1,2,i}) \\ \text{Rot}(\mathbf{x}_{2,1,i}) & \text{Rot}(\mathbf{x}_{2,2,i}) \end{pmatrix} \mathbf{S}^{-1}$, we have

$$\mathbf{k} = (\mathbf{M}')^{-1} \cdot \begin{pmatrix} \mathbf{x}_{1,1,i} \\ \mathbf{x}_{1,2,i} \\ \mathbf{x}_{2,1,i} \\ \mathbf{x}_{2,2,i} \end{pmatrix} \text{ and } \mathbf{M}' \cdot \mathbf{k} = \begin{pmatrix} \mathbf{x}_{1,1,i} \\ \mathbf{x}_{1,2,i} \\ \mathbf{x}_{2,1,i} \\ \mathbf{x}_{2,2,i} \end{pmatrix}$$

Thus, we derive the following expression:

$$\sum_{j=1}^{4n} k_j \begin{pmatrix} \mathbf{m}_{1,1,j} \mathbf{q} \\ \mathbf{m}_{1,2,j} \mathbf{q} \\ \mathbf{m}_{2,1,j} \mathbf{q} \\ \mathbf{m}_{2,2,j} \mathbf{q} \end{pmatrix} = \begin{pmatrix} \mathbf{x}_{1,1,i} \\ \mathbf{x}_{1,2,i} \\ \mathbf{x}_{2,1,i} \\ \mathbf{x}_{2,2,i} \end{pmatrix}.$$

Namely,

$$\sum_{j=1}^{4n} k_j \begin{pmatrix} \mathbf{m}_{1,1,j} \mathbf{q} & \mathbf{m}_{1,2,j} \mathbf{q} \\ \mathbf{m}_{2,1,j} \mathbf{q} & \mathbf{m}_{2,2,j} \mathbf{q} \end{pmatrix} = \begin{pmatrix} \mathbf{x}_{1,1,i} & \mathbf{x}_{1,2,i} \\ \mathbf{x}_{2,1,i} & \mathbf{x}_{2,2,i} \end{pmatrix}.$$

Since the j -th column of $\text{Rot}(\mathbf{x})$ is $\mathbf{x} \cdot x^j$, then we have

$$\sum_{j=1}^{4n} k_j \begin{pmatrix} \text{Rot}(\mathbf{m}_{1,1,j} \mathbf{q}) & \text{Rot}(\mathbf{m}_{1,2,j} \mathbf{q}) \\ \text{Rot}(\mathbf{m}_{2,1,j} \mathbf{q}) & \text{Rot}(\mathbf{m}_{2,2,j} \mathbf{q}) \end{pmatrix} = \begin{pmatrix} \text{Rot}(\mathbf{x}_{1,1,i}) & \text{Rot}(\mathbf{x}_{1,2,i}) \\ \text{Rot}(\mathbf{x}_{2,1,i}) & \text{Rot}(\mathbf{x}_{2,2,i}) \end{pmatrix}.$$

Thus, $\mathbf{X}_i = \sum_{j=1}^{4n} k_j \mathbf{M}_j$ and $\mathbf{X}_i \in \text{Space}(\mathbf{M})$. Similarly, $\mathbf{Y}_i \in \text{Space}(\mathbf{N})$. ■

Lemma 3.5 Given the public parameter par , suppose

$$(w_1 \mathbf{X}_{i_1} + w_2 \mathbf{X}_{i_2}) \bmod \mathbf{M} = \mathbf{S} \begin{pmatrix} \text{Rot}(\mathbf{d}_{1,1}) & \text{Rot}(\mathbf{d}_{1,2}) \\ \text{Rot}(\mathbf{d}_{2,1}) & \text{Rot}(\mathbf{d}_{2,2}) \end{pmatrix} \mathbf{S}^{-1}. \text{ Then}$$

$$(1) \mathbf{d}_{t,t} = w_1 \mathbf{x}_{t,t,i_1} + w_2 \mathbf{x}_{t,t,i_2} = (w_1 \mathbf{h} \mathbf{a}_{t,t,i_1} + w_2 \mathbf{h} \mathbf{a}_{t,t,i_2}) \bmod \mathbf{f}, t \in [2];$$

$$(2) \mathbf{d}_{1,2} \bmod \mathbf{f} = \mathbf{d}_{2,1} \bmod \mathbf{f} = 0.$$

Proof. By the method generated \mathbf{X}_i and \mathbf{M}_j , we have

$$\begin{aligned} & (w_1 \mathbf{X}_{i_1} + w_2 \mathbf{X}_{i_2}) \bmod \mathbf{M} \\ &= w_1 \mathbf{X}_{i_1} + w_2 \mathbf{X}_{i_2} - \sum_{j=1}^{4n} k_j \mathbf{M}_j, \\ &= \mathbf{S} \begin{pmatrix} \text{Rot}(w_1 \mathbf{x}_{1,1,i_1} + w_2 \mathbf{x}_{1,1,i_2} - \mathbf{r}_{1,1} \mathbf{q}) & \text{Rot}(w_1 \mathbf{x}_{1,2,i_1} + w_2 \mathbf{x}_{1,2,i_2} - \mathbf{r}_{1,2} \mathbf{q}) \\ \text{Rot}(w_1 \mathbf{x}_{2,1,i_1} + w_2 \mathbf{x}_{2,1,i_2} - \mathbf{r}_{2,1} \mathbf{q}) & \text{Rot}(w_1 \mathbf{x}_{2,2,i_1} + w_2 \mathbf{x}_{2,2,i_2} - \mathbf{r}_{2,2} \mathbf{q}) \end{pmatrix} \mathbf{S}^{-1} \end{aligned}$$

where $\mathbf{r}_{t,s} = \sum_{j=1}^{4n} k_j \mathbf{m}_{t,s,j}$, $t \in [2]$, $s \in [2]$.

By the method generated $\mathbf{x}_{t,s,i}$, $t \in [2]$, $s \in [2]$ and $\mathbf{q} = \mathbf{h} \mathbf{q}_0 \mathbf{f}$, we have

$$\mathbf{d}_{t,t} = w_1 \mathbf{x}_{t,t,i_1} + w_2 \mathbf{x}_{t,t,i_2} - \mathbf{r}_{t,t} \mathbf{q} = (w_1 \mathbf{h} \mathbf{a}_{t,t,i_1} + w_2 \mathbf{h} \mathbf{a}_{t,t,i_2}) \bmod \mathbf{f}_1,$$

$$\mathbf{d}_{1,2} = w_1 \mathbf{x}_{1,2,i_1} + w_2 \mathbf{x}_{1,2,i_2} - \mathbf{r}_{1,2} \mathbf{q} = 0 \bmod \mathbf{f},$$

$$\mathbf{d}_{2,1} = w_1 \mathbf{x}_{2,1,i_1} + w_2 \mathbf{x}_{2,1,i_2} - \mathbf{r}_{2,1} \mathbf{q} = 0 \bmod \mathbf{f}.$$

So, the proof is complete. ■

Lemma 3.6 Given the public parameter par , suppose

$(\mathbf{X}_{i_1} \mathbf{X}_{i_2}) \bmod \mathbf{M} = \mathbf{S} \begin{pmatrix} \text{Rot}(\mathbf{d}_{1,1}) & \text{Rot}(\mathbf{d}_{1,2}) \\ \text{Rot}(\mathbf{d}_{2,1}) & \text{Rot}(\mathbf{d}_{2,2}) \end{pmatrix} \mathbf{S}^{-1}$. Then

- (1) $\mathbf{d}_{t,t} = \mathbf{x}_{1,1,i_1} \mathbf{x}_{1,1,i_2} = (\mathbf{h}^2 \mathbf{a}_{t,t,i_1} \mathbf{a}_{t,t,i_2}) \bmod \mathbf{f}_t, t \in \llbracket 2 \rrbracket$;
- (2) $\mathbf{d}_{1,2} \bmod \mathbf{f} = \mathbf{d}_{2,1} \bmod \mathbf{f} = 0$.

Proof. By the method generated \mathbf{X}_i and \mathbf{M}_j , we have

$$\begin{aligned} & (\mathbf{X}_{i_1} \mathbf{X}_{i_2}) \bmod \mathbf{M} \\ &= \mathbf{X}_{i_1} \mathbf{X}_{i_2} - \sum_{j=1}^{4n} k_j \mathbf{M}_j \\ &= \mathbf{S} \begin{pmatrix} \text{Rot}(\mathbf{x}_{1,1,i_1} \mathbf{x}_{1,1,i_2} + \mathbf{x}_{1,2,i_1} \mathbf{x}_{2,1,i_2} - \mathbf{r}_{1,1} \mathbf{q}) & \text{Rot}(\mathbf{x}_{1,1,i_1} \mathbf{x}_{1,2,i_2} + \mathbf{x}_{1,2,i_1} \mathbf{x}_{2,2,i_2} - \mathbf{r}_{1,2} \mathbf{q}) \\ \text{Rot}(\mathbf{x}_{2,1,i_1} \mathbf{x}_{1,1,i_2} + \mathbf{x}_{2,2,i_1} \mathbf{x}_{2,1,i_2} - \mathbf{r}_{2,1} \mathbf{q}) & \text{Rot}(\mathbf{x}_{2,1,i_1} \mathbf{x}_{1,2,i_2} + \mathbf{x}_{2,2,i_1} \mathbf{x}_{2,2,i_2} - \mathbf{r}_{2,2} \mathbf{q}) \end{pmatrix} \mathbf{S}^{-1} \end{aligned}$$

where $\mathbf{r}_{t,s} = \sum_{j=1}^{4n} k_j \mathbf{m}_{t,s,j}, t \in \llbracket 2 \rrbracket, s \in \llbracket 2 \rrbracket$.

By the method generated $\mathbf{x}_{t,s,i}, t \in \llbracket 2 \rrbracket, s \in \llbracket 2 \rrbracket$ and $\mathbf{q} = \mathbf{h} \mathbf{q}_0 \mathbf{f}$, we have

$$\begin{aligned} \mathbf{d}_{1,1} &= \mathbf{x}_{1,1,i_1} \mathbf{x}_{1,1,i_2} + \mathbf{x}_{1,2,i_1} \mathbf{x}_{2,1,i_2} - \mathbf{r}_{1,1} \mathbf{q} = (\mathbf{h}^2 \mathbf{a}_{1,1,i_1} \mathbf{a}_{1,1,i_2}) \bmod \mathbf{f}_1, \\ \mathbf{d}_{2,2} &= \mathbf{x}_{2,1,i_1} \mathbf{x}_{1,2,i_2} + \mathbf{x}_{2,2,i_1} \mathbf{x}_{2,2,i_2} - \mathbf{r}_{2,2} \mathbf{q} = (\mathbf{h}^2 \mathbf{a}_{2,2,i_1} \mathbf{a}_{2,2,i_2}) \bmod \mathbf{f}_2. \\ \mathbf{d}_{1,2} &= \mathbf{x}_{1,1,i_1} \mathbf{x}_{1,2,i_2} + \mathbf{x}_{1,2,i_1} \mathbf{x}_{2,2,i_2} - \mathbf{r}_{1,2} \mathbf{q} = 0 \bmod \mathbf{f}, \\ \mathbf{d}_{2,1} &= \mathbf{x}_{2,1,i_1} \mathbf{x}_{1,1,i_2} + \mathbf{x}_{2,2,i_1} \mathbf{x}_{2,1,i_2} - \mathbf{r}_{2,1} \mathbf{q} = 0 \bmod \mathbf{f}. \end{aligned}$$

So, the proof is complete. \blacksquare

Lemma 3.7 Given the public parameter par , suppose

$$(w_1 \mathbf{Y}_{i_1} + w_2 \mathbf{Y}_{i_2}) \bmod \mathbf{N} = \mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\mathbf{u}_{1,1}) & \text{Rot}(\mathbf{u}_{1,2}) \\ \text{Rot}(\mathbf{u}_{2,1}) & \text{Rot}(\mathbf{u}_{2,2}) \end{pmatrix} \mathbf{T}. \text{ Then}$$

- (1) $\mathbf{u}_{t,t} = w_1 \mathbf{y}_{t,t,i_1} + w_2 \mathbf{y}_{t,t,i_2} = (w_1 \mathbf{h} \mathbf{a}_{t,t,i_1} \mathbf{g}_t + w_2 \mathbf{h} \mathbf{a}_{t,t,i_2} \mathbf{g}_t) \bmod \mathbf{f}_t, t \in \llbracket 2 \rrbracket$;
- (2) $\mathbf{u}_{1,2} \bmod \mathbf{f} = \mathbf{u}_{2,1} \bmod \mathbf{f} = 0$.

Proof. The proof is identical to that of Lemma 3.5. \blacksquare

Lemma 3.8 Given the public parameter par , suppose

$$(\mathbf{Y}_{i_1} \mathbf{Y}_{i_2}) \bmod \mathbf{N} = \mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\mathbf{u}_{1,1}) & \text{Rot}(\mathbf{u}_{1,2}) \\ \text{Rot}(\mathbf{u}_{2,1}) & \text{Rot}(\mathbf{u}_{2,2}) \end{pmatrix} \mathbf{T}. \text{ Then}$$

- (1) $\mathbf{u}_{t,t} = \mathbf{y}_{t,t,i_1} \mathbf{y}_{t,t,i_2} = (\mathbf{h}^2 \mathbf{g}_t \mathbf{a}_{t,t,i_1} \mathbf{a}_{t,t,i_2}) \bmod \mathbf{f}_t, t \in \llbracket 2 \rrbracket$;
- (2) $\mathbf{u}_{1,2} \bmod \mathbf{f} = \mathbf{u}_{2,1} \bmod \mathbf{f} = 0$.

Proof. The proof is same as that of Lemma 3.6. \blacksquare

Lemma 3.9 Encoding $\mathbf{U} \leftarrow \text{Enc}(\text{par}, k, \{w_i\}_{i \in \llbracket \tau \rrbracket}})$ is a level- k encoding.

Proof. Given that $\mathbf{D} = \sum_{i=1}^{\tau} w_i \cdot (\mathbf{X}_i)^k \bmod \mathbf{M}$, we derive the following expressions by Lemma 3.5, 3.6:

$$\begin{aligned}
\mathbf{D} &= \sum_{i=1}^{\tau} w_i \cdot (\mathbf{X}_i)^k \bmod \mathbf{M} \\
&= \mathbf{S} \begin{pmatrix} \text{Rot}(\sum_{i=1}^{\tau} w_i \cdot (\mathbf{x}_{1,1,i})^k + \mathbf{r}_{1,1} \mathbf{f}_1) & \text{Rot}(\mathbf{r}_{1,2} \mathbf{f}) \\ \text{Rot}(\mathbf{r}_{2,1} \mathbf{f}) & \text{Rot}(\sum_{i=1}^{\tau} w_i \cdot (\mathbf{x}_{2,2,i})^k + \mathbf{r}_{2,2} \mathbf{f}_2) \end{pmatrix} \mathbf{S}^{-1} \\
&= \mathbf{S} \begin{pmatrix} \text{Rot}(\mathbf{d}_{1,1}) & \text{Rot}(\mathbf{d}_{1,2}) \\ \text{Rot}(\mathbf{d}_{2,1}) & \text{Rot}(\mathbf{d}_{2,2}) \end{pmatrix} \mathbf{S}^{-1}
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\mathbf{d}_{t,t} \bmod \mathbf{f}_t &= (\sum_{i=1}^{\tau} w_i \cdot (\mathbf{x}_{t,t,i})^k + \mathbf{r}_{t,t} \mathbf{f}_t) \bmod \mathbf{f}_t \\
&= (\sum_{i=1}^{\tau} w_i \cdot (\mathbf{h}\mathbf{a}_{t,t,i})^k) \bmod \mathbf{f}_t \\
\mathbf{d}_{1,2} \bmod \mathbf{f} &= \mathbf{d}_{2,1} \bmod \mathbf{f} = \mathbf{0}.
\end{aligned}$$

Given that $\mathbf{U} = \sum_{i=1}^{\tau} w_i \cdot (\mathbf{Y}_i)^k \bmod \mathbf{N}$, we derive the following expressions by Lemma 3.7, 3.8:

$$\begin{aligned}
\mathbf{U} &= \sum_{i=1}^{\tau} w_i \cdot (\mathbf{Y}_i)^k \bmod \mathbf{N} \\
&= \mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\sum_{i=1}^{\tau} w_i \cdot (\mathbf{y}_{1,1,i})^k + \mathbf{s}_{1,1} \mathbf{f}_1) & \text{Rot}(\mathbf{s}_{1,2} \mathbf{f}) \\ \text{Rot}(\mathbf{s}_{2,1} \mathbf{f}) & \text{Rot}(\sum_{i=1}^{\tau} w_i \cdot (\mathbf{y}_{2,2,i})^k + \mathbf{s}_{2,2} \mathbf{f}_2) \end{pmatrix} \mathbf{T}, \\
&= \mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\mathbf{u}_{1,1}) & \text{Rot}(\mathbf{u}_{1,2}) \\ \text{Rot}(\mathbf{u}_{2,1}) & \text{Rot}(\mathbf{u}_{2,2}) \end{pmatrix} \mathbf{T}
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\mathbf{u}_{t,t} \bmod \mathbf{f}_t &= (\sum_{i=1}^{\tau} w_i \cdot (\mathbf{y}_{t,t,i})^k + \mathbf{s}_{t,t} \mathbf{f}_t) \bmod \mathbf{f}_t \\
&= (\sum_{i=1}^{\tau} w_i \cdot (\mathbf{h}\mathbf{a}_{t,t,i})^k \cdot \mathbf{g}_t^k) \bmod \mathbf{f}_t \\
\mathbf{u}_{1,2} \bmod \mathbf{f} &= \mathbf{u}_{2,1} \bmod \mathbf{f} = \mathbf{0}.
\end{aligned}$$

As such, \mathbf{U} is a level- k encoding of \mathbf{D} . ■

Lemma 3.10 Encoding $\mathbf{U} \leftarrow \text{Add}(\text{par}, k, \mathbf{U}_1, \dots, \mathbf{U}_s)$ is a level- k encoding.

Proof. By Lemma 3.5 and Lemma 3.9, the sum \mathbf{U} of level- k encodings $\mathbf{U}_r, r \in \llbracket s \rrbracket$ is a level- k encoding. ■

Lemma 3.11 $\mathbf{U} \leftarrow \text{Mul}(\text{par}, 1, \mathbf{U}_1, \dots, \mathbf{U}_k)$ is a level- k encoding.

Proof. Given that $\mathbf{U}_r, r \in \llbracket k \rrbracket$ are level-1 encodings, we obtain

$$\mathbf{U}_r = \mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\mathbf{u}_{1,1,r}) & \text{Rot}(\mathbf{u}_{1,2,r}) \\ \text{Rot}(\mathbf{u}_{2,1,r}) & \text{Rot}(\mathbf{u}_{2,2,r}) \end{pmatrix} \mathbf{T},$$

where $\mathbf{u}_{t,t,r} = \mathbf{d}_{t,t,r} \mathbf{h}\mathbf{g}_t + \mathbf{r}_{t,t,r} \mathbf{f}_t$, $\mathbf{u}_{1,2,r} = \mathbf{s}_{1,2,r} \mathbf{f}$, $\mathbf{u}_{2,1,r} = \mathbf{s}_{2,1,r} \mathbf{f}$.

As such, we derive the following expressions by Lemma 3.8:

$$\begin{aligned}
\mathbf{U} &= \prod_{t=1}^k \mathbf{U}_t \bmod \mathbf{N} \\
&= \mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\prod_{r=1}^k \mathbf{u}_{1,1,r} + \mathbf{s}_{1,1} \mathbf{f}_1) & \text{Rot}(\mathbf{s}_{1,2} \mathbf{f}) \\ \text{Rot}(\mathbf{s}_{2,1} \mathbf{f}) & \text{Rot}(\prod_{r=1}^k \mathbf{u}_{2,2,r} + \mathbf{s}_{2,2} \mathbf{f}_2) \end{pmatrix} \mathbf{T}, \\
&= \mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\mathbf{u}_{1,1}) & \text{Rot}(\mathbf{u}_{1,2}) \\ \text{Rot}(\mathbf{u}_{2,1}) & \text{Rot}(\mathbf{u}_{2,2}) \end{pmatrix} \mathbf{T}
\end{aligned}$$

So, we get

$$\begin{aligned}
\mathbf{u}_{t,t} \bmod \mathbf{f}_t &= (\prod_{r=1}^k \mathbf{u}_{t,t,r} + \mathbf{s}_{t,t} \mathbf{f}_t) \bmod \mathbf{f}_t \\
&= (\prod_{r=1}^k \mathbf{h} \mathbf{d}_{t,t,r} \mathbf{g}_t) \bmod \mathbf{f}_t, \\
&= (\prod_{r=1}^k \mathbf{h} \mathbf{d}_{t,t,r}) (\mathbf{g}_t)^k \bmod \mathbf{f}_t \\
\mathbf{u}_{1,2} \bmod \mathbf{f} &= \mathbf{u}_{2,1} \bmod \mathbf{f} = \mathbf{0}.
\end{aligned}$$

Thus, \mathbf{U} is a level- k encoding. ■

Lemma 3.12 For an arbitrary integer $k > 0$, the zero-testing algorithm $\text{isZero}(\text{par}, \mathbf{D}, \mathbf{U})$ correctly determines whether \mathbf{U} is an encoding of zero.

Proof. Given a level-0 encoding \mathbf{D} and an arbitrary level- k encoding \mathbf{U} , we have

$$\begin{aligned}
\mathbf{D} &= \mathbf{S} \begin{pmatrix} \text{Rot}(\mathbf{d}_{1,1}) & \text{Rot}(\mathbf{d}_{1,2}) \\ \text{Rot}(\mathbf{d}_{2,1}) & \text{Rot}(\mathbf{d}_{2,2}) \end{pmatrix} \mathbf{S}^{-1} = \mathbf{S} \cdot \mathbf{D}' \cdot \mathbf{S}^{-1}, \\
\mathbf{U} &= \mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\mathbf{u}_{1,1}) & \text{Rot}(\mathbf{u}_{1,2}) \\ \text{Rot}(\mathbf{u}_{2,1}) & \text{Rot}(\mathbf{u}_{2,2}) \end{pmatrix} \mathbf{T} = \mathbf{T}^{-1} \cdot \mathbf{U}' \cdot \mathbf{T},
\end{aligned}$$

where $\mathbf{d}_{1,2} \bmod \mathbf{f} = \mathbf{d}_{2,1} \bmod \mathbf{f} = \mathbf{0}$ and $\mathbf{u}_{1,2} \bmod \mathbf{f} = \mathbf{u}_{2,1} \bmod \mathbf{f} = \mathbf{0}$.

Hence, $\|\mathbf{D}'\| \leq n^2 \|\mathbf{S}\| \|\mathbf{D}\| \|\mathbf{S}^{-1}\|$ and $\|\mathbf{U}'\| \leq n^2 \|\mathbf{T}\| \|\mathbf{U}\| \|\mathbf{T}^{-1}\|$.

On the basis of $\|\mathbf{D}\| \leq \|\mathbf{M}\|$ and $\|\mathbf{U}\| \leq \|\mathbf{N}\|$, we obtain $\|\mathbf{D}'\| \leq n^6 \xi$ and $\|\mathbf{U}'\| \leq n^6 \xi$. As a result, $\|\mathbf{d}_{t,s}\| \leq n^6 \xi$ and $\|\mathbf{u}_{t,s}\| \leq n^6 \xi$, $t \in [2]$, $s \in [2]$.

We first compute $v = [\mathbf{s}^* \cdot \mathbf{D} \cdot \mathbf{P}_{zt} \cdot \mathbf{U} \cdot \mathbf{t}^*]_q$ as follows:

$$\begin{aligned}
v &= [\mathbf{s}^* \cdot \mathbf{D} \cdot \mathbf{P}_{zt} \cdot \mathbf{U} \cdot \mathbf{t}^*]_q \\
&= \left[\mathbf{s}^T \cdot \mathbf{D}' \cdot \begin{pmatrix} \text{Rot}(\mathbf{p}_{zt,1}) & \text{Rot}(\mathbf{r}_1) \\ \text{Rot}(\mathbf{r}_2) & \text{Rot}(\mathbf{p}_{zt,2}) \end{pmatrix} \cdot \mathbf{U}' \cdot \mathbf{t} \right]_q, \\
&= \left[\mathbf{s}^T \cdot \begin{pmatrix} \text{Rot}(\mathbf{v}_{1,1}) & \text{Rot}(\mathbf{v}_{1,2}) \\ \text{Rot}(\mathbf{v}_{2,1}) & \text{Rot}(\mathbf{v}_{2,2}) \end{pmatrix} \cdot \mathbf{t} \right]_q
\end{aligned}$$

where $\mathbf{v}_{1,1} = [\mathbf{d}_{1,1} \mathbf{p}_{zt,1} \mathbf{u}_{1,1} + \mathbf{d}_{1,2} \mathbf{r}_2 \mathbf{u}_{1,1} + \mathbf{d}_{1,1} \mathbf{r}_1 \mathbf{u}_{2,1} + \mathbf{d}_{1,2} \mathbf{p}_{zt,2} \mathbf{u}_{2,1}]_q = [\mathbf{d}_{1,1} \mathbf{p}_{zt,1} \mathbf{u}_{1,1} + \mathbf{w}_{1,1}]_q$,

$\mathbf{v}_{1,2} = [\mathbf{d}_{1,1} \mathbf{p}_{zt,1} \mathbf{u}_{1,2} + \mathbf{d}_{1,2} \mathbf{r}_2 \mathbf{u}_{1,2} + \mathbf{d}_{1,1} \mathbf{r}_1 \mathbf{u}_{2,2} + \mathbf{d}_{1,2} \mathbf{p}_{zt,2} \mathbf{u}_{2,2}]_q = [\mathbf{w}_{1,2}]_q$,

$\mathbf{v}_{2,1} = [\mathbf{d}_{2,1} \mathbf{p}_{zt,1} \mathbf{u}_{1,1} + \mathbf{d}_{2,2} \mathbf{r}_2 \mathbf{u}_{1,1} + \mathbf{d}_{2,1} \mathbf{r}_1 \mathbf{u}_{2,1} + \mathbf{d}_{2,2} \mathbf{p}_{zt,2} \mathbf{u}_{2,1}]_q = [\mathbf{w}_{2,1}]_q$,

$\mathbf{v}_{2,2} = [\mathbf{d}_{2,1} \mathbf{p}_{zt,1} \mathbf{u}_{1,2} + \mathbf{d}_{2,2} \mathbf{r}_2 \mathbf{u}_{1,2} + \mathbf{d}_{2,1} \mathbf{r}_1 \mathbf{u}_{2,2} + \mathbf{d}_{2,2} \mathbf{p}_{zt,2} \mathbf{u}_{2,2}]_q = [\mathbf{d}_{2,2} \mathbf{p}_{zt,2} \mathbf{u}_{2,2} + \mathbf{w}_{2,2}]_q$.

By $\mathbf{d}_{1,2} \bmod \mathbf{f} = \mathbf{d}_{2,1} \bmod \mathbf{f} = \mathbf{0}$, $\mathbf{u}_{1,2} \bmod \mathbf{f} = \mathbf{u}_{2,1} \bmod \mathbf{f} = \mathbf{0}$, and

$\mathbf{p}_{z,t} = \left[\mathbf{h}_t \mathbf{f}_t^{-1} \right]_q$, $t \in [2]$, we have $\left\| \left[\mathbf{w}_{t,s} \right]_q \right\| = \left\| \mathbf{w}_{t,s} \right\| \leq 4n^{12} \xi^2$.

(1) If \mathbf{U} is an encoding of zero, then $\mathbf{u}_{t,t} \bmod \mathbf{f}_t = \mathbf{0}$, $t \in [2]$. That is,

$$\left\| \left[\mathbf{d}_{t,t} \mathbf{p}_{z,t} \mathbf{u}_{t,t} \right]_q \right\| \leq n^{12} \xi^2, t \in [2].$$

As such, we derive the following expression:

$$\begin{aligned} \|v\| &= \left\| \left[\mathbf{s}^T \cdot \begin{pmatrix} \text{Rot}(\mathbf{v}_{1,1}) & \text{Rot}(\mathbf{v}_{1,2}) \\ \text{Rot}(\mathbf{v}_{2,1}) & \text{Rot}(\mathbf{v}_{2,2}) \end{pmatrix} \cdot \mathbf{t} \right]_q \right\| \\ &\leq n^2 \|\mathbf{s}\| \left\| \begin{pmatrix} \text{Rot}(\mathbf{v}_{1,1}) & \text{Rot}(\mathbf{v}_{1,2}) \\ \text{Rot}(\mathbf{v}_{2,1}) & \text{Rot}(\mathbf{v}_{2,2}) \end{pmatrix} \right\| \|\mathbf{t}\| \\ &< n^2 \|\mathbf{s}\| (\|\mathbf{v}_{1,1}\| + \|\mathbf{v}_{1,2}\| + \|\mathbf{v}_{2,1}\| + \|\mathbf{v}_{2,2}\|) \|\mathbf{t}\|. \\ &\leq n^4 \times 4 \times 5n^{12} \xi^2 \\ &\leq q / 2^n \end{aligned}$$

(2) If \mathbf{U} is not an encoding of zero, then $\mathbf{u}_{t,t} \bmod \mathbf{f}_t \neq \mathbf{0}$ for at least one $t \in [2]$.

That is, by Lemma 4 [GGH13], $\left\| \left[\mathbf{d}_{t,t} \mathbf{p}_{z,t} \mathbf{u}_{t,t} \right]_q \right\| \geq q^{1-\varepsilon}$ with high probability for at least one $t \in [2]$. Without loss of generality, assume $\mathbf{d}_{t,t} \bmod \mathbf{f}_t \neq \mathbf{0}$, $t \in [2]$, otherwise \mathbf{D} can be considered as an encoding of zero.

As such, we derive the following expression:

$$\begin{aligned} \|v\| &= \left\| \left[\mathbf{s}^T \cdot \begin{pmatrix} \text{Rot}(\mathbf{v}_{1,1}) & \text{Rot}(\mathbf{v}_{1,2}) \\ \text{Rot}(\mathbf{v}_{2,1}) & \text{Rot}(\mathbf{v}_{2,2}) \end{pmatrix} \cdot \mathbf{t} \right]_q \right\| \\ &\geq \left\| \left[\mathbf{s}^T \begin{pmatrix} \text{Rot}(\mathbf{d}_{1,1} \mathbf{p}_{z,1} \mathbf{u}_{1,1}) & \\ & \text{Rot}(\mathbf{d}_{2,2} \mathbf{p}_{z,2} \mathbf{u}_{2,2}) \end{pmatrix} \cdot \mathbf{t} \right]_q \right\| - q / 2^n, \\ &\geq q^{1-\varepsilon} - q / 2^n \\ &\geq q^{1-\varepsilon'} \end{aligned}$$

where $\varepsilon, \varepsilon'$ are small positive constants.

As such, $\text{isZero}(\text{par}, \mathbf{D}, \mathbf{U})$ correctly decides whether \mathbf{U} is an encoding of zero. ■

Lemma 3.13 Suppose that \mathbf{D} is a level-0 encoding and

$$\mathbf{U}_r = \mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\mathbf{u}_{1,1,r}) & \text{Rot}(\mathbf{u}_{1,2,r}) \\ \text{Rot}(\mathbf{u}_{2,1,r}) & \text{Rot}(\mathbf{u}_{2,2,r}) \end{pmatrix} \mathbf{T}, r \in [2]$$

two level- k encodings. If $\mathbf{U}_r, r \in [2]$ encode the same level-0 elements, namely, $\mathbf{u}_{t,t,1} = \mathbf{u}_{t,t,2} = \mathbf{a}_t(\mathbf{g}_t)^k \bmod \mathbf{f}_t, t \in [2]$, then we derive the following expression:

$$\text{Ext}(\text{par}, \mathbf{D}, \mathbf{U}_1) = \text{Ext}(\text{par}, \mathbf{D}, \mathbf{U}_2).$$

Proof. Given that $\mathbf{u}_{t,t,1} = \mathbf{u}_{t,t,2} = \mathbf{a}_t(\mathbf{g}_t)^k \bmod \mathbf{f}_t$, $t \in [2]$, we obtain $\mathbf{u}_{t,t,r} = \mathbf{a}_t(\mathbf{g}_t)^k + \mathbf{r}_t \mathbf{f}_t, r \in [2]$. As such, we derive the following expressions using similar notations of Lemma 3.12:

$$\begin{aligned}
v_r &= \left[\mathbf{s}^* \cdot \mathbf{D} \cdot \mathbf{P}_{zt} \cdot \mathbf{U}_1 \cdot \mathbf{t}^* \right]_q \\
&= \left[\mathbf{s}^T \cdot \begin{pmatrix} \text{Rot}(\mathbf{v}_{1,1,r}) & \text{Rot}(\mathbf{v}_{1,2,r}) \\ \text{Rot}(\mathbf{v}_{2,1,r}) & \text{Rot}(\mathbf{v}_{2,2,r}) \end{pmatrix} \cdot \mathbf{t} \right]_q \\
&= \left[\mathbf{s}^T \begin{pmatrix} \text{Rot}(\mathbf{d}_{1,1} \mathbf{p}_{z,1} \mathbf{u}_{1,1,r} + \mathbf{w}_{1,1,r}) & \text{Rot}(\mathbf{w}_{1,2,r}) \\ \text{Rot}(\mathbf{w}_{2,1,r}) & \text{Rot}(\mathbf{d}_{2,2} \mathbf{p}_{z,2} \mathbf{u}_{2,2,r} + \mathbf{w}_{2,2,r}) \end{pmatrix} \mathbf{t} \right]_q, \\
&= [v + w_r]_q
\end{aligned}$$

$$\begin{aligned}
\text{where } v &= \left[\mathbf{s}^T \begin{pmatrix} \text{Rot}(\mathbf{d}_{1,1} \mathbf{p}_{z,1} \mathbf{a}_1(\mathbf{g}_1)^k) & \\ & \text{Rot}(\mathbf{d}_{2,2} \mathbf{p}_{z,2} \mathbf{a}_2(\mathbf{g}_2)^k) \end{pmatrix} \mathbf{t} \right]_q, \\
w_r &= \left[\mathbf{s}^T \begin{pmatrix} \text{Rot}(\mathbf{d}_{1,1} \mathbf{p}_{z,1} \mathbf{r}_1 \mathbf{f}_1 + \mathbf{w}_{1,1,r}) & \text{Rot}(\mathbf{w}_{1,2,r}) \\ \text{Rot}(\mathbf{w}_{2,1,r}) & \text{Rot}(\mathbf{d}_{2,2} \mathbf{p}_{z,2} \mathbf{r}_2 \mathbf{f}_2 + \mathbf{w}_{2,2,r}) \end{pmatrix} \mathbf{t} \right]_q, r \in \llbracket 2 \rrbracket.
\end{aligned}$$

According to our parameters, $\left\| \left[\mathbf{d}_{t,t} \mathbf{p}_{z,t} \mathbf{r}_t \mathbf{f}_t \right]_q \right\| \leq n^{12} \xi^2$. On the basis of Lemma 3.12, we obtain $\left\| [w_r]_q \right\| \leq q/2^\eta$, $r \in \llbracket 2 \rrbracket$. Thus, the $\gamma = \eta - \lambda$ most significant bits of v_1, v_2 , which are the same with high probability, are decided on the basis of the first term v . ■

3.3 Security

The security of our constructions depends on new hardness assumptions and cannot be reduced to classic hard problems, such as lattice hard problem or LWE.

We adaptively extend the definition of ext-GCDH/ext-GDDH in [LSS14] to our construction. Consider the following process:

- (1) $(\text{par}) \leftarrow \text{InstGen}(1^\lambda)$.
- (2) Select an arbitrary positive integer k .
- (3) For $t = 0$ to k :
 - Sample $w_{t,i} \leftarrow D_{\mathbb{Z}, \sigma^i}$, $i \in \llbracket \tau \rrbracket$
 - Generate level-1 encoding of $\mathbf{D}_t = (\sum_{i=1}^{\tau} w_{t,i} \cdot \mathbf{X}_i) \bmod \mathbf{M}$:
 - $\mathbf{U}_t = (\sum_{i=1}^{\tau} w_{t,i} \cdot \mathbf{Y}_i) \bmod \mathbf{N}$.
- (4) Sample $r_{0,i} \leftarrow D_{\mathbb{Z}, \sigma^i}$, $i \in \llbracket \tau \rrbracket$ and generate $\mathbf{R}_0 = (\sum_{i=1}^{\tau} r_{0,i} \cdot \mathbf{X}_i) \bmod \mathbf{M}$.
- (5) Compute $\mathbf{U} = \prod_{t=1}^k \mathbf{U}_t \bmod \mathbf{N}$.
- (6) Set $v_C = v_D = \text{Ext}(\text{par}, \mathbf{D}_0, \mathbf{U})$.
- (7) Set $v_R = \text{Ext}(\text{par}, \mathbf{R}_0, \mathbf{U})$.

Definition 3.14 (ideal-ext-GCDH/ideal-ext-GDDH). The extraction k -graded computational Diffie-Hellman problem (ideal-ext-GCDH) is on input $\{\text{par}, \mathbf{U}_0, \dots, \mathbf{U}_k\}$ to output an extraction encoding $v_C \in \mathbb{Z}$. The extraction k -graded decisional Diffie-Hellman problem (ideal-ext-GDDH) distinguishes between v_D and v_R , that is, between the distributions $D_{\text{GDDH}} = \{\text{par}, \mathbf{U}_0, \dots, \mathbf{U}_k, v_D\}$ and $D_{\text{RAND}} = \{\text{par}, \mathbf{U}_0, \dots, \mathbf{U}_k, v_R\}$.

In this study, we assume that the ideal-ext-GCDH/ideal-ext-GDDH is hard.

3.4 Cryptanalysis

3.4.1 Easily Computable Quantities

Given $\mathbf{N}_j, \mathbf{M}_j, j \in \llbracket N \rrbracket$ in the public parameters, we derive the following expressions:

$$\mathbf{M}_j = \mathbf{S} \begin{pmatrix} \text{Rot}(\mathbf{q}) & \\ & \text{Rot}(\mathbf{q}) \end{pmatrix} \begin{pmatrix} \text{Rot}(\mathbf{m}_{1,1,j}) & \text{Rot}(\mathbf{m}_{1,2,j}) \\ \text{Rot}(\mathbf{m}_{2,1,j}) & \text{Rot}(\mathbf{m}_{2,2,j}) \end{pmatrix} \mathbf{S}^{-1},$$

$$\mathbf{N}_j = \mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\mathbf{q}) & \\ & \text{Rot}(\mathbf{q}) \end{pmatrix} \begin{pmatrix} \text{Rot}(\mathbf{n}_{1,1,j}) & \text{Rot}(\mathbf{n}_{1,2,j}) \\ \text{Rot}(\mathbf{n}_{2,1,j}) & \text{Rot}(\mathbf{n}_{2,2,j}) \end{pmatrix} \mathbf{T}.$$

By unimodular matrices \mathbf{S}, \mathbf{T} , we compute their determinants

$$\det(\mathbf{M}_j) = \det \begin{pmatrix} \text{Rot}(\mathbf{m}_{1,1,j}) & \text{Rot}(\mathbf{m}_{1,2,j}) \\ \text{Rot}(\mathbf{m}_{2,1,j}) & \text{Rot}(\mathbf{m}_{2,2,j}) \end{pmatrix} (\det(\text{Rot}(\mathbf{q})))^2,$$

$$\det(\mathbf{N}_j) = \det \begin{pmatrix} \text{Rot}(\mathbf{n}_{1,1,j}) & \text{Rot}(\mathbf{n}_{1,2,j}) \\ \text{Rot}(\mathbf{n}_{2,1,j}) & \text{Rot}(\mathbf{n}_{2,2,j}) \end{pmatrix} (\det(\text{Rot}(\mathbf{q})))^2.$$

As a result, the determinants $\det(\text{Rot}(\mathbf{q}))$ can be computed by using the GCD algorithm.

Given $\mathbf{Y}_i, \mathbf{X}_i$ in the public parameters, we can generate their determinants

$$\det(\mathbf{Y}_i) = \det \begin{pmatrix} \text{Rot}(\mathbf{a}_{1,1,i}\mathbf{g}_1 + \mathbf{d}_{1,1,i}\mathbf{f}_1) & \text{Rot}(\mathbf{d}_{1,2,i}\mathbf{f}) \\ \text{Rot}(\mathbf{d}_{2,1,i}\mathbf{f}) & \text{Rot}(\mathbf{a}_{2,2,i}\mathbf{g}_2 + \mathbf{d}_{2,2,i}\mathbf{f}_2) \end{pmatrix} (\det(\text{Rot}(\mathbf{h})))^2,$$

$$\det(\mathbf{X}_i) = \det \begin{pmatrix} \text{Rot}(\mathbf{a}_{1,1,i} + \mathbf{e}_{1,1,i}\mathbf{f}_1) & \text{Rot}(\mathbf{e}_{1,2,i}\mathbf{f}) \\ \text{Rot}(\mathbf{e}_{2,1,i}\mathbf{f}) & \text{Rot}(\mathbf{a}_{2,2,i} + \mathbf{e}_{2,2,i}\mathbf{f}_2) \end{pmatrix} (\det(\text{Rot}(\mathbf{h})))^2.$$

Similarly, the determinants $\det(\text{Rot}(\mathbf{h}))$ can be computed by using the GCD algorithm.

Given that $\mathbf{N}_j, \mathbf{M}_j, j \in \llbracket N \rrbracket$ are also encodings of zero, the following quantities are not reduced modulo q :

$$\mu_j = [\mathbf{s}^* \mathbf{P}_{zt} \mathbf{N}_j \mathbf{t}^*]_q, \nu_j = [\mathbf{s}^* \mathbf{M}_j \mathbf{P}_{zt} \mathbf{t}^*]_q, \theta_{j_1, j_2} = [\mathbf{s}^* \mathbf{M}_{j_1} \mathbf{P}_{zt} \mathbf{N}_{j_2} \mathbf{t}^*]_q.$$

It is easy to verify that $\pi_{i_1, i_2} = [\mathbf{s}^* (\mathbf{X}_{i_1} \mathbf{P}_{zt} \mathbf{Y}_{i_2} - \mathbf{X}_{i_2} \mathbf{P}_{zt} \mathbf{Y}_{i_1}) \mathbf{t}^*]_q, i_1, i_2 \in \llbracket \tau \rrbracket$ are not reduced modulo q .

For these quantities generated from encodings of zero, building a system of equations is possible. In fact, if we define a function $f_{\mathbf{s}, \mathbf{t}}(\mathbf{w}_i) = \mathbf{s}^T \cdot \text{Rot}(\mathbf{w}_i) \cdot \mathbf{t}$, then integers $\mu_j, \nu_j, \theta_{j_1, j_2}$, and π_{i_1, i_2} are values of $f_{\mathbf{s}, \mathbf{t}}$. Given only several values of $f_{\mathbf{s}, \mathbf{t}}$, we cannot solve \mathbf{s}, \mathbf{t} when unknown \mathbf{w}_i . Currently, we do not find efficient attack for the function $f_{\mathbf{s}, \mathbf{t}}$.

In addition, we notice the decisional linear (DLIN) and the subgroup membership (SubM) problems are easy in our construction since $\mathbf{M}_i = \mathbf{S} \text{Rot}(\mathbf{m}_i, \mathbf{q}) \mathbf{S}^{-1}$ and $\mathbf{N}_i = \mathbf{T}^{-1} \text{Rot}(\mathbf{n}_i, \mathbf{q}) \mathbf{T}$ are encodings of zero.

3.4.2 Attack of Pellet-Mary and Stehlé

The key point of Pellet-Mary and Stehlé attack [PS15] is to find inverse element of an encoding over modulo encoding of zero. Given the public parameters par , one requires to find $(\mathbf{Y}_i)^{-1} \bmod \mathbf{N}_j$ for $i \in \llbracket \tau \rrbracket, j \in \llbracket 4n \rrbracket$. It is easy to see that when $\mathbf{h} = 1$ in our scheme, one can compute $(\mathbf{Y}_i)^{-1} \bmod \mathbf{N}_j$. Without loss of generality, assume that $\det(\mathbf{Y}_i), \det(\mathbf{N}_j)$ are coprime. One first computes the adjacent matrices $\mathbf{Y}_i^*, \mathbf{N}_j^*$ of $\mathbf{Y}_i, \mathbf{N}_j$ such that $\mathbf{Y}_i \times \mathbf{Y}_i^* = \det(\mathbf{Y}_i) \mathbf{I}$, $\mathbf{N}_j \times \mathbf{N}_j^* = \det(\mathbf{N}_j) \mathbf{I}$, where \mathbf{I} is identity matrix. Then, one finds two integer a, b using Euclid algorithm such that $a \det(\mathbf{Y}_i) \mathbf{I} + b \det(\mathbf{N}_j) \mathbf{I} = \mathbf{I}$, namely $\mathbf{Y}_i \times (a \mathbf{Y}_i^*) + \mathbf{N}_j \times (b \mathbf{N}_j^*) = \mathbf{I}$. Finally, one inputs inverse element $(\mathbf{Y}_i)^{-1} = (a \mathbf{Y}_i^*) \bmod \mathbf{N}$, where modulo \mathbf{N} is to reduce the size of $a \mathbf{Y}_i^*$.

However, when $\mathbf{h} \neq 1$, we currently do not know how to compute $(\mathbf{Y}_i)^{-1} \bmod \mathbf{N}$. If one uses the above method, then one can only obtains $\mathbf{Y}_i \times (a \mathbf{Y}_i^*) + \mathbf{N}_i \times (b \mathbf{N}_i^*) = (\det(\text{Rot}(\mathbf{h})))^2 \mathbf{I}$. Thus, the Pellet-Mary and Stehlé attack [PS15] does not work in our construction.

3.4.3 Lattice Reduction Attack

Given that $\mathbf{N}_j, \mathbf{M}_j \in \mathbb{Z}^{2n \times 2n}, j \in \llbracket 4n \rrbracket$, one can attempt to use the lattice reduction algorithm to determine the secret elements in our construction. By using \mathbf{N}_j and \mathbf{M}_j , one generates the following lattices:

$$L_1(j_1, j_2) = \begin{pmatrix} \mathbf{M}_{j_1} \\ \mathbf{M}_{j_2} \end{pmatrix}, \quad L_2(j_1, j_2) = \begin{pmatrix} \mathbf{N}_{j_1} \\ \mathbf{N}_{j_2} \end{pmatrix}.$$

By applying the lattice reduction algorithm [LLL82], one obtains $\mathbf{E}_1 \begin{pmatrix} \text{Rot}(\mathbf{m}_{1,1,j} \mathbf{q}) & \text{Rot}(\mathbf{m}_{1,2,j} \mathbf{q}) \\ \text{Rot}(\mathbf{m}_{2,1,j} \mathbf{q}) & \text{Rot}(\mathbf{m}_{2,2,j} \mathbf{q}) \end{pmatrix} \mathbf{S}^{-1}$ and $\mathbf{E}_2 \begin{pmatrix} \text{Rot}(\mathbf{n}_{1,1,j} \mathbf{q}) & \text{Rot}(\mathbf{n}_{1,2,j} \mathbf{q}) \\ \text{Rot}(\mathbf{n}_{2,1,j} \mathbf{q}) & \text{Rot}(\mathbf{n}_{2,2,j} \mathbf{q}) \end{pmatrix} \mathbf{T}$.

However, for large enough n , \mathbf{E}_1 and \mathbf{E}_2 are not sure identity matrices according to current lattice reduction algorithm. If $\mathbf{E}_1 = \mathbf{E}_2 = \mathbf{I}$, one can solve \mathbf{T} and \mathbf{S} . When \mathbf{T} and \mathbf{S} are known, our construction can be broken. Thus, the dimension n of ring in our construction requires sufficiently large to prevent lattice reduction attack.

4 Variants

4.1 Asymmetric Variant

Asymmetric ideal multilinear maps with different groups are required in some applications. Similar to [GGH13], we briefly describe the asymmetric variant as follows:

In this variant, we use different ideal generators $\mathbf{g}_{t,l} \leftarrow D_{\mathbb{Z}^n, \sigma}, l \in \llbracket \beta \rrbracket$ to represent asymmetric maps. Similarly, one can generate the parameters of the inner layer $\mathbf{x}_{t,s,i,l}, \mathbf{y}_{t,s,i,l}$, for $t \in \llbracket 2 \rrbracket, s \in \llbracket 2 \rrbracket, i \in \llbracket \tau \rrbracket, l \in \llbracket \beta \rrbracket$ and the parameters of the outer layer

$$\mathbf{X}_{i,l} = \mathbf{S} \begin{pmatrix} \text{Rot}(\mathbf{x}_{1,1,i,l}) & \text{Rot}(\mathbf{x}_{1,2,i,l}) \\ \text{Rot}(\mathbf{x}_{2,1,i,l}) & \text{Rot}(\mathbf{x}_{2,2,i,l}) \end{pmatrix} \mathbf{S}^{-1} \quad \text{and} \quad \mathbf{Y}_{i,l} = \mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\mathbf{y}_{1,1,i,l}) & \text{Rot}(\mathbf{y}_{1,2,i,l}) \\ \text{Rot}(\mathbf{y}_{2,1,i,l}) & \text{Rot}(\mathbf{y}_{2,2,i,l}) \end{pmatrix} \mathbf{T}.$$

Thus, the public parameters become $\text{par}_1 = \left\{ q, \mathbf{M}, \mathbf{N}, \{ \mathbf{X}_{i,l}, \mathbf{Y}_{i,l} \}_{i \in [\tau], l \in [\beta]}, \mathbf{P}_z, \mathbf{s}^*, \mathbf{t}^* \right\}$.

4.2 Commutative Variant

In the commutative variant, we switch from the integer ring to the polynomial ring to decrease the size of the public parameters. On one hand, the dimension n in our construction must be sufficiently large to guarantee security. On the other hand, the number of level-1 encodings in the public parameter $\tau \geq n^2 + O(\lambda)$ should be able to prevent algebraic equation attack. Thus, the size of the public parameters is too large to be practical. As such, we use $R^{(y)} = \mathbb{Z}[y] / \langle y^m + 1 \rangle$ and $R_q^{(y)} = R^{(y)} / qR^{(y)}$ instead of \mathbb{Z} and \mathbb{Z}_q . We set $m = O(\lambda)$ and $n = O(1)$. Now, the variant works in polynomial ring $R = \mathbb{Z}[y, x] / \langle y^m + 1 \rangle \langle x^n + 1 \rangle$. In this case, the size of the public parameters is relative practical.

This commutative variant is the same as our construction in Section 3, except all operations are conducted over the ring $R^{(y)}$ and $R_q^{(y)}$ instead of \mathbb{Z} and \mathbb{Z}_q . It is not difficult to verify that this variant construction is correct.

4.3 Variant without Noise

According to analysis in 3.4.2, we can further construct new variant. Let

$$\mathbf{Y} = \mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\mathbf{y}_{1,1}) & \text{Rot}(\mathbf{y}_{1,2}) \\ \text{Rot}(\mathbf{y}_{2,1}) & \text{Rot}(\mathbf{y}_{2,2}) \end{pmatrix} \mathbf{T}, \quad \mathbf{N}_j = \mathbf{T}^{-1} \begin{pmatrix} \text{Rot}(\mathbf{n}_{1,1,j}) & \text{Rot}(\mathbf{n}_{1,2,j}) \\ \text{Rot}(\mathbf{n}_{2,1,j}) & \text{Rot}(\mathbf{n}_{2,2,j}) \end{pmatrix} \mathbf{T}, \quad j \in [4n]$$

such that for $t \in [2], s \in [2]$, $\mathbf{y}_{t,s} = \mathbf{h}\mathbf{a}_{t,s}$, $\mathbf{n}_{t,s,j} = \mathbf{h}\mathbf{b}_{t,s,j}$ with $\mathbf{h}, \mathbf{a}_{t,s}, \mathbf{b}_{t,s,j} \leftarrow D_{\mathbb{Z}^n, 2^\lambda}$.

Assume that the public parameters $\text{par}_2 = \left\{ \mathbf{Y}, \{ \mathbf{N}_j \}_{j \in [4n]} \right\}$. Given a random integer a , one generates an encoding $\mathbf{U} = (a\mathbf{Y}) \bmod \mathbf{N}$. The security of the variant depends on new hardness assumption. That is, we suppose that computing $\mathbf{Y}^{-1} \bmod \mathbf{N}$ is hard. Here, we assume that b_0, b_j satisfied to $b_0\mathbf{Y} + \sum_{j=1}^{4n} b_j\mathbf{N}_j = \mathbf{0}^{n \times n}$ are large enough.

Given $k+1$ encodings $\mathbf{U}_j = (a_j\mathbf{Y}) \bmod \mathbf{N}$, it is easy to verify that $\mathbf{C}_i = (a_i \prod_{j \neq i} \mathbf{U}_j) \bmod \mathbf{N}$, $i \in [k+1]$ are same. However, only given $\left\{ \text{par}_2, \{ \mathbf{U}_j \}_{j \in [k+1]} \right\}$, one currently cannot obtain \mathbf{C}_i .

Possible attacks. (1) Since $\mathbf{h} \neq 1$, there currently is efficient algorithm to solve $\mathbf{Y}^{-1} \bmod \mathbf{N}$ over the integers. (2) Given an encoding $\mathbf{U} = (a\mathbf{Y}) \bmod \mathbf{N}$, one can compute $\mathbf{U} \cdot \mathbf{Y}^{-1} = a\mathbf{I} + \sum_{j=1}^{4n} r_j(\mathbf{N}_j \cdot \mathbf{Y}^{-1})$ over \mathbb{Q} . However, one cannot remove the noise term and obtain the secret number a .

5 Applications

Using our construction of ideal multilinear maps, we describe two applications: the one-round

multipartite Diffie–Hellman key exchange protocol and witness encryption. Their security relies on the hardness assumption of the ideal-ext-GDDH.

5.1 Multipartite Key Exchange Protocol

Setup(1^λ). The output $(\text{par}) \leftarrow \text{InstGen}(1^\lambda)$ is used as the public parameter.

Publish(par, j). We let m be the number of participants. For $j \in \llbracket m \rrbracket$, each party j samples random elements $w_{j,i} \leftarrow D_{\mathbb{Z}, \sigma}, i \in \llbracket \tau \rrbracket$, which are used as secret keys. Thereafter, level-1 encoding $\mathbf{U}_j \leftarrow \text{Enc}\left(\text{par}, 1, \{w_{j,i}\}_{i \in \llbracket \tau \rrbracket}\right)$ is computed and published as a public key.

KeyGen($\text{par}, j, \{w_{j,i}\}_{i \in \llbracket \tau \rrbracket}, \{\mathbf{U}_j\}_{j \neq i}$). Each party j computes a level- $m-1$ encoding $\mathbf{C}_j = \prod_{r \neq j} \mathbf{U}_r$ and a level-0 encoding $\mathbf{D}_j = (\sum_{i=1}^{\tau} w_{j,i} \cdot \mathbf{X}_i) \bmod \mathbf{M}$, and extracts the common secret key $sk = \text{Ext}(\text{par}, \mathbf{D}_j, \mathbf{C}_j)$.

Remark 5.1 Given that each party merely requires the level-1 encoding be generated in the MPKE protocol, the parameters $\mathbf{s}^*, \mathbf{X}_i, \mathbf{P}_{zt}$ in par can be combined into a vector $\mathbf{p}_{zt,i} = \left[\mathbf{s}^* \cdot \mathbf{X}_i \cdot \mathbf{P}_{zt} \right]_q$. As such, the public parameters can be $\text{par}_2 = \left\{ q, \mathbf{N}, \{\mathbf{p}_{zt,i}, \mathbf{Y}_i\}_{i \in \llbracket \tau \rrbracket}, \mathbf{t}^* \right\}$.

Theorem 5.2 Suppose that the ideal-ext-GDDH is hard. Then our protocol is a one-round multipartite Diffie–Hellman key exchange protocol.

Proof. The proof is similar to the proof presented in Theorem 2 in [GGH13]. \blacksquare

5.2 Witness Encryption

5.2.1 Construction

Let integer K be a multiple of 3. An instance of 3-exact cover problem consists of a number K and a collection Set of subsets $S_1, S_2, \dots, S_\beta \subset \llbracket K \rrbracket$. The problem is to find a 3-exact cover of $\llbracket K \rrbracket$. For an instance of witness encryption, the public key is a collection Set and the public parameters par in our ideal construction, the secret key is a hidden 3-exact cover of $\llbracket K \rrbracket$.

Encrypt($1^\lambda, \text{par}, M$):

(1) For each $k \in \llbracket K \rrbracket$, generate a level-1 encoding $\mathbf{U}_k = \sum_{i=1}^{\tau} d_{k,i} \mathbf{Y}_i \bmod \mathbf{N}$, where $\mathbf{d}_k \leftarrow D_{\mathbb{Z}^\tau, \sigma}$.

(2) Generate an encryption key $sk = \text{Ext}(\text{par}, \mathbf{I}, \mathbf{U})$, where $\mathbf{U} = (\prod_{k=1}^K \mathbf{U}_k) \bmod \mathbf{N}$, and encrypt a message M into ciphertext C , where \mathbf{I} is the $2n \times 2n$ identity matrix.

(3) For each subset $S_i = \{i_1, i_2, i_3\}$ of Set , generate a level-3 encoding $\mathbf{U}_{S_i} = (\mathbf{U}_{i_1} \mathbf{U}_{i_2} \mathbf{U}_{i_3}) \bmod \mathbf{N}$.

(4) Output the ciphertext C and all level-3 encodings $E = \{\mathbf{U}_{S_i}, S_i \in Set\}$

corresponding to Set .

Decrypt(C, E, W):

- (1) Given C, E and a witness set W , compute $\mathbf{U} = (\prod_{S_i \in W} \mathbf{U}_{S_i}) \bmod \mathbf{N}$.
- (2) Generate $sk = \text{Ext}(\text{par}, \mathbf{I}, \mathbf{U})$, and decrypt C to get the plaintext M .

Correctness. The correctness of witness encryption directly follows that of our ideal construction.

Security. Similar to [GGSW13], the security of our construction depends on the assumption of the Decision Graded Encoding No-Exact-Cover.

Theorem 5.3 Suppose that the Decision Graded Encoding No-Exact-Cover is hard. Then our construction is a witness encryption scheme.

Proof. The proof is identical to one presented in Theorem 5.2 in [GGSW13]. \blacksquare

5.2.2 The Hu-Jia Attacks

(1) The Hu-Jia attack in [HJ15b] does not work in our construction. This is because one cannot compute the inverse $(\mathbf{U}_{S_i})^{-1} \bmod \mathbf{N}$ of $\mathbf{U}_{S_i} = (\mathbf{U}_{i_1} \mathbf{U}_{i_2} \mathbf{U}_{i_3}) \bmod \mathbf{N}$. Given \mathbf{U}_{S_i} and \mathbf{N} , one cannot find $(\mathbf{U}_{S_i})^{-1} \bmod \mathbf{N}$ by analysis in 3.4.2.

(2) The Hu-Jia attack in [HJ15a] does not apply to our construction. To attack the GGH-based WE [GGH13], Hu and Jia [HJ15a] first generate a combined 3-exact cover EC and compute the collection of positive factors CPF and the collection of negative factors CNF of EC (see [HJ15a]). Then, they compute $\mathbf{v}_{PPF} = \prod_{pf \in CPF} \mathbf{v}^{(pf)}$ and $\mathbf{v}_{PNF} = \prod_{nf \in CNF} \mathbf{v}^{(nf)}$, where $\mathbf{v}^{(pf)}, \mathbf{v}^{(nf)}$ are equivalent secrets. Finally, they solve $\mathbf{v}_{PTS} = \prod_{k=1}^K \mathbf{v}^{(k)}$ by using equation $(\mathbf{v}_{PPF} - \mathbf{v}_{PTS} \times \mathbf{v}_{PNF}) \in \langle \mathbf{g} \rangle$ and a basis of \mathbf{g} .

Now, we adapt their notation to our construction to obtain the following equation

$$\left(\prod_{pf \in CPF} \mathbf{U}_{(pf)} \right) \bmod \mathbf{N} = \left(\prod_{k=1}^K \mathbf{U}_k \right) \bmod \mathbf{N} \times \left(\prod_{nf \in CNF} \mathbf{U}_{(nf)} \right) \bmod \mathbf{N},$$

Given $\mathbf{U}_{PPF} = \left(\prod_{pf \in CPF} \mathbf{U}_{(pf)} \right) \bmod \mathbf{N}$ and $\mathbf{U}_{PNF} = \left(\prod_{nf \in CNF} \mathbf{U}_{(nf)} \right) \bmod \mathbf{N}$, find $\mathbf{U}_{PTS} = \left(\prod_{k=1}^K \mathbf{U}_k \right) \bmod \mathbf{N}$. Similar as the above (1), one cannot solve the inverse $(\mathbf{U}_{PNF})^{-1} \bmod \mathbf{N}$ of \mathbf{U}_{PNF} .

Remark 5.4 We select an element $\varphi \in \llbracket K \rrbracket$ such that $|S^{(\varphi)}| = \max \left\{ |S^{(\pi)}|, \pi \in \llbracket K \rrbracket \right\}$, where $S^{(\pi)} = \{S_i \mid (\pi \in S_i) \cap (S_i \in Set)\}$. For $S_i \in S^{(\varphi)}$, we modify $\mathbf{U}_{S_i} = (\mathbf{U}_{i_1} \mathbf{U}_{i_2} \mathbf{U}_{i_3}) \bmod \mathbf{N}$ into $\mathbf{u}_{S_i} = \left[\mathbf{s}^* \cdot \mathbf{I} \cdot \mathbf{P}_{z_t} \cdot \mathbf{U}_{S_i} \right]_q$. In this case, we do not increase the size of q since the level-0 encoding \mathbf{I} (identity matrix) compensates an increase multiplied by \mathbf{U}_{S_i} . When decrypting, we compute the secret key as follows:

$$sk = \text{Extract}_s \left(\text{msbs}_\gamma \left(\left[\mathbf{u}_{S_i \in W \cap S_i \in S^{(\varphi)}} \cdot \left(\prod_{S_i \in W \cap S_i \notin S^{(\varphi)}} \mathbf{U}_{S_i} \bmod \mathbf{N} \right) \cdot \mathbf{t}^* \right]_q \right) \right).$$

Using this countermeasure, we merely increase the difficulty that adversary attacks our witness encryption using a combined 3-exact cover. Since any subset in $S^{(\varphi)}$ cannot be used in any combined subset.

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