

On Algebraic Immunity of Trace Inverse Functions over Finite Fields with Characteristic Two

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Abstract—The trace inverse function $\text{Tr}(\lambda x^{-1})$ over the finite field \mathbb{F}_{2^n} is a class of very important Boolean functions and has been used in many stream ciphers, for example, SFINKS, RAKAPOSHI, the simple counter stream cipher presented by W. Si and C.S. Ding, etc. In order to evaluate the security of those algorithms in assistance to (fast) algebraic attacks, it is essential to algebraic properties of $\text{Tr}(\lambda x^{-1})$. However, currently only some bounds on algebraic immunity of $\text{Tr}(\lambda x^{-1})$ are given in public literature. In this work we give the exact value of $\text{Tr}(\lambda x^{-1})$ over finite fields \mathbb{F}_{2^n} , that is, $\text{AI}(\text{Tr}(\lambda x^{-1})) = \lfloor \sqrt{n} \rfloor + \lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \rceil - 2 = \lfloor 2\sqrt{n} \rfloor - 2$, where $n \geq 2$, $\lambda \in \mathbb{F}_{2^n}$ and $\lambda \neq 0$, which is just the upper bound given by Y. Nawaz et al. And at the same time our result shows that D.K. Dalai's conjecture on the algebraic immunity of $\text{Tr}(\lambda x^{-1})$ is correct. What is more, we further demonstrate some weak properties of $\text{Tr}(\lambda x^{-1})$ in resistance to fast algebraic attacks.

Index Terms—Trace inverse function; Algebraic immunity; Fast algebraic attacks.

I. INTRODUCTION

Boolean functions have wide applications in cryptography [1]. An important topic in symmetric ciphers is the discussion of cryptographic properties of Boolean functions and constructions of Boolean functions with good cryptographic properties, which are mainly motivated by nonlinear filter/combiner generators (mainly corresponding to single-output Boolean functions) in stream ciphers and S-boxes (mainly corresponding to multi-output Boolean functions) in block ciphers. Since 70's in the 20th century, Boolean functions have been paid attention widely, and so far many fruitful and profound results on Boolean functions have been achieved [2].

The inverse function x^{-1} over the finite field \mathbb{F}_{2^n} is an important multi-output Boolean function, which is first introduced by K. Nyberg [3]. The inverse function x^{-1} possesses many good cryptographic properties, including permutation, high algebraic degree, high nonlinearity, almost optimal differential uniformity, etc, and has been adopted in many symmetric algorithms, for example, AES [4] in block ciphers, SNOW 2.0/3G [5], [6], ZUC [7] in stream ciphers, and so on.

The trace function $\text{Tr}(\lambda x)$ over the finite field \mathbb{F}_{2^n} is another important Boolean function, which characters all linear functions from extension fields to base fields [8]. The composite $\text{Tr}(\lambda x^{-1})$ of these two functions has been adopted by many stream ciphers, including SFINKS [9](eStream project),

RAKAPOSHI [10], the simple counter stream cipher proposed by W. Si and C.S. Ding [11].

Recently an important progress in cryptanalysis areas is algebraic attacks and fast algebraic attacks presented by N. Courtois and W. Meier [12], [13]. Algebraic attacks and fast algebraic attacks are very powerful analysis tools and can be applied to almost all cryptographic algorithms [14], [15], [16]. In order to resist against algebraic attacks, the concept of the algebraic immunity is introduced [12] and has been got attention widely [17], [18], [19], [20]. Therefore it is necessary to discuss the algebraic immunity of $\text{Tr}(\lambda x^{-1})$ in the sense of both itself cryptographic properties and security evaluations of those adopting them as a building component in resistance to algebraic attacks. In this paper we mainly deal with the algebraic immunity of $\text{Tr}(\lambda x^{-1})$.

A. Known results on the algebraic immunity of $\text{Tr}(\lambda x^{-1})$

In FSE 2006 Y. Nawaz et al [21] gave an upper bound of the algebraic immunity of $\text{Tr}(\lambda x^{-1})$ over \mathbb{F}_{2^n} by multiplying a very special Boolean function, that is,

$$\text{AI}(\text{Tr}(\lambda x^{-1})) \leq \lfloor \sqrt{n} \rfloor + \lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \rceil - 2, \quad (1)$$

where $\lambda \in \mathbb{F}_{2^n}$ and $\lambda \neq 0$, $\text{AI}(\text{Tr}(\lambda x^{-1}))$ denotes the algebraic immunity of $\text{Tr}(\lambda x^{-1})$, which will be defined in the next section.

In 2008 V.V. Bayev [22] further provided a lower bound of the algebraic immunity of $\text{Tr}(\lambda x^{-1})$ when $n \geq 5$ and constructed a large class of Boolean functions defined by their trace form with algebraic immunity $O(\sqrt{n})$, that is,

$$\text{AI}(\text{Tr}(\lambda x^{-1})) \geq \lfloor 2\sqrt{n+4} \rfloor - 4. \quad (2)$$

It is easy to see that the lower bound given by V.V. Bayev is bounded by a constant difference no more than 4 compared to the upper bound given by Y. Nawaz et al.

Recently D.K. Dalai [23] (See IACR eprint 2013/273) presented a method of computing algebraic immunities by means of incident matrices and utilized it to verify the upper bound given by Y. Nawaz et al when $n \leq 21$. On the basis of experiments, he further conjectured the algebraic immunity of $\text{Tr}(\lambda x^{-1})$ just arrives the upper bound given by Y. Nawaz et al, that is,

Conjecture 1 (Dalai's Conjecture):

$$\text{AI}(\text{Tr}(\lambda x^{-1})) = \lfloor \sqrt{n} \rfloor + \lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \rceil - 2. \quad (3)$$

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B. Main works in the paper

Inspired by V.V. Bayev's work [22], we introduce the concept of mono-integers in this paper. Based on mono-integers, two main contributions are made: one is to give the exact value of algebraic immunity of $\text{Tr}(\lambda x^{-1})$, see Theorem 1, which just arrives the upper bound given by Y. Nawaz et al, and at the same time shows that Dalai's conjecture is correct; the other is to demonstrate some weak properties of $\text{Tr}(\lambda x^{-1})$ in resistance to fast algebraic attacks, see Propositions 3 and 4 in Section 5.

Theorem 1 (Main Theorem): Let $\text{Tr}(\lambda x^{-1})$ be the trace inverse function over finite fields \mathbb{F}_{2^n} , where $n \geq 2$, $\lambda \in \mathbb{F}_{2^n}$ and $\lambda \neq 0$. Then

$$\text{AI}(\text{Tr}(\lambda x^{-1})) = \lfloor \sqrt{n} \rfloor + \lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \rceil - 2 = \lceil 2\sqrt{n} \rceil - 2. \quad (4)$$

C. The organization of the paper

The rest of the paper is organized as below: In Section 2 some preliminaries including basic concepts and notations are provided. In Section 3 a key concept, i.e., mono-integers, is introduced, and then some properties on mono-integers are derived. Based on mono-integers, an entire proof of the main theorem is given in Section 4, and further some weak properties of $\text{Tr}(\lambda x^{-1})$ in resistance against fast algebraic attacks are demonstrated in Section 5.

II. PRELIMINARIES

A. Boolean functions

Let \mathbb{F}_2 be the binary field with elements 0 and 1. For a given integer $n \geq 2$, we denote by \mathbb{Z}_n , \mathbb{F}_2^n and \mathbb{F}_{2^n} the residue class ring modulo n , the vector space of dimension n over \mathbb{F}_2 and the finite field with 2^n elements respectively.

Let $f(x_0, x_1, \dots, x_{n-1})$ be a mapping from \mathbb{F}_2^n to \mathbb{F}_2 , which is called an n -variables Boolean function. Denote by \mathcal{B}_n the set of all possible n -variables Boolean functions. For any $f(x_0, x_1, \dots, x_{n-1}) \in \mathcal{B}_n$, $f(x_0, x_1, \dots, x_{n-1})$ can be uniquely represented as a multivariate polynomial over \mathbb{F}_2 :

$$f(x_0, x_1, \dots, x_{n-1}) = \sum_{I \subseteq \mathbb{Z}_n} c_I \left(\prod_{i \in I} x_i \right), \quad c_I \in \mathbb{F}_2, \quad (5)$$

which is called the algebraic normal form of $f(x_0, x_1, \dots, x_{n-1})$. The algebraic degree of $f(x_0, x_1, \dots, x_{n-1})$, denoted by $\text{deg}(f(x_0, x_1, \dots, x_{n-1}))$, is defined as

$$\text{deg}(f(x_0, x_1, \dots, x_{n-1})) = \max\{|I| \mid I \subseteq \mathbb{Z}_n \text{ and } c_I \neq 0\},$$

where $|I|$ means the size of the set I .

Let ϕ be an arbitrary isomorphism from \mathbb{F}_2^n onto \mathbb{F}_{2^n} . Then

$$f(\phi^{-1}(x)) = \sum_{k=0}^{2^n-1} c_k x^k, \quad c_k \in \mathbb{F}_{2^n}$$

is called a polynomial representation of $f(x_0, x_1, \dots, x_{n-1})$ over \mathbb{F}_{2^n} . We write $f(\phi^{-1}(x))$ as $f(x)$ in short without confusion. For distinguishing from the degree of the polynomial $f(x)$, we call the algebraic degree of $f(\phi^{-1}(x))$ the Boolean

algebraic degree of $f(x)$, denoted by $\text{deg}_B(f(x))$. It is easy to see that

$$\begin{aligned} \text{deg}_B(f(x)) &= \text{deg}(f(\phi^{-1}(x))) \\ &= \max\{w_H(k) \mid c_k \neq 0, 0 \leq k \leq 2^n - 1\}, \end{aligned}$$

where $w_H(k)$ denotes the Hamming weight of k in the binary representation.

B. Trace inverse function

Let $\text{Tr}(x)$ be the trace function over \mathbb{F}_{2^n} , which is defined as

$$\text{Tr}(x) = \sum_{i=0}^{n-1} x^{2^i}.$$

The trace function $\text{Tr}(x)$ is a linear Boolean function from \mathbb{F}_{2^n} to \mathbb{F}_2 . Let x^{-1} be the inverse function over \mathbb{F}_{2^n} . In this paper we make convention that $\mathbf{0}^{-1} = \mathbf{0}$ and $\mathbf{0}^0 = \mathbf{0}$ (here $\mathbf{0}$ denotes the zero element in \mathbb{F}_2 or \mathbb{F}_{2^n}). Then the trace inverse function $\text{Tr}(\lambda x^{-1})$ is written as

$$\text{Tr}(\lambda x^{-1}) = \sum_{i=0}^{n-1} (\lambda x^{-1})^{2^i} = \sum_{i=0}^{n-1} \lambda^{2^i} x^{-2^i}.$$

For any nonzero $\lambda \in \mathbb{F}_{2^n}$, it is easy to verify that $\text{Tr}(\lambda x^{-1})$ is a Boolean function over \mathbb{F}_{2^n} and has the Boolean algebraic degree $n - 1$.

One of main works in this paper is to determine the algebraic immunity of $\text{Tr}(\lambda x^{-1})$ over \mathbb{F}_{2^n} , which is defined as the minimal Boolean algebraic degree of nonzero annihilators $f(x)$ of $\text{Tr}(\lambda x^{-1})$ or $\text{Tr}(\lambda x^{-1}) + 1$, and denoted by $\text{AI}(\text{Tr}(\lambda x^{-1}))$. For any given Boolean function $f(x)$ with $\text{deg}_B(f(x)) = d$, where d is a positive integer, denote

$$\mathcal{D}_t(f) = \{0 \leq k \leq 2^n - 1 \mid w_H(k) = t, c_k \neq 0\}, \quad 0 \leq t \leq d.$$

So $f(x)$ can be written as

$$f(x) = \sum_{t=0}^d \sum_{k \in \mathcal{D}_t(f)} c_k x^k.$$

Note that $x^{2^n} = x$ for any $x \in \mathbb{F}_{2^n}$, we have

$$\begin{aligned} &(\text{Tr}(\lambda x^{-1}) + \delta)f(x) \\ &= \sum_{t=0}^d \sum_{k \in \mathcal{D}_t(f)} c_k \left(\delta x^k + \sum_{i=0}^{n-1} \lambda^{2^i} x^{k-2^i \bmod (2^n-1)} \right), \end{aligned} \quad (6)$$

where $\delta \in \mathbb{F}_2$.

Our main idea is to observe when the monomial $x^{k-2^i \bmod (2^n-1)}$ occurs only one time in the expansion of $(\text{Tr}(\lambda x^{-1}) + \delta)f(x)$. If there exists such an integer k such that $x^{k-2^i \bmod (2^n-1)}$ occurs exactly one time in equality (6), then we have $(\text{Tr}(\lambda x^{-1}) + \delta)f(x) \neq 0$, that is, $f(x)$ must be not an annihilator of both $\text{Tr}(\lambda x^{-1})$ and $\text{Tr}(\lambda x^{-1}) + 1$.

C. Loop runs of integers

For a given integer $0 < k < 2^n - 1$, we are interested in its loop runs in the binary representation. Set $k = k_{[n-1]}k_{[n-2]} \cdots k_{[1]}k_{[0]} = \sum_{i=0}^{n-1} k_{[i]}2^i$, where $k_{[i]}$ means the i -th bit of k in the binary representation and takes 0 or 1 for $0 \leq i \leq n-1$. Joint all bits of k head to tail and form a circle. We call some consecutive 1s (or 0s) in the circle a loop one

(or zero) run of k , and denote by $\overbrace{\bar{b}b \cdots b\bar{b}}^x$ the x successive b 's, where $b = 0$ or 1 , \bar{b} means the complement of b , and x is called the length of the loop b run. For any given integer $0 < k < 2^n - 1$, we only focus on the number of loop one (or zero) runs, the maximal length of loop one runs and the maximal length of loop zero runs of k , denoted by $\tau(k)$, $r(k)$ and $s(k)$ respectively.

Example 1: Set $n = 8$ and $k = 26 = (00011010)_2$. Then 11 and 1 are two loop one runs of 26, and 0000 and 0 are two loop zero runs of 26. So $\tau(26) = 2$, $r(26) = 2$ and $s(26) = 4$.

III. MONO-INTEGERS

For two given integers $n \geq 4$ and $1 \leq d < n$, in this section we will be interested in which integers $0 < k < 2^n - 1$ of weight d and $i \in \mathbb{Z}_n$ make the monomial $x^{k-2^i \bmod(2^n-1)}$ occur exactly one time in the expansion of $(\text{Tr}(\lambda x^{-1}) + \delta)f(x)$. In order to character the above feature, we introduce a concept on mono-integers.

Definition 1: Let n and d be two integers such that $n \geq 4$ and $1 \leq d < n$. An integer $0 < k < 2^n - 1$ is called a *mono-integer* of size n and weight d if it meets the following two conditions:

- 1) $w_H(k) = d$; and
- 2) there exists an integer $i \in \mathbb{Z}_n$ such that for any $0 < k' < 2^n - 1$ with $w_H(k') \leq d$ and $j \in \mathbb{Z}_n$, if

$$k - 2^i \equiv k' - 2^j \pmod{2^n - 1},$$

then $k = k'$ and $i = j$.

For any given n and d , denote by $\mathcal{M}_{n,d}$ the set of all mono-integers of size n and weight d . Here we provide some properties on k and i mentioned in Definition 1, which will be used in the next section. For simplicity, throughout the paper we will make convention that the addition and subtraction on the exponents of monomials are done in the residue class ring modulo $(2^n - 1)$. So $k - 2^i \bmod(2^n - 1)$ can be written as $k - 2^i$ in short, similarly $k - 2^i + 2^j \bmod(2^n - 1)$ for $k - 2^i + 2^j$, and so on.

Lemma 1: Let $k \in \mathcal{M}_{n,d}$ and i be mentioned as Definition 1. When $d \geq 2$, we have

- 1) $k_{[i+1]} = k_{[i]} = k_{[i-1]} = 0$; and
- 2) $w_H(k - 2^i) > d$.

Proof: Since Item 2 can be derived directly from Item 1, we only prove Item 1. By Condition 2 in Definition 1, we have $w_H(k - 2^i + 2^j) > d$ for any $i \neq j \in \mathbb{Z}_n$. First we claim $k_{[i]} = 0$. This is because that if $k_{[i]} = 1$, we take j such that $k_{[j]} = 0$, then $w_H(k - 2^i + 2^j) = d$. A contradiction. Second, if $k_{[i+1]} = 1$, we can always take $j \neq i$ such that $(k - 2^i)_{[j]} = 1$ due to $d \geq 2$. Then $w_H(k - 2^i + 2^j) \leq d$. A contradiction. Finally, if $k_{[i-1]} = 1$, take $j = i - 1$, and

then we have $w_H(k - 2^i + 2^j) = w_H(k - 2^{i-1}) = d - 1$. A contradiction. So Item 1 holds. \blacksquare

Corollary 1: When $d \geq 2$, we have $s(k) \geq 3$ for any $k \in \mathcal{M}_{n,d}$.

In order to check $k \in \mathcal{M}_{n,d}$, in practice we only need to check whether some i 's indicating the second zero position in the maximal loop zero runs of k meet Condition 2 in Definition 1 or not, which is illustrated in the following figure:

$$\cdots 1 \overbrace{0 \cdots 00}^i 1 \cdots .$$

Roughly speaking, if one of such i 's does meet Condition 2 in Definition 1, then $k \in \mathcal{M}_{n,d}$, otherwise, $k \notin \mathcal{M}_{n,d}$. More precisely, we have the following conclusion on k and i .

Lemma 2 ($n \geq 4$): For any given $0 < k < 2^n - 1$ with $w_H(k) = d$, $k \in \mathcal{M}_{n,d}$ if and only if either of the following two conditions holds

- 1) $s(k) \geq r(k) + 2$; or
- 2) $s(k) = r(k) + 1$, $r(k) \geq 2$ and k contains exactly one maximal loop one run which is just the left neighbor of some maximal loop zero run of k .

Proof: When $d = 1$, since $n \geq 4$, thus $s(k) \geq 3$, and when $d \geq 2$, by Corollary 1 we have $s(k) \geq 3$ as well if $k \in \mathcal{M}_{n,d}$. Therefore below we only consider the case $s(k) \geq 3$. In this case we can take i as indicated below

$$\cdots 1 \overbrace{0 \cdots 00}^i 1 \cdots .$$

Then $k - 2^i$ can be written as

$$\cdots 0 \overbrace{1 \cdots 1}^{s(k)-1} 01 \cdots ,$$

and

$$w_H(k - 2^i) = (d - 1) + (s(k) - 1) = d + s(k) - 2.$$

Note that for any $j \in \mathbb{Z}_n$, we have

$$\begin{aligned} w_H(k - 2^i + 2^j) &\geq w_H(k - 2^i) - (r(k - 2^i) - 1) \\ &= d + s(k) - 1 - r(k - 2^i), \end{aligned}$$

where the equality holds if and only if j 's are taken as indicated below:

$$\frac{k - 2^i}{r(k-2^i)} \quad \frac{k - 2^i + 2^j}{r(k-2^i)} \\ \cdots 0 \underbrace{1 \cdots 1}_j 0 \cdots \Rightarrow \cdots 1 \underbrace{0 \cdots 0}_j 0 \cdots$$

Since Condition 2 in Definition 1 holds if and only if $k - 2^i$ has exactly one maximal loop one run and i just locates at the starting position among this maximal loop one run. It is easy to check that the above event occurs only under the following two cases: 1) $s(k) \geq r(k) + 2$; 2) $s(k) = r(k) + 1$ and k contains exactly one maximal loop one run which is just the left neighbor of some maximal loop zero run of k . The latter is illustrated in the following figure

$$\frac{k}{r(k) \quad r(k)+1} \quad \frac{k - 2^i}{r(k)-1 \quad r(k)} \\ \cdots 0 \underbrace{1 \cdots 1}_{i} 0 \cdots 001 \cdots \Rightarrow \cdots 0 \underbrace{1 \cdots 1}_{i} 01 \cdots 1 \cdots 01 \cdots$$

Note that $r(k) = s(k) - 1 \geq 2$, the desired conclusion follows. ■

For any given positive integers n and d , denote by $\mathcal{W}_{n,d}$ the set of integers $0 < k < 2^n - 1$ of weight d . The following corollary further gives an efficient and necessary condition on $\mathcal{M}_{n,d} = \mathcal{W}_{n,d}$, that is,

Corollary 2: Let $n \geq 4$ and $1 \leq d < n$. Then $\mathcal{M}_{n,d} = \mathcal{W}_{n,d}$ if and only if for any $1 \leq \tau \leq d$, we have

$$\lceil \frac{n-d}{\tau} \rceil \geq d - \tau + 3. \quad (7)$$

Proof: By the definitions of $s(k)$ and $r(k)$, we have

$$s(k) \geq \lceil \frac{n-d}{\tau(k)} \rceil \quad \text{and} \quad r(k) \leq d - \tau(k) + 1$$

for any $k \in \mathcal{W}_{n,d}$. Since $\mathcal{M}_{n,d} = \mathcal{W}_{n,d}$ is equivalent to $s(k) \geq r(k) + 2$ for any $k \in \mathcal{W}_{n,d}$, that is, inequality (7) holds. ■

Corollary 3: 1) $\mathcal{M}_{4,1} = \mathcal{W}_{4,1}$;

2) $\mathcal{M}_{n,d} = \mathcal{W}_{n,d}$ for any $n \geq 5$ and $1 \leq d \leq \lfloor 2\sqrt{n+4} \rfloor - 5$.

Proof: Item 1 is trivial, and we only prove Item 2. Obviously, by Corollary 2, if d meets $\frac{n-d}{\tau} \geq d - \tau + 3$ for any $1 \leq \tau \leq d$, then d meets inequality (7) as well, which implies that $\mathcal{M}_{n,d} = \mathcal{W}_{n,d}$. Note that $(d - \tau + 3)\tau \leq (\frac{d+3}{2})^2$ for any $1 \leq \tau \leq d$, further if d meets $n - d \geq (\frac{d+3}{2})^2$, we have $\mathcal{M}_{n,d} = \mathcal{W}_{n,d}$ as well. Note that the latter is equivalent to $d \leq \lfloor 2\sqrt{n+4} \rfloor - 5$, thus Item 2 holds. ■

Remark 1: By Item 2 of Corollary 3 a lower bound of the algebraic degree of $\text{Tr}(\lambda x^{-1})$ can be derived directly, which is just the one given by V.V. Bayev in [22].

IV. PROOF OF MAIN THEOREM

Note that $\lfloor \sqrt{n} \rfloor + \lceil \frac{n}{\lfloor \sqrt{n} \rfloor} \rceil = \lfloor 2\sqrt{n} \rfloor$ for all positive integers n , we denote $\mathbf{d}_0 = \lfloor 2\sqrt{n} \rfloor - 2$ for simplicity. In order to prove the main theorem, we only need to prove $d \geq \mathbf{d}_0$ for any nonzero annihilator $f(x)$ of $\text{Tr}(\lambda x^{-1})$ or $\text{Tr}(\lambda x^{-1}) + 1$ with Boolean algebraic degree d . Based on mono-integers introduced in the previous section we can do it. Before the proof of the main theorem we provide some conclusions related to mono-integers.

Proposition 1: Let $f(x)$ be a Boolean function with $\deg_B(f(x)) = d$ over \mathbb{F}_2^n and $\mathcal{D}_d(f)$ be defined as above, where $1 \leq d < n$. If $\mathcal{D}_d(f) \cap \mathcal{M}_{n,d} \neq \emptyset$, then $(\text{Tr}(\lambda x^{-1}) + \delta)f(x) \neq 0$ for $\delta = 0, 1$.

Proof: Let $k \in \mathcal{D}_d(f) \cap \mathcal{M}_{n,d}$. By the definition of mono-integers, the monomial $x^{k-2^i \bmod (2^n-1)}$ occurs exactly one time in the expansion of $(\text{Tr}(\lambda x^{-1}) + \delta)f(x)$ for some $i \in \mathbb{Z}_n$. So $(\text{Tr}(\lambda x^{-1}) + \delta)f(x) \neq 0$. ■

Corollary 4: Let $f(x)$ be an annihilator of $\text{Tr}(\lambda x^{-1}) + \delta$ for some $\delta \in \mathbb{F}_2$ with $\deg_B(f(x)) = d$. Then for any $k \in \mathcal{D}_d(f)$, we have $k \notin \mathcal{M}_{n,d}$.

Below we always assume that $f(x)$ is an annihilator of $\text{Tr}(\lambda x^{-1})$ or $\text{Tr}(\lambda x^{-1}) + 1$ with $\deg_B(f(x)) = d$. By Corollary 3, we have $d \geq 2$.

Lemma 3: Let $d \geq 2$ and $\mathcal{D}_t(f)$ be defined as above, where $1 \leq t \leq d$. Then

1) for any $k \in \mathcal{D}_d(f)$, we have $s(k) \leq r(k) + 1$; and

2) for any $k \in \mathcal{D}_{d-1}(f)$, we have $s(k) \leq r(k) + 3$. In particular, when k contains exactly one maximal loop one runs which is just the left neighbor of some maximal loop zero runs, we have $s(k) \leq r(k) + 2$.

Proof: Item 1 follows directly from Lemma 2. Below we consider Item 2. Suppose that there exists an $k \in \mathcal{D}_{d-1}(f)$ such that $s(k) \geq r(k) + 4$. Without loss of generality, let k have the form

$$\dots 1 \overbrace{0 \dots 000}^{\geq r(k)+4} 1 \dots$$

and take i and j as indicated above. Set $k' = k - 2^i + 2^j$. Then $k' \in \mathcal{D}_d(f)$ and has the form

$$\dots 1 \overbrace{0 \dots 0}^{\geq r(k)+2} 101 \dots$$

By Lemma 2, $k' \in \mathcal{M}_{n,d}$. A contradiction. The second part of Item 2 can be deduced directly by Item 2 of Lemma 2. So the conclusion follows. ■

Since it is easy to verify the main theorem for $n = 2, 3$, below we assume that $n \geq 4$ and prove the main theorem under the cases $r(k) \geq 2$ and $r(k) = 1$ respectively.

A. The case $r(k) \geq 2$

In this section we will prove Theorem 1 under the case $r(k) \geq 2$ for some $k \in \mathcal{D}_d(f)$. First when $s(k) < r(k)$, we have the following conclusion:

Lemma 4 ($s(k) < r(k)$): Let $k \in \mathcal{D}_d(f)$. If $s(k) < r(k)$, then $d \geq \mathbf{d}_0$.

Proof: By the definition of $s(k)$ and $r(k)$, we have

$$s(k) \geq \frac{n-d}{\tau} \quad \text{and} \quad r(k) \leq d - \tau + 1,$$

where $\tau(k) = \tau$. Thus we have

$$\begin{aligned} \frac{n-d}{\tau} &\leq s(k) \leq r(k) - 1 \leq d - \tau \\ \Rightarrow \quad n - d &\leq (d - \tau)\tau \leq \frac{d^2}{4} \\ \Rightarrow \quad d &\geq \lfloor 2\sqrt{n+1} \rfloor - 2 \geq \mathbf{d}_0. \end{aligned}$$

Lemma 4 shows that we only need to consider the cases $s(k) = r(k) + 1$ and $s(k) = r(k)$ when $r(k) \geq 2$.

Lemma 5: Let $k \in \mathcal{D}_d(f)$. If $s(k) = r(k) + 1 \geq 3$, then the length of the loop zero run after any maximal loop one run in k must be 1.

Proof: By the position relation of the maximal loop one runs and maximal loop zero runs of k , we subdivided k into two cases: adjacent and non-adjacent, which are illustrate in the following figure:

$$\begin{array}{c} k \qquad \qquad \qquad k' = k - 2^i + 2^j \\ \overbrace{\dots 1 \overbrace{0 \dots 00}^{r(k)+1} 1 \dots 0 \overbrace{1 \dots 10}^{r(k)} \dots 0}^x \quad \Rightarrow \quad \dots 0 \overbrace{1 \dots 10}^{r(k)} 101 \dots 1 \overbrace{0 \dots 00}^{r(k)+x} 1 \dots \\ \dots 1 \overbrace{0 \dots 00}^{r(k)+1} 1 \dots 0 \overbrace{1 \dots 10}^{r(k)} \dots 0 \quad \Rightarrow \quad \dots 0 \overbrace{1 \dots 10}^{r(k)+1} 10 \overbrace{0 \dots 00}^{r(k)+x} 1 \dots \end{array}$$

and

$$\begin{array}{c} k \qquad \qquad \qquad k' = k - 2^i + 2^j \\ \overbrace{\dots 1 \overbrace{0 \dots 00}^{r(k)+1} 1 \dots 0 \overbrace{1 \dots 10}^{r(k)} \dots 0}^x \quad \Rightarrow \quad \dots 0 \overbrace{1 \dots 10}^{r(k)+1} 10 \overbrace{0 \dots 00}^{r(k)+x} 1 \dots \\ \dots 1 \overbrace{0 \dots 00}^{r(k)+1} 1 \dots 0 \overbrace{1 \dots 10}^{r(k)} \dots 0 \quad \Rightarrow \quad \dots 0 \overbrace{1 \dots 10}^{r(k)+1} 10 \overbrace{0 \dots 00}^{r(k)+x} 1 \dots \end{array}$$

In order to eliminate the monomial x^{k-2^i} of $\text{Tr}(x^{-1}+\delta)f(x)$, $f(x)$ must contain another monomial $c_{k'}x^{k'}$, that is, $k' \in \mathcal{D}_d(f)$. Note that when $x \geq 2$, we have $k' \in \mathcal{M}_{n,d}$ by Lemma 2, which contradicts with $\mathcal{D}_d(f) \cup \mathcal{M}_{n,d} = \emptyset$. So the conclusion follows. ■

Lemma 6 ($s(k) = r(k) + 1$): Let $k \in \mathcal{D}_d(f)$. If $s(k) = r(k) + 1$ and $r(k) < d - \tau + 1$, then $d \geq \mathbf{d}_0$.

Proof: By Lemma 5, it is known that k contains at least one loop zero run of length 1, thus we have

$$s(k) \geq \frac{n-d-1}{\tau-1}.$$

Thus we have

$$\begin{aligned} \frac{n-d-1}{\tau-1} &\leq s(k) = r(k) + 1 \leq d - \tau + 1 \\ \Rightarrow n-d-1 &\leq (d-\tau+1)(\tau-1) \leq \frac{d^2}{4} \\ \Rightarrow d &\geq \lceil 2\sqrt{n} \rceil - 2 = \mathbf{d}_0. \end{aligned}$$

Lemma 7 ($s(k) = r(k) + 1$): Let $k \in \mathcal{D}_d(f)$. If $s(k) = r(k) + 1$ and $r(k) = d - \tau + 1 \geq 2$, then $d \geq \mathbf{d}_0$.

Proof: The condition $r(k) = d - \tau + 1$ implies that k contains exactly one maximal loop one run and all except this maximal loop one run are loop one runs of length 1. Without loss of generality, let k have the form:

$$\dots 0 \overbrace{1 \dots 1}^{r(k)} 0 \overbrace{1 \dots 0}^x 1 \dots$$

We first claim $x \leq r(k)$. Assume that $x = r(k) + 1$, we take i and j as indicated below:

$$\frac{\overbrace{1 \dots 1}^{r(k)} \overbrace{0 \dots 0}^{r(k)+1}}{k} \Rightarrow \overbrace{1 \dots 1}^{r(k)+2} \overbrace{0 \dots 0}^{r(k)} \overbrace{1 \dots 1}^{r(k)} \overbrace{0 \dots 0}^{r(k)+1}$$

and set $k' = k - 2^i + 2^j$. Then $k' \in \mathcal{D}_d(f)$. But by Lemma 2, $k' \in \mathcal{M}_{n,d}$. A contradiction. So $x \leq r(k)$.

Below we subdivide k into two cases:

- Case 1: k has the form:

$$\dots 0 \overbrace{1 \dots 1}^{r(k)} 0 \overbrace{1 \dots 0}^x 0 \overbrace{1 \dots 1}^{r(k)+1} 0 \overbrace{1 \dots 0}^y 1 \dots,$$

and i and j are taken as indicated above. Set $k' = k - 2^i + 2^j$. Then $k' \in \mathcal{D}_{d-1}(f)$ and has the form

$$\dots 0 \overbrace{1 \dots 1}^{r(k)} 0 \overbrace{1 \dots 0}^x 0 \overbrace{1 \dots 1}^{r(k)+2+y} 0 \dots$$

By Lemma 3 we have $y = 1$. The above result shows that k contains at least two loop zero runs of length 1 and one loop zero runs with length no more than $r(k)$, thus we have

$$s(k) \geq \frac{n-d-2+1}{\tau-2} = \frac{n-d-1}{\tau-2}.$$

So

$$\frac{n-d-1}{\tau-2} \leq s(k) = r(k) + 1 = d - \tau + 2,$$

which follows $d \geq \mathbf{d}_0$.

- Case 2: k has the form:

$$\dots 1 \overbrace{0 \dots 0}^i 0 \overbrace{1 \dots 1}^j 0 \overbrace{1 \dots 0}^x 10 \dots,$$

and i and j are taken as indicated above. Set $k' = k - 2^i + 2^j$. Then $k' \in \mathcal{D}_d(f)$ and has the form

$$\dots 0 \overbrace{1 \dots 1}^i 0 \overbrace{1 \dots 0}^j 0 \overbrace{1 \dots 0}^x 10 \dots.$$

Take i' and j' as indicated below:

$$\dots 0 \overbrace{1 \dots 1}^{i'} 0 \overbrace{1 \dots 0}^{j'} 0 \overbrace{1 \dots 0}^x 10 \dots,$$

and set $k'' = k' - 2^{i'} + 2^{j'}$. Then $k'' \in \mathcal{D}_{d-1}(f)$ and has the form

$$\dots 0 \overbrace{1 \dots 1}^{r(k)+1} 0 \overbrace{0 \dots 0}^{r(k)+2+x} 0 \dots 10 \dots.$$

By Item 2 of Lemma 3, we have $x = 1$. So

$$s(k') \geq \frac{n-d-1}{\tau(k')-1} = \frac{n-d-1}{\tau-2}.$$

Note that $s(k') = r(k') = r(k) + 1 = d - \tau + 2$, thus

$$\frac{n-d-1}{\tau-2} \leq d - \tau + 2,$$

which follows $d \geq \mathbf{d}_0$.

Combine the above two cases, we can get the desired conclusion. ■

Finally we give a proof of the case $s(k) = r(k) \geq 2$.

Lemma 8 ($s(k) = r(k)$): Let $k \in \mathcal{D}_d(f)$. If $s(k) = r(k) \geq 2$, then $d \geq \mathbf{d}_0$.

Proof: First, if $r(k) < d - \tau + 1$, we have

$$\frac{n-d}{\tau} \leq s(k) = r(k) \leq d - \tau,$$

which follows $d \geq \mathbf{d}_0$.

Second, when $r(k) = d - \tau + 1$, we divide k into two cases:

- Case 1: k contains exactly one maximal loop zero run. Then we have

$$s(k) \geq \frac{n-d+(\tau-1)}{\tau}.$$

Since $s(k) = r(k) \leq d - \tau + 1$, thus

$$\frac{n-d+(\tau-1)}{\tau} \leq d - \tau + 1,$$

which follows $d \geq \mathbf{d}_0$.

- Case 2: k contains at least two maximal loop zero runs. Without loss of generality, let k have the form

$$\dots 0 \overbrace{1 \dots 1}^{r(k)} 0 \overbrace{1 \dots 0}^x 0 \overbrace{1 \dots 1}^{r(k)} 0 \overbrace{1 \dots 0}^y 1 \dots.$$

Since we have proved the case $x = r(k)$ (see the proof on k' at Case 2 in Lemma 7), thus below we always assume $x < r(k)$. Take i as indicated below:

$$\dots 0 \overbrace{1 \dots 1}^{r(k)} \overbrace{0 \dots 0}^x 1 \dots 1 \overbrace{0 \dots 0}^{r(k)} \overbrace{1 \dots 0}^{j_2} \overbrace{0 \dots 0}^y 1 \dots,$$

and here j can be taken two possible values which are indicated above, that is, j_1 and j_2 . When $j = j_1$, set $k' = k - 2^i + 2^{j_1}$. Then $k' \in \mathcal{D}_d(f)$ and has the form

$$\dots 1 \overbrace{0 \dots 0}^{r(k)+x} \overbrace{0 \dots 0}^i 1 \dots 0 \overbrace{1 \dots 1}^{r(k)+1} \overbrace{0 \dots 0}^y 1 \dots.$$

By Item 1 of Lemma 3, we have $x \leq 2$. When $x = 2$, by Lemma 7, we can obtain $d \geq \mathbf{d}_0$. When $x = 1$, note that

$$\frac{n-d-1}{\tau-1} \leq s(k) = r(k) \leq d - \tau + 1,$$

we have $d \geq \mathbf{d}_0$ as well.

When $j = j_2$, set $k' = k - 2^i + 2^{j_2}$. Then we have $k' \in \mathcal{D}_{d-1}(f)$ and

$$\dots 0 \overbrace{1 \dots 1}^{r(k)} \overbrace{0 \dots 0}^x 1 \dots 1 \overbrace{0 \dots 0}^{r(k)+1+y} 1 \dots.$$

By Item 2 of Lemma 3, we have $y \leq 2$. Note that $x < r(k)$, thus we have

$$\frac{n-d-y+1}{\tau-1} \leq s(k) = r(k) \leq d - \tau + 1,$$

which follows $d \geq \mathbf{d}_0$.

Combine the above all cases, we can get the desired conclusion. \blacksquare

B. The case $r(k) = 1$

In this section we will prove Theorem 1 under the case $r(k) = 1$ for some $k \in \mathcal{D}_d(f)$. Note that $r(k) = 1$ implies $\tau = \tau(k) = d$ and $r_i = 1$ for all $1 \leq i \leq \tau$. By Lemma 2, we have $s(k) \leq 2$. The following conclusion shows that k does not contain two adjacent loop zero runs of length 2.

Lemma 9: For any $k \in \mathcal{D}_d(f)$, if $r(k) = 1$, then k does not contain the substring 100100.

Proof: Suppose that k contains the substring 100100 and i and j are taken as indicated above. Set $k' = k - 2^i + 2^j$. Then $k' \in \mathcal{D}_{d-1}(f)$ and contains the substring 100000 , which contradicts with Item 2 in Lemma 3. So the conclusion holds. \blacksquare

Proposition 2: Let $n \geq 4$ and $f(x)$ be an annihilator of $\text{Tr}(\lambda x^{-1}) + \delta$ for some $\delta \in \mathbb{F}_2$ with $\deg_B(f(x)) = d \geq 2$. If there exists an $k \in \mathcal{D}_d(f)$ such that $r(k) = 1$, then $d \geq \mathbf{d}_0$.

Proof: We will adopt *reductio ad absurdum* to prove the above conclusion, that is, assume that there exists an annihilator of $\text{Tr}(\lambda x^{-1})$ or $\text{Tr}(\lambda x^{-1}) + 1$ whose Boolean algebraic degree d is less than \mathbf{d}_0 . It is noticed that if the Boolean algebraic degree of $f(x)$ is less than $\mathbf{d}_0 - 1$, we always find another function $g(x)$, which is a multiple of $f(x)$ and has the Boolean

algebraic degree $\mathbf{d}_0 - 1$, such that $(\text{Tr}(\lambda x^{-1}) + \delta)g(x) = 0$ for some $\delta \in \mathbb{F}_2$. Therefore below we always assume $d = \mathbf{d}_0 - 1$.

Table I lists the values of the 3-tuples $(n, n-d, d)$ for $4 \leq n \leq 25$. The case $n = 4$ is trivial, and it is easy to verify that for any $k \in \mathcal{D}_d(f)$, when $n = 6, 8, 9, 11, 12, 13, 14, 15, 17, 18, 21$, k always contains two adjacent loop zero runs of length 2, and when $n = 16, 19, 20$ and $n \geq 22$, k has at least one loop zero runs with length no less than 3, which contradicts with Lemmas 9 and 3 respectively. Thus we have $d \geq \mathbf{d}_0$.

TABLE I
THE VALUES OF THE 3-TUPLES $(n, n-d, d)$ FOR $4 \leq n \leq 25$

n	4	5	6	7	8	9	10	11	12	13	14
$n-d$	3	3	4	4	5	6	6	7	8	8	9
d	1	2	2	3	3	3	4	4	4	5	5
n	15	16	17	18	19	20	21	22	23	24	25
$n-d$	10	11	11	12	13	14	14	15	16	17	17
d	5	5	6	6	6	6	7	7	7	7	8

When $n = 5$, k has the form 10010 for any $k \in \mathcal{D}_2(f)$. Take i and j as indicated below: 10010 , and set $k' = k - 2^i + 2^j$. Then $k' \in \mathcal{D}_1(f)$ and has the form 10000, which contradicts with Item 2 of Lemma 3. So $d \geq \mathbf{d}_0$.

When $n = 7$, k has the form 1001010 for any $k \in \mathcal{D}_3(f)$. Take i and j as indicated below: 1001010 , and set $k' = k - 2^i + 2^j$. Then $k' \in \mathcal{D}_2(f)$ and has the form 1000010. Take i' as indicated below: 1000010 . Then j' can be taken two possible values, that is, j_1 and j_2 , indicated as below: 1000010 . When $j' = j_1$, set $k'' = k' - 2^{i'} + 2^{j_1}$, then $k'' \in \mathcal{D}_3(f)$ and has the form 1000110, which is corresponding to the case $r(k'') \geq 2$ and has been proven in Section IV-A. When $j = j_2$, set $k^{(3)} = k' - 2^{i'} + 2^{j_2}$, then $k^{(3)} \in \mathcal{D}_1(f)$ and has the form 1000000. Take i'' as indicated below: 1000000 . Then $k^{(3)} - 2^{i''}$ has the form 0111111. It is easy to verify that for any $i'' \neq j_3 \in \mathbb{Z}_7$, we have $r(k^{(3)} - 2^{i''} + 2^{j_3}) \geq 2$, which implies that $d \geq \mathbf{d}_0$.

When $n = 10$, since $s(k) \leq 2$, k has just the form 1001010010 by Lemma 9. Similarly to the case $n = 7$, we have $d \geq \mathbf{d}_0$ as well and do not repeat it here.

Combine the above all cases, and we can get the desired conclusion. \blacksquare

V. WEAK PROPERTIES IN RESISTANCE TO FAST ALGEBRAIC ATTACKS

When n is a bit larger, it is easy to see that \mathbf{d}_0 is far smaller than $\lceil \frac{n}{2} \rceil$, which shows that $\text{Tr}(\lambda x^{-1})$ has weak properties in resistance to algebraic attacks. In this section we will further demonstrate that $\text{Tr}(\lambda x^{-1})$ has weak properties in resistance to fast algebraic attacks as well.

Proposition 3: Let $n \geq 5$ and $2 \leq d < \mathbf{d}_0$, and $f(x)$ be a Boolean function of the form $\sum_{k \in \mathcal{W}_{n,d}} c_k x^k$ over \mathbb{F}_{2^n} . Denote $g_\delta(x) = (\text{Tr}(\lambda x^{-1}) + \delta)f(x)$ for $\delta = 0, 1$. Then we have

$$\deg_B(g_\delta(x)) = d + \max_{k \in \mathcal{D}_d(f)} s(k) - 1. \quad (8)$$

In particular, we have

$$\deg_B(g_\delta(x)) \geq d + \lceil \frac{n}{d} \rceil - 2, \quad (9)$$

where the equality holds if and only if $s(k) = \lceil \frac{n}{d} \rceil - 1$ for all $k \in \mathcal{D}_d(f)$.

Proof: We first prove equality (8). Since $s(k) \geq 2$ for any $k \in \mathcal{D}_d(f)$ when $d < d_0$, it is easy to see that

$$\begin{aligned} \deg_B(g_\delta(x)) &\leq \max_{k \in \mathcal{D}_d(f), i \in \mathbb{Z}_n} \{w_H(k), w_H(k - 2^i)\} \\ &= d + \max_{k \in \mathcal{D}_d(f)} s(k) - 1. \end{aligned}$$

On the other hand, without loss of generality, let $k \in \mathcal{D}_d(f)$ with maximal $s(k)$ among $\mathcal{D}_d(f)$, and take $i \in \mathbb{Z}_n$ as

indicated below: $\dots 1 \overbrace{0 \dots 0}^{s(k)} 1 \dots$. Then $k - 2^i$ has the form $\dots 0 \overbrace{1 \dots 1}^{s(k)+1} \dots$ and $w_H(k - 2^i) = d + s(k) - 1 > d$.

By Lemma 4, we have $s(k) \geq r(k)$. When $s(k) > r(k)$ or $s(k) = r(k)$ but k has exactly one maximal loop one run which is just at the left neighbor of the maximal loop zero run of k , it is easy to see that $k - 2^i$ occurs exactly one time in $g_\delta(x)$, thus $\deg_B(g_\delta(x)) \geq w_H(k - 2^i) = d + s(k) - 1$. Otherwise, we take j as indicated below:

$$\dots 0 \overbrace{1 \dots 1}^{r(k)} 0 \dots 1 \overbrace{0 \dots 0}^{s(k)} 1 \dots,$$

and set $k' = k - 2^i + 2^j$. Then $w_H(k') = d$ and has the form:

$$\dots 1 \overbrace{0 \dots 0}^{\geq r(k)+1} \dots 0 \overbrace{1 \dots 1}^{\geq s(k)+1} \dots$$

If $k' \in \mathcal{D}_d(f)$, since $s(k') \geq r(k) + 1 = s(k) + 1 > s(k)$, it is a contradiction with the pick of k , which shows that $k - 2^i$ occurs exactly one time in $g_\delta(x)$. So $\deg_B(g_\delta(x)) \geq w_H(k - 2^i) = d + s(k) - 1$. So equality (8) follows.

Second, note that for any $k \in \mathcal{W}_{n,d}$, we have

$$s(k) \geq \lceil \frac{n-d}{\tau(k)} \rceil \geq \lceil \frac{n-d}{d} \rceil = \lceil \frac{n}{d} \rceil - 1.$$

So the conclusion follows. \blacksquare

Further for a general Boolean function $f(x)$ with $\deg_B(f(x)) = d$, we have

Proposition 4: Let $n \geq 7$ and $n \neq 9$, and $f(x)$ be a Boolean function with $\deg_B(f(x)) = d$, where $2 \leq d \leq \lfloor \sqrt{n} \rfloor$. Denote $g_\delta(x) = (\text{Tr}(\lambda x^{-1}) + \delta)f(x)$ for $\delta = 0, 1$. Then we have

$$\deg_B(g_\delta(x)) \geq d + \max_{k \in \mathcal{D}_d(f)} s(k) - 2. \quad (10)$$

In particular, we have

$$\deg_B(g_\delta(x)) \geq d + \lceil \frac{n}{d} \rceil - 3. \quad (11)$$

Proof: By Corollaries 2 and 3, it is easy to verify that $\mathcal{M}_{n,d} = \mathcal{W}_{n,d}$ for $n \geq 7$, $n \neq 9$ and $2 \leq d \leq \lfloor \sqrt{n} \rfloor$. For any $k \in$

$\mathcal{D}_d(f)$, take i as indicated below: $\dots 1 \overbrace{0 \dots 0}^{s(k)} 0 \dots$. Then $w_H(k) =$

$d + s(k) - 2$ and $x^{k-2^i \bmod(2^n-1)}$ only occurs one time in $g_\delta(x)$. So the conclusion follows. \blacksquare

Example 2: Let $n \geq 5$ and $2 \leq k \leq \lfloor \sqrt{n} \rfloor$. For any given nonzero $\xi \in \mathbb{F}_{2^n}$, denote $f_\xi(x) = \text{Tr}(\xi x^k) \neq 0$, where $w_H(k) = d$ and $s(k) = \lceil \frac{n}{d} \rceil - 1$. Set $g_{\delta,\xi}(x) = (\text{Tr}(\lambda x^{-1}) + \delta)f_\xi(x)$ for $\delta = 0, 1$. By Proposition 3, we have $\deg_B(g_{\delta,\xi}(x)) = d + \lceil \frac{n}{d} \rceil - 2$.

When $\text{Tr}(\lambda x^{-1})$ is used as a component in stream ciphers, an attacker can achieve good trade-off of time-memory-data by choosing carefully different d , k and ξ . For example, take $n = 128$, and some possible combination of d and $\deg_B(g_{\delta,\xi}(x))$ are listed in Table 2.

Table 2 Possible combination of d and $\deg_B(g_{\delta,\xi}(x))$

d	$\deg_B(g_{\delta,\xi}(x))$
2	64
4	34
8	22

Remark 2: By Proposition 4, it is possible to achieve $g_{\delta,\xi}(x)$ with lower Boolean algebraic degree by choosing more complex $f_\xi(x)$ when $2 \leq d < \lfloor \sqrt{n} \rfloor$. At this time the Boolean algebraic degree of $g_{\delta,\xi}(x)$ will reduce at most 1 than those listed in Table 2. When $d = 8$, if there exists $f_\xi(x)$ such that $\deg_B(g_{\delta,\xi}(x)) = 21$, which is just equal to $\text{AI}(\text{Tr}(\lambda x^{-1}))$, then the attacker utilizes $f_\xi(x)$ and $g_{\delta,\xi}(x)$ to launch fast algebraic attacks and will take the same cost in the off-line phase as that by algebraic attacks, however the cost taken by the attacker in the on-line phase will be reduced dramatically down.

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