

# Revisiting the Cryptographic Hardness of Finding a Nash Equilibrium

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## Abstract

The exact hardness of computing a Nash equilibrium is a fundamental open question in algorithmic game theory. This problem is complete for the complexity class PPAD. It is well known that problems in PPAD cannot be NP-complete unless  $NP = coNP$ . Therefore, a natural direction is to reduce the hardness of PPAD to the hardness of problems used in cryptography.

Bitansky, Paneth, and Rosen [FOCS 2015] prove the hardness of PPAD assuming the existence of quasi-polynomially hard indistinguishability obfuscation and sub-exponentially hard one-way functions. This leaves open the possibility of basing PPAD hardness on simpler, polynomially hard, computational assumptions.

We make further progress in this direction and reduce PPAD hardness directly to polynomially hard assumptions. Our first result proves hardness of PPAD assuming the existence of *polynomially hard* indistinguishability obfuscation ( $i\mathcal{O}$ ) and one-way permutations. While this improves upon Bitansky et al.'s work, it does not give us a reduction to simpler, polynomially hard computational assumption because constructions of  $i\mathcal{O}$  inherently seems to require assumptions with sub-exponential hardness. In contrast, *public key functional encryption* is a much simpler primitive and does not suffer from this drawback. Our second result shows that PPAD hardness can be based on *polynomially hard* compact public key functional encryption and one-way permutations. Our results further demonstrate the power of polynomially hard compact public key functional encryption which is believed to be weaker than indistinguishability obfuscation. Our techniques are general and we expect them to have various applications.

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# 1 Introduction

The problem of computing a *Nash equilibrium* is fundamental to algorithmic game theory. The hardness of this problem has attracted significant attention. Since a mixed Nash equilibrium is guaranteed to exist for every game [Nas51], the problem belongs to the complexity class TFNP [MP91]. In a series of works, originating from Papadimitriou [Pap94], the problem was established to be complete for the complexity class PPAD [DGP09, CDT09]. PPAD is a subclass of TFNP containing problems that reduce (in polynomial time) to a special problem called as END-OF-LINE (or EOL in short). Informally, EOL instance includes a “succinct” description of an exponential sized directed graph where each node has in-degree and out-degree at most 1 and a source node having in-degree 0 and out-degree 1. The goal is to find another source or a sink (having in-degree 1 and out-degree 0). It is easy to observe that such a node is guaranteed to exist by a simple parity argument.

The exact hardness of this problem, however, is still not fully understood. Since the class PPAD is *total*, it is unlikely to contain NP-complete problems unless polynomial hierarchy collapses to the first level [MP91, Pap94]. This is similar to the status of hardness assumptions in cryptography which are not believed to be NP-complete, but nevertheless, hard. Due to this similarity, cryptographic problems were suggested as natural candidates in [Pap94] for studying the hardness of PPAD. Indeed, the hardness of some total super-classes of PPAD, such as PPA and PPP, can already be reduced to “standard” cryptographic problems like factoring and collision-resistant hashing [Jer12]. However, such a reduction is not known for PPAD.

A natural extension of this idea is to consider cryptographic problems with a richer and more powerful structure. One of the richest cryptographic structure is *program obfuscation* as formulated by Barak, Goldreich, Impagliazzo, Rudich, Sahai, Vadhan, and Yang [BGI<sup>+</sup>12]. It is a compiler to transform any computer program into an “unintelligible one” while preserving its functionality. Ideally, the obfuscation of a program should be a “virtual black-box” (VBB), i.e., access to the obfuscated program should be no better than access to a black-box implementing the program [BGI<sup>+</sup>12]. Abbot, Kane and Valiant [AKV04] show that PPAD-hardness can be based on VBB obfuscation of a natural pseudo random function. Unfortunately, VBB obfuscation is impossible in general [BGI<sup>+</sup>12], and there are strong limitations to obfuscating pseudorandom functions [GK05, BCC<sup>+</sup>14], including the one in [AKV04].

A natural relaxation of VBB obfuscation is *indistinguishability obfuscation* ( $i\mathcal{O}$ ) [BGI<sup>+</sup>12]. Informally,  $i\mathcal{O}$  guarantees that the obfuscation of a circuit looks indistinguishable from the obfuscation of any other, functionally equivalent, circuit of same size. Starting from the work of Garg, Gentry, Halevi, Raykova, Sahai and Waters [GGH<sup>+</sup>13b], several candidate constructions [BR14, BGK<sup>+</sup>14, PST14, GLSW15, Zim15, AB15, GMS16] for  $i\mathcal{O}$  have been suggested based on various assumptions on multilinear maps [GGH13a] and public key functional encryption [AJ15, BV15a, AJS15].

Motivated by the progress on obfuscation, Bitansky, Paneth and Rosen [BPR15] revisit the hardness of PPAD and provide an elegant reduction to the hardness of  $i\mathcal{O}$ . This is the first reduction of its kind which reduces PPAD-hardness to the security of a concrete and plausible cryptographic primitive. This, together with the progress on  $i\mathcal{O}$ , gives hope to the possibility of basing PPAD-hardness on simpler, more standard cryptographic primitives.

## 1.1 Our contribution

In this work, we revisit the problem of reducing PPAD-hardness to rich and expressive cryptographic systems. We build upon the work of [BPR15] with two specific goals:

- **Rely on polynomial-hardness of  $i\mathcal{O}$ :** One drawback of the BPR reduction is that it requires  $i\mathcal{O}$  schemes with at least quasi-polynomial security. It is not clear if such a large loss in the reduction is necessary. Our first goal is to obtain an improved, polynomial time reduction.
- **Rely on simpler, polynomially hard, assumptions:** While tremendous progress has been made on justifying the security of current  $i\mathcal{O}$  schemes, ultimately the security of the resulting constructions still either relies on an exponential number of assumptions (basically, one per pair of circuits), or a polynomial set of assumptions with exponential loss in the reduction. Our second goal is thus to completely get rid of  $i\mathcal{O}$  or any other component with non-polynomial time flavor, and reduce PPAD-hardness to simpler, polynomially hard, assumptions.

With respect to our first goal, we prove the following theorem:

**Theorem 1** *Assuming the existence of polynomially hard one-way permutations and indistinguishability obfuscation for P/poly, the END-OF-LINE problem is hard for polynomial-time algorithms.*

This polynomially reduces the hardness of PPAD to  $i\mathcal{O}$  since PPAD is the class of problems that are reducible to the END-OF-LINE problem.

With respect to our second goal, we show that PPAD-hardness can be reduced to the security of *compact public-key functional encryption* ( $\mathcal{FE}$ ) in polynomial time. We note that polynomially hard public key functional encryption is a polynomially falsifiable assumption [Nao03].

A public key functional encryption ( $\mathcal{FE}$ ) scheme for general circuits [BSW11, O’N10] is similar to an ordinary (public-key) encryption scheme with the crucial difference that there are many decryption keys, each of which has an associated function  $f$ ; when an encryption of a message  $m$  is decrypted with a key for function  $f$ , it decrypts to the value  $f(m)$ . The intuitive security guarantee is that given the secret key corresponding to  $f$  and a ciphertext encrypting  $m$ , an adversary would not be able to get any information about  $m$  except  $f(m)$ . Our second result proves the following theorem:

**Theorem 2** *Assuming the existence of polynomially-hard one-way permutations and compact public key functional encryption for general circuits, the END-OF-LINE problem is hard for polynomial-time algorithms.*

Compact functional encryption, as demonstrated by the recent results of Bitansky and Vaikuntanathan [BV15b] and Ananth, Jain and Sahai [AJS15], can be generically constructed from the so called “collusion-resistant function encryption with collusion-succinct ciphertexts,” which in turn can be constructed from simpler polynomial hardness assumptions over multi-linear maps, as shown by Garg, Gentry, Halevi, and Zhandry [GGHZ16]. This is in sharp contrast to  $i\mathcal{O}$  where all constructions still inherently seem to require exponential loss in the security reduction.<sup>1</sup> Combined

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<sup>1</sup>An informal explanation of this observation appears in [GLSW15].

with the results of [GGHZ16, BV15b, AJS15], theorem 2 bases PPAD-hardness on simpler polynomial hardness assumptions. It is interesting to note that compact public key functional encryption implies indistinguishability obfuscators [AJ15, BV15a] but with sub-exponential security loss.

## 1.2 Our Techniques

We now present a technical overview of our approach. Building upon the work of [BPR15], it suffices to show a sampling procedure that samples hard instances of SINK-OF-VERIFIABLE-LINE problem. We will first show how to generate such instances using polynomially-hard  $i\mathcal{O}$  and then discuss how to do the same using polynomially-hard  $\mathcal{FE}$ .

### 1.2.1 PPAD Hardness from Indistinguishability Obfuscation

Let us start by recalling the definition of PPAD. The class PPAD is defined to be the set of all *total* search problems that are polynomial time reducible to the END-OF-LINE (EOL) problem. Intuitively, an EOL instance includes a *succinct* description of an exponential sized directed graph with each node having in-degree and out-degree at most 1. Given a source node (which has in-degree 0 and out-degree 1), the goal is to find another source or a sink (which has in-degree 1 and out-degree 0). By a simple parity argument one can observe that such a node is guaranteed to exist.

The hardness of PPAD was proven in [BPR15] by considering a different problem, proposed in [AKV04], called SINK-OF-VERIFIABLE-LINE problem (SVL) in [BPR15]. It was shown that SVL reduces to the EOL problem [AKV04, BPR15], and therefore hardness of SVL implies hardness of EOL and PPAD.

An instance of the SVL problem is specified by a tuple  $(x_s, \text{Succ}, \text{Ver}, T)$  where  $x_s$  is called the source node,  $\text{Succ}$  and  $\text{Ver}$  are called successor and verification circuits respectively, and  $T$  is a target index.  $\text{Succ}$  succinctly defines an (exponential sized) directed *line graph* starting from the source node  $x_s$ . That is, a node  $x$  is connected to a node  $y$  in the graph through an outgoing edge if and only if  $y = \text{Succ}(x)$ .  $\text{Ver}$  is used to verify whether a given node is the  $i^{\text{th}}$  node (starting from the source node  $x_s$ ) on the path defined by  $\text{Succ}$ . To be more precise,  $\text{Ver}(x, i) = 1$  if and only if  $x = \text{Succ}^{i-1}(x_s)$ . The goal, given the instance, is to find the  $T$ -th node (Target) on the path. We want to construct an efficiently samplable distribution over instances of SVL for which no polynomial time algorithm can find the  $T$ -th node with non-negligible probability.

**BPR approach.** Bitansky et al., building upon [AKV04], consider a line graph where the  $i$ -th node is defined by the output of pseudorandom function (PRF) on  $i$ , i.e., the  $i$ -th node is  $(i, \sigma)$  such that  $\sigma = \text{PRF}_S(i)$  for a randomly chosen key  $S$ . Intuitively,  $\sigma$  is a signature on  $i$ . The successor circuit of the hard SVL instance,  $\text{Succ}$ , is then defined by *obfuscating* a “verify and sign” circuit,  $\text{VS}_S$ , using general purpose  $i\mathcal{O}$ ;  $\text{VS}_S$  simply outputs the next point  $(i + 1, \text{PRF}_S(i + 1))$  if the input is a valid point  $(i, \sigma)$  and rejects otherwise. The verification circuit  $\text{Ver}$  simply tests that a given input will not be rejected by the successor circuit. The source node is given by  $(1, \text{PRF}_S(1))$  and the target index  $T$  is set to a super-polynomial value in the security parameter.

Intuitively, the hardness of the above instance relies on the fact that it is impossible to obtain a signature on a node before obtaining the signature on the previous node in the path. Since  $T$  is super-polynomial in the security parameter, it follows that no polynomial time algorithm can obtain a signature on  $T$ . While the underlying idea of this reduction is intuitive, reducing its hardness to  $i\mathcal{O}$  is more involved. This is shown by first changing the obfuscated circuit  $\text{Succ}$  so that it does not

behave correctly on a randomly chosen point  $u$ , and simply outputs  $\perp$ . One can think of the Succ circuit being “punctured” at point  $u$ . This would also imply that the “punctured” circuit does not output a signature on  $u + 1$  unlike the original circuit. The next step uses *this fact* to “puncture” the circuit at the point  $u + 1$ . This step is realized through the “punctured” programming approach of Sahai and Waters [SW14]. At a high level, this process is then repeated for the next point  $u + 2$ , and then for  $u + 3$ , and so on, until the circuit does not have the ability to sign on any point in the interval  $[u, T]$ . Once the circuit is “punctured” at  $T$ , it can be observed that no algorithm can find the  $T^{\text{th}}$  node with non-zero probability. Performing these changes however, requires more care since the number of points in  $[u, T]$  is not polynomial. In hindsight, the primary reason for sub-exponential loss in this approach is because it is not possible to “puncture” a larger interval in a “single shot.” In particular, to be able to use the security of  $i\mathcal{O}$ , this approach must increase the “punctured” interval by one point at a time.

**Our approach: many chains of varying length.** Our main idea is to introduce a richer structure to the nodes in the graph, that avoids the need to increase the “punctured” interval by one point at a time. Instead, we want to make longer “jumps,” sometimes of exponential length, in the proof strategy. Specifically, we aim to make only polynomially many jumps in total to travel from  $u$  to  $T$ .

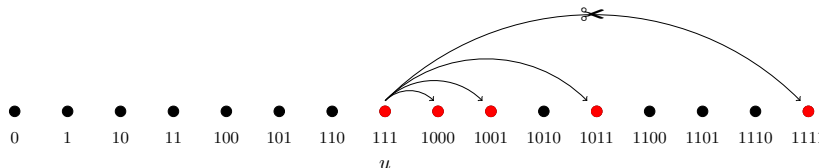
In particular, instead of considering one signature per node, we consider  $\kappa$  signatures for every node where  $2^\kappa$  is the total number of nodes on the line. That is, a node in our graph is of the form  $(i, \sigma_1, \dots, \sigma_\kappa)$  where  $\sigma_j$  is a signature on the first  $j$  bits of  $i$  computed using a key  $S_j$  (different for each index) for every  $j \in [\kappa]$ . The successor circuit is obfuscation of a program which simply checks each signature on appropriate prefixes of  $i$ , and if so, it signs all  $\kappa$  prefixes of  $i + 1$  using appropriate keys. The verification circuit is as before, the source node is simply the signatures on the first node, i.e.  $(0^\kappa, \text{PRF}_{S_1}(0), \dots, \text{PRF}_{S_\kappa}(0^\kappa))$ , and  $T = 2^\kappa - 1$ . Observe that the BPR reduction is equivalent to having only  $\sigma_\kappa$ .

We now explain how this structure on the nodes helps us in achieving a polynomial loss in the reduction. As before, we start by “puncturing” the successor circuit on a random point  $u$ . To illustrate the main idea, let us assume that the binary representation of  $u$  has  $k$  trailing 1s, i.e.,  $u$  is of the form:  $u_1 \cdots u_{\kappa-k-1} \| 01^k$  where  $1 \leq k \leq \kappa$ . Then,  $u + 1 = u_1 \cdots u_{\kappa-k-1} \| 10^k$ , i.e., it has  $k$  trailing 0s. Observe that:

1. The first  $\kappa - k$  prefix bits of  $u + 1$  are *identical* to the first  $\kappa - k$  prefix bits of *all points* in the interval  $[u + 1, u + 2^k]$ .
2. Signature  $\sigma_{\kappa-k}$  (corresponding to the prefix of length  $\kappa - k$ ) for the node  $u + 1$  is not needed (for checking and signing) anywhere else on the line graph except for nodes in the interval  $[u + 1, u + 2^k]$ .

As before, suppose that we have punctured the successor circuit at a random node  $u$ . Then, the fact that the punctured circuit does not output *any* signature on  $u + 1$  means that it does not output the signature  $\sigma_{\kappa-k}$  on the first  $\kappa - k$  bits of  $u + 1$ ; consequently, and most importantly, this means that it does not output this signature on the first  $\kappa - k$  bits of *any point in the interval*  $[u + 1, u + 2^k]$ . This allows us to increase the interval from  $[u + 1, u + 2^k]$  by considering only a constant number of hybrids. We then repeat this process by considering  $u + 2^k$  as our next point and iterate until we reach  $T$ .

Metaphorically, the signatures can be thought of as “virtual chains” emanating from each node and connecting to other nodes. The first chain coming out of a node  $i$  is connected to  $i$ ’s immediate neighbor which is  $i + 1$ . The second chain is connected to a node two hops away from  $i$  and the  $j$ -th chain is connected to a node  $2^j$  hops away from  $i$  and so on. The number of chains coming out from a node  $i$  is one more than the number of trailing ones in the binary representation of  $i$ . Equivalently, the number of chains coming out of  $i$  is the number of bits that change from  $i$  to  $i + 1$ . Puncturing the circuit is viewed as cutting chains of appropriate lengths between points. While BPR strategy always cuts a chain of length 1, our proof strategy cuts the longest possible chain it can and then iterates the process again until it reaches the target  $T$ . See Figure 1 for an illustration.



**Figure 1:** Illustration of cutting a chain for  $u = 0111$

While implementing the above idea we face the difficulty that for a random  $u$  the number of chains coming out of  $u$  could be very small (as small as 1). We get over this difficulty by initially cutting “smaller” length chains until we have the ability to cut “larger” length chains. Intuitively, this is made possible since the number of trailing 1s in  $u + 2^k$  is strictly larger than the number of trailing 1s (given by  $k$ ) in  $u$ . We show that we need to cut no more than a linear (in the security parameter  $\kappa$ ) number of chains to reach  $T$  and hence our reduction suffers only a polynomial (in fact linear) loss in the security parameter.

### 1.2.2 PPAD Hardness from Functional Encryption

We now give a technical overview of our hardness result for PPAD from *compact* functional encryption with *polynomial loss*. As noted earlier, although  $i\mathcal{O}$  can be reduced to compact  $\mathcal{FE}$  [AJ15, BV15a], we cannot directly rely on this reduction since it suffers sub-exponential security loss. Instead, we try to directly reduce PPAD-hardness to compact  $\mathcal{FE}$ .

To directly reduce PPAD-hardness to  $\mathcal{FE}$ , we follow the same approach as before, and generate hard on average instances of SVL using functional encryption. To demonstrate the technical challenges while proving the result from  $\mathcal{FE}$  we will be considering a single PRF key, as in BPR [BPR15], instead of our idea of using  $\kappa$  keys to implement “multiple chains of varying length.” The scenario with a single PRF key already captures the main technical challenges while keeping the exposition simple. Later, we will explain how to combine the two ideas together to obtain a direct polynomial reduction to  $\mathcal{FE}$ .

The line graph implicitly defined by this successor circuit will be similar to the BPR reduction as before. The successor circuit encodes a pseudo random function  $\text{PRF}_S : \{0, 1\}^\kappa \rightarrow \{0, 1\}^\kappa$  in its description. The source node is given by  $(0^\kappa, \text{PRF}_S(0^\kappa))$ . A node  $(x, \sigma)$  is present on the line graph if and only if  $\sigma = \text{PRF}_S(x)$ . The successor circuit takes as input  $(x, \sigma)$ , checks the validity of the node and if the node is valid outputs  $(x + 1, \text{PRF}_S(x + 1))$ . The target index is given by  $2^\kappa - 1$ .

Our goal is to produce an “obfuscated” (or encrypted) version of this successor circuit using  $\mathcal{FE}$ . To do this, we will rely on the “binary tree construction” idea of [AJ15, BV15a] for constructing  $i\mathcal{O}$  from  $\mathcal{FE}$ . Note that though this reduction suffers from sub-exponential loss and we tailor the construction of our successor circuit so that it suffers only from a polynomial loss.

**Binary tree based evaluation [AJ15, BV15a].** Let us first recall the main ideas of [AJ15, BV15a] for constructing  $i\mathcal{O}$  from  $\mathcal{FE}$ . We present an “over-simplified” version of their construction which is actually sufficient for our purposes but is not sufficient for achieving  $i\mathcal{O}$  security.

An “obfuscation” for a circuit  $C : \{0, 1\}^\kappa \rightarrow \{0, 1\}^*$  is a sequence of  $\kappa + 1$  functional keys  $\text{FSK}_1, \dots, \text{FSK}_{\kappa+1}$  generated using independently sampled master secret keys  $\text{MSK}_1, \dots, \text{MSK}_{\kappa+1}$  along with a ciphertext  $c_\phi$  encrypting the empty string under public-key  $\text{PK}_1$  (corresponding to  $\text{MSK}_1$ ). The first  $\kappa$  function keys implement the “bit-extension” functionality. That is, the  $i^{\text{th}}$  function key corresponds to a function that takes in an  $(i - 1)$ -bit string  $y \in \{0, 1\}^{i-1}$  and outputs functional encryptions of  $y\|0$  and  $y\|1$  under  $\text{PK}_{i+1}$ .<sup>2</sup> The function key  $\text{FSK}_{\kappa+1}$  corresponds to the circuit  $C$ .

To evaluate the obfuscated circuit on an input  $x \in \{0, 1\}^\kappa$ , one does the following: decrypt  $c_\phi$  under  $\text{FSK}_1$  to obtain encryptions of 0 and 1. Depending on the bit  $x_1$ , choose either the left or right encryption and decrypt it using  $\text{FSK}_2$  and so on. Thus, in  $\kappa$  steps one can obtain an encryption of  $x$  under  $\text{PK}_{\kappa+1}$  which can be used to compute  $C(x)$  using  $\text{FSK}_{\kappa+1}$ . One can think of the construction as having a binary tree structure where evaluating the circuit on an input  $x$  corresponds to traversing along the path labeled  $x$ .

**Sub-exponential loss.** An intuitive reason for why this construction requires sub-exponential loss to achieve  $i\mathcal{O}$  is that the behavior of the obfuscated circuit should be changed on all  $\kappa$ -bit inputs which are  $2^\kappa$  in number. The key insight in our reduction is that we can achieve our goals by changing the behavior of the obfuscated circuit at only polynomial many inputs and thus incurring only a polynomial security loss.

**Our Construction.** We will motivate our construction through a series of attempts and fixes.

**First attempt.** Our first attempt was to mimic the construction of [AJ15, BV15a]. We generate  $2\kappa + 1$  functional keys  $\text{FSK}_1, \dots, \text{FSK}_{2\kappa+1}$  where the first  $2\kappa$  of them correspond to the bit-extension function used for encrypting  $(x, \sigma)$  under  $\text{PK}_{2\kappa+1}$  and  $\text{FSK}_{2\kappa+1}$  corresponds to the circuit `Next` that checks the validity of the node  $(x, \sigma)$  and outputs the next node in the graph if  $(x, \sigma)$  is valid. The main question with this approach is: How does the circuit `Next` check the validity of the input node and output the next node in the path? The circuit `Next` must somehow have access to the PRF key  $S$  but this access should not be “visible” to the outside world.

We definitely cannot hardwire the PRF key  $S$  in the circuit as the current constructions of public key functional encryption schemes do not provide any meaningful notions of “function-privacy.” One possible approach is to “propagate” the key  $S$  along the entire tree. That is, encrypt the key  $S$  in the ciphertext  $c_\phi$  and the bit extension functions output encryptions that also includes  $S$ . Though this approach sounds promising, we are unable to use the “punctured” programming techniques of Sahai and Waters that were crucial in the reduction of PPAD hardness to  $i\mathcal{O}$ . In particular, to

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<sup>2</sup>The randomness needed for generating the encryptions is obtained using a PRF.

puncture the key  $S$  at a point  $x$  we need to puncture the key along every path thus incurring a sub-exponential loss that we wanted to avoid. To fix this issue, we develop “fine-grained” puncturing techniques.

**Second attempt: “prefix puncturing.”** To solve the problem explained earlier, we develop techniques to “surgically” puncture the PRF key  $S$  along a path  $x$  without affecting the distribution on rest of the paths. We now explain the details.

Every string  $y \in \{0, 1\}^{\leq \kappa}$  has a natural association with a node in the binary tree where the root is associated with the empty string  $\phi$ . At a high level, we want the set of keys  $K_y$  appearing in node  $y$  to have the following properties:

- The keys derived from  $K_y$  can be used for checking the validity of every node in the subtree rooted at  $y$ . This translates to be able to compute the PRF value at  $x$  for every  $(x, \sigma)$  that appears in the subtree rooted at  $y$ . We denote this property as *prefix puncturability*.
- The keys derived from  $K_y$  can be used for computing the next node for every node in the subtree rooted at  $y$ . This would translate to the ability to compute the PRF value at  $x + 1$  for every  $(x, \sigma)$  appearing at the subtree rooted at  $y$ .

A pseudorandom function that has a natural binary tree structure and has the prefix-puncturable property is the construction due to Goldreich, Goldwasser and Micali [GGM86]. We exploit this property in the GGM construction to propagate the “prefix-punctured” keys along the binary tree.

At every node  $y \in \{0, 1\}^{\leq \kappa}$ , we propagate two keys  $S_y, S_{y+1}$  where  $S_y$  denotes the key  $S$  prefix-punctured at string  $y$ . Intuitively,  $S_y$  is the key used for checking the input node is valid and  $S_{y+1}$  is used for generating the next node on the path.<sup>3</sup> The bit extension function generates  $S_{y||0}, S_{y||0+1}$  and  $S_{y||1}, S_{y||1+1}$  from  $S_y, S_{y+1}$  and propagates these values along with  $y||0$  and  $y||1$  respectively. The circuit Next receives  $S_x, S_{x+1}$  where  $x \in \{0, 1\}^{\leq \kappa}$  and checks the validity of the input signature using  $S_x$  and generates the next node in the path if the input is valid using  $S_{x+1}$ .

Note that the puncturing of the keys does not happen after the level  $\kappa$  as by this time we have parsed the  $x$  which completely determines the the key  $S_x, S_{x+1}$ . Therefore, we need to propagate  $S_x, S_{x+1}$  along the entire subtree rooted at  $x$  where we parse  $\sigma$ . This creates the following problem: consider a scenario where the successor circuit already outputs  $\perp$  on the point  $x$  and we are trying to extend the interval to include  $x + 1$ . Recall that the crucial idea behind the ability to increase the interval is that  $S_{x+1}$  does not occur anywhere else in the computation of the circuit. We observe that  $S_{x+1}$  gets propagated along the entire subtree (of exponential size) rooted at  $x$  where the input  $\sigma$  is parsed. Hence, to “remove all traces” of  $S_{x+1}$  along the subtree rooted at  $x$ , we need to incur a sub-exponential loss.

**Final construction: “encrypt the next signature.”** We solve the above problem by “implicitly” checking whether the given node is valid. This implicit checking is facilitated by encrypting the signature on the next node by using the signature on the current node. Intuitively, an evaluator can obtain the signature on the next node if and only if he holds a valid signature on the current node.

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<sup>3</sup>Note that instead of  $S_{y+1}$  it is enough to propagate  $S_{y+1||0^{\kappa-|y|}}$ . It is in fact crucial for our reduction that we propagate  $S_{y+1||0^{\kappa-|y|}}$  instead of  $S_{y+1}$ . But we will use  $S_{y+1}$  for ease of notation and exposition.



Instead of propagating the keys  $S_x, S_{x+1}$  in clear in the subtree parsing  $\sigma$ , we “cut-short” the tree at level where  $x$  is parsed. Once  $x$  is parsed (and hence we have the values  $S_x$  and  $S_{x+1}$ ), we apply a length doubling injective pseudo random generator PRG on the signature  $S_x$  to obtain two halves  $\text{PRG}_0(S_x)$  and  $\text{PRG}_1(S_x)$ . We encrypt  $S_{x+1}$  under  $\text{PRG}_1(S_x)$  and output the encryption along with  $\text{PRG}_0(S_x)$ . The Next circuit takes  $\sigma, \text{PRG}_0(S_x)$  and the encrypted version of  $S_{x+1}$  and checks whether  $\text{PRG}_0(\sigma) = \text{PRG}_0(S_x)$  <sup>4</sup> and if yes it decrypts using  $\text{PRG}_1(\sigma)$  to obtain  $S_{x+1}$ . Notice that now we don’t run into the same problem while trying to increase the interval to include  $S_{x+1}$ . This is because we can first change  $S_x$  to a random string by relying on pseudo randomness at punctured point property of GGM PRF and then relying on semantic security of secret key encryption we can change the encryption under  $\text{PRG}_1(S_x)$  to some junk value. Implementing these two steps is non-trivial and we rely on “hidden trapdoor” technique of Ananth et al. [ABSV15] while generating the function keys to achieve this.

Note that we still haven’t explained how the successor circuit is “punctured” at a random point in the first place. To this end, we “artificially” change the honest execution of the circuit to have a hardwired random value  $v$  and the circuit checks if  $\text{PRG}(x) = v$  and if so outputs  $\perp$ . The honest execution does not output  $\perp$  for any input  $x$  with overwhelming probability since PRG has sparse images. We then change this random  $v$  to  $\text{PRG}(u)$  for a random  $u$  relying on the security of the PRG. A consequence of this fix is that even our honest evaluation of the successor circuit looks somewhat “artificial.” This seems necessary to circumvent the sub-exponential loss incurred while constructing obfuscation from functional encryption.

**Putting it all together.** To show hardness of PPAD from  $\mathcal{FE}$  by incurring polynomial loss in the security reduction we need to combine the above ideas with that of “multiple-chains of varying length”. As explained in the chain-cutting technique we generate  $\kappa$  GGM keys  $S_1, \dots, S_\kappa$ . We propagate the “prefix-punctured” keys corresponding to every index  $i \in [\kappa]$  along every node in the binary tree. A careful reader might have noticed that though it is necessary to check the validity of the input signatures for every prefix, it is actually sufficient to generate signatures on the next node on the path only for those bit positions that change when incrementing by 1. This is because for the rest of the bit positions that share the same prefix with the input node and we can just output those input signatures along with those newly computed ones, provided the input is valid. This observation is in fact crucial to prove the security of our construction. We need to ensure that the Next circuit must have the ability to check the validity of every signature but it has access only to those prefix punctured keys corresponding to the bit positions that change when incrementing by 1.

We satisfy these two “conflicting” properties by decoupling the process of checking the input signatures and the process of generating the next node on the path. In order to check the input signatures we propagate  $\text{PRG}_0(S_{i,x})$  for every  $i \in [\kappa]$  and to generate the signatures on the next node on the path we propagate an encrypted version of  $S_{j,x+1}$  under  $\text{PRG}_1(S_{j,x})$  only for those bits  $j$  that change when incrementing  $x$ .

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<sup>4</sup>We need this explicit check for the verification circuit to decide if a particular node is an  $i^{\text{th}}$  node or not. Also, we need a stronger property on pseudo random generator called as left half injectivity for this check to be correct always.

### 1.3 Subsequent Work

Garg, Pandey, Srinivasan and Zhandry in [GPSZ16] extended our techniques to base Trapdoor Permutations on polynomial hardness of compact Functional Encryption. In the same work, they also showed how to base Non-Interactive Key Exchange (NIKE) for unbounded parties from polynomially hard compact Functional Encryption. Recently, Garg and Srinivasan [GS16] extended our techniques to construct adaptively secure Functional Encryption against unbounded collusions from single-key, selectively secure Functional encryption with weakly compact ciphertexts.

Rosen, Segev and Shahaf [RSS16] investigated the possibility of basing average-case PPAD hardness on standard cryptographic assumptions. They showed that average-case PPAD hardness does not imply one-way functions in a black-box manner and average-case SVL hardness cannot be based on injective trapdoor functions in a black-box manner. An implication of this work is that it might be possible to base PPAD hardness on one-way functions but such a result has to use techniques that significantly deviate from Bitansky et al. [BPR15] and our work.

Hubáček and Yogev [HY16] extended our result to base hardness of a complexity class CLS on compact Functional Encryption. CLS is a sub-class of PPAD and captures Continuous Local Search problems. They showed a reduction between the SVL problem and a problem called as END-OF-METERED-LINE which is contained in CLS. This allowed them to base hardness of CLS on polynomially hard compact Functional Encryption.

## 2 PPAD

A large part of this section is taken verbatim from [BPR15]. A search problem is given by a tuple  $(I, R)$ .  $I$  defines the set of instances and  $R$  is an NP relation. Given  $x \in I$ , the goal is to find a witness  $w$  (if it exists) such that  $R(x, w) = 1$ . We say that a search problem  $(I_1, R_1)$  polynomial time reduces to another search problem  $(I_2, R_2)$  if there exists polynomial time algorithms  $P, Q$  such that for every  $x_1 \in I_1$ ,  $P(x_1) \in I_2$  and given  $w_2$  such that  $(P(x_1), w_2) \in R_2$ ,  $R_1(x_1, Q(w_2)) = 1$ .

A search problem is said to be *total* if for any  $x \in \{0, 1\}^*$ , there exists a polynomial time procedure to test whether  $x \in I$  and for all  $x \in I$ , the set of witnesses  $w$  such that  $R(x, w) = 1$  is non-empty. The class of total search problems is denoted by TFNP. PPAD [Pap94] is a subset of TFNP and is defined by its complete problem called as END-OF-LINE (abbreviated as EOL).

**Definition 3 ([Pap94])**  $EOL = \{I_{EOL}, R_{EOL}\}$  where  $I_{EOL} = \{(x_s, \text{Succ}, \text{Pred}) : \text{Succ}(x_s) \neq x_s = \text{Pred}(x_s)\}$  and  $R_{EOL}((x_s, \text{Succ}, \text{Pred}), w) = 1$  iff  $(\text{Pred}(\text{Succ}(w)) \neq w) \vee (\text{Succ}(\text{Pred}(w)) \neq w \wedge w \neq x_s)$ .

**Definition 4 ([Pap94])** The complexity class PPAD is the set of all search problems  $(I, R)$  such that  $(I, R) \in \text{TFNP}$  and  $(I, R)$  polynomial time reduces to EOL.

A related problem to EOL is the SINK-OF-VERIFIABLE-LINE (abbreviated as SVL) which is defined as follows:

**Definition 5 ([AKV04, BPR15])**  $SVL = \{I_{SVL}, R_{SVL}\}$  where  $I_{SVL} = \{(x_s, \text{Succ}, \text{Ver}, T)\}$  and  $R_{SVL}((x_s, \text{Succ}, \text{Ver}, T), w) = 1$  iff  $(\text{Ver}(w, T) = 1)$ .

SVL instance defines a single directed path with the source being  $x_s$ . Succ is the successor circuit and there is a directed edge between  $u$  and  $v$  if and only if  $\text{Succ}(u) = v$ . Ver is the *verification*

circuit and is used to test whether a given node is the  $i^{\text{th}}$  node from  $x_s$ . That is,  $\text{Ver}(x, i) = 1$  iff  $x = \text{Succ}^{i-1}(x_s)$ . The goal is to find the  $T^{\text{th}}$  node in the path. It is easy to observe that for every *valid* SVL instance the set of witness  $w$  is not empty. But SVL may not be total since there is no known efficient procedure to test whether the instance is valid or not. But it was shown in [AKV04, BPR15] that SVL polynomial time reduces to EOL.

**Lemma 6** ([AKV04, BPR15]) *SVL polynomial time reduces to EOL.*

### 3 Preliminaries

$\kappa$  denotes the security parameter. A function  $\mu(\cdot) : \mathbb{N} \rightarrow \mathbb{R}^+$  is said to be negligible if for all polynomials  $\text{poly}(\cdot)$ ,  $\mu(\kappa) < \frac{1}{\text{poly}(\kappa)}$  for large enough  $\kappa$ . For a probabilistic algorithm  $\mathcal{A}$ , we denote by  $\mathcal{A}(x; r)$  the output of  $\mathcal{A}$  on input  $x$  with the content of the random tape being  $r$ . We will omit  $r$  when it is implicit from the context. We denote  $y \leftarrow \mathcal{A}(x)$  as the process of sampling  $y$  from the output distribution of  $\mathcal{A}(x)$  with a uniform random tape. For a finite set  $S$ , we denote  $x \stackrel{\$}{\leftarrow} S$  as the process of sampling  $x$  uniformly from the set  $S$ . We model non-uniform adversaries  $\mathcal{A} = \{\mathcal{A}_\kappa\}$  as circuits such that for all  $\kappa$ ,  $\mathcal{A}_\kappa$  is of size  $p(\kappa)$  where  $p(\cdot)$  is a polynomial. We will drop the subscript  $\kappa$  from the adversary's description when it is clear from the context. We will also assume that all algorithms are given the unary representation of security parameter  $1^\kappa$  as input and will not mention this explicitly when it is clear from the context. We will use PPT to denote Probabilistic Polynomial Time algorithm. We denote  $[k]$  to be the set  $\{1, \dots, k\}$ . We will use  $\text{negl}(\cdot)$  to denote an unspecified negligible function and  $\text{poly}(\cdot)$  to denote an unspecified polynomial.

A binary string  $x \in \{0, 1\}^\kappa$  is represented as  $x_1 \cdots x_\kappa$ .  $x_1$  is the most significant (or the highest order bit) and  $x_\kappa$  is the least significant (or the lowest order bit). The  $i$ -bit prefix  $x_1 \cdots x_i$  of the binary string  $x$  is denoted by  $x_{[i]}$ . We use  $x||y$  to denote concatenation of binary strings  $x$  and  $y$ . We say that a binary string  $y$  is a prefix of  $x$  if and only if there exists a string  $z \in \{0, 1\}^*$  such that  $x = y||z$ .

**Injective Pseudo Random Generator.** We give the definition of an injective Pseudo Random Generator PRG.

**Definition 7** *An injective pseudo random generator PRG is a deterministic polynomial time algorithm with the following properties:*

- **Expansion:** *There exists a polynomial  $\ell(\cdot)$  (called as the expansion factor) such that for all  $\kappa$  and  $x \in \{0, 1\}^\kappa$ ,  $|\text{PRG}(x)| = \ell(\kappa)$ .*

- **Pseudo randomness:** *For all  $\kappa$  and for all poly sized adversaries  $\mathcal{A}$ ,*

$$|\Pr[\mathcal{A}(\text{PRG}(U_\kappa)) = 1] - \Pr[\mathcal{A}(U_{\ell(\kappa)}) = 1]| \leq \text{negl}(\kappa)$$

*where  $U_i$  denotes the uniform distribution on  $\{0, 1\}^i$ .*

- **Injectivity:** *For every  $\kappa$  and for all  $x, x' \in \{0, 1\}^\kappa$  such that  $x \neq x'$ ,  $\text{PRG}(x) \neq \text{PRG}(x')$ .*

We in fact need an additional property from an injective PRG. Let us consider PRG where the expansion factor (or the output length) is given by  $2 \cdot \ell(\cdot)$ . Let us denote the first  $\ell(\cdot)$  bits of the output of the PRG by the function  $\text{PRG}_0$  and the next  $\ell(\cdot)$  bits of the output of the PRG by  $\text{PRG}_1$ .

**Definition 8** A pseudo random generator PRG is said to be left half injective if for every  $\kappa$  and for all  $x, x' \in \{0, 1\}^\kappa$  such that  $x \neq x'$ .  $\text{PRG}_0(x) \neq \text{PRG}_0(x')$ .

Note that left half injective PRG is also an injective PRG. We note that the standard construction of pseudo random generator for arbitrary polynomial stretch from one-way permutations is left half injective. For completeness, we state the construction:

**Lemma 9** Assuming the existence of one-way permutations, there exists a pseudo random generator that is left half injective.

**Proof** Let  $f : \{0, 1\}^\kappa \rightarrow \{0, 1\}^\kappa$  be a one-way permutation with hardcore predicate  $B : \{0, 1\}^\kappa \rightarrow \{0, 1\}$  [GL89]. Let  $G$  be an algorithm defined as follows: On input  $x \in \{0, 1\}^\kappa$ ,  $G(x) = f^n(x) \| B(x) \| B(f(x)) \cdots B(f^{n-1}(x))$  where  $n = 2\ell(\kappa) - \kappa$ . Clearly,  $|G(x)| = 2\ell(\kappa)$ . The pseudo randomness property of  $G(\cdot)$  follows from the security of hardcore bit. The left half injectivity property follows from the observation that  $f^n$  is a permutation. ■

**Puncturable Pseudo Random Function.** We recall the notion of puncturable pseudo random function from [SW14]. The construction of pseudo random function given in [GGM86] satisfies the following definition [BW13], [KPTZ13],[BGI14].

**Definition 10** A puncturable pseudo random function  $\mathcal{PRF}$  is a tuple of PPT algorithms  $(\text{KeyGen}_{\mathcal{PRF}}, \text{PRF}, \text{Punc})$  with the following properties:

- **Efficiently Computable:** For all  $\kappa$  and for all  $S \leftarrow \text{KeyGen}_{\mathcal{PRF}}(1^\kappa)$ ,  $\text{PRF}_S : \{0, 1\}^{\text{poly}(\kappa)} \rightarrow \{0, 1\}^\kappa$  is polynomial time computable.
- **Functionality is preserved under puncturing:** For all  $\kappa$ , for all  $y \in \{0, 1\}^\kappa$  and  $\forall x \neq y$ ,

$$\Pr[\text{PRF}_{S\{y\}}(x) = \text{PRF}_S(x)] = 1$$

where  $S \leftarrow \text{KeyGen}_{\mathcal{PRF}}(1^\kappa)$  and  $S\{y\} \leftarrow \text{Punc}(S, y)$ .

- **Pseudo randomness at punctured points:** For all  $\kappa$ , for all  $y \in \{0, 1\}^\kappa$ , and for all poly sized adversaries  $\mathcal{A}$

$$|\Pr[\mathcal{A}(\text{PRF}_S(y), S\{y\}) = 1] - \Pr[\mathcal{A}(U_\kappa, S\{y\}) = 1]| \leq \text{negl}(\kappa)$$

where  $S \leftarrow \text{KeyGen}_{\mathcal{PRF}}(1^\kappa)$ ,  $S\{y\} \leftarrow \text{Punc}(S, y)$  and  $U_\kappa$  denotes the uniform distribution over  $\{0, 1\}^\kappa$ .

**Indistinguishability Obfuscator.** We now define Indistinguishability obfuscator from [BGI<sup>+</sup>12, GGH<sup>+</sup>13b].

**Definition 11** A PPT algorithm  $i\mathcal{O}$  is an indistinguishability obfuscator for a family of circuits  $\{C_\kappa\}_\kappa$  that satisfies the following properties:

- **Correctness:** For all  $\kappa$  and for all  $C \in C_\kappa$  and for all  $x$ ,

$$\Pr[i\mathcal{O}(C)(x) = C(x)] = 1$$

where the probability is over the random choices of  $i\mathcal{O}$ .

- **Security:** For all  $\mathcal{C}_0, \mathcal{C}_1 \in \mathcal{C}_\kappa$  such that for all  $x$ ,  $\mathcal{C}_0(x) = \mathcal{C}_1(x)$  and for all poly sized adversaries  $\mathcal{A}$ ,

$$|\Pr[\mathcal{A}(i\mathcal{O}(\mathcal{C}_0)) = 1] - \Pr[\mathcal{A}(i\mathcal{O}(\mathcal{C}_1)) = 1]| \leq \text{negl}(\kappa)$$

**Functional Encryption.** We recall the notion of functional encryption with selective indistinguishability based security [BSW11, O’N10].

A functional encryption  $\mathcal{FE}$  is a tuple of PPT algorithms (FE.Setup, FE.Enc, FE.KeyGen, FE.Dec) with the message space  $\{0, 1\}^*$  having the following syntax:

- FE.Setup( $1^\kappa$ ): Takes as input the unary encoding of the security parameter  $\kappa$  and outputs a public key  $PK$  and a master secret key  $MSK$ .
- FE.Enc $_{PK}(m)$ : Takes as input a message  $m \in \{0, 1\}^*$  and outputs an encryption  $C$  of  $m$  under the public key  $PK$ .
- FE.KeyGen( $MSK, f$ ): Takes as input the master secret key  $MSK$  and a function  $f$  (given as a circuit) as input and outputs the function key  $FSK_f$ .
- FE.Dec( $FSK_f, C$ ): Takes as input the function key  $FSK_f$  and the ciphertext  $C$  and outputs a string  $y$ .

**Definition 12 (Correctness)** The functional encryption scheme  $\mathcal{FE}$  is correct if for all  $\kappa$  and for all messages  $m \in \{0, 1\}^*$ ,

$$\Pr \left[ y = f(m) \left| \begin{array}{l} (PK, MSK) \leftarrow \text{FE.Setup}(1^\kappa) \\ C \leftarrow \text{FE.Enc}_{PK}(m) \\ FSK_f \leftarrow \text{FE.KeyGen}(MSK, f) \\ y \leftarrow \text{FE.Dec}(FSK_f, C) \end{array} \right. \right] = 1$$

**Definition 13 (Selective Security)** For all  $\kappa$  and for all poly sized adversaries  $\mathcal{A}$ ,

$$|\Pr[\text{Expt}_{1^\kappa, 0, \mathcal{A}} = 1] - \Pr[\text{Expt}_{1^\kappa, 1, \mathcal{A}} = 1]| \leq \text{negl}(\kappa)$$

where  $\text{Expt}_{1^\kappa, b, \mathcal{A}}$  is defined below:

- **Challenge Message Queries:** The adversary  $\mathcal{A}$  outputs two messages  $m_0, m_1$  such that  $|m_0| = |m_1|$  to the challenger.
- The challenger samples  $(PK, MSK) \leftarrow \text{FE.Setup}(1^\kappa)$  and generates the challenge ciphertext  $C \leftarrow \text{FE.Enc}_{PK}(m_b)$ . It then sends  $(PK, C)$  to  $\mathcal{A}$ .
- **Function Queries:**  $\mathcal{A}$  submits function queries  $f$  to the challenger. The challenger responds with  $FSK_f \leftarrow \text{FE.KeyGen}(MSK, f)$ .
- If  $\mathcal{A}$  makes a query  $f$  to functional key generation oracle such that  $f(m_0) \neq f(m_1)$ , output of the experiment is  $\perp$ . Otherwise, the output is  $b'$  which is the output of  $\mathcal{A}$ .

**Remark 14** We say that the functional encryption scheme  $\mathcal{FE}$  is **single-key, selectively secure** if the adversary  $\mathcal{A}$  in  $\text{Expt}_{1^\kappa, b, \mathcal{A}}$  is allowed to query the functional key generation oracle  $\text{FE.KeyGen}(MSK, \cdot)$  on a single function  $f$ .

**Definition 15 (Compactness, [AJS15, BV15a, AJ15])** *The functional encryption scheme  $\mathcal{FE}$  is said to be compact if for all  $\kappa \in \mathbb{N}$  and for all  $m \in \{0, 1\}^*$  the running time of the encryption algorithm  $\text{FE.Enc}$  is  $\text{poly}(\kappa, |m|)$ .*

Bitansky et al. in [BV15b] and Ananth et al. in [AJS15] show a generic transformation from any collusion-resistant  $\mathcal{FE}$  for general circuits where the ciphertext size is independent of the number of collusions (but may depend arbitrarily on the circuit parameters) to a compact  $\mathcal{FE}$  for general circuits. The property that the ciphertext size does not depend on the number of collusion is referred as *collusion-succinctness*.

**Lemma 16 ([BV15b, AJS15])** *Assuming the existence of selectively secure collusion-resistant functional encryption with collusion-succinct ciphertexts, there exists a selectively secure compact functional encryption scheme.*

**Symmetric Key Encryption.** A Symmetric-Key Encryption scheme  $\mathcal{SK}\mathcal{E}$  is a tuple of algorithms  $(\text{SK.KeyGen}, \text{SK.Enc}, \text{SK.Dec})$  with the following syntax:

- $\text{SK.KeyGen}(1^\kappa)$  : Takes as input an unary encoding of the security parameter  $\kappa$  and outputs a symmetric key  $SK$ .
- $\text{SK.Enc}_{SK}(m)$  : Takes as input a message  $m \in \{0, 1\}^*$  and outputs an encryption  $C$  of the message  $m$  under the symmetric key  $SK$ .
- $\text{SK.Dec}_{SK}(C)$ : Takes as input a ciphertext  $C$  and outputs a message  $m'$ .

We say that  $\mathcal{SK}\mathcal{E}$  is *correct* if for all  $\kappa$  and for all messages  $m \in \{0, 1\}^*$ ,  $\Pr[\text{SK.Dec}_{SK}(C) = m] = 1$  where  $SK \leftarrow \text{SK.KeyGen}(1^\kappa)$  and  $C \leftarrow \text{SK.Enc}_{SK}(m)$ .

**Definition 17** *For all  $\kappa$  and for all polysized adversaries  $\mathcal{A}$ ,*

$$\left| \Pr[\text{Expt}_{1^\kappa, 0, \mathcal{A}} = 1] - \Pr[\text{Expt}_{1^\kappa, 1, \mathcal{A}} = 1] \right| \leq \text{negl}(\kappa)$$

where  $\text{Expt}_{1^\kappa, b, \mathcal{A}}$  is defined below:

- **Challenge Message Queries:** *The adversary  $\mathcal{A}$  outputs two messages  $m_0$  and  $m_1$  such that  $|m_0| = |m_1|$  for all  $i \in [n]$ .*
- *The challenger samples  $SK \leftarrow \text{SK.KeyGen}(1^\kappa)$  and generates the challenge ciphertext  $C$  where  $C \leftarrow \text{SK.Enc}_{SK}(m_b)$ . It then sends  $C$  to  $\mathcal{A}$ .*
- *Output is  $b'$  which is the output of  $\mathcal{A}$ .*

**Prefix Puncturable Pseudo Random Functions.** We now define the notion of prefix puncturable pseudo random function PPRF which is satisfied by the construction of the pseudo random function in [GGM86].

**Definition 18** *A prefix puncturable pseudo random function  $\mathcal{PPRF}$  is a tuple of PPT algorithms  $(\text{KeyGen}_{\mathcal{PPRF}}, \text{PrefixPunc})$  satisfying the following properties:*

- **Functionality is preserved under repeated puncturing:** For all  $\kappa$ , for all  $y \in \cup_{k=0}^{\text{poly}(\kappa)} \{0, 1\}^k$  and for all  $x \in \{0, 1\}^{\text{poly}(\kappa)}$  such that there exists a  $z \in \{0, 1\}^*$  s.t.  $x = y||z$ ,

$$\Pr[\text{PrefixPunc}(\text{PrefixPunc}(S, y), z) = \text{PrefixPunc}(S, x)] = 1$$

where  $S \leftarrow \text{KeyGen}_{\mathcal{P}\mathcal{R}\mathcal{F}}(1^\kappa)$ .

- **Pseudorandomness at punctured prefix:** For all  $\kappa$ , for all  $x \in \{0, 1\}^{\text{poly}(\kappa)}$ , and for all poly sized adversaries  $\mathcal{A}$

$$|\Pr[\mathcal{A}(\text{PrefixPunc}(S, x), \text{Keys}) = 1] - \Pr[\mathcal{A}(U_\kappa, \text{Keys}) = 1]| \leq \text{negl}(\kappa)$$

where  $S \leftarrow \text{KeyGen}_{\mathcal{P}\mathcal{R}\mathcal{F}}(1^\kappa)$  and  $\text{Keys} = \{\text{PrefixPunc}(S, x_{[i-1]} || (1 - x_i))\}_{i \in [\text{poly}(\kappa)]}$ .

## 4 Hardness from Indistinguishability Obfuscation

In this section, we prove that SVL is hard on average assuming polynomial hardness of indistinguishability obfuscation, injective PRGs and puncturable pseudo random functions. Coupled with the fact that SVL reduces to EOL (Lemma 6) we have the following theorem.

**Theorem 19** *Assume the existence of one-way permutations and indistinguishability obfuscation against polynomial time adversaries then we have that EOL problem is hard for polynomial time algorithms.*

### 4.1 Hard on Average SVL Instances

In this section, we describe an efficient sampler that provides hard on average instances  $(x_s, \text{Succ}, \text{Ver}, 1^\kappa)$  of SVL. Here  $x_s$  is the source node and  $\text{Succ}$  is the successor circuit. We define a directed edge between  $u$  and  $v$  if and only if  $\text{Succ}(u) = v$ .  $\text{Ver}$  is the verification circuit and is used to test whether a given node is the  $k^{\text{th}}$  node from  $x_s$ . That is,  $\text{Ver}(x, k) = 1$  iff  $x = \text{Succ}^{k-1}(x_s)$ . For the generated instances, we argue that it is hard to find the  $1^\kappa$  node in the path.

The formal description of hard on average SVL instance sampler is provided in Figure 3. Internally this sampler generates an obfuscation of the  $\text{Next}$  circuit provided in Figure 2. Next we describe the SVL instances which we consider informally.

The instance we generate defines a line graph. The nodes in the graph are of the form:  $(x, \sigma_1, \dots, \sigma_\kappa)$  where  $x \in \{0, 1\}^\kappa$ . The nodes satisfy the following relation: for all  $i \in [\kappa]$ ,  $\text{PRF}_{S_i}(x_{[i]}) = \sigma_i$  and in that case we say that  $(x, \sigma_1, \dots, \sigma_\kappa)$  is *valid*. The node  $(x, \sigma_1, \dots, \sigma_\kappa)$  is connected to  $(x+1, \sigma'_1, \dots, \sigma'_\kappa)$  through an outgoing edge and is connected to  $(x-1, \sigma''_1, \dots, \sigma''_\kappa)$  through an incoming edge where  $\sigma'_1, \dots, \sigma'_\kappa$  and  $\sigma''_1, \dots, \sigma''_\kappa$  satisfy the above described PRF relationship. The source node is given by  $(0^\kappa, \text{PRF}_{S_1}(0), \dots, \text{PRF}_{S_\kappa}(0^\kappa))$ .

At a very high level successor circuit of our SVL instances provides a method for moving forward from one node to the next. The successor circuit in our instances corresponds to an obfuscation of the  $\text{Next}$  circuit. This circuit on input a node of the form  $(x, \sigma_1, \dots, \sigma_\kappa)$  checks for the validity of the input. If it is valid, it outputs the next node  $(x+1, \sigma'_1 \dots \sigma'_\kappa)$  where  $\sigma'_i = \text{PRF}_{S_i}((x+1)_{[i]})$  in the path. On an invalid input, it outputs  $\perp$ .

For the hard SVL instances we additionally need to provide a verification circuit. The verification circuit just uses the successor circuit in a very natural manner. The verification circuit on input  $(x, \sigma_1, \dots, \sigma_\kappa, j)$  outputs 1 if and only if  $x = j-1$  and  $\text{Next}_{S_1, \dots, S_\kappa}(x, \sigma_1, \dots, \sigma_\kappa) \neq \perp$ .

**Input:**  $(x, \sigma_1, \dots, \sigma_\kappa)$

**Hardcoded Parameters:**  $S_1, \dots, S_\kappa$

1. For any  $i \in [\kappa]$ , if  $\sigma_i \neq \text{PRF}_{S_i}(x_{[i]})$  then output  $\perp$ .
2. If  $x = 1^\kappa$ , then output SOLVED.
3. Else output  $(x + 1, \sigma'_1, \dots, \sigma'_\kappa)$ , where for all  $i \in [\kappa]$  compute  $\sigma'_i = \text{PRF}_{S_j}((x + 1)_{[i]})$ .

**Padding:** This circuit is padded so that total size of the circuit is  $p(\kappa)$ , for some polynomial  $p(\cdot)$  specified later.

**Figure 2:**  $\text{Next}_{S_1, \dots, S_\kappa}$

- **Sampled Ingredients:** Sample  $\{S_i\}_{i \in [\kappa]} \leftarrow \text{KeyGen}_{\mathcal{P}\mathcal{R}\mathcal{F}}(1^\kappa)$ . For all  $i \in [\kappa]$ ,  $S_i$  is a seed for a PRF mapping  $i$  bits to  $\kappa$  bits. That is,  $\text{PRF}_{S_i} : \{0, 1\}^i \rightarrow \{0, 1\}^\kappa$ .
- **Source Node:** The source node  $x_s = (0^\kappa, \text{PRF}_{S_1}(0), \dots, \text{PRF}_{S_\kappa}(0^\kappa))$ .
- **Successor Circuit:** The successor circuit is given by  $i\mathcal{O}(\text{Next}_{S_1, \dots, S_\kappa})$  where the circuit  $\text{Next}_{S_1, \dots, S_\kappa}$  is described in Figure 2.
- **Verification Circuit:** The verification circuit, given by  $\text{Ver}$ , on input  $((x, \sigma_1 \dots \sigma_\kappa), j)$  checks if  $x = j - 1$  and  $i\mathcal{O}(\text{Next}_{S_1, \dots, S_\kappa})((x, \sigma_1 \dots \sigma_\kappa)) \neq \perp$ .

**Figure 3:** Sampler for hard on average instances of SVL based on hardness of  $i\mathcal{O}$

## 4.2 Proof of Hardness

We start by showing that our sampler generates SVL instances, satisfying constraints in Definition 5. It suffices to show that the verification circuit  $\text{Ver}$  on input  $((x, \sigma_1, \dots, \sigma_\kappa), j)$  outputs 1 if and only if  $(x, \sigma_1, \dots, \sigma_\kappa) = \text{Succ}^{j-1}(x_s)$ . This follows immediately from our construction. Our verification circuit  $\text{Ver}$  outputs 1 if and only if  $x = j - 1$  and  $\text{Next}_{S_1, \dots, S_\kappa}(x, \sigma_1, \dots, \sigma_\kappa) \neq \perp$ . The later requirement means implies that  $(x, \sigma_1, \dots, \sigma_\kappa)$  must be valid. Therefore, by design, we have that  $(x, \sigma_1, \dots, \sigma_\kappa) = \text{Succ}^{j-1}(x_s)$ .

Next we show that no polynomial time adversary can output a valid value  $(1^\kappa, \sigma_1, \dots, \sigma_\kappa)$ . We will show this via a hybrid argument. Starting with providing SVL instance as in distribution from Figure 3, namely hybrid  $\text{Hyb}_0$ , we move to a hybrid  $\text{Hyb}_{4, \delta(u_0)}$  where  $\delta(u_0) \leq \kappa$  is a number specified later. In the final hybrid, the successor circuit returns  $\perp$  on all inputs of the form  $(1^\kappa, \cdot, \dots, \cdot)$ . Observe that the advantage of any adversary in solving the SVL instance in this final hybrid is 0. This proves our claim. We make this change by going through a polynomial (in fact linear) in the security parameter number of intermediate hybrids.



**Circuit Next\*.** In our proof we will extensively use the circuit  $\text{Next}_{S_1, \dots, S_\kappa, u, u'}^*$  which is a modification of  $\text{Next}_{S_1, \dots, S_\kappa}$  again padded to size  $p(\kappa)$ . The circuit  $\text{Next}_{S_1, \dots, S_\kappa, u, u'}^*$  is identical to  $\text{Next}_{S_1, \dots, S_\kappa}$  except that on inputs  $(x, \cdot, \dots, \cdot)$  such that  $u \leq x \leq u'$  it outputs  $\perp$ .

**Our hybrids.** Next we describe our hybrids.

- **Hyb<sub>0</sub>:** This hybrid corresponds to SVL instance generation as given in Figure 3.
- **Hyb<sub>1</sub>:** In the hybrid we change how the successor circuit is generated. In particular, the successor circuit is generated as  $i\mathcal{O}(\text{Next}_{S_1, \dots, S_\kappa, v}^1)$  instead of  $i\mathcal{O}(\text{Next}_{S_1, \dots, S_\kappa})$ . The circuit  $\text{Next}_{S_1, \dots, S_\kappa, v}^1$  is identical to  $\text{Next}_{S_1, \dots, S_\kappa}$  except that on input  $(x, \cdot, \dots, \cdot)$  such that  $\text{PRG}(x) = v$ , it outputs  $\perp$ . (Again this circuit is padded to size  $p(\kappa)$ .) The value  $v$  itself is chosen uniformly from  $\{0, 1\}^{2^\kappa}$ .

Since  $v$  is chosen uniformly at random and PRG is length doubling, with overwhelming probability we have that for all  $x \in \{0, 1\}^\kappa$   $\text{PRG}(x) \neq v$ . Hence the circuits  $\text{Next}_{S_1, \dots, S_\kappa}$  and  $\text{Next}_{S_1, \dots, S_\kappa, v}^1$  are functionally equivalent with overwhelming probability. Therefore, indistinguishability obfuscation implies computational indistinguishability between **Hyb<sub>0</sub>** and **Hyb<sub>1</sub>**.

- **Hyb<sub>2</sub>:** In this hybrid we change how the value  $v$ , hard-coded in  $\text{Next}_{S_1, \dots, S_\kappa, v}^1$  is generated. In particular, instead of sampling  $v$  uniformly at random from  $\{0, 1\}^{2^\kappa}$ , we generate  $v$  as  $\text{PRG}(u_0)$  where  $u_0$  is sampled uniformly from  $\{0, 1\}^\kappa$ . Here PRG is a length doubling injective pseudorandom generator.

The indistinguishability between **Hyb<sub>1</sub>** and **Hyb<sub>2</sub>** follows from the pseudorandomness property of the PRG.

- **Hyb<sub>3</sub>:** In this hybrid we change how the successor circuit is generated. Recall that in hybrid **Hyb<sub>2</sub>**, the successor circuit was the obfuscated circuit  $i\mathcal{O}(\text{Next}_{S_1, \dots, S_\kappa, \text{PRG}(u_0)}^1)$ . In **Hyb<sub>3</sub>** we change it to the obfuscated circuit  $i\mathcal{O}(\text{Next}_{S_1, \dots, S_\kappa, u_0, u_0}^*)$ .

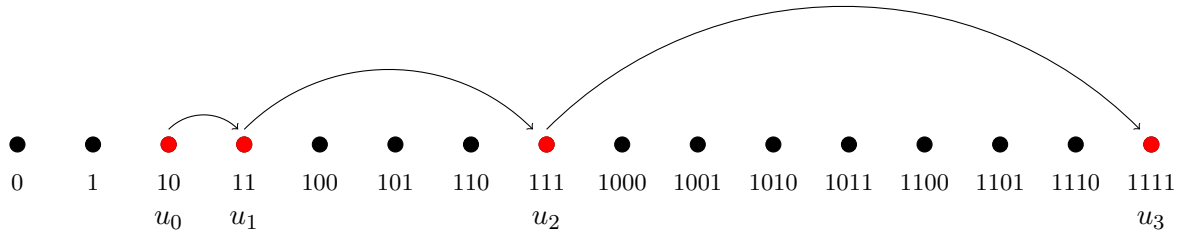
Note that the circuits  $\text{Next}_{S_1, \dots, S_\kappa, \text{PRG}(u_0)}^1$  and  $\text{Next}_{S_1, \dots, S_\kappa, u_0, u_0}^*$  are functionally equivalent because of the injectivity of the PRG. Hence, indistinguishability obfuscation implies computational indistinguishability between **Hyb<sub>2</sub>** and **Hyb<sub>3</sub>**.

- **Hyb<sub>4,j</sub>:** In hybrid **Hyb<sub>4,j</sub>** for  $j \in \{0, \dots, \delta(u_0)\}$ , the successor circuit is generated as an obfuscation of the circuit  $\text{Next}_{S_1, \dots, S_\kappa, u_0, u_j}^*$  where  $u_j$  values are described below. Here  $\delta(u_0)$  is the number of 0 bits in the binary representation of  $u_0$ .

**Defining  $u_j$  values.** For any string  $u \in \{0, 1\}^\kappa$ , let  $f(u)$  denote the index of the lowest order bit of  $u$  that is 0 (with the index of the highest order bit being 1). More formally,  $f(u)$

is the smallest  $j$  such that  $u = u_{[j]} \parallel 1^{\kappa-j}$ . For example, if  $u = \overbrace{100}^3 11$  then  $f(u) = 3$ . Also let  $\delta(u)$  be the number of 0 bits in the binary representation of  $u$ .

Starting with a value  $u_0 \in \{0, 1\}^\kappa$  we define a sequence of values such that  $u_{j+1}$  is the value  $u_j$  with the  $f(u_j)^{\text{th}}$  bit set to 1. More formally,  $u_{j+1} = u_j + 2^{\kappa-f(u_j)}$ . It is easy to see that for all  $j \in \{0, \dots, \delta(u_0)\}$  we have that  $\delta(u_{j+1}) < \delta(u_j)$  and  $u_{\delta(u_0)} = 1^\kappa$ . Moreover,  $\delta(u_0) \leq \kappa$ . The sequence of  $u_i$  values starting with  $u_0 = 0010$  are illustrated in Figure 4.



**Figure 4:** Illustration of the steps starting with  $u_0 = 0010$ .

**Indistinguishability Argument.** Observe that the hybrid  $\text{Hyb}_{4,0}$  is identical to hybrid  $\text{Hyb}_3$  and  $\text{Hyb}_{4,\delta(u_0)}$  is such that the successor outputs  $\perp$  on all inputs of the form  $(1^\kappa, \cdot, \dots, \cdot)$ . Consequently, no adversary can solve the SVL instance in this final hybrid with probability better than 0. Hence, in order to complete the proof, it suffices to argue that the hybrids  $\text{Hyb}_{4,j}$  and  $\text{Hyb}_{4,j+1}$  are computationally indistinguishable for each  $j \in \{0, \dots, \delta(u_0) - 1\}$ . We argue this next.

**Indistinguishability between  $\text{Hyb}_{4,j}$  and  $\text{Hyb}_{4,j+1}$ .** We prove indistinguishability via sequence of sub-hybrids, where in each successive hybrid we make a small change to the successor circuit. We let  $f_j$  as the shorthand for  $f(u_j)$  and  $t_j = u_{j[f_j]} + 1$ .

- $\text{Hyb}_{4,j,1}$ : Let  $S'_{f_j} = \text{Punc}(S_{f_j}, t_j)$  and let  $\sigma^* = \text{PRF}_{S_{f_j}}(t_j)$ . Replace the successor circuit from an obfuscation of  $\text{Next}_{S_1, \dots, S_\kappa, u_0, u_j}^*$  to an obfuscation of  $\text{Next}_{S_1, \dots, S'_{f_j}, \dots, S_\kappa, u_0, u_j, \sigma^*}^2$ .

$\text{Next}_{S_1, \dots, S'_{f_j}, \dots, S_\kappa, u_0, u_j, \sigma^*}^2$  is identical to  $\text{Next}_{S_1, \dots, S_\kappa, u_0, u_j}^*$  except that it cannot compute  $\text{PRF}_{S_{f_j}}(t_j)$  because it is provided with a punctured key  $S'_{f_j}$ . However, the exact same value  $\sigma^*$  is hardwired in it which it uses whenever needed instead of computing  $\text{PRF}_{S_{f_j}}(t_j)$ . (The circuit is appropriately padded to size  $p(\kappa)$ .)

Computational indistinguishability between hybrids  $\text{Hyb}_{4,j}$  and  $\text{Hyb}_{4,j,1}$  follows from the security of the indistinguishability obfuscation scheme.

- $\text{Hyb}_{4,j,2}$ : The successor circuit is still an obfuscation of  $\text{Next}_{S_1, \dots, S'_{f_j}, \dots, S_\kappa, u_0, u_j, \sigma^*}^2$  with  $S'_{f_j} = \text{Punc}(S_{f_j}, t_j)$  just as in hybrid  $\text{Hyb}_{4,j,1}$ . However, instead of generating  $\sigma^* = \text{PRF}_{S_{f_j}}(t_j)$ , we sample  $\sigma^* \leftarrow \{0, 1\}^\kappa$ . (The circuit is appropriately padded to size  $p(\kappa)$ .)

Computational indistinguishability between hybrids  $\text{Hyb}_{4,j,1}$  and  $\text{Hyb}_{4,j,2}$  follows from the pseudorandom at punctured point property of the puncturable PRF.

**Important Observations.** Let's understand the role of  $\sigma^*$  in the circuit obfuscated above.

Observe that the successor circuit on input  $(x, \sigma_1, \dots, \sigma_{f_j}, \dots, \sigma_\kappa)$  checks if  $\sigma_{f_j} = \sigma^*$  for all  $x \in \{u_j + 1, \dots, u_{j+1}\}$ . Furthermore, if this check passes then  $\sigma^*$  is output for all  $x \in \{u_j + 1, \dots, u_{j+1} - 1\}$ .

Next note that  $\sigma^*$  is not used for any other purpose. This relies on the fact that in this hybrid the successor is set to output  $\perp$  when  $x = u_j$ . At this point it is instructive to recall that

$x = u_j$  is the only other input on which the successor circuit in  $\text{Hyb}_{4,j-1}$  was expected to output this value without being providing the same value in the input.

- $\text{Hyb}_{4,j,3}$ : In this circuit instead of generating the successor circuit as an obfuscation of  $\text{Next}_{S_1, \dots, S'_{f_j}, \dots, S_\kappa, u_0, u_j, \sigma^*}^2$  we generate it as an obfuscation of  $\text{Next}_{S_1, \dots, S'_{f_j}, \dots, S_\kappa, u_0, u_j, \tau^*}^3$  where  $\tau^* = \text{PRG}(\sigma^*)$ . Note that  $\text{Next}^2$  used  $\sigma^*$  to only check if  $\sigma_{f_j} = \sigma^*$  for certain choices of inputs. In these cases,  $\text{Next}^3$  is modified to check if  $\text{PRG}(\sigma_{f_j}) = \tau^*$  and outputs the provided input value  $\sigma_{f_j}$  if the check passes. (The circuit is appropriately padded to size  $p(\kappa)$ .)

Note that by the injectivity property of the PRG we have that the circuits considered here are functionally equivalent and hence computational indistinguishability between hybrids  $\text{Hyb}_{4,j,2}$  and  $\text{Hyb}_{4,j,3}$  follows from the security of indistinguishability obfuscation.

- $\text{Hyb}_{4,j,4}$ : In this hybrid we still obfuscate  $\text{Next}_{S_1, \dots, S'_{f_j}, \dots, S_\kappa, u_0, u_j, \tau^*}^3$  but instead of generating  $\tau^* = \text{PRG}(\sigma^*)$ , we generate  $\tau^*$  as a random string in  $\{0, 1\}^{2\kappa}$ . (The circuit is appropriately padded to size  $p(\kappa)$ .)

Indistinguishability between  $\text{Hyb}_{4,j,3}$  and  $\text{Hyb}_{4,j,4}$  follows from the security of the PRG.

- $\text{Hyb}_{4,j,5}$ : In this hybrid we change the successor circuit to now be an obfuscation of  $\text{Next}_{S_1, \dots, S'_{f_j}, \dots, S_\kappa, u_0, u_{j+1}, \tau^*}^3$ . (The circuit is appropriately padded to size  $p(\kappa)$ .)

Observe that with overwhelming probability the circuits obfuscated in hybrids  $\text{Hyb}_{4,j,4}$  and  $\text{Hyb}_{4,j,5}$  are functionally equivalent and hence computational indistinguishability follows from indistinguishability obfuscation.

- $\text{Hyb}_{4,j,6}$ : Instead of using the puncture key  $S'_{f_j}$  we use the unpunctured key  $S_{f_j}$ . More specifically, we generate the sampler as an obfuscation of  $\text{Next}_{S_1, \dots, S_\kappa, u_0, u_{j+1}}^*$ .

Again computational indistinguishability follows by indistinguishability obfuscation.

We set  $p(\cdot)$  to be the least polynomial such that all circuits considered in the construction and the proof are of size at most  $p(\kappa)$ . Note that hybrid  $\text{Hyb}_{4,j,6}$  is same as  $\text{Hyb}_{4,j+1}$ . This concludes the proof.

## 5 Hardness Result based on Functional Encryption

In this section we show that SVL is hard on average assuming polynomially hard functional encryption and one-way permutations. Coupled with the fact that SVL reduces to EOL (Lemma 6) we have the following theorem.

**Theorem 20** *Assume the existence of one-way permutations and functional encryption against polynomial time adversaries then we have that EOL problem is hard for polynomial time algorithms.*

Recall that hard SVL instance based on  $i\mathcal{O}$  (Section 4), required  $\kappa$  puncturable PRF keys. Basing hardness on polynomially hard functional encryption requires us to still maintain  $\kappa$  keys. However, now we need to use prefix-puncturing (see Definition 18) which is more delicate and needs to be handled carefully. Consequently the construction ends up being complicated. However, the

special mechanism of prefix-puncturing that we use is crucial to understanding our construction. So towards simplifying exposition, we start by abstracting out the details of this puncturing and present a special tree structure and some properties about it next.

## 5.1 Special Tree Key Structure

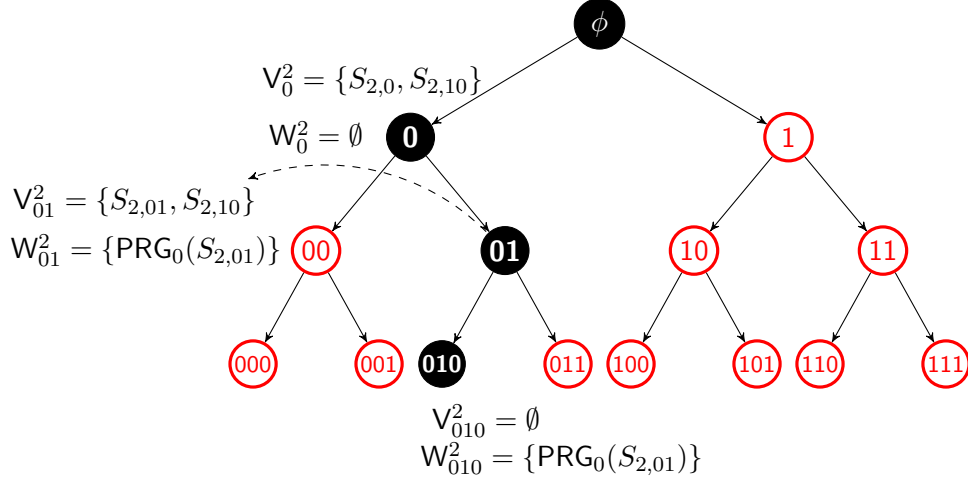
Let  $x_{[i]}$  denote the first  $i$  (higher order) bits of  $x$  i.e  $x_1 \cdots x_i$ . Now note that any  $y \in \{0, 1\}^i$  can be identified with a node in a binary tree for which nodes at depth  $i$  correspond to strings  $\{0, 1\}^i$ . Note that the root of the tree corresponds to the empty string  $\phi$ . As previously mentioned our construction needs  $\kappa$  PPRF keys, namely  $S_1, \dots, S_\kappa$ . The key  $S_i$  works on inputs of length  $i$ . We use  $S_{i,x}$  to denote the key  $S_i$  prefix punctured at a string  $x \in \{0, 1\}^{\leq i}$ .

Looking ahead, in our hard-on-average instances of SVL each  $x \in \{0, 1\}^\kappa$  will be attached with *associated signature* values  $\sigma_1, \dots, \sigma_\kappa$  where for each  $i \in [\kappa]$  we have that  $\sigma_i = \text{PrefixPunc}(S_i, x_{[i]})$ . Furthermore in our construction given  $x$  and the associated signature values, we will need to verify these values and provide the associated signature values for  $x + 1$ , but this has to be done in a circuitous manner because of several security reasons. We do not delve into the security arguments right away, but focus on describing the prefix-puncturing that we need to perform.

We next describe the set  $V_x^i$  where  $x \in \{0, 1\}^{\leq i}$ , which contains suitable prefix-puncturings of the key  $S_i$ . Intuitively, we want this set to contain all keys that will allow us to perform the task of *checking the validity* of the  $i^{\text{th}}$  associated signature on any input of the form  $x||y$  where  $y \in \{0, 1\}^{\kappa-|x|}$  as well as *computing* the  $i^{\text{th}}$  associated signature for  $(x||y) + 1$ . Furthermore, it should suffice to generate  $V_{x||y}^i$  for all  $y$ . For any node  $x \in \{0, 1\}^{\leq i}$ , this very naturally translates to the keys  $S_{i,x}$  and  $S_{i,x+1}$ . A careful reader might have noticed that instead of  $S_{i,x+1}$ , it in fact suffices to just have  $S_{i,(x+1)||0^{i-|x|}}$ . As it turns out we must only include  $S_{i,(x+1)||0^{i-|x|}}$ . Including  $S_{i,x+1}$  prevents the Derivability Lemma (Lemma 22) from going through.

Recall that the key  $S_i$  corresponds to a PPRF key for inputs of length  $i$ . Therefore, for  $x||y$  such that  $|x| = i$ , the key  $S_i$  can be prefix-punctured only for the prefix  $x = (x||y)_{[i]}$ . This raises the following question. Should we include  $S_{i,x}$  and  $S_{i,x+1}$  in all  $V_{x||y}^i$ ? As we will see later, in our construction, we carefully decouple the checking of associated signatures from the generation of new associated signatures. An important consequence, relevant here is that, even though the checks need to be performed for all  $x||y$ , a new  $i^{\text{th}}$  associated signature needs to be generated for only one choice of  $y$ , namely  $1^{\kappa-|x|}$  (the all 1 string of length  $\kappa - |x|$ ). This design choice (which is crucial for polynomial security loss) also allows us to set  $V_{x||y}^i$  for all other choices of  $y$  to be  $\emptyset$ . In terms of the binary tree structure one can think of this as  $V_x^i$  getting passed only along the rightmost path in the subtree rooted at  $x$ . At a very high level, this allows us to argue that the key  $S_i$  (proved formally in Lemma 22) can be punctured at a special point by removing keys from  $V_x^i$  for only a polynomial number of choices of  $x$  and  $i$ . This is crucial for ensuring that our proof of security has only a polynomial number of hybrids.

Next note that dropping keys from  $V_{x||y}^i$  (such that  $|x| = i$ ) hinders the checking of associated signatures provided along with inputs  $x||y$  where  $y \neq 1^{\kappa-i}$ . We tackle this issue by introducing a vestigial set  $W_{x||y}^i$  corresponding to each  $V_{x||y}^i$ . This vestigial set contains remnants of the keys that were dropped from  $V_x^i$ . We craft these remnants to be such that they suffice for performing the necessary checks. In particular, we set these remnants to be the left half of an left half injective PRG evaluation on the dropped key. More formally,  $V_x^i$  and  $W_x^i$  are defined as follows. In the following, for any  $i \in [\kappa]$  we treat  $1^i + 1$  as  $1^i$ , and  $\phi + 1$  as  $\phi$ . Here  $1^i$  is a string of  $i$  1s and  $\phi$  is



**Figure 5:** Example of values contained in  $V_x^2$  for  $x \in \{0, 1\}^{\leq 3}$ .

the empty string.

$$\begin{aligned}
 V_x &= \bigcup_{i \in [\kappa]} V_x^i & V_x^i &= \begin{cases} \{S_{i,x_{[i]}}, S_{i,x_{[i]}+1}\} & \text{if } |x| > i \text{ and } x = x_{[i]} \parallel 1^{|x|-i} \\ \{S_{i,x}, S_{i,(x+1)} \parallel 0^{i-|x|}\} & \text{if } |x| \leq i \\ \emptyset & \text{otherwise} \end{cases} \\
 W_x &= \bigcup_{i \in [\kappa]} W_x^i & W_x^i &= \begin{cases} \{\text{PRG}_0(S_{i,x_{[i]}})\} & \text{if } |x| \geq i \\ \emptyset & \text{otherwise} \end{cases}
 \end{aligned}$$

For the empty string  $x = \phi$ , these sets can be initialized as follows.

$$\begin{aligned}
 V_\phi &= \bigcup_{i \in [\kappa]} V_\phi^i & V_\phi^i &= \{S_i\} \\
 W_\phi &= \bigcup_{i \in [\kappa]} W_\phi^i & W_\phi^i &= \emptyset
 \end{aligned}$$

**Illustration with an example.** Finally we explain what sets  $V_x^2, W_x^2$  contain when  $x$  is a prefix of 010 in Figure 5. At the root node we have  $V_\phi^2 = \{S_2\}$  and  $W_\phi^2 = \emptyset$ . The set  $V_0^2$  contains  $S_{2,0}$  and  $S_{2,10}$  and the set  $W_0^2$  is still empty. Next note that  $V_{01}^2$  contains  $S_{2,01}, S_{2,10}$  and  $W_{01}^2$  contains  $\text{PRG}_0(S_{2,01})$ . Finally set  $V_{010}^2 = \emptyset$  and  $W_{010}^2$  continues to contain  $\text{PRG}_0(S_{2,01})$ .

**Properties of the special tree key structure.** We now prove several properties about the special tree key structure. Intuitively speaking the crux of the lemmas is the claim V-set for can a node can be used to derive its children. Furthermore each element in V-set for any node can only be derived from the V-set of nodes in exactly two different paths.

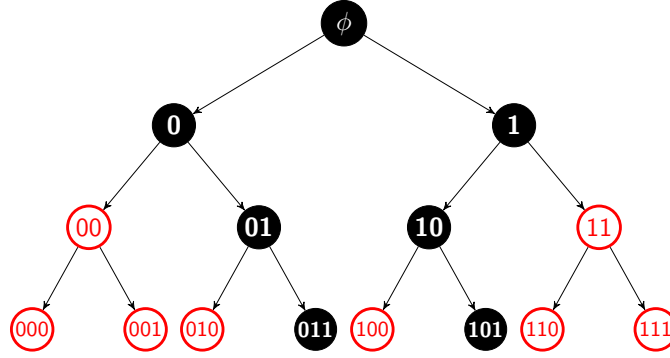
**Lemma 21 (Computability Lemma)** *There exists an explicit efficient procedure that given  $V_x, W_x$  computes  $V_{x \parallel 0}, W_{x \parallel 0}$  and  $V_{x \parallel 1}, W_{x \parallel 1}$ .*

**Proof** We start by noting that it suffices to show that for each  $i$ , given  $V_x^i, W_x^i$  one can compute  $V_{x\|0}^i, W_{x\|0}^i$  and  $V_{x\|1}^i, W_{x\|1}^i$ . We argue this next. Observe that two cases arise either  $|x| < i$  or  $|x| \geq i$ . We deal with the two cases:

- $|x| < i$ : In this case  $V_x^i$  is  $\{S_{i,x}, S_{i,(x+1)\|0^{i-|x|}}\}$  and these values can be used to compute  $S_{i,x\|0}, S_{i,x\|1}, S_{i,(x\|0)+1} = S_{i,x\|1}$  and  $S_{i,((x\|1)+1)\|0^{i-|x|-1}} = S_{i,(x+1)\|0\|0^{i-|x|-1}} = S_{i,(x+1)\|0^{i-|x|}}$ . Observe by case by case inspection that these values are sufficient for computing  $V_{x\|0}^i, W_{x\|0}^i$  and  $V_{x\|1}^i, W_{x\|1}^i$  in all cases.
- $|x| \geq i$ : Note that according to the constraints placed on  $x$  by the definition, if  $V_x^i = \emptyset$  then both  $V_{x\|0}^i$  and  $V_{x\|1}^i$  must be  $\emptyset$  as well. On the other hand if  $V_x^i \neq \emptyset$  then  $V_{x\|0}^i$  is still  $\emptyset$  while  $V_{x\|1}^i = V_x^i$ . Additionally,  $W_{x\|0}^i = W_{x\|1}^i = W_x^i$ .

This concludes the proof. ■

**Lemma 22 (Derivability Lemma)** *For every  $i \in [\kappa], x \in \{0, 1\}^i$  and  $x \neq 1^i$  we have that,  $S_{i,x+1}$  can be derived from keys in  $V_y^i$  if and only if  $y$  is a prefix of  $x\|1^{\kappa-i}$  or  $(x+1)\|1^{\kappa-i}$ . Additionally,  $S_{i,0^i}$  can be derived from keys in  $V_y^i$  if and only if  $y$  is a prefix of  $0^i\|1^{\kappa-i}$ .*



**Figure 6:** Black nodes represent the choices of  $x \in \{0, 1\}^{\leq 3}$  such that  $V_x^2$  can be used to derive  $S_{2,10}$ .

**Proof** We start by noting that for any  $y \in \{0, 1\}^{>i} \cap \{0, 1\}^{\leq \kappa}$ , by definition of V-sets we have that  $V_y^i = V_{y\|0^i}^i$  or  $V_y^i = \emptyset$ . Hence it suffices to prove the above lemma for  $y \in \{0, 1\}^{\leq i}$ .

We first prove that if  $y$  is a prefix of  $x$  or  $(x+1)$  then we can derive  $S_{i,x+1}$  from  $V_y^i$ . Two cases arise:

- Observe that if  $y$  is a prefix of  $x$  then we must have that either  $y$  is a prefix of  $x+1$  or  $x+1 = (y+1)\|0^{i-|y|}$ . Next note that by definition of V-sets we have that  $V_y^i = \{S_{i,y}, S_{i,(y+1)\|0^{i-|y|}}\}$ , and one of these values can be used to compute  $S_{i,x+1}$ .
- On the other hand if  $y$  is a prefix of  $x+1$  then again by definition of V-sets we have that  $V_y^i = \{S_{i,y}, S_{i,(y+1)\|0^{i-|y|}}\}$ , and  $S_{i,y}$  can be used to compute  $S_{i,x+1}$ .

Next we show that no other  $y \in \{0, 1\}^{\leq i}$  allows for such a derivation. Note that by definition of V-sets we have that  $V_y^i = \{S_{i,y}, S_{i,(y+1)\|0^{i-|y|}}\}$ . We will argue that neither  $S_{i,y}$  nor  $S_{i,(y+1)\|0^{i-|y|}}$  can be used to derive  $S_{i,x+1}$ .

- We are given that  $y$  is not a prefix of  $x + 1$ . This implies that  $S_{i,y}$  cannot be used to derive  $S_{i,x+1}$ .
- Now we need to argue that  $S_{i,(y+1)\|0^{i-|y|}}$  cannot be used to compute  $S_{i,x+1}$ . For this, it suffices to argue that  $x + 1 \neq (y + 1)\|0^{i-|y|}$ . If  $x + 1 = (y + 1)\|0^{i-|y|}$  then  $y$  must be prefix of  $x$ . However, we are given that this is not the case. This proves our claim.

The argument for the value  $S_{i,0^i}$  follows analogously. This concludes the proof.  $\blacksquare$

## 5.2 Hard on Average SVL Instances

In this section, we describe our construction for hard on average instance of SVL. In particular, we describe our sampler that samples hard on average instances  $(x_s, \text{Succ}, \text{Ver}, 1^\kappa)$ . Here  $x_s$  is the source node and Succ is the successor circuit. We define a directed edge between  $u$  and  $v$  if and only if  $\text{Succ}(u) = v$ . Ver is the verification circuit and is used to test whether a given node is the  $k^{\text{th}}$  node from  $x_s$ . That is,  $\text{Ver}(x, k) = 1$  iff  $x = \text{Succ}^{k-1}(x_s)$ . For the generated instances, we argue that it is hard to find the  $1^\kappa$  node in the path.

In our construction we use a selectively secure functional encryption scheme (FE.Setup, FE.KeyGen, FE.Enc, FE.Dec), a prefix-puncturable PRF (Definition 18), a semantically secure symmetric key encryption (SK.KeyGen, SK.Enc, SK.Dec) and injective PRGs having the left half injectivity property Definition 8.  $\text{PRG}_0$  and  $\text{PRG}_1$  denote the left and the right part of the output of this PRG.

The formal description of hard on average SVL instance sampler is provided in Figure 7. Internally this sampler generates the successor circuit to include functional encryption secret keys for circuits provided in Figure 8. Next we informally describe the SVL instances considered.

A sampled instance implicitly defines a line graph where each node in the graph is of the form  $(x, \sigma_1, \dots, \sigma_\kappa)$  where  $\sigma_i = \text{PrefixPunc}(S_i, x_{[i]})$  for all  $i \in [\kappa]$ . We say a node is *valid* if the above condition holds. The node  $(x, \sigma_1, \dots, \sigma_\kappa)$  is connected to  $(x + 1, \sigma'_1, \dots, \sigma'_\kappa)$  by an outgoing edge and to  $(x - 1, \sigma''_1, \dots, \sigma''_\kappa)$  by an incoming edge. The successor circuit on input  $(x, \sigma_1, \dots, \sigma_\kappa)$  checks for the validity of the node and if the node is valid it outputs  $(x + 1, \sigma'_1, \dots, \sigma'_\kappa)$ . The verification circuit on input  $(x, \sigma_1, \dots, \sigma_\kappa, j)$  outputs 1 if and only if  $x = j - 1$  and  $(x, \sigma_1, \dots, \sigma_\kappa)$  is valid.

We now explain how the successor circuit works. The successor circuit is described by a sequence of  $\kappa + 1$  secret keys  $\text{FSK}_1, \dots, \text{FSK}_{\kappa+1}$  for appropriate functions. These keys are generated corresponding to independent instances of functional encryption. Along with the keys the successor circuit also contains a ciphertext  $c_\phi$  that encrypts the empty string,  $\phi$ , under  $PK_1$  along with the key values  $V_\phi$  and  $W_\phi$ . Intuitively, the function key  $\text{FSK}_i$  corresponds to a function  $F_i$  that takes as input a binary string  $x$  of length  $i$  and outputs an encryption of  $x\|0$  and  $x\|1$  under  $PK_{i+1}$ . Additionally these ciphertexts, in addition to  $x\|0$  and  $x\|1$ , also contain key values  $V_{x\|0}, W_{x\|0}$  and  $V_{x\|1}, W_{x\|1}$  respectively. Recall from Section 5.1 that the keys in these sets are used to test validity of signatures provides as input and to generate the new ones.

The successor circuit on an input of the form  $(x, \sigma_1, \dots, \sigma_\kappa)$  does the following. It first obtains an encryption of  $x$  along with key values  $V_x$  and  $W_x$  under the public key  $PK_{\kappa+1}$ . This is done as follows. Start with  $c_\phi$  and decrypt it using key  $\text{FSK}_1$  to obtain encryptions of 0 and 1. Choose one of them based on which one is a prefix of  $x$  and continue the process. Repeating this process  $\kappa$  times results in the desired ciphertext. Next decrypt the obtained ciphertext using  $\text{FSK}_{\kappa+1}$  and it provides some information essential for checking validity of provided input signatures and additional

- **Sampled Ingredients:**

1. Sample  $\{S_i\}_{i \in [\kappa]}$  and  $K_\phi$  from  $\text{KeyGen}_{\mathcal{PPRF}}(1^\kappa)$ . Here  $S_i$ 's is a key that works for  $i$  bit inputs, namely  $\text{PPRF}_{S_i} : \{0, 1\}^i \rightarrow \{0, 1\}^\kappa$  for all  $i \in [\kappa]$ . Similarly,  $K_\phi$  works on inputs of length  $\text{rand}(\kappa)$  where  $\text{rand}(\cdot)$  would be specified later. Initialize  $V_\phi^i = S_i$ ,  $V_\phi = \bigcup_{i \in [\kappa]} V_\phi^i$  and  $W_\phi = \emptyset$ .
2. Sample  $(PK_i, MSK_i) \leftarrow \text{FE.Setup}(1^\kappa)$  for all  $1 \leq i \leq \kappa + 1$ .
3. Sample  $sk \leftarrow \text{SK.KeyGen}(1^\kappa)$  and let  $\Pi \leftarrow \text{SK.Enc}_{sk}(\pi)$  and  $\Lambda \leftarrow \text{SK.Enc}_{sk}(\lambda)$  where  $\pi = 0^{\ell(\kappa)}$  and  $\lambda = 0^{\ell'(\kappa)}$ . Here  $\ell(\cdot)$  and  $\ell'(\cdot)$  are appropriate length functions specified later.
4. Sample  $v \leftarrow \{0, 1\}^{2\kappa}$ .

- **Functional encryption ciphertext and keys to simulate obfuscation:**

1. For each  $i \in [\kappa]$  generate  $\text{FSK}_i \leftarrow \text{FE.KeyGen}(MSK_i, F_{i, PK_{i+1}, \Pi})$  and  $\text{FSK}_{\kappa+1} \leftarrow \text{FE.KeyGen}(MSK_{\kappa+1}, G_{v, \Lambda})$ , where  $F_{i, PK_{i+1}, \Pi}$  and  $G_{v, \Lambda}$  are circuits described in Figure 8.
2. Let  $c_\phi = \text{FE.Enc}_{PK_1}(\phi, V_\phi, W_\phi, 0^\kappa, 0)$

- **Source node:** The source node  $x_s$  is given by  $(0^\kappa, \sigma_1, \dots, \sigma_\kappa)$  where  $\sigma_i = \text{PPRF}_{S_i}(0^i)$  for all  $i \in [\kappa]$ .

- **Successor Circuit:** The successor circuit **Succ** in our setting takes as input  $x, \sigma_1, \dots, \sigma_\kappa$  and outputs  $x + 1, \sigma'_1, \dots, \sigma'_\kappa$  if the associated signatures  $\sigma_1, \dots, \sigma_\kappa$  are valid. It proceeds as follows:

1. For  $i \in [\kappa]$  compute  $c_{x_{[i-1]}\|0}, c_{x_{[i-1]}\|1} := \text{FE.Dec}(\text{FSK}_i, c_{x_{[i-1]}})$ .
2. Obtain  $d_x = ((\alpha_1, \dots, \alpha_\kappa), (\beta_j, \dots, \beta_\kappa))$  as output of  $\text{FE.Dec}(\text{FSK}_{\kappa+1}, c_x)$ . Here  $j = f(x)$  where  $f(x)$  is the smallest  $j$  such that  $x = x_{[j]}\|1^{\kappa-j}$ .
3. Output  $\perp$  if  $\text{PRG}_0(\sigma_i) \neq \alpha_i$  for any  $i \in [\kappa]$  or if  $d_x = \perp$ .
4. If  $x = 1^\kappa$ , output SOLVED.
5. For each  $i \in [j - 1]$  set  $\sigma'_i = \sigma_i$ .
6. For each  $i \in \{j, \dots, \kappa\}$  set  $\gamma_i = \text{PRG}_1(\sigma_i)$  and  $\sigma'_i$  as  $\text{SK.Dec}_{\gamma_j, \dots, \gamma_\kappa}(\beta_i)$ , decrypting  $\beta_i$  encrypted under  $\gamma_j, \dots, \gamma_\kappa$ .
7. Output  $(x + 1, \sigma'_1, \dots, \sigma'_\kappa)$ .

- **Verification Circuit:** The verification circuit **Ver** on input  $x, \sigma_1, \dots, \sigma_\kappa, j$  outputs 1 if **Succ** on input  $x, \sigma_1, \dots, \sigma_\kappa$  doesn't output  $\perp$  and  $x = j - 1$  and 0 otherwise.

**Figure 7:** Hard on average instance for SVL based on hardness of FE.



$$F_{i, PK_{i+1}, \Pi}$$

**Hardcoded Values:**  $i, PK_{i+1}, \Pi$ .

**Input:**  $(x \in \{0, 1\}^{i-1}, V_x, W_x, K_x, sk, \text{mode})$

1. If  $(\text{mode} = 0)$  then output  $\text{FE.Enc}_{PK_{i+1}}(x||0, V_{x||0}, W_{x||0}, K_{x||0}, sk, \text{mode}; K'_{x||0})$  and  $\text{FE.Enc}_{PK_{i+1}}(x||1, V_{x||1}, W_{x||1}, K_{x||1}, sk, \text{mode}; K'_{x||1})$ , where for  $b \in \{0, 1\}$ ,  $K_{x||b} = \text{PrefixPunc}(K_x, b||0)$  and  $K'_{x||b} = \text{PrefixPunc}(K_x, b||1)$  and  $(V_{x||0}, W_{x||0}), (V_{x||1}, W_{x||1})$  are computed using the efficient procedure from the Computability Lemma (Lemma 21).
2. Else recover  $(x||0, c_{x||0})$  and  $(x||1, c_{x||1})$  from  $\text{SK.Dec}_{sk}(\Pi)$  and output  $c_{x||0}$  and  $c_{x||1}$ .

$$G_{v, \Lambda}$$

**Hardcoded Values:**  $v, \Lambda$

**Input:**  $x \in \{0, 1\}^\kappa, V_x, W_x, K_x, sk, \text{mode}$

1. If  $(\text{PRG}(x) = v)$  then output  $\perp$ .
2. If  $\text{mode} = 0$ , (Below  $j = f(x)$  where  $f(x)$  is the largest  $j$  such that  $x = x_{[j]}||1^{\kappa-j}$ .)
  - (a) For each  $i \in [\kappa]$ , set  $\alpha_i = \text{PRG}_0(\sigma_i)$  (obtained from  $W_x^i$  for  $i \leq j$  and from  $V_x^i$  for  $i > j$ ).
  - (b) For each  $i \in \{j, \dots, \kappa\}$  set  $\gamma_i = \text{PRG}_1(\sigma_i)$  and  $\beta_i = \text{SK.Enc}_{\gamma_j, \dots, \gamma_\kappa}(S_{i, x_{[i]}+1})$ , encrypting  $S_{i, x_{[i]}+1}$  under  $\gamma_j, \dots, \gamma_\kappa$ . (Using randomness obtained by expanding  $K_x$  sufficiently.)
  - (c) Output  $((\alpha_1, \dots, \alpha_\kappa), (\beta_j, \dots, \beta_\kappa))$
3. Else recover  $(x, d_x)$  from  $\text{SK.Dec}_{sk}(\Lambda)$  and output  $d_x$ .

**Figure 8:** Circuits for which functional encryption secret keys are given out.

information to generate the signatures for the next node. More details are provided in Figures 7 and 8.

**Setting  $\text{rand}(\cdot)$**  We set  $\text{rand}(\kappa) = 2\kappa + r(\kappa)$  where  $r(\kappa)$  is the maximum number of random bits used for generating encryptions of  $S_{i, x_{[i]}+1}$  under  $\gamma_j, \dots, \gamma_\kappa$  for every  $i \in [j, \kappa]$ .

**Correctness.** The correctness of our construction follows for the correctness of the underlying primitives. Here we note that since  $v \xleftarrow{\$} \{0, 1\}^{2\kappa}$ , therefore the probability that  $v$  is in the image of the PRG is negligible and hence this step of  $G_{v, \Lambda}$  is not triggered. Similarly since the provided

ciphertext  $c_\phi$  is set to be in mode 0 the other mode is never triggered.

Given that the above case do not arise we have that  $\alpha_i = \text{PRG}_0(S_{i,x_{[i]}})$ . Recall that the successor check if  $\text{PRG}_0(\sigma_i) = \alpha_i$ . Now from the left half injectivity property of the PRG we have that these checks pass if and only if  $\sigma_i = S_{i,x_{[i]}}$ . Hence, if successor does not output  $\perp$  then the input  $x, \sigma_1, \dots, \sigma_\kappa$  must be valid. Additionally from the correctness of  $\text{SK.Dec}$  we have that the successor recovers the correct values  $\sigma'_i = S_{i,x+1_{[i]}}$  corresponding to prefixes of  $x + 1$  which are different from the prefixes of  $x$ . This provides the full set of associated signatures on  $x + 1$ .

The correctness of the verification circuit  $\text{Ver}$  directly follows from the correctness of the successor circuit.

### 5.3 Proof of Hardness

In this section, we will prove that the SVL instances generated in our construction are hard to solve. The proof of security mimics the proof for the setting of  $i\mathcal{O}$  with two crucial differences. Specifically:

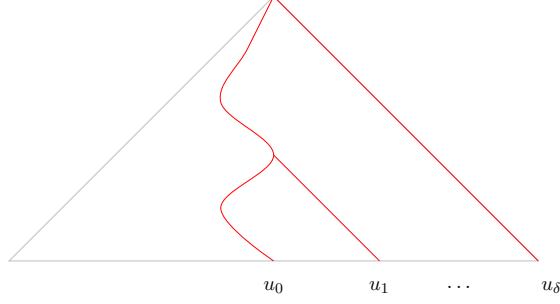
1. In our setting the honest distribution has the PRG check in-built and so we do not need to introduce it as was done in going from  $\text{Hyb}_0$  to  $\text{Hyb}_1$  as done in Appendix 4.2. In particular, this check is built into the circuit  $G_{v,\Lambda}$ .
2. Since we are using functional encryption, the puncturing, checking of input signatures and generating on new ones has to be done differently and a bit more carefully. We highlight these differences as we go along the proof.

**Hybrid Structure.** We describe our proof in a way analogous to the proof from Appendix 4.2. In particular, we show that no polynomial time adversary can output a valid value  $(1^\kappa, \sigma_1, \dots, \sigma_\kappa)$ . We will show this via a hybrid argument. Starting with providing SVL instance as in distribution from Figure 7, namely hybrid  $\text{Hyb}_0$ , we move to a hybrid  $\text{Hyb}_{4,\delta}$ , where  $\delta$  is defined below. In the final hybrid, the successor circuit returns  $\perp$  on all inputs of the form  $(1^\kappa, \cdot, \dots, \cdot)$ . Observe that the advantage of any adversary in solving the SVL instance in this final hybrid is 0. This proves our claim. We make this change by going through a polynomial in the security parameter number of intermediate hybrids. Next we describe our hybrids.

- $\text{Hyb}_0$  : This hybrid corresponds to SVL instance generation as given in Figure 7.
- $\text{Hyb}_1$  : In this hybrid we change how the value  $v$ , hard-coded in  $G_{v,\Lambda}$  is generated. In particular, instead of sampling  $v$  uniformly at random from  $\{0, 1\}^{2\kappa}$ , we generate  $v$  as  $\text{PRG}(u_0)$  where  $u_0$  is sampled uniformly from  $\{0, 1\}^\kappa$ . Here PRG is a length doubling injective pseudorandom generator.

Computational indistinguishability between  $\text{Hyb}_0$  and  $\text{Hyb}_1$  follows from the pseudorandomness property of the PRG.

**Recalling notation for  $u_j$ .** Recall from Appendix 4.2, for any string  $u \in \{0, 1\}^\kappa$ , let  $f(u)$  denote the index of the lowest order bit of  $u$  that is 0 (with the index of the highest order bit being 1). More formally,  $f(u)$  is the smallest  $j$  such that  $u = u_{[j]} || 1^{\kappa-j}$ . For example, if



**Figure 9:** Starting with  $u_0$ ,  $u_{j+1}$  is obtained by setting the lowest order 0-bit in  $u_j$  to 1.

$u = \overbrace{100}^3 11$  then  $f(u) = 3$ . Also let  $\delta(u)$  be the number of 0 bits in the binary representation of  $u$ .

Starting with a value  $u_0 \in \{0, 1\}^\kappa$  we define a sequence of values such that  $u_{j+1}$  is the value  $u_j$  with the  $f(u_j)^{th}$  bit set to 1. More formally,  $u_{j+1} = u_j + 2^{\kappa-f(u_j)}$ . We will use the shorthand  $\delta$  to denote  $\delta(u_0)$ .

**New  $\pi, \lambda$  values.** As in Appendix 4.2 we process the hybrids according to  $U_j$  values. However, here we need to set the stage before these can be done. For this we define some additional notation. For any  $x \in \{0, 1\}^{\leq \kappa}$ , let  $c_x^*$  denote the ciphertext and  $d_x^*$  the clear output in execution of the successor in the  $\text{Hyb}_1$  when Steps 1 and 2 of the successor circuit are executed on input  $x$ . We let  $P$  be the set of all prefixes of  $u_0, u_1, \dots, u_\delta$  including the empty string  $\phi$ . Note that  $|P| \leq (\kappa + 1)^2$ . Additionally we define  $Q$  as follows. For every  $x \in P$ , let  $y$  be the value with the last bit of  $x$  flipped. We add  $y$  to  $Q$  if  $y \notin P$ . We set:

$$\begin{aligned} \pi^* &= \parallel_{x \in P \cup Q} (x, c_x^*) \\ \lambda^* &= \parallel_{x \in P \cap \{0, 1\}^\kappa} (x, d_x^*) \end{aligned}$$

We set  $\ell(\kappa)$  and  $\ell'(\kappa)$  to be the polynomials that describe an upper bound on the lengths of  $\pi^*$  and  $\lambda^*$  over all choices of  $u_0 \in \{0, 1\}^\kappa$ .

- **Hyb<sub>2</sub>:** In this hybrid we change how the hardcoded values  $\Pi$  and  $\Lambda$  are generated. Unlike hybrids  $\text{Hyb}_1$  and  $\text{Hyb}_2$  where these values were generated as encryptions of  $0^{\ell(\kappa)}$  and  $0^{\ell'(\kappa)}$ , in this hybrid we generate them as encryptions  $\pi^*$  and  $\lambda^*$  describe above, respectively.

Computational indistinguishability between  $\text{Hyb}_1$  and  $\text{Hyb}_2$  follows from the semantic security of the encryption scheme.

- **Hyb<sub>3</sub>:** In this hybrid, for  $x \in P$  we change the the  $c_x^*$  values embedded in  $\pi^*$ . Recall that in hybrid  $\text{Hyb}_2$  for each  $x$ ,  $c_x^*$  is generated as  $\text{FE.Enc}_{PK_{|x|+1}}(x, V_x, W_x, K_x, 0^\kappa, 0; K'_x)$ . We change the  $c_x^*$  to be now generated as  $\text{FE.Enc}_{PK_{|x|+1}}(x, 0^{2\kappa}, 0^{2\kappa}, 0^\kappa, sk, 1; \omega_x)$  using fresh randomness  $\omega_x$ .<sup>5</sup>

<sup>5</sup>Note that we do not change ciphertexts corresponding to  $x \in Q$ .

Computational indistinguishability between  $\text{Hyb}_2$  and  $\text{Hyb}_3$  follows by a sequence of sub-hybrids. We define an ordering on elements of  $P$  as follows. For  $x, y \in P$  we say that  $x < y$  if either  $|x| < |y|$ , or  $|x| = |y|$  and  $x < y$ <sup>6</sup>. Next we define the hybrid  $\text{Hyb}_{2,y}$  to be a modification of  $\text{Hyb}_2$  where for all  $x \in P$  such that  $x \leq y$  we have that  $c_x^*$  is generated as  $\text{FE.Enc}_{PK_{|x|+1}}(x, 0^{2\kappa}, 0^{2\kappa}, 0^\kappa, sk, 1; \omega_x)$  using fresh randomness  $\omega_x$ .

Note that it suffices to argue that  $\text{Hyb}_{2,x'}$  and  $\text{Hyb}_{2,x}$  are indistinguishable for any two adjacent values  $x'$  and  $x$  in  $P$  such that  $x' < x$ . We argue this via a two step hybrid argument.

1.  $\text{Hyb}_{2,x,1}$ : In this hybrid we change  $c_x^*$  to  $\text{FE.Enc}_{PK_{|x|+1}}(x, \mathbb{V}_x, W_x, K_x, 0^\kappa, 0; \omega_x)$  using fresh randomness  $\omega_x$ .

Note that for all prefixes  $x''$  of  $x$  we have that  $x'' < x$ . Therefore for all such  $x''$  we have that  $c_{x''} = \text{FE.Enc}_{PK_{|x''|+1}}(x'', 0^{2\kappa}, 0^{2\kappa}, 0^\kappa, sk, 1; \omega_{x''})$ . This fact along with the pseudorandom at punctured point property implies computational indistinguishability from the previous hybrid, namely  $\text{Hyb}_{2,x}$ .

2.  $\text{Hyb}_{2,x,2}$ : In this hybrid we change  $c_x^*$  to  $\text{FE.Enc}_{PK_{|x|+1}}(x, 0^{2\kappa}, 0^{2\kappa}, 0^\kappa, sk, 1; \omega_x)$  using fresh randomness  $\omega_x$ .

The computational indistinguishability of this hybrid from  $\text{Hyb}_{2,x,1}$  relies on the selective security of the functional encryption scheme with public-key  $PK_{|x|+1}$ . Note that we can invoke security of functional encryption as the change in the messages being encrypted does not change the output of the decryption using key  $\text{FSK}_{|x|+1}$ .

- $\text{Hyb}_{4,j+1}$ : In hybrid  $\text{Hyb}_{4,j+1}$  for  $j \in \{0, \dots, \delta - 1\}$ , we make two changes with respect to  $\text{Hyb}_{4,j}$ . Just like in Appendix 4.2 we let  $f_j$  as the shorthand for  $f(u_j)$  and  $t_j = u_{j[f_j]} + 1$ .

- We change the set  $W_{t_j}^{\nu_j}$  to be a uniformly random string  $z \leftarrow \{0, 1\}^\kappa$  rather than containing the value  $\text{PRG}_0(S_{\nu_j, t_j})$ . Note that this change needs to be made at two places. Namely we set  $W_{t_j}^{\nu_j} = \{z\}$  in  $c_p^*$  where  $p$  is a sibling path of  $u_{j+1}$  and from there on this value will be percolated to all its descendents as well. Additionally we set  $\alpha_{f_j}$  included in  $d_{u_{j+1}}^*$  to be  $z$ .
- We now generate encryptions  $\beta_{f_j}, \dots, \beta_\kappa$  included in  $d_{u_{j+1}}^*$  with encryption of  $0^\kappa$ .

Note that as a consequence of this change the successor circuit now starts to output  $\perp$  additionally on all inputs in  $\{u_j + 1, \dots, u_{j+1}\}$ . This is because for every input  $\sigma_{f_j}$  we have that  $z \neq \text{PRG}_0(\sigma_{f_j})$  with overwhelming probability. Since in hybrid  $\text{Hyb}_{4,j}$  the successor was already outputting  $\perp$  on inputs  $\{u_0, \dots, u_j\}$  we have that the successor outputs  $\perp$  on all inputs in  $\{u_0, \dots, u_{j+1}\}$ .

Now we argue computational indistinguishability between  $\text{Hyb}_{4,j}$  and  $\text{Hyb}_{4,j+1}$ .

- $\text{Hyb}_{4,j,1}$ : In this hybrid instead we replace the key  $S_{f_j, t_j}$  with a random string  $S' \leftarrow \{0, 1\}^\kappa$ .

Computational indistinguishability follows from the pseudorandom at punctured point property. This argument relies on that fact that no  $\mathbb{V}$  set that hasn't been removed can be used to obtain  $S_{f_j, t_j}$ . This follows from the following two facts.

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<sup>6</sup>Note that  $\phi$  is the smallest value in  $P$  by this ordering.

- $V_y^{f_j}$  values have been removed whenever  $y$  is a prefix of  $u_j$  or  $u_{j+1}$ . Note that it follows from the Derivability Lemma (Lemma 22) that these were the only  $V$ -sets that could be used to derive  $S_{f_j, t_j}$ .
- Additionally  $\sigma_{f_j}$  is encrypted in  $\beta_{f_j}$  and this value is included in  $d_{u_j}^*$ . But this has already been replaced with an encryption of  $0^\kappa$  except  $d_{u_0}^*$  which is set to  $\perp$ .
- **Hyb<sub>4,j,2</sub>**: In this hybrid instead we replace the  $\beta_{f_j}, \dots, \beta_\kappa$  in  $d_{u_{j+1}}^*$  to be generated using fresh randomness.  
Computational indistinguishability follows from the pseudorandom at punctured point property.
- **Hyb<sub>4,j,3</sub>**: In this hybrid, we change  $\text{PRG}_0(S')$  and  $\text{PRG}_1(S')$  to be random strings  $z, z'$ . Change of  $\text{PRG}_0(S')$  to  $z$  implies that the set  $W_{t_j}^{f_j}$  is  $\{z\}$ . Similarly  $\gamma_{f_j}$  will be  $z'$ . Indistinguishability between **Hyb<sub>4,j</sub>** and **Hyb<sub>4,j,1</sub>** follows from the pseudorandomness property of the PRG.
- **Hyb<sub>4,j,4</sub>**: In this hybrid we set encryption of all  $\beta_{f_j}, \dots, \beta_\kappa$  in  $d_{u_{j+1}}^*$  with encryption of  $0^\kappa$ .  
Next by semantic security we have this hybrid is computationally indistinguishable from the previous. Here we rely on the fact that one of the keys  $\gamma_{f_j}$  has been replaced with random.

Note that hybrid **Hyb<sub>4,j,4</sub>** is same as hybrid **Hyb<sub>4,j+1</sub>**.

**Concluding the proof.** Observe that the hybrid **Hyb<sub>4,0</sub>** is identical to hybrid **Hyb<sub>3</sub>** and **Hyb<sub>4, $\delta$</sub>**  is such that the successor outputs  $\perp$  on all inputs of the form  $(1^\kappa, \cdot, \dots, \cdot)$ . Consequently, no adversary can solve the SVL instance in this final hybrid with probability better than 0. We summarize our hybrids in Table 1.

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Hybrids	Description	Consequence
Hyb <sub>0</sub>	Real World	
Hyb <sub>1</sub>	Change $v := \text{PRG}(u_0)$ .	Successor circuit starts outputting $\perp$ at $u_0$
Hyb <sub>2</sub>	Changing the encrypted values to $\Pi^*$ and $\Lambda^*$ .	Ready to puncture the successor circuit on the paths defined by $u_0, \dots, u_{\delta(u_0)}$ .
Hyb <sub>3</sub>	Punctured the successor circuit on the paths defined by $u_0, \dots, u_{\delta(u_0)}$ using the “hidden” trapdoor mechanism.	Ready to make successor circuit output $\perp$ on the range $[u_0 + 1, u_1]$
Hyb <sub>4,j+1</sub>	Change $\text{PRG}_0(S_{\nu_j, t_j})$ to uniformly random value. Change encryptions $\beta_{f_j}, \dots, \beta_\kappa$ included in $d_{u_{j+1}}^*$ with encryption of $0^\kappa$	Successor circuit now outputs $\perp$ additionally on all points in the range $[u_j + 1, u_{j+1}]$ . Ready to change the successor circuit to output $\perp$ on the interval $[u_{j+1} + 1, u_{j+2}]$

**Table 1:** Summary of Hybrids

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