

# Constructing Elliptic Curves with Prescribed Embedding Degrees

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**Abstract.** Pairing-based cryptosystems depend on the existence of groups where the Decision Diffie-Hellman problem is easy to solve, but the Computational Diffie-Hellman problem is hard. Such is the case of elliptic curve groups whose embedding degree is large enough to maintain a good security level, but small enough for arithmetic operations to be feasible. However, the embedding degree for most elliptic curves is enormous, and the few previously known suitable elliptic curves have embedding degree  $k \leq 6$ . In this paper, we examine criteria for curves with larger  $k$  that generalize prior work by Miyaji *et al.* based on the properties of cyclotomic polynomials, and propose efficient representations for the underlying algebraic structures.

## 1 Introduction

A subgroup  $G$  of (the group of points of) an elliptic curve  $E(\mathbb{F}_q)$  is said to have *embedding degree* or *security multiplier*  $k$  if the subgroup order  $r$  divides  $q^k - 1$ , but does not divide  $q^i - 1$  for all  $0 < i < k$ . The Tate pairing [3, 9, 11] (or the Weil pairing [15, 18, 25]) maps the discrete logarithm in  $G$  to the discrete logarithm in  $\mathbb{F}_{q^k}$ , and this is the basis for the Frey-Rück attack [10].

An important open problem in pairing-based cryptography [5, 6, 12, 14, 22, 23, 27, 29] is to build curves containing a subgroup with embedding degree  $k$  that is at once big enough to prevent the Frey-Rück attack, but small enough that the Tate pairing is efficiently computable, which in turn means that arithmetic in  $\mathbb{F}_{q^k}$  is feasible. The embedding degree is known to be usually enormous [2, 19], and for a long time, the only elliptic curves known to admit subgroups with reasonable  $k$  were supersingular curves, particularly over  $\mathbb{F}_{3^m}$  where  $k = 6$  [18]. Such curves are constructed over fields of low characteristic, making them more susceptible to discrete logarithm algorithms [24].

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Recently, Miyaji, Nakabayashi and Takano [19] showed, using certain properties of the cyclotomic polynomial of order  $k$ , how to build non-supersingular curves over  $\mathbb{F}_q$  of prime order with  $k = 3, 4, 6$  using the complex multiplication (CM) method [16, 20], as long as certain conditions, which we call the MNT criteria, hold for the field size  $q$ , the trace of Frobenius  $t$  [25, III.4.6], and the curve order  $n$ . However, no technique was known for systematically building curves where  $k > 6$  but not enormous.

In this paper we investigate generalizations of the MNT criteria for curves with general embedding degree  $k$ , and address the actual construction of such curves. We also discuss representations of the involved fields and groups that lead to efficient implementations of the Tate pairing. Another method for building curves with arbitrary  $k$  has been recently proposed by Dupont, Enge and Morain [8].

This paper is organized as follows. Section 2 describes generalizations of the MNT criteria. Section 3 deals with the problem of solving the resulting CM equation to obtain suitable field and curve parameters. Section 4 discusses techniques for efficient implementation of the finite field arithmetic and the Tate pairing. We present our conclusions in section 5.

## 2 Generalizing the MNT criteria

Any elliptic curve  $E$  over  $\mathbb{F}_q$  of order  $n$  satisfies Hasse's theorem [25, V.1.1], which states that the trace  $t$  of the Frobenius endomorphism on  $E$ , related to  $q$  and  $n$  by the equation  $n = q + 1 - t$ , is restricted to  $|t| \leq 2\sqrt{q}$ .

Given an embedding degree  $k > 0$ , our goal is to find a prime<sup>4</sup>  $q$ , an integer  $t$  such that  $|t| \leq 2\sqrt{q}$ , a large prime  $r$  satisfying  $r \mid q^k - 1$  but  $r \nmid q^i - 1$  for all  $0 < i < k$ , and a curve  $E(\mathbb{F}_q)$  whose trace of Frobenius is  $t$  and whose order  $n = q + 1 - t$  satisfies  $r \mid n$ , that is,  $n = hr$  for some  $h$ .

We begin by noticing that  $q = (t - 1) + hr$  implies  $q^u - 1 \equiv (t - 1)^u - 1 \pmod{r}$  for all  $u > 0$ . Therefore, any suitable  $r$  must satisfy  $r \mid (t - 1)^k - 1$  and  $r \nmid (t - 1)^i - 1$  for all  $0 < i < k$ . Let  $\Phi_m$  be the  $m$ -th cyclotomic polynomial [17, definition 2.44]. It is well known [17, theorem 2.45(i)] that  $x^u - 1 = \prod_{d \mid u} \Phi_d(x)$  for any  $u > 0$ . This leads to the following lemma.

**Lemma 1.** *Any suitable prime  $r$  satisfies  $r \mid \Phi_k(t - 1)$  and  $r \nmid \Phi_i(t - 1)$  for all  $0 < i < k$ .*

*Proof.* It is necessary that  $r \nmid \Phi_i(t - 1)$  for all  $0 < i < k$ , as otherwise  $r \mid (t - 1)^i - 1$  for some  $0 < i < k$ . But since  $r$  is prime,  $r \mid (t - 1)^k - 1$  implies  $r \mid \Phi_d(t - 1)$  for some  $d$  dividing  $k$ , and the only remaining possibility is  $d = k$ . Hence, necessarily  $r \mid \Phi_k(t - 1)$ .  $\square$

The basic MNT strategy is to choose a trace  $t$  of suitable size, finding a prime  $r$  in the conditions of the above lemma, computing from  $t$  and  $r$  a prime of form

<sup>4</sup> Actually,  $q$  may be a prime power, but for simplicity we will only refer to the prime case as this is the most relevant in practice.

$q = hr + t - 1$  for some small cofactor<sup>5</sup>  $h$ , and finally using the CM method to build the desired curve.

## 2.1 Constraining the parameters

We now derive explicit constraints on the form of  $t$  and  $q$ , for any  $h$  and any  $k$ , generalizing the original approach by Miyaji *et al.*

Let  $\ell$  be an integer with  $|\ell| > 1$ , let  $r$  be a prime factor of  $\Phi_k(\ell)$ , and let  $d$  be an integer satisfying  $1 \leq d \leq \deg \Phi_k/2$ . Set  $n = hr$  for some  $h$ ,  $q = n + \ell^d$ , and  $t = \ell^d + 1$ . It follows from lemma 1 that  $r$  satisfies  $r \mid \ell^{kd} - 1$ , and in general  $r \nmid \ell^{id} - 1$  for  $0 < i < k$ . The restriction  $d \leq \deg \Phi_k/2$  is imposed to ensure the Hasse bound is satisfied.

**Theorem 1.** *The choice of parameters proposed above leads to curves containing a subgroup of order  $r$  with embedding degree at most  $k$ .*

*Proof.* From the condition  $q = hr + \ell^d$  it follows that  $q^k - 1 \equiv \ell^{dk} - 1 \pmod{r}$ . Since  $\Phi_k(\ell) \mid \ell^{dk} - 1$  [17, theorem 2.45(i)], the restriction  $r \mid \Phi_k(\ell)$  implies  $r \mid \ell^{dk} - 1$ , that is,  $\ell^{dk} - 1 \equiv 0 \pmod{r}$ . Therefore,  $q^k - 1 \equiv 0 \pmod{r}$ , that is,  $r \mid q^k - 1$ .  $\square$

An important observation here is that we still must verify that  $r \nmid q^i - 1$  for  $0 < i < k$ , even for such a special case as  $r = \Phi_k(\ell)$ . Since  $\ell^i - 1 = \prod_{u \mid i} \Phi_u(\ell)$ , it is obvious that  $\Phi_k(\ell) \nmid \ell^i - 1$ , and hence, apparently  $r \nmid \ell^i - 1$ . However, this reasoning is wrong: the relation  $\Phi_k(\ell) \nmid \ell^i - 1$  only holds for the polynomials themselves, not necessarily for some *specific* argument  $\ell$ .

In contrast to the original work by Miyaji *et al.*, the above criteria are not exhaustive, and it is not difficult to find other conditions leading to perfectly valid parameters. For instance, an obvious generalization is  $q = n + \Phi_k(\ell)g(\ell) + \ell^d$ , for any polynomial  $g(\ell)$ . This does not help much because in general  $\Phi_k(\ell)g(\ell)$  makes the trace  $t$  too big to satisfy the Hasse bound, except when the term  $\ell^d$  cancels the term of highest degree in  $\Phi_k(\ell)g(\ell)$  and the remaining terms are of suitably low degree (for example, for  $k = 9$ , by picking an appropriate  $g$ , one can obtain  $q = n - \ell^3 - 1$ ). Also, if  $d$  is even,  $k$  is even, and  $k/2$  is odd, it can be verified that setting  $t = -\ell^d + 1$  and  $q = hr - \ell^d$  is equally possible, that is,  $r \mid q^k - 1$  (the restriction to even  $k$  such that  $k/2$  is odd ensures that  $q^{k/2} - 1 \equiv -(\ell^{dk} + 1) \not\equiv 0 \pmod{r}$ ). Sometimes, fortuitous solutions barely resembling (but related to) the MNT criteria can be found for particular choices of  $k$ . However, for simplicity we will focus on the parameters considered in theorem 1; extending the discussion below to other parameters should not be difficult, and would be hardly necessary in practice.

## 3 Solving the CM equation

The strategy to build curves given the above criteria seems straightforward: choose  $\ell$  and  $h$ , find a prime  $q$  and the corresponding trace  $t$  according to the

<sup>5</sup> Miyaji *et al.* consider only  $h = 1$ , that is, curves of prime order.

proposed relations, solve for the CM discriminant  $D$  (and for  $V$ ) the CM equation  $DV^2 = 4q - t^2$ , or equivalently,  $DV^2 = 4n - (t - 2)^2$ , and use the CM method to compute the curve equation coefficients. Since  $n = hr$  and  $r \mid \Phi_k(\ell)$ , we can write  $n = m\Phi_k(\ell)$  for some  $m$ . Thus, the CM equations for the parameter criteria given in section 2.1 has the form:

$$DV^2 = 4m\Phi_k(\ell) - (\ell^d - 1)^2. \quad (1)$$

Unfortunately, this approach is not practical, because in general the CM discriminant  $D$  is too large (comparable to  $q$ ), and cryptographically significant parameters would have  $q \approx 2^{160}$  at least. It is possible to find toy examples using this direct approach, though. For instance, the curve  $E : y^2 = x^3 - 3x + 183738738969463$  over  $\mathbb{F}_{449018176625659}$  has  $4r$  points, where  $r = 112254544155601$ . The subgroup of order  $r$  has embedding degree  $k = 12$ . This curve satisfies  $m = 4$ ,  $r = \Phi_k(\ell)$ ,  $t = \ell + 1$ , and  $q = 4\Phi_k(\ell) + \ell$  for  $\ell = 3255$ . The CM equation is  $DV^2 = 4q - t^2$  where  $D = 13188099$  and  $V = 11670$ , and the class number is 2940.

Miyaji et al. solve this problem for  $k = 3, 4, 6$  by noticing that the CM equation leads, in these cases, to a quadratic Diophantine equation reducible to Pell's equation, whose solution is well known [26]. The case of arbitrary  $k$  is much harder, since no general method is known to solve Diophantine equations of degree  $\deg(\Phi_k) \geq 4$ .

However, Tzanakis [28] describes how to solve quartic elliptic Diophantine equations of form

$$V^2 = a\ell^4 + b\ell^3 + c\ell^2 + d\ell + e^2,$$

where  $a > 0$  (notice the squared independent term). For  $k = 5, 8, 10, 12$ , the degree of  $\Phi_k$  is 4, so that equation 1 has the form  $DV^2 = a\ell^4 + b\ell^3 + c\ell^2 + d\ell + f$ . If a solution to this equation in small integers is known (as can often be found by exhaustive search), this equation reduces to Tzanakis form by multiplying both sides by  $D$  and applying a linear transformation involving the known solution, so that the independent term of the transformed equation is a perfect square.

Unfortunately again, this approach has proven unsuccessful in practice. Using the Magma implementation of Tzanakis method, we were not able to find any cryptographically significant examples of curves with  $k = 5, 8, 10, 12$ , the only such cases being those where  $D$  is too large for traditional CM methods.

We have not tried more recent variants of the CM method like that of [1], so there is still hope that solutions of equation 1 with large  $D$  are actually practical. But even if this direct approach remains out of reach, there are successful ways to generate suitable field and curve parameters, as we show next.

### 3.1 A particular case

Let  $p$  be a prime (not to be confused with the finite field size  $q$ ). We describe how to find algebraic solutions of equation 1 for the special case  $D = 3$ ,  $d = 1$ , and  $k = 3^i 2^j p^s$ , for certain exponents  $i, j, s$  and prime  $p > 3$ . In principle, this method enables the ratio  $m/r$  to get arbitrarily small for large  $k$  if  $s = 0$ .

The cyclotomic polynomials are known [21] to satisfy the following properties. If  $v$  is any prime dividing  $u$ , then  $\Phi_{uv}(x) = \Phi_u(x^v)/\Phi_u(x)$ . On the other hand, if  $v \nmid u$ , then  $\Phi_{uv}(x) = \Phi_u(x^v)$ .

Using these properties, it is easy to show by induction that for all  $i > 0$ ,

$$\Phi_{3^i}(\ell) = \ell^{2 \cdot 3^{i-1}} + \ell^{3^{i-1}} + 1 \text{ and } \Phi_{2^i \cdot 3}(\ell) = \ell^{2^i} - \ell^{2^{i-1}} + 1.$$

Restrict  $\ell$  so that  $\ell \equiv 1 \pmod{3}$ . Thus,  $4\Phi_{3^i}(\ell) - 1 = 3[(2\ell^{3^{i-1}} + 1)/3]^2$  and  $4\Phi_{2^i \cdot 3}(\ell) - 3 = [(2\ell^{2^{i-1}} - 1)]^2$ . In the first case, multiplying both sides by  $(\ell - 1)^2$  leads to  $4 \cdot (\ell - 1)^2 \Phi_{3^i}(\ell) - (\ell - 1)^2 = 3[(\ell - 1)(2\ell^{3^{i-1}} + 1)/3]^2$ , which gives the solution

$$\begin{aligned} k &= 3^i, r = \Phi_{3^i}(\ell)/3, \\ t &= \ell + 1, m = (\ell - 1)^2, \\ V &= (\ell - 1)(2\ell^{3^{i-1}} + 1)/3. \end{aligned}$$

In the second case, multiplying both sides by  $(\ell - 1)^2/3$  leads to  $4 \cdot [(\ell - 1)^2/3] \Phi_{2^i \cdot 3}(\ell) - (\ell - 1)^2 = 3[(\ell - 1)(2\ell^{2^{i-1}} - 1)/3]^2$ , which gives the solution

$$\begin{aligned} k &= 2^i \cdot 3, r = \Phi_{2^i \cdot 3}(\ell), \\ t &= \ell + 1, m = (\ell - 1)^2/3, \\ V &= (\ell - 1)(2\ell^{2^{i-1}} - 1)/3. \end{aligned}$$

In both cases, we assume that  $q = mr + \ell$  is prime.

Similarly, one can show by induction that, for any prime  $p > 3$  and for all  $i, j > 0$

$$\Phi_{3^i p^j}(\ell) = [(2\ell^{3^{i-1} p^j} + 1)^2 + 3]/[(2\ell^{3^{i-1} p^{j-1}} + 1)^2 + 3].$$

Multiplying both sides by  $12z^2[(2\ell^{3^{i-1} p^{j-1}} + 1)^2 + 3]$  for any  $z$  leads to  $4 \cdot 3z^2[(2\ell^{3^{i-1} p^{j-1}} + 1)^2 + 3] \Phi_{3^i p^j}(\ell) - (6z)^2 = 3[2z(2\ell^{3^{i-1} p^j} + 1)]^2$ . Choosing  $r$  to be any large factor of  $\Phi_{3^i p^j}(\ell)$ , this gives the solution

$$\begin{aligned} k &= 3^i p^j, \ell = 6z + 1, \\ n &= 3z^2[(2\ell^{3^{i-1} p^{j-1}} + 1)^2 + 3] \Phi_{3^i p^j}(\ell), \\ V &= z(2\ell^{3^{i-1} p^j} + 1). \end{aligned}$$

It is also straightforward (but tedious) to show that

$$\Phi_{3^i 2^j p^s}(\ell) = [(2\ell^{3^{i-1} 2^{j-1} p^s} - 1)^2 + 3]/[(2\ell^{3^{i-1} 2^{j-1} p^{s-1}} - 1)^2 + 3]$$

hence  $4 \cdot 3z^2[(2\ell^{3^{i-1} 2^{j-1} p^{s-1}} - 1)^2 + 3] \Phi_{3^i 2^j p^s}(\ell) - (6z)^2 = 3[2z(2\ell^{3^{i-1} 2^{j-1} p^s} - 1)]^2$ , which gives the solution

$$\begin{aligned} k &= 3^i 2^j p^s, \ell = 6z + 1, \\ m &= 3z^2[(2\ell^{3^{i-1} 2^{j-1} p^{s-1}} - 1)^2 + 3], \\ V &= 2z(2\ell^{3^{i-1} 2^{j-1} p^s} - 1). \end{aligned}$$

In all cases, it is necessary to ensure that  $q = m\Phi_k(\ell) + \ell$  is prime.

Appendix A contains a detailed example of this method.

It is unclear whether this strategy can be extended to more general  $k$ , since the corresponding expressions get very involved. Moreover, this method only produces solutions for  $D = 3$ , which could potentially have a lower security level, even though no specific vulnerability based on the small  $D$  value is known at the time of this writing [4, section VIII.2]. The next method we describe is suitable for much more general  $D$ .

### 3.2 A general method

The general form of the criteria we have been considering is  $n = m\Phi_k(\ell)$ ,  $t = \ell^d + 1$ ,  $q = n + t - 1 = m\Phi_k(\ell) + \ell^d$ , where  $1 \leq d \leq \deg \Phi_k / 2$ . Usually one wants  $m$  to be small and  $\Phi_k(\ell)$  to contain a large prime factor  $r$  (the best case is then  $\Phi_k(\ell)$  itself being a prime). However, finding solutions under these conditions is hard for any  $k$  such that  $\deg \Phi_k > 2$ . Therefore, we relax the restrictions on  $m$ , say, by allowing  $m$  to be comparable to  $r$ . It turns out that obtaining suitable parameters becomes quite easy.

Consider the CM equation  $DV^2 = 4m\Phi_k(\ell) - (\ell^d - 1)^2$ . We assume that both  $D$  and  $\ell$  are chosen and  $t \nmid D$ ; we want to find a solution  $m$  (and  $V$ ) to this equation. For convenience, let  $A = 4\Phi_k(\ell)$  and  $B = (\ell^d - 1)^2$ , so that the equations reads  $DV^2 = Am - B$ .

Initially, find the smallest  $m_0 \geq 0$  such that  $D \mid Am_0 - B$ , that is,  $Am_0 - B = z_0D$  for some  $z_0$ . If  $A$  is invertible modulo  $D$ , then  $m_0 = B/A \pmod{D}$  and  $z_0 = (Am_0 - B)/D$ . If  $A$  is *not* invertible modulo  $D$  but  $B$  is a multiple of  $D$ , then  $m_0 = 0$  and  $z_0 = -B/D$ . Otherwise, there is no solution for this choice of  $\ell$  and  $D$ . Thus we ensure that  $m_0$  is never larger than  $D$ .

Define:

$$\begin{aligned} m_i &= m_0 + iD, \\ z_i &= z_0 + iA. \end{aligned}$$

Substituting these into the CM equation gives  $Dz_i = Am_i - B$ . This means that any solution to this equation involves  $z_i$  that is a perfect square. Therefore, solve  $V^2 = z_0 + iA$  for  $V$  and  $i$ , and pick up the smallest solution  $i$  such that  $q = m_i\Phi_k(\ell) + \ell^d$  is a prime. This requires  $z_0$  to be a quadratic residue (QR) modulo  $A$ . If so, all solutions  $i_\alpha$  can be written as  $i_\alpha = i_0 + \alpha A$ , where  $i_0 = (V_0^2 - z_0)/A$  and  $V_0 = \sqrt{z_0} \pmod{A}$ . A neat strategy to obtain a tight ratio between  $\log q$  and  $\log r$  is to restrict the search to  $i_0$  alone and vary only  $\ell$ .

Experiments we conducted showed that, in practice,  $m$  tends to be close to  $r$ . Nevertheless, such solutions are perfectly suitable for most pairing-based cryptosystems, the only exception being the short signature scheme of [6].

Appendix B contains examples of this method.

## 4 Implementation issues

Since curves with medium-sized embedding degree  $k$  can be effectively constructed as described above, the natural question to ask is how to efficiently implement the underlying arithmetic and, particularly, the Tate pairing.

We restrict the discussion to even  $k$ . Let  $\text{even}(\mathbb{F}_{q^k})$  denote the subset of  $\mathbb{F}_{q^k}$  consisting of polynomials whose component monomials are all of even degree, i.e.  $\text{even}(\mathbb{F}_{q^k}) = \{u \in \mathbb{F}_{q^k} : u(x) = a_{k-2}x^{k-2} + a_{k-4}x^{k-4} + \dots + a_0\}$ . Similarly, let  $\text{odd}(\mathbb{F}_{q^k})$  denote the subset of  $\mathbb{F}_{q^k}$  consisting of polynomials whose component monomials are all of odd degree, i.e.  $\text{odd}(\mathbb{F}_{q^k}) = \{u \in \mathbb{F}_{q^k} : u(x) = a_{k-1}x^{k-1} + a_{k-3}x^{k-3} + \dots + a_1x\}$ .

We propose representing  $\mathbb{F}_{p^k}$  as  $\mathbb{F}_p[x]/R_k(x)$  with a reduction polynomial of form  $R_k(x) = x^k + x^2 + \omega$  for some  $\omega \in \mathbb{F}_p$ . This choice is motivated by the following analysis.

**Lemma 2.** *If  $R(x) = x^k + x^2 + \omega$  is irreducible over  $\mathbb{F}_q$ , then  $r(x) = x^{k/2} + x + \omega$  is irreducible over  $\mathbb{F}_q$ .*

*Proof.* By contradiction. If  $r(x) = f(x)g(x)$  for some  $f, g \in \mathbb{F}_q[x]$ , then  $R(x) = r(x^2) = f(x^2)g(x^2)$ , against the hypothesis that  $R(x)$  is irreducible.  $\square$

This establishes that the mapping  $\psi : \mathbb{F}_q[x]/r(x) \rightarrow \mathbb{F}_q[x]/R(x)$ ,  $\psi(f) = F$  such that  $F(x) = f(x^2)$ , induces an isomorphism between  $\mathbb{F}_{q^{k/2}}$  and  $\text{even}(\mathbb{F}_{q^k})$ .

Notice that this lemma would remain valid if  $R$  contained more even-degree monomials, but a trinomial is better suited for efficient implementations. However, it is possible that an irreducible *binomial*  $R(x) = x^k + \omega$  exists, in which case it would be an even better choice.

**Lemma 3.** *Let  $Q = (u, v) \in E(\mathbb{F}_{q^k})$  where  $E : v^2 = f(u)$ ,  $u \in \text{even}(\mathbb{F}_{q^k})$ , and  $f(u)$  is a quadratic nonresidue. Then  $v \in \text{odd}(\mathbb{F}_{q^k})$ .*

*Proof.* Notice that  $f(u) \in \text{even}(\mathbb{F}_{q^k})$ . Let  $v(x) = \alpha(x) + x\beta(x)$ , where  $\alpha, \beta \in \text{even}(\mathbb{F}_{q^k})$ . Then  $v^2 = (\alpha^2 + \beta^2) + x(2\alpha\beta) \in \text{even}(\mathbb{F}_{q^k})$ , so that either  $\alpha = 0$  or  $\beta = 0$ . But  $\beta = 0$  would mean that  $f(u) = \alpha^2$  is a quadratic residue, against the hypothesis. Therefore,  $\alpha = 0$ , that is,  $v \in \text{odd}(\mathbb{F}_{q^k})$ .  $\square$

We are now prepared to address the main theorem:

**Theorem 2.** *Let  $Q = (u, v) \in E(\mathbb{F}_{q^k})$  where  $E : v^2 = f(u)$ ,  $u \in \text{even}(\mathbb{F}_{q^k})$ , and  $f(u)$  is a quadratic nonresidue. If  $S = (s, t) \in \langle Q \rangle$ , then  $s \in \text{even}(\mathbb{F}_{q^k})$  and  $t \in \text{odd}(\mathbb{F}_{q^k})$ .*

*Proof.* This is a consequence of the elliptic addition rules [25, algorithm 2.3]. It is straightforward but tedious to show that the point addition formula and the doubling formula do satisfy the theorem. Thus, we only have to let  $S = mQ$ , and proceed by induction on  $m$ .  $\square$

This way, curve points  $Q = (u, v) \in E/\mathbb{F}_{p^k}$ , where  $f(u) \in \text{even}(\mathbb{F}_{q^k})$  is a quadratic nonresidue, are not only suitable for the computation of the Tate pairing  $e(P, Q)$  for any  $P \in E/\mathbb{F}_p$  (because obviously  $Q$  is linearly independent from  $P$ ); they also have the nice property that the technique of denominator elimination [3, section 5.1] is applicable, thus nearly doubling the performance of Miller’s algorithm.

## 5 Conclusion

We have shown how to effectively solve the problem of constructing elliptic curves with prescribed embedding degree, and suggested ways to efficiently implement the resulting curves so that pairing-based cryptosystems are practical.

A natural line of further research is to investigate whether curves of any given embedding degree can be constructed for general abelian varieties. We also point out that the problem of generating elliptic curves of prime or near-prime order remains open; such curves are important in certain cryptosystems, like the BLS signature scheme [6].

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## References

1. A. Agashe, K. Lauter, and R. Venkatesan, “Constructing elliptic curves with a known number of points over a prime field,” High Primes and Misdemeanours: lectures in honour of the 60th birthday of Hugh Cowie Williams, Fields Institute Communications Series, Vol. 42, 2002, pp. 1–17.
2. R. Balasubramanian, N. Koblitz, “The improbability that an Elliptic Curve has Subexponential Discrete Log Problem under the Menezes-Okamoto-Vanstone Algorithm,” Journal of Cryptology, Vol. 11, No. 2, 1998, pp. 141–145.
3. P. S. L. M. Barreto, H. Y. Kim, B. Lynn, and M. Scott, “Efficient algorithms for pairing-based cryptosystems,” Advances in Cryptology – Crypto’2002, Lecture Notes in Computer Science **2442**, pp. 354–368, Springer-Verlag, 2002.
4. I. Blake, G. Seroussi, and N. Smart, “Elliptic Curves in Cryptography,” Cambridge University Press, 1999.
5. D. Boneh and M. Franklin, “Identity-based encryption from the Weil pairing,” Advances in Cryptology – Crypto’2001, Lecture Notes in Computer Science **2139**, pp. 213–229, Springer-Verlag, 2001.
6. D. Boneh, B. Lynn, and H. Shacham, “Short signatures from the Weil pairing,” Asiacrypt’2001, Lecture Notes in Computer Science **2248**, pp. 514–532, Springer-Verlag, 2002.



7. R. Crandall and C. Pomerance, "Prime Numbers: a Computational Perspective," Springer-Verlag, 2001.
8. R. Dupont, A. Enge, and F. Morain "Building curves with arbitrary small MOV degree over finite prime fields," Cryptology ePrint Archive, Report 2002/094, available at <http://eprint.iacr.org/2002/094>.
9. G. Frey, M. Müller, and H. Rück, "The Tate Pairing and the Discrete Logarithm Applied to Elliptic Curve Cryptosystems," IEEE Transactions on Information Theory, 45(5), pp. 1717–1719, 1999.
10. G. Frey and H. Rück, "A Remark Concerning  $m$ -Divisibility and the Discrete Logarithm in the Divisor Class Group of Curves," Mathematics of Computation, 62 (1994), pp. 865–874.
11. S. D. Galbraith, K. Harrison, and D. Soldara, "Implementing the Tate pairing," Algorithmic Number Theory – ANTS V, Lecture Notes in Computer Science **2369**, pp. 324–337, Springer-Verlag, 2002.
12. F. Hess, "Exponent Group Signature Schemes and Efficient Identity Based Signature Schemes Based on Pairings," Cryptology ePrint Archive, Report 2002/012, available at <http://eprint.iacr.org/2002/012/>.
13. IEEE Std 2000–1363, "Standard Specifications for Public Key Cryptography," 2000.
14. A. Joux, "A one-round protocol for tripartite Diffie-Hellman," Algorithm Number Theory Symposium – ANTS IV, Lecture Notes in Computer Science **1838**, pp. 385–394, Springer-Verlag, 2000.
15. A. Joux and K. Nguyen, "Separating Decision Diffie-Hellman from Diffie-Hellman in Cryptographic Groups," Journal of Cryptology, Vol. 16, No. 4, 2003, pp. 239–247.
16. G. J. Lay and H. G. Zimmer, "Constructing Elliptic Curves with Given Group Order over Large Finite Fields," Algorithmic Number Theory Symposium – ANTS I, Lecture Notes in Computer Science **877** (1994), pp. 250–263.
17. R. Lidl and H. Niederreiter, "Introduction to finite fields and their applications," Cambridge University Press, 1986.
18. A. Menezes, T. Okamoto, and S. Vanstone, "Reducing elliptic curve logarithms to logarithms in a finite field," IEEE Transactions on Information Theory 39(1993), pp. 1639–1646.
19. A. Miyaji, M. Nakabayashi, and S. Takano, "New explicit conditions of elliptic curve traces for FR-reduction," IEICE Trans. Fundamentals, Vol. E84 A, no. 5, May 2001.
20. F. Morain, "Building cyclic elliptic curves modulo large primes," Advances in Cryptology – Eurocrypt'91, Lecture Notes in Computer Science **547** (1991), pp. 328–336.
21. T. Nagell, "Introduction to Number Theory," 2nd reprint edition, Chelsea Publishing, 2001.
22. K. G. Paterson, "ID-based signatures from pairings on elliptic curves," Electronics Letters, Vol. 38, No. 18, 2002, pp. 1025–1026.
23. R. Sakai, K. Ohgishi, and M. Kasahara, "Cryptosystems based on pairing," 2000 Symposium on Cryptography and Information Security (SCIS2000), Okinawa, Japan, Jan. 26–28, 2000.
24. O. Schirokauer, D. Weber, and T. Denny, "Discrete Logarithms: the Effectiveness of the Index Calculus Method," ANTS, pp. 337–361, 1996.
25. J. H. Silverman, "Elliptic curve discrete logarithms and the index calculus," Workshop on Elliptic Curve Cryptography (ECC'98), September 14–16, 1998.
26. N. P. Smart, "The Algorithmic Resolution of Diophantine Equations," London Mathematical Society Student Text **41**, Cambridge University Press, 1998.

27. N. P. Smart, “An Identity Based Authenticated Key Agreement Protocol Based on the Weil Pairing,” *Electronics Letters*, Vol. 38, 2002, pp. 630–632.
28. N. Tzanakis, “Solving elliptic diophantine equations by estimating linear forms in elliptic logarithms. The case of quartic equations,” *Acta Arithmetica* **75** (1996), pp. 165–190.
29. E. Verheul, “Self-blindable Credential Certificates from the Weil Pairing,” *Advances in Cryptology – Asiacrypt’2001, Lecture Notes in Computer Science* **2248** (2002), pp 533–551.

## A An example of the closed-form construction

This simple construction implements the method of section 3.1 and quickly yields a curve and a point of large prime order  $r$ , with embedding degree  $k = 12$ .

1. Choose  $z$  of an appropriate size at random.
2. Set  $w = 3 \cdot z$ ,  $t = w + 2$ .
3. Set  $r = w^4 + 4w^3 + 5w^2 + 2w + 1$ . If  $r$  is not prime return to step 1.
4. Set  $q = (w^6 + 4w^5 + 5w^4 + 2w^3 + w^2 + 3w + 3)/3$ . If  $q$  is not prime, return to step 1.
5. Use the CM method to find the curve of the form  $y^2 = x^3 + B$  with discriminant  $D = 3$  of order  $n = q + 1 - t$ . Find a point of order  $r$  on the curve using the method described in [13, section A11.3].

Note that  $n = mr$  and  $m = 3z^2$ . Rather than using the CM method in step 5, in practice small values of  $B$  can be tested to find the correct curve [7].

An example run of this algorithm yields

```

z = 67749197969
r = 1706481765729006378056715834692510094310238833
q = 23498017525968473690296083113864677063688317873484513641020158425447
n = 23498017525968473690296083113864677063688317873484513640816910831539

```

Here  $r$  is a 151-bit prime, and  $q$  is a 224-bit prime. The curve is quickly found as  $E : y^2 = x^3 + 4$  over  $\mathbb{F}_q$ .

## B An example of the general construction

Let  $s$  be the approximate desired size (in bits) of the subgroup order  $r$ , let  $D$  be the chosen CM discriminant, and let  $k$  be the desired embedding degree. The following procedure implements a simplified subset of the general construction method described in section 3.2, and yields a suitable field size  $q$ , as prime subgroup order  $r$ , the curve order  $n$  (it also indirectly provides the cofactor  $m$ , which it seeks to minimize, and the trace of Frobenius  $t$ ).

1. Choose  $\ell \approx 2^{s/g}$  at random, where  $g \equiv \deg(\Phi_k)$ .

2. Compute  $r \leftarrow \Phi_k(\ell)$ ,  $t \leftarrow \ell + 1$ ,  $A \leftarrow 4r$ , and  $B \leftarrow (\ell - 1)^2$ .
3. Check that  $r$  is prime and also that  $A$  is invertible mod  $D$  and not a perfect square. If these conditions fail, choose another  $\ell$  in step 1.
4. Find the smallest  $m_0 \geq 0$  such that  $Am_0 - B = z_0D$  for some  $z_0$ , namely, set  $m_0 \leftarrow B/A \pmod{D}$  and  $z_0 \leftarrow (Am_0 - B)/D$ .
5. Check that  $z_0$  is a QR mod  $r$ , and then compute  $V \leftarrow \sqrt{z_0} \pmod{r}$ . If  $z_0$  is not a QR mod  $r$ , or if  $V^2 - z_0 \not\equiv 0 \pmod{4}$ , choose another  $\ell$  in step 1. This ensures that  $V^2 - z_0 \equiv 0 \pmod{A}$ .
6. Let  $i_0 \leftarrow (V^2 - z_0)/A$ ,  $m \leftarrow m_0 + i_0D$ ,  $n \leftarrow mr$ , and  $q \leftarrow n + t - 1$ . If  $q$  is not prime, restart with another  $\ell$  at step 1. Otherwise, we have the solution.

An example run of this algorithm for  $k = 7$  and  $D = 500003$  yields

```

q = 125070141847460013396986527273692733814291536913611095852428963052461\
  4109630975056367228761343097
r = 93161485761743186136191195699326539602148725131
m = 13425090940189806839398998187415093504886695170332
n = 125070141847460013396986527273692733814291536913611095852428963052461\
  4109630975056367228694013492
t = 67329606

```

Here  $r$  is a 157-bit prime, and  $q$  is a 320-bit prime. The curve is quickly found as  $E : y^2 = x^3 - 3x + b$  over  $\mathbb{F}_q$ , where

```

b = 315283565391589690418903185062076693159181569566876474809008162248459\
  256213526466473404332175506.

```

Another example, this time for  $k = 11$  and  $D = 500003$ :

```

q = 645793306563485513812965048035098778963201537968134813236427213716936\
  868831560525236938964558029767656530135495362724707835601050941159
r = 33237721806329292477733472892286817383477632299281817794659481922677
m = 19429529807320017250929519781158178098446838731085667916658667094871
n = 645793306563485513812965048035098778963201537968134813236427213716936\
  868831560525236938964558029767656530135495362724707835601045289667
t = 5651493

```

Here  $r$  is a 225-bit prime,  $q$  is a 448-bit prime, and the curve is  $E : y^2 = x^3 - 3x + b$  over  $\mathbb{F}_q$ , where

```

b = 647031214884002531107159243711613572471318050068528980921500459546489\
  30431134872280193466957817854171112529642288979049370687597648933.

```