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Identification of Multiple Invalid Signatures in Pairing-based Batched Signatures

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Abstract. This paper describes new methods in pairing-based signature schemes for identifying the invalid digital signatures in a batch, after batch verification has failed. These methods efficiently identify non-trivial numbers of invalid signatures in batches of (potentially large) numbers of signatures.
Our methods use "divide-and-conquer" search to identify the invalid signatures within a batch, but prune the search tree to substantially reduce the number of pairing computations required. The methods presented in this paper require computing on average $O(w)$ products of pairings to identify w invalid signatures within a batch of size N, compared with the $O(w(\log_2(N/w)+1))$ [for $w < N/2$] that traditional divide-and-conquer methods require. Our methods avoid the problem of *exponential growth* in expected computational cost that affect earlier proposals which, on average, require computing $O(w)$ products of pairings.
We compare the expected performance of our batch verification methods with previously published divide-and-conquer and exponential cost methods for Cha-Cheon identity-based signatures [6]. However, our methods also apply to a number of short signature schemes and as well as to other identity-based signature schemes.
Keywords Pairing-based signatures, Identity-based signatures, Batch verification, Short signatures, Wireless networks

1 Introduction

Public-key digital signatures have frequently been used in proposals for securing wireless network protocols. Proposals include methods for performing the following: combating SPAM [11]; securing routing protocols [19, 30]; providing secure accounting and charging for use of the wireless network, or securely giving incentives to nodes for desirable (to the network) behavior [22, 4]; protecting location and safety messages in vehicular networks [23, 21]; and securely transporting ordinary messages in delay (or disruption) tolerant networks [7, 26]. Even in wireless networks that have significant performance constraints such as sensor networks, it has been

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argued on efficiency grounds that signature schemes should be used for message authentication rather than symmetric cryptographic techniques [10, 27].

When protocol designers need to select a bandwidth efficient signature scheme, they will be drawn to schemes based on bilinear pairings, such as the short signature schemes [2, 5] or the bandwidth efficient identity-based signature schemes [6, 5]. However, the computational cost of such schemes, especially the cost of their verification algorithms, can negatively impact the performance of wireless networks (e.g., increased delay, CPU utilization, energy consumption). Therefore, whenever circumstances allow, designers will employ batch verification methods for such pairing-based signature schemes [29, 5, 14]. Adversaries can attempt to negate the advantages of batch verification by corrupting messages or signatures within a batch. To counter such attacks, efficient methods are needed to identify the valid signatures within a batch that has failed initial batch verification.

To discover the valid signatures in an invalid batch, rather than verifying each signature individually, "divide-and-conquer" (DQ) techniques have been proposed [20, 14]. These methods can be significantly faster than verifying individually whenever the ratio of the number of invalid signatures to the batch size is low. These methods require only $O(w(\log_2(N/w) + 1))$ batch verifications and, for pairing-based signatures, product of pairings computations [13]. Recent methods for identification of invalid pairing-based signatures require $O(w)$ batch verifications [14]. When the ratio w/N is very low these methods can provide significant performance improvements over DQ methods; however, the cost of performing the batch verifications used in these methods grows exponentially, limiting their use to very small batches, and to batches with only very few invalid signatures.

Our contribution. In this paper, we present two new methods for finding invalid signatures in pairing-based schemes. These methods are based on divide-and-conquer searching, but differ from previous methods in how the (sub-)batches are verified. The average number of product of pairings computations required in our methods is $O(w)$, which is a substantial improvement over previous divide-and-conquer methods when the ratio w/N is low, and is the same complexity as the exponential cost methods. The expected number of multiplications in $\mathbb{F}_{q^d}$ required of the new methods is $O(w\sqrt{N})$, and $O(wN)$, compared to estimates of the cost of the two exponential cost methods, $O(N^{w-1}/(w-1)!)$ and $O(w^{w-1}N^{\frac{w-1}{2}}/(w-1)!)$ [14]. We have specified these methods and compared their performance for Cha-Cheon signatures [6]; however, these methods can be applied to several other pairing-based signature schemes, specifically the batched identity-based and batched short signature schemes discussed in [8].

2 Notation

In this paper we assume that pairing-based schemes use bilinear pairings on an elliptic curve E, defined over $\mathbb{F}_q$, where q is a large prime. $\mathbb{G}_1$ and $\mathbb{G}_2$ are distinct subgroups of prime order r on this curve, where $\mathbb{G}_1$ is a subset of the points on E with coordinates in $\mathbb{F}_q$, and $\mathbb{G}_2$ is a subset of the points on E with coordinates in $\mathbb{F}_{q^d}$, for a small integer d (the embedding degree). The pairing e is a map from $\mathbb{G}_1 \times \mathbb{G}_2$ into $\mathbb{G}_T$ where $\mathbb{G}_T$ is a multiplicative group of order r in the field $\mathbb{F}_{q^d}$.

We use the following notation for the components of the costs of (batch) signature verification and invalid signature identification methods for Cha-Cheon signatures. $\mathsf{CstDblPair}$ is the cost of a double product of pairings computation [13]. $\mathsf{CstMult}\mathbb{G}_1(\mathrm{t}_1)$ is the cost of multiplying an element of $\mathbb{G}_1$ by a scalar s of size $|s|$ and $t_1 = \lceil \log_2(|s|) \rceil$; likewise $\mathsf{CstDlbMult}\mathbb{G}_1(\mathrm{t}_1, \mathrm{t}_2)$ is the cost of a pair of multiplications of elements of $\mathbb{G}_1$ by scalars of size t_1 and t_2 simultaneously. $\mathsf{CstAdd}\mathbb{G}_1$ is the cost of adding two elements of $\mathbb{G}_1$, and $\mathsf{CstSub}\mathbb{G}_1$ is the cost of subtracting an element of $\mathbb{G}_1$ from another element. $\mathsf{CstInv}\mathbb{G}_\mathrm{T}$ is the cost of computing an inverse of an element in $\mathbb{G}_T$; $\mathsf{CstMult}\mathbb{G}_\mathrm{T}$ is the cost of multiplying two elements of $\mathbb{G}_T$; and $\mathsf{CstExpt}\mathbb{G}_\mathrm{T}(\mathrm{t}_1)$ is the cost of raising an element of $\mathbb{G}_T$ to the power s.

3 Background

Batch cryptography was introduced by Fiat [9], and the first batch signature scheme was that of Naccache *et al.* [18] for a variant of DSA signatures. Bellare *et al.* [1] presented three generic methods for batching modular exponentiations: *the random subset test*, the *small exponents test* (SET), and the *bucket test*, which are related to techniques in [18, 28].

The inputs to the small exponents test are a security parameter l, a generator g of the group G of prime order q, and (x_1, y_1), $(x_2, y_2), \ldots, (x_N, y_N)$ with $x_i \in Z_q$ and $y_i \in G$. The verifier 1) checks that $g^{x_i} = y_i$ for all i, $1 \leq i \leq N$; 2) chooses n random integers $r_1, \ldots, r_N$ in the range $[0, 2^l - 1]$; 3) computes $x = \sum_{i=1}^{N} x_i r_i$ and $y = \prod_{i=1}^{N} {y_i}^{r_i}$; and 4) tests whether $g^x = y$ and accepts the batch if true, else rejects. If the test rejects a batch, then there is at least one invalid pair (x_i, y_i); the probability that the test accepts a batch containing invalid signatures is at most 2^{-l} [1], if the order of G is a prime [3]. One of the r's can be set to one without loss of security [14]. The small exponents test has appeared in pairing-based signature schemes [2, 5] including, in [14], as the batch verifier for the Cha-Cheon signature scheme [6].[1]

Cha-Cheon identity-based signature scheme. $H(m, U)$ is a cryptographic hash that maps a bit string m and a point $U \in \mathbb{G}_1$ to an integer between 1 and r.

1. *Setup:* The system manager selects an order r point $T \in \mathbb{G}_2$ and randomly selects an integer s in the range $[1, r-1]$. The manager computes $S = sT$. The public system parameters are T and S. The system manager's secret key is s.
2. *Extract:* Each user is given a key pair. The user's public key, Q, is a point in $\mathbb{G}_1$ that is derived from the user's identity using a public algorithm. The user's private key is $C = sQ$.
3. *Sign:* To sign a message m, the signer randomly generates an integer t in the range $[1, r-1]$ and outputs a signature (U, V) where $U = tQ$ and $V = (t + H(m, U))C$
4. *Verify:* To verify a signature (U, V) of message m, the verifier derives the signer's public key Q from the alleged signer's identity and computes $h = H(m, U)$. If $e(U + hQ, S) = e(V, T)$ then the signature is accepted. Otherwise, the signature is rejected. This test can be rewritten as $1 = e(U + hQ, S) \cdot e(V, -T)$ which can be computed more efficiently [13].

In [14] the following batch verifier was presented. The verifier obtains N messages m_i, for $i = 1$ to N, and the signatures (U_i, V_i) and signer's identity for each message. The verifier derives each public key Q_i from the signer's identity and checks that U_i and V_i are elements of $\mathbb{G}_1$.

[1] For Cha-Cheon signatures with the cost parameters in Section 5, SET always verifies a valid batch more efficiently than the random subset test, and the bucket test verifies a valid batch more efficiently than SET when the batch size exceeds 512.

The verifier sets $r_1 = 1$ and generates random values r_i from $[0, 2^l - 1]$, for $i = 2$ to N. The batch is valid if

$$1 = \alpha_0 = e\left(\sum_{i=1}^{N} B_i, S\right) \cdot e\left(\sum_{i=1}^{N} D_i, -T\right)$$

where $B_i = r_i \left(U_i + H(m_i, U_i) \cdot Q_i\right)$, $D_i = r_i V_i$, and 1 is the identity in $\mathbb{F}_{q^d}$.

3.1 Identifying Invalid Signatures

The problem of identifying invalid signatures within a batch has only recently been investigated. Work in this area generally falls into three categories: divide-and-conquer methods [20, 14], identification code based methods [20], and expo-nent testing methods [15, 16, 25, 14].

Divide-and-Conquer Methods Pastuszak *et al.* [20] first investigated methods for identifying invalid signatures within a batch. They explored "divide-and-conquer" methods for generic batch verifiers, i.e., methods that work with any of the three batch verifiers studied by Ballare *et al.* In these methods the set of signatures in an invalid batch is repeatedly divided into $d \geq 2$ smaller sub-batches to verify. The most efficient of their techniques, the *Fast DC Verifier* method, exploits knowledge of the results of the first $d - 1$ sub-batch verifications to determine whether the verification of the d^{th} sub-batch is necessary, i.e., if a (sub-)batch *batch* is invalid and the first $d-1$ sub-batches of *batch* are all valid, then the d'th sub-batch must be invalid, and the batch verifier for that sub-batch is not computed. Performance measurements of one of the methods of [20] for the Boneh, Lynn and Shacham (BLS) [2] signature scheme have been reported [8]. The authors observed that the divide-and-conquer method they studied outperformed verifying each signature individually when $w/N < .15$ in batches of 1024 BLS signatures using 160-bit MNT curves.

In [14] a more efficient divide-and-conquer method called Binary Quick Search (BQS) was presented; BQS is applicable to small exponents test based verifiers. In this method a batch verifier that compares two quantities, X and Y, is replaced with an equivalent test $A = XY^{-1}$, and the batch is accepted if $A = 1$.[2] The BQS algorithm is always[3] more efficient than any $d = 2^n$ary DC Verifier; When it is necessary to verify the d^{th} child sub-batch, in BQS the sub-batch can be verified by simply computing a single inverse operation and a single multi-term multiplication (or $d - 1$ ordinary multiplications) rather than the much more expensive batch verification required by the Fast DC verifier. The upper bound of the number of batch verifications required by BQS is half that of the Fast DC Verifier for $d = 2$ [14].

Identification Code based Methods Pastuszak *et al.* [20] investigated using a Hamming identification code and a two-layer Hamming identification code for identifying invalid signatures in generic batch verification. The Hamming code verifier can identify a single error in a batch of size $2^n - 1$ using $n + 2$ batch verifications, and the two-layer verifier can identify 2 invalid signatures in a batch of $2^n - 2$ signatures using $3n + 3$ batch verifications.

[2] For the initial batch verification, if it is more efficient to do so compute X and Y, compare them, and compute $A = XY^{-1}$ if the comparison fails; otherwise A is computed directly, e.g., in Cha-Cheon where $A = \alpha_0$.

[3] Except when $w = 1$ and the invalid signature is located in the rightmost position in the batch then Fast DC verifier and BQS have equal costs.

Exponent Testing Methods The first exponent testing method, developed by Lee *et al.* [15], was capable of finding a single invalid signature within a batch of "DSA-type" signatures. Signatures of this type have verification equations of the form "$g^m = s \mod p$" where m is the message, s is the signature, the generator g has order q, and p and q are primes where $q \mid (p-1)$. To identity an invalid signature, compute $X = \prod_{i=1}^{N} s_i / g^{\sum_{i=1}^{N} m_i}$ and $Y = \prod_{i=1}^{N} s_i^i / g^{\sum_{i=1}^{N} i \cdot m_i}$ and test whether $Y = X^z$ for $z \in [1, N]$. The Exponentiation method of Law and Matt [14], for the special case of identifying a single invalid signature, is similar to the above method.

Lee et al. [16] applied their approach for DSA-type signatures to identifying a single invalid signature in batches of RSA signatures. They addressed the problem of identifying multiple invalid RSA signatures by using their RSA method in a divide-and-conquer method that is somewhat similar to the Single Pruning Search we present in Section 4. However, Stanek showed in [25] that their approach for RSA signatures is not secure.

In [14] two exponent testing methods for pairing-based batch signatures, the Exponentiation method and the Exponentiation with Sectors method, were presented. Both methods require computing a number of batch verifications that are proportional to the number of invalid signatures w in the batch. The Exponentiation method requires $w + 1$ verifications (including the initial batch verification) and the same number of product of pairings computations. Exponentiation with Sectors requires at most $2w + 1$ product of pairings computations. Both methods use exhaustive search during batch verification, resulting in exponential cost.

Exponentiation Method. For the Cha-Cheon signature scheme, compute α_0 and test whether α_0 is equal to the identity. If so, the batch is valid. Otherwise compute $\alpha_j, w \geq j \geq 1$,

$$\alpha_j = e\left(\sum_{i=1}^{N} i^j B_i, S\right) e\left(\sum_{i=1}^{N} i^j D_i, -T\right), \tag{1}$$

and perform a test on the values $\alpha_j, \alpha_{j-1}, \ldots \alpha_0$. For $j = 1$, test whether $\alpha_1 = \alpha_0^{z_1}$ has a solution for $1 \leq z_1 \leq N$ using Shanks' giant-step baby-step algorithm [24]. If successful, $w = 1$ and z_1 is the position of the invalid signature. In general the method tests whether

$$\alpha_j = \prod_{t=1}^{j} (\alpha_{j-t})^{(-1)^{t-1} p_t} \tag{2}$$

has a solution where p_t is the tth elementary symmetric polynomial in $1 \leq z_1 \leq \ldots \leq z_j \leq N$. The authors show that the tests can be performed in $O(\sqrt{N})$ for $j = 1$ and $O(N^{j-1}/(j-1)!)$ for $j \geq 2$ multiplications in $\mathbb{F}_{q^d}$. If a test fails increment j, compute α_j, and test. When $j = w$ the test will succeed, and the values of $z_1, \ldots, z_w$ are the positions of the invalid signatures.

Exponentiation with Sectors Method. The Exponentiation with Sectors Method uses two stages. In the first stage, the batch is divided into approximately $\sqrt{N}$ sectors of approximately equal size and the Exponentiation method is used, where each B_i, and D_i within a sector is multiplied by the same constant, to identify the v invalid sectors. In the second stage, the Exponentiation method is used to find the invalid signatures within a batch consisting of the signatures from the v invalid sectors. This method requires $w+v+1$ product of pairings computations, including the initial verification, where $v \leq \min(w, \sqrt{N})$. During the first stage the tests can be performed in $O(N^{\frac{1}{4}})$ for $j = 1$ and $O(\sqrt{N}^{j-1}/(j-1)!)$ for $j \geq 2$ multiplications in $\mathbb{F}_{q^d}$. During the second stage the number of multiplications required for $w \leq j \geq v$ is $O(\sqrt{v\sqrt{N}})$ for $j = 1$, and $O(v\sqrt{N}^{j-1}/(j-1)!)$ for each $j \geq 2$.

4 An Alternate Approach to Divide-and-Conquer Methods

Divide-and-conquer methods can be viewed as operating on (for simplicity) a binary tree T with $w \geq 1$ invalid signatures whose root node, $root_T$, is the batch, and each pair of child nodes represents the two nearly equal size sub-batches of their parent. Previously published methods such as Binary DC Verifier and BQS identify the invalid signatures within the initial batch by descending through the tree, performing verifications on the sub-batches of the nodes they encounter. When one of the methods reaches a node whose sub-batch is valid, the methods do not visit its descendants, if any. The methods identify the invalid signatures by identifying those nodes that are either the ancestors of the leaf nodes of T that represent invalid signatures, or the leaf nodes themselves. The difference between the published methods are 1) the degree of the tree and 2) how efficiently the nodes of the tree are verified.

The methods we propose view T as consisting of a parent sub-tree PT with root node $root_{PT} = root_T$, and the leaves of PT are the roots of the w maximal sub-trees ST_i, $i = 1, .., w$, of T which represent sub-batches that have a single invalid signature. If the $w = 1$, then $T = ST_1$ and PT is the node $root_T$. The new methods identify invalid signatures by descending through PT and identifying its leaves, and concurrently identifying the single invalid signature in each of the sub-batches these leaves represent.

For signature schemes such as the Cha-Cheon, *node* of T is the root of some ST_i if there exists a value z, $lb \leq z \leq ub$, that is a solution to $\alpha_{1,node} = \alpha_{0,node}^z$. The values lb and ub are the lower and upper bounds of the sub-batch represented by *node* within B, and $\alpha_{j,node} = e\left(\sum_{i=lb}^{ub} i^j B_i, S\right) \cdot e\left(\sum_{i=lb}^{ub} i^j D_i, -T\right)$. Shanks' giant-step baby-step algorithm can determine if such a solution exists in time (multiplications in $\mathbb{F}_{q^d}$) proportional to the square root of the size of the sub-batch. If no solution is found, then *node* is in PT but is not a leaf; hence its children must be tested. We refer to this approach as *single pruning*.

When the children of an interior node p in PT, l and its sibling r, are leaves of PT, there exist values z_l in the range of indexes of the signatures in the left sub-batch and z_r in the range of signatures in the right sub-batch, such that $\alpha_{1,p} = \alpha_{0,l}^{z_l} \cdot \alpha_{0,r}^{z_r}$. The values z_l and z_r can be determined using an algorithm, $PairSolver(Left, Right)$, with cost proportional to the size of the (sub-)batch represented by p, see Appendix A.1. If the algorithm fails, then at least one of the child nodes is not a leaf of PT and they are tested individually using Shanks' algorithm. We refer to this approach as *paired single pruning*.

4.1 Single Pruning Search (SPS) Method.

The recursive algorithm below describes the Single Pruning Search (SPS) method on a batch B which is a list of $N = 2^h$, $h \geq 1$, randomly ordered message / signature pairs $((m_1, s_1), \ldots, (m_N, s_N))$ where the signature components for Cha-Cheon are verified elements of $\mathbb{G}_1$. On the initial call to $SPS(X, \alpha_{0,P}, \alpha_{1,P})$, $X = B$, $\alpha_{0,P} = 1$, $\alpha_{1,P} = 1$. SPS uses the following algorithms:

1. $Get_0(X)$ – checks whether α_0 has been computed for X and if so returns it; otherwise it computes α_0 by the most efficient method available, and it may compute α_0^{-1}. See Appendix A.
2. $Get_1(X)$ – checks whether α_1 has been computed for X and if so returns it; otherwise it computes α_1 by the most efficient method available,and it may compute α_1^{-1}. See Appendix A.
3. $Shanks(X)$ – if X has a single invalid signature, the algorithm returns the position of the invalid signature; otherwise the algorithm returns 0. See Appendix A.1.
4. $Left(X)$ – returns a sub-batch with the first $len/2$ pairs in X, or $\varnothing$ if $X = \varnothing$.
5. $Right(X)$ – returns a sub-batch with the later $len/2$ pairs in X, or $\varnothing$ if $X = \varnothing$.
6. $Len(X)$ – returns the number of pairs in X, or 0 if $X = \varnothing$.

Algorithm $SPS(X, \alpha_{0,P}, \alpha_{1,P})$ *(Single Pruning Search)*

Input: X a list of message / signature pairs, $\alpha_{0,P}$ and $\alpha_{1,P}$ in $\mathbb{G}_T$.
Output: A list of the invalid pairs in the batch.
Return: A boolean.

```
if (Len(X) = 1) then
    output X                                        //The signature pair X = (m_i, s_i)
    return (true)
else
    α_{0,N} ← Get_0(X)
    if (α_{0,N} = 1) then
        return (true)
    elseif (X = B) then
        invα_{0,[B]} ← α_{0,N}^{-1}          // Since α_0 of B is not equal to 1, compute and store α_0^{-1}
    endif
    α_{1,N} ← Get_1(X)                       // If α_{0,N} = α_{0,P} then Get_1 only copies values
    if (α_{0,N} ≠ α_{0,P}) then
        z ← Shanks(X)
        if (z ≠ 0) then
            output (m_z, s_z)
            return (true)
        elseif (X = B) then
            invα_{1,[B]} ← α_{1,N}^{-1}      // Since Shanks(B) failed, compute and store α_1^{-1}
        endif
    endif
    if (SPS(Left(X), α_{0,N}, α_{1,N})) then
        SPS(Right(X), α_{0,N}, α_{1,N})
    endif
    return (α_{0,N} ≠ α_{0,P})
endif
```

$Get_0(X)$ computes products of pairings only for the root node and left children nodes that are tested by SPS. $Get_1(X)$ only computes products of pairings for the root and for each left child X tested by SPS when the α_0 of X is not equal to α_0 of the parent of X.

4.2 Paired Single Pruning Search Method

The recursive algorithm below describes the Paired Single Pruning Search (PSPS) method on a batch B, which is a list of $N = 2^h$, $h \geq 1$, randomly ordered message / signature pairs $((m_1, s_1), \ldots, (m_N, s_N))$ where the signature components for Cha-Cheon are verified elements of $\mathbb{G}_1$. On the initial call to $PSPS(X, \alpha_{0,P}, \alpha_{1,P})$, $X = B$, $\alpha_{0,P} = 1$, $\alpha_{1,P} = 1$.

Algorithm $PSPS(X, \alpha_{0,P}, \alpha_{1,P})$ *(Paired Single Pruning Search)*
Input: X a list of message / signature pairs, $\alpha_{0,P}$ and $\alpha_{1,P}$ in $\mathbb{G}_T$.
Output: A list of the invalid pairs in the batch.
Return: A boolean.

```
if (Len(X) = 1) then
    output X                                    //The signature pair X = (m_i, s_i)
    return (true)
else
    α_{0,N} ← Get_0(X)
    if α_{0,N} = 1 then
        return (true)
    elseif (X = B) then
        invα_{0,[B]} ← α_{0,N}^{-1}       // Since α_0 of B is not equal to 1, compute and store α_0^{-1}
    endif
    if (α_{0,N} = α_{0,P}) then
        α_{1,N} ← Get_1(X)                 // If α_{0,N} = α_{0,P} then Get_1 only copies values
    else
        if (X ≠ B and X = Left(Parent(X))) then          // X is a left child node
            (z_l, z_r) ← PairSolver(X, Right(Parent(X)))
            if z_l ≠ 0 then
                output (m_{z_l}, s_{z_l}), (m_{z_r}, s_{z_r}); return (false)
            end
        end
        α_{1,N} ← Get_1(X)
        z ← Shanks(X)
        if z ≠ 0 then
            output (m_z, s_z); return (true)
        elseif (X = B) then
            invα_{1,[B]} ← α_{1,N}^{-1}    // Since Shanks(B) failed, compute and store α_1^{-1}
        endif
    endif
    if (PSPS(Left(X), α_{0,N}, α_{1,N})) then
        PSPS(Right(X), α_{0,N}, α_{1,N})
    endif
    return (α_{0,N} ≠ α_{0,P})
endif
```

PSPS uses the following algorithms:

1. $PairSolver(Left, Right)$ – uses the double pruning technique to return the positions of two invalid signatures, one in $Left$ and one in $Right$, or returns $(0, 0)$. $PairSolver(Left, Right)$ alternately computes a value from one of two series, and terminates when a newly computed value from one series is equal to one of the values already computed for the other series, or when both series have been computed. See Appendix A.1.
2. $Parent(X)$ – returns the parent of X, or $\varnothing$ if X is the initial batch B.

5 Performance

For Cha-Cheon signatures, the divide-and-conquer methods and the exponentiation methods batch verify by first checking that the signature components are in $\mathbb{G}_1$, then computing α_0 for B, and testing whether $\alpha_0 = 1$. With the exception of BQS and the DC Verifiers, they compute their α_0s, as shown in $Get_0(B)$ in Appendix A. The cost (not including the membership tests) is $N \cdot \mathsf{CstDlbMult}\mathbb{G}_1(\mathrm{t}_1, \mathrm{t}_2) + N \cdot \mathsf{CstMult}\mathbb{G}_1(\mathrm{t}_1) + 2(N-1)\,\mathsf{CstAdd}\mathbb{G}_1 + \mathsf{CstDblPair}$ with $t_1 = \lceil \log_2(r)/2 \rceil$ and $t_2 = \lceil \log_2(r) \rceil$.

BQS and the DC Verifiers can compute α_0 with the same cost. In BQS the B_is and D_is of sibling leaf nodes in T are added together to obtain the corresponding values for their parent nodes. This process is repeated for the interior nodes until the corresponding values for the root node of T, $\sum\limits_{i=1}^{N} B_i$ and $\sum\limits_{i=1}^{N} D_i$ are obtained, see Appendix B.1.

5.1 Cost of the New Methods when $w \geq 1$

Single Pruning Search Performance If $w = 1$, the cost of SPS increases by the cost of computing α_0^{-1} ($\mathsf{CstInv}\mathbb{G}_\mathrm{T}$) and α_1 for B ($2(N-1)\,\mathsf{CstAdd}\mathbb{G}_1 + \mathsf{CstDblPair}$), plus the expected cost of a successful $Shanks(B)$ call, which is approximately $\frac{4}{3}\sqrt{N}\,\mathsf{CstMult}\mathbb{G}_\mathrm{T}$. $Shanks(X)$ tests whether the equation $\alpha_{1,n} = \alpha_{0,n}^{z}$ has a solution z in the range of l up to u, the bounds of the (sub-)batch X within the original batch. The algorithm alternately computes a value from the series $\alpha_1 \cdot (\alpha_0^{-1})^i$ and the series $(\alpha_0^s)^j \cdot \alpha_0^l$, $s = \lfloor \sqrt{|X|} \rfloor$, stopping when a match is found (single invalid signature) or when both series have been computed.

If $w \geq 2$, the average cost of SPS is the sum of the costs of computing α_0, α_0^{-1}, α_1, a failed $Shanks(B)$ call ($2\sqrt{N}\,\mathsf{CstMult}\mathbb{G}_\mathrm{T}$), α_1^{-1}, plus the sum of the costs generated as SPS investigates the descendents of $root_T$.

The following recurrence relation generates these costs:

$\mathsf{R}_{(\mathsf{S})}(w, M) =$

$$\begin{cases} 0, & w = 0, 1, \\ & w > M \text{ or} \\ & w = 2 \text{ and } M = 2; \\ \dfrac{\begin{bmatrix} 2\binom{M/2}{2}(\mathsf{R}_{(\mathsf{S})}(2, M/2) + \mathsf{C}_{(\mathsf{S})}(2, 0, M/2)) + \\ \binom{M/2}{1}^2(\mathsf{C}_{(\mathsf{S})}(1, 1, M/2)) \end{bmatrix}}{\binom{M}{2}}, & w = 2 \text{ and } M > 2; \\ \dfrac{\begin{bmatrix} 2\binom{M/2}{w}(\mathsf{R}_{(\mathsf{S})}(w, M/2) + \mathsf{C}_{(\mathsf{S})}(w, 0, M/2)) + \\ 2\binom{M/2}{w-1}\binom{M/2}{1}(\mathsf{R}_{(\mathsf{S})}(w-1, M/2) + \mathsf{C}_{(\mathsf{S})}(w-1, 1, M/2)) + \\ \sum_{i=2}^{w-2} \binom{M/2}{w-i}\binom{M/2}{i}(\mathsf{R}_{(\mathsf{S})}(w-i, M/2) + \mathsf{R}_{(\mathsf{S})}(i, M/2) + \mathsf{C}_{(\mathsf{S})}(w-i, i, M/2)) \end{bmatrix}}{\binom{M}{w}}, & w \geq 3, \end{cases}$$

where for Cha-Cheon:

	Costs		
Argument	CstDblPair	CstInv$\mathbb{G}_T$	CstMult$\mathbb{G}_T$
$\mathsf{C}_{(\mathsf{S})}(2, 0, M/2)$	1		
$\mathsf{C}_{(\mathsf{S})}(1, 1, M/2)$	2	2	$\frac{8}{3}\sqrt{M/2}$
$\mathsf{C}_{(\mathsf{S})}(w, 0, M/2)$	1		
$\mathsf{C}_{(\mathsf{S})}(w-1, 1, M/2)$	2	2	$\frac{10}{3}\sqrt{M/2}$
$\mathsf{C}_{(\mathsf{S})}(w-i, i, M/2)$	2	2	$4\sqrt{M/2}$

For $\mathsf{C}_{(\mathsf{S})}(2, 0, M/2)$ and $\mathsf{C}_{(\mathsf{S})}(w, 0, M/2)$, $Get_0(X)$ is called for the left child node and no inverse is computed, with cost CstDblPair. For $\mathsf{C}_{(\mathsf{S})}(1, 1, M/2)$, both $Get_0(X)$ and $Get_1(X)$ are called for the left child, combined cost is $2\,\mathsf{CstDblPair} + 2\,\mathsf{CstInv}\mathbb{G}_T$; for the right child, cost is zero, and two successful calls are made to $Shanks(X)$ with combined cost of $\frac{8}{3}\sqrt{M/2}\,\mathsf{CstMult}\mathbb{G}_T$. $\mathsf{C}_{(\mathsf{S})}(w-1, 1, M/2)$ is similar to $\mathsf{C}_{(\mathsf{S})}(1, 1, M/2)$ except that one of the calls to $Shanks(X)$ fails to find a solution. Both calls to $Shanks(X)$ fail for $\mathsf{C}_{(\mathsf{S})}(w-i, i, M/2)$.

Paired Single Pruning Search Performance If $w \leq 1$, PSPS has the same average cost as SPS. If $w \geq 2$, the average cost of PSPS is the sum of the costs of computing α_0, α_0^{-1}, α_1, α_1^{-1}, the cost of the failed *Shanks* test on the batch B, and the sum of the costs generated as

PSPS investigates the descendants of $root_T$. The following recurrence relation generates these costs:

$\mathsf{R}_{(\mathsf{P})}(w, M) =$

$$\begin{cases} 0, & w = 0, 1, \\ & w > M \text{ or} \\ & w = 2 \text{ and } M = 2; \\ \dfrac{\left[\begin{array}{l} 2\binom{M/2}{2}(\mathsf{R}_{(\mathsf{P})}(2, M/2) + \mathsf{C}_{(\mathsf{P})}(2, 0, M/2)) + \\ \binom{M/2}{1}^2(\mathsf{C}_{(\mathsf{P})}(1, 1, M/2)) \end{array}\right]}{\binom{M}{2}}, & w = 2 \text{ and } M > 2; \\ \dfrac{\left[\begin{array}{l} 2\binom{M/2}{w}(\mathsf{R}_{(\mathsf{P})}(w, M/2) + \mathsf{C}_{(\mathsf{P})}(w, 0, M/2)) + \\ 2\binom{M/2}{w-1}\binom{M/2}{1}(\mathsf{R}_{(\mathsf{P})}(w-1, M/2) + \mathsf{C}_{(\mathsf{P})}(w-1, 1, M/2)) + \\ \sum\limits_{i=2}^{w-2} \binom{M/2}{w-i}\binom{M/2}{i}(\mathsf{R}_{(\mathsf{P})}(w-i, M/2) + \mathsf{R}_{(\mathsf{P})}(i, M/2) + \mathsf{C}_{(\mathsf{P})}(w-i, i, M/2)) \end{array}\right]}{\binom{M}{w}}, & w \geq 3, \end{cases}$$

where for Cha-Cheon:

	Costs		
Argument	CstDblPair	CstInv$\mathbb{G}_{\mathrm{T}}$	CstMult$\mathbb{G}_{\mathrm{T}}$
$\mathsf{C}_{(\mathsf{P})}(2, 0, M/2)$	1		
$\mathsf{C}_{(\mathsf{P})}(1, 1, M/2)$	1	1	$M/2$
$\mathsf{C}_{(\mathsf{P})}(w, 0, M/2)$	1		
$\mathsf{C}_{(\mathsf{P})}(w-1, 1, M/2)$	2	2	$M + \frac{10}{3}\sqrt{M/2}$
$\mathsf{C}_{(\mathsf{P})}(w-i, i, M/2)$	2	2	$M + 4\sqrt{M/2}$

For $\mathsf{C}_{(\mathsf{P})}(2, 0, M/2)$ and $\mathsf{C}_{(\mathsf{P})}(w, 0, M/2)$, $Get_0(X)$ is called for the left child node and no inverse is computed. For $\mathsf{C}_{(\mathsf{P})}(1, 1, M/2)$, $Get_0(X)$ is called for the left child, the cost is $\mathsf{CstDblPair} + \mathsf{CstInv}\mathbb{G}_{\mathrm{T}}$, and a successful call is made to $PairSolver(Left, Right)$ with expected cost of $M/2\,\mathsf{CstMult}\mathbb{G}_{\mathrm{T}}$. For the argument $\mathsf{C}_{(\mathsf{P})}(w-1, 1, M/2)$, both $Get_0(X)$ and $Get_1(X)$ are called for both the left and right child, one failed call is made to $PairSolver(Left, Right)$ with cost $M\,\mathsf{CstMult}\mathbb{G}_{\mathrm{T}}$, and one successful and one failed call are made to $Shanks(X)$. $\mathsf{C}_{(\mathsf{P})}(w-i, i, M/2)$ is similar to $\mathsf{C}_{(\mathsf{P})}(w-1, 1, M/2)$ except that both calls to $Shanks(X)$ fail.

5.2 Number of Product of Pairings Computations of SPS and PSPS

Let T be a perfect binary tree, and $\mathrm{PT}_{(2)}$ be the sub-tree of PT, where each node represents 2 or more invalid signatures. For each node in $\mathrm{PT}_{(2)}$, SPS computes an α_0 for its left child, unless the child is a leaf node of T. For each node in the $\mathrm{PT}_{(2)}$ with both child nodes in PT, SPS also computes α_1 for its left child, unless the child is a leaf node of T. This computation occurs $w-1$ times. SPS also computes a pair of αs for the root. Therefore, including the initial batch verification, SPS requires $|\,\mathrm{PT}_{(2)}\,| + (w+1)$ product of pairings computations, if none of the leaf nodes of PT are leaves of T. SPS requires two fewer α computations whenever a pair of leaves of PT are leaves of T.

For a perfect binary tree, the number of ways j pairs leaves of PT can be leaves of T is $\binom{N/2}{j}$, and the number of ways the remaining $w-2j$ invalid signatures can be in the remaining $N/2-j$ distinct 3 node subtrees at the lowest level of T is $\binom{N/2-j}{w-2j}2^{w-2j}$. Therefore, the expected number of occurrences of two sibling leaf nodes of T both representing invalid signatures is

$$\frac{1}{\binom{N}{w}} \sum_{j=1}^{\lfloor w/2 \rfloor} \binom{N/2}{j} \binom{N/2-j}{w-2j} 2^{w-2j}.$$

Since $\binom{N-2}{w-2} = \sum_{j=1}^{\lfloor w/2 \rfloor} \binom{N/2}{j}\binom{N/2-j}{w-2j}2^{w-2j}$ the expression simplifies to $\frac{w(w-1)}{2(N-1)}$, and the expected number of αs computed by Single Pruning Search when the batch size is a power of 2 is

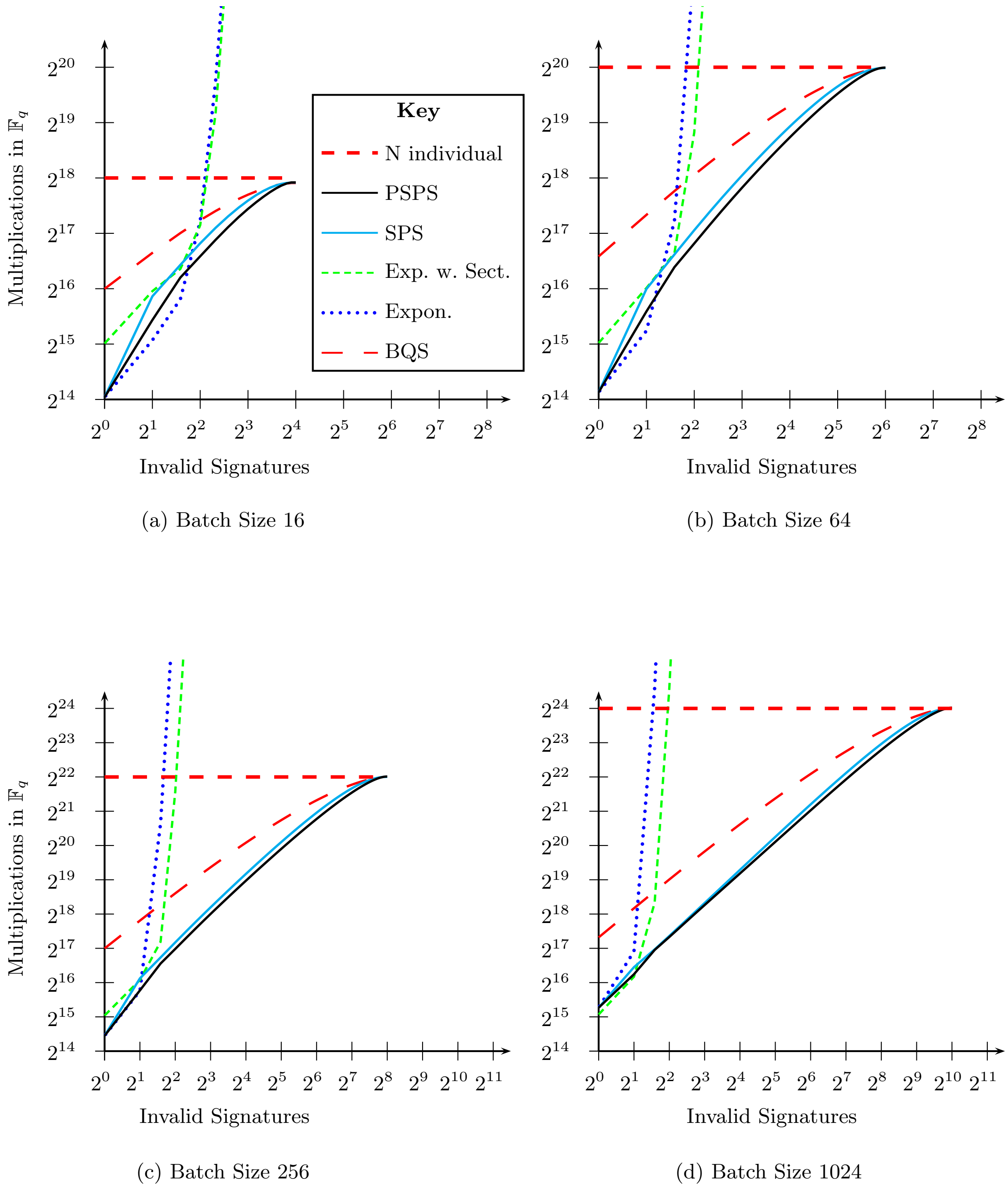
$$|\,\mathrm{PT}_{(2)}\,| + (w+1) - \frac{w(w-1)}{N-1}\ .$$

In Appendix C we show that $|\,\mathrm{PT}_{(2)}\,| < 2w-1$; therefore the expected number of product of pairings computations required by SPS is less than $3w$. Since the number of product of pairings computations in the cost functions of PSPS are all less than or equal to the corresponding functions of SPS , the average number of product of pairings used by PSPS is also $O(w)$.

5.3 Number of Multiplications in F_q

Figure 1 and Figure 2 compare methods analyzed in Section 5.1 and Appendix B for finding invalid signatures in a batch once the initial batch verification has failed for Cases A and C of [12]. In Case A, the group order r is a 160-bit value, the elliptic curve E is defined over $\mathbb{F}_q$, where q is a 160-bit value, and the embedding degree $d = 6$. In Case C, the group order r is a 256-bit value, q is a 256-bit value, and the embedding degree $d = 12$. All costs are given in terms of the number of multiplications (m) in $\mathbb{F}_q$, assuming that squaring has the same cost as multiplication, using the following estimates from Granger, Page and Smart [12], and Granger and Smart [13].

- For Case A, 1 double product of pairings $= 16,355m$, 1 multiplication in $F_{q^6} = 15m$, 1 inverse in $F_{q^6} = 44m$ (assuming 1 inverse in $F_q = 10m$), and 1 elliptic curve addition $= 11m$.
- For Case C, 1 double product of pairings $= 62,797m$, 1 multiplication in $F_{q^{12}} = 45m$, 1 inverse in $F_{q^{12}} = 104m$, and 1 elliptic curve addition $= 11m$.



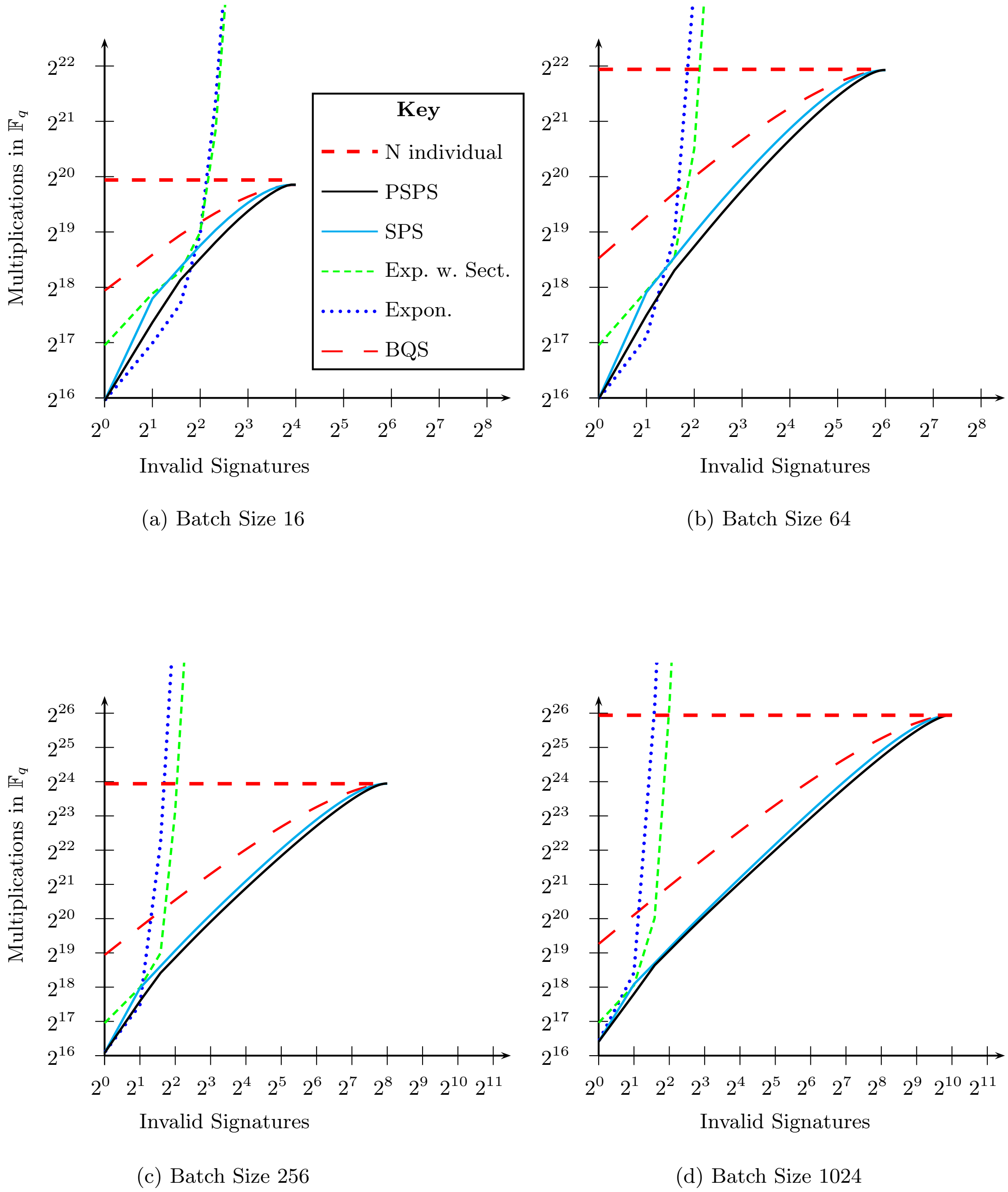
(a) Batch Size 16

(b) Batch Size 64

(c) Batch Size 256

(d) Batch Size 1024

Fig. 1: Number of multiplies in F_q, where r and q are 160-bit values and $d = 6$.



(a) Batch Size 16

(b) Batch Size 64

(c) Batch Size 256

(d) Batch Size 1024

Fig. 2: Number of multiplies in F_q, where r and q are 256-bit values and $d = 12$.

6 Conclusion

We have presented two new methods, Single Pruning Search and Paired Single Pruning Search, for identifying invalid signatures in pairing-based batch signature schemes using the small exponents test, and have analyzed their average case performance. These new methods require $O(w)$ product of pairings computations and $O(w\sqrt{N})$ and $O(wN)$ number of multiplications in $\mathbb{F}_{q^d}$. The methods are described for Cha-Cheon signatures, but are applicable to other batch verified signature schemes such as the batch verifiers presented in [8].

These new methods, like BQS and earlier divide-and-conquer methods, can be used when there is uncertainty in the number of invalid signatures in a batch. As shown in the figures in Section 5.3, the new methods significantly outperform the Binary Quick Search method when $w \ll N$, and perform as well as or better than the exponentiation methods except when N and w are small. Unlike the exponentiation methods, with the new methods a batch verifier is not forced to switch methods when tests for small w fail.

In [14] the authors suggested that the exponentiation methods can be used with BQS to provide improved performance after tests for small values of w fail. While this is certainly true, the result can be expensive. For example, with $N = 64$, a batch verifier that assumes that the number of invalid signature in the batch is small would start with the Exponentiation Method, but if the tests for $w = 1$ and $w = 2$ both fail, the verifier would switch to Exponentiation with Sectors Method to test if $w = 3$. If the test for $w = 3$ fails, then the verifier would switch to BQS. If $w = 4$, the cost of this sequence for Case A (ignoring the common cost of signature component validation and α_0 computation) is at least $\approx 3.70 \times 10^5$ multiplications in $\mathbb{F}_q$, compared to $\approx 2.73 \times 10^5$ multiplications if only BQS was used, and $\approx 1.16 \times 10^5$ multiplications with the Paired Single Pruning Search Method.

Ideally, we would have a single efficient method for finding the invalid signatures in a batch that always has the lowest expected cost no matter how many signatures are invalid. Such a method would be especially useful when an adversary is occasionally able to inject bursts of several invalid signatures into some batches. Short of that ideal, but a practical alternative, would be a small set of methods, each of which for some range of batch sizes of interest always provides the lowest expected cost. Currently the Paired Single Pruning Search, provides the lowest expected cost when the batch size is in the range 128 to 512. For batches larger than 512, we would expect batch verifiers to utilize the bucket test for Cha-Cheon and related signature schemes rather than the small exponents test. Finding such a minimal cost method for batches smaller than 128 is as an open problem. Another open problem is to find more efficient methods than the generic DC verifiers of [20] for identifying the invalid signatures in a batch when an initial bucket test verifier fails.

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8 Disclaimer

The views and conclusions contained in this paper are those of the author and should not be interpreted as representing the official policies, either expressed or implied, of the Army Research Laboratory or the U. S. Government.

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A Auxiliary Algorithms for SPS and PSPS

The algorithms in Section 4 for the SPS and PSPS methods call $Get_0(X)$ to obtain α_0 (and α_0^{-1}) for X, and $Get_1(X)$ to obtain α_1 (and α_1^{-1}) for X. In this section we describe these algorithms for Cha-Cheon signatures. $Get_0(X)$ and $Get_1(X)$ use the following algorithms:

1. $Lowerindex(X)$ – returns the index within the batch B of the message / signature pair in the lowest position in X.
2. $Upperindex(X)$ – returns the index in B of the pair in the highest position in X.

Algorithm $Get_0(X)$ *(Obtain α_0 (and α_0^{-1}) for Cha-Cheon)*
Input: X a list of message / signature pairs.
Output: None.
Return: The value $\alpha_{0,[X]}$ for X.

```
P ← Parent(X); L ← Left(P); R ← Right(P)
if (α_{0,[X]}) then                          // True if α_{0,[X]} has been computed
      return (α_{0,[X]})
elseif (X = R) then                          // Right child
      α_{0,[R]} ← α_{0,[P]} · invα_{0,[L]}
      invα_{0,[R]} ← invα_{0,[P]} · α_{0,[L]}
      return (α_{0,[R]})
elseif (X = L) then                          // Left child
      l ← Lowerindex(X); u ← Upperindex(X)
      α_{0,[L]} ← e(VB_l − VB_{u+1}, S) · e(VD_l − VD_{u+1}, −T)
      if ( α_{0,[L]} ≠ α_{0,[P]}) then
            invα_{0,[L]} ← α_{0,[L]}^{−1}
      else
            invα_{0,[L]} ← invα_{0,[P]}
      endif
      return (α_{0,[L]})
else                                         // Root node, compute VBs, VDs, and α_{0,[B]}
      VB_{len(X)} ← B_{len(X)}
      VD_{len(X)} ← D_{len(X)}
      for i = Len(X) − 1 downto 1 do
            VB_i ← VB_{i+1} + B_i
            VD_i ← VD_{i+1} + D_i
      endfor
      α_{0,[B]} ← e(VB_1, S) · e(VD_1, −T)
      return (α_{0,[B]})
endif
```

When X is a left child, the cost is at most $2 \cdot \mathsf{CstSub}\mathbb{G}_1 + \mathsf{CstDblPair} + \mathsf{CstInv}\mathbb{G}_\mathrm{T}$. If X is a right child, the cost is at most $2\,\mathsf{CstMult}\mathbb{G}_\mathrm{T}$. Since $\mathsf{CstDblPair} \gg \mathsf{CstSub}\mathbb{G}_1$ and $\mathsf{CstDblPair} \gg \mathsf{CstMult}\mathbb{G}_\mathrm{T}$, we estimate the cost of Get_0 for the siblings as $\mathsf{CstDblPair} + \mathsf{CstInv}\mathbb{G}_\mathrm{T}$ in Section 5.

Algorithm $Get_1(X)$ *(Obtain α_1 (and α_1^{-1}) for Cha-Cheon)*
Input: X a list of message / signature pairs.
Output: None.
Return: The value $\alpha_{1,[X]}$ for X.

```
P ← Parent(X); L ← Left(P); R ← Right(P)
if (α_{1,[X]}) then                              // True if α_{1,[X]} has been computed
      return (α_{1,[X]})
elseif (α_{0,[X]} = α_{0,[P]}) then
      α_{1,[X]} ← α_{1,[P]}
      invα_{1,[X]} ← invα_{1,[P]}
      return (α_{1,[X]})
elseif (X = R) then                              // Right child
      α_{1,[R]} ← α_{1,[P]} · invα_{1,[L]}
      invα_{1,[R]} ← invα_{1,[P]} · α_{1,[L]}
      return (α_{1,[R]})
elseif (X = L) then                              // Left child
      l ← Lowerindex(X); u ← Upperindex(X)
      WB_{l,u} ← UB_u − (u · VB_{u+1} + WB_{1,l−1})
      WD_{l,u} ← UD_u − (u · VD_{u+1} + WD_{1,l−1})
      if (l ≠ 1) then
            WB_{1,u} ← WB_{1,l−1} + WB_{l,u}
            WD_{1,u} ← WD_{1,l−1} + WD_{l,u}
      endif
      α_{1,[L]} ← e(WB_{l,u}, S) · e(WD_{l,u}, −T)
      if ( α_{1,[L]} ≠ α_{1,[P]}) then
            invα_{1,[L]} ← α_{1,[L]}^{−1}
      else
            invα_{1,[L]} ← invα_{1,[P]}
      endif
      return (α_{1,[L]})
else                                             // Root node, compute α_{1,[B]}
      WB_{1,0} ← ∞
      WD_{1,0} ← ∞
      UB_1 ← VB_1
      UD_1 ← VD_1
      for i = 2 upto len(X) do
            UB_i ← UB_{i−1} + VB_i
            UD_i ← UD_{i−1} + VD_i
      endfor
      α_{1,[B]} ← e(UB_{len(X)}, S) · e(UD_{len(X)}, −T)
      return (α_{1,[B]})
endif
```

To compute α_1 and its inverse for a left child costs no more than $4 \cdot \mathsf{CstAdd}\mathbb{G}_1 + 2 \cdot \mathsf{CstSub}\mathbb{G}_1 + 2 \cdot \mathsf{CstMult}\mathbb{G}_1(\mathrm{t}_1) + \mathsf{CstInv}\mathbb{G}_\mathrm{T} + \mathsf{CstDblPair}$ with $t_1 = \lceil log_2(Len(X)) \rceil < \lceil log_2(N) \rceil$. If X is a right child, the cost is at most $2\,\mathsf{CstMult}\mathbb{G}_\mathrm{T}$. We estimate the cost of Get_1 for the pair of siblings as $\mathsf{CstDblPair} + \mathsf{CstInv}\mathbb{G}_\mathrm{T}$ in Section 5.[4]

A.1 Testing functions for SPS and PSPS

Algorithm $Shanks(X)$ *(Test whether there is a single invalid signature in X)*

Input: X a list of message / signature pairs.
Output: None.
Return: The index in B of the invalid signature or 0

$l \leftarrow Lowerindex(X)$; $u \leftarrow Upperindex(X)$; $s \leftarrow \lfloor \sqrt{u-l+1} \rfloor$; $t \leftarrow \lceil (u-l+1)/s \rceil$
$sqrt\alpha_0 \leftarrow \alpha^s_{0,[X]}$; $leftside[0] \leftarrow \alpha_{1,[X]}$; $rightside[0] \leftarrow \alpha^l_{0,[X]}$
if $(leftsize[0] = rightside[0])$ then
    return (l)
endif
for $i = 1$ upto $s-1$ do
    $leftside[i] \leftarrow leftside[i-1] \cdot inv\alpha_{0,[X]}$
    for $j = 0$ upto $i-1$ do
        if $(leftside[i] = rightside[j])$ then
            return $(i + j * s + l)$
        endif
    endfor
    $rightside[i] \leftarrow rightside[i-1] \cdot sqrt\alpha_0$
    for $j = 0$ upto i do
        if $(leftsize[j] = rightside[i])$ then
            return $(j + i * s + l)$
        endif
    endfor
endfor
$i \leftarrow s$
while $(j + i * s + l \leq u + 1)$ do
    $rightside[i] \leftarrow rightside[i-1] \cdot sqrt\alpha_0$
    for $j = 0$ upto s do
        if $(leftsize[j] = rightside[i])$ then
            return $(j + i * s + l)$
        endif
    endfor
    $i \leftarrow i + 1$
endwhile
return (0)

[4] Since $\mathsf{CstDblPair} + \mathsf{CstInv}\mathbb{G}_\mathrm{T} \gg 4 \cdot \mathsf{CstAdd}\mathbb{G}_1 + 2 \cdot \mathsf{CstSub}\mathbb{G}_1 + 2 \cdot \mathsf{CstMult}\mathbb{G}_1(\mathrm{t}_1) + 2\,\mathsf{CstMult}\mathbb{G}_\mathrm{T}$ for $N \leq 1024$.

The algorithm $Shanks(X)$ shown above tests whether the equation $\alpha_{1,n} = \alpha_{0,n}^{z}$ has a solution z in the range of l up to u, the bounds of the (sub-)batch X within the original batch. If X has a single invalid signature the algorithm returns the position of the invalid signature, otherwise the algorithm returns 0. The algorithm checks for a solution of

$$\alpha_1 \cdot (\alpha_0^{-1})^d = (\alpha_0^s)^c \cdot \alpha_0^l$$

with either $0 \leq d < s$ and $0 \leq c < t$ or $d = s$ and $c = t{-}1$ where $s = \lfloor \sqrt{u-l} \rfloor, t = \lceil (u{-}l{+}1)/s \rceil$. This version of Shanks baby-step giant-step algorithm has better average performance than the textbook version [17], $\approx \frac{4}{3}s$ vs $\approx \frac{3}{2}s$ multiplications. For simplicity's sake we do not sort the values of $leftside$ and $rightside$.

When there is single invalid signature in X the average cost of the algorithm is at most $2\,\mathsf{CstExpt}\mathbb{G}_{\mathrm{T}}(\mathrm{t}_1)$,[5] and $mShank(m)$ multiplications in $\mathbb{G}_T$ where $m = u - l + 1$. When $t = s$, i.e., when M is a square number

$$mShank(m) = \frac{4}{3}\sqrt{m} - \frac{3}{2} + \frac{1}{6\sqrt{m}} = \frac{1}{m} \cdot \left(\sum_{i=0}^{\sqrt{m}-2} (4i^2 + 3i + 1) + \sum_{i=0}^{\sqrt{m}-1} (2i^2 + 2i) \right).$$

The approximation $mShank(m) = \frac{4}{3}\sqrt{m}$ is used in Section 5. When $lowerindex(X) = 1$ only one exponentiation is required. When the test fails the cost is at most $2\,\mathsf{CstExpt}\mathbb{G}_{\mathrm{T}}(\mathrm{t}_1) + 2\sqrt{m}\,\mathsf{CstMult}\mathbb{G}_{\mathrm{T}}$. Since $\mathsf{CstDblPair} >> \mathsf{CstExpt}\mathbb{G}_{\mathrm{T}}(\mathrm{t}_1)$ we can ignore the contribution of the $\mathsf{CstExpt}\mathbb{G}_{\mathrm{T}}(\mathrm{t}_1)$s to the costs.

The algorithm $PairSolver(Left, Right)$ shown below tests whether the equation $\alpha_{1,parent} = \alpha_{0,left}^{z_{left}} \cdot \alpha_{0,right}^{z_{right}}$ has a solution z_{left}, z_{right} in the range of l_{left} up to u_{left} and l_{right} up to u_{right}, the bounds of the $Left$ and $Right$ sub-batches. If both $Left$ and $Right$ have a single invalid signature the algorithm returns the positions of the invalid signatures, otherwise the algorithm returns the pair $0, 0$. The algorithm searches for a solution of

$$\alpha_{1,parent} \cdot (\alpha_{0,left}^{-1})^{l_{left}} \cdot (\alpha_{0,left}^{-1})^d = (\alpha_{0,right})^c \cdot \alpha_{0,right}^{l_{right}}$$

with $0 \leq d \leq s$, and $0 \leq c \leq t$ where $s = u_{left} - l_{left} + 1$, $t = u_{right} - l_{right} + 1$ and $t = s$ $(N = 2^h)$. For simplicity's sake we do not sort the values of $leftside$ and $rightside$.

When there is single invalid signature in both $Left$ and $Right$ the average cost of the algorithm is at most $2\,\mathsf{CstExpt}\mathbb{G}_{\mathrm{T}}(\mathrm{t}_1)$,[6] plus $s\,\mathsf{CstMult}\mathbb{G}_{\mathrm{T}}$. When $lowerindex(Left) = 1$ only one exponentiation is required. Otherwise the cost is at most $2\,\mathsf{CstExpt}\mathbb{G}_{\mathrm{T}}(\mathrm{t}_1)$ and $2s\,\mathsf{CstMult}\mathbb{G}_{\mathrm{T}}$. Since $\mathsf{CstDblPair} >> \mathsf{CstExpt}\mathbb{G}_{\mathrm{T}}(\mathrm{t}_1)$, in Section 5 we use $s\,\mathsf{CstMult}\mathbb{G}_{\mathrm{T}}$ as the cost of a successful call and as $2s\,\mathsf{CstMult}\mathbb{G}_{\mathrm{T}}$ for a failed call.

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[5] With $t_1 = \lceil log_2(s) \rceil$ and when $l > 0$, $t_1 = \lceil log_2(l) \rceil$.

[6] With $t_1 = \lceil log_2(l_{right}) \rceil$ and when $l_{left} > 0$, $t_1 = \lceil log_2(l_{left}) \rceil$.

Algorithm $PairSolver(Left, Right)$

Input: A pair of lists $Left$ and $Right$ where $len_{left} \geq len_{right}$.
Output: None.
Return: The index in B of the invalid signature in $Left$ and in $Right$ or the pair $0, 0$

$P \leftarrow Parent(Left)$
$l_{left} \leftarrow Lowerindex(Left)$; $u_{left} \leftarrow Upperindex(Left)$
$s \leftarrow u_{left} - l_{left} + 1$
$l_{right} \leftarrow Lowerindex(Right)$; $u_{right} \leftarrow Upperindex(Right)$
$t \leftarrow u_{right} - l_{right} + 1$
$leftside[0] \leftarrow \alpha_{1,[P]} \cdot inv\alpha_{0,[Left]}^{l_{left}}$
$rightside[0] \leftarrow Get_0(Right)$
$rightside[0] \leftarrow rightside[0]^{l_{right}}$
if $(leftsize[0] = rightside[0])$ then
    return (l_{left}, l_{right})
endif
for $i = 1$ upto t do
    $leftside[i] \leftarrow leftside[i-1] \cdot inv\alpha_{0,[Left]}$
    for $j = 0$ upto $i - 1$ do
        if $(leftside[i] = rightside[j])$ then
            return $(i + l_{left}, j + l_{right})$
        endif
    endfor
    $rightside[i] \leftarrow rightside[i-1] \cdot \alpha_{0,[Right]}$
    for $j = 0$ upto i do
        if $(leftsize[j] = rightside[i])$ then
            return $(j + l_{left}, i + l_{right})$
        endif
    endfor
endfor
return $(0, 0)$

B Expected Costs of Earlier Methods

B.1 Binary Quick Search

The Binary Quick Search Method can compute the can compute the values $\sum_{i=1}^{N} B_i$ and $\sum_{i=1}^{N} D_i$ for Cha-Cheon signatures by using the following algorithm (assuming $N = 2^h$).

```
for i = 1 to N do                              // This loop computes the N B_is and D_is
    B[h,i] ← B_i; D[h,i]← D_i
endfor
for j = h downto 1 do
    for i = 1 upto 2^j by 2 do
        B[j − 1,(i+1)/2] ← B[j,i] + B[j,i + 1]
        D[j − 1,(i+1)/2] ← D[j,i] + D[j,i + 1]
    endfor
endfor
```

For Cha-Cheon signatures the cost to compute α_0 is the sum of 1) the cost to compute the B_is and D_is which is $N\cdot$ $\mathsf{CstDlbMult}\mathbb{G}_1(\mathrm{t}_1,\mathrm{t}_2)$ $+N\cdot$ $\mathsf{CstMult}\mathbb{G}_1(\mathrm{t}_1)$ with $t_1 = \lceil \log_2(r)/2\rceil$ and $t_2 = \lceil \log_2(r)\rceil$, 2) the cost of the algorithm above, $2(N-1)\,\mathsf{CstAdd}\mathbb{G}_1$, and 3) a double product of pairings computation. If $w \geq 1$ the average cost of BQS is increased by the sum of the costs generated as BQS investigates the descendants of $root_T$. The following recurrence relation generates these costs.

$$\mathbf{R}_{(\mathbf{B})}(w,M) = \begin{cases} 0, & w = 0 \text{ or } w > M; \\ \dfrac{\binom{M/2}{1}}{\binom{M}{1}} 2(\mathbf{R}_{(\mathbf{B})}(1,M/2) + \mathrm{C}_{(\mathrm{B})}(l)), & w = 1; \\ \dfrac{\left[\begin{array}{l} \binom{M/2}{w} 2(\mathbf{R}_{(\mathbf{B})}(w,M/2) + \mathrm{C}_{(\mathrm{B})}(b)) + \\ \sum_{i=1}^{w-1} \binom{M/2}{w-i}\binom{M/2}{i}(\mathbf{R}_{(\mathbf{B})}(w-i,M/2) + \mathbf{R}_{(\mathbf{B})}(i,M/2) + \mathrm{C}_{(\mathrm{B})}(b)) \end{array}\right]}{\binom{M}{w}}, & w \geq 2; \end{cases}$$

where for Cha-Cheon $\mathrm{C}_{(\mathrm{B})}(l) = \mathsf{CstDblPair}$ and $\mathrm{C}_{(\mathrm{B})}(b) = \mathsf{CstDblPair} + \mathsf{CstInv}\mathbb{G}_\mathrm{T}$.

B.2 Exponentiation Method

The algorithms used to compute the components of the Exponentiation Method are described in [14]. The authors' analysis of the algorithms focused on bounding their worse case performance when $w << N$. In this section we examine their average case performance after an initial batch verification has failed. When $w \geq 1$ the expected additional cost is sum of the cost of each additional α_j for B, $1 \leq j \leq w$, plus the cost of the failed tests for a solution to equation (2) for $1 \le j < w$ and the expected cost of a successful test for $j = w$. Note that the Exponentiation Method only computes at most w inverses in $\mathbb{F}_{q^d}$, $\alpha_0^{-1}, \alpha_1^{-1}, ..., \alpha_{w-1}^{-1}$, when performing these tests.

Computing $\boldsymbol{\alpha_j, j \ge 1}$. The cost of computing each α_j of equation (1) for $j \ge 1$ is approximately $\mathsf{CstDblPair} + (2(N-1)+j)\,\mathsf{CstAdd}\mathbb{G}_1 + j\,\mathsf{CstSub}\mathbb{G}_1$ plus $2j\,\mathsf{CstMult}\mathbb{G}_1(\mathrm{t}_1)$, where each $t_1 = \lceil \log_2(k) \rceil$ for $1 \le k \le j$, plus an additional $2j\,\mathsf{CstMult}\mathbb{G}_1(\mathrm{t}_1)$, $t_1 = \lceil \log_2(s_{j,k} \rfloor) \rceil$ where each $s_{j,k}$ for $1 \le k \le j$ is a Stirling number of the second kind. However, since the exponentiation methods are only useful for very small w and both the ks and the Stirling numbers are very small when w is small, we approximate this cost by $\mathsf{CstDblPair} + 2N\,\mathsf{CstAdd}\mathbb{G}_1$ in Section 5.

Testing equation (2) for $\boldsymbol{w = 1}$. The method searches for a solution to $\alpha_1 = \alpha_0^{x_1}$, for $1 \le x_1 \le N$. If a solution exists then $w = 1$. Textbook Shank's giant-step baby-step algorithm [17] is used in [14], it has an average cost for a successful test of about $\frac{3}{2}\sqrt{N}\,\mathsf{CstMult}\mathbb{G}_\mathrm{T} + \mathsf{CstExpt}\mathbb{G}_\mathrm{T}(\mathrm{t}_1) + \mathsf{CstInv}\mathbb{G}_\mathrm{T}$ where $t_1 \approx \log_2(\sqrt{N})$. Since the number of multiplications in $\mathbb{G}_T$ used in $\mathsf{CstExpt}\mathbb{G}_\mathrm{T}(\mathrm{t}_1)$ is small compared to $\frac{3}{2}\sqrt{N}\,\mathsf{CstMult}\mathbb{G}_\mathrm{T}$, we use $\frac{3}{2}\sqrt{N}\,\mathsf{CstMult}\mathbb{G}_\mathrm{T} + \mathsf{CstInv}\mathbb{G}_\mathrm{T}$ as the average cost of a successful test, and $2\sqrt{N}\,\mathsf{CstMult}\mathbb{G}_\mathrm{T} + \mathsf{CstInv}\mathbb{G}_\mathrm{T}$ when the test fails in Section 5 .

Testing equation (2) for $\boldsymbol{w = 2}$. In [14] the Factor Method was developed to search for a solution to $\alpha_2 = \alpha_1^{x_1+x_2}\alpha_0^{x_1 x_2}$ for $1 \le x_1 < x_2 \le N$. The method searches for a solution to the equation

$$\alpha_2^4(\alpha_1^{-4})^{(x_2+x_1)}(\alpha_0^{-1})^{(x_2+x_1)^2} = (\alpha_0^{-1})^{(x_2-x_1)^2},$$

which in the worse case costs $8N\,\mathsf{CstMult}\mathbb{G}_\mathrm{T}$ plus an inverse. The cost of computing the $N-1$ values of the right hand side is $2N\,\mathsf{CstMult}\mathbb{G}_\mathrm{T}$. Computing the values for the left side is done in two stages the first stage uses an inverse computation followed by 2 multiplications to produce each value of the left hand side for $3 \le (s = x_2 + x_1) \le N - 1$ (except for $s = 3$). The second stage computes the values $N \le s \le 2N - 1$. Computing each of these values requires 4 multiplications. If no solution is found then the cost of the left hand side is one inverse computation and $6N$ multiplications. If each value for the left hand side is compared against the values for the right hand side when it is computed, and the computations are halted if a solution is found, then the average cost for the Factor Method is one inverse and $mFM(N,2)$ multiplications where

$$mFM(N,2) = \frac{sLHS(N)}{\binom{N}{2}} + 2N \tag{3}$$

and

$$sLHS(N) \approx \sum_{s=3}^{N-1}(2s\lfloor\frac{s-1}{2}\rfloor) + (2N+2)\lfloor\frac{N-1}{2}\rfloor + \sum_{s=N+1}^{2N-1}(4(s-N)+(2N+2))\lfloor\frac{2N+1-s}{2}\rfloor$$

(N even). $sLHS(N)/(N\binom{N}{2}) = 2.827,\ 2.456,\ 2.364$, and 2.341. for $N = 16,\ 64,\ 256$, and 1024; compared to 6 for a failed test. The average cost of the multiplications in a successful test is

77.23 $\mathsf{CstMult}\mathbb{G}_\mathrm{T}$, 285.2 $\mathsf{CstMult}\mathbb{G}_\mathrm{T}$, 1117 $\mathsf{CstMult}\mathbb{G}_\mathrm{T}$, and 4445 $\mathsf{CstMult}\mathbb{G}_\mathrm{T}$ for $N = 16,\ 64,\ 256,$ and 1024.

Testing equation (2) for $w \geq 3$. For $j \geq 3$ the values α_j through α_0 and the inverses of α_{j-1} through α_0 are used to compute the quantities δ_0, δ_1, and δ_2 for all possible values of $x_3, \ldots, x_j$ and then the Factor Method is used to search for a solution an equation of the following form

$$\delta_0 \cdot \delta_1^{(x_2+x_1)} \delta_2^{(x_2+x_1)^2} = \delta_2^{(x_2-x_1)^2} \tag{4}$$

for some valid combination of values of the δs, where $1 \leq z_1 < z_2 < z_3$ and $3 \leq z_3 \leq N-(j-3)$.

In [14] the authors determined that the number of multiplications in $\mathbb{G}_T$ required to compute all of the δs is

$$cDeltas(N,j) = 3\left(\sum_{t=1}^{j-2}\binom{j-2}{t}N^t + (j-3)\binom{N}{j-2}\right).$$

After the δs have been computed, the Factor method is executed repeatedly to test equation (4) until a solution is found or all of the possible solutions have been tested. In [14] the authors determined that if no solution is found the total number of multiplications used in the $\binom{N-2}{j-2}$ calls to the Factor Method is

$$fFM(N,j) = 8\sum_{z_3=3}^{N-(j-3)}\binom{N-z_3}{j-3}(z_3-1).$$

The efficiency of a execution of the Factor Method is the ratio of the number of possible solutions tested to the number of multiplications used, which is $z_3/16 = \binom{z_3-1}{2}/(8(z_3-1))$. Since the efficiency increases as the value of z_3 increases, it is more efficient to test for solutions where z_3 is maximal first, then proceed to test for smaller values of z_3. This approach does not alter the worse case performance of the algorithm.

If this approach is used, the expected number of multiplications in $\mathbb{G}_T$ used by calls to the Factor Method during a successful test when there are $w \geq 3$ invalid signatures is:

$$\begin{aligned} mFM(N,w) = \frac{1}{\binom{N}{w}} \sum_{i=2}^{N-(w-2)} \binom{w+i-5}{w-3} \Bigg[& sLHS(N+4-w-i) \\ & + 8\binom{N+4-w-i}{2}\Bigg[\frac{(N+4-w-i)}{4} \\ & + \sum_{l=1}^{i-2}\binom{w+l-4}{w-3}(N+3-w-l) + (N+4-w-i)\frac{\binom{w+i-5}{w-3}-1}{2}\Bigg]\Bigg] \end{aligned}$$

and the approximate number of multiplications in $\mathbb{G}_T$ used to find the invalid signatures is

$$2\sqrt{N} + 8N + cDeltas(N,w) + \sum_{j=3}^{w-1} fFM(N,j) + mFM(N,w). \tag{5}$$

which plus $w\,\mathsf{CstInv}\mathbb{G}_\mathrm{T}$ is the cost used in Section 5 for testing equation (2) when $w \geq 3$.

B.3 Exponentiation with Sectors Method

If the batch is divided into Z sectors each of size S, then the probability that $\lceil \frac{w}{S} \rceil \leq l \leq \min(Z, w)$ sectors are invalid is

$$v(l) = \frac{1}{\binom{N}{w}} \sum_{p_i(l,S)} \prod_{j=1}^{S} \binom{Z - \sum_{k=1}^{j-1} s_{i,k}}{s_{i,j}} \cdot \binom{S}{j}^{s_{i,j}}$$

where the partition $p_i(l, S)$ is an element of the set of partitions of w of length l, i.e., the number of non-zero elements in the partition, and each part of $p_i(l, S)$ is less than or equal to S. $s_{i,k}$ is the number of parts of $p_i(l, S)$ with value k. $w = \sum_{k=1}^{S} s_{i,k} \cdot k$. The average number of pairing computations, multiplications in $\mathbb{G}_T$, and elliptic curve operations used in the Exponentiation with Sectors Method can be computed using the formulas for the Exponentiation Method with the the appropriate substitution of variables. The average number of product of pairings computations required is $w + \sum_{l=\lceil \frac{w}{S} \rceil}^{\min(Z,w)} v(l)\ l + 1$.

In stage 1 of Exponentiation with Sectors, the number of sectors $Z = \sqrt{N}$ and the values of l are substituted for N and w in equation (1), in the average cost of the Shanks test (for $l = 1$), in equation (3) plus the cost of a failed Shanks test (for $l = 2$), and in equation (5) (for each $l \geq 3$). The sum of the results, each weighted by the appropriate $v(l)$, is the expected cost of this stage.

During stage 2 the values of $l\sqrt{N}$, are substituted for N in the cost equations; w is unchanged. The weighted sum of the results is the expected cost of this stage.

C $|\,\mathrm{PT}_{(2)}\,| < 2w - 1$

We show that $|\,\mathrm{PT}_{(2)}\,| < 2w - 1$ whenever $N = 2^i$ for $i = 1, 2, \ldots$. Let $S_{(2)}(i, w) = |\,\mathrm{PT}_{(2)}\,|$ for $N = 2^i$ and $0 \leq w \leq N$. Note that $S_{(2)}(i, w) = 0$ when $w = 0, 1$, and $S_{(2)}(1, 2) = 1$. Assume that $S_{(2)}(i, w) < 2w - 1$.

For $w = 2$

$$\begin{aligned}
S_{(2)}(i+1, 2) &= \sum_{j=0}^{2} \frac{\binom{2^i}{2-j}\binom{2^i}{j}}{\binom{2^{i+1}}{2}} (S_{(2)}(i, 2-j) + S_{(2)}(i, j) + 1) \\
&< \frac{2\binom{2^i}{2}\binom{2^i}{0}}{\binom{2^{i+1}}{2}}(4) + \frac{\binom{2^i}{1}\binom{2^i}{1}}{\binom{2^{i+1}}{2}}(1) \\
&< 2 + \frac{2^i - 2}{2^{i+1} - 1} \\
&< 3.
\end{aligned}$$

For $w \geq 3$

$$\begin{aligned}
S_{(2)}(i+1,w) &= \sum_{j=0}^{w} \frac{\binom{2^i}{w-j}\binom{2^i}{j}}{\binom{2^{i+1}}{w}}(S_{(2)}(i,w-j)+S_{(2)}(i,j)+1) \\
&< \frac{2\binom{2^i}{w}\binom{2^i}{0}}{\binom{2^{i+1}}{w}}((2w) + \frac{2\binom{2^i}{w-1}\binom{2^i}{1}}{\binom{2^{i+1}}{w}}(2w-2) + \sum_{j=2}^{w-2} \frac{\binom{2^i}{w-j}\binom{2^i}{j}}{\binom{2^{i+1}}{w}}(2w-1) \\
&< 2w-1-\frac{2\binom{2^i}{w-1}}{w\binom{2^{i+1}}{w}}\ \big((2^i+1)(w-1)\big) \\
&< 2w-1.
\end{aligned}$$