

Protocols for Multiparty Coin Toss With Dishonest Majority*

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Abstract

Coin-tossing protocols are protocols that generate a random bit with uniform distribution. These protocols are used as a building block in many cryptographic protocols. Cleve [STOC 1986] has shown that if at least half of the parties can be malicious, then, in any r -round coin-tossing protocol, the malicious parties can cause a bias of $\Omega(1/r)$ to the bit that the honest parties output. However, for more than two decades the best known protocols had bias $\frac{t}{\sqrt{r}}$, where t is the number of corrupted parties. Recently, in a surprising result, Moran, Naor, and Segev [TCC 2009] have shown that there is an r -round two-party coin-tossing protocol with the optimal bias of $O(1/r)$. We extend Moran et al. results to the *multiparty model* when less than $2/3$ of the parties are malicious. The bias of our protocol is proportional to $1/r$ and depends on the gap between the number of malicious parties and the number of honest parties in the protocol. Specifically, for a constant number of parties or when the number of malicious parties is somewhat larger than half, we present an r -round m -party coin-tossing protocol with optimal bias of $O(1/r)$.

Keywords. Fair Coin Tossing, Multiparty Computation, Dishonest Majority, Secure with Abort; Cheat Detection.

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Contents

1	Introduction	2
1.1	Our Results	3
1.2	The Idea of Our Protocol	4
1.3	Follow-up Works	4
2	Preliminaries	5
2.1	The Real vs. Ideal Paradigm	5
2.1.1	$1/p$ -Indistinguishability and $1/p$ -Secure Computation	6
2.2	Security with Abort and Cheat Detection	7
2.3	Cryptographic Tools	8
2.4	The Two-Party Protocol of Moran et al.	9
3	Coin Tossing with Dishonest Majority – A Warm-Up	10
3.1	Multiparty Coin Tossing When Half of the Parties Can Be Malicious	10
3.2	A 5-Party Protocol that Tolerates up to 3 Malicious Parties	11
3.2.1	Eliminating the Dealer for the 5-Party Protocol	12
4	Our Main Construction	16
5	A Protocol with an On-Line Dealer	16
5.1	The Simulator for the On-Line Dealer Protocol	19
5.2	Proof of the Correctness of the Simulation	20
6	Omitting the On-Line Dealer	27
6.1	The Protocol Without the Dealer	27
6.1.1	Proving the Feasibility of the Reconstruction	28
6.2	The Simulator for the Protocol Without a Dealer	31
6.3	Proof of the Correctness of the Simulation	34
6.3.1	The Initialization Step	35
6.3.2	The Interaction Rounds of the Protocol	36
6.3.3	The Termination Step	37
6.3.4	Concluding the Proof of Theorem 6	39
7	Coin-Tossing Protocol for any Constant Fraction of Corrupted Parties	39
	References	43
A	On Achieving Cheat Detection	43

1 Introduction

Secure multiparty computation in the malicious model allows distrustful parties to compute a function securely. Designing such secure protocols is a delicate task with a lot of subtleties. An interesting and basic functionality for secure computation is coin tossing – generating a random bit with uniform distribution. This is a simple task where the parties have no inputs. However, already this task raises questions of fairness and how malicious parties can bias the output. Understanding what can be achieved for coin tossing in various settings can be considered as a step towards understanding general secure and fair multiparty computation. Indeed, some of the early works on secure computation were on coin tossing, e.g., [5, 6, 9]. Furthermore, coin tossing is used as a basic tool in constructing many protocols that are secure in the malicious model. Secure protocols for coin tossing are a digital analogue of physical coin tossing, which have been used throughout history to resolve disputes.

The main problem in designing coin-tossing protocols is the prevention of a bias of the output. The bias of a coin-tossing protocol measures the maximum influence of the adversary controlling a subset of malicious parties on the output of the honest parties, where the bias is 0 if the output is always uniformly distributed and the bias is $1/2$ if the adversary can force the output to be always (say) 1. To demonstrate the problems of designing a coin-tossing protocol, we describe Blum’s two-party coin-tossing protocol [5].

Example 1.1 (Blum’s coin-tossing protocol [5]). *To toss a coin, Alice and Bob execute the following protocol.*

- Alice chooses a random bit a and sends a commitment $c = \text{commit}(a)$ to Bob.
- Bob chooses a random bit b and sends it to Alice.
- Alice sends the bit a to Bob together with $\text{de-commit}(c)$.
- If Bob does not abort during the protocol, Alice outputs $a \oplus b$, otherwise she outputs a random bit.
- If Alice does not abort during the protocol and c is a commitment to a , then Bob outputs $a \oplus b$, otherwise he outputs a random bit.

If Alice is malicious, then she can bias the output toward (say) 1. If $a \oplus b = 1$, she opens the commitment and Bob outputs 1. However, if $a \oplus b = 0$, Alice aborts, and Bob outputs 1 with probability $1/2$. Altogether, the probability that the honest Bob outputs 1 is $3/4$. It can be proved that this is the best that Alice can do in this protocol, and, hence, the bias of the protocol is $1/4$. This protocol demonstrates the problems caused by parties aborting the protocol and the need to define how the output of the other parties is computed after such aborts.

While the above protocol is a significant improvement over naive protocols whose bias is $1/2$, the protocol still has a constant bias. If *more than half* of the parties are honest, then, using general secure multiparty protocols, there are constant-round protocols with negligible bias (assuming a broadcast channel), e.g., the protocol of [20]. Cleve [9] proved that when at least half of the parties can be malicious, the bias of every protocol with r rounds is $\Omega(1/r)$. In particular, this proves that without honest majority no protocol with polynomial number of rounds (in the security parameter) can have negligible bias. On the positive side, it was shown in [2, 9] that there is a two-party protocol with bias $O(1/\sqrt{r})$. This can be generalized to an m -party protocol that tolerates any number of malicious parties and has bias $O(t/\sqrt{r})$, where t is a bound on the number of malicious parties. Cleve and Impagliazzo [10] have shown that, in a model where commitments are available only as black-box (and no other assumptions are made), the bias of every coin-tossing protocol is $\Omega(1/\sqrt{r})$.¹ The protocols of [5, 2, 9] are in this model.

¹The lowerbound of [10] holds in a stronger model which we will not discuss in this paper.

The question if there is a coin-tossing protocol without an honest majority that has bias $O(1/r)$ was open for many years. Recently, in a breakthrough result, Moran, Naor, and Segev [18] have shown that there is an r -round two-party coin-tossing protocol with bias $O(1/r)$. Moran et al. ask the following question:

“An interesting problem is to identify the optimal trade-off between the number of parties, the round complexity, and the bias. Unfortunately, it seems that several natural variations of our approach fail to extend to the case of more than two parties. Informally, the main reason is that a coalition of malicious parties can guess the threshold round with a pretty good probability by simulating the protocol among themselves for any possible subset.”

1.1 Our Results

Our main contribution is a multi-party coin-tossing protocol that has small bias when less than $2/3$ of the parties are malicious.

Theorem 1 (Informal). *Let m, t , and r be integers such that $m/2 \leq t < 2m/3$. There exists an r -round m -party coin-tossing protocol tolerating t malicious parties that has bias $O(2^{2^{k+1}}/r')$, where $k = 2t - m$ and $r' = r - O(k + 1)$.*

The above protocol nearly has the desired dependency on r , i.e., the dependency implied by the lower bound of Cleve [9]. However, its dependency on k (where k is the difference between the number of malicious parties and the number of honest parties) has, in general, a prohibitive cost. Nevertheless, there are interesting cases where the bias is $O(1/r)$.

Corollary 2 (Informal). *Let m and t be constants such that $m/2 \leq t < 2m/3$ and r be an integer. There exists an r -round m -party coin-tossing protocol tolerating t malicious parties that has bias $O(1/r)$.*

For example, we construct an r -round 5-party coin-tossing protocol tolerating 3 malicious parties that has bias $8/(r - O(1))$ (this follows from our general construction in Sections 4–6).

Notice that the protocol of Theorem 4 depends on k and not on the number of malicious parties t . Thus, it is efficient when k is small.

Corollary 3 (Informal). *Let m, r be integers and $t = m/2 + O(1)$. There exists an r -round m -party coin-tossing protocol tolerating t malicious parties that has bias $O(1/r)$.*

For example, for any even m we construct an r -round m -party coin-tossing protocol tolerating $m/2$ malicious parties that has bias $1/(2r - O(1))$. Furthermore, even when $t = 0.5m + 0.5 \log \log m - 1$, the bias of our protocol is small, namely, $O(m/(r - O(\log \log m)))$.

We also reduce the bias compared to previous protocols when more than $2/3$ of the parties are malicious. The bias of the m -party protocol of [2, 9] is $O(t/\sqrt{r})$. We present in Section 7 a protocol whose bias is $O(1/\sqrt{r})$ when t/m is constant, that is, when the fraction of malicious parties is constant we get rid of the factor t in the bias.

Communication Model. We consider a communication model where all parties can only communicate through an authenticated broadcast channel. On one hand, if a party broadcasts a message, then all other parties see *the same* message. This ensures some consistency between the information the parties have. On the other hand, there are no private channels and all parties see all messages. We assume a synchronous model; however, the adversary is rushing.² That is, in this model, in each round each party broadcasts a

²If there is synchronous broadcast without a rushing adversary then coin tossing is trivial.

message. However, the adversary can wait until it sees the messages sent by the honest parties before it broadcasts the messages of the malicious parties (that is, these messages can depend on the honest parties' messages).

We note that our results can be translated to a model with authenticated point-to-point channels with a PKI infrastructure (in an m -party protocol, the translation will increase the number of rounds by a multiplicative factor of $O(m)$). Thus, our results hold in the two most common models for secure multiparty computation.

1.2 The Idea of Our Protocol

We generalize the two-party protocol of Moran et al. [18] to the multi-party setting. In the protocol of [18], in each round Alice and Bob get bits that are their output if the other party aborts: If a party aborts in round i , then the other party outputs the bit it got in round $i - 1$. Furthermore, there is a special round i^* ; prior to round i^* the bits given to Alice and Bob are random independent bits and from round i^* onward the bits given to Alice and Bob are the same fixed bit. The adversary can bias the output only if it guesses i^* . In our protocol, in each round there are many bits. We define a collection of subsets of the parties and each subset gets a bit. The bits are chosen similarly to [18]: prior to i^* they are independent; from i^* onward they are fixed. In our case we cannot give the bits themselves to the parties. We rather use a few layers of secret-sharing schemes to store these bits. For every subset in the collection, we use a first secret-sharing scheme to share the bit of the subset among the parties of the subset. We use an additional secret-sharing scheme to share the shares of the first secret-sharing scheme. The threshold in the latter secret sharing scheme is chosen such that the protocol can proceed until enough parties aborted. In the round when the number of aborted parties ensures that there is an honest majority, an appropriate subset in the collection is chosen, its bit is reconstructed, and this bit is the output of the honest parties. The description of how to implement these ideas appears in Sections 4–6.

The construction of Moran et al. [18] is presented in two phases. In the first phase they present a protocol with a trusted dealer, for which an adversary can only inflict bias $O(1/r)$. Then, they show how to implement this protocol in the real-world, using a constant round secure-with-abort multiparty protocol, as well as secret-sharing and authentication schemes. This can be seen as a general transformation from any two-party coin-tossing protocol with a trusted dealer, into a real world two-party coin-tossing protocol. We observe that the transformation of Moran et al. to a real-world protocol requires some further care for the multiparty case generalization. We show how this can be achieved by adopting the definition of secure multiparty computation of [1], which requires the protocol to detect a cheating party, that is, at the end of the protocol either the honest parties hold a correct output or all honest parties agree on a party (or parties) that cheated during the protocol.

1.3 Follow-up Works

1/p-secure protocols. In a follow-up work, Beimel et al. [4] study multiparty $1/p$ -secure protocols for general functionalities. Informally, a protocol is $1/p$ -secure if every adversary cannot harm the protocol with probability greater than $1/p$ (e.g., it cannot bias the output with probability greater than $1/p$). The formal definition of $1/p$ -secure computation appears in Definition 2.3. The main result in [4] is constructions of $1/p$ -secure protocols that are resilient against *any* number of corrupt parties provided that the number of parties is constant and the size of the range of the functionality is at most polynomial in the security parameter n . In addition, Beimel et al. provide some impossibility result that states that the feasibility of the results for $1/p$ -secure computation are essentially tight. The protocols in [4] combine ideas from the

protocols presented in this paper and these presented in [16]. In addition, the bias of the protocol of [4] is $O(1/\sqrt{r})$.

Multiparty protocols for coin-tossing in the preprocessing model. Ishai et al. [15] constructed a new type of secret-sharing scheme in which the reconstruction algorithm informs each honest player of the correct set of cheaters, i.e., each honest player gets a list of all the parties that input a wrong share in the reconstruction phase. This scheme can be used to convert our multiparty coin-tossing protocols into *unconditional* multiparty coin-tossing protocols in the *preprocessing model*, namely, assuming that there exist an off-line dealer which computes some values in a preprocessing phase, gives them to the parties, and halts. Such a transformation eliminates the cryptographic assumptions and the need of a broadcast channel, while the bias of the resulted protocols is not effected.

Organization. In Section 2, we provide security definitions in multi-party protocols, a description of the cryptographic tools used in this paper, and a brief overview of the result by Moran et al. [18]. In Section 3, we present two warm-up constructions for multiparty coin-tossing with bias $O(1/r)$ where r is the number of rounds in the protocol. In Sections 4–6 we present our main result – a coin-tossing protocol that has nearly optimal bias and can tolerate up to $2/3$ fraction of malicious parties. In Section 7 we describe an r -round m -party coin-tossing protocol that tolerates up to t dishonest parties, where t is some constant fraction of m .

2 Preliminaries

A multi-party coin-tossing protocol with m parties is defined using m probabilistic polynomial-time Turing machines p_1, \dots, p_m having the security parameter 1^n as their only input. The coin-tossing computation proceeds in rounds, in each round, the parties broadcast and receive messages on a broadcast channel. The number of rounds in the protocol is typically expressed as some polynomially-bounded function r in the security parameter. At the end of protocol, the (honest) parties should hold a common bit w . We denote by $\text{CoinToss}()$ the ideal functionality that gives the honest parties the same uniformly distributed bit w , that is, $\Pr[w = 0] = \Pr[w = 1] = 1/2$. In our protocol, the output bit w might have some bias.

In this work we consider a malicious static computationally-bounded adversary, i.e., a non-uniform that runs in a polynomial-time. The adversary is allowed to corrupt some subset of the parties. That is, before the beginning of the protocol, the adversary corrupts a subset of the parties that may deviate arbitrarily from the protocol, and thereafter the adversary sees the messages sent to the corrupt parties and controls the messages sent by the corrupted parties. The honest parties follow the instructions of the protocol.

The parties communicate in a synchronous network, using only a broadcast channel. The adversary is rushing, that is, in each round the adversary hears the messages sent by the honest parties before broadcasting the messages of the corrupted parties for this round (thus, the messages broadcast by corrupted parties can depend on the messages of the honest parties broadcast in this round).

2.1 The Real vs. Ideal Paradigm

The security of multiparty computation protocols is defined using the real vs. ideal paradigm. In this paradigm, we consider the real-world model, in which protocols are executed. We then formulate an ideal model for executing the task at hand. This ideal model involves a trusted party whose functionality captures the security requirements of the task. Finally, we show that the real-world protocol “emulates” the ideal-world protocol: For any real-life adversary \mathcal{A} there should exist an ideal-model adversary \mathcal{S} (also called

simulator) such that the global output of an execution of the protocol with \mathcal{A} in the real-world model is distributed similarly to the global output of running \mathcal{S} in the ideal model. In the coin-tossing protocol, the parties do not have inputs. Thus, to simplify the definitions, we define secure computation without inputs (except for the security parameters).

The Real Model. Let Π be an m -party protocol computing \mathcal{F} . Let \mathcal{A} be a non-uniform probabilistic polynomial time adversary with auxiliary input aux . Let $\text{REAL}_{\Pi, \mathcal{A}(\text{aux})}(1^n)$ be the random variable consisting of the view of the adversary (i.e., its random input and the messages it got) and the output of the honest parties, following an execution of Π , where each party p_j begins by holding the input 1^n .

The Ideal Model. The basic ideal model we consider is a model without abort. Specifically, there are parties $\{p_1, \dots, p_m\}$, and an adversary \mathcal{S} who has corrupted a subset I of them. An ideal execution for the computing \mathcal{F} proceeds as follows:

Inputs: Party p_j holds a security parameter 1^n . The adversary \mathcal{S} has some auxiliary input aux .

Trusted party sends outputs: The trusted party computes $\mathcal{F}(1^n)$ with uniformly random coins and sends the appropriate outputs to the parties.

Outputs: The honest parties output whatever they received from the trusted party, the corrupted parties output nothing, and \mathcal{S} outputs an arbitrary probabilistic polynomial-time computable function of its view (the outputs of the corrupted parties).

Let $\text{IDEAL}_{\mathcal{F}, \mathcal{S}(\text{aux})}(1^n)$ be the random variable consisting of the output of the adversary \mathcal{S} in this ideal world execution and the output of the honest parties in the execution.

In this work we consider a few formulations of the ideal-world, and consider composition of a few protocols, all being executed in the same real-world, however, each secure with respect to a different ideal-world. We prove the security of the resulting protocol, using the hybrid model techniques of Canetti [7].

2.1.1 $1/p$ -Indistinguishability and $1/p$ -Secure Computation

As explained in the introduction, the ideal functionality $\text{CoinToss}()$ cannot be implemented when there is no honest majority. We use $1/p$ -secure computation, defined by [14, 16], to capture the divergence from the ideal world. This notion applies to general secure computation. We start with some notation.

A function $\mu(\cdot)$ is *negligible* if for every positive polynomial $q(\cdot)$ and all sufficiently large n it holds that $\mu(n) < 1/q(n)$. A *distribution ensemble* $X = \{X_{a,n}\}_{a \in \{0,1\}^*, n \in \mathbb{N}}$ is an infinite sequence of random variables indexed by $a \in \{0, 1\}^*$ and $n \in \mathbb{N}$.

Definition 2.1 (Statistical Distance and $1/p$ -indistinguishability). *We define the statistical distance between two random variables A and B as the function*

$$\text{SD}(A, B) = \frac{1}{2} \sum_{\alpha} \left| \Pr[A = \alpha] - \Pr[B = \alpha] \right|.$$

For a function $p(n)$, two distribution ensembles $X = \{X_{a,n}\}_{a \in \{0,1\}^, n \in \mathbb{N}}$ and $Y = \{Y_{a,n}\}_{a \in \{0,1\}^*, n \in \mathbb{N}}$ are computationally $1/p$ -indistinguishable, denoted $X \stackrel{1/p}{\approx} Y$, if for every non-uniform polynomial-time algorithm D there exists a negligible function $\mu(\cdot)$ such that for every n and every $a \in \{0, 1\}^*$,*

$$\left| \Pr[D(X_{a,n}) = 1] - \Pr[D(Y_{a,n}) = 1] \right| \leq \frac{1}{p(n)} + \mu(n).$$

Two distribution ensembles are *computationally indistinguishable*, denoted $X \stackrel{c}{\equiv} Y$, if for every $c \in \mathbb{N}$ they are computationally $\frac{1}{n^c}$ -indistinguishable.

We next define the notion of $1/p$ -secure computation [14, 16, 4]. The definition uses the standard real/ideal paradigm [11, 7], except that we consider a completely fair ideal model (as typically considered in the setting of honest majority), and require only $1/p$ -indistinguishability rather than indistinguishability.

Definition 2.2 (perfect $1/p$ -secure computation). *An m -party protocol Π is said to perfectly $1/p$ -securely compute a functionality \mathcal{F} if for every non-uniform adversary \mathcal{A} in the real model, there exists a polynomial-time adversary \mathcal{S} in the ideal model such that for every $n \in \mathbb{N}$ and for every $\text{aux} \in \{0, 1\}^*$*

$$\text{SD}(\text{IDEAL}_{\mathcal{F}, \mathcal{S}(\text{aux})}(1^n), \text{REAL}_{\Pi, \mathcal{A}(\text{aux})}(1^n)) \leq \frac{1}{p(n)}.$$

Definition 2.3 ($1/p$ -secure computation [14, 16, 4]). *Let $p = p(n)$ be a function. An m -party protocol Π is said to $1/p$ -securely compute a functionality \mathcal{F} if for every non-uniform probabilistic polynomial-time adversary \mathcal{A} in the real model, there exists a non-uniform probabilistic polynomial-time adversary \mathcal{S} in the ideal model corrupting the same parties as \mathcal{A} such that the following two distribution ensembles are computationally $1/p(n)$ -indistinguishable*

$$\{\text{IDEAL}_{\mathcal{F}, \mathcal{S}(\text{aux})}(1^n)\}_{\text{aux} \in \{0, 1\}^*, n \in \mathbb{N}} \stackrel{1/p}{\approx} \{\text{REAL}_{\Pi, \mathcal{A}(\text{aux})}(1^n)\}_{\text{aux} \in \{0, 1\}^*, n \in \mathbb{N}}.$$

We next define the notion of *secure computation* and notion of *bias of a coin-tossing protocol* by using the previous definition.

Definition 2.4 (Secure computation). *An m -party protocol Π is said to securely compute a functionality \mathcal{F} , if for every $c \in \mathbb{N}$, the protocol Π is $(1/n^c)$ -securely compute the functionality \mathcal{F} .*

Definition 2.5. *We say that a protocol is a coin-tossing protocol with bias $1/p$ if it is a $1/p$ -secure protocol for the functionality $\text{CoinToss}()$.*

2.2 Security with Abort and Cheat Detection

We present here a definition of secure multiparty computation that is more stringent than standard definitions of secure computation with abort. This definition extends the definition for secure computation as given by Aumann and Lindell [1]. Roughly speaking, our definition requires that one of two events is possible: If at least one party deviates from the prescribed protocol, then the adversary obtains the outputs of these parties (but nothing else), and all honest parties are notified by the protocol that these parties have aborted. Otherwise, the protocol terminates normally, and all parties receive their outputs. Again, we consider the restricted case where parties hold no private inputs. The formal definition uses the real vs. ideal paradigm as discussed in Section 2.1. We next describe the appropriate ideal model.

Execution in the ideal model. Let $D \subseteq [m]$ denote the indices of the corrupted parties, controlled by an adversary \mathcal{A} . An ideal execution proceeds as follows:

Inputs: Each party obtains a security parameter 1^n . The adversary \mathcal{A} receives an auxiliary input denoted aux .

Trusted party sends outputs to adversary: The trusted party computes $\mathcal{F}(1^n)$ with uniformly random coins and sends the appropriate outputs to the parties p_j such that $j \in D$.

Adversary instructs the trusted party to continue or halt: \mathcal{A} sends either a “continue” message or $\{\text{“abort}_j\text{”}\}_{j \in J}$ to the trusted party on behalf of a set of corrupt parties indexed by J . If it sends a “continue” message, the trusted party sends the appropriate output to the parties p_j for $j \notin D$ (i.e., to all honest parties). Otherwise, if it sends $\{\text{“abort}_j\text{”}\}_{j \in J}$ for some set of indices of corrupt parties $J \subseteq D$, then the trusted party sends the set J to all parties p_j for $j \notin D$.

Outputs: An honest party always outputs the message it obtained from the trusted party. The corrupted parties output nothing. The adversary \mathcal{A} outputs an probabilistic polynomial-time computable function of the auxiliary input aux , and the outputs obtained from the trusted party.

We let $\text{IDEAL}_{\mathcal{F}, \mathcal{S}(\text{aux})}^{\text{CD}}(1^n)$ and $\text{REAL}_{\Pi, \mathcal{A}(\text{aux})}(1^n)$ be defined as in Section 2.1 (where in this case $\text{IDEAL}_{\mathcal{F}, \mathcal{S}(\text{aux})}^{\text{CD}}(1^n)$ refers to the above execution with cheat detection of \mathcal{F}). This ideal model is different from the ideal world without cheat detection (see, e.g., [11]); without cheat detection, the trusted party sends a \perp symbol when the trusted party gets an abort message. With cheat detection, the honest parties *know* the identities of the corrupted parties that cause the abort. This cheat detection is achieved by most multiparty protocols, including that of [12], but not all (e.g., the protocol of [13] does not meet this requirement). Using this notation we define secure computation with abort and cheat detection.

Definition 2.6. *Let \mathcal{F} and Π be as above. A protocol Π is said to securely compute \mathcal{F} with abort and cheat detection if for every non-uniform polynomial-time adversary \mathcal{A} for the real model, there exists a non-uniform polynomial-time adversary \mathcal{S} for the ideal model corrupting the same parties as \mathcal{A} , such that*

$$\left\{ \text{IDEAL}_{\mathcal{F}, \mathcal{S}(\text{aux})}^{\text{CD}}(1^n) \right\}_{\text{aux} \in \{0,1\}^*, n \in \mathbb{N}} \stackrel{c}{\equiv} \left\{ \text{REAL}_{\Pi, \mathcal{A}(\text{aux})}(1^n) \right\}_{\text{aux} \in \{0,1\}^*, n \in \mathbb{N}}.$$

2.3 Cryptographic Tools

We next informally describe two cryptographic tools that we use in our protocols.

Signature Schemes. A signature on a message proves that the message was created by its presumed sender, and its content was not altered. A signature scheme is a triple $(\text{Gen}, \text{Sign}, \text{Ver})$ containing the key generation algorithm Gen , which gets as input a security parameter 1^n and outputs a pair of keys, the signing key K_S and the verification key K_V , the signing algorithm Sign , and the verifying algorithm Ver . We assume that it is infeasible to produce signatures without holding the signing key.

Secret-Sharing Schemes. An α -out-of- m secret-sharing scheme is a mechanism for sharing data among a set of parties such that every set of parties of size α can reconstruct the secret, while any smaller set knows nothing about the secret. In this paper, we use Shamir’s α -out-of- m secret-sharing scheme [21]. In this scheme, for every $\alpha - 1$ parties, the shares of these parties are uniformly distributed and independent of the secret. Furthermore, given at most such $\alpha - 1$ shares and a secret s , one can *efficiently* complete them to m shares of the secret s .

In order to simplify the presentation of our protocols, we construct the following way to share a secret in a threshold secret-sharing scheme *with respect to a certain party*.

Construction 2.7. *Let s be some secret taken from some finite field \mathbb{F} . We share s among m parties with respect to a special party p_j in an α -out-of- m secret-sharing scheme as follows:*

1. Choose shares $(s^{(1)}, s^{(2)})$ of the secret s in a two-out-of-two secret-sharing scheme, that is, select $s^{(1)} \in \mathbb{F}$ uniformly at random and compute $s^{(2)} = s - s^{(1)}$. Denote these shares by $\text{mask}_j(s)$ and $\text{comp}(s)$, respectively.
2. Generate shares $(\lambda^{(1)}, \dots, \lambda^{(j-1)}, \lambda^{(j+1)}, \dots, \lambda^{(m)})$ of the secret $\text{comp}(s)$ in an $(\alpha - 1)$ -out-of- $(m - 1)$ Shamir's secret-sharing scheme. For each $\ell \neq j$, denote $\text{comp}_\ell(s) = \lambda^{(\ell)}$.

Output:

- The share of party p_j is $\text{mask}_j(s)$. We call this share, p_j 's masking share.
- The share of each party p_ℓ , where $\ell \neq j$, is $\text{comp}_\ell(s)$. We call this share, p_ℓ 's complement share.

In the above, we shared the secret s among the parties in P in a secret-sharing scheme such that any set of size at least α that contains p_j can reconstruct the secret. In addition, similarly to the Shamir secret-sharing scheme, the following property holds: for any set of $\beta < \alpha$ parties (regardless if the set contains p_j), the shares of these parties are uniformly distributed and independent of the secret. Furthermore, given such $\beta < \alpha$ shares and a secret s , one can *efficiently* complete them to m shares of the secret s and *efficiently* select uniformly at random one vector of shares competing the β shares to m shares of the secret s .

2.4 The Two-Party Protocol of Moran et al.

Moran, Naor, and Segev [18] present a two-party coin-tossing protocol with optimal bias with respect to the round complexity (i.e., meeting the lowerbound of Cleve [9]). We next briefly review their protocol, which later serves as the basis for our construction. Following the presentation of [18], we first describe a construction that uses an on-line trusted party called the dealer. Later, we describe how the dealer can be eliminated.

The main underlying idea is that the dealer chooses a special round during which the parties actually unknowingly learn the output of the protocol. If the adversary guesses this round, it can bias the output by aborting. If the adversary aborts (or behaves maliciously) in any other time, then there is no bias. However, this special round is uniformly selected (out of r possible rounds) and then concealed such that the adversary is unable to guess it with probability exceeding $1/r$. Therefore, the bias of the protocol is $O(1/r)$.

More specifically, for two parties Alice and Bob to jointly toss a random coin, the protocol proceeds as follows. In a preprocessing phase, the dealer selects a special round number $i^* \in \{1, \dots, r\}$, uniformly at random, and selects bits $a_1, \dots, a_{i^*-1}, b_1, \dots, b_{i^*-1}$, independently, uniformly at random. It then uniformly selects a bit $w \in \{0, 1\}$ and sets $a_i = b_i = w$ for all $i^* \leq i \leq r$. Thereafter, the protocol proceeds in rounds: In round i , the dealer gives Alice the bit a_i and Bob the bit b_i . If none of the parties abort, then at the end of the protocol both output $a_r = b_r = w$. If a party prematurely aborts in some round i , the honest party outputs the bit it received in the previous round (i.e., a_{i-1} or b_{i-1} respectively). If one party aborts before the other party received its first bit (i.e., a_1 or b_1), then the other party outputs a random bit.

The security of the protocol follows from the fact that, unless the adversary aborts in round i^* , it cannot bias the output of the protocol. The view of any of the parties up to round $i < i^*$ is independent of the value of i^* , hence, any adversary corrupting a single party can guess i^* with probability at most $1/r$.

To eliminate the trusted party, Moran et al. first turn the trusted party from an on-line dealer into an off-line dealer, i.e., one that computes some values in a preprocessing phase, gives them to the parties, and halts. To achieve this, they use a 2-out-of-2 secret-sharing scheme and an authentication scheme. The trusted party selects i^* , bits $a_1, \dots, a_{i^*-1}, b_1, \dots, b_{i^*-1}$, and a bit $w \in \{0, 1\}$ as before. It then selects random shares for a_i and b_i for each $i \in \{1, \dots, r\}$. That is, it computes shares $a_i^{(A)} \oplus a_i^{(B)} = a_i$ and $b_i^{(A)} \oplus b_i^{(B)} = b_i$. At

the beginning of the protocol, the trusted party sends to Alice her shares of the a_i 's, that is, $a_i^{(A)}$, and the shares $b_i^{(A)}$ together with an authentication of the bit $b_i^{(A)}$ with a key k_A (i.e., authenticated shares of the b_i 's), and sends to Bob his shares of the b_i 's and authenticated shares of the a_i 's with a key k_B . The key k_A is given to Bob and the key k_B is given to Alice. The protocol now proceeds in rounds. In each round i Bob sends to Alice his authenticated share of a_i , so Alice can reconstruct the bit a_i . Alice then sends to Bob her authenticated share of b_i . An adversary cannot forge with noticeable probability an authentication, and is, thus, essentially limited to aborting in deviating from the prescribed protocol.

The off-line dealer is then replaced by a (constant round) secure-with-abort two-party protocol of [17] for computing the preprocessing functionality. That is, at the end of the initialization protocol, the parties get the authenticated shares of the a_i 's and the b_i 's, while the underlying i^* stays secret. The security of the 2-party preprocessing protocol guarantees that a polynomial-time adversary is essentially as powerful as in a computation with an off-line dealer.

3 Coin Tossing with Dishonest Majority – A Warm-Up

In this section we present two warm-up constructions of multiparty protocols coin-tossing with bias $O(1/r)$ where r is the number of rounds in the protocol. The first construction considers the case that at most half of the parties are malicious (however, there is no majority of honest parties). The second construction solves the problem of coin tossing with 5 parties, where at most 3 are malicious. These two protocols demonstrate the main difficulties in constructing multiparty coin-tossing protocols with dishonest majority, alongside the techniques we use to overcome these difficulties. In Sections 4–6, we present a construction for the general case that combines ideas from the two constructions presented in this section.

The main issue of any coin-tossing protocol is how to deal with premature aborts. The protocol must instruct any large enough subset of parties (i.e., at least as large as the set of honest parties) how to jointly reconstruct a bit if all other parties abort the protocol. Since there is no honest majority, an adversary controlling some set of parties can compute the output of this set assuming that the other parties abort. The problem in designing a protocol is how to ensure that this information does not enable the adversary to bias the output.

3.1 Multiparty Coin Tossing When Half of the Parties Can Be Malicious

In this section we present a protocol with optimal (up to a small constant) bias with respect to round complexity, when up to half the parties may be corrupt. We next give an informal description of the protocol with an on-line trusted party who acts as a dealer. In this protocol, the parties simulate the 2-party protocol of [18]. That is, we partition the parties into two sets A and B , one will simulate Alice and the other will simulate Bob. The main difficulty is that the adversary is not restricted to corrupting parties only in one of these sets. To overcome this problem, in our partition A contains a single party p_1 , and the set B contains the parties p_2, \dots, p_m . If the adversary corrupts p_1 , it gains full access to the view of Alice in the 2-party protocol; however, in this case a strict majority of the parties simulating Bob is honest, and the adversary will gain no information about the bits of Bob, i.e., the b_i 's.

We next describe the protocol. We assume for simplicity that m is even. In a preprocessing phase, the dealer uniformly selects $i^* \in \{1, \dots, r\}$ and then uniformly and independently selects a_1, \dots, a_{i^*-1} , b_1, \dots, b_{i^*-1} . Finally, it uniformly selects $w \in \{0, 1\}$ and sets $a_i = b_i = w$ for all $i^* \leq i \leq r$. In each round i , the dealer sends a_i to A , selects random shares of b_i in Shamir's $m/2$ -out-of- $(m-1)$ secret-sharing scheme, and sends each share to the appropriate party in B . We stress that formally (to model a rushing

adversary), the dealer first sends the malicious parties their messages, allows them to abort, and proceeds as described below.

During the execution, some parties might abort; we say that a party is active if it has not aborted. If a party p_j prematurely aborts, then the trusted party notifies all currently active parties that p_j has aborted. We next describe the actions when a party aborts:

- If p_1 aborts in round i , then the parties in B reconstruct b_{i-1} , output it, and halt. In this case, since p_1 is corrupt, at least $m/2$ honest parties exist in B and, thus, they will be able to reconstruct the output.
- If in round i less than $m/2$ active parties remain in B , then p_1 broadcasts a_{i-1} to the remaining $m/2 - 1$ parties in B and all (honest) parties output a_{i-1} and halt. In this case p_1 must be honest, and hence, can be trusted to broadcast a_{i-1} .
- While there are still at least $m/2$ active parties in B (i.e., at most $m/2 - 1$ of them abort) and p_1 is active, the protocol proceeds without a change.

To prevent cheating, the dealer needs to sign the messages given to the parties in B . We omit these details in this subsection.

Recall that at most $m/2$ out of the m parties are malicious. Thus, if p_1 is corrupted, then at most $(m/2) - 1$ parties in B are corrupted, and they cannot reconstruct b_i . The argument that the above protocol is $O(1/r)$ -secure is now straightforward. An adversary wishing to bias the protocol must cause premature termination. To do so, it must either corrupt p_1 (and gain no information on the b_i 's) or otherwise corrupt $m/2$ parties in B (hence, leaving p_1 uncorrupted). Thus, for any adversary in the multi-party protocol there is an adversary corrupting Alice or Bob in the two party protocol with dealer of [18] that is, essentially, as powerful. An important feature that we exploit in our protocol is the fact that in the two-party protocol Bob does not need its bit b_{i-1} if Alice does not abort. Thus, in our protocol the parties in B do not reconstruct b_{i-1} unless p_1 aborts in round i .

More work is required in order to eliminate the trusted dealer; the details are a special case of those described in Section 6.

3.2 A 5-Party Protocol that Tolerates up to 3 Malicious Parties

In this section we consider the case where $m = 5$ and $t = 3$, i.e., a 5-party protocol where up to 3 of the parties may be malicious. The protocol we present in this section inferior compared to the protocol in Sections 4–6. As in previous protocols, we first sketch our construction assuming there is a special on-line trusted dealer. This dealer interacts with the parties in rounds, sharing bits to subsets of parties, and proceeds with the normal execution as long as at least 4 of the parties are still active.

Denote the trusted dealer by T and the parties by p_1, \dots, p_5 . In this protocol, for each interaction round, the dealer produces a bit for each one of the 10 possible triplets of parties; this bit is recovered if in case of a premature termination of at least 2 parties occurs. Formally, for each triplet $J \subset \{1, \dots, 5\}$ (i.e., $|J| = 3$), denote $P_J = \{p_j : j \in J\}$ and by σ_J^i a bit to be recovered by 2 or 3 active parties in P_J if the protocol terminates in round $i + 1$. In a preprocessing phase, the dealer T selects uniformly at random $i^* \in \{1, \dots, r\}$, indicating the special round in this five-party protocol. Then, for every $0 \leq i < i^*$ it selects σ_J^i independently and uniformly at random for each triplet $J \subset \{1, \dots, 5\}$. Finally, it independently and uniformly selects a random bit w and sets $\sigma_J^i = w$, for every $i \in \{i^*, \dots, r\}$ and for every triplet $J \subset \{1, \dots, 5\}$.

The dealer T interacts with p_1, \dots, p_5 in rounds, where round i , for $1 \leq i \leq r$ consists of three phases:

The peeking phase. The dealer sends to the adversary all the bits σ_j^i such that there is a majority of corrupted parties in P_J , i.e., at least 2 parties p_j s.t. $j \in J$ are controlled by the adversary.

The abort and premature termination. The adversary sends to T a list of parties that abort in the current round. If there are less than 4 active parties (i.e., there are either 2 or 3 active parties),³ T sends σ_J^{i-1} to the active parties, where J is the lexicographically first triplet that contains all the indices of the active parties. The honest parties output this bit and halt.

The main phase. If at least 4 parties are active, T notifies the active parties that the protocol proceeds normally.

If after r rounds, there are at least 4 active parties, T simply sends w to all active parties and the honest parties output this bit.

As an example of a possible execution of the protocol, assume that p_1 aborts in round 4 and p_3 and p_4 abort in round 37. In this case, T sends $\sigma_{\{1,2,5\}}^{36}$ to p_2 and p_5 , which output this bit.

Recall that the adversary obtains the bit of the triplet J if at least two parties in P_J are malicious. If the adversary causes the dealer to halt, then, either there are two remaining active parties, both of them must be honest, or there are three active parties and at most one of them is malicious. In either case, the adversary does not know σ_J^{i-1} in advance. Furthermore, the dealer reveals the appropriate bit σ_J^{i-1} to the active parties. Jumping ahead, these properties are later preserved in a real world protocol by using a 2-out-of-3 secret-sharing scheme.

We next argue that any adversary can bias the output of the above protocol by at most $O(1/r)$. As in the protocol of Moran et al., the adversary can only bias the output by causing the protocol to terminate in round i^* . In contrast to the protocol of [18], in our protocol if in some round there are two bits σ_j^i and $\sigma_{j'}^i$ that the adversary can obtain such that $\sigma_j^i \neq \sigma_{j'}^i$, then the adversary can deduce that $i \neq i^*$. However, there are at most 7 bits that the adversary can obtain in each round (i.e., the bits of sets P_J containing at least two malicious parties). For a round i such that $i < i^*$, the probability that all these bits are equal to (say) 0 is $(1/2)^7$. Such rounds are indistinguishable to the adversary from round i^* . Intuitively, the best an adversary can do is guess one of these rounds, and therefore cannot succeed guessing with probability better than $1/2^7$. Thus, the bias the adversary can cause is $2^7/r$.

3.2.1 Eliminating the Dealer for the 5-Party Protocol

We eliminate the trusted on-line dealer in a few steps using a few layers of secret-sharing schemes. First, we change the on-line dealer, so that in each round i it shares the bit σ_j^i among the parties in P_J using a 2-out-of-3 secret-sharing scheme – called *inner* secret-sharing scheme. In fact, the dealer first signs the bit share to prevent cheating. The same requirements on σ_j^i as in the above protocol are preserved using this inner secret-sharing scheme. That is, the adversary is able to obtain information on σ_j^i only if it controls at least two of the parties in P_J . On the other hand, if the adversary does not control at least two parties in some P_J (i.e., there is an honest majority in P_J), then, in round i , the honest parties can reconstruct σ_j^{i-1} (if so instructed by the protocol).

Next we turn the on-line dealer into an off-line dealer. That is, we show that it is possible for the dealer to only interact with the parties once, sending each party some input, so that thereafter, the parties interact in rounds (without the dealer) and in each round i each party learns its shares in the i th inner secret-sharing scheme. That is, in each round i , each party p_j learns a share of σ_j^i in a 2-out-of-3 secret-sharing scheme,

³The reason for requiring that the dealer does not continue when at least two parties abort will become clear when we transform the protocol to a protocol with an off-line dealer.

for every triplet J such that $j \in J$. For this purpose, the dealer computes, in a preprocessing phase, the appropriate shares for the inner secret-sharing scheme. The shares of each round for each party p_j are then shared in a 2-out-of-2 secret-sharing scheme, where p_j gets one of the two shares (this serves as a mask, allowing only p_j to later reconstruct its shares of the appropriate σ_j^i 's while other shares are sent on a broadcast channel). All parties except p_j get shares in a 3-out-of-4 Shamir secret-sharing scheme of the other share of the 2-out-of-2 secret sharing. See Construction 2.7 for a formal description. We call the resulting secret-sharing scheme the *outer* scheme.

The use of the 4-out-of-5 secret-sharing scheme plays a crucial role in eliminating the on-line dealer. It guarantees that an adversary, corrupting at most three parties, cannot reconstruct the shares of round i before round i . However, if at least two parties do not reveal their shares, then they prevent a reconstruction of the outer scheme (this is why we cannot proceed after 2 parties aborted); in this case there is an honest majority. Hence, the protocol proceeds normally as long as at least 4 parties are active. If, indeed, at least two parties abort (in round i), then the remaining parties use their shares of the inner scheme to reconstruct the bit σ_J^{i-1} for the appropriate triplet J .

To prevent malicious parties from cheating, by say, sending false shares and causing reconstruction of wrong secrets, every message that a party should send during the execution of the protocol is signed in the preprocessing phase (together with the appropriate round number and with the party's index). Furthermore, all shares in the inner secret-sharing scheme are signed (as they are used as messages if reconstruction is required). In addition, the dealer sends a verification key to each of the parties. To conclude, the off-line dealer gives each party the signed shares for the outer secret sharing scheme together with the verification key. A formal description of the functionality of the off-line dealer, called ShareGenForFive_r , is given in Figure 1. In order to simplify the presentation, we use slightly different notations compared to the notations used in the presentation of the general construction in Sections 4–6.

The protocol with the off-line dealer proceeds in rounds. In round i of the protocol, all parties broadcast their (signed) shares in the outer (4-out-of-5) secret-sharing scheme. Thereafter, each party can unmask the message it receives (with its share in the appropriate 2-out-of-2 secret-sharing scheme) to obtain its signed shares in the 2-out-of-3 sharing of the bits σ_j^i (for the appropriate triplets J 's to which the party belongs). If a party stops broadcasting messages or broadcasts improperly signs messages, then all other parties consider it as aborted and will ignore all messages it will send in the future. If two or more parties abort, the remaining parties reconstruct the bit of the lexicographically first triplet that contains all of them, as described above. In the special case of premature termination already in the first round, the remaining parties engage in a fully secure protocol (with honest majority) to toss a completely random coin. In Figure 2 we formally define the 5-party coin-tossing protocol tolerating up-to 3 malicious parties.

Finally, we replace the off-line dealer by using a secure-with-abort protocol with cheat-detection computing the functionality computed by the dealer, that is, $\text{Functionality ShareGenForFive}_r$. Obtaining the outputs of this computation, an adversary is unable to infer any information regarding the output of the coin-tossing protocol (as the functionality does not give any information to an adversary controlling at most 3 parties). The adversary, however, can prevent the execution, at the price of at least one malicious party being detected by all other parties. In such an event, the remaining parties will start over. This can go on at most twice, before there is a honest majority and a completely fair coin can be obtained. The details of this final step are given in Section 6.

The above construction can be generalized in a straightforward manner to any number m of parties and any number t of malicious parties such that $t < 2m/3$ (where there is a bit for every set Q of size t ; the outer secret-sharing is a $(t + 1)$ -out-of- m scheme and the inner secret-sharing scheme is $(m - t)$ -out-of- t scheme). However, in the protocol described in Sections 4–6 the bias on the output is substantially smaller;

Functionality ShareGenForFive_r

Computing default bits and signing keys

1. Choose $w \in \{0, 1\}$ and $i^* \in \{1, \dots, r\}$ uniformly at random.
2. For each $i \in \{1, \dots, r\}$ and for each triplet $J \subset \{1, \dots, 5\}$ (i.e., $|J| = 3$),
 - (a) if $i < i^* - 1$, then choose independently and uniformly at random $\sigma_J^i \in \{0, 1\}$.
 - (b) if $i \geq i^*$, then set $\sigma_J^i = w$.
3. Compute $(K_{\text{sign}}, K_{\text{ver}}) \leftarrow \text{Gen}(1^n)$.

Computing signed shares of the inner secret-sharing scheme

4. For each $i \in \{1, \dots, r\}$ and for each triplet $J \subset \{1, \dots, 5\}$,
 - (a) Choose random secret shares of σ_J^i in an 2-out-of-3 Shamir's secret-sharing scheme for the parties in P_J . For each party $p_j \in P_J$, let $S_j^{i,J}$ be its share of σ_J^i .
 - (b) Sign each share $S_j^{i,J}$:

$$R_j^{i,J} \leftarrow (S_j^{i,J}, i, J, j, \text{Sign}((S_j^{i,J}, i, J, j), K_{\text{sign}})).$$

Computing shares of the outer secret-sharing scheme

5. For each $i \in \{1, \dots, r\}$, for each triplet $J \subset \{1, \dots, 5\}$, and for each $j \in J$: share p_j 's signed share $R_j^{i,J}$ using a 4-out-of-5 secret-sharing scheme with respect to p_j as defined in Construction 2.7; that is, one masking share $\text{mask}_j(R_j^{i,J})$ that only p_j obtains and 4 complement shares for the rest of the parties are produced.

Signing the messages of all parties

6. For every $1 \leq q \leq 5$, compute the message $m_{(q,i)}$ that $p_q \in \{p_1, \dots, p_5\}$ broadcasts in round i by concatenating (1) q , (2) the round number i , and (3) the complement shares $\text{comp}_q(R_j^{i,J})$ for all triplets J and for all $j \neq q$ such that $j \in J$ produced in Step (5) for p_q .
7. Compute $M_{q,i} \leftarrow (m_{q,i}, \text{Sign}(m_{q,i}, K_{\text{sign}}))$.

Outputs: Each party p_j receives

- The verification key K_{ver} .
- The messages $M_{j,1}, \dots, M_{j,r}$ that p_j broadcasts during the protocol.
- p_j 's private masks $\text{mask}_j(R_j^{i,J})$ which were produced in Step (5) for each $1 \leq i \leq r$ and each triplet $J \subset \{1, \dots, 5\}$ such that $j \in J$.

Figure 1: The initialization functionality ShareGenForFive_r.

A 5-party protocol $\text{FivePartyCoinToss}_r$

Joint input: Security parameter 1^n .

Preliminary phase:

- The parties engage in a secure-with-abort and cheat-detection protocol in order to computing the functionality ShareGenForFive_r . If a single party aborts, then this phase is repeated. This may happen at most twice before an honest majority is guaranteed. If at least two parties aborted (and detected), the parties use multiparty secure computation (with full security) to toss a fair coin, output this resulting bit, and halt.

In each round $i = 1, \dots, r$ do:

1. Each active party $p_j \in \{p_1, \dots, p_5\}$ broadcasts $M_{(j,i)}$ (its share of the p_q 's share of the bit σ_j^i for each triplet $J \subset \{1, \dots, 5\}$ (i.e., $|J| = 3$) and each $q \in J$ where $q \neq j$)
2. For each active party $p_j \in \{p_1, \dots, p_5\}$, if $\text{Ver}(M_{(j,i)}, K_{\text{ver}}) = \text{Fail}$ or if p_j broadcasts an invalid or no message, then all parties mark p_j as inactive.
3. If at least 2 parties were marked inactive:
 - If $i = 1$ (i.e., the parties aborted during the first round), then the active parties use a multiparty secure protocol (with full security) to toss a fair coin, output this resulting bit, and halt.
 - Otherwise, the active parties reconstruct the bit σ_J^{i-1} where J is the lexicographically first triplet that contains all the indices of the active parties, output it, and halt.

At the end of round r : Each active party p_j broadcasts the signed share $R_j^{r,J}$, for each triplet $J \subset \{1, \dots, 5\}$, such that, $j \in J$, in order to reconstruct $\sigma_j^r = w$. All honest parties output w .

Figure 2: The $\text{FivePartyCoinToss}_r$ protocol.

this is done using a better way for distributing bits to subsets.

4 Our Main Construction

In Sections 4–6 we present our main result – a coin-tossing protocol that has nearly optimal bias and can tolerate up to $2/3$ fraction of malicious parties. More specifically, we prove the following theorem:

Theorem 4. *If enhanced trap-door permutations exist, then for any m , t , and $r = r(n)$ such that $m/2 \leq t < 2m/3$, there is an r -round m -party coin-tossing protocol tolerating up to t malicious parties that has bias $O\left(2^{2^{k+1}}/r'\right)$, where $k = 2t - m$ and $r' = r - O(k + 1)$.*

In the above theorem, k is the difference between the number of malicious parties and the number of honest parties, i.e., $k = t - (m - t) = 2t - m$. For example, if $t = m/2$, then the number of honest and malicious parties are equal and $k = 0$,

Following [18], we describe our protocol in two steps. In Section 5, we describe Protocol CTWithD_r that uses an online trusted party, called dealer. In Section 6, we get rid of the on-line dealer. This simplifies the description and understanding of our protocols. More importantly, we prove the security of our main protocol in a modular way. In Section 5, we prove

Theorem 5. *Protocol CTWithD_r is an r -round m -party coin-tossing protocol with an on-line dealer tolerating up to t malicious parties and has bias $O\left(2^{2^{k+1}}/r\right)$.*

We then consider the on-line dealer of Protocol CTWithD_r as an ideal functionality. In this protocol, the honest parties do not send any messages and in each round the dealer sends messages to the parties; we consider an interactive functionality sending the messages that the dealer sends. We prove

Theorem 6. *Let $t < 2m/3$. If enhanced trap-door permutations exist, then Protocol $\text{CoinToss}_{r'}$ presented in Section 6, is a computationally-secure implementation (with full security) of the dealer functionality in Protocol CTWithD_r . Protocol $\text{CoinToss}_{r'}$ has $r' = r + O(k + 1)$ rounds.*

Proof of Theorem 4. This theorem follows from Theorem 5 and Theorem 6 by the composition theorem of Canetti [7]: Consider the (ideal) functionality \mathcal{F}_D of the dealer in Protocol CTWithD_r . By Theorem 6, Protocol CTWithD_r is a computationally-secure implementation of \mathcal{F}_D (with full security). Consider the protocol $\Pi^{\mathcal{F}_D}$ which simply executes the functionality \mathcal{F}_D . By Theorem 5, $\Pi^{\mathcal{F}_D}$ is a $1/p$ -secure implementation of $\text{CoinToss}()$. By the composition theorem of Canetti [7], Π^{CTWithD_r} is a $1/p$ -secure implementation of $\text{CoinToss}()$. \square

We stress that constructing fair coin-tossing protocols assuming a trusted dealer is an easy task, e.g., the trusted party can choose a random bit and send it to each party. However, when considering a rushing adversary, one cannot eliminate the trusted party in this protocol. The coin-tossing protocol we describe, Protocol CTWithD_r , is designed such that it is possible to transform it to a protocol with no trusted party.

5 A Protocol with an On-Line Dealer

In this section we describe a protocol with a special trusted party T that acts as an on-line dealer interacting with the parties in rounds. The protocol consists of r rounds of interaction. In a preliminary phase, the

trusted party T selects the prescribed output bit w and a special round number i^* uniformly at random. In each round of interaction, the trusted party T selects a bit for each subset in a predetermined collection (this collection is part of the design of the protocol). The bits of all subsets for any round before round i^* are uniformly and independently chosen; the bits of all subsets for all rounds from round i^* and on all equal w . These bits are not given to honest parties. However, if the number of corrupted parties in a subset Q is larger than some predefined threshold, then the corrupted parties get the bit of Q .

In the real world these bits are secret-shared. Since in the real world the adversary is rushing, the corrupted parties in the real world receive their shares before the honest parties do. Hence, in the protocol with the dealer, interaction between the parties and T in each round has three phases. In the first phase, for each subset Q that contains enough malicious parties, the trusted party sends the bit of the subset to the malicious parties in the subset Q . In the second phase, malicious parties may abort the computation (and by that prevent later reconstruction of some of the information). To do so, these parties send to T an “abort” message. Finally, in the third phase, a “proceed” message is sent to the parties (this can be considered as an “ideal” secret sharing).

The protocol proceeds normally as long as less than $m - t$ parties abort. If at most t parties are active in round i , the trusted party reveals the value of the bit of some subset Q in round $i - 1$, i.e., σ_S^{i-1} . The subset Q is chosen in a way that guarantees that the adversary does not know the value of this bit before instructing parties to abort. Jumping ahead, in the real world Q will also hold the additional property that the adversary cannot prevent the reconstruction of the bit of Q from round $i - 1$.

In order to simplify the presentation of this protocol, we assume that there are private channels between each party and the on-line dealer. The formal description of the protocol called CTWithD $_r$ is given in Figure 3. In Sections 5.1–5.2, we prove that Protocol CTWithD $_r$, described in Figure 3, is an $O(2^{2^{k+1}}/r)$ -secure protocol for computing the CoinToss() functionality. Our proof follows the real vs. ideal paradigm. We consider a malicious adversary \mathcal{A} that corrupts at most $t < 2m/3$ of the parties in an execution of Protocol CTWithD $_r$. In Section 5.1 we present a simulator \mathcal{S} that corrupts the same subset of parties in the ideal world with access to the trusted party T_{CoinToss} for computing the CoinToss() functionality (that is, T_{CoinToss} gives the same uniformly chosen bit to all parties). The simulator \mathcal{S} has black-box access to \mathcal{A} and at the end of the computation it outputs a view of the adversary \mathcal{A} . In Section 5.2 we bound the statistical distance between the random variable describing the view of the real-world adversary \mathcal{A} concatenated with the output of the honest parties in the appropriate execution and the random variable describing the output of the simulator \mathcal{S} concatenated with the bit selected by the ideal-world trusted party.

The bias of the protocol – an informal argument. We next informally explain why the protocol has small bias, that is, we give a sketch of the proof of Theorem 5. First, we claim that the adversary can bias the output only if the premature termination occurs in round i^* :

1. If the premature termination occurs after round i^* (or does not occur at all), then the output is already fixed.
2. If the premature termination occurs before round i^* , then the adversary does not know the random bit σ_J^{i-1} , which is the output of the honest parties, as there are less than o_J malicious parties in Q_J (as shown in the formal analysis of this event in Section 5.2).

Thus, the adversary can bias the output only if it guesses i^* . If $\sigma_J^i \neq \sigma_{J'}^i$ for two bits that the adversary gets from the trusted party, then it can learn that $i < i^*$. It can be shown that the adversary gets 2^{k+1} such bits (out of the 2^{k+2} bits). With probability $1/2^{2^{k+1}}$, all these bits are all equal in a round prior to i^* and

Joint Input: The security parameter 1^n , a polynomial $r = r(n)$ specifying the number of rounds in the protocol, and a bound t on the number of corrupted parties.

Underlying Subsets: Let $P_j = \{p_j\}$ for $1 \leq j \leq k+1$ and $P_{k+2} = \{p_{k+2}, \dots, p_m\}$. Define $Q_J = \cup_{j \in J} P_j$ for each $J \subset \{1, \dots, k+2\}$.

For each subset P_j define a reconstruction threshold value o_j : For $1 \leq j \leq k+1$ define $o_j = 1$ and define $o_{k+2} = m - t$. Finally, define $o_J = \sum_{j \in J} o_j$ for each $J \subset \{1, \dots, k+2\}$.

Instructions for the (trusted) dealer:

The preprocessing phase: Select a bit σ_J^i for each subset Q_J and each round i as follows:

1. Select $i^* \in \{1, \dots, r\}$ and $w \in \{0, 1\}$ independently with uniform distribution.
2. For each $J \subset \{1, \dots, k+2\}$, select $\sigma_J^0, \dots, \sigma_J^{i^*-1}$ independently with uniform distribution.
3. For each $J \subset \{1, \dots, k+2\}$, set $\sigma_J^{i^*} = \dots = \sigma_J^r = w$.

Protocol initialization: Send a “start” message to all parties. For each party p_j , upon receiving an “abort _{j} ” from party p_j , notify all parties that party p_j has aborted. If at least $m - t$ parties aborted, go to premature termination with $i = 1$. Otherwise, send “proceed” to all parties.

Interaction rounds: In each round $1 \leq i \leq r$, interact with the parties in three phases:

- **The peeking phase:** For each $J \subset \{1, \dots, k+2\}$, if Q_J contains at least o_J malicious parties, send the bit σ_J^i to all malicious parties in Q_J .
- **The abort phase:** For each party p_j , upon receiving an “abort _{j} ” message from party p_j , notify all parties that party p_j has aborted. (Ignore all other types of messages.)
If at least $m - t$ parties have aborted so far, go to premature termination step.
- **The main phase:** Send “proceed” to all parties.

Premature termination step: This step consists of two phases, after which the protocol terminates and all honest parties hold the same output.

- **The abort phase:** For each party p_j , upon receiving an “abort _{j} ” message from party p_j , remove party p_j from the list of active parties.
- **The default output phase:** Let D be the set of indices of parties that aborted the protocol thus far, i.e., $D = \{j : p_j \text{ has aborted}\}$.
 - If $|D \cap \{k+2, \dots, m\}| \geq m - t$ then $J = \{1, \dots, k+1\} \setminus D$.
 - If $|D \cap \{k+2, \dots, m\}| < m - t$ then $J = (\{1, \dots, k+1\} \setminus D) \cup \{k+2\}$.
 - Send $w' = \sigma_J^{i-1}$ to all parties and halt.

Normal termination: This phase is executed if the last round of the protocol is completed. Send w to all parties.

Instructions for honest parties: Upon receiving output y from the dealer, output y . (Honest parties do not send any message throughout the protocol.)

Figure 3: Protocol CTWithD _{r} .

the adversary cannot distinguish such round from i^* . By [14, Lemma 2], this implies that the adversary can guess i^* with probability at most $2^{2^{k+1}}/r$. Therefore, the bias is $O(2^{2^{k+1}}/r)$.

5.1 The Simulator for the On-Line Dealer Protocol

We next describe the simulator \mathcal{S} for Protocol CTWithD $_r$. Let B be the subset of corrupted parties in the execution. On input $(1^n, \text{aux})$, the simulator \mathcal{S} obtains $w_S = \text{CoinToss}()$ from the trusted party T_{CoinToss} (i.e., w_S is uniformly selected from $\{0, 1\}$). The simulator \mathcal{S} internally apply \mathcal{A} with input $(1^n, \text{aux})$. The simulator \mathcal{S} will interact with \mathcal{A} playing the role of T , by proceeding as follows:

Simulating the preprocessing phase: \mathcal{S} selects a bit σ_J^i for every subset Q_J for which the corrupted parties get σ_J^i (i.e., for which $|B \cap Q_J| \geq o_J$), and every round i as follows:

1. Selects $i^* \in \{1, \dots, r\}$ with uniform distribution.
2. For each $J \subset \{1, \dots, k+2\}$, the simulator \mathcal{S} selects uniformly and independently at random $\sigma_J^1, \dots, \sigma_J^{i^*-1}$.
3. For each $J \subset \{1, \dots, k+2\}$, the simulator \mathcal{S} sets $\sigma_J^{i^*} = \dots = \sigma_J^r = w_S$.

Simulating protocol initialization: \mathcal{S} sends a “start” message to all corrupt parties (that is, \mathcal{S} sends the messages for the corrupt parties to the adversary \mathcal{A}). For each party p_j , upon receiving an “abort $_j$ ” message from party p_j , the simulator notifies all corrupted parties that party p_j has aborted. If at least $m - t$ parties aborted, \mathcal{S} sets $i = 1$ and proceeds to simulate the premature termination step. Otherwise, \mathcal{S} sends “proceed” to all corrupt parties.

Simulating interaction rounds: In each round $1 \leq i \leq r$, the simulator \mathcal{S} interacts with the corrupt parties, in three phases:

- **The peeking phase:** For each $J \subset \{1, \dots, k+2\}$, if Q_J contains at least o_J corrupt parties, then \mathcal{S} sends the bit σ_J^i to all corrupt parties in Q_J (i.e., to the adversary).
- **The abort phase:** For each party p_j , upon receiving an “abort $_j$ ” message from party p_j , the simulator notifies all parties that party p_j is inactive.
If at least $m - t$ parties have aborted so far, \mathcal{S} simulates the premature termination step.
- **The main phase:** \mathcal{S} sends “proceed” to all parties.

Simulating the premature termination step: This round consists of two phases, after which the simulation terminates.

- **The abort phase:** For each party p_j , upon receiving an “abort $_j$ ” message from party p_j , remove party p_j from the list of active parties.
- **The default output phase:** Regardless of the subset of set of aborted parties, the simulator sends w_S to all parties and halts the interaction with \mathcal{A} .

Simulating normal termination: If the last round of the protocol is completed, then \mathcal{S} sends w_S to all parties.

At the end of the interaction with \mathcal{A} , the simulator will output the sequence of messages exchanged between the simulator and the corrupted parties.

5.2 Proof of the Correctness of the Simulation

We consider the two random variables from Section 2.1, both of the form (V, C) , where V describes a possible view of \mathcal{A} , and C describes a possible output of the honest parties (i.e., $C \in \{0, 1\}$). The first random variable $\text{REAL}_{\text{CTWithD}_r, \mathcal{A}(\text{aux})}(1^n)$ describes the real world – an execution of the CTWithD protocol, where V describes the view of the adversary \mathcal{A} in this execution, and C is the output of the honest parties in this execution. The second random variable $\text{IDEAL}_{\text{CoinToss}(), \mathcal{S}(\text{aux})}(1^n)$ describes the ideal world – an execution with the trusted party T_{CoinToss} , where V describes the output of the simulator \mathcal{S} in this execution, and C is the output of the honest parties in this execution. Our goal here is to show that the statistical distance between these two random variables is $O(2^{2^{k+1}}/r)$, that is, to show that the protocol is *perfectly* $O(2^{2^{k+1}}/r)$ secure. For the rest of this proof, we simplify notations by omitting notations that are clear from the context. In particular, we denote the above two random variables by $\text{REAL} = (V_{\text{REAL}}, C_{\text{REAL}})$ and $\text{IDEAL} = (V_{\text{IDEAL}}, C_{\text{IDEAL}})$ respectively.

Lemma 5.1. *For every non-uniform polynomial-time adversary \mathcal{A} corrupting $t < 2m/3$ of the parties in Protocol CTWithD_r , for the simulator \mathcal{S} described in Section 5.1, controlling the same parties as \mathcal{A} , for every $n \in \mathbb{N}$ and for every $\text{aux} \in \{0, 1\}^*$, the following two random variables are $1/p(n) = O(2^{2^{k+1}}/r)$ -close:*

$$\text{SD} \left(\text{IDEAL}_{\text{CoinToss}(), \mathcal{S}(\text{aux})}(1^n), \text{REAL}_{\text{CTWithD}_r, \mathcal{A}(\text{aux})}(1^n) \right) \leq 1/p.$$

Proof. The flow of our proof is as follows. We first bound the statistical distance between the two random variables by the probability that the adversary \mathcal{A} guesses the special round i^* . We do this by showing that, conditioned on the event that the adversary fails to guess round i^* , the two random variables are identically distributed. Then, we bound the probability of guessing i^* . We show that the adversary learns at most 2^{k+1} new bits in each round. Hence, the probability that all these bits are equal in round $i < i^*$ is at least $\frac{1}{2^{2^{k+1}}}$. Finally, we show that this implies that the probability that of guessing i^* correctly is bounded by $2^{2^{k+1}}/r$.

We consider the probability of a given pair (v, c) according to the two different random variables. For thus we recall that the honest parties are deterministic in both models. We compare the two following probabilities:

1. The probability that v is the view of the adversary \mathcal{A} in an execution of Protocol CTWithD_r and c is the output of the honest parties in this execution, i.e.,

$$\Pr[\text{REAL} = (v, c)] = \Pr[V_{\text{REAL}} = v \quad \wedge \quad C_{\text{REAL}} = c],$$

where the probability is taken over the random coins of the dealer T .

2. The probability that v is the output of the simulator \mathcal{S} in an ideal-world execution with the trusted party T_{CoinToss} and c is the output of the honest parties in this execution, i.e.,

$$\Pr[\text{IDEAL} = (v, c)] = \Pr[V_{\text{IDEAL}} = v \quad \wedge \quad C_{\text{IDEAL}} = c],$$

where the probability is taken over the random coins of the simulator \mathcal{S} and the random coins of the ideal-world trusted party T_{CoinToss} .

Observe that in the simulation \mathcal{S} follows the same instructions as the trusted party T in Protocol CTWithD_r , except for two changes. First, \mathcal{S} does not flip an internal coin to obtain w_S (in contrast T chooses w at random), but rather gets w_S externally from T_{CoinToss} . However, since both w_S and w are

uniformly distributed, it is practically the same operation. The major difference is that in the case of a premature termination step, \mathcal{S} always uses w_S as its message to the corrupt parties, while T uses the bit of the appropriate subset Q_J as its message. In both cases, this last message sent to the adversary is simply the output bit of the honest parties, that is, in both of the above random variables, the last message in v equals c . Hence, it is equivalent to consider the two random variables as above, where in both cases, the last message sent to the corrupt parties in the premature termination round is omitted from c . Thus, we consider the possible views v of the adversary without the last message sent to the corrupted parties.

We define a random variable $\text{PT} \in \{\text{“before”}, \text{“during”}, \text{“after”}\}$ specifying whether premature termination occurred before, during, or after the special round i^* . I.e., let i be the round number in which premature termination occurs (if ever); if $i < i^*$ (or if premature termination occurs during the protocol initialization step), then $\text{PT} = \text{“before”}$, if $i = i^*$, then $\text{PT} = \text{“during”}$, and $\text{PT} = \text{“after”}$ if $i > i^*$ or premature termination never occurs. The random variable is defined both in the ideal and the real world.

In the next claim, we bound the statistical distance between the two random variables according to Definition 2.1.

Claim 5.2.

$$\text{SD}(\text{IDEAL}, \text{REAL}) \leq \frac{1}{2} \Pr[\text{PT} = \text{“during”}].$$

In order to prove this claim, we analyze the above probabilities conditioned on these three possible events:

PT = “before” – premature termination occurs before round i^* . We argue that in this case, both in the real protocol and in the simulation, the actions of \mathcal{A} (according to V) and its view before the premature termination step are independent of C (and, hence, also of the message sent to corrupt parties in the premature termination) and, thus, the probabilities are exactly the same.

In a real-world execution, the output of the honest parties is the random bit for the appropriate set Q_J , i.e., $C_{\text{REAL}} = \sigma_J^{i-1}$. This bit is independent of all other bits (since $i - 1 < i^*$). Hence, if \mathcal{A} does not receive σ_J^{i-1} from T , then this bit is completely unknown to \mathcal{A} (before the last message is sent to the corrupted parties). This is clearly true if premature termination occurs during the protocol initialization. To see that it is also the case if premature termination occurs during round $i < i^*$, consider the following two cases:

1. If $|D \cap \{k+2, \dots, m\}| \geq m - t$, then $J = \{1, \dots, k+1\} \setminus D$ and $o_J = |J|$. There are at most t corrupt parties and at least $m - t$ of them are in $Q_{\{k+2\}}$, thus, at most $t - (m - t) = k$ corrupt parties among the $k+1$ parties p_1, \dots, p_{k+1} . In other words, there is at least one honest (and therefore active) party in Q_J , and the adversary does not learn σ_J^{i-1} during the peeking phase.
2. If $|D \cap \{k+2, \dots, m\}| < m - t$, then $J = (\{1, \dots, k+1\} \setminus D) \cup \{k+2\}$. In this case, let $\alpha = |D \cap \{1, \dots, k+1\}|$, therefore, $o_J = (k+1-\alpha) + (m-t) = 2t - m + 1 - \alpha + m - t = t + 1 - \alpha$. The set Q_J contains at most $t - \alpha < o_J$ corrupt parties, thus, these parties do not get the bit σ_J^{i-1} from the trusted party.

Thus, given $\text{PT} = \text{“before”}$, the output of the honest parties is independent of the view of the adversary,

hence,

$$\begin{aligned}
& \Pr[V_{\text{REAL}} = v \wedge C_{\text{REAL}} = c \mid \text{PT} = \text{“before”}] \\
&= \Pr[V_{\text{REAL}} = v \mid \text{PT} = \text{“before”}] \cdot \Pr[C_{\text{REAL}} = c \mid \text{PT} = \text{“before”}] \\
&= \Pr[V_{\text{REAL}} = v \mid \text{PT} = \text{“before”}] \cdot \frac{1}{2}.
\end{aligned}$$

In the ideal-world execution, all the messages sent by the simulator in any round $i < i^*$ are independent of w_S , which was the value given to \mathcal{S} by T_{CoinToss} . Thus,

$$\begin{aligned}
& \Pr[V_{\text{IDEAL}} = v \wedge C_{\text{IDEAL}} = c \mid \text{PT} = \text{“before”}] \\
&= \Pr[V_{\text{IDEAL}} = v \mid \text{PT} = \text{“before”}] \cdot \Pr[C_{\text{IDEAL}} = c \mid \text{PT} = \text{“before”}] \\
&= \Pr[V_{\text{IDEAL}} = v \mid \text{PT} = \text{“before”}] \cdot \Pr[w_S = c \mid \text{PT} = \text{“before”}] \\
&= \Pr[V_{\text{IDEAL}} = v \mid \text{PT} = \text{“before”}] \cdot \frac{1}{2}.
\end{aligned}$$

As explained above, \mathcal{S} follows the same random process in interacting with \mathcal{A} (before sending the last message in the premature termination) as does T in the real-world execution. Hence,

$$\Pr[V_{\text{REAL}} = v \mid \text{PT} = \text{“before”}] = \Pr[V_{\text{IDEAL}} = v \mid \text{PT} = \text{“before”}],$$

and, thus,

$$\begin{aligned}
\Pr[V_{\text{REAL}} = v \wedge C_{\text{REAL}} = c \mid \text{PT} = \text{“before”}] &= \\
\Pr[V_{\text{IDEAL}} = v \wedge C_{\text{IDEAL}} = c \mid \text{PT} = \text{“before”}]. &
\end{aligned} \tag{1}$$

PT = “after” – premature termination occurs after round i^* or never occurs. Here v must contain $\sigma_J^{i^*}$ for some J , which, in the real-world execution, is equal to the output bit of all sets for any round $i > i^*$ (recall that the output bit of the honest parties is determined by one such bit), and in the simulation it equals w_S . Thus, in both scenarios, v must be consistent with i^* and with c , hence, v completely determines c . Again, since \mathcal{S} follows the same random process in interacting with \mathcal{A} as does T in the real-world execution, we have,

$$\Pr[V_{\text{REAL}} = v \mid \text{PT} = \text{“after”}] = \Pr[V_{\text{IDEAL}} = v \mid \text{PT} = \text{“after”}],$$

and, thus,

$$\begin{aligned}
\Pr[V_{\text{REAL}} = v \wedge C_{\text{REAL}} = c \mid \text{PT} = \text{“after”}] &= \\
\Pr[V_{\text{IDEAL}} = v \wedge C_{\text{IDEAL}} = c \mid \text{PT} = \text{“after”}]. &
\end{aligned} \tag{2}$$

PT = “during” – premature termination occurs in round i^* . This is the interesting case which causes the statistical distance. In the real world, the output of the honest parties is $\sigma_J^{i^*-1}$ for some J , while in the ideal world their output is w_S . In the first case the output is independent of the adversary’s view, while in the second case, the view determining the output.

In both scenarios, premature termination occurs in round i^* and all the bits $\sigma_j^{i^*}$ given to the adversary in round i^* must be the same (otherwise both probabilities are zero). If, however, c does not equal $\sigma_j^{i^*}$ that the adversary sees according to v , then we have,

$$\begin{aligned} \Pr[V_{\text{REAL}} = v \wedge C_{\text{REAL}} = c \mid \text{PT} = \text{“during”}] \\ &= \Pr[V_{\text{REAL}} = v \mid \text{PT} = \text{“during”}] \cdot \Pr[C_{\text{REAL}} = c \mid \text{PT} = \text{“during”}] \\ &= \Pr[V_{\text{REAL}} = v \mid \text{PT} = \text{“during”}] \cdot \frac{1}{2} \end{aligned}$$

and

$$\Pr[V_{\text{IDEAL}} = v \wedge C_{\text{IDEAL}} = c \mid \text{PT} = \text{“during”}] = 0.$$

Otherwise, v is consistent both with i^* and with c , and we have,

$$\Pr[V_{\text{REAL}} = v \wedge C_{\text{REAL}} = c \mid \text{PT} = \text{“during”}] = \Pr[V_{\text{REAL}} = v \mid \text{PT} = \text{“during”}] \cdot \frac{1}{2}$$

and

$$\begin{aligned} \Pr[V_{\text{IDEAL}} = v \wedge C_{\text{IDEAL}} = c \mid \text{PT} = \text{“during”}] \\ &= \Pr[V_{\text{IDEAL}} = v \mid \text{PT} = \text{“during”}] \\ &= \Pr[V_{\text{REAL}} = v \mid \text{PT} = \text{“during”}]. \end{aligned}$$

Hence, for any pair (v, c) , it holds that,

$$\begin{aligned} &\left| \Pr[V_{\text{IDEAL}} = v \wedge C_{\text{IDEAL}} = c \mid \text{PT} = \text{“during”}] \right. \\ &\quad \left. - \Pr[V_{\text{REAL}} = v \wedge C_{\text{REAL}} = c \mid \text{PT} = \text{“during”}] \right| \\ &\leq \frac{1}{2} \Pr[V_{\text{REAL}} = v \mid \text{PT} = \text{“during”}]. \end{aligned} \tag{3}$$

Bounding the statistical distance between the global outputs of the real-world and ideal world. We next use the above analysis of the three possible cases to bound the statistical distance between the two random variables describing the global outputs of the ideal and real models. For every pair (v, c) , the following holds,

$$\begin{aligned} \Pr[V_{\text{IDEAL}} = v \wedge C_{\text{IDEAL}} = c] \\ &= \Pr[V_{\text{IDEAL}} = v \wedge C_{\text{IDEAL}} = c \mid \text{PT} = \text{“before”}] \cdot \Pr[\text{PT} = \text{“before”}] \\ &\quad + \Pr[V_{\text{IDEAL}} = v \wedge C_{\text{IDEAL}} = c \mid \text{PT} = \text{“during”}] \cdot \Pr[\text{PT} = \text{“during”}] \\ &\quad + \Pr[V_{\text{IDEAL}} = v \wedge C_{\text{IDEAL}} = c \mid \text{PT} = \text{“after”}] \cdot \Pr[\text{PT} = \text{“after”}]. \end{aligned}$$

Similarly,

$$\begin{aligned}
\Pr[V_{\text{REAL}} = v \wedge C_{\text{REAL}} = c] &= \Pr[V_{\text{REAL}} = v \wedge C_{\text{REAL}} = c \mid \text{PT} = \text{“before”}] \cdot \Pr[\text{PT} = \text{“before”}] \\
&+ \Pr[V_{\text{REAL}} = v \wedge C_{\text{REAL}} = c \mid \text{PT} = \text{“during”}] \cdot \Pr[\text{PT} = \text{“during”}] \\
&+ \Pr[V_{\text{REAL}} = v \wedge C_{\text{REAL}} = c \mid \text{PT} = \text{“after”}] \cdot \Pr[\text{PT} = \text{“after”}].
\end{aligned}$$

Hence, by Equations (1), (2), and (3), for every pair (v, c) , we have,

$$\begin{aligned}
&\left| \Pr[V_{\text{IDEAL}} = v \wedge C_{\text{IDEAL}} = c] - \Pr[V_{\text{REAL}} = v \wedge C_{\text{REAL}} = c] \right| \\
&= \left| \Pr[V_{\text{IDEAL}} = v \wedge C_{\text{IDEAL}} = c \mid \text{PT} = \text{“before”}] \cdot \Pr[\text{PT} = \text{“before”}] \right. \\
&\quad - \Pr[V_{\text{REAL}} = v \wedge C_{\text{REAL}} = c \mid \text{PT} = \text{“before”}] \cdot \Pr[\text{PT} = \text{“before”}] \\
&\quad + \Pr[V_{\text{IDEAL}} = v \wedge C_{\text{IDEAL}} = c \mid \text{PT} = \text{“during”}] \cdot \Pr[\text{PT} = \text{“during”}] \\
&\quad - \Pr[V_{\text{REAL}} = v \wedge C_{\text{REAL}} = c \mid \text{PT} = \text{“during”}] \cdot \Pr[\text{PT} = \text{“during”}] \\
&\quad + \Pr[V_{\text{IDEAL}} = v \wedge C_{\text{IDEAL}} = c \mid \text{PT} = \text{“after”}] \cdot \Pr[\text{PT} = \text{“after”}] \\
&\quad \left. - \Pr[V_{\text{REAL}} = v \wedge C_{\text{REAL}} = c \mid \text{PT} = \text{“after”}] \cdot \Pr[\text{PT} = \text{“after”}] \right| \\
&= \frac{1}{2} \Pr[V_{\text{REAL}} = v \mid \text{PT} = \text{“during”}] \cdot \Pr[\text{PT} = \text{“during”}]. \tag{4}
\end{aligned}$$

This is true since the probability that premature termination occurs in round i^* , i.e., $\Pr[\text{PT} = \text{“during”}]$, is the same in both models (again, since \mathcal{S} follows the same random process in interacting with \mathcal{A} , before sending the last message in the premature termination, as does T in the real-world execution.)

Now, we are ready to conclude the proof of Claim 5.2.

Proof of Claim 5.2.

$$\begin{aligned}
\text{SD}(\text{IDEAL}, \text{REAL}) &= \\
&\frac{1}{2} \sum_{(v,c)} \left| \Pr[V_{\text{IDEAL}} = v \wedge C_{\text{IDEAL}} = c] - \Pr[V_{\text{REAL}} = v \wedge C_{\text{REAL}} = c] \right| \\
&\leq \frac{1}{2} \sum_{(v,c)} \frac{1}{2} \Pr[V_{\text{REAL}} = v \mid \text{PT} = \text{“during”}] \cdot \Pr[\text{PT} = \text{“during”}] \quad \text{by Equation (4)} \\
&\leq \frac{1}{2} \Pr[\text{PT} = \text{“during”}] \cdot \sum_{(v,c)} \frac{1}{2} \Pr[V_{\text{REAL}} = v \mid \text{PT} = \text{“during”}] \\
&\leq \frac{1}{2} \Pr[\text{PT} = \text{“during”}].
\end{aligned}$$

Where the last inequality is due to the properties of the probability function and the fact that $c \in \{0, 1\}$. \square

Hence, the statistical distance is bounded by the probability of the adversary guessing i^* correctly (before the abort phase of round i^*). We next bound this probability by first showing that the view of the adversary in each round prior to round i^* has probability $1/2^{2^{k+1}}$ to be indistinguishable from i^* , and then by showing that this implies that the overall probability for any adversary to guess i^* is at most $2^{2^{k+1}}/r$.

If $\sigma_J^i \neq \sigma_{J'}^i$ for two bits that the adversary receives during the peeking phase of round i , then it can learn that $i < i^*$. The information that the adversary obtains in each round are the bits σ_J^i for J 's such that there are enough corrupt parties in Q_J (i.e., for which $|B \cap Q_J| \geq o_J$). We next claim that the adversary only receives half of the 2^{k+2} bits.

We remark that to obtain a total bias of $O(2^{2^{k+2}}/r)$ (instead of the $O(2^{2^{k+1}}/r)$ bound of Theorem 5), one may omit the following claim and use the trivial bound of 2^{k+2} on the number of bits that the adversary sees in each round. The rest of our proof is independent of this claim. Hence, the reader may choose to skip the technical details of the proof of Claim 5.3 below.

Claim 5.3. *Let $B \subset P$ such that $|B| = t < 2m/3$ be the subset of corrupt parties, then there exists at most 2^{k+1} different subsets $J \subseteq \{1, \dots, k+2\}$ such that $|B \cap Q_J| \geq o_J$ (i.e., set that the adversary receives the bits of the associated Q_J in the peeking phase of any round).*

Proof. Recall that $Q_{\{j\}} = P_j$ is a singleton (and $o_{\{j\}} = o_j = 1$) for all $1, \dots, k+1$, $Q_{\{k+2\}} = P_{k+2} = \{p_{k+2}, \dots, p_m\}$ (and $o_{\{k+2\}} = o_{k+2} = m - t$), and $Q_J = \cup_{j \in J} P_j$. We consider two cases regarding the size of the intersection between the set of corrupted parties B and $Q_{\{k+2\}}$:

1. $|B \cap Q_{\{k+2\}}| < o_{\{k+2\}} = m - t$. Let $J' \subseteq \{1, \dots, k+1\}$ be any subset and let $J = J' \cup \{k+2\}$. Thus, $o_J = o_{\{k+2\}} + o_{J'} = m - t + |J'| > |B \cap Q_{\{k+2\}}| + |B \cap Q_{J'}| = |B \cap Q_J|$, since $|B \cap Q_{J'}| \leq |Q_{J'}| = |J'|$. In other words, every such set Q_J contains less than o_J corrupt parties. Hence, there are at least 2^{k+1} subsets J such that $o_J > |B \cap Q_J|$.
2. $|B \cap Q_{\{k+2\}}| \geq o_{\{k+2\}}$. In this case, $|B \cap Q_{\{k+2\}}| = m - t + d$ for some $0 \leq d \leq k$. Thus, there are $t - (m - t + d) = 2t - m - d = k - d$ corrupt parties in $\{p_1, \dots, p_{k+1}\}$, and, therefore, there are $d + 1$ honest parties in $\{p_1, \dots, p_{k+1}\}$. Without loss of generality, we assume that p_1, \dots, p_{d+1} are honest.

We argue that there are at least 2^{k+1} bits that the adversary does not see by presenting two types of subsets whose bits the adversary cannot see.

- (a) Subsets J such that, $J \subseteq \{1, \dots, k+1\}$. In this case, $o_J = |Q_J|$ and the adversary can only see that bits of subsets Q_J such that $J \subseteq \{d+2, \dots, k+1\}$, i.e., 2^{k-d+1} bits. That is, there are 2^{k-d} bits that the adversary can see for sets $J \subseteq \{1, \dots, k+1\}$.
- (b) Subsets J , such that $k+2 \in J$. In this case, $o_J = |J \cap \{1, \dots, k+1\}| + o_{k+2} = |J| - 1 + m - t$. If $\{1, \dots, d+1\} \subset J$ (where p_1, \dots, p_{d+1} are honest), then Q_J contains $(|J| - 1) - (d+1)$ corrupt parties from p_1, \dots, p_{k+1} and $m - t + d$ corrupt parties from $\{p_{k+2}, \dots, p_m\}$. That is, Q_J contains $|J| - 2 + m - t < o_J$ corrupt parties, and the adversary does not know σ_J^{i-1} . There are 2^{k-d} sets J that contain $\{1, \dots, d+1\} \cup \{k+2\}$, for these sets the adversary cannot see the bits. Thus, the adversary sees at most $2^{k+1} - 2^{k-d}$ bits for sets J that contain $\{k+2\}$.

Altogether, that are at least $2^{k-d-1} + (2^{k+1} - 2^{k-d-1}) = 2^{k+1}$ bits that the adversary does not see and the adversary sees at most 2^{k+1} bits. \square

Hence, for each $i < i^*$, with probability $1/2^{2^{k+1}}$ the bits that are available to the adversary are all equal, and the adversary cannot distinguish this round from i^* . It is left to show that this implies that the adversary can guess i^* with probability at most $2^{2^{k+1}}/r$. Lemma 2 in [14] proves a generalization of this fact. For the sake of completeness, we include here a proof for our special case.

Claim 5.4. *Let \mathcal{A} be an adversary in Protocol CTWithD $_r$, such that, in each round i , there are α bits of the form σ_j^i that \mathcal{A} sees. The probability that \mathcal{A} guess i^* is $O(2^\alpha/r)$.*

Proof. First, recall that \mathcal{A} is deterministic. In addition, the exact sequence of aborts by corrupted parties is immaterial. Thus, for the rest of this proof, whenever we say that \mathcal{A} aborts in round i we mean that \mathcal{A} instructs a large enough subset of corrupt parties to abort the computation after having seen the bits of round i or in prior rounds (and at least one party in round i).

We denote by \mathcal{V} the set of all possible views of \mathcal{A} ending with a premature termination. Clearly,

$$\sum_{v \in \mathcal{V}} \Pr[V_{\text{REAL}} = v \mid i^* = r] \leq 1. \quad (5)$$

For $1 \leq i \leq r$ let $\mathcal{V}_i \subseteq \mathcal{V}$ be the set of all views for which the deterministic \mathcal{A} aborts in round i . For a view $v \in \mathcal{V}_i$ denote by v_{i-1} the prefix of v that \mathcal{A} sees until round $i-1$ and denote by γ_i the bits that \mathcal{A} sees in the peeking phase of round i (i.e., $v = v_{i-1} \circ \gamma_i$). Without loss of generality, we assume, that for any $v \in \mathcal{V}_i$, the bits that the adversary sees in round i (i.e., γ_i) are all the same (otherwise, there exists a more successful adversary that never guesses i^* with this view, because if the bits are not the same, then $i \neq i^*$). For any round i , let V_{REAL}^i be a random variable describing the view of the adversary until the end of round i .

We next consider the probabilities of the above views when $i^* = i$ and when $i^* = r$. First, note that until round i both views happen with exactly the same probability, i.e.,

$$\Pr[V_{\text{REAL}}^{i-1} = v_{i-1} \mid i^* = i] = \Pr[V_{\text{REAL}}^{i-1} = v_{i-1} \mid i^* = r]. \quad (6)$$

When $i^* = i$, given the prefix of the view, there are only two possible events for the bits of round i each occurring with probability $1/2$, that is, the bits the adversary sees are either all 1 or they are all 0. On the other hand, if $i^* > i$ (specifically, if $i^* = r$), then each of these events happens with probability $1/2^\alpha$. Hence, we have for every $v \in \mathcal{V}_i$ that

$$\Pr[V_{\text{REAL}} = v \mid V_{\text{REAL}}^{i-1} = v_{i-1} \wedge i^* = i] = 2^{\alpha-1} \cdot \Pr[V_{\text{REAL}} = v \mid V_{\text{REAL}}^{i-1} = v_{i-1} \wedge i^* = r]. \quad (7)$$

Combining Equation (6) and Equation (7) we get for every $v \in \mathcal{V}_i$

$$\begin{aligned} \Pr[V_{\text{REAL}} = v \mid i^* = i] &= \Pr[V_{\text{REAL}} = v \mid V_{\text{REAL}}^{i-1} = v_{i-1} \wedge i^* = i] \cdot \Pr[V_{\text{REAL}}^{i-1} = v_{i-1} \mid i^* = i] \\ &= 2^{\alpha-1} \cdot \Pr[V_{\text{REAL}} = v \mid V_{\text{REAL}}^{i-1} = v_{i-1} \wedge i^* = r] \cdot \Pr[V_{\text{REAL}}^{i-1} = v_{i-1} \mid i^* = r] \\ &= 2^{\alpha-1} \cdot \Pr[V_{\text{REAL}} = v \mid i^* = r]. \end{aligned}$$

Finally, the probability that \mathcal{A} aborts in i^* is

$$\begin{aligned} \Pr[\text{PT} = \text{“during”}] &= \sum_{i=1}^r \Pr[i = i^*] \Pr[\text{PT} = \text{“during”} \mid i^* = i] \\ &= \sum_{i=1}^r \Pr[i = i^*] \sum_{v \in \mathcal{V}_i} \Pr[V_{\text{REAL}} = v \mid i^* = i] \\ &= \frac{1}{r} \sum_{i=1}^r \sum_{v \in \mathcal{V}_i} 2^{\alpha-1} \cdot \Pr[V_{\text{REAL}} = v \mid i^* = r] \\ &= \frac{2^{\alpha-1}}{r} \sum_{v \in \mathcal{V}} \Pr[V_{\text{REAL}} = v \mid i^* = r] \leq \frac{2^{\alpha-1}}{r} \quad \text{by Equation (5)}. \end{aligned}$$

□

Concluding the proof of Lemma 5.1. By Claim 5.2 the statistical distance between the global outputs of an ideal-world simulation and a real-world execution of the protocol is bounded by the probability that the adversary guesses the special round i^* in time. In Claim 5.4 we bound this probability by $2^\alpha/r$, where α is the number of bits that the adversary sees in each round of the protocol. By Claim 5.3, $\alpha \leq 2^{k+1}$, thus, the statistical distance between the two random variables is bounded by $2^{2^{k+1}}/r$ and

$$\text{SD} \left(\text{IDEAL}_{\text{CoinToss}(), \mathcal{S}(\text{aux})}(1^n), \text{REAL}_{\text{CTWithD}_r, \mathcal{A}(\text{aux})}(1^n) \right) \leq 2^{2^{k+1}}/r.$$

□

6 Omitting the On-Line Dealer

6.1 The Protocol Without the Dealer

In this section we show how Protocol CTWithD_r , presented in Section 5, can be transformed into a real-world protocol. That is, we present a fully secure m -party protocol implementing the ideal functionality described in Protocol CTWithD_r . The resulting protocol has r' rounds, where $r' = r + c(k + 1)$, for some constant c . The protocol is executed in a network where the parties communicate via an authenticated broadcast channel.

In Figure 4 we describe the initialization functionality, Functionality MultiShareGen_r , which chooses i^* , prepares bits for each set for each round, shares those bits, and signs them. In Figure 5, we present Protocol CoinToss_r , which proceeds in rounds and emulates Protocol CTWithD_r . Protocol CoinToss_r uses a Functionality Reconstruction, described in Figure 6, which reconstructs the output bit if many malicious parties have aborted. Before formally describing our construction, we outline its main components.

The inner secret-sharing scheme. To implement the “ideal secret sharing functionality” of the trusted party T in the CTWithD_r protocol to share the bits σ_j^i , we use an o_J -out-of- $|Q_J|$ Shamir secret-sharing scheme. That is, in each round i , each party $p_j \in Q_J$ obtains a share $S_j^{i,J}$ in an o_J -out-of- $|Q_J|$ secret-sharing of σ_j^i . The same requirement on σ_j^i as in the protocol with the on-line protocol are preserved using this inner secret-sharing scheme. That is, the adversary is able to obtain information on σ_j^i only if it controls at least o_J of the parties in Q_J . On the other hand, if, in a premature termination in round i , at least o_J parties in Q_J cooperate, then they can reconstruct σ_j^{i-1} from their shares.

The outer secret-sharing scheme. In the protocol with the on-line dealer, the adversary never learns anything about the bits σ_j^i before round i begins. To achieve this property in the real-world protocol, the shares of the inner secret-sharing schemes of all rounds are shared, in a preliminary phase, using a $(t + 1)$ -out-of- m secret-sharing scheme. The $t + 1$ threshold guarantees that the adversary, controlling at most t parties, cannot see the shares of the inner secret-sharing scheme for a given round i without the honest parties, which will not occur before round i .

In each round i the parties send messages so that each party can reconstruct its shares in the inner secret-sharing schemes of round i . Since all messages are broadcast and all parties can see them, the shares that party p_j receives in round i are masked by using yet another layer of secret sharing. Specifically, a share

$S_j^{i,J}$ to be reconstructed by p_j in round i is signed and shared (already in the preliminary phase) in a 2-out-of-2 secret-sharing scheme, such that one share is given to p_j and the other is shared among all parties in a t -out-of- m secret-sharing scheme; for details see Construction 2.7. We refer to the combination of these two layers of secret sharing as the outer secret-sharing scheme.

Premature Termination. The $t + 1$ threshold of the outer secret-sharing scheme enables reconstruction of the shares of the inner scheme as long as at least $t + 1$ parties participate in the reconstruction. This allows the real-world protocol to proceed with normal interaction rounds as long as less than $m - t$ parties have aborted (as does the ideal-world protocol). If during round i the number of parties that have aborted is at least $m - t$, then an honest majority is guaranteed (since $t < 2m/3$). Thus, in a premature termination in round $i > 1$, the active parties can engage in a fully secure multiparty computation of a reconstruction functionality.

Signatures. In order to confine adversarial strategies to premature aborts, the messages that the parties send are signed (together with the appropriate round number and the index of the sending party), and a verification key is given to all parties. Furthermore, all shares in the inner secret-sharing scheme are signed (as they are used as messages if reconstruction is required). Any message failing to comply with the prescribed protocol is considered an abort message. Since all messages are publicly broadcast, all parties can keep record of all aborts.

The preliminary phase. The goal of the preliminary phase is to compute the MultiShareGen_r functionality, which computes the bits for the underlying sets and the signed shares for the inner and outer secret-sharing schemes. As an honest majority is not guaranteed, it is not possible to implement this functionality by a secure protocol with fairness. That is, we cannot implement an ideal functionality where a trusted party computes the MultiShareGen_r functionality and sends the appropriate output to each party. However, since the outputs of the MultiShareGen_r functionality do not reveal any information regarding the output of the protocol to any subset of size at most t , fairness is not essential for this part of the computation. We use a protocol with cheat detection, that is, if the protocol fails, then at least one corrupt party is identified by all honest parties. The computation is then repeated without the detected malicious parties.

More formally, we compute the MultiShareGen_r functionality using a multiparty protocol that is secure-with-abort and cheat-detection. Informally, this means that we use a protocol that implements the following ideal model: the trusted party computes the MultiShareGen_r functionality and gives the outputs of the corrupted parties to the adversary; the adversary either sends “proceed”, in this case, the trusted party sends the appropriate output to each honest party; otherwise, the adversary sends $\{\text{“abort}_j\}_{j \in J}$ for some non-empty set of indices J of corrupt parties to the trusted party. The trusted party, in turn sends the set J to the honest parties. Using methods from Pass [19], one can obtain a constant-round multiparty protocol secure-with-abort and cheat-detection. See details in Appendix A. Since this protocol is repeated at most $k + 1$ times before an honest majority is guaranteed, the round complexity of the preliminary phase is $O(k)$.

6.1.1 Proving the Feasibility of the Reconstruction

We next claim that Functionality Reconstruction described in Figure 6 is well-defined, that is, if the functionality is computed (after premature termination in round $i > 1$), then, indeed, σ_J^{i-1} can be reconstructed. To see this, observe that the number of parties in the appropriate set Q_J that participate in the computation (i.e., not in D) is at least the reconstruction threshold o_J .

Functionality MultiShareGen_r

Computing default bits and signing keys

1. Choose $w \in \{0, 1\}$ and $i^* \in \{1, \dots, r\}$ uniformly at random.
2. For each $i \in \{1, \dots, r\}$ and for each $J \subset \{1, \dots, k+2\}$,
 - (a) if $i \leq i^* - 1$, then choose independently and uniformly at random $\sigma_J^i \in \{0, 1\}$.
 - (b) if $i \geq i^*$, then set $\sigma_J^i = w$.
3. Compute $(K_{\text{sign}}, K_{\text{ver}}) \leftarrow \text{Gen}(1^n)$.

Computing signed shares of the inner secret-sharing scheme

4. For each $i \in \{1, \dots, r\}$ and for each $J \subset \{1, \dots, k+2\}$
 - (a) Choose random secret shares of σ_J^i in an o_J -out-of- $|Q_J|$ Shamir's secret-sharing scheme for the parties in Q_J . For each party $p_j \in Q_J$, let $S_j^{i,J}$ be its share of σ_J^i .
 - (b) Sign each share $S_j^{i,J}$:

$$R_j^{i,J} \leftarrow (S_j^{i,J}, i, J, j, \text{Sign}((S_j^{i,J}, i, J, j), K_{\text{sign}})).$$

Computing shares of the outer secret-sharing scheme

5. For each $i \in \{1, \dots, r\}$, for each $J \subset \{1, \dots, k+2\}$, and for each $p_j \in Q_J$: share p_j 's signed share $R_j^{i,J}$ using a $(t+1)$ -out-of- m secret-sharing scheme with respect to p_j as defined in Construction 2.7; that is, one masking share $\text{mask}_j(R_j^{i,J})$ and $m-1$ complement shares $(\text{comp}_1(R_j^{i,J}), \dots, \text{comp}_{j-1}(R_j^{i,J}), \text{comp}_{j+1}(R_j^{i,J}), \dots, \text{comp}_m(R_j^{i,J}))$ are produced.

Signing the messages of all parties

6. For every $1 \leq q \leq m$, compute the message $m_{(q,i)}$ that $p_q \in P$ broadcasts in round i by concatenating (1) q , (2) the round number i , and (3) the complement shares $\text{comp}_q(R_j^{i,J})$ for all J and for all $j \neq q$ such that $p_j \in Q_J$ produced in Step (5) for p_q .
7. Compute $M_{q,i} \leftarrow (m_{q,i}, \text{Sign}(m_{q,i}, K_{\text{sign}}))$.

Outputs: Each party p_j receives

- The verification key K_{ver} .
- The messages $M_{j,1}, \dots, M_{j,r}$ that p_j broadcasts during the protocol.
- p_j 's private masks $\text{mask}_j(R_j^{i,J})$ which were produced in Step (5) for each $1 \leq i \leq r$ and each $J \subset \{1, \dots, k+2\}$ such that $p_j \in Q_J$.

Figure 4: The initialization functionality MultiShareGen_r.

An m -party protocol CoinToss_r

Joint input: Security parameter 1^n .

Preliminary phase:

1. $D \leftarrow \emptyset$
2. If $|D| < m - t$,
 - (a) The parties execute a secure-with-abort and cheat-detection protocol computing Functionality $\text{ShareGenWithAbort}_r$.
 - (b) If a set of parties abort, that is, the output of the honest parties is $\{\text{"abort}_j\}_{j \in J}$ for some non-empty set J , then $D \leftarrow D \cup J$ and Step (2) is repeated without the parties whose indices are in D .
3. Else, if $|D| \geq m - t$, then the premature termination step is executed with $i = 1$.

In each round $i = 1, \dots, r$ do:

4. Each party $p_j \in P$ broadcasts $M_{j,i}$ (containing its shares in the outer secret-sharing scheme).
5. For each p_j , if $\text{Ver}(M_{(j,i)}, K_{\text{ver}}) = 0$ or if p_j broadcasts an invalid or no message, then all parties mark p_j as inactive, i.e., set $D \leftarrow D \cup \{j\}$. If $|D| \geq m - t$, then the premature termination step is executed.

Premature termination step

6. If $i = 1$, then the active parties (parties p_j s.t. $j \in D$) use a multiparty secure protocol (with fairness) to toss a fair coin, output the resulting bit, and halt.
7. Otherwise,
 - (a) Each party p_j reconstructs $R_j^{i-1, J}$, the signed share of the inner secret-sharing scheme produced in Step (4) of Functionality MultiShareGen_r , for every $J \subset \{1, \dots, k+2\}$ such that $p_j \in Q_J$.
 - (b) The active parties execute a secure multiparty protocol (with an honest majority) computing Functionality Reconstruction , where the input of each party p_j is $R_j^{i-1, J}$ for every $J \subset \{1, \dots, k+2\}$ such that $p_j \in Q_J$.
 - (c) The active parties return the output of this protocol, and halt.

At the end of round r :

8. Each active party p_j broadcasts the signed shares $R_j^{r, J}$ for each J such that $p_j \in Q_J$.
9. Let J be the lexicographically first set such that at least o_J parties broadcast properly signed shares $R_j^{r, J}$. Each active party reconstructs the bit σ_J^r of J , outputs σ_J^r , and halts.

Figure 5: Protocol CoinToss_r .

Functionality Reconstruction

Joint Input: The indices of inactive parties, D , and the verification key, K_{ver} .

Private Input of p_j : A set of signed shares $R_j^{i-1, J}$ for each $J \subset \{1, \dots, k+2\}$ such that $p_j \in Q_J$.

Computation:

1. For each p_j , if p_j 's input is not appropriately signed or malformed, then $D \leftarrow D \cup \{j\}$.
2. Define the set J :
 - If $|D \cap \{k+2, \dots, m\}| \geq m-t$ then $J = \{1, \dots, k+1\} \setminus D$.
 - If $|D \cap \{k+2, \dots, m\}| < m-t$ then $J = (\{1, \dots, k+1\} \setminus D) \cup \{k+2\}$.
3. Reconstruct σ_J^{i-1} from the shares of the active parties in Q_J .

Outputs: Each honest party p_j outputs the value σ_J^{i-1} , the secret reconstructed from the signed shares $R_j^{i-1, J}$.

Figure 6: The functionality for reconstruction the output bit if the premature termination step is executed in round $i > 1$.

Claim 6.1. *There are at least o_J active parties in Q_J .*

Proof. Let D be the set of aborted parties and J as defined in Functionality Reconstruction.

- If $|D \cap \{k+2, \dots, m\}| \geq m-t$, then $Q_J = \{p_j : j \in \{1, \dots, k\} \setminus D\}$ contains only active parties and $|Q_J| = o_J$. Notice that we already proved that in this case $|Q_J| \geq 1$.
- If $|D \cap \{k+2, \dots, m\}| \leq m-t-1$, then $o_J = |J| - 1 + m-t$ and $|Q_J \setminus \{p_j : j \in D\}| = |J| - 1 + |\{k+2, \dots, m\} \setminus D|$. To prove that $|Q_J \setminus \{p_j : j \in D\}| \geq o_J$, it suffices to show that $|\{k+2, \dots, m\} \setminus D| \geq m-t$:

$$|\{k+2, \dots, m\} \setminus D| \geq (m-k-1) - (m-t-1) = t-k = t - (2t-m) = m-t.$$

□

6.2 The Simulator for the Protocol Without a Dealer

We next prove the security of Protocol CoinToss_r and, hence, the correctness of Theorem 6. As explained in Section 4 we prove that this protocol is a real-world implementation of the (ideal) functionality of Protocol CTWithD_r , the protocol with the on-line dealer. We analyze Protocol CoinToss_r in a hybrid model with the following 3 ideal implementations of the functionalities that we described before:

ShareGenWithAbort $_r$. This is an implementation of Functionality MultiShareGen_r in a secure-with-abort and cheat-detection model. That is, first Functionality MultiShareGen_r is executed. Then, the adversary gets the outputs of the malicious parties of the functionality. Next, the adversary decides whether to halt or to continue; If the adversary decides to continue, it sends a “proceed” message to

Functionality $\text{ShareGenWithAbort}_r$, which sends the honest parties their outputs. Otherwise, the adversary sends $\{\text{“abort}_j\text{”}\}_{j \in J}$ for some set of malicious parties indexed by J , and this message is sent to the honest parties.

FairCoinToss. This is an implementation with full security (with fairness) of the functionality that computes a uniformly distributed random bit. That is, the functionality simply chooses a random bit and gives it to all parties.

Reconstruction. This is an implementation with full security (with fairness) of the functionality described in Figure 6; this functionality is used in the premature termination step in Protocol CoinToss_r for reconstructing the output bit from the shares of the previous round.

When the last two functionalities are executed, an honest majority is guaranteed, hence, these functionalities can be implemented with full security.

We consider an adversary \mathcal{A} in the hybrid model described above, corrupting $t < 2m/3$ of the parties that engage in Protocol CoinToss_r . We next describe a simulator \mathcal{S} interacting with the honest parties in the ideal-world via a trusted party T_{CTWithD} executing Functionality CTWithD_r . The simulator \mathcal{S} runs the adversary \mathcal{A} internally with black-box access. Simulating \mathcal{A} in an execution of the protocol, \mathcal{S} corrupts the same subset of parties as does \mathcal{A} , denoted $B = \{p_{i_1}, \dots, p_{i_t}\}$. At the end of the computation it outputs a possible view of the adversary \mathcal{A} . To start the simulation, \mathcal{S} invokes \mathcal{A} with the auxiliary input aux and the security parameter 1^n .

Simulating the preliminary phase: Upon receiving a “start” message from the ideal-world trusted party T_{CTWithD} , the simulator \mathcal{S} plays the role of Functionality $\text{ShareGenWithAbort}_r$ (possibly, up to $m - t$ times), simulating the interaction between the adversary \mathcal{A} and this functionality.

1. $D \leftarrow \emptyset$.
2. The simulator \mathcal{S} prepares outputs for the corrupted parties in a single interaction with Functionality $\text{ShareGenWithAbort}_r$. The simulator \mathcal{S} does this by first setting $\sigma_j^i = 1$, for all $J \subset \{1, \dots, k + 2\}$ and for all $i \in \{1, \dots, r\}$. Then, \mathcal{S} follows Steps 3–7 in the computation of Functionality MultiShareGen_r (skipping Step (1) and Step (2)) to obtain uniformly distributed shares for the parties.⁴
3. For each $p_j \in B$, the simulator \mathcal{S} sends to \mathcal{A} :
 - The verification key K_{ver} .
 - The masking shares $\text{mask}_j(R_j^{i,J})$ for each $i \in \{1, \dots, r\}$ and for each $J \subset \{1, \dots, k + 2\}$ such that $p_j \in Q_J$.
 - The messages $M_{j,1}, \dots, M_{j,r}$.
4. For each $p_j \in B$, if \mathcal{A} sends an “abort_j” message behalf of p_j to \mathcal{S} , then \mathcal{S} sends “abort_j” to the trusted party T_{CTWithD} .
 - (a) $D \leftarrow D \cup \{j\}$.
 - (b) If $|D| < m - t$, then Steps 2–3 are repeated.
 - (c) Otherwise ($|D| \geq m - t$), the simulator \mathcal{S} executes the simulation of the premature termination step with $i = 1$.

⁴These shares are temporary and will later be opened to the actual bits obtained from T_{CTWithD} during the interaction rounds using the properties of Shamir’s secret-sharing scheme; specifically, they can alternatively be computed by setting σ_j^i to 0.

5. If \mathcal{A} sends a “continue” message to \mathcal{S} (simulating Functionality ShareGenWithAbort $_r$), then \mathcal{S} sends an “abort $_j$ ” message for each party $p_j \in D$ to the trusted party T_{CTWithD} (and gets in return a “proceed” message). The simulator \mathcal{S} denotes by $B_0 = B \setminus D$ the set of active corrupt parties after the simulation of the execution of ShareGenWithAbort $_r$ (from here on, \mathcal{S} will interact with parties in B_0).

Simulating interaction rounds:

Let \mathcal{J}_{B_0} be the collection of subsets $J \subset \{1, \dots, k+2\}$ such that there are enough corrupted parties in Q_J to reconstruct the bits associated with J , i.e., $|B_0 \cap Q_J| \geq o_J$.

To simulate round i for $i = 1, \dots, r$, the simulator \mathcal{S} proceeds as follows:

1. \mathcal{S} gets from the trusted party T_{CTWithD} the bits that the corrupted parties see, that is, \mathcal{S} gets a bit τ_J^i for each $J \in \mathcal{J}_{B_0}$.⁵
2. The simulator \mathcal{S} selects the shares in the inner secret-sharing scheme for corrupted parties: For every $J \in \mathcal{J}_{B_0}$, the simulator \mathcal{S} selects uniformly at random shares of τ_J^i in an o_J -out-of- $|Q_J|$ Shamir secret-sharing scheme. Denote these shares by $\{X_j^{i,J} : p_j \in Q_J\}$. For each $p_j \in Q_J$, let $Y_j^{i,J} \leftarrow (X_j^{i,J}, i, J, j, \text{Sign}((X_j^{i,J}, i, J, j), K_{\text{sign}}))$.
3. The simulator \mathcal{S} selects complementary shares for all honest parties, with the requirement that, together with the masking share $\text{mask}_j(R_j^{i,J})$ held by p_j and the complementary shares held by the corrupted parties in $B_0 \setminus \{p_j\}$, they are a sharing of $Y_j^{i,J}$.

To do so, for every $J \in \mathcal{J}_{B_0}$ and for each $p_j \in B_0$,

- (a) \mathcal{S} calculates $\alpha_j = \text{mask}_j(R_j^{i,J}) \oplus Y_j^{i,J}$.
- (b) \mathcal{S} generates $m-t$ shares of α_j uniformly at random from all possible vectors of $m-t$ shares in a t -out-of- $(m-1)$ Shamir secret-sharing scheme, where together with the t given shares $\{\text{comp}_q(R_j^{i,J}) : p_q \in B \setminus p_j\}$ which were produced in Step (2) in the simulation of the preliminary phase, they will form a secret sharing of $Y_j^{i,J}$. (This is possible according to Construction 2.7, since the shares held by the corrupted parties before round i do not determine the reconstructed value). Denote by $\text{comp}_q(Y_j^{i,J})$ the complementary share that \mathcal{S} generates for the honest party p_q for $p_j \in B_0 \cap Q_J$, where $J \in \mathcal{J}_{B_0}$.
4. For party p_j and subset J , such that, either $p_j \notin B$ or $J \notin \mathcal{J}_{B_0}$, let $\text{comp}_q(R_j^{i,J})$ be the complementary share which was produced in Step (2) in the simulation of the preliminary phase, i.e., $\text{comp}_q(R_j^{i,J})$.
5. Construct signed messages for the honest parties: To construct the message $M'_{q,i}$ for an honest party p_q in round i , the simulator computes $m'_{q,i}$ by concatenating the following strings:
 - (a) q ,
 - (b) The round number i ,
 - (c) The complement shares which were described in Step (4) above,
 - (d) The complement shares $\text{comp}_q(Y_j^{i,J})$ for all $J \in \mathcal{J}_{B_0}$ and for all $j \neq q$ such that $p_j \in Q_J$ produced in Step (3) for p_q .

⁵In Steps 2–5, the simulator \mathcal{S} constructs the messages of the honest parties in order to allow the corrupted parties in each $J \in \mathcal{J}_{B_0}$ to reconstruct $Y_j^{i,J}$.

- Then, \mathcal{S} signs $m'_{q,i}$, i.e., \mathcal{S} computes $M'_{q,i} \leftarrow (m'_{q,i}, \text{Sign}(m'_{q,i}, K_{\text{sign}}))$.
6. The simulator \mathcal{S} sends all the message $M'_{q,i}$ on behalf of each honest party p_q to \mathcal{A} .
 7. If \mathcal{A} sends an invalid or no message on behalf of p_j , then \mathcal{S} sends “abort $_j$ ” to T_{CTWithD} .
 - (a) $D \leftarrow D \cup \{j\}$.
 - (b) If $|D| \geq m - t$, then \mathcal{S} executes the simulation of the premature termination step.
 - (c) Otherwise, the simulator \mathcal{S} proceeds to the next round.

Simulating the premature termination step:

- If $i = 1$, \mathcal{S} simulates \mathcal{A} 's interaction with Functionality FairCoinToss as follows:
 1. The simulator \mathcal{S} gets from T_{CTWithD} the bit σ . Then, \mathcal{S} sends σ to \mathcal{A} .
- If $i > 1$, \mathcal{S} simulates \mathcal{A} 's interaction with Functionality Reconstruction as follows:
 1. \mathcal{S} receives from \mathcal{A} the inputs of the active malicious parties in the current round denoted by B_i . If the input for $p_j \in B_i$ is not appropriately signed, then \mathcal{S} sends “abort $_j$ ” to T_{CTWithD} .
 2. \mathcal{S} gets from T_{CTWithD} a bit σ and sends it to \mathcal{A} .

Simulating normal termination at the end of round r :

1. The simulator gets w from the trusted party T_{CTWithD} .
2. \mathcal{S} constructs all the signed shares of the inner secret-sharing scheme for each $J \subset \{1, \dots, k+2\}$ and for each honest party $p_j \in Q_J$ as follows.
 - For each $J \in \mathcal{J}_{B_0}$, the signed shares $\{Y_j^{r,J}\}$ were produced in Step (2) of the simulation of the interaction of round r .
 - For each $J \notin \mathcal{J}_{B_0}$, the simulator \mathcal{S} selects uniformly at random secret shares of w in an o_J -out-of- $|Q_J|$ Shamir secret-sharing scheme. Denote these shares by $\{X_j^{r,J}\}$. For each share $X_j^{r,J}$, the simulator concatenates the corresponding identifying details, and signs them to obtain: $Y_j^{r,J} \leftarrow (X_j^{r,J}, r, J, j, \text{Sign}((X_j^{r,J}, r, J, j), K_{\text{sign}}))$. ($Y_j^{r,J}$ is the share of inner secret-sharing scheme that party $p_j \in Q_J$ should reconstruct.)
3. For each honest party p_j , the simulator \mathcal{S} sends to \mathcal{A} the shares $Y_j^{r,J}$ for all subsets J , such that $p_j \in Q_J$.

The Output of the Simulator: The simulator \mathcal{S} outputs $r_{\mathcal{A}}$ and the sequence of messages exchanged between \mathcal{S} and the adversary \mathcal{A} and halts.

6.3 Proof of the Correctness of the Simulation

In this section we prove that the simulator described in Section 6.2 simulates Protocol CoinToss $_r$ described in Section 6.1. We consider the two random variables from Section 2.1, both of the form (V, C) , where V describes a possible view of \mathcal{A} , and C describes a possible output of the honest parties (i.e., $C \in \{0, 1\}$). The first random variable $\text{REAL}_{\text{CoinToss}_r, \mathcal{A}(\text{aux})}(1^n)$ describes the hybrid world, i.e., an execution of the CoinToss $_r$ protocol in the above hybrid model, where V describes the view of the adversary \mathcal{A} in this execution, and C is the output of the honest parties in this execution. The second random variable

$\text{IDEAL}_{\text{CTWithD}_r, \mathcal{S}(\text{aux})}(1^n)$ describes the output of the simulator and the honest parties in the ideal world, i.e., an execution of the r -round m -party coin-tossing protocol CTWithD_r with an on-line dealer, where V describes the output of the simulator \mathcal{S} in this execution, and C is the output of the honest parties in this execution. Our goal here is to show that these two random variables are computationally indistinguishable. For the rest of this proof, we simplify notations by omitting all notations that are clear from the context; we denote the above two random variables by $\text{REAL} = (V_{\text{REAL}}, C_{\text{REAL}})$ and $\text{IDEAL} = (V_{\text{IDEAL}}, C_{\text{IDEAL}})$ respectively.

In the main part of the proof, we assume that an adversary never (successfully) forges a signature during the execution of the protocol. Under this assumption, we prove that these two random variables are identically distributed. Since the probability that a polynomial-time adversary successfully forges a signature for any message is negligible, the statistical distance between these two random variables is negligible.

We next prove a slightly stronger claim than the one required for the above. We define two random variables $\text{KEY}_{\text{IDEAL}}$ and KEY_{REAL} representing the sequence of signing keys used in the (repeated) iterations of the preliminary phase to sign the messages. That is, $\text{KEY}_{\text{IDEAL}}$ represents the sequence of keys that were used to sign the messages in the computations of Step (2) during a given execution of the simulation, and KEY_{REAL} represents the sequence of keys that were used to sign the messages in the computations of Step (2a) (i.e., in the executions of $\text{Functionality ShareGenWithAbort}_r$) in an execution of Protocol CoinToss_r in the hybrid model. The signing keys are not part of the view of the adversary. However, we prove that the joint distributions $(V_{\text{REAL}}, C_{\text{REAL}}, \text{KEY}_{\text{REAL}})$ and $(V_{\text{IDEAL}}, C_{\text{IDEAL}}, \text{KEY}_{\text{IDEAL}})$ are identically distributed makes the proof easier and at the same time proves that REAL and IDEAL are identically distributed. Let $\text{REAL}_+ = (V_{\text{REAL}}, C_{\text{REAL}}, \text{KEY}_{\text{REAL}})$ and $\text{IDEAL}_+ = (V_{\text{IDEAL}}, C_{\text{IDEAL}}, \text{KEY}_{\text{IDEAL}})$

We consider $r+2$ random variables $\{\text{REAL}_+^i\}_{i=0}^{r+1}$ that are related to REAL_+ and $r+2$ random variables $\{\text{IDEAL}_+^i\}_{i=0}^{r+1}$ that are related to IDEAL_+ . Each of these variables describes a partial execution until a given point as next described. The random variable REAL_+^0 describes the execution until the end of the preliminary phase in Protocol CoinToss_r . That is, REAL_+^0 contains the sequence of signing keys used by $\text{Functionality ShareGenWithAbort}_r$ and the messages that the adversary \mathcal{A} exchanges with $\text{Functionality ShareGenWithAbort}_r$ in all interactions with this functionality, i.e., until either \mathcal{A} sends a “continue” message to the functionality or until \mathcal{A} interacts with the functionality $m-t$ times. Similarly, the random variable IDEAL_+^0 describes the simulation until the end of all iterations of the simulator \mathcal{S} . In addition, IDEAL_+^{r+1} contains the interaction in the termination step (normal or premature termination).

For each $1 \leq i \leq r$, the random variable REAL_+^i contains REAL_+^{i-1} and, in addition, contains the messages that were exchanged between the adversary (i.e., the corrupt parties) and the honest parties in the first i rounds in the execution of Protocol CoinToss_r (in the hybrid model). Specifically, if the execution was prematurely terminated before round i , then we have $\text{REAL}_+^i = \text{REAL}_+^{i-1}$. The random variable IDEAL_+^i is defined similarly.

We next prove that for every $0 \leq i \leq r+1$, it holds that $\text{REAL}_+^i \equiv \text{IDEAL}_+^i$ (assuming no signatures were forged). Towards this goal, we fix a triple $\lambda = (v, c, \text{keys})$ and show that for every i the probability that IDEAL_+^i is consistent with λ is the same as the probability that REAL_+^i is consistent with λ , where in both scenarios we invoke \mathcal{A} on 1^n and the same auxiliary information aux .

6.3.1 The Initialization Step

Claim 6.2. *The probability that REAL_+^0 is consistent with λ is the same as the probability that IDEAL_+^0 is consistent with λ .*

Proof. The simulator follows the same procedure for generating the messages to malicious parties as the pro-

tol with one main difference. The simulator chooses $\sigma_j^i = 1$ for every $1 \leq i \leq r$ and $J \subset \{1, \dots, k+2\}$, while the protocol chooses these values at random. However, since the adversary holds at most t shares of Construction 2.7 with threshold $t+1$ (i.e., $\alpha = t+1$), these shares are uniformly distributed (as explained in Construction 2.7). Thus, the probability that the adversary sees the shares that are consistent with λ is the same in both models. As the messages sent to the corrupt parties are these shares and signatures on these shares, the claim follows. \square

6.3.2 The Interaction Rounds of the Protocol

Claim 6.3. *For every $1 \leq i \leq r$, the following two probabilities are equal:*

1. *The probability that REAL_+^i is consistent with λ , given that REAL_+^{i-1} is consistent with λ .*
2. *The probability that IDEAL_+^i is consistent with λ , given that IDEAL_+^{i-1} is consistent with λ .*

Proof. If the execution prematurely terminated in round $i' < i$, then there is nothing to prove, since by definition $\text{REAL}_+^i = \text{REAL}_+^{i-1}$ and $\text{IDEAL}_+^i = \text{IDEAL}_+^{i-1}$ and, thus, are both consistent with λ . We, thus, assume that round i does take place according to λ .

The messages broadcast in round i are shares in the outer secret-sharing schemes of shares generated in an inner secret-sharing scheme. We will show that in both worlds the hybrid and the ideal, they are distributed according to the same distribution. The selection of the bits that the adversary sees throughout the protocol is done by the same process in the real protocol (by Functionality $\text{ShareGenWithAbort}_r$) and in the ideal world (by the trusted dealer). That is, in both cases the bits are selected according to the algorithm defined by Functionality MultiShareGen_r . Furthermore, given these values, each bit is shared independently of the other bits. Thus, it suffices to show that the outer shares of each bits are equally distributed. There are two cases:

Outer shares for $p_j \in B_0$ and $J \in \mathcal{J}_{B_0}$ such that $j \in J$. Since $J \in \mathcal{J}_{B_0}$, the simulator gets from the trusted party T_{CTWithD} a bit τ_j^i . The simulator then generates shares of an inner secret-sharing scheme of τ_j^i , and signs these shares. Let $Y_j^{i,J}$ be the share of p_j generated by the simulator. This share is generated in the same process as the share $R_j^{i,J}$ generated in the hybrid world.

The process of generating the outer shares of $Y_j^{i,J}$ and $R_j^{i,J}$ (in the ideal and hybrid worlds, respectively) are different. In the hybrid world, $R_j^{i,J}$ is simply shared using Construction 2.7. In the ideal world, the values $\left\{ \text{comp}_q(R_j^{i,J}) \right\}_{q \in B_0}$ (generated in the simulation of the preliminary phase) are completed to m shares of Construction 2.7 for the secret $Y_j^{i,J}$. As explain in Construction 2.7, the set of at most t shares of $R_j^{i,J}$ are uniformly distributed, thus, their completion to shares of $Y_j^{i,J}$ is distributed as in the hybrid world.

Outer shares for $p_j \notin B_0$ or $J \notin \mathcal{J}_{B_0}$. In the hybrid world, first the value σ_j^i is chosen in Functionality MultiShareGen_r , then this bit is shared in an inner secret-sharing scheme, signed, and shared in an outer secret-sharing scheme. In the ideal world, the process is similar, however for the fixed bit $\sigma_j^i = 1$.

- If $p_j \notin B_0$, the parties in B_0 miss the mask of $R_j^{i,J}$, thus, they cannot see the difference between the worlds.

- If $J \notin \mathcal{J}_{B_0}$, the number of parties in B_0 that are in Q_J is less than o_J . Thus, the inner shares given to the parties in $Q_J \cap B_0$ are equally distributed regardless of the value of σ_J^i , and their sharing in the outer secret-sharing scheme is equally distributed in the two worlds.

□

6.3.3 The Termination Step

We next prove that the random variables are equal distributed after the termination (premature or normal).

Claim 6.4. *The following two probabilities are equal:*

1. *The probability that REAL_+ is consistent with λ , given that REAL_+^r is consistent with λ .*
2. *The probability that IDEAL_+ is consistent with λ , given that IDEAL_+^r is consistent with λ .*

Proof. We need to show that the probabilities that in both worlds, the view of the adversary \mathcal{A} in the termination phase is consistent with λ , given that everything else was consistent with λ is the same in both worlds. We consider the two possible types of termination of the protocol, i.e., normal termination and premature termination. The type of termination we need to consider is specified by the view in previous round, and, thus, by REAL_+^r and IDEAL_+^r . We therefore consider the type of termination as specified by λ .

A premature termination step is specified by λ : We separate the analysis of this case into two. First, we consider the case where the premature termination takes place with $i = 1$. Then, we consider the case where $1 < i \leq r$. In all of the cases discussed below, at least $m - t$ parties have aborted, and, thus, an honest majority is guaranteed among the remaining active parties.

1. The case where $i = 1$ (according to λ): This case is possible either if according to λ at least $m - t$ parties aborted in the preliminary phase, or $m - t$ parties aborted before the first round of interaction was completed. By the definition of the premature termination step, in both the above cases the output of the protocol should be a uniform bit that is independent of the view of the adversary. We next claim that both in the hybrid-world and in the simulation this is indeed the case. Formally, we claim that given that the view of the adversary (as well as the sequence of signing keys used thus far) is consistent with λ , the view of the adversary in the premature termination step will consist of an independent uniform bit, both in the hybrid-world and in the simulation.

In the premature termination step in the hybrid-world, the parties use Functionality FairCoinToss to toss a completely fair coin (the functionality is fully secure, since an honest majority is guaranteed). In the simulation, the simulator \mathcal{S} sends an “ abort_j ” message to T_{CTWithD} for each party p_j that has aborted. Then, \mathcal{S} gets in return a bit σ and sends σ to \mathcal{A} . By the definition of Protocol CTWithD_r the trusted party will return $w' = \sigma_J^0$, for some subset J . Regardless of the exact subset, this bit is uniform and independent of the view of the adversary.

2. The case where $1 < i \leq r$ (according to λ): We claim that in this case the output of the honest parties (which is part of the view of the adversary) is the bit σ_J^{i-1} for the same subset J , both in the hybrid-world and in the simulation. This suffices, since, as explained above, the bits for all sets are selected according to the same randomized process in the real protocol (by Functionality $\text{ShareGenWithAbort}_r$) and in the ideal world (by the trusted dealer). Thus, given that the same view for the adversary.

In the hybrid-world the parties execute Functionality Reconstruction to select the appropriate bit (the functionality is fully secure, since an honest majority is guaranteed). In the ideal-world the simulator turns to the trusted party T_{CTWithD} for the output bit. In both cases the bit is chosen according to the set D of parties that have aborted the protocol. By the assumption that the view of \mathcal{A} thus far is consistent with λ , we have that the set of parties that aborted before the beginning of the premature termination step has started is the same in both models. Now, all inputs of corrupted party to Functionality Reconstruction are going to be the same both in the hybrid-world and in the simulation, thus the subset D is going to be the same. Therefore, in both models, the same subset J is going to be selected, and by the above the same bit is going to be output.

A normal termination step is specified by λ : According to specification of Protocol CTWithD_r , at the end of round r , each party p_j broadcasts its (signed) shares in the inner secret-sharing scheme of round r , i.e., party p_j broadcasts $R_j^{r,J}$ for each J such that $p_j \in Q_J$. Thus, in the hybrid-world, the adversary will receive the shares sent by the honest parties. To simulate the messages sent by the honest parties in this case, the simulator \mathcal{S} gets the output bit w from the trusted party T_{CTWithD} , selects inner secret-sharing schemes shares for the honest parties, and signs them. That is, \mathcal{S} selects for each honest party p_j shares $\{X_j^{r,J}\}$ and then concatenates the corresponding identifying details, and signs them to obtain: $\{Y_j^{r,J}\}$.

We first consider the issue of the output bit itself. If $\mathcal{J}_{B_0} \neq \emptyset$, that is, that the adversary learns at least on bit in each round, then, the output bit is completely determined by the bits that the adversary saw in round r , hence, REAL_+^r and IDEAL_+^r both determine (and agree on) the output bit w . If the adversary does not learn any bit in round r (and, hence, it does not learn any bits in any other round), then the view of the adversary is completely independent of the value of the output bit and since in both worlds this bit is uniform, the probability that it agrees with λ is $1/2$ in both cases. We continue our analysis under the added assumption that (besides REAL_+^r and IDEAL_+^r being consistent with λ) the prescribed output bit in both worlds is consistent with λ . That is, we assume that the bit w that \mathcal{S} obtains from the trusted party T_{CTWithD} is the same as the output bit selected by Functionality $\text{ShareGenWithAbort}_r$, and that they both agree with λ .

We next claim that, under the above assumptions (i.e., conditioned on REAL_+^r and IDEAL_+^r and the output bit in each model being consistent with λ), the probability in both models that the adversary sees shares (of honest parties) for the output bit that are according to λ is the same.

By the properties of the Shamir secret-sharing scheme, the shares of the honest parties for the bits of sets $J \in \mathcal{J}_{B_0}$ (i.e., such that $|B_0 \cap Q_J| \geq o_J$) are completely determined by the view of the adversary, i.e., by the shares of corrupted parties for these bits (since any o_J shares determine the other shares in a Shamir o_J -out-of- $|Q_J|$ scheme). Hence, these shares will be consistent with λ with probability 1 in both models. We note that this is true, since the selection of shares is done properly in both model (specifically, in the simulation, the shares for honest parties $X_j^{r,J}$ were already selected in the simulation of round r , together with the shares of corrupt parties). For the remaining shares, i.e., shares of honest parties p_j and subsets $J \notin \mathcal{J}_{B_0}$, we argue that in both models they are uniformly distributed over all shares that complete those seen by the adversary to be secret shares of the output bit w . This is true since the shares that the adversary sees for this bit do not determine its value and by the properties of secret-sharing schemes. Thus, the distribution over the shares of honest parties p_j and subsets $J \notin \mathcal{J}_{B_0}$ is the same in both models, given the set of shares of corrupt parties for these bits. That is, selecting uniformly over all shares $X_j^{r,J}$ for honest parties that complete the shares of the

corrupt parties (that are specified by λ) to a sharing of w (as is done by the simulator) is equivalent to uniformly selecting shares for all parties to share w and then selecting uniformly over those for which the shares of corrupt parties agree with λ (which describes the distribution over shares, implied by the hybrid world random variable).

□

So far we considered the correctness of the simulation process, however, in order to argue that the protocol is secure, we also need to verify that the simulator can be implemented by a probabilistic polynomial time algorithm.

Claim 6.5. *The simulator \mathcal{S} described in Section 6.2 is efficient.*

Proof. The simulator uses the adversary in a black-box manner and never rewinds it. Furthermore, the adversary is itself a (non-uniform) polynomial time algorithm. In addition, the simulator needs to simulate Functionality ShareGenWithAbort _{r} (at most $m - t$ times), Functionality FairCoinToss (at most once), and Functionality Reconstruction (at most once). Since all three functionalities are polynomial time computable, their simulation is efficient. Finally, in each round of the simulation (and in the simulation of a normal termination) the simulator needs to select uniformly distributed shares of the bits that the adversary sees. By the properties of Construction 2.7, this can be done efficiently. In addition, in each round of the simulation the simulator needs to sign these shares, which is also done in polynomial time since it holds the signing key. Hence, since the number of rounds in the protocol is polynomial in the security parameter, the whole simulation is done in polynomial time. □

6.3.4 Concluding the Proof of Theorem 6

Proof of Theorem 6. In Section 6.2 we describe an ideal-world simulator for any hybrid-world adversary for Protocol CoinToss _{r} . By the combination of Claim 6.2, Claim 6.3, and Claim 6.4, we have that, under the assumption that the adversary never forges a signature of a message, for any pair (v, c) , the probability that REAL = (v, c) is exactly the same as the probability that IDEAL = (v, c) . That is, the global outputs of an ideal-world simulation and of a hybrid-world execution are identically distributed under the above assumption. As explained above, since the probability to successfully forge a signature for any message is negligible, the above two random variables are statistically close even without the assumption that the adversary never successfully forges any message. Finally, by Claim 6.5 the simulator is efficient.

The analysis of the security of Protocol CoinToss _{r} is done relative to an hybrid-world with three ideal functionalities. By Canetti [7], the real-world protocol is computationally secure if computationally secure real-world implementations of these three functionalities exist. This is indeed the case if enhanced trap-door permutations exist. Hence, the theorem follows. □

7 Coin-Tossing Protocol for any Constant Fraction of Corrupted Parties

In this section we describe a coin-tossing protocol for any constant fraction of corrupted parties. More specifically, we prove the following theorem:

Theorem 7. *If enhanced trap-door permutations exist, then for any m , t , and $r = r(n)$ such that $t = (1 - \epsilon)m$, for some constant $\epsilon > 0$, there is an r -round m -party coin-tossing protocol tolerating up to t malicious parties that has bias $O(\epsilon/\sqrt{r-t})$.*

Before our work, the best known protocol for this scenario is an extension of Blum’s two-party coin-tossing protocol [5] to an r -round m -party protocol that has bias $O(t/\sqrt{r-t})$ [2, 9]. In this protocol, in each round i of the protocol, the parties jointly select a random bit σ_i in two phases. In the first phase, each party commits to a “private” random bit, and in the second phase the private bits are all revealed and the output bit σ_i is taken to be the XOR of all private bits. The output of the whole protocol is the value of the majority of the σ_i ’s. When there is an abort in round i , the remaining parties repeat the computation of round i and continue with the prescribed computation.

Intuitively, the best strategy for a rushing adversary to bias the output of the protocol, say toward 0, is in each round i to instruct a corrupted party to abort before the completion of the revealing phase if $\sigma_i = 1$. This is possible, since the rushing adversary learns σ_i before the completion of round i , specifically, a corrupted party can delay its message until all honest parties reveal their bit. This can go on at most t times, resulting in a total bias of $O(t/\sqrt{r})$, whenever $r = \Omega(t^2)$.

In our protocol, we follow the general structure of the above protocol in computing the σ_i ’s and taking the majority over them. However, we compute each σ_i using a secure-with-abort and cheat-detection protocol, such that either the computation is completed or at least a constant fraction of the malicious parties abort (specifically, $m - t$ malicious parties abort).

Next, we briefly describe the computation of each σ_i in our protocol. We show how to obtain a constant round secure-with-abort and cheat-detection protocol to compute a random bit that identifies at least $m - t$ cheating parties. Let m_i be the number of active parties at the beginning of round i and t_i be an upper bound on the number of active corrupted parties at the beginning of round i (that is, if t' parties have aborted in rounds $1, \dots, i-1$, then $t_i = t - t'$). We use a constant round secure-with-abort and cheat-detection protocol. Such a protocol exists under reasonable cryptographic assumptions. More details appear in Appendix A.

In the first phase, a preprocessing phase, the active parties execute a constant round secure-with-abort and cheat-detection protocol to compute a $(t_i + 1)$ -out-of- m_i secret-sharing of a random bit σ_i . That is, at the end of this phase, each party holds a share in a $(t_i + 1)$ -out-of- m_i Shamir secret-sharing scheme of σ_i . To confine adversarial strategies to aborts, the share that each party receives is signed and a verification key is given to all parties. In a second phase, a revealing phase, all parties reveal their shares and reconstruct σ_i . In the reveal phase, broadcasting anything other than a signed share is treated as abort. The formal description of the protocol appears in Figure 7.

To see that the above protocol achieves the required properties, observe that after the first phase the adversary cannot reconstruct σ_i . Thus, by aborting the preprocessing round, malicious parties cannot bias the output. We stress that they are able to cause the preprocessing phase to fail, at the cost of at least one malicious party being detected by all honest parties. In such a case, the preprocessing stage is repeated without the detected party. This, however, can only happen at most t times in total, throughout the whole protocol.

In the revealing phase, a rushing adversary can learn σ_i before the corrupted parties broadcast their messages and, thus, can bias the output by not broadcasting these messages. However, by the properties of the secret-sharing scheme, at least $m_i - t_i$ parties have to not broadcast their message, and, hence, effectively abort the computation. Hence, the adversary cannot do this too many times throughout the protocol. The next claim bounds the maximal number of bits the adversary could learn during the protocol.

Claim 7.1. *The adversary can prevent the honest parties of learning the bit of $\frac{1-2\epsilon}{\epsilon}$ rounds throughout the protocol, before an honest majority among active parties is guaranteed.*

Proof. In each round i , a $(t_i + 1)$ -out-of- m_i Shamir secret-sharing scheme is used for sharing σ_i , therefore, $m_i - t_i$ parties have to abort in order to prevent revealing σ_i . However, recall that $m_i - t_i$ is a lower bound on

An m -party protocol CoinTossForConstantFraction $_r$

Joint input: The security parameter 1^n , a polynomial $r = r(n)$ specifying the number of rounds in the protocol, and a bound t on the number of corrupted parties.

Initialization: $D \leftarrow \emptyset, t_1 \leftarrow t, m_1 \leftarrow m$.

In each round $i = 1, \dots, r$ **do:**

1. If $t_i \geq m_i/2$,
 - (a) The parties execute a secure-with-abort and cheat-detection protocol computing a functionality in which:
 - Signing and verification keys are produced.
 - A bit σ_i is uniformly selected at random.
 - The bit σ_i is shared in a $(t_i + 1)$ -out-of- m_i Shamir secret-sharing.
 - Each share is signed using the signing key.

The output of each party is a signed share and the verification key.

 - i. If some parties abort, that is, the output of the honest parties is $\{\text{"abort}_j\}_{j \in J}$ for some non-empty set J , then $t_i \leftarrow t_i - |J|$, $m_i \leftarrow m_i - |J|$, $D \leftarrow D \cup J$, and Step (1) is repeated without the parties whose indices are in D .
 - ii. Else (no party aborts), set $t_{i+1} \leftarrow t_i$ and $m_{i+1} \leftarrow m_i$.
 - (b) The parties reconstruct the bit for this round σ_i that was secret-shared in Step (1a):
 - i. Each active party broadcasts its signed share.
 - ii. If $t_i + 1$ valid shares were sent, each party reconstructs σ_i .
 - iii. Else, a set of aborted parties indexed by J is identified and
 - A. Set $t_i \leftarrow t_i - |J|$, $m_i \leftarrow m_i - |J|$, and $D \leftarrow D \cup J$.
 - B. Step (1) is repeated without the parties whose indices are in D .
2. Else ($t_i < m_i/2$), the active parties execute a secure multiparty protocol with an honest majority in which the bits $\sigma_i, \dots, \sigma_r$ are selected uniformly at random and each active party receives them. Next, the active parties skip to the end of round r .

At the end of round r : Each active party calculates the majority of the reconstructed bits, outputs it, and halts.

Figure 7: Protocol CoinTossForConstantFraction $_r$.

the number of honest parties, therefore, $m - t = m_1 - t_1 = m_2 - t_2 = \dots$. Hence, the number of parties that have to abort in order to prevent the honest parties from reviling the bit is constant and equal to the number of honest parties. Assume that the adversary prevented the honest parties of learning the bits of x rounds and after it an honest majority is achieved. Therefore, $t_x \leq t - x(m - t)$ and $m_x \leq m - x(m - t)$. By solving the next inequality,

$$t - x(m - t) \leq \frac{m - x(m - t)}{2}$$

we get and the number of these rounds is bounded by $\frac{1-2\epsilon}{\epsilon}$. \square

Thus, the majority function is applied to $\Omega(r - t)$ random bits, of which the adversary can bias $\frac{1-2\epsilon}{\epsilon}$. Thus, the total bias of the protocol is $O\left(\frac{1}{\epsilon\sqrt{r-t}}\right)$.

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References

- [1] Y. Aumann and Y. Lindell. Security against covert adversaries: Efficient protocols for realistic adversaries. In S. Vadhan, editor, *Proc. of the Fourth Theory of Cryptography Conference – TCC 2007*, volume 4392 of *Lecture Notes in Computer Science*, pages 137–156. Springer-Verlag, 2007.
- [2] B. Averbuch, M. Blum, B. Chor, S. Goldwasser, and S. Micali. How to implement Bracha’s $O(\log n)$ Byzantine agreement algorithm, 1985. Unpublishe manuscript.
- [3] D. Beaver, S. Micali, and P. Rogaway. The round complexity of secure protocols. In *Proc. of the 22nd ACM Symp. on the Theory of Computing*, pages 503–513, 1990.
- [4] A. Beimel, Y. Lindell, E. Omri, and I. Orlov. $1/p$ -secure multiparty computation without honest majority and the best of both worlds. In P. Rogaway, editor, *Advances in Cryptology – CRYPTO 2011*, volume 6841 of *Lecture Notes in Computer Science*, pages 277–296. Springer-Verlag, 2011.
- [5] M. Blum. Coin flipping by telephone a protocol for solving impossible problems. *SIGACT News*, 15(1):23–27, 1983.
- [6] M. Blum and S. Micali. How to generate cryptographically strong sequences of pseudo-random bits. *SIAM J. on Computing*, 13:850–864, 1984.
- [7] R. Canetti. Security and composition of multiparty cryptographic protocols. *J. of Cryptology*, 13(1):143–202, 2000.
- [8] R. Canetti, Y. Lindell, R. Ostrovsky, and A. Sahai. Universally composable two-party and multi-party secure computation. In *Proc. of the 34th ACM Symp. on the Theory of Computing*, pages 494–503, 2002.
- [9] R. Cleve. Limits on the security of coin flips when half the processors are faulty. In *Proc. of the 18th ACM Symp. on the Theory of Computing*, pages 364–369, 1986.

- [10] R. Cleve and R. Impagliazzo. Martingales, collective coin flipping and discrete control processes. Manuscript, 1993.
- [11] O. Goldreich. *Foundations of Cryptography, Volume II Basic Applications*. Cambridge University Press, 2004.
- [12] O. Goldreich, S. Micali, and A. Wigderson. How to play any mental game. In *Proc. of the 19th ACM Symp. on the Theory of Computing*, pages 218–229, 1987.
- [13] S. Goldwasser and Y. Lindell. Secure computation without agreement. In *DISC '02: Proceedings of the 16th International Conference on Distributed Computing*, pages 17–32. Springer-Verlag, 2002.
- [14] S. D. Gordon and J. Katz. Partial fairness in secure two-party computation. In Henri Gilbert, editor, *Advances in Cryptology – EUROCRYPT 2010*, volume 6110 of *Lecture Notes in Computer Science*, pages 157–176. Springer-Verlag, 2010.
- [15] Y. Ishai, R. Ostrovsky, and H. Seyalioglu. Identifying cheaters without an honest majority. In *Proc. of the Ninth Theory of Cryptography Conference – TCC 2012*, pages 21–38, 2012.
- [16] J. Katz. On achieving the “best of both worlds” in secure multiparty computation. In *Proc. of the 39th ACM Symp. on the Theory of Computing*, pages 11–20, 2007.
- [17] Y. Lindell. Parallel coin-tossing and constant-round secure two-party computation. *J. of Cryptology*, 16(3):143–184, 2003.
- [18] T. Moran, M. Naor, and G. Segev. An optimally fair coin toss. In *Proc. of the Sixth Theory of Cryptography Conference – TCC 2009*, pages 1–18, 2009.
- [19] R. Pass. Bounded-concurrent secure multi-party computation with a dishonest majority. In *Proc. of the 36th ACM Symp. on the Theory of Computing*, pages 232–241, 2004.
- [20] T. Rabin and M. Ben-Or. Verifiable secret sharing and multiparty protocols with honest majority. In *Proc. of the 21st ACM Symp. on the Theory of Computing*, pages 73–85, 1989.
- [21] A. Shamir. How to share a secret. *Communications of the ACM*, 22:612–613, 1979.

A On Achieving Cheat Detection

As discussed in Section 2.2, a key requirement in all our constructions is the existence of constant-round secure with abort protocols with the ability to detect a cheating party. That is, the protocol we use does not only guarantee security with abort, but that in case of abort (or any cheating for that matter), honest parties will be able to detect at least one cheating party. Standard definitions of secure multiparty computation with abort do not guarantee the ability to detect cheating parties. That is, they require that honest parties are notified by the protocol upon an abort, however, there is no concept of pointing out a cheating party. Such a definition of cheat detection was given by Aumann and Lindell [1] in their work on the covert adversary model. Roughly speaking, their definition requires that in the ideal world upon abort the trusted party gives all honest parties an ID of one of the corrupted parties (as chosen by the adversary). The formal definition is given in Section 2.2.

A generic way of obtaining secure protocols is taking a protocol that is secure against a semi-honest adversary and compiling it into a protocol that is secure in the malicious model. Goldreich, Micali, and Wigderson [12] offer a specific paradigm for achieving such compilation. Very roughly, their technique is based on each party proving to the other parties in zero-knowledge that every step it makes is as prescribed in the protocol. In our setting, any protocol obtained by this compilation, will have the ability to detect cheating if the zero-knowledge sub-protocols it uses have two properties:

Perfect completeness. The verifier has no way to incriminate an honest prover.

Public coins. Any party (not only the verifier) can verify the correctness of a proof, given the transcript of messages.

Recall that we assume that all communication is sent on an authenticated broadcast channel. Thus, any party that cheats must fail to prove its correctness using a zero-knowledge proof. By the above public coin property, all other parties can verify that the proof is incorrect, and, hence, detect this party's cheating. By the perfect correctness, a malicious verifier cannot incriminate the prover.

For our needs, it is essential that the protocol has *constant number of rounds*. One can obtain a multi-party constant-round secure-with-abort protocol with the ability to detect cheating by using methods from Pass [19]. Pass presents a construction of bounded concurrent simulation-sound zero-knowledge protocols (that is, the soundness of each one of the protocols is preserved even when all the other protocols are simulated at the same time with the role of the prover and verifier reversed). He shows how these protocols can be used to realize the ideal one-to-many zero-knowledge proof of knowledge functionality, $\text{IDEAL}_{ZK}^{1:M}$ (that is, a functionality where a single prover should prove to all other parties the correctness of some statement). Realizing this functionality, Pass can apply the compilation technique of Canetti, Lindell, Ostrovsky, and Sahai [8] (which, in turn, follows the paradigm of [12]) to the constant-round semi-honest protocol of [3]. Thus, Pass's results [19] imply a constant-round multiparty secure-with-abort computation for any feasible functionality. However, this protocol does not immediately imply the ability to detect cheating.

Pass [19] considers a setting where the parties are connected through a point-to-point asynchronous public network. To realize the $\text{IDEAL}_{ZK}^{1:M}$ functionality, he reduces the usage of the $\text{IDEAL}_{ZK}^{1:M}$ functionality to the usage of the more standard two-party ideal zero-knowledge proof of knowledge functionality, IDEAL_{ZK} (i.e., a functionality that models interaction between a single prover and single verifier). The idea underlying this reduction is similar to the idea used by Goldwasser and Lindell [13] to implement a broadcast channel in a similar setting. However, as is the case with the protocol of [13], this step does not yield the ability to detect cheating since a corrupted party can cause the protocol to terminate without revealing its identity. Fortunately, we can overcome this problem by using a broadcast channel as well as a synchronous network, thus, simplifying the reduction to IDEAL_{ZK} to requiring each party to prove in zero-knowledge the same statement to each of the other parties (while other parties monitor the interaction). Furthermore, to verify that the resulting protocol has the ability to detect cheating, it suffices to observe that the zero-knowledge protocols of Pass satisfy the two above-mentioned properties of perfect completeness and public coins.⁶

⁶Canetti et al. [7] use common random string. As we use a broadcast channel and we do not require universally composable security, we do not need to use such a string.