Generating Pairing-friendly Parameters for the CM Construction of Genus 2 Curves over Prime Fields

Kristin Lauter and Ning Shang

Abstract. We present two contributions in this paper. First, we give a quantitative analysis of the scarcity of pairing-friendly genus 2 curves. This result is an improvement relative to prior work which estimated the density of pairing-friendly genus 2 curves heuristically. Second, we present a method for generating pairing-friendly parameters for which $\rho \approx 8$, where ρ is a measure of efficiency in pairing-based cryptography. This method works by solving a system of equations given in terms of coefficients of the Frobenius element. The algorithm is easy to understand and implement.

1 Introduction

In order to use the Jacobian variety of a curve over a finite field for discrete logarithm based cryptography, suitable parameters must be chosen, and a curve with those parameters must be found. One such parameter is the underlying finite field $\mathbb{F}_p$ over which the curve is defined. Another important parameter is the cardinality N of the group of $\mathbb{F}_p$-rational points on the Jacobian of the curve. For many implementations of discrete logarithm based cryptographic protocols, $\mathbb{F}_p$ is a prime field, i.e., p is a prime number, and N is prime number or a prime times a small cofactor, to resist the Pohlig-Hellman attack [19] on the discrete logarithm problem. Pairing-based cryptography poses further restrictions on the curves since in addition a small embedding degree is required.

Genus 2 point-counting methods ([12], [10]) choose random curve equations over a finite field and compute the number of points on the Jacobian of the curve until one that is good for discrete logarithm-based cryptography is found. An alternative to point counting is to use the genus 2 Complex Multiplication (CM) algorithm ([24]) to construct curves with a given number of points on its Jacobian. Like the case of the elliptic curve CM method, the genus 2 CM method is very efficient once the class polynomials of the CM field are computed. The hard problem is to find CM fields such that the class polynomials can be computed *and* such that the order of the Jacobian of the curve N and the embedding degree are suitable. For a history of the genus 2 CM method, the reader can refer to [5]. In brief, the algorithm works as follows: Let K be a quartic CM field with primitive CM type.

1. Find a prime p such that there exists $\omega \in K$ with $\omega\bar{\omega} = p$, and an integer N depending on p and $\mathcal{O}_K$ which will be the group order of the Jacobian of the genus 2 curve having CM by $\mathcal{O}_K$. Such p and N can be identified by using a method in [24].
2. Compute the Igusa class polynomials $H_i(x), i = 1, 2, 3$ of K. This step can be done using the methods as described in one of [22], [24], [5], [13].
3. Construct a curve C from a set of roots of $H_i(x)$ over $\mathbb{F}_p$ via the Mestre-Cardona-Quer Algorithm [18], [4], and check if the Jacobian of the curve has order N.

In practice to use the CM method, the quartic CM field K must have small discriminant. So it is desirable to have algorithms which take as input a given field K, and output good cryptographic parameters p and N for a curve C over $\mathbb{F}_p$ with $\#\text{Jac}(C, \mathbb{F}_p) = N$, where $\text{Jac}(C, \mathbb{F}_p)$ denotes the $\mathbb{F}_p$-rational points of the Jacobian of the curve C.

The genus 2 CM method is a useful alternative to point counting, since genus 2 point counting methods are still slow, and the low density of pairing-friendly curves among cryptographically strong ones, as we will see in Section 4, makes it extremely hard to find suitable curves for pairing-based cryptography via point counting. This indicates that the CM method is probably the only suitable method for finding pairing-friendly genus 2 curves currently available. In this paper, we present a method for generating pairing-friendly parameters for the CM construction of genus 2 curves.

The rest of the paper is organized as follows: Section 2 reviews related work. Section 3 gives background on CM fields and pairings. Section 4 shows quantitatively the scarcity of pairing-friendly genus 2 curve among all those that are suitable for discrete-logarithm-based cryptography. Sections 5 and 6 propose two methods, without and with polynomial parameterization, for generating pairing-friendly genus 2 curves. Some sample numerical data can be found in the appendices.

This paper has been published as part of a PhD thesis [21].

2 Related work

In 2002, Rubin and Silverberg [20] showed that supersingular Jacobians of genus 2 hyperelliptic curves have small embedding degrees (≤ 12). In 2007, Hitt [14] presented, for characteristic 2, the construction of families of genus 2 curves with small embedding degree. Freeman [6] gave a method in 2007 for constructing genus 2 curves with ordinary Jacobians over prime fields, which uses parameterization of the CM fields to obtain conditions that lead to the result, and produces a value $\rho \approx 8$.[1] In 2008, Kawazoe and Takahashi [16] suggested a way to find pairing-friendly parameters to generate curves of the form $y^2 = x^5 + ax$ over $\mathbb{F}_p$ for a prime p written as $p = c^2 + 2d^2$, by exploiting the closed formulas for the order of the Jacobian of such curves. This method produces curves with $\rho \leq 4$, whose Jacobians are however not absolutely simple. In 2008, Freeman, Stevenhagen and Streng [9] and Freeman [7] proposed methods for generating parameters for more general pairing-friendly ordinary abelian varieties. The former constructs a suitable Frobenius element which leads to a pairing-friendly abelian variety by extending a method of Cocks and Pinch [8]. The latter finds suitable polynomials parameterizing key elements and generates good parameters by evaluating such polynomials at many different input values. When applied to the case of genus 2, [9] produces $\rho \approx 8$ and [7] is able to further reduce the value to $\rho < 8$.

Although it is known to some extent (see [11]) that pairing-friendly parameters are very rare, among all the work generating such parameters for genus 2 curves, this is the first paper that analyzes quantitatively how unlikely cryptographically strong pairing-friendly parameters are.

The algorithms presented in this paper, together with those in [6], [9], and [7], are the only known methods that generate pairing-friendly parameters for ordinary genus 2 curves over prime fields, which have absolutely simple Jacobians. Unlike [6], we do not need to parameterize the CM field. Our algorithms are also more concrete and more explicit when compared to [9] and [7]. Therefore, these algorithms are easier to understand and implement.

3 Background

3.1 The CM field and the Frobenius element

Let $K := \mathbb{Q}(\eta)$, where

$$\eta = \begin{cases} i\sqrt{a + b\sqrt{d}} & \text{if } d \equiv 2, 3 \pmod 4 \\ i\sqrt{a + b\frac{-1+\sqrt{d}}{2}} & \text{if } d \equiv 1 \pmod 4 \end{cases},$$

be a fixed primitive quartic CM field, where $d > 0$ is squarefree and $\mathbb{Q}(\sqrt{d})$ has class number 1. The condition that K is primitive is equivalent to $\Delta > 0$ is not a square, where $\Delta = a^2 - b^2 d$, if $d \equiv 2, 3 \pmod 4$, and $\Delta = a^2 - a \cdot b - b^2 \left(\frac{d-1}{4}\right)$, if $d \equiv 1 \pmod 4$. We want to construct a genus 2 hyperelliptic curve C over a finite field $\mathbb{F}_p$ of prime order such that $\mathrm{End}(\mathrm{Jac}(C, \mathbb{F}_p)) \otimes \mathbb{Q} = K$,

[1] The definition of ρ can be found later in Section 5. It is a measure of efficiency in pairing-based cryptography. In general, the smaller ρ is, the more efficient the pairing is for cryptography.

and $N := \#\text{Jac}(C, \mathbb{F}_p)$ is "almost prime", meaning that N is a product of a large prime number and a small cofactor.

If such a curve C is found, then there exists an element, called the Frobenius element, $\pi \in \text{End}(\text{Jac}(C, \mathbb{F}_p))$ that satisfies the condition $|\pi| = \sqrt{p}$, where $|\pi|$ is the usual absolute value of the complex number π.

Assume for simplicity that the Frobenius element π is in an order

$$\mathcal{O} := \begin{cases} \mathbb{Z} + \sqrt{d}\mathbb{Z} + \eta\mathbb{Z} + \eta\sqrt{d}\mathbb{Z} & \text{if } d \equiv 2, 3 \pmod 4 \\ \mathbb{Z} + \frac{-1+\sqrt{d}}{2}\mathbb{Z} + \eta\mathbb{Z} + \eta\frac{-1+\sqrt{d}}{2}\mathbb{Z} & \text{if } d \equiv 1 \pmod 4 \end{cases}.$$

We first look at the case $d \equiv 2, 3 \pmod 4$ and write

$$\pi = c_1 + c_2\sqrt{d} + \eta(c_3 + c_4\sqrt{d}), \quad c_i \in \mathbb{Z}.$$

The relationship $\pi\bar{\pi} = p$ gives us

$$(c_1^2 + c_2^2 d + c_3^2 a + c_4^2 ad + 2c_3c_4bd) + (2c_1c_2 + 2c_3c_4a + c_3^2 b + c_4^2 bd)\sqrt{d} = p.$$

Since 1 and $\sqrt{d}$ are linearly independent over $\mathbb{Q}$ we must have

$$c_1^2 + c_2^2 d + c_3^2 a + c_4^2 ad + 2c_3c_4bd = p \tag{1}$$

$$2c_1c_2 + 2c_3c_4a + c_3^2 b + c_4^2 bd = 0 \tag{2}$$

Let $\bar{\alpha}$ and α^σ denote the imaginary and real embeddings of K into $\overline{K}$. The characteristic polynomial of π is

$$\begin{aligned} h(x) &= (x - \pi)(x - \bar{\pi})(x - \pi^\sigma)(x - \bar{\pi}^\sigma) \\ &= x^4 - 4c_1x^3 + (2p + 4(c_1^2 - c_2^2 d))x^2 - 4c_1px + p^2 \end{aligned}$$

The fact that $\#\text{Jac}_{\mathbb{F}_p}(C) = h(1)$ gives the condition

$$N = (p+1)^2 - 4(p+1)c_1 + 4(c_1^2 - c_2^2 d). \tag{3}$$

We want N to be almost prime, i.e., $N = c \cdot r$ with r prime and c small (say, $c < 2000$).

We have $p \sim N^{\frac{1}{2}}$. Based on the discussions above, Weng ([24]) gives a probabilistic method for searching for parameters for discrete logarithm based cryptography, which produces a prime p and an almost prime N.

3.2 Weil and Tate-Lichtenbaum pairings

An excellent survey of the best known implementations of pairings on Jacobians of hyperelliptic curves is given in [1]. In this section we give only some basic information that we need about pairings on general abelian varieties.

For an abelian variety $\mathcal{A}$ over a finite field F and an integer r coprime to the characteristic of F, the Weil pairing is a nondegenerate, skew-symmetric bilinear map

$$e_r^W : \mathcal{A}(\bar{F})[r] \times \mathcal{A}(\bar{F})[r] \to \mu_r(\bar{F}),$$

where $\bar{F}$ is an algebraic closure of F and $\mu_r(\bar{F})$ is the group of r^{th} roots of unity in $\bar{F}$; the Tate-Lichtenbaum pairing is a nondegenerate bilinear map

$$e_r^{TL} : \mathcal{A}(F)[r] \times \mathcal{A}(F)/r\mathcal{A}(F) \to F^*/(F^*)^r.$$

$F^*/(F^*)^r$ is isomorphic to $\mu_r(\bar{F})$ if and only if $\mu_r(\bar{F}) \subseteq F$.

Definition 1 (Embedding degree). *Let $\mathcal{A}$ be an abelian variety over a finite field $F = \mathbb{F}_p$. Let r be an integer coprime to p which divides $\#\mathcal{A}(F)$. The field $F(\mu_r(\bar{F}))$ is a finite extension $\mathbb{F}_{p^k}$ of F. The number k is called the* **embedding degree of $\mathcal{A}$ with respect to** *r, and it is the smallest integer such that $r|(p^k - 1)$.*

We also call the embedding degree of the Jacobian of a nonsingular projective curve C the "embedding degree of the curve C." For pairing-based cryptography, we need an abelian variety $\mathcal{A}$ with $\#\mathcal{A}$ almost prime, i.e., $\#\mathcal{A} = h \cdot r$, where h is a small positive integer and r is a prime number, and the embedding degree k of $\mathcal{A}$ with respect to r which is not too large.

Definition 2 (Pairing-friendly abelian variety). *Let H and K be positive integers. Let $\mathcal{A}$ be an abelian variety over a finite field $\mathbb{F}_p$. We say $\mathcal{A}$ is* **pairing-friendly with respect to parameters** *H* **and** *K if $\#\mathcal{A} = h \cdot r$ for some positive integer $h \leq H$ and a prime number r, and the embedding degree k of $\mathcal{A}$ with respect to r is no larger than K.*

By convention, we call an abelian variety "pairing-friendly" if H and K are "small." We also say a nonsingular projective curve C is "pairing-friendly" if C has a pairing-friendly Jacobian. We also call the parameters $(p, \#\mathcal{A})$ "pairing-friendly".

4 Pairing-friendly genus 2 curves are rare: a quantitative analysis

In this section, we shall show quantitatively that there are very few pairing-friendly parameters for genus 2 hyperelliptic curves among all possible almost prime group orders for Jacobians of genus 2 hyperelliptic curves over prime fields. Inspired by [2], in which elliptic curves of prime orders over finite fields are considered, we generalize its result to the genus 2 case to also deal with Jacobians of *almost prime* orders. A heuristic estimation of the density of pairing-friendly genus 2 curves was performed earlier in [11]. Our result shows a more explicit improvement to this prior work. The main result of this section is Theorem 1. Before proving it, we first introduce several lemmas.

Let p be an odd prime number, and let $\log(\cdot)$ denote the natural logarithm. Let $\alpha_0 = 4/5$.

Lemma 1. *For positive c, M and a, $a \in \mathbb{Z}$, let $\mathcal{S}_{a,c,M}$ denote the set of pairs of primes (x, y) such that $\frac{M}{2} \leq x \leq M$ and $|x^2 - a \cdot y| \leq c \cdot x^{3/2}$. Then $\forall c$, $\forall 0 < \alpha < \alpha_0$ $\exists M_0(c, \alpha) > 0$ such that $\forall M > M_0(c, \alpha)$, $\forall a < M^{\alpha}$, we have*

$$|\mathcal{S}_{a,c,M}| \geq \tilde{c} \cdot \frac{c}{a} \cdot \frac{M^{5/2}}{(\log M)^2}$$

for an effectively computable constant $\tilde{c}$.

Proof. Let $\pi(x)$ be number of primes in the interval $[1, x]$. Let $N = \pi(M) - \pi(\frac{M}{2})$ be the number of primes in $(M/2, M]$. The Prime Number Theorem (P.N.T.) implies $N > \frac{1}{3} \cdot \frac{M}{\log M}$ when $M > M_1$ for some $M_1 > 0$.

By a result of Huxley [15] (suggested by Igor Shparlinski), we have

$$\pi(A) - \pi(A - B) \sim \frac{B}{\log A} \quad (A^{\Theta} < B < \frac{1}{2}A), \tag{4}$$

for any constant $\Theta > 7/12$.

Now let p be a prime number in $(M/2, M]$. We look at the number of primes y such that $\left|p^2 - a \cdot y\right| \leq c \cdot p^{3/2}$, i.e., $\frac{1}{a}\left(p^2 - c \cdot p^{3/2}\right) \leq y \leq \frac{1}{a}\left(p^2 + c \cdot p^{3/2}\right)$. Denote this number by N_p.

Let c be fixed. In (4), let $A = 1/a \cdot p^2$ and $B = c/a \cdot p^{3/2}$. Let $M_2(c, \alpha) = 8c^2$. Then it is clear that $B < \frac{1}{2}A$ for $M/2 \leq p \leq M$, when $M > M_2(c, \alpha)$.

For $0 < \alpha < \alpha_0$, write $\alpha = (3/2 - 2\theta - \epsilon)/(1 - \theta)$, where $7/12 < \theta < 3/4$, and $\epsilon > 0$ are constant (this can always be done for such a constant α). Note that $A^{\theta} < B \iff a^{1-\theta} < c \cdot p^{3/2-2\theta}$. Let $M_3(c, \alpha) = \left((2^{3/2} - 2\theta)/c\right)^{1/\epsilon}$. Then $\forall M > M_3(c, \alpha)$, $M/2 \leq p \leq M$, and $a < M^{\alpha}$, we have

$$c \cdot p^{3/2-2\theta} \geq c(M/2)^{3/2-2\theta},$$
$$a^{1-\theta} \leq (M^{\alpha})^{1-\theta} = M^{3/2-2\theta-\epsilon}.$$

Note that

$$\begin{aligned} M > M_3(c,\alpha) &\iff M > \left((2^{3/2-2\theta})/c\right)^{1/\epsilon} \\ &\iff c \cdot M^{\epsilon} > 2^{3/2-2\theta} \\ &\iff c(M/2)^{3/2-2\theta} > M^{3/2-2\theta-\epsilon}. \end{aligned}$$

It implies $c \cdot p^{3/2-2\theta} > a^{1-\theta}$, and thus $B > A^{\theta}$.

Let $M_4(c,\alpha) > \max\{M_2(c,\alpha), M_3(c,\alpha)\}$ be large enough such that (4) holds with

$$\pi\left(\frac{1}{a}\cdot p^2\right) - \pi\left(\frac{1}{a}\left(p^2 - c\cdot p^{3/2}\right)\right) > \frac{1}{2}\cdot\frac{c/a\cdot p^{3/2}}{\log(1/a\cdot p^2)},$$

for all $M > M_4(c,\alpha)$, $M/2 \le p \le M$. Let $M > M_4(c,\alpha)$, $M/2 \le p \le M$, and $a < M^{\alpha}$. We have

$$\begin{aligned} N_p &\ge \pi\left(\frac{1}{a}\left(p^2 + c\cdot p^{3/2}\right)\right) - \pi\left(\frac{1}{a}\left(p^2 - c\cdot p^{3/2}\right)\right) \\ &> \pi\left(\frac{1}{a}\cdot p^2\right) - \pi\left(\frac{1}{a}\left(p^2 - c\cdot p^{3/2}\right)\right) \\ &> \frac{1}{2}\cdot\frac{c/a\cdot p^{3/2}}{\log(1/a\cdot p^2)} \\ &> \frac{1}{2}\cdot\frac{c/a\cdot (M/2)^{3/2}}{2\log(M)} \\ &> \frac{1}{12}\cdot\frac{c}{a}\cdot\frac{M^{3/2}}{\log M}. \end{aligned}$$

Note that the value p does not appear in the resulting inequality above. Let $M_0(c) = \max\{M_1, M_4(c)\}$. When $M > M_0(c)$, $a < M^{\alpha}$, summing over all suitable primes p, $M/2 \le p \le M$, we obtain

$$|\mathcal{S}_{a,c,M}| = \sum_{\substack{\frac{M}{2} \le p \le M \\ p \text{ prime}}} N_p \ge \frac{1}{12}\cdot\frac{c}{a}\cdot\frac{M^{3/2}}{\log M}\cdot\frac{1}{3}\cdot\frac{M}{\log M} = \frac{1}{36}\cdot\frac{c}{a}\cdot\frac{M^{\frac{5}{2}}}{(\log M)^2}.$$

Let $\tilde{c} = 1/36$. Then the result follows. □

Remark 1. If the Riemann Hypothesis is true, then the constant α_0 in Lemma 1 can be relaxed to $\alpha_0 = 1$.

Remark 2. If we take $a = 1$, the result of Lemma 1 is comparable to the heuristic result in [11] (estimate of the volume of S in Section 4.2 of [11]).

Lemma 2. *For positive K, M and a, $K \in \mathbb{Z}$, $a \in \mathbb{Z}$, let $\mathcal{T}_{a,M,K}$ denote the set of pairs of primes (x,y) such that $\frac{M}{2} \le x \le M$, $|x^2 - a\cdot y| \le 5x^{3/2}$ and $y|(x^k-1)$ for some $k \le K$. Then $|\mathcal{T}_{a,M,K}| < \frac{1}{4}MK(K+1)\log M$.*

Proof. For every integer x with $M/2 \le x \le M$, let $\mathcal{B}_x$ be the set of primes y such that $y|(x^k-1)$ for some integer k with $0 < k \le K$. Since $x^k - 1$ has fewer than $\log(x^k)$ distinct prime divisors, we have

$$\mathcal{B}_x < \sum_{k=1}^{K} k\log x \le \frac{1}{2}K(K+1)\log x.$$

Summing over all such integer x and note that $\frac{M}{2} \le x \le M$, we have

$$|\mathcal{T}_{a,M,K}| \le \sum_{M/2 < x \le M} |\mathcal{B}_x| < \frac{1}{4}MK(K+1)\log M.$$

□

Remark 3. It is worth noting that the result in Lemma 2 does not require M to be large. Igor Shparlinski pointed out that the result of Lemma 2 can be further improved to

$$|\mathcal{T}_{a,M,K}| = O(MK^2/\log M)$$

when M is large and $a < M^{\alpha}, \alpha > 0$, by noting that when the prime y is close to x^2/a, the number of y such that $y|(x^k - 1)$ is at most about $k/(2-\alpha)$ and that there are $O(M/\log M)$ primes x in the interval $[M/2, M]$. When $0 < \alpha < 1$, this improved result can be written as

$$|\mathcal{T}_{a,M,K}| < \frac{1}{2}MK^2/\log M. \tag{5}$$

Remark 4. It is possible that the result of (5) may be further refined to be closer to the heuristic result in [11] (the estimate of the volume of S' in Section 4.2 of [11]). However, such a refinement would likely require techniques different from those used in the proof of Lemma 2.

Lemma 3. *Let c, H, M and K be positive, $K \in \mathbb{Z}$. Let $\widetilde{\mathcal{S}}_{H,c,M}$ denote the set of pairs of primes (x, y) such that $\frac{M}{2} \le x \le M$ and $\left|x^2 - a \cdot y\right| \le c \cdot x^{3/2}$ for some $a \in \mathbb{Z}$, $1 \le a \le H$. Let $\widetilde{\mathcal{T}}_{H,M,K}$ denote the set of pairs of primes (x, y) such that $\frac{M}{2} \le x \le M$, $\left|x^2 - a \cdot y\right| \le 5x^{3/2}$ for some $a \in \mathbb{Z}$, $1 \le a \le H$, and $y|(x^k - 1)$ for some $k \le K$. Then for any $c > 0$, for any $0 < \alpha < \alpha_0$, when M is sufficiently large and $H < M^{\alpha}$, we have*

$$\frac{\widetilde{\mathcal{T}}_{H,M,K}}{\widetilde{\mathcal{S}}_{H,c,M}} < c' \frac{H \cdot K^2 \cdot \log M}{c \cdot M^{3/2}}$$

for an effectively computable positive constant c'. A possible choice of such a constant is $c' = 18$.

Proof. Let a be an integer such that $1 \le a \le H$. By Lemma 1 and Remark 3, when M is sufficiently large, we have

$$\begin{aligned}\frac{\mathcal{T}_{a,M,K}}{\mathcal{S}_{a,c,M}} &< \frac{1/2 \cdot MK^2/\log M}{\frac{1}{36} \cdot \frac{c}{a} \cdot \frac{M^{\frac{5}{2}}}{(\log M)^2}} \\ &< 18 \cdot \frac{a \cdot K^2 \cdot \log M}{c \cdot M^{3/2}} \\ &< 18 \cdot \frac{H \cdot K^2 \cdot \log M}{c \cdot M^{3/2}}.\end{aligned}$$

Note that $\widetilde{\mathcal{T}}_{H,M,K} = \sum_{1 \le a \le H} \mathcal{T}_{a,M,K}$ and $\widetilde{\mathcal{S}}_{H,c,M} = \sum_{1 \le a \le H} \mathcal{S}_{a,c,M}$. Hence we have

$$\frac{\widetilde{\mathcal{T}}_{H,M,K}}{\widetilde{\mathcal{S}}_{H,c,M}} < 18 \cdot \frac{H \cdot K^2 \cdot \log M}{c \cdot M^{3/2}}$$

for large M and $H < M^{\alpha}$. □

Theorem 1. *Let H and K be positive integers. Let α be any constant such that $0 < \alpha < \alpha_0$. Let (p, N) be a randomly (w.r.t. uniform distribution) chosen pair in which p is a prime in the interval $[\frac{M}{2}, M]$ and N is the group order of the Jacobian of a genus 2 curve C defined over $\mathbb{F}_p$ such that $N = \#\mathrm{Jac}(C, \mathbb{F}_p) = h \cdot r$, with $h \in \mathbb{Z}$, $1 \le h \le H < M^{\alpha}$, and r prime. For M large enough, the probability that (p, N) is pairing-friendly with respect to parameters H and K is less than*

$$c'' \frac{H \cdot K^2 \cdot \log M}{M^{3/2}}$$

for an effectively computable positive constant c''.

Proof. The Riemann Hypothesis for abelian varieties over finite fields, proved by Weil in [23], implies the Hasse-Weil bound for genus 2 curves, i.e.,

$$\#\mathrm{Jac}(C, \mathbb{F}_p) \in \left[(\sqrt{p}-1)^4, (\sqrt{p}+1)^4\right].$$

For p large enough, we have $\#\mathrm{Jac}(C, \mathbb{F}_p) \in \left[p^2 - 5p^{3/2}, p^2 + 5p^{3/2}\right]$. Let $c = 1/9$. By Proposition 2.4 of [17], almost all integers $z \in \left[p^2 - cp^{3/2}, p^2 + cp^{3/2}\right]$ can be assumed to be the cardinality of the Jacobian of a genus 2 hyperelliptic curve (given by a quintic or sextic polynomial) over $\mathbb{F}_p$. In Lemma 3, let $c = 1/9$, $x = p$, $y = r$ and $a = h$. The conclusion then follows, observing that $c = 1/9$ is small enough so that the total number of pairs (p, N) in the statement of Theorem 1 is strictly larger than $\widetilde{\mathcal{S}}_{H,c,M}$. Note that we can choose $c'' = 10c'$, where c' is the constant from Lemma 3. □

Theorem 1 says there are very few pairing-friendly parameters for genus 2 hyperelliptic curves when H and K are much smaller than p.

5 Algorithms for generating pairing-friendly genus 2 curves over prime fields

Let k be a desired embedding degree. Let C be a genus 2 hyperelliptic curve defined over a finite field $\mathbb{F}_p$ whose Jacobian over $\mathbb{F}_p$ has a subgroup of order r such that $\mathrm{Jac}(C, \mathbb{F}_p)$ has embedding degree k with respect to r. The ratio of the bit length of $\#\mathrm{Jac}(C, \mathbb{F}_p)$ to the bit length of r is a good measure of efficiency in pairing-based cryptography. Define

$$\rho = 2\log(p)/\log(r).$$

In many pairing-based cryptographic applications, we prefer this value to be close to 1.

In [6], a method to generate genus 2 curves with ordinary Jacobians over prime fields with low embedding degrees is proposed. An important part of this method is a parameterization of the CM field. The method generates curves with value $\rho \approx 8$. We propose another way of generating good parameters, without parameterizing the CM field, which gives a similar ρ value.

Let $K := \mathbb{Q}(\eta)$ be a fixed quartic CM field. We want to construct a genus 2 hyperelliptic curve C over a prime field $\mathbb{F}_p$ such that $\mathrm{Jac}(C, \mathbb{F}_p)$ has CM by K, and such that $\mathrm{Jac}(C, \mathbb{F}_p)$ has a subgroup of prime order r, and $\mathrm{Jac}(C, \mathbb{F}_p)$ has a prescribed embedding degree k with respect to r. For cryptographic applications, we need p and r to be large. We will present the algorithm for the case $d \equiv 2, 3 \pmod 4$ in this paper, where d is as defined in Section 3.1. The case $d \equiv 1 \pmod 4$ can be treated similarly.

In the case $d \equiv 2, 3 \pmod 4$, such a curve can be constructed if we can find a simultaneous integral solution $(c_1, c_2, c_3, c_4, p, r)$, in which p and r are large prime numbers, to the following system of equations:

$$c_1^2 + c_2^2 d + c_3^2 a + c_4^2 ad + 2c_3c_4bd = p \quad (6)$$

$$2c_1c_2 + 2c_3c_4a + c_3^2 b + c_4^2 bd = 0 \quad (7)$$

$$(p+1)^2 - 4c_1(p+1) + 4(c_1^2 - dc_2^2) \equiv 0 \pmod r \quad (8)$$

$$\Phi_k(p) \equiv 0 \pmod r. \quad (9)$$

Here a, b, d and k are fixed, and $\Phi_k(x)$ is the k^{th} cyclotomic polynomial. Equations (6) and (7) mean that the prime p corresponds to a good Weil number, as discussed in Section 3.1. Equation (8) ensures that the Jacobian has a subgroup of prime order r. Equation (9) guarantees that the Jacobian of the curve the embedding degree with respect to r is at most k. Note that Equation 9 implies $p^k \equiv 1 \pmod r$. Given that $p^{r-1} \equiv 1 \pmod r$, we must have $k|(r-1)$, i.e., $r \equiv 1 \pmod k$.

Theorem 2. *If $(c_1, c_2, c_3, c_4, p, r)$ is returned by Algorithm 1, then it provides a solution to the system of equations (6), (7), (8), (9).*

Algorithm 1 Generating pairing parameters for $K = \mathbb{Q}(\eta)$, $d \equiv 2, 3 \pmod 4$

Require: Integers a, b, d with $d > 0$ squarefree, $d \equiv 2, 3 \pmod 4$, $a^2 - b^2 d > 0$ not a square; a prescribed embedding degree k; a bit size n of the desired subgroup order; maximum numbers of trials, M_1 and M_2.

Ensure: Integers c_1, c_2, c_3, c_4, prime numbers p and r, where r has n bits, satisfying Equations (6), (7), (8), (9); or "Not found."

1: Let $c_1 = \pm 1$.
2: **repeat**
3:    Choose a prime number r of n bits such that $r \equiv 1 \pmod k$.
4:    With c_1 fixed as above, try to solve the system of equations given by (6), (7), (8), (9) over the finite field $\mathbb{F}_r$ for a simultaneous solution $(\bar{c}_2, \bar{c}_3, \bar{c}_4, \bar{p})$.
5:    **if** such a solution exists **then**
6:       **repeat**
7:          Choose lifts c_3 and c_4 of $\bar{c}_3$ and $\bar{c}_4$ to $\mathbb{Z}$ such that $f := bc_3^2 + 2ac_3c_4 + bdc_4^2$ is even. Set $c_2 = -c_1 f/2$.
8:          Let $p = ac_3^2 + 2bdc_3c_4 + 2adc_4^2 + 1 + dc_2^2$.
9:          **if** p is prime **then**
10:             Return $(c_1, c_2, c_3, c_4, p, r)$.
11:          **end if**
12:       **until** Lines 7 through 11 have been tried M_2 times.
13:    **end if**
14: **until** M_1 primes r have been tried.
15: Return "Not found."

Proof. It is clear that if $(c_1, c_2, c_3, c_4, p, r)$ is returned, then Equations (8) and (9) are automatically satisfied. Equations (6) and (7) are satisfied by the constructions in Step 7 and 8. Step 9 ensures that p is prime. □

Depending on p and $\mathcal{O}_K$, there are 2 or 4 possibilities for the group order $\#\mathrm{Jac}(C, \mathbb{F}_q)$ [24] [5]. However, for a demonstration purpose, in the algorithm above we are only interested in curves C whose Jacobian has exact group order given by

$$N = (p+1)^2 - 4c_1(p+1) + 4(c_1^2 - dc_2^2).$$

Algorithm 1 looks difficult to analyze because we do not know how likely it is that a solution is found in Step 4. However, experimental results show that the algorithm returns valid parameters quickly and with high probability.

Example 1. Using Algorithm 1 in the case of $a = 2, b = -1, d = 2$, some suitable pairing parameters are found in Appendix A, where r are 160, 256, 512 and 1024 bits, respectively. The computations were performed by the computer algebra system MAGMA [3]. Note that $K = \mathbb{Q}(i\sqrt{2-\sqrt{2}}) \neq \mathbb{Q}(\zeta_5)$ is Galois, so there are only two possibilities for the group order $\#\mathrm{Jac}(C, \mathbb{F}_p)$ [24], namely,

$$N_1 = (p+1)^2 - 4c_1(p+1) + 4(c_1^2 - dc_2^2),$$

or the group order for a quadratic twist of the curve:

$$N_2 = 2(p+1)^2 + 8(c_1^2 - c_2^2 d) - N_1.$$

6 Generating parameters with polynomial parameterization of coefficients c_i

The parameter c_1 produced by Algorithm 1 is always ± 1 and the size of c_2 dominates that of c_1, c_3 and c_4. In fact, this is not necessary. We can modify the search method using the idea of

Algorithm 2 Generating pairing parameters for $K = \mathbb{Q}(\eta)$, $d \equiv 2, 3 \pmod 4$ with polynomial parameterization

Require: Integers a, b, d with $d > 0$ squarefree, $d \equiv 2, 3 \pmod 4$, $a^2 - b^2 d > 0$ not a square; a prescribed embedding degree k; a bit size n of the desired subgroup order; maximum numbers of trials, M_1 and M_2.

Ensure: Integers c_1, c_2, c_3, c_4, prime numbers p and r, where r has n bits, satisfying Equations (6), (7), (8), (9); or "Not found."

1: Choose degree 2 bivariate polynomials $C_3(x, y)$ and $C_4(x, y) \in \mathbb{Z}[x, y]$ such that there is a factorization in $\mathbb{Z}[x, y]$

$$bC_3^2 + 2aC_3C_4 + bdC_4^2 = U \cdot V,$$

where U and V are bivariate polynomials of degree 2. Let $C_1(x, y) = U(x, y)$ and $C_2(x, y) = -\frac{1}{2}V(x, y)$.

2: **repeat**

3: Choose a prime number r of n bits such that $r \equiv 1 \pmod k$.

4: Try to solve the system of equations given by (7), (8), (9), with c_i replaced by $C_i(x, y), i = 1, 2, 3, 4$, over the finite field $\mathbb{F}_r$ for a simultaneous solution $(\bar{x}, \bar{y}, \bar{p})$.

5: **if** Such a solution exists **then**

6: **repeat**

7: Choose lifts x and y of $\bar{x}$ and $\bar{y}$ to $\mathbb{Z}$ such that $c_i := C_i(x, y), i = 1, 2, 3, 4$ are all integers. Let $p = ac_3^2 + 2bdc_3c_4 + 2adc_4^2 + c_1^2 + dc_2^2$.

8: **if** p is prime **then**

9: Return $(c_1, c_2, c_3, c_4, p, r)$.

10: **end if**

11: **until** Lines 7 through 10 have been tried M_2 times.

12: **end if**

13: **until** M_1 primes r have been tried.

14: Return "Not found."

polynomial parameterization and produce pairing parameters with c_1, c_2, c_3 and c_4 roughly of the same size. The algorithm is stated as Algorithm 2.

Similarly to Theorem 2, we have

Theorem 3. *If $(c_1, c_2, c_3, c_4, p, r)$ is returned by Algorithm 2, then it provides a solution to the system of equations (6), (7), (8), (9).*

In Algorithm 2, it is clear that we need $\gcd(C_1, C_2, C_3, C_4) = 1 \in \mathbb{Z}[x, y]$ so that a prime p can be found.

Example 2. Let $C_3(x, y) = C_4(x, y) = xy$, $C_1(x, y) = x^2$ and $C_2(x, y) = -(a + b(1 + d)/2)y^2$. Then they satisfy $bC_3^2 + 2aC_3C_4 + bdC_4^2 + 2C_1C_2 = 0$. Using these polynomials in the above algorithm, we have found for $K = \mathbb{Q}(i\sqrt{2 - \sqrt{2}})$ (i.e., $a = 2, b = -1, d = 2$) parameters in which r are 160, 256, 512 and 1024 bits, respectively. Some of these parameters are presented in Appendix B.

Since x and y are roughly the same size as r, the value of p obtained by this method is $\approx r^4$. It is thus a natural thought that if we parameterize the polynomials $C_i(x, y)$ with degree 1 polynomials in $\mathbb{Z}[x, y]$, then the size of p may be reduced to $\approx r^2$. Unfortunately, the following Proposition 1 shows that such parameterizations will not succeed in achieving this goal.

Proposition 1. *Let a, b, d be integers such that d is squarefree and $a^2 - b^2 d > 0$ is not a square. Let $f(X, Y) = bX^2 + 2aXY + bdY^2$ be a bivariate polynomial in $\mathbb{Q}[X, Y]$. Let F, G be polynomials of total degree 1 in $\mathbb{Q}[X_1, X_2, \ldots, X_n]$ such that F and G are not associated with one another. Then $f(F, G)$ is irreducible in $\mathbb{Q}[X_1, X_2, \ldots, X_n]$.*

Proof. First we note that $b \neq 0$, as indicated by the condition that $a^2 - b^2 d > 0$ is not a square. Let $D = a^2 - b^2 d$. Let $\alpha = -a/b + \sqrt{D}/b$ and $\beta = -a/b - \sqrt{D}/b$. Then $f(X, Y)$ can be factored over $\bar{\mathbb{Q}}$ as

$$f(X, Y) = bX^2 + 2aXY + bdY^2 = b(X - \alpha Y)(X - \beta Y),$$

where $\bar{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$.

Let F and G be polynomials of total degree 1 in $\mathbb{Q}[X_1, X_2, \ldots, X_n]$. Write

$$F(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} f_i X_i + f_0,$$
$$G(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} g_i X_i + g_0,$$

where $f_i, g_i \in \mathbb{Q}$. Suppose $f(F, G)$ is reducible in $\mathbb{Q}[X_1, X_2, \ldots, X_n]$. Then we can write

$$f(F, G) = bH_1 \cdot H_2,$$

where $H_j = \sum_{i=1}^{n} h_i^{(j)} X_i + h_0^{(j)} \in \mathbb{Q}[X_1, X_2, \ldots, X_n]$, $j = 1, 2$, both of total degree 1.

Now we have

$$b(F - \alpha G)(F - \beta G) = f(F, G) = bH_1 \cdot H_2.$$

Note that $\mathbb{Q}(\sqrt{D})[X_1, X_2, \ldots, X_n]$ is a unique factorization domain. Because $F - \alpha G$, $F - \beta G$, H_1 and H_2 are of degree 1, they are irreducible. without of loss of generality, we may assume

$$F - \alpha G = \gamma H_1, \tag{10}$$

for some $\gamma \in \mathbb{Q}(\sqrt{D})^{\times}$. We can write $\gamma = s + t\sqrt{D}$ with $s, t \in \mathbb{Q}$ and $t \neq 0$. Here we require $t \neq 0$ as the polynomial on the left hand side of Equation (10) is in $\mathbb{Q}(\sqrt{D})[X_1, X_2, \ldots, X_n] \backslash \mathbb{Q}[X_1, X_2, \ldots, X_n]$.

Equation (10) gives

$$F - (-a/b + \sqrt{D}/b)G = (s + t\sqrt{D})H_1.$$

Equating the coefficients of X_i and the constant terms on both sides of the above equation, we obtain

$$f_i + (a/b)g_i + (g_i/b)\sqrt{D} = s \cdot h_i^{(1)} + t \cdot h_i^{(1)}\sqrt{D}, \quad 0 \leq i \leq n.$$

This in turn gives

$$f_i + (a/b)g_i = s \cdot h_i^{(1)}, \tag{11}$$
$$g_i/b = t \cdot h_i^{(1)}. \tag{12}$$

If $g_i = 0$ for some i, we must have $h_i^{(1)} = 0$ by (12), which again implies $f_i = 0$ by (11). Otherwise, if $g_i \neq 0$, we can divide both sides of (11) and (12) to obtain

$$b(f_i/g_i) = s/t,$$

thus

$$f_i/g_i = s/(b \cdot t).$$

Therefore, for all $0 \leq i \leq n$, we have $f_i = c \cdot g_i$, where the constant $c = s/(b \cdot t) \in \mathbb{Q}$. Hence $F = c \cdot G$, i.e., F and G are associated. □

An alternative way to do polynomial parameterization in Step 1 of Algorithm 2 is to use degree 1 and degree 2 polynomials for $C_3(x, y)$ and $C_4(x, y)$. This will produce different kinds of c_i's, but the resulting ρ value is still approximately 8 in general. On-going research is aiming at reducing further the value of ρ.

Acknowledgements

The authors would like to thank Igor Shparlinski and Sam Wagstaff, Jr. for their valuable comments.

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A Parameters produced by Algorithm 1

Here are some parameters found by Algorithm 1 for the CM field $K = \mathbb{Q}\left(i\sqrt{2-\sqrt{2}}\right)$ and embedding degree $k = 5$. Corresponding to this CM field there is a genus 2 curve defined over the rationals [22].

$$C : y^2 = -x^5 + 3x^4 + 2x^3 - 6x^2 - 3x + 1.$$

The curves over prime fields corresponding to these parameters are either C reduced modulo p, or its quadratic twist C'.

On average, a MAGMA script found one set of parameters with $r = 160, 256, 512$ and 1024 bits in 0.0918, 0.3486, 2.9938, and 46.5615 seconds, respectively. The computations were performed on an AMD Quad-Core Opteron(TM) 2.4GHz computer running Linux kernel release 2.6.9-34.0.1.ELsmp; only one processor was used for computation.

$\underline{r : 160 \text{ bits. } k = 5}$.
$p =$ 252823257935282285362732638695054084330470208363294037922085422639242974021428617016685256858478396063171049776321146642543762678397966294736679271737114219377482492730434694368080216503567747137

$r =$ 1461501637330902918203684832716283019655932544881

$N =$ 63919599753010277074371937583511654240318456396799666644013838461562311041359420067669494611780522533031261231082704491098182528779928522366939854055782191379965677314562703378699008278543675026648680068400692359055695472813113539589727797257635464036783573538469958621972108837801425046905165205437534564314478956666193424293380483508555554755117650959335536265110336972288875552378947584

$c_1 = 1$

$c_2 =$ 11243292621276079848206331730630023731174251699959569954973786210137165821520551831056883188430192

$c_3 = -64248144848395594424557829122788871673183688623832$

$c_4 = -109802017909327381229794505154259988889529711346380$

$\rho \approx 8.072$

The equation of the curve over $\mathbb{F}_p$ is $y^2 = -x^5 + 3x^4 + 2x^3 - 6x^2 - 3x + 1$.

$\underline{r : 256 \text{ bits. } k = 5}$.

$p =$ 704881071480907162078296670102869074389758456316878620976045254499487753057018612511712201735014180524772377962473016939310167112744621549084701281800977311922475243532026678663444416777984086642261820360878053209107260269920646366156330351242218700528276622717003991911130319025660067745840160149952389932917329

$r =$ 115792089237316195423570985008687907853269984665640564039457584007913129642241

$N =$ 496857324932071752145912383893889169489835622033784989598880614229969960057380528145341182621544436360674179722969415484955886684347872770026411053244140018566049974700076815541374371031592611720892555014703585816910913734818476522890003367060634939104658599174570132609823174216276573137

866957202831985326892972974643475849712058075634522614506805458611699021204439299923123514578344182885280717576928922896637801778010790956345539296480701514721219823943376856364544844490404257431312550838391605233331165209132474804644712415449375768349765769814512250344721171550541443831388350786300229054528190120614531020814267875552

$c_1 = 1$

$c_2 = -593667024299357207475224021693404867559353586749362364291192910163117377314091174679730494164377377555124836261959845126549114759751896733965375133869149502$

$c_3 = -354880931356668387362428709913319025744571268059526422587605882999030905852987$4

$c_4 = -593697948081387184889577965812434116409665571501180864734898731859616318106416$8

$\rho \approx 8.093$

The equation of the curve over $\mathbb{F}_p$ is $y^2 = 3(-x^5 + 3x^4 + 2x^3 - 6x^2 - 3x + 1)$.

B Parameters produced by Algorithm 2

Below are some examples of the parameters found by Algorithm 2 for $K = \mathbb{Q}(i\sqrt{2-\sqrt{2}})$ and embedding degree $k = 3$. Here, we choose $C_3(x, y) = C_4(x, y) = xy$, $C_1(x, y) = x^2$ and $C_2(x, y) = -(a + b(1 + d)/2)y^2$ in Step 1 of Algorithm 2.

On average, our MAGMA implementation found one set of parameters with $r = 160, 256, 512$ and 1024 bits in 0.1092, 0.4468, 4.1718, and 50.0140 seconds, respectively. The computations were performed on an AMD Quad-Core Opteron(TM) 2.4GHz computer running Linux kernel release 2.6.9-34.0.1.ELsmp; only one processor was used for computation.

$\underline{r : 160 \text{ bits. } k = 3}$.

$p = 276032206782791857604308501919988591136740885931343898740256384866241646755370297962312472363405383281006525389401749509877968225746849762659605462196860012810902927696872985980055896486816238781048$1

$r = 1461501637330902918203684832716283019655932543447$

$N = 761937791813779631994733941106633708154739036303135746201414612683681374022951126862517606109944088144225942806086156441245392989328784595634034161547380138187778862280883378421865820312039814035229710820316286444508345243160595796537771020027471372909123195630278485253513049270650615256435136442386120895901675012299462125369911866209880438172735833621377815629134260417168291854627897831493756$8

$c_1 = 8534137516742463259606559105420332781926440781378518072065318554603358974825609017627770035655463$21

$c_2 = -467312771754171603865894820458465529298297100229438686497717835334951148694691783854304471959958498$

$c_3 = c_4 = -893097022442711268703148300906455700266481456199004270995167374051672546438742749426798352836518846$

$\rho \approx 8.2401$

The equation of the curve over $\mathbb{F}_p$ is $y^2 = 3(-x^5 + 3x^4 + 2x^3 - 6x^2 - 3x + 1)$.

$\underline{r : 256 \text{ bits. } k = 3}$.

$p = 822920761971611209794051125149779261868007917105814333422807428702492483230083267137722107007539895222282160142127021544643255654790661296929370353893229675700191477216018550151093614656582383928029105989773078845819666993126278663824378978346229524223744879456228542389848372082725722442158288715575434737334633$7

$r = 11579208923731619542357098500868790785326998466564056403945758400791\
3129640743$

$N = 6771985804839371942637307533597848073765705721625192468898693422808253032215444487859365278749079347589549730845666733117453777198238279219494528067898898802444337872521971715298664355377109626744303642701670738909572492483970382806444921112182297078703529019972656022670120081903677992042490892895555013596712575651692176016210908268738361775620639618631060792503322957268647411120627219341692712631035265600931543321649702304993088353733186022177113837635426687931704695261041122831639155388140714003673423775883028281057290061738442630720051414075948315034087299281022702814170148521555266833233821764657269729790825740 48$

$c_1 = 8995673873914792173814769472743515847127808746498390024090608840436917034478629557785770257234423972877031276763948663931761267676699233257997627484142748 89$

$c_2 = -37991623628115110376463338097314310242107491290686099464180935183342377361666157361851641817813389652802954347538621691112444090127229546877853722663935 38$

$c_3 = c_4 = 826752993461818687372977161424676277826795982340834314841144222880879064934057527406278242014856452108248795365051952735073888493606158382570327029793767 42$

$\rho = 8.0950$

The equation of the curve over $\mathbb{F}_p$ is $y^2 = -x^5 + 3x^4 + 2x^3 - 6x^2 - 3x + 1$.