NTRU-KE: A Lattice-based Public Key Exchange Protocol

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Abstract. Public key exchange protocol is identified as an important application in the field of public-key cryptography. Most of the existing public key exchange schemes are Diffie-Hellman (DH)-type, whose security is based on DH problems over different groups. Note that there exists Shor’s polynomial-time algorithm to solve these DH problems when a quantum computer is available, we are therefore motivated to seek for a non-DH-type and quantum resistant key exchange protocol. To this end, we turn our attention to lattice-based cryptography. The higher methodology behind our roadmap is that in analogy to the link between ElGamal, DSA, and DH, one should expect a NTRU lattice-based key exchange primitive in related to NTRU-ENCRYPT and NTRU-SIGN. However, this expected key exchange protocol is not presented yet and still missing. In this paper, this missing key exchange protocol is found, hereafter referred to as NTRU-KE, which is studied in aspects of security and key-mismatch failure. In comparison with ECDH (Elliptic Curve-based Diffie-Hellman), NTRU-KE features faster computation speed, resistance to quantum attack, and more communication overhead. Accordingly, we come to the conclusion that NTRU-KE is currently comparable with ECDH. However, decisive advantage of NTRU-KE will occur when quantum computers become a reality.

Keywords: Key exchange, Diffie-Hellman, NTRU, quantum resistance

1 Introduction

1.1 Application Background

The first practical public key exchange protocol was introduced by Diffie and Hellman in their epoch-making paper [1], allowing two parties that have never met to establish a shared secret key by exchanging information over an untrusted and insecure channel. An adversary is unable to retrieve the key even with the ability to eavesdrop the communication channel. In practical applications, public key exchange protocols play an important role in the use of shared-key cryptography to protect transmitted data over insecure networks. For instance, they are a central building block of today’s most commonly used cryptographic protocols (such as SSL [2], IPSec [3], and SSH [4]), which have enabled today’s proliferation of secure electronic commerce over the Internet. Consequently, public key exchange, since its proposal, has stimulated considerable research efforts.

1.2 Related Work

Let us begin with the classic Diffie-Hellman (DH) primitive protocol. A prime $p$ and a generator $g \in \mathbb{Z}_p^*$ are public generated, they are known not only to all parties but also the adversary. The detailed steps of DH are as follows.

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Step 1: entity $i$ picks $x \in [1, p-2]$ at random and let $X = g^x \mod p$. Then, he sends $X$ to entity $j$.

Step 2: entity $j$ picks $y \in [1, p-2]$ at random and let $Y = g^y \mod p$. Then, he sends $Y$ to entity $i$.

It holds that $X^y = (g^x)^y = (g^y)^x = Y^x$ over group $\mathbb{Z}_p^\times$. Entity $i$ computes $Y^x \mod p$, and entity $j$ computes $X^y \mod p$, then both entities come up with the common secret key $K = g^{xy} \mod p$.

After the invention of the DH protocol, most of the subsequent public key exchange protocols appear to be DH-type protocols. By DH-type we mean that these protocols are essentially the DH protocol, but works over different groups. These DH-type protocols are summarized below.

- **Elliptic curve-based DH.** Elliptic curve-based DH (ECDH) is a variant of the DH protocol using group that consists of points on an elliptic curve (see [5] for details). Compared with DH, the primary benefit possessed by ECDH is a smaller key size, thereby reducing storage and transmission requirements.

- **Recurrence equations-based DH.** This class of DH-type public key exchange protocols are based on recurrence equations, see e.g., GH [6], LUC [7], and Chebyshev-based [8, 9]. These cryptosystems are all based on recurrence equations, but with different forms. We explicitly note that there always exists a DH-type key exchange protocol in each cryptosystem. In these cryptosystems, the basic group operation is iteration of a certain recurrence equation, and the group members are generated by iterations of the recurrence equation from certain initial values.

- **XTRDH.** In XTR [10] cryptosystem, it follows that the corresponding key exchange protocol XTRDH is a DH-type protocol. In essence, XTRDH is also a recurrence equation-based DH. We list it separately because the significant feature of XTRDH is that it uses the trace over $\text{GF}(p^2)$ to represent elements of a subgroup of $\text{GF}(p^6)^\times$. This leads to a smaller key size and substantial savings both in communication and computation overhead in applications.

We are now concerned with the hard problems, on which the security of these DH-type protocols depends. Let $g$ denote a generator of a group (e.g., the multiplicative group $\mathbb{Z}_p^\times$, an elliptic curve group, or a group generated by iterations of a certain recurrence equation). Let $x$ and $y$ be randomly chosen elements in this group.

**Definition 1.** i) The computational DH (CDH) problem is informally stated as follows: given a generator $g$ and the values of $g^x$ and $g^y$ in a group, compute the value of $g^{xy}$.

ii) The CDH assumption: for appropriate parameters, it is computational intractable to solve CDH problem.

iii) The discrete logarithm (DL) problem is informally stated as follows: given a generator $g$ and the value of $g^x$ in a group, compute the value of $x$.

iv) The DL assumption: for appropriate parameters, it is computational intractable to solve DL problem.

The security of these DH-type protocols relies on the CDH assumption (in stronger semantic security notion, decisional DH (DDH) assumption is required). Notice that we can break CDH assumption if we already know an algorithm to break DL assumption. We denote this relationship by CDH assumption $> DL$ assumption. To date, the most efficient means known to solve the DH problem is to solve the DL problem, and there is a strong heuristic argument showing that DL problem and DH problem are very likely to be equivalent [11]. Accordingly, we may also say that the security of these DH-type protocols relies on the DL assumption.
1.3 Research Motivations

Two motivations of our research are identified.

- **Motivation 1.** It is disappointed to observe that over three decades after the discovery of the DH protocol, the cryptographer’s toolbox still contains very few public key exchange primitive schemes. Despite the fact that a tremendous amount of key exchange protocols have been proposed and analyzed, many of these protocols are dependent on DH problem, see, e.g., MTI [12], Unified Model [13], MQV [14], HMQV [15], to just list a few. This motivates us to seek for high-performance non-DH-type key exchange primitives.

- **Motivation 2.** Shor in [16,17] showed that quantum computers, when they become a reality, will render DH-type protocols insecure. It is shown that the quantum computer is developed rapidly [18], leading to that quantum resistant key exchange protocols are urgently needed.

Target at designing a non-DH-type and quantum resistant key exchange primitive, we should turn our attention to new hard problems. Ironically, over the past years, very few convincingly efficient key exchange protocols were well designed by using other hard problems despite of considerable research efforts. The seek for new schemes appears sometimes hopeless as new schemes are immediately broken or, if they survive, are compared with the DH-type protocols, which are obviously elegant, simple, and efficient. Consequently, the search for new key exchange protocols still remains a major challenge.

1.4 Roadmap

In the complexity-theoretic-based modern cryptography, the security of a public-key protocol should depend on a hard problem. To design a non-DH-type scheme, we start by review the existing cryptosystems and the underlying hard problems. Three of the most successful practical cryptosystems are taken into account.

1. DL cryptosystem that is based on DL problem in the multiplicative group \( \mathbb{Z}_p^* \);
2. ECDL cryptosystem that is based on ECDL problem in an elliptic curve group; and
3. NTRU cryptosystem that is related to shortest vector problem (SVP) and closest vector problem (CVP) in a NTRU lattice.

Generally speaking, a public cryptosystem primarily consists of three important members: 1) public key encryption, 2) digital signature, and 3) public key exchange. The representative members in the above three cryptosystems are displayed in Table. 1. It is clearly observed from Table. 1 that the public key exchange protocol in the NTRU cryptosystem is missing. In analogy to the link between ElGamal, DSA, and DH, one should expect a NTRU lattice-based key exchange primitive in related to NTRU-ENCRIPT and NTRU-SIGN. Following the naming rule, this key exchange protocol, if exists, should be named as NTRU-KE. Our roadmap is now clear, namely exploring this potentially existed NTRU-KE.

<table>
<thead>
<tr>
<th>Table 1. Representative Members in DL, ECDL, and NTRU Cryptosystems</th>
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<tr>
<td>protocol</td>
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<td>public key exchange</td>
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**Note:** the prefix "EC" denotes the corresponding elliptic curves-based hard problem or protocols. The symbol "?" denotes the item is missing currently.
1.5 Main Contributions

We regard our main contributions as three-fold:

1. contributing towards a successful construction of a non-DH-type and quantum resistant key exchange primitive scheme NTRU-KE, whose security is related to hard problems in a NTRU lattice;
2. estimating the key-mismatch failure of NTRU-KE theoretically; and
3. performing a detailed comparative study between NTRU-KE and ECDH.

1.6 Paper Organization

The remainder of this paper is organized as follows. Section 2 introduces technical and mathematical preliminaries. In Section 3, we describe NTRU-KE, formalize the underlying hard problem, and analyze the lattice attacks. In Section 4, the probability of key-mismatch failure of NTRU-KE is estimated. Then, the performance of NTRU-KE is theoretically compared with ECDH in Section 5. Finally, some conclusions are drawn in Section 6.

2 Notions, Mathematical and NTRU-ENCRYPT Background

A generic execution of a key exchange protocol between two entities is called a run of the protocol. Any particular run of a protocol is called a session. The keying information exchanged in the course of a protocol run is referred to as a session key. The individual messages that form a protocol run are called flows.

We denote by \( \mathbb{Z} \) the integer ring and by \( \mathbb{Z}_q \) the residue class ring \( \mathbb{Z}/\mathbb{Z}_q \). The truncated polynomial ring \( R_q = \mathbb{Z}_q[x]/(x^N - 1) \) consists of all polynomials with degree less than \( N \) and coefficients in \( \mathbb{Z}_q \). An element \( f \in R_q \) can be written as a polynomial or a vector,

\[
    f = \sum_{i=0}^{N-1} f_i x^i = [f_0, \ldots, f_{N-1}].
\]

Elements in \( R_q \) are written in bold type, if the vector description is needed. Two polynomials \( f, g \in R_q \) are multiplied by the ordinary convolution,

\[
    (f * g)_k \equiv \sum_{i+j \equiv k \pmod{N}} (f_i \cdot g_j), \quad k = 0, \ldots, N - 1,
\]

which is commutative and associative. The convolution product is presented by \( * \) to distinguish it from the multiplication in \( \mathbb{Z}_q \). We define a center \( l_2 \)-norm of an element \( f \in R_q \) by

\[
    |f|_2 = (\sum_{i=0}^{N-1} (f_i - \bar{f})^2)^{1/2},
\]

where \( \bar{f} = \frac{1}{N} \sum_{i=0}^{N-1} f_i \). Let \( a \leftarrow A \) denote the process of picking an element \( a \) from the set \( A \) uniformly at random. A real-valued function \( \epsilon(k) < k^{-c} \) is negligible if for every \( c > 0 \) there exists \( k_c > 0 \) such that \( \epsilon(k) < k^{-c} \) for all \( k > k_c \).

The NTRU-ENCRYPT depends on three global integers \( (N, p, q) \) and four set \( \mathcal{L}_f, \mathcal{L}_q, \mathcal{L}_r, \mathcal{L}_m \) of polynomials of degree \( N - 1 \) with small integer coefficients. The previous parameter choices for NTRU-ENCRYPT often take the small integer coefficients of polynomials to be binary \{0, 1\} or
trinary \{-1,0,1\}. We adopt the trinary polynomials because they are better to resist the recent hybrid meet-in-the-middle (MITM) and lattice reduction attack [25]. We set the notation:

\[ \mathcal{L}(d_1, d_2) = \begin{cases} F \in \mathbb{R}_q : & \text{F has } d_1 \text{ coefficients equal to 1}, \\
& d_2 \text{ coefficients equal to } -1, \text{ the rest 0.} \end{cases} \]

With this notation, we proceed to set

\[ \mathcal{L}_f = \mathcal{L}(d_f, d_f - 1), \mathcal{L}_g = \mathcal{L}(d_g, d_g), \mathcal{L}_r = \mathcal{L}(d_r, d_r). \] (1)

We take \( \mathcal{L}_f = \mathcal{L}(d_f, d_f - 1) \) instead of \( \mathcal{L}_f = \mathcal{L}(d_f, d_f) \) because it is required that \( f \) is invertible and a polynomial satisfying \( f(1) = 0 \) can never be invertible.

The NTRU-ENCRYPT algorithm starts by setting up three global integers \((N, p, q)\), such that:
- \( N \) is prime,
- \( p \) and \( q \) are coprime, \( \gcd(p, q) = 1 \), and
- \( q \) is considerably larger than \( p \).

**Key Generation.**

Step 1: Choose \( f \leftarrow \mathcal{L}_f \text{ and } g \leftarrow \mathcal{L}_g \) such that there exist \( F_q, F_p \in \mathbb{R}_q \) satisfying

\[ F_q \ast f \equiv 1 \pmod{q}, \ F_p \ast f \equiv 1 \pmod{p}. \]

Step 2: Let

\[ h \equiv f_q^{-1} \ast g \pmod{q}. \]

public key: \( h \), private key: \( f, F_p \).

**Encryption.**

Step 1: To encrypt \( m \in \mathcal{L}_m \), we first choose an \( r \leftarrow \mathcal{L}_r \), then compute the ciphertext as

\[ c \equiv pr \ast h + m \pmod{q}. \]

**Decryption.**

Step 1: We first compute

\[ a = f \ast c \pmod{q} \]

\[ = pr \ast g + f \ast m \pmod{q}. \]

For appropriate parameter choices, the absolute value of coefficients in \((pr \ast g + f \ast m)\) is small, together with the fact that \( q \) is chosen to be large, we almost derive

\[ a = pr \ast g + f \ast m. \]

Step 2: We then recover the message \( m \) by computing

\[ m = F_p \ast a \pmod{p}. \]

Note that, the latest parameters selection for NTRU-ENCRYPT always set \( f = 1 + pF \), where \( F \in \mathcal{L}_f \) in the key generation process, then the Step 2 in the decryption process is eliminated since \( F_p \equiv 1 \pmod{p} \).

The security of NTRU-ENCRYPT depends on the following computational assumption.

**Definition 2. ([26])** i) **The NTRU-ENCRYPT inversion problem:** For a given security parameter \( k \), which specifies \((N, p, q)\) and the spaces \( \mathcal{L}_f, \mathcal{L}_g, \mathcal{L}_m, \) and \( \mathcal{L}_r \), as well as a random public key \( h \) and ciphertext \( c \equiv h \ast r + m \pmod{q} \), where \( m \in \mathcal{L}_m \) and \( r \in \mathcal{L}_r \), find \( m \).

ii) **The NTRU-ENCRYPT assumption:** for appropriate parameters, it is computational intractable to solve the NTRU-ENCRYPT inversion problem.
3 NTRU-KE Construction

The classic DH protocol and the subsequent DH-type protocols have given cryptographers a strong impression that a key exchange primitive protocol typically consists of two flows. However, it seems impossible to design a two-flow-based key exchange primitive in relation to NTRU-ENCRYPT, this is because the special structure of the public key in NTRU-ENCRYPT. This, from our point of view, is the main reason why the NTRU-KE protocol has been missing up till now. Notably, there is no restriction that a key exchange primitive must consist of two flows, and indeed, most of applicable key exchange protocols consist of more than two flows. In view of this consideration, the three-flow-based NTRU-KE primitive protocol is proposed.

The NTRU-KE protocol starts by setting up three global integers \((N, p, q)\) at first, this process is the same as that of NTRU-ENCRYPT. The data flows of NTRU-KE are depicted in Fig. 1. Here is a detailed verbal description of these steps.

- **Step 1**: entity \(i\) picks \(f_i \leftarrow \mathcal{L}_f\) such that there exists the inverse of \(f_i\) in \(R_q\), and picks \(g_i \leftarrow \mathcal{L}_g\). Then, he computes \(h_i \equiv f_i^{-1} \cdot g_i \pmod{q}\) and sends \(h_i\) to \(j\).

- **Step 2**: upon receipt of \(h_i\), entity \(j\) picks \(f_j \leftarrow \mathcal{L}_f\) such that there exists the inverse of \(f_j\) in \(R_q\), and picks \(g_j \leftarrow \mathcal{L}_g\), \(r_j \leftarrow \mathcal{L}_r\). Then, he computes \(h_j \equiv f_j^{-1} \cdot g_j \pmod{q}\) and \(e_j \equiv pr_j \cdot h_i + f_j \pmod{q}\). Afterwards, he sends \(h_j\) and \(e_j\) to \(i\).

- **Step 3**: upon receipt of \(h_j\) and \(e_j\), entity \(i\) picks \(r_i \leftarrow \mathcal{L}_r\) and computes \(e_i \equiv pr_i \cdot h_j + f_i \pmod{q}\). Then, he sends \(e_i\) to \(j\) and computes \(a_i = f_i \cdot e_j \pmod{q}\), \(K_i = a_i \pmod{p} = f_i \cdot f_j \pmod{p}\).

- **Step 4**: upon receipt of \(e_i\), entity \(j\) computes \(a_j = f_j \cdot e_i \pmod{q}\), \(K_j = a_j \pmod{p} = f_i \cdot f_j \pmod{p}\).

It can be easily checked that the resultant common session key is

\[
K_i = K_j = f_i \cdot f_j \pmod{p}.
\] (2)
The NTRU-KE protocol consists of three flows, which is one more flow than these DH-type protocols. Compared with NTRU-ENCRYPT, it does not need to compute the inverse of \( f \) modulo \( p \) in NTRU-KE. Like DH-type protocols, the NTRU-KE protocol features elegant and simple.

### 3.1 Security

When a new public key protocol is proposed, in very few instances is there a rigorous proof that breaking this public key protocol is equivalent to solving an existing well studied hard problem. What frequently occurs is a definition of a new hard problem, which may relate to a previously well-defined hard problem. In the same manner, the proposed NTRU-KE follows this routine. The security of NTRU-KE depends on the following NTRU-KE assumption.

**Definition 3.** i) The **NTRU-KE problem**: For a given security parameter \( k \), which specifies \((N, p, q)\) and the spaces \( \mathcal{L}_f, \mathcal{L}_g, \mathcal{L}_m, \) and \( \mathcal{L}_r \), as well as \( h_i, h_j, e_i, \) and \( e_j \), where these variables are clearly defined in Fig. 1, find \( f_i \ast f_j \pmod{p} \).

ii) The **NTRU-KE assumption**: for appropriate parameters, it is computational intractable to solve the NTRU-KE problem.

Despite it is hard to prove this assumption, the relationship between the NTRU-ENCRYPT assumption and the NTRU-KE assumption can be investigated. Suppose that there exists an efficient algorithm \( A \) to solve the NTRU-ENCRYPT inversion problem. Then, \( A \) can be reduced to an efficient algorithm to find \( f_j \) from \( e_j \) and find \( f_i \) from \( e_i \). Finally, \( A \) can derive the session key by computing \( K = f_i \ast f_j \pmod{p} \). This implies NTRU-KE assumption > NTRU-ENCRYPT assumption (in analogy to CDH assumption > DL assumption).

### 3.2 Lattice Attacks

The security of NTRU-ENCRYPT is related to CVP and SVP in a convolution modular lattice (CML) \( \mathcal{L}_h \) (also called as a NTRU lattice). \( \mathcal{L}_h \) is the \( 2N \) dimensional lattice with basis given by the rows of the matrix,

\[
\mathcal{L}_h = \begin{bmatrix} I & H \\ 0 & I \end{bmatrix},
\]

where \( I \) is an \( N \times N \) identity matrix, \( 0 \) is an \( N \times N \) zero matrix, and \( H \) is a circulant matrix generated from \( h \),

\[
H_{i,j} = h_{j-i \pmod{N}}.
\]

Another way to describe CML \( \mathcal{L}_h \) (3) is the set of vectors

\[
\mathcal{L}_h = \{ [f, g] \in \mathbb{Z}^{2N} \mid f \ast h \equiv g \pmod{q} \}.
\]

For appropriate parameters and a padding scheme, the best known attack on NTRU-ENCRYPT appears to be lattice attacks. Likewise, based on current knowledge, these lattice attacks are also the best known attack on NTRU-KE. Two lattice attacks on NTRU-KE are discussed below.

1) **Lattice attack on \( h_i \) (or \( h_j \)).** Suppose that a passive adversary \( A \) tries to recover secret information \([f_i, g_i]\) from \( h_i \). We here drop the subscript \( i \), because it is the same case for entity \( j \). By \( h \equiv f^{-1} \ast g \pmod{q} \), there exists \( u \in \mathbb{Z} \) with

\[
u = -f \ast h + g \pmod{q},
\]

(5)
such that
\[ [f, u] \cdot L_h = [f, g]. \]
Therefore, \( L_h \) contains the vector \([f, g]\). By choosing appropriate parameters, then the vector \([f, g]\) is quite short, so it can be found by solving SVP (or approximate SVP) in \( L_h \). Indeed, this lattice attack corresponds to the lattice attack on a NTRU-ENCRYPT private key.

2) **Lattice attack on** \( e_i \) (or \( e_j \)). In the light of

\[ e_i \equiv pr_i \ast h_j + f_i \mod q, \]  

we can rewrite this relation in the vector form as

\[ [0, e_i] \equiv [r_i, r_i \ast (ph_j) \mod q] + [-r_i, f_i]. \]

Recall the description of CML in (4), we can deduce from (7) that \([r_i, r_i \ast (ph_j) \mod q]\) is in the lattice \( L_{ph_j} \). The point \([0, e_i]\) is only \([-r_i, f_i]\) away from a lattice point of \( L_{ph_j} \). By choosing appropriate parameters, then the vector \([-r_i, f_i]\) is quite short, so \([-r_i, f_i]\) can be recovered if we can find a vector in \( L_{ph_j} \) that is closest to the vector \([0, e_i]\). Indeed, this lattice attack corresponds to the lattice attack on a NTRU-ENCRYPT message.

It can be easily found that either of the above two lattice attacks, if succeed, will lead to a recovery of the session key. They indicate how NTRU-KE relate to SVP and CVP. The ideal lattice basis reduction algorithms can be used to solve SVP and CVP in CML, and therefore, they can be applied to carry out the above two lattice attacks. However, the lattice attacks do not necessarily imply that NTRU-KE is insecure, as currently known lattice reduction algorithms (such as LLL [27] algorithm or its improved variants, e.g., BKZ-LLL [28]) still remain exponential time algorithms in terms of security parameters.

### 4 Key-mismatch Failure Analysis

It is well-known that NTRU-ENCRYPT algorithm has a probability of description failure. A similar failure is also found in NTRU-KE, hereafter referred to as key-mismatch failure. We now proceed to estimate the probability of key-mismatch failure in NTRU-KE. We first set

\[
\begin{align*}
    P_i &= \text{Prob}[(f_i \ast e_j \mod q) \neq f_i \ast e_j], \\
    P_j &= \text{Prob}[(f_j \ast e_i \mod q) \neq f_j \ast e_i].
\end{align*}
\]

Because of symmetry, \( P_i = P_j \) holds immediately. Some parameter choices for NTRU-KE may cause occasional key-mismatch failure. The probability of key-mismatch failure \( P_{KM} \) is formally defined as

\[ P_{KM} = \text{Prob}[K_i \neq K_j]. \]

For appropriate parameters, \( P_i \) and \( P_j \) can be very small quantities, this leads to

\[ P_{KM} \approx 2P_i = 2P_j. \]

The probability \( P_i \) can be conservatively bounded by the probability that one or more coefficients of

\[ a_i = f_i \ast e_j = pr_j \ast g_i + f_i \ast f_j \]

have an absolute value greater than \( q/2 \). We set

\[ P(c) = \text{Prob}\left[\begin{array}{c}
    \text{a given coefficient of } pr_j \ast g_i + \\
    f_i \ast f_j \text{ has absolute value } \geq c.
\end{array}\right] \]

(9)
Let $X_j$ denote a coefficient of $pr_j * g_i + f_i * f_j$, then $X_j$ is a sum of $N$ terms,

$$X_j = \sum_{i=1}^{N} (pg_i + z_i),$$

with each $y_i = r_k g_l$ and $z_i = f_s f_t$ for some $k, l, s, t$. Because of (1) and (6), the mathematical expectation of $X_j$ can be obtained. For sufficiently large $N$, it holds that $E(X_j) = 0$. Then, the variance of $X_j$ is

$$\sigma^2 = D(X_j) = E(X_j^2) - E^2(X_j)$$

$$= E(X_j^2) = \sum_{i=1}^{N} (p^2 E(y_i^2) + E(z_i^2))$$

$$= \frac{4}{N} (p^2 r d g + d_j^2). \quad (10)$$

Combining (6) and (10) yields

$$\sigma = \sqrt{\frac{4}{N} (p^2 d^2 r + d_j^2)}. \quad (11)$$

Suppose that $N$ is sufficiently large and $X_j$ are independent variables. Note that $X_j$ are identically distributed, Central Limit Theorem is applied to $X_j$, one has

$$\text{Prob}[X_j \geq c] = \text{Prob}[X_j \leq -c] < \frac{1}{\sqrt{2\pi}} \int_{c/\sigma}^{\infty} e^{-x^2/2} dx. \quad (12)$$

We can rewrite (12) as a single inequality,

$$\text{Prob}[|X_j| \geq c] < \frac{2}{\sqrt{2\pi}} \int_{c/\sigma}^{\infty} e^{-x^2/2} dx. \quad (13)$$

Translating this into the notation of the complementary error function, we arrive at

$$P(c) = \text{Prob}[|X_j| \geq c] < \text{erfc}(c/(\sqrt{2}\sigma)). \quad (14)$$

Recall that one or more $X_j$ have an absolute value greater than $c = q/2$ will lead to $f_i \ast e_j(\mod q) \neq f_i \ast e_j$, it holds that

$$P_l = \sum_{i=1}^{N} \binom{N}{i} P^i(c)(1 - P(c))^{N-i}, \quad c = q/2. \quad (15)$$

We can always ensure that $P(c)$ is a very small quantity via choosing appropriate parameters, therefore ignoring the high order small quantities in (15) yields

$$P_l \approx N \cdot P(c). \quad (16)$$

Combining (8), (14), (16), and ignoring approximation errors leads to

$$P_{KM} < U, \quad (17)$$

where

$$U = 2N \cdot \text{erfc}(c/(\sqrt{2}\sigma)), \quad (18)$$
with \( c = q/2 \), and \( \sigma \) is given by (11). It is indicated from (17) that the probability of key-
mismatch failure in NTRU-KE is approximately bounded above by \( U \).

Since several assumptions and approximations are made in deriving (17), a simulation is
accompanied to validate the correctness of (17). In this simulation, we let

\[
p = 3, \quad d_r = \lfloor N/3 \rfloor, \quad d_f = \lfloor N/4 \rfloor,
\]

with \( p \) and \( N \) being x- and y-coordinates in Fig. 2. Simulation results in Fig. 2 show that the
bound in (18) is a little conservative estimation of reality. Accordingly, (18) can be applied to
have an estimation of key-mismatch failure in NTRU-KE.

5 Comparison

It is currently impossible to have an experimental comparison of performance for NTRU-KE
and ECDH (the state-of-the-art benchmark). This is because how to choose secure parameters
of NTRU-KE remains unsolved and this is a nontrivial task. According to the performance
report of NTRU-ENCRYPT \([20, 29]\), NTRU-KE shares an identical performance with that of
NTRU-ENCRYPT since they have an identical underlying basic operation, i.e., convolution
multiplication in \( R_q \). Several merits and drawbacks are summarized as follows.

**Merits:**

1. *Computation speed.* NTRU-KE is faster than ECDH.
2. *Space cost of domain parameters.* NTRU-KE is smaller than ECDH.
3. *Domain parameters selection.* NTRU-KE is easier than ECDH. This is because the selection
   of secure elliptic curves is always cumbersome \([30]\).
4. *Hardware implementation.* The convolution multiplication in the NTRU cryptosystem offers
   better performance in hardware implementation than the basic operation in ECDH \([31]\).
5. *Resistance to parallelized and distributed attack.* NTRU-KE is better resistance to paral-
   lelized and distributed attack. This is because lattice reduction algorithms (e.g., LLL) are
   highly sequential.
6. *Resistance to quantum attack.* NTRU-KE is considered to be quantum resistant, whereas
   ECDH is not. This is because lattice-based cryptography is currently quantum resistant \([32]\).
Drawbacks:

1. **Communication Overhead.** NTRU-KE is heavier than ECDH. Moreover, NTRU-KE needs one more communication flow.
2. **Space cost for private key.** NTRU-KE is larger than ECDH.
3. **Key-mismatch failure.** NTRU-KE has a probability of key-mismatch failure. This probability can be restricted to be negligible via choosing appropriate parameters.

According to the above summary, we come to the conclusion that NTRU-KE is comparable with ECDH based on current knowledge. The decisive advantage of NTRU-KE, however, will occur when a quantum computer is available.

6 Conclusions

In this paper, we have helped the NTRU cryptosystem to find its long missing member NTRU-KE, which has been studied in aspects of security and key-mismatch failure. It worth noting that the proposed NTRU-KE is only an anonymous key exchange primitive, and therefore, it does not provide authentication of the entities, making it vulnerable to man-in-the-middle attacks. A wide variety of cryptographic authentication schemes and protocols have been developed to provide authenticated key agreement to prevent man-in-the-middle and related attacks. These methods generally mathematically bind the agreed key to other agreed-upon data, such as public/private key pairs, shared secret keys, and passwords. We will not elaborate how to enable NTRU-KE to provide authentication. One meaningful future work is to investigate the parameter choices of NTRU-KE.

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References