

# Two-faced processes and existence of RNG with proven properties

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## Abstract

Random and pseudorandom number generators (RNG and PRNG) are used for many purposes including cryptographic, modeling and simulation applications. For such applications a generated bit sequence should mimic true random, i.e., by definition, such a sequence could be interpreted as the result of the flips of a fair coin with sides that are labeled 0 and 1. It is known that the Shannon entropy of this process is 1 per letter, whereas for any other stationary process with binary alphabet the Shannon entropy is strictly less than 1. On the other hand, the entropy of the PRNG output should be much less than 1 bit (per letter), but the output sequence should look like truly random. We describe random processes for which those, in a first glance contradictory properties, are valid.

More precisely, it is shown that there exist binary-alphabet random processes whose entropy is less than 1 bit (per letter), but a frequency of occurrences of any word  $|u|$  goes to  $2^{-|u|}$ , where  $|u|$  is the length of  $u$ . In turn, it gives a possibility to construct RNG and PRNG which possess theoretical guarantees. This, in turn, is important for applications such

as those in cryptography.

**keywords:** random number generator, pseudorandom number generator, Shannon entropy, random process, true randomness

## 1 Introduction

Random numbers are widely used in cryptographic, simulation (e.g., in Monte Carlo methods) and modeling (e.g., computer games) applications. A generator of truly random binary digits generates such sequences  $x_1x_2\dots$  that, with probability one, for any binary word  $u$  the following property is valid:

$$\lim_{t \rightarrow \infty} \nu_t(u)/(t - |u|) = 2^{-|u|} \quad (1)$$

where  $\nu_t(u)$  is a number of occurrences of the word  $u$  in the sequence  $x_1\dots x_{|u|}$ ,  $x_2\dots x_{|u|+1}$ , ...,  $x_{t-|u|+1}\dots x_t$ . (As in most studies in this field, for brevity, we will consider the case when processes generate letters from the binary alphabet  $\{0, 1\}$ , but the results can be extended to the case of any finite alphabet.) The RNG and PRNG attract attention of many researchers due to its importance to practice and interest of theory, because, in a certain sense, this problem is close to foundations of probability theory, see, for example, [2, 5].

There are two types of methods for generating sequences of random digits: so called RNG and PRNG. The RNGs are based on digitizing of physical processes (like noises in electrical circuits), whereas PRNGs can be considered as computer programs whose input is a (short) word (called a seed) and the output is a long sequence (compared to the input). As a rule, the seed is a truly random sequence and the PRNG can be viewed as an expander of randomness which stretches a short truly random seed into a long sequence that is supposed to appear and behave as a true random sequence [4]. So, the purpose of RNG and PRNG is

to use low-entropy sources for generating sequences which look truly random. Note that the Shannon entropy of the truly random process (i.e., the Bernoulli with  $p(0) = p(1) = 1/2$ ) is 1 per letter, whereas for any other stationary process the entropy is strictly less than 1; see [3]. That is why, the properties of truly randomness and low entropy are, in a certain sense, contradictory.

There are a lot of papers devoted to RNG and PRNG, because they are widely used in cryptography and other fields. For example, the National Institute of Standards and Technology (NIST, USA) published a recommendation specifying mechanisms for the generation of random bits using deterministic methods [1]. Nowadays, quality of almost all practically used RNG and PRNG is estimated by statistical tests intended to find deviations from true randomness (see, for ex., NIST Statistical Test Suite [6]). Nevertheless, researchers look for RNG and PRNG with provable guarantees on their randomness because methods with proven properties are of great interest in cryptography.

In this paper we describe several kinds of random processes whose entropy can be much less than one, but, in a certain sense, they generate sequences for which the property of true randomness (1) is valid either for any integer  $k$  or for  $ks$  from a certain interval (i.e.  $1 < k < K$ , where  $K$  is an integer). It shows the existence of low-entropy RNGs and PRNGs which generate sequences satisfying the property (1). Besides, the description of the suggested processes show how they can be used to construct RNGs and PRNGs for which the property (1) is valid. Note that so-called two-faced processes, for which the property (1) is valid for a given  $k$  were described in [8, 7]. Here those processes are generalized and some new results concerning their properties are established.

More precisely, in this paper we describe the following two processes. First, we describe so-called two-faced process of order  $k$ ,  $k \geq 1$ , which is the  $k$ -order Markov chain and, with probability 1, for any sequence  $x_1 \dots x_t$  and any binary

word  $u \in \{0, 1\}^k$  the frequency of occurrence of the word  $u$  in the sequence  $x_1 \dots x_{|u|}$ ,  $x_2 \dots x_{|u|+1}$ ,  $\dots$ ,  $x_{t-|u|+1} \dots x_t$  goes to  $2^{-|u|}$ , where  $t$  grows. Second, we describe so-called twice two-faced processes for which this property is valid for any integer  $k$ . Besides, we show how such processes can be used to construct RNG and PRNG for which the property (1) is valid.

The paper is organized as follows: the next part contains descriptions of two-faced processes and transformations. The third part gives definitions of the so-called twice two-faced processes for which the property (1) valid for every integer  $k$ . In the conclusion we briefly discuss possible application of two-faced processes to RNG and PRNG.

## 2 Two-faced processes

First, we describe two families of random processes  $T_{k,\pi}$  and  $\bar{T}_{k,\pi}$ , where  $k = 1, 2, \dots$ , and  $\pi \in (0, 1)$  are parameters. The processes  $T_{k,\pi}$  and  $\bar{T}_{k,\pi}$  are Markov chains of the connectivity (memory)  $k$ , which generate letters from  $\{0, 1\}$ . It is convenient to define their transitional matrices inductively. The process matrix of  $T_{k,\pi}$  is defined by conditional probabilities  $P_{T_{1,\pi}}(0/0) = \pi$ ,  $P_{T_{1,\pi}}(0/1) = 1 - \pi$  (obviously,  $P_{T_{1,\pi}}(1/0) = 1 - \pi$ ,  $P_{T_{1,\pi}}(1/1) = \pi$ ). The process  $\bar{T}_{1,\pi}$  is defined by  $P_{\bar{T}_{1,\pi}}(0/0) = 1 - \pi$ ,  $P_{\bar{T}_{1,\pi}}(0/1) = \pi$ . Assume that transitional matrices  $T_{k,\pi}$  and  $\bar{T}_{k,\pi}$  are defined and describe  $T_{k+1,\pi}$  and  $\bar{T}_{k+1,\pi}$  as follows

$$\begin{aligned}
 P_{T_{k+1,\pi}}(0/0u) &= P_{T_{k,\pi}}(0/u), \\
 P_{T_{k+1,\pi}}(1/0u) &= P_{T(k,\pi)}(1/u), \\
 P_{T(k+1,\pi)}(0/1u) &= P_{\bar{T}(k,\pi)}(0/u), \\
 P_{T(k+1,\pi)}(1/1u) &= P_{\bar{T}(k,\pi)}(1/u),
 \end{aligned} \tag{2}$$

and, vice versa,

$$\begin{aligned}
P_{\bar{T}(k+1,\pi)}(0/0u) &= P_{\bar{T}(k,\pi)}(0/u), \\
P_{\bar{T}(k+1,\pi)}(1/0u) &= P_{\bar{T}(k,\pi)}(1/u), \\
P_{\bar{T}(k+1,\pi)}(0/1u) &= P_{T(k,\pi)}(0/u), \\
P_{\bar{T}(k+1,\pi)}(1/1u) &= P_{T(k,\pi)}(1/u)
\end{aligned} \tag{3}$$

for each  $u \in \{0,1\}^k$  (here  $vu$  is a concatenation of the words  $v$  and  $u$ ). For example,

$$\begin{aligned}
P_{T(2,\pi)}(0/00) &= \pi, P_{T(2,\pi)}(0/01) = 1 - \pi, \\
P_{T(2,\pi)}(0/10) &= 1 - \pi, P_{T(2,\pi)}(0/11) = \pi.
\end{aligned} \tag{4}$$

To define a process  $x_1x_2\dots$  the initial probability distribution needs to be specified. We define the initial distribution of the processes  $T(k, \pi)$  and  $\bar{T}(k, \pi)$ ,  $k = 1, 2, \dots$ , to be uniform on  $\{0,1\}^k$ , i.e.  $P\{x_1\dots x_k = u\} = 2^{-k}$  for any  $u \in \{0,1\}^k$ . On the other hand, sometimes processes with a different (or unknown) initial distribution will be considered; that is why, in both cases the initial state will be mentioned in order to avoid misunderstanding.

Let us define the Shannon entropy of a stationary process  $\mu$ . The conditional entropy of order  $m$ ,  $m = 1, 2, \dots$ , is defined by

$$h_m = - \sum_{u \in \{0,1\}^{m-1}} \mu(u) \sum_{v \in \{0,1\}} \mu(v/u) \log \mu(v/u) \tag{5}$$

and the limit Shannon entropy is defined by

$$h_\infty = \lim_{m \rightarrow \infty} h_m, \tag{6}$$

see [3].

The following theorem describes the main properties of the processes defined above.

**Theorem 1.** *Let a sequence  $x_1x_2\dots$  be generated by the process  $T(k, \pi)$  (or  $\bar{T}(k, \pi)$ ),  $k \geq 1$  and  $u$  be a binary word of length  $k$ . Then,*

*i) If the initial state obeys the uniform distribution over  $\{0, 1\}^k$ , then for any  $j \geq 0$*

$$P(x_{j+1}\dots x_{j+k} = u) = 2^{-|u|}. \quad (7)$$

*ii) for any initial state of the Markov chain  $T(k, \pi)$  (or  $\bar{T}(k, \pi)$ )*

$$\lim_{j \rightarrow \infty} P(x_{j+1}\dots x_{j+k} = u) = 2^{-|u|}. \quad (8)$$

*iii) For each  $\pi \in (0, 1)$  the  $k$ -order Shannon entropy ( $h_k$ ) of the processes  $T(k, \pi)$  and  $\bar{T}(k, \pi)$  equals 1 bit per letter whereas the limit Shannon entropy ( $h_\infty$ ) equals  $-(\pi \log_2 \pi + (1 - \pi) \log_2 (1 - \pi))$ .*

The proof of the theorem is given in the Appendix, but here we consider examples of “typical” sequences of the processes  $T(1, \pi)$  and  $\bar{T}(1, \pi)$  for  $\pi$ , say,  $1/5$ . Such sequences could be as follows: 010101101010100101... and 00001111100011111000.....

We can see that each sequence contains approximately one half of 1’s and one half of 0’s. (That is why the first order Shannon entropy is 1 per a letter.) On the other hand, both sequences do not look like truly random, because they, obviously, have too long subwords like either 101010.. or 000..11111... (In other words, the second order Shannon entropy is much less than 1 per letter.) So, informally, we can say that those sequences mimic truly random, if one takes into account only frequencies of words of the length one.

Due to Theorem 1, we give the following

**Definition 1.** *A random process is called asymptotically two-faced of order  $k$ , if the equation (8) is valid for all  $u \in \{0, 1\}^k$ . If the equation (7) is valid, the*

process is called two-faced of order  $k$ .

Theorem 1 shows that the processes  $T(k, \pi)$  and  $\bar{T}(k, \pi)$  are two-faced. The statements i) and ii) show that the processes look like truly random if we consider blocks whose length is less than the process order  $k$ . On the other hand, if we take into consideration blocks whose length is greater, the statement iii) shows that their distribution is far from uniform (if  $\pi$  is either small or large). Those properties explain the name “two-faced”.

The following theorem shows that, in a certain sense, there exist quite many two-faced processes.

**Theorem 2.** *Let  $X = x_1x_2\dots$  and  $Y = y_1y_2\dots$  be random processes. We define the process  $Z = z_1z_2\dots$  by equations  $z_1 = x_1 \oplus y_1$ ,  $z_2 = x_2 \oplus y_2, \dots$  where  $x_1x_2\dots$  and  $y_1y_2\dots$  are distributed according to  $X$  and  $Y$  and  $a \oplus b = (a + b) \bmod 2$ . Then, if  $X$  is a  $k$ -order two-faced process ( $k \geq 1$ ), then  $Z$  is a  $k$ -order two-faced process. If  $X$  is an asymptotically  $k$ -order two-faced process then  $Z$  is asymptotically  $k$ -order two-faced, too.*

### 3 Two-faced transformation

Earlier we described two-faced processes which, in a certain sense, mimic truly random. In this section we show how any Bernoulli process can be converted to a two-faced process. Informally, any sequence  $X = x_1x_2\dots$  created by Bernoulli process with  $P(x_i = 0) = \pi$ ,  $P(x_i = 1) = 1 - \pi$ , will be transformed into a sequence  $y_1y_2\dots$  of “letters”  $\pi$  and  $(1 - \pi)$  by a map  $0 \rightarrow \pi$ ,  $1 \rightarrow (1 - \pi)$ . Then this sequence can be considered as an input of the transition matrix  $T_{k,\pi}$  and a new sequence  $Y = y_1y_2\dots$  can be generated according to  $k$ -order two-faced process, if we have an initial state, i.e. a binary word of length  $k$ . For example, let  $k = 2$ , the initial state be 01 and  $x_1x_2\dots x_5 = 10010$ . Then,  $y_1\dots y_5$

$= (1 - \pi)\pi\pi(1 - \pi)\pi$  and, according to (4), we obtain a new sequence 01110. In fact, the output sequence is generated by the transition matrix  $T_{k,\pi}$ ; that is why the output process is 2-order two-faced.

Now we formally describe a family of transformations which, in a certain sense, convert random processes into two-faced ones. For this purpose we first define two families of matrices  $M_k$  and  $\bar{M}_k$ ,  $k \geq 1$ , which are connected with transition matrices  $T_{k,\pi}$  and  $\bar{T}_{k,\pi}$ .

**Definition 2.** For any  $k \geq 1$ ,  $v \in \{0, 1\}^k$ ,  $w \in \{0, 1\}$ , the matrix  $M_k$  is defined as follows:

$$M_k(w, v) = \begin{cases} 0, & \text{if } T_{k,\pi}(w, v) = \pi \\ 1, & \text{if } T_{k,\pi}(w, v) = 1 - \pi \end{cases} \quad (9)$$

$\bar{M}_k$  is obtained from  $\bar{T}_{k,\pi}$  analogously.

Informally, these matrices combine the two steps from the previous example. Namely, a transition from  $x_1x_2\dots$  to a sequence of symbols  $\pi, 1 - \pi$  and, second, transition from it to the new sequence of zeros and ones.

**Definition 3.** Let  $X = x_1x_2\dots$  be an infinite binary word,  $k > 0$  be an integer and  $v \in \{0, 1\}^k$ . The two-faced conversion  $\tau^k$  maps a pair  $(X, v)$  into an infinite binary sequence  $Y$  as follows:

$$y_{-k+1}y_{k+2}\dots y_0 = v ,$$

$$y_i = M_k(x_i, y_{i-k}y_{i-k+1}\dots y_{i-1}) \quad \text{if } 1 \leq i \quad (10)$$

where  $i = 1, 2, \dots$

It can be seen from definitions that the  $y_1y_2\dots$  is generated according to the transition matrix  $T_{k,\pi}$  if  $x_1x_2\dots$  generated by Bernoulli process with  $P(0) = \pi$ ,  $P(1) = (1 - \pi)$ . From this and Theorem 1 we obtain the following statement:



**Claim 1.** Let  $X = x_1x_2\dots$  be any Bernoulli process,  $k \geq 1$  be an integer and  $\tau_k$  be a two-faced transformation. If  $v$  is a word from  $\{0,1\}^k$ , then  $\tau^k(X, v)$  is asymptotically two-faced of order  $k$ . If, additionally,  $v$  obeys the uniform distribution on  $\{0,1\}^k$ , then  $\tau^k(X, v)$  is two-faced of order  $k$ .

## 4 Generalization

The  $k$ -order two-faced processes mimic true random ones for block lengths  $1, 2, \dots, k$ . Here we describe such processes that mimic true randomness for blocks of every length. By analogy with so-called twice universal codes known in information theory, we call such processes twice two-faced.

**Definition 4.** A random process is called (asymptotically) twice two-faced, if the equation (7) (8) is valid for every integer  $k$  and  $u \in \{0,1\}^k$ .

Now we describe a family of such processes.

Let  $n^* = n_1, n_2, \dots$  be an infinite sequence of integers such that  $n_1 < n_2 < n_3 \dots$  and  $X^1 = x_1^1x_2^1\dots$ ,  $X^2 = x_1^2x_2^2\dots$ ,  $X^3 = x_1^3x_2^3\dots$ , ... be (asymptotically) two-faced processes of order  $n_1, n_2, \dots$ , correspondingly. Define a process  $W = w_1w_2 \dots$  by

$$w_i = \begin{cases} x_i^1 & i \leq n_1, \\ x_i^1 \oplus x_i^2 & n_1 < i \leq n_2, \\ x_i^1 \oplus x_i^2 \oplus x_i^3 & n_2 < i \leq n_3, \\ \dots\dots\dots & \dots \end{cases} \quad (11)$$

and denote this process as  $\bigoplus_{i=1}^{\infty} X^i$ .

**Theorem 3.** If all  $X^i$   $i = 1, 2, \dots$  are two-faced then the process  $\bigoplus_{i=1}^{\infty} X^i$  is twice two-faced, i.e. for any binary word  $u$  the equation (7) is valid. If all  $X^i$  are asymptotically two faced, then the process  $\bigoplus_{i=1}^{\infty} X^i$  is asymptotically twice two-faced, i.e. equation (8) is valid for any word  $u$ .

It is worth noting that the total entropy of the processes  $X^1, X^2, \dots$  can be arbitrarily small, hence, the “input randomness” of the process  $\bigoplus_{i=1}^{\infty} X^i$  can be very small, whereas, in a certain sense, the process looks like a truly random.

## 5 Conclusion

In this paper we focus on the existence of processes whose entropy can be arbitrary small, but they mimic truly randomness in the sense that the frequency of occurrences of any word  $u$  asymptotically equals  $2^{-|u|}$ . In Conclusion we note how such processes can be directly used in order to construct (or “improve ” ) RNGs and PRNGs. For example, Theorems 2 and 3 shows that output sequence of any RNG and PRNG will, in a certain sense, looks like truly random, if it is summed with a (low-entropy) two-faced (or twice two-faced) process.

The possibility to transform Bernoulli processes into two-faced ones gives a possibility to create low-entropy two-faced processes. Indeed, schematically, it can be done as follows: Imagine, that one has a short word  $v$  (it corresponds to the seed of a PRNG) and wants to create a sequence  $V$ ,  $|V| > |v|$ , which could be considered as generated by a  $k$ -order two-faced process. Now denote  $h = |v|/|V|$  and let  $\pi$  be a solution of the equation  $-(\pi \log \pi + (1-\pi) \log(1-\pi)) = h$ . It is well-known in information theory that there exists a lossless code  $\varphi$  which compresses sequences generated by a Bernoulli process with probability  $(\pi, 1 - \pi)$  in such a way that the (average) length of output words is close to the Shannon entropy  $h$ , see [3]. Denote the decoder by  $\varphi^{-1}$  and let the sequence  $U$  be  $\varphi^{-1}(v)$ . Informally, this sequence will look like generated by a Bernoulli source with probabilities  $(\pi, 1 - \pi)$  and the final sequence  $V$  can be obtained from  $U$  by the transformation as described in the Definition 3. (We did not consider the initial  $k$ -bit words, which can be obtained, for example, as a part of the seed  $v$ . In such a case  $h$  can be defined as  $(|v| - k)/|V|$ .)

## 6 Appendix

*Proof* of Theorem 1. We prove the theorem for the process  $T_{k,\pi}$ , but this proof is valid for  $\bar{T}_{k,\pi}$ , too. First we show that

$$p^*(x_1 \dots x_k) = 2^{-k}, \quad (12)$$

$(x_1 \dots x_k) \in \{0, 1\}^k$ , is a stationary (or limit) distribution for the processes  $T_{k,\pi}$ . For any values of  $k, k \geq 1$ , (12) will be proved if we show that the system of equations

$$\begin{aligned} P_{T(k,\pi)}(x_1 \dots x_k) &= P_{T(k,\pi)}(0x_1 \dots x_{k-1}) P_{T(k,\pi)}(x_k/0x_1 \dots x_{k-1}) \\ &+ P_{T(k,\pi)}(1x_1 \dots x_{k-1}) P_{T(k,\pi)}(x_k/1x_1 \dots x_{k-1}); \\ \sum_{v \in \{0,1\}^k} p(v) &= 1 \end{aligned}$$

has the solution  $p(x_1 \dots x_k) = 2^{-k}$ ,  $(x_1 \dots x_k) \in \{0, 1\}^k$ . It can be easily seen, if we take into account that, by definitions (2) and (3), the equality  $P_{T(k,\pi)}(x_k/0x_1 \dots x_{k-1}) + P_{T(k,\pi)}(x_k/1x_1 \dots x_{k-1}) = 1$  is valid for all  $(x_1 \dots x_k) \in \{0, 1\}^k$ . From this equality and the law of total probability we immediately obtain (12). Having taken into account that the initial distribution matches the stationary (limit) one, we obtain the first claim of the theorem (7). From definitions (2), (3), we can see that all transition probabilities are nonzero (they are either  $\pi$  or  $1 - \pi$ ). Hence, the Markov chain  $T(k, \pi)$  is ergodic and the equations (7) are valid due to ergodicity.

Let us prove the third claim of the theorem. From the definitions (2), (3) we can see that either  $P_{T(k,\pi)}(0/x_1 \dots x_k) = \pi$ ,  $P_{T(k,\pi)}(1/x_1 \dots x_k) = 1 - \pi$  or  $P_{T(k,\pi)}(0/x_1 \dots x_k) = 1 - \pi$ ,  $P_{T(k,\pi)}(1/x_1 \dots x_k) = \pi$ . From this and (5) we can

see that  $h_{k+1} = -(\pi \log_2 \pi + (1 - \pi) \log_2(1 - \pi))$  and, taking into account (6), we obtain  $h_\infty = -(\pi \log_2 \pi + (1 - \pi) \log_2(1 - \pi))$ . The theorem is proved.

*Proof* of Theorem 2. The following chain of equations proves the first claim of the theorem:

$$\begin{aligned}
& P\{z_{j+1} \dots z_{j+k} = u\} = \\
& \sum_{v \in \{0,1\}^k} P\{x_{j+1} \dots x_{j+k} = v\} P\{y_{j+1} \dots y_{j+k} = v \oplus u\} \quad (13) \\
& = 2^{-k} \sum_{v \in \{0,1\}^k} P\{y_{j+1} \dots y_{j+k} = u \oplus v\} = 2^{-k} \times 1 = 2^{-k}.
\end{aligned}$$

(Here we took into account (7) and the obvious equation  $v \oplus u \oplus v = u$ .) In order to prove the second statement, we note that by definitions,

$$\lim_{j \rightarrow \infty} P(x_{j+1} \dots x_{j+k} = u) = 2^{-|u|}$$

for any  $u \in \{0, 1\}^k$ , see (8). Hence, for any  $\delta$ ,  $\delta > 0$ , there exists  $J$  such that

$$|P(x_{j+1} \dots x_{j+k} = u) - 2^{-|u|}| < \delta \quad u \in \{0, 1\}^k$$

if  $j > J$ . From this inequality and the equation (13) we obtain

$$\begin{aligned}
& (2^{-k} - \delta) \sum_{v \in \{0,1\}^k} P\{y_{j+1} \dots y_{j+k} = u \oplus v\} \\
& \leq P\{z_{j+1} \dots z_{j+k} = u\} \leq \\
& (2^{-k} + \delta) \sum_{v \in \{0,1\}^k} P\{y_{j+1} \dots y_{j+k} = u \oplus v\}.
\end{aligned}$$

Taking into account that this sum equals 1, we obtain the following inequalities:

$$(2^{-k} - \delta) \leq P\{z_{j+1} \dots z_{j+k} = u\} \leq (2^{-k} + \delta).$$

It is true for any  $\delta > 0$ , hence (7) is valid and the process  $Z$  is asymptotically  $k$ -order two-faced. Theorem is proven.

*Proof of Theorem 3.* Let  $u$  be any binary word and  $|u| = k$ . Take such an integer  $n_i$  that  $k \leq n_i$  and consider the process  $S = \bigoplus_{j=1}^{i-1} X^j \oplus \bigoplus_{j=i+1}^{\infty} X^j$ . (Here  $U \oplus V = \{u_1 \oplus v_1 \ u_2 \oplus v_2 \ u_3 \oplus v_3 \ \dots\}$ ). Obviously,  $\bigoplus_{j=1}^{\infty} X^j = X^i \oplus S$ . The process  $X^i$  is (asymptotically)  $n_i$ -order two faced. Having taken into account Theorem 2 we can see that  $\bigoplus_{j=1}^{\infty} X^j$  is  $n_i$ -order two faced and, hence,  $k$ -order (asymptotically) two-faced (because  $k \leq n_i$ , hence (7) ( (8) ) is valid. . Theorem is proven.

## Acknowledgment

Research was supported by Russian Foundation for Basic Research (grant no. 15-29-07932).

## References

- [1] E. Barker and J. Kelsey. *Recommendation for Random Bit Generator (RBG) Constructions (DRAFT NIST Special Publication 800-90C)*. National Institute of Standards and Technology, 2012.
- [2] C. S. Calude. *Information and Randomness - An Algorithmic Perspective. 2nd Edition*. Springer-Verlag, 2002.
- [3] Thomas M. Cover and Joy A. Thomas. *Elements of information theory*. Wiley-Interscience, New York, NY, USA, 2006.
- [4] Pierre L'Ecuyer. *Random Number Generation and Quasi-Monte Carlo*. Wiley, 2014.

- [5] M. Li and P.M.B. Vitanyi. *An Introduction to Kolmogorov Complexity and Its Applications, 3rd edition*. Springer, New York, 2008.
- [6] Andrew Rukhin, Juan Soto, James Nechvatal, Miles Smid, Elaine Barker, Stefan Leigh, Mark Levenson, Mark Vangel, David Banks, Alan Heckert, James Dray, and San Vo. *A Statistical Test Suite for Random and Pseudorandom Number Generators for Cryptographic Applications*. National Institute of Standards and Technology, 2010.
- [7] B. Ryabko and A. Fionov. *Basics of Contemporary Cryptography for IT Practitioners*. World Scientific Publishing Co, 2005.
- [8] B. Ryabko and V. Monarev. Using information theory approach to randomness testing. *Journal of Statistical Planning and Inference*, 133(1):95–110, 2005.