Physically Observable Cryptography

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Abstract

Complexity-theoretic cryptography considers only abstract notions of computation, and hence cannot protect against attacks that exploit the information leakage (via electromagnetic fields, power consumption, etc.) inherent in the *physical* execution of any cryptographic algorithm. Such "physical observation attacks" *bypass* the impressive barrier of mathematical security erected so far, and successfully *break* mathematically impregnable systems. The great practicality and the inherent availability of physical attacks threaten the very relevance of complexity-theoretic security.

To respond to the present crisis, we put forward *physically observable cryptography*: a powerful, comprehensive, and precise model for defining and delivering cryptographic security against an adversary that has access to information leaked from the physical execution of cryptographic algorithms.

Our general model allows for a variety of adversaries. In this paper, however, we focus on the strongest possible adversary, so as to capture what is cryptographically possible in the worst possible, physically observable setting. In particular, we

- consider an adversary that has full (and indeed adaptive) access to any leaked information;
- show that some of the basic theorems and intuitions of traditional cryptography no longer hold in a physically observable setting; and
- construct pseudorandom generators that are provably secure against *all* physical-observation attacks.

Our model makes it easy to meaningfully restrict the power of our general physically observing adversary. Such restrictions may enable schemes that are more efficient or rely on weaker assumptions, while retaining security against meaningful physical observations attacks.

1 Introduction

"NON-PHYSICAL" ATTACKS. A *non-physical attack* against a cryptographic algorithm A is one in which the adversary is given some access to (at times even full control over) A's explicit inputs (e.g., messages and plaintexts) and some access to A's outputs (e.g., ciphertexts and digital signatures). The adversary is also given full knowledge of A —except, of course, for the secret key— but absolutely no "window" into A's internal state during a computation: he may know every single line of A's code, but whether A's execution on a given input results in making more multiplications

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than additions, in using lots of RAM, or in accessing a given subroutine, remains totally unknown to him. In a non-physical attack, A's execution is essentially a *black box*. Inputs and outputs may be visible, but what occurs within the box cannot be observed at all.

For a long time, due to lacking cryptographic theory and the consequent naive design of cryptographic algorithms, adversaries had to search no further than non-physical attacks for their devious deeds. (For instance, an adversary could often ask for and obtain the digital signature of a properly chosen message and then forge digital signatures at will.) More recently, however, the sophisticated reduction techniques of complexity-theoretic cryptography have shut the door to such attacks. For instance, if one-way functions exist, fundamental tools such as pseudorandom generation [17] and digital signatures [27, 24] can be implemented so as to be *provably* secure against *all* non-physical attacks.

Unfortunately, other realistic and more powerful attacks exist.

"Physical-Observation" Attacks. In reality, a cryptographic algorithm A must be run in a *physical* device P, and, quite outside of our control, the laws of Nature have something to say on whether P is reducible to a black box during an execution of A. Indeed, like for other physical processes, a real algorithmic execution generates all kinds of physical *observables*, which may thus fall into the adversary's hands, and be quite informative at that. For instance, Kocher et al. [20] show that monitoring the electrical power consumed by a smart card running the DES algorithm [25] is enough to retrieve the very secret key! In another example, a series of works [26, 2] show that sometimes the electromagnetic radiation emitted by a computation, even measured from a few yards away with a homemade antenna, could suffice to retrieve a secret key.

Physically Observable Cryptography. Typically, physical-observation attacks are soon followed by defensive measures (e.g., [9, 19]), giving us hope that at least *some* functions could be securely computed in our physical world. However, no rigorous theory currently exists that identifies *which* elementary functions need to be secure, and to *what extent*, so that we can construct complex cryptographic systems *provably* robust against *all* physical-observation attacks. This paper puts forward such a theory.

Our theory is not about "shielding" hardware (neither perfectly[1] nor partially[2]) but rather about how to *use partially shielded hardware in a provably secure manner.* That is, we aim at providing rigorous answers to questions of the following *relative* type:

(1) *Given a piece of physical hardware $\mathcal{P}$ that is guaranteed to compute a specific, elementary function $f(x)$ so that only some information $L_{\mathcal{P},f}(x)$ leaks to the outside,*

is it possible to construct

(2) *a physical pseudorandom generator, encryption scheme, etc., provably secure against all physically-observing adversaries?*

Notice that the possibility of such reductions is far from guaranteed: hardware $\mathcal{P}$ is assumed "good" only for computing f, while any computation outside $\mathcal{P}$ (i.e., beyond f) is assumed to be fully observable by the adversary.

[1] Perfectly shielded hardware, so that all computation performed in it leaks nothing to the outside, might be impossible to achieve and is much more than needed.

[2] We are after a computational theory here, and constructing totally or partially shielded hardware is not a task for a computational theorist.

Providing such reductions is important even with the current, incomplete knowledge about shielding hardware.[3] In fact, physically observable cryptography may properly *focus* the research in hardware protection by identifying which specific and elementary functions need to be protected and how much.

A New and General Model. Physically observable cryptography is a new and fascinating world defying our traditional cryptographic intuition. (For example, as we show, such fundamental results as the equivalence of unpredictability and indistinguishability for pseudorandom generators [30] fail to hold.) Thus, as our first (and indeed main) task, we construct a precise model, so as to be able to reason rigorously.

There are, of course, many possible models for physically observable cryptography, each rigorous and meaningful in its own right. How do we choose? We opted for the most pessimistic model of the world that still leaves room for cryptography. That is, we chose a very general model for the interplay of physical computation, information leakage, and adversarial power, trying to ensure that security in our model implies security in the real world, no matter how unfriendly the latter turns out to be (unless it disallows cryptographic security altogether).

First Results in the General Model. A new model is of interest only when non-trivial work can be done within its confines. We demonstrate that this is the case by investigating the fundamental notion of pseudorandom generation. In order to do so, we provide physically-observable variants of the traditional definitions of one-way functions, hardcore bits, unpredictability and indistinguishability. Already in the definitions stage, our traditional intuition is challenged by the unexpected behavior of these seemingly familiar notions, which is captured by several (generally easy to prove) claims and observations.

We then proceed to the two main theorems of this work. The first theorem shows that *unpredictable* physically observable generators with arbitrary expansion can be constructed from any (properly defined) physically observable one-way permutation. It thus provides a physically observable analogue to the results of [13, 7] in the traditional world. Unfortunately, this construction does not result in *indistinguishable* physically observable generators.

Our second main theorem shows that indistinguishable physically observable generators with *arbitrary expansion* can be constructed from such generators with *1-bit expansion*. It is thus the equivalent of the hybrid argument (a.k.a. "statistical walk") of [15].

Both of these theorems require non-trivial proofs that differ in significant ways from their traditional counterparts, showing how different the physically observable world really is.

Specialized Models. The generality of our model comes at a price: results in it require correspondingly strong assumptions. We wish to emphasize, however, that in many settings (e.g., arising from advances in hardware manufacturing) it will be quite meaningful to consider specialized models of physically observable cryptography, where information leakage or adversarial power are in some way restricted. It is our expectation that more efficient results, or results relying on lesser assumptions, will be awaiting in such models.

Passive vs. Active Physical Adversaries. Traditional cryptography has benefited from a thorough understanding of computational security against passive adversaries before tackling computational security against active adversaries. We believe similar advantages can be gained for physical security. Hence, for now, we consider *physically observing* adversaries only. Note, however, that our adversary has a traditional computational component and a novel physical one, and we do not start from scratch in its computational component. Indeed, our adversary will be

[3]Had complexity-theoretic cryptography waited for a proof of existence of one-way functions, we would be waiting still!

computationally quite active (e.g., it will be able to adaptively choose inputs to the scheme it attacks), but will be passive in its physical component (i.e., it will observe a physical computation without tampering with it). Attacks (e.g., [4, 8, 6, 5, 28]), defenses (e.g., [26, 23]), and models (e.g., [12]) for physically active adversaries are already under investigation, but their full understanding will ultimately depend on a full understanding of the passive case.

OTHER RELATED WORK. We note that the question of building protected hardware has been addressed before with mathematical rigor. In particular, Chari, Jutla, Rao and Rohatgi [9] consider how to protect a circuit against attackers who receive a noisy function of its state (their motivation is protection against power analysis attacks). Ishai, Sahai and Wagner [18] consider how to guarantee that adversaries who can physically probe a limited number of wires in a circuit will not be able to learn meaningful information from it. This line of research is complementary to ours: we consider reductions among physical computing devices in order to guarantee security against all physical observation attacks under some assumptions, whereas the authors of [9] and [18] consider how to build particular physical computing devices secure against a particular class of physical observations attacks. In a way, this distinction is analogous to the distinction in traditional cryptography between research on cryptographic reductions on the one hand, and research on finding instantiations of secure primitives (one-way functions, etc.) on the other.

2 Intuition for Physically Observable Computation

Our model for physically observable (PO for short) computation is based on the following (overlapping)

Informal Axioms

1. *Computation, and only computation, leaks information*

   Information may leak whenever bits of data are accessed and computed upon. The leaking information actually depends on the particular operation performed, and, more generally, on the configuration of the currently active part of the computer. However, there is no information leakage in the absence of computation: data can be placed in some form of storage where, when not being accessed and computed upon, it is totally secure.

2. *Same computation leaks different information on different computers*

   Traditionally, we think of algorithms as carrying out computation. However, an algorithm is an abstraction: a set of general instructions, whose physical implementation may vary. In one case, an algorithm may be executed in a physical computer with lead shielding hiding the electromagnetic radiation correlated to the machine's internal state. In another case, the same algorithm may be executed in a computer with a sufficiently powerful inner battery hiding the power utilized at each step of the computation. As a result, the same elementary operation on 2 bits of data may leak different information: e.g., (for all we know) their XOR in one case and their AND in the other.

3. *Information leakage depends on the chosen measurement*

   While much may be observable at any given time, not all of it can be observed simultaneously (either for theoretical or practical reasons), and some may be only observed in a probabilistic sense (due to quantum effects, noise, etc.). The specific information leaked depends on the actual measurement made. Different measurements can be chosen (adaptively and adversarially) at each step of the computation.

4. *Information leakage is local*

   The information that may be leaked by a physically observable device is the same in any execution with the same input, independent of the computation that takes place before the device is invoked or after it halts. In particular, therefore, *measurable information dissipates:* though an adversary can choose what information to measure at each step of a computation, information not measured is lost. Information leakage depends on the *past* computational history only to the extent that the *current* computational configuration depends on such history.

5. *All leaked information is efficiently computable from the computer's internal configuration.*

   Given an algorithm and its physical implementation, the information leakage is a polynomial-time computable function of (1) the algorithm's internal configuration, (2) the chosen measurement, and possibly (3) some randomness (outside anybody's control).

Remarks

As expected, the real meaning of our axioms lies in the precise way we use them in our model and proofs. However, it may be worthwhile to clarify here a few points.

- *Some form of security for unaccessed memory is mandatory.* For instance, if a small amount of information leakage from a stored secret occurs at every unit of time (e.g., if a given bit becomes 51% predictable within a day) then a patient enough adversary will eventually reconstruct the entire secret.

- *Some form of security for unaccessed memory is possible.* One may object to the requirement that only computation leaks information on the grounds that in modern computers, even unaccessed memory is refreshed, moved from cache and back, etc. However, as our formalization below shows, all we need to assume is that there is *some* storage that does not leak information when not accessed. If regular RAM leaks, then such storage can be the hard drive; if that also leaks, use flash memory; etc.

- *Some form of locality for information leakage is mandatory.* The hallmark of modern cryptography has been constructing complex systems out of basic components. If the behavior of these components changed depending on the context, then no general principles for modular design could arise. Indeed, if corporation A produced a properly shielded device used in computers build by corporation B, then corporation B should not damage the shielding on the device when assembling its computers.

- *The restriction of a single adversarial measurement per step should not misinterpreted.* If two measurements M_1 and M_2 can be "fruitfully" performed one after the other, our model allows the adversary to perform the single measurement $M = (M_1, M_2)$.

- *The polynomial-time computability of leaked information should not be misinterpreted.* This efficient computability is quite orthogonal to the debate on whether physical (e.g., quantum) computation could break the polynomial-time barrier. Essentially, our model says that *the most* an adversary may obtain from a measurement is the entire current configuration of the cryptographic machine. And such configuration is computable in time linear in the number of steps executed by the crypto algorithm. For instance, if a computer stores a Hamiltonian graph but not its Hamiltonian tour, then performing a breadth-first search on the graph should not leak its Hamiltonian tour.

(Of course, should an adversary more powerful than polynomial-time be considered, then the power of the leakage function might also be increased "accordingly.")

Of course, we do not know that these axioms are "exactly true", but definitely hope to live in a world that "approximates" them to a sufficient degree: life without cryptography would be rather dull indeed!

3 Models and Goals of Physically Observable Cryptography

Section 3.1 concerns itself with abstract computation, not yet its physical implementation. Section 3.2 describes how we model physical implementations of such abstract computation. Section 3.3 defines what it means, in our model, to build high-level constructions out of low-level primitives.

3.1 Computational Model

Motivation. Axiom 1 guarantees that unaccessed memory leaks no information. Thus we need a computing device that clearly separates memory that is actively being used from memory that is not. The traditional Turing machine, which accesses its tape sequentially, is not a suitable computational device for the goal at hand: if the reading head is on one end of the tape, and the machine needs to read a value on the other end, it must scan the entire tape, thus accessing every single memory value. We thus must augment the usual Turing machine with random access memory, where each bit can be addressed individually and independently of other bits, and enable the resulting machine to copy bits between this random-access memory and the usual tape where it can work on them. (Such individual random access can be realistic implemented.)

Axiom 4 guarantees that the leakage of a given device is the same, independent of the computation that follows or precedes it. Thus we need a model that can properly segregate one portion of a computation from another. The traditional notion of computation as carried out by a *single* Turing machine is inadequate for separating computation into multiple independent components, because the configuration of a Turing machine must incorporate (at a minimum) all future computation. To enable the modularity of physically observable cryptography, our model of computation will actually consist of *multiple* machines, each with its own physical protection, that may call each other as subroutines. In order to provide true independence, each machine must "see" its own memory space, independent of other machines (this is commonly known as virtual memory). Thus our multiple machines must be accompanied by a *virtual memory manager* that would provide for parameter passing while ensuring memory independence that is necessary for modularity. (Such virtual memory management too can be realistically implemented.)

Formalization Without Loss of Generality. Let us now formalize this model of computation (without yet specifying how information may leak). A detailed formalization is of course necessary for proofs to be meaningful. This is particularly true in the case of a new theory, where no strong intuition has yet been developed. However, the particular choice of these details is not crucial. Our theorems are robust enough to hold also for different reasonable instantiations of this model.

Abstract Virtual-Memory Computers. An *abstract virtual-memory computer*, or abstract computer for short, consists of a *collection* of special Turing machines, which invoke each other as subroutines and share a special common memory. We call each member of our collection an *abstract virtual-memory Turing machine* (abstract VTM or simply VTM for short). We write $\mathcal{A} = (A_1, \ldots, A_n)$ to mean that an abstract computer $\mathcal{A}$ consists of abstract VTMs $A_1, \ldots, A_n$, where A_1 is a distinguished VTM: the one invoked first and whose inputs and outputs coincide with those of $\mathcal{A}$. Note that abstract computers and VTMs are *not* physical devices: they represent logical computation, may have many different physical implementations. We consider physical computers in Section 3.2, after fully describing logical computation.

In addition to the traditional input, output, work and random tapes of a probabilistic Turing machine, a VTM has random access to its own *virtual address space* (VAS): an unbounded array of bits that starts at address 1 and goes on indefinitely.

The salient feature of an abstract virtual memory computer is that, while each VTM "thinks" it has its own individual VAS, in reality all of them, via a proper memory manager, share a single *physical address space (PAS).*

Virtual-Memory Management. As it is common in modern operating systems, a single *virtual-memory manager* (working in polynomial time) supervises the mapping between individual VASes and the unique PAS. The virtual-memory manager also allows for parameter passing among the different VTMs.

When a VTM is invoked, from its point of view every bit in its VAS is initialized to 0, except for those locations where the caller placed the input. The virtual-memory manager ensures that the VAS of the caller is not modified by the callee, except for the callee's output values (that are mapped back into the caller's VAS).

Virtual-memory management is a well studied subject (outside the scope of cryptography), and we shall refrain from discussing it in detail. The only explicit requirement that we impose onto our virtual-memory manager is that it should only *remap* memory addresses, but never *access* their content. (As we shall discuss in later sections, this requirement is crucial to achieving cryptographic security in the *physical world*, where each memory access may result in a leakage of sensitive information to the adversary.)

Accessing Virtual Memory. If A is a VTM, then we denote by m_A the content of A's VAS, and, for a positive integer j, we denote by $m_A[j]$ the bit value stored at location j. Every VTM has an additional, special *VAS-access tape*. To read the bit $m_A[j]$, A writes down j on the VAS-access tape, and enters a special state. Once A is in that state, the value $m_A[j]$ appears on the VAS-access tape at the current head position (the mechanics of this are the same as for an oracle query). To write a bit b in location j in its VAS, A writes down (j, b) on the VAS-access tape, and enters another special state, at which point $m_A[j]$ gets set to b.

Note that this setup allows each machine to work almost entirely in VAS, and use its work tape for merely computing addresses and evaluating simple gates.

Inputs and Outputs of a VTM. All VTM inputs and outputs are binary strings always residing in virtual memory. Consider a computation of a VTM A with an input i of length ℓ and an output o of length L. Then, at the start of the computation, the input tape of A contains 1^ℓ, the unary representations of the input length. The input i itself is located in the first ℓ bit positions of A's VAS, which will be read-only to A. At the end of the computation, A's output tape will contain a sequence of L addresses, $b_1, \ldots, b_L$, and o itself will be in A's VAS: $o = m_A[b_1] \ldots m_A[b_L]$. (The reason for input length to be expressed in unary is the preservation of the notion of polynomial running time with respect to the length of the *input tape*.)

Calling VTMs as Subroutines. Each abstract VTM in the abstract virtual-memory computer has a unique name and a special *subroutine-call tape*. When a VTM A' makes a subroutine call to a VTM A, A' specifies where A' placed the input bits to A and where A' wants the output bits of A, by writing the corresponding addresses on this tape. The memory manager remaps locations in the VAS of A' to the VAS of A and vice versa. Straightforward details are provided in Appendix B.

3.2 Physical Security Model

Physical Virtual-Memory Computers. We now formally define what information about the operation of a machine can be learned by the adversary. Note, however, that an abstract virtual-memory computer is an abstract object that may have different physical implementations. To model information leakage of any particular implementation, we introduce a *physical* virtual-memory computer (physical computer for short) and a *physical* virtual-memory Turing machine (physical VTM for short). A physical VTM $\mathcal{P}$ is a pair (L, A), where A is an abstract VTM and L is the *leakage function* described below. A physical VTM is meant to model a single shielded component that can be combined with others to form a computer. If $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ is an abstract computer and $P_i = (L_i, A_i)$, then we call P_i a *physical implementation* of A_i and $\mathcal{P} = (P_1, P_2, \ldots P_n)$ a *physical implementation* of $\mathcal{A}$.

If a physical computer $\mathcal{P}$ is deterministic (or probabilistic, but Las Vegas), then we denote by $f_{\mathcal{P}}(x)$ the function computed by $\mathcal{P}$ on input x.

The Leakage Function. The leakage function L of a physical VTM $P = (L, A)$ is a function of three inputs, $L = L(\cdot, \cdot, \cdot)$.

- The first input is the current internal configuration C of A, which incorporates everything that is in principle measurable. More precisely, C is a binary string encoding (in some canonical fashion) the information of all the tapes of A, the locations of all the heads, and the current state (but *not* the contents of its VAS m_A). We require that only the "touched" portions of the tapes be encoded in C, so that the space taken up by C is polynomially related to the space used by T (not counting the VAS space).
- The second input M is the setting of the measuring apparatus, also encoded as a binary string (in essence, a specification of what the adversary chooses to measure).
- The third input R is a sufficiently long random string to model the randomness of the measurement.

By specifying the setting M of its measuring apparatus, while A is in configuration C, the adversary will receive information $L(C, M, R)$, for a fresh random R (unknown to the adversary).

Because the adversary's computational abilities are restricted to polynomial time, we require the function $L(C, M, R)$ to be computable in time that is polynomial in the lengths of C and M.

The Adversary. Adversaries for different cryptographic tasks can be quite different (e.g., compare a signature scheme adversary to a pseudorandom generator distinguisher). However, we will augment all of the them in the same way with the ability to observe computation. We formalize this notion below.

Definition 1. We say that the adversary F *observes* the computation of a physical computer $\mathcal{P} = (P_1, P_2, \ldots, P_n)$, where $P_i = (L_i, A_i)$ if:

1. F is invoked before each step of a physical VTM of $\mathcal{P}$, with configuration of F preserved between invocations.

2. F has a special read-only *name tape* that contains the name of the physical VTM P_i of $\mathcal{P}$ that is currently active.

3. At each invocation, upon performing some computation, F writes down a string M on a special *observation tape*, and then enters a special state. Then the value $L_i(C, M, R)$, where P_i is the currently active physical VTM and R is a sufficiently long fresh random string unknown to F, appears on the observation tape, and $\mathcal{P}$ takes its next step.

4. This process repeats until $\mathcal{P}$ halts. At this point F is invoked again, with its name tape containing the index 0 indicating that $\mathcal{P}$ halted.

Notice that the above adversary is adaptive: while it cannot go back in time, its choice of what to measure in each step can depend on the results of measurements chosen in the past. Moreover, while at each step the adversary can measure only one quantity, to have a strong security model, we give the adversary all the time it needs to obtain the result of the previous measurement, decide what to measure next, and adjust its measuring apparatus appropriately.

Suppose the adversary F running on input x_F observes a physical computer $\mathcal{P}$ running on input $x_{\mathcal{P}}$, then $\mathcal{P}$ halts and produces output $y_{\mathcal{P}}$, and then F halts and produces output y_F. We denote this by

$$y_{\mathcal{P}} \leftarrow \mathcal{P}(x_{\mathcal{P}}) \rightsquigarrow F(x_F) \rightarrow y_F .$$

Note that F sees neither $x_{\mathcal{P}}$ nor $y_{\mathcal{P}}$ (unless it can deduce these values indirectly by observing the computation).

3.3 Assumptions, Reductions, and Goals

In addition to traditional, complexity-theoretic assumptions (e.g., the existence of one-way permutations), physically observable cryptography also has *physical assumptions.* Indeed, the very existence of a machine that "leaks less than complete information" is an assumption about the physical world. Let us be more precise.

Definition 2. A physical VTMs is *trivial* if its leakage function reveals its entire internal configuration[4] and *non-trivial* otherwise.

Fundamental Premise. *The very existence of a non-trivial physical VTM is a physical assumption.*

Just like in traditional cryptography, the goal of physically observable cryptography is to rigorously derive desirable objects from simple (physical and computational) assumptions. As usual, we refer to such rigorous derivations as *reductions.* Reductions are expected to use stated assumptions, *but should not themselves consist of assumptions!*

Definition 3. Let $\mathcal{P}'$ and $\mathcal{P}$ be physical computers. We say that $\mathcal{P}'$ *reduces to* $\mathcal{P}$ (alternatively, $\mathcal{P}$ *implies* $\mathcal{P}'$) if every non-trivial physical VTM of $\mathcal{P}'$ is also a physical VTM of $\mathcal{P}$.

[4] It suffices, in fact, to reveal only the current state and the characters observed by the reading heads—the adversary can infer the rest by observing the leakage at every step.

4 Definitions and Observations

Having put forward the rules of physically observable cryptography, we now need to gain some experience in distilling its first assumptions and constructing its first reductions.

We start by quickly recalling basic notions and facts from traditional cryptography that we use in this paper.

4.1 Traditional Building Blocks

We assume familiarity with the traditional GMR notation (recalled in our Appendix A).

We also assume familiarity with the notions of one-way function [10] and permutation; with the notion of of hardcore bits [7]; with the fact that all one-way functions have a Goldreich-Levin hardcore bit [13]; and with the notion of a *natural* hardcore bit (one that is simply a bit of the input, such as the last bit of the RSA input [3]). Finally, recall the well-known *iterative generator* of Blum and Micali [7], constructed as follows:

iterate a one-way permutation on a random seed, outputting the hardcore bit at each iteration.

(All this traditional material is more thoroughly summarized in Appendix C.)

4.2 Physically Observable One-Way Functions and Permutations

Avoiding a Logical Trap. In traditional cryptography, the *existence* of a one-way function is currently an assumption, while the *definition* of a one-way function does not depend on any assumption. We wish that the same be true for physically observable one-way functions. Unfortunately, the most obvious attempt to defining physically observable one-way functions does not satisfy this requirement. The attempt consists of replacing the Turing machine T in the one-way function definition of Appendix C with a physical computer $\mathcal{P}$ observed by F. Precisely,

Definition Attempt: A *physically observable (PO) one-way functions* is a function $f : \{0,1\}^* \to \{0,1\}^*$ such that there exists a polynomial-time physical computer $\mathcal{P}$ that computes f and, for any polynomial-time adversary F, the following probability is negligible as a function of k:

$$\Pr[x \stackrel{R}{\leftarrow} \{0,1\}^k \, ; \, y \leftarrow \mathcal{P}(x) \rightsquigarrow F(1^k) \to \mathit{state} \, ; \, z \leftarrow F(\mathit{state}, y) : f(z) = y].$$

Intuitively, physically observable one-way functions should be "harder to come by" than traditional ones: unless no traditional one-way functions exist, we expect that only some of them may also be PO one-way. Recall, however, that *mathematically* a physical computer $\mathcal{P}$ consists of pairs (L, A), where L is a leakage function and A an abstract VTM, in particular a single Turing machine. Thus, by setting L be the constant function 0, and $A = \{T\}$, where T is the Turing machine computing f, we obtain a non-trivial computer $\mathcal{P} = \{(L, A)\}$ that ensures that f is PO one-way as soon as it is traditionally one-way. The relevant question, however, is not whether such a computer can be mathematically defined, but whether it can be *physically* built. As we have said already, the mere existence of a non-trivial physical computer is in itself an assumption, and *we do not want the definition of a physically observable one-way function to rely on an assumption.* Therefore, we do not define what it means for a *function* f to be physically observable one-way. Rather, we define what it means for a particular *physical computer computing* f to be one-way.

We shall actually introduce, in order of strength, three physically observable counterparts of traditional one-way functions and one-way permutations.

Minimal One-Way Functions and Permutations. Avoiding the logical trap discussed above, the first way of defining one-way functions (or permutations) in the physically observable world is to say that $\mathcal{P}$ is a one-way function if it computes a permutation $f_{\mathcal{P}}$ that is hard to invert despite the leakage from $\mathcal{P}$'s computation. We call such physically observable one-way functions and permutations "minimal" in order to distinguish them from the other two counterparts we are going to discuss later on.

Definition 4. A polynomial-time deterministic physical computer $\mathcal{P}$ is *minimal one-way function* if for any polynomial-time adversary F, the following probability is negligible as a function of k:

$$\Pr[x \xleftarrow{R} \{0,1\}^k \,;\, y \leftarrow \mathcal{P}(x) \rightsquigarrow F(1^k) \rightarrow \mathit{state} \,;\, z \leftarrow F(\mathit{state}, y) : f_{\mathcal{P}}(z) = y].$$

Furthermore, if $f_{\mathcal{P}}$ is length-preserving and bijective, we call $\mathcal{P}$ a *minimal one-way permutation.*

Durable Functions and Permutations. A salient feature of an abstract permutation is that the output is random for a random input. The following definition captures this feature, even in the presence of computational leakage.

Definition 5. A *durable* function (permutation) is a minimal one-way function (permutation) $\mathcal{P}$ such that, for any polynomial-time adversary F, the value $|p_k^P - p_k^R|$ is negligible in k, where

$$\begin{aligned} p_k^P &= \Pr[x \xleftarrow{R} \{0,1\}^k \,;\, y \leftarrow \mathcal{P}(x) \rightsquigarrow F(1^k) \rightarrow \mathit{state} : F(\mathit{state}, y) = 1] \\ p_k^R &= \Pr[x \xleftarrow{R} \{0,1\}^k \,;\, y \leftarrow \mathcal{P}(x) \rightsquigarrow F(1^k) \rightarrow \mathit{state} \,;\, z \xleftarrow{R} \{0,1\}^k : F(\mathit{state}, z) = 1]\,. \end{aligned}$$

Maximal One-Way Functions and Permutations. We now define physically observable one-way functions that leak nothing at all.

Definition 6. A *maximal* one-way function (permutation) is a minimal one-way function (permutation) $\mathcal{P}$ such that the leakage functions of its component physical VTMs are independent of the input x of $\mathcal{P}$ (in other words, x has no effect on the distribution of information that leaks).

One can also define *statistically maximal* functions and permutations, where for any two inputs x_1 and x_2, the observed leakage from $\mathcal{P}(x_1)$ and $\mathcal{P}(x_2)$ is statistically close; and *computationally maximal* functions and permutations, where for any two inputs x_1 and x_2, what $\mathcal{P}(x_1)$ leaks is indistinguishable from what $\mathcal{P}(x_2)$ leaks. We postpone defining these formally.

4.3 Physically Observable Pseudorandomness

One of our goals in the sequel will be to provide a physically observable analogue to the Blum-Micali [7] construction of pseudorandom generators. To this end, we provide here physically observable analogues of the notions of indistinguishability [30] and unpredictability [7].

Unpredictability. The corresponding physically observable notion replaces "unpredictability of bit $i+1$ from the first i bits" with "unpredictability of bit $i+1$ from the first i bits and the leakage from their computation."

Definition 7. Let p be a polynomially bounded function such that $p(k) > k$ for all positive integers k. Let $\mathcal{G}$ be a polynomial-time deterministic physical computer that, on a k-bit input, produces $p(k)$-bit output, one bit at a time (i.e., it writes down on the output tape the VAS locations of the output bits in left to right, one a time). Let $\mathcal{G}^i$ denote running $\mathcal{G}$ and aborting it after it outputs the i-th bit. We say that $\mathcal{G}$ is a *PO unpredictable generator with expansion* p if for any polynomial-time adversary F, the value $|p_k - 1/2|$ is negligible in k, where

$$p_k = \Pr[(i, \mathit{state}_1) \leftarrow F(1^k)\ ;\ x \xleftarrow{R} \{0,1\}^k\ ;\ y_1 y_2 \ldots y_i \leftarrow G^i(x) \rightsquigarrow F(\mathit{state}_1) \to \mathit{state}_2 : F(\mathit{state}_2, y_1 \ldots y_i) = y_{i+1}]\,,$$

(where y_j denotes the j-th bit of $y = \mathcal{G}(x)$).

Indistinguishability. The corresponding physically observable notion replaces "indistinguishability" by *"indistinguishability in the presence of leakage."* That is, a polynomial-time adversary F first observes the computation of a pseudorandom string, and then receives either that same pseudorandom string or a totally independent random string, and has to distinguish between the two cases.

Definition 8. Let p be a polynomially bounded function such that $p(k) > k$ for all positive integers k. We say that a polynomial-time deterministic physical computer $\mathcal{G}$ is a *PO indistinguishable generator with expansion* p if for any polynomial-time adversary F, the value $|p_k^G - p_k^R|$ is negligible in k, where

$$\begin{aligned} p_k^G &= \Pr[x \xleftarrow{R} \{0,1\}^k\ ;\ y \leftarrow \mathcal{G}(x) \rightsquigarrow F(1^k) \to \mathit{state} : F(\mathit{state}, y) = 1] \\ p_k^R &= \Pr[x \xleftarrow{R} \{0,1\}^k\ ;\ y \leftarrow \mathcal{G}(x) \rightsquigarrow F(1^k) \to \mathit{state}\ ;\ z \xleftarrow{R} \{0,1\}^{p(k)} : F(\mathit{state}, z) = 1]\,. \end{aligned}$$

4.4 First Observations

Reductions in our new environment are substantially more complex than in the traditional setting, and we have chosen a very simple one as our first example. Namely, we prove that minimal one-way permutations compose just like traditional one-way permutations.

Claim 1. *A minimal one-way permutation $\mathcal{P}$ implies a minimal one-way permutation $\mathcal{P}'$ such that $f_{\mathcal{P}'}(\cdot) = f_{\mathcal{P}}(f_{\mathcal{P}}(\cdot))$.*

Proof. To construct $\mathcal{P}'$, build a trivial physical VTM that simply runs $\mathcal{P}$ twice. See Appendix D for details. We wish to emphasize that, though simple, the details of the proof of Claim 1 illustrate exactly how our axioms for physically observable computation (formalized in our model) play out in our proofs. □

Despite this good news about our simplest definition, minimal one-way permutations are not suitable for the Blum-Micali construction due to the following observation.

Observation 1. Minimal one-way permutations do not chain. That is, an adversary observing the computation of $\mathcal{P}'$ from Claim 1 and receiving $f_{\mathcal{P}}(f_{\mathcal{P}}(x))$ may well be able to compute the intermediate value $f_{\mathcal{P}}(x)$.

This is so because $\mathcal{P}$ may leak its entire output while being minimal one-way.

Unlike minimal one-way permutations, maximal one-way permutations do suffice for the Blum-Micali construction.

Claim 2. *A maximal one-way permutation $\mathcal{P}$ implies a PO unpredictable generator.*

Proof. The proof of this claim, whose details are omitted here, is fairly straightforward: simply mimic the Blum-Micali construction, computing $x_1 = \mathcal{P}(x_0)$, $x_2 = \mathcal{P}(x_1)$, $\ldots$, $x_n = \mathcal{P}(x_{n-1})$ and outputting the Goldreich-Levin bit of x_n, of x_{n-1}, $\ldots$, of x_1. Note that the computation of Goldreich-Levin must be done on a trivial physical VTM (because to do otherwise would involve another assumption), which will result in full leakage of x_n, x_{n-1}, $\ldots$, x_0. Therefore, for unpredictability, it is crucial that the bits be computed and output one at a time and in reverse order like in the original Blum-Micali construction. □

Observation 2. Using maximal (or durable or minimal) one-way permutations in the Blum-Micali construction does not yield PO indistinguishable generators.

Indeed, the output from the above construction is easily distinguishable from random in the presence of leakage, because of the eventual leakage of $x_0, x_1, \ldots, x_n$.

The above leads to the following observation.

Observation 3. A PO unpredictable generator is not necessarily PO indistinguishable.

However, indistinguishability still implies unpredictability, even in this physically observable world.

If the maximal one-way permutation satisfies an additional property, we can obtain PO indistinguishable generators. Recall that a (traditional) hardcore bit of x is *natural* if it is a bit in some fixed location of x.

Claim 3. *A maximal one-way permutation $\mathcal{P}$ for which $f_{\mathcal{P}}$ has a (traditional) natural hardcore bit implies a PO indistinguishable generator.*

Proof. Simply use the previous construction, but output the natural hardcore bit instead of the Goldreich-Levin one. Because all parameters (including inputs and outputs) are passed through memory, this output need not leak anything. Thus, the result is indistinguishable from random in the presence of leakage, because there is no meaningful leakage. □

The claims and observations so far have been fairly straightforward. We now come to the two main theorems.

5 Theorems

Our first main theorem demonstrates that the notion of a durable function is in some sense the "right" analogue of the traditional one-way permutation: when used in the Blum-Micali construction, with Goldreich-Levin hardcore bits, it produces a PO unpredictable generator; moreover, the proof seems to need all of the properties of durable functions. (Identifying the *minimal* physically observable assumption for pseudorandom generation is a much harder problem, not addressed here.)

Theorem 1. *A durable function implies a PO unpredictable generator (with any polynomial expansion).*

Proof. Utilize the Blum-Micali construction, outputting (in reverse order) the Goldreich-Levin bit of each x_i, just like in Claim 2. The hard part is to show that this is unpredictable. Durable functions, in principle, could leak their own hardcore bits—this would not contradict the indistinguishability of the output from random (indeed, by the very definition of a hardcore bit). However, what helps us here is that we are using specifically the Goldreich-Levin hardcore bit, computed as $r \cdot x_i$ for a random r. Note that r will be leaked to the adversary before the first output bit is even produced, during its computation as $r \cdot x_n$. But crucially, the adversary will not yet know r during the iterated computation of the durable function, and hence will be unable to tailor its measurement to the particular r. We can then show (using the same error-correcting code techniques for reconstructing x_i as in [13]) that $r \cdot x_i$ is unpredictable given the leakage obtained by the adversary. More details of the proof are deferred to Appendix E. □

Our second theorem addresses the stronger notion of PO indistinguishability. We have already seen that PO indistinguishable generators can be built out of maximal one-way permutations with natural hardcore bits. However, this assumption may be too strong. What this theorem shows is that as long as there is some way to a build the simplest possible PO indistinguishable generator—the one with one-bit expansion—there is a way to convert it to a PO indistinguishable generator with arbitrary expansion.

Theorem 2. *A PO indistinguishable generator that expands its input by a single bit implies a PO indistinguishable generator with any polynomial expansion.*

Proof. The proof consists of a hybrid argument, but such arguments are more complex in our physically observable setting (in particular, rather than a traditional single "pass" through n intermediate steps —where the first is pseudorandom and the last is truly random— they now require two passes: from 1 to n and back). Details can be found in Appendix F.

□

6 Some Further Directions

A New Role for Older Notions. In traditional cryptography, in light of the Goldreich-Levin construction [13], it seemed that finding natural hardcore bits of one-way functions became a nearly pointless endeavor (from which only minimal efficiency could be realized). However, Claim 2 changes the state of affairs dramatically. This shows how physically observable cryptography may provide new impetus for research on older subjects.

(Another notion from the past that seemed insignificant was the method of outputting bits backwards in the Blum-Micali generator. It was made irrelevant by the equivalence of unpredictability and indistinguishability. In our new world, however, outputting bits backwards is crucially important for Claim 2 and Theorem 1.)

Inherited vs. Generated Randomness. Our definitions in the physically observable model do not address the origin of the secret input x for a one-way function $\mathcal{P}$: according to the definitions, nothing about x is observable by F before $\mathcal{P}$ starts running. One may take another view of a one-way function, however: one that includes the generation of a random input x as the first step. While in traditional cryptography this distinction seems unimportant, it is quite crucial in physically observable cryptography: the very generation of a random x may leak information about x. It is conceivable that some applications require a definition that includes the generation of a random x as part of the functionality of $\mathcal{P}$. However, we expect that in many instances it is possible to "hardwire" the secret randomness before the adversary has a chance to observe the machine, and then rely on pseudorandom generation.

Deterministic Leakage and Repeated Computations. Our definitions allow for repeated computation to leak new information each time. However, the case can be made (e.g., due to proper hardware design) that some devices computing a given function f may leak the same information whenever f is evaluated at the same input x. This is actually implied by making the leakage function deterministic and independent of the adversary measurement. *Fixed-leakage physically observable cryptography* promises to be a very useful restriction of our general model (e.g., because, for memory efficiency, crucial cryptographic quantities are often reconstructed from small seeds, such as in the classical pseudorandom function of [16]).

Signature Schemes. In a forthcoming paper we shall demonstrate that digital signatures provide another example of a crucial cryptographic object constructible in our general model. Interestingly, we shall obtain our result by relying on some old constructions (e.g., [21] and [22]), highlighting once more how old research may play a role in our new context.

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A Minimal GMR Notation

- *Random assignments.* If S is a probability space, then "$x \leftarrow S$" denotes the algorithm which assigns to x an element randomly selected according to S. If F is a finite set, then the notation "$x \leftarrow F$" denotes the algorithm which assigns to x an element selected according to the probability space whose sample space is F and uniform probability distribution on the sample points.

- *Probabilistic experiments.* If $p(\cdot, \cdot, \cdots)$ is a predicate, we use $\Pr[x \leftarrow S; y \leftarrow T; ... : p(x, y, \cdots)]$ to denote the probability that $p(x, y, \cdots)$ will be true after the ordered execution of the algorithms $x \leftarrow S$, $y \leftarrow T, ...$

B Calling VTMs as Subroutines

If A' wants to call A on the ℓ-bit input $i = m_{A'}[a'_1] \ldots m_{A'}[a'_\ell]$, and if A returns an L-bit output on an ℓ-bit input, then the VTM A' has to write down on its subroutine-call tape

1. name of A;
2. a sequence of ℓ addresses in its own VAS, $a'_1, \ldots, a'_\ell$;
3. a sequence of L distinct addresses in its own VAS, $b'_1, \ldots, b'_L$.

Then A' enters a special "call" state and suspends its computation. At this point, the memory manager creates a new VAS for A, ensuring that

- location i in the VAS of A, for $1 \leq i \leq \ell$, is mapped to the same PAS location as a'_i in the VAS of A', and
- all the other locations in the VAS of A map to blank and unassigned PAS locations. (Namely, in case of nested calls, any VAS location of any machine in the call stack —i.e., A', the caller of A', etc.— must not map to these PAS locations.)

Then the computation of A begins in the "start" state, with a blank work tape and the input tape containing 1^ℓ. When A halts, the memory manager remaps location b'_i, for $1 \leq i \leq L$, in the VAS of A' to the same PAS location as b_i in the VAS of A. (Recall that b_i appears on the output tape of A, and that all the b'_i are distinct, so the remapping is possible.) The output value of A is taken to be the value $o = m_{A'}[b'_1] \ldots m_{A'}[b'_\ell]$, and A' resumes operation.

Note that the input locations a'_i in the caller's VAS do not need to be distinct; nor do the output locations b_i in the callee's VAS. Therefore, it is possible that the memory manager will need to map two or more locations in a VTM's VAS to the same PAS location (indeed, because accessing memory may cause leakage, remapping memory is preferable to copying it). When a VAS location is written to, however, the memory manager ensures that only one PAS location is affected: if the VAS location is mapped to the same physical address as another VAS location, it gets remapped to a fresh physical address.

C Traditional Building Blocks

- *One-way functions* [10]. A one-way function is a function $f : \{0,1\}^* \to \{0,1\}^*$ such that there exists a polynomial-time Turing machine T that computes f and, for any polynomial-time adversary F, the following probability is negligible as a function of k:

$$\Pr[x \stackrel{R}{\leftarrow} \{0,1\}^k \,;\, y \leftarrow T(x) \,;\, z \leftarrow F(1^k, y) \,:\, f(z) = y].$$

- *One-way permutations.* A one-way permutation is a one-way function that is length-preserving and bijective.

- *One-way permutations are composable.* For all n, if f is a one-way permutation, so is f^n.

- *One-way permutations are chainable.* For all $0 \le i < n$ and for all polynomial-time adversary F, the following probability is negligible as a function of k:

$$\Pr[x \stackrel{R}{\leftarrow} \{0,1\}^k \,;\, y \leftarrow f^n(x) \,;\, (i, \mathit{state}) \leftarrow F(y) \,;\, z \leftarrow F(\mathit{state}, f^{n-i}(x)) \,:\, f^{i+1}(z) = y].$$

- *Hardcore Bits* [7]. Let f be a one-way permutation, and B a polynomial-time computable predicate. We say that B is a hardcore bit (for f) if, for any polynomial-time adversary F, the value $|p_k - 1/2|$ is negligible in k, where

$$p_k = \Pr[x \stackrel{R}{\leftarrow} \{0,1\}^k \,;\, y \leftarrow f(x) \,;\, g \leftarrow F(1^k, y) \,:\, g = B(x)].$$

  The first hardcore bit was exhibited for the discrete-log function [7].

- *All one-way permutations have a hardcore bit* [13]. Let f be a one-way permutation, and let $r_1, \ldots, r_k$ be a sequence of random bits. Then, informally, the randomly chosen predicate B_r is overwhelmingly likely a hardcore bit for f, where B_r is the predicate so defined: for a k-bit string $x = x_1 \cdots x_k$, $B_r(x) = x_1 \times r_1 + \ldots x_k \times r_k \bmod 2$.

- *Natural hardcore bits.* We call a hardcore bit B natural if $B(x)$ returns the bit in a fixed location of the bit string x. Some specific one-way permutations possess natural hardcore bits —for instance, the last bit is hardcore for the RSA function [3].

- *Unpredictable pseudorandom generators* [7]. Let p be a polynomially bounded function such that $p(k) > k$ for all positive integers k. Let G be a polynomial-time deterministic algorithm that, on a k-bit input, produces a $p(k)$-bit output. We say that G is an *unpredictable pseudorandom generator with expansion* p if for any polynomial-time adversary F, the value $|p_k - 1/2|$ is negligible in k, where

$$p_k \quad = \quad \Pr[(i, \mathit{state}) \leftarrow F(1^k) \,;\, x \stackrel{R}{\leftarrow} \{0,1\}^k \,;\, y \leftarrow G(x) \,:\, F(\mathit{state}, y_1 \ldots y_i) = y_{i+1}]\,,$$

  (where y_j denotes the j-th bit of y).

- *Indistinguishable pseudorandom generators* [30]. Unpredictable pseudorandom generators are provably the same as indistinguishable generators, defined as follows. Let G, again, be a polynomial-time deterministic algorithm that, on a k-bit input, produces a $p(k)$-bit output. We say that G is an *indistinguishable pseudorandom generator with expansion* p if for any polynomial-time adversary F, the value $|p_k^G - p_k^R|$ is negligible in k, where

  $$\begin{aligned} p_k^G &= \Pr[x \xleftarrow{R} \{0,1\}^k \,;\, y \leftarrow G(x) : F(\mathit{state}, y) = 1] \\ p_k^R &= \Pr[x \xleftarrow{R} \{0,1\}^k \,;\, y \leftarrow G(x) \,;\, z \xleftarrow{R} \{0,1\}^{p(k)} : F(\mathit{state}, z) = 1] \end{aligned}$$

  Because every unpredictable pseudorandom generator is indistinguishable and vice versa, we refer to them as simply "pseudorandom generators" or "PRGs."

- *The iterative PRG construction* [7].

  For any one-way permutation f, the following is a pseudorandom generator:

  > *choose a random secret seed, and iterate f on it, outputting the hardcore bit at each iteration.*

D Proof of Claim 1

Proof of Claim 1. Let $\mathcal{P} = (P_1, \ldots, P_n)$ be a minimal one-way permutation, where each physical VTM P_i is a pair consisting of a leakage function L_i and an abstract VTM A_i. Intuitively, $\mathcal{P}'$ simply runs $\mathcal{P}$ twice (i.e., it calls twice P_1 which is the entry point to all other P_i of $\mathcal{P}$). Formally this is accomplished by creating a new trivial physical VTM P_0 that twice calls P_1. Define P_0 to be the (new) trivial physical VTM (L_0, A_0), where L_0 is the trivial leakage function (i.e., the one leaking everything) and A_0 is the following abstract VTM:

> On input a k-bit value x in VAS locations $1, 2, \ldots, k$, call $A_1(x)$ as a subroutine specifying that the returned value y_1 be placed in VAS locations $k+1, k+2, \ldots, 2k$.
>
> Then, run A_1 again on input y_1, specifying that the returned value y_2 be placed in VAS locations $2k+1, 2k+2, \ldots, 3k$.
>
> Output y_2 (i.e., place the addresses $2k+1, 2k+2, \ldots, 3k$ on the output tape) and halt.

Consider now the physical computer $\mathcal{P}'$ that has the above specified P_0 as the first machine, together with all the machines of $\mathcal{P}$, that is, $\mathcal{P}' = (P_0, P_1, \ldots, P_n)$. It is clear that $\mathcal{P}'$ is implied by $\mathcal{P}$ and that $\mathcal{P}'$ computes $f_\mathcal{P}(f_\mathcal{P}(x))$ in polynomial time. Therefore, all that is left to prove is the "one-wayness" of $\mathcal{P}'$: that is, that the adversary will not succeed in finding z such that $f_\mathcal{P}(f_\mathcal{P}(z)) = y_2$ as described in the experiment of Definition 4. This is done by the following elementary reduction.

Because $f'_\mathcal{P} = f^2_\mathcal{P}$ is a permutation, finding any inverse z of y_2 means finding the original input x. Suppose there exists an adversary F' that succeeds in finding x after observing the computation of $\mathcal{P}'$ and receiving $y_2 = f_\mathcal{P}(f_\mathcal{P}(x))$. Then, in the usual style of cryptographic reductions, we derive a contradiction by showing that there exists another adversary F that (using F') succeeds in finding x after observing the computation of $\mathcal{P}$ and receiving $y_1 = f_\mathcal{P}(x)$.

$F(1^k)$ "virtually" executes $\mathcal{P}'(x) \rightsquigarrow F'(1^k)$: at at each (virtual) step of $\mathcal{P}'$, F receives the measurement that F' wishes to make, and responds with the appropriately distributed leakage. In so doing, however, F is only entitled to observe $\mathcal{P}(x)$ once.

Recall that F' expects to observe a five-stage computation:

1. P_0 prepares the tapes for the subroutine call to $P_1(x)$
2. P_1 and its subroutines compute $y_1 = f_{\mathcal{P}}(x)$
3. P_0 prepares the tapes for the subroutine call to $P_1(y_1)$
4. P_1 and its subroutines compute $y_2 = f_{\mathcal{P}}(y_1)$
5. P_0 places the address of y_2 on the output tape

During Stage 1, F can very easily answer any measurement made by F'. In fact, (1) because P_0 trivial, any measurement of F' should be answered with the entire configuration of P_0 and, (2) because P_0 just reassigns VAS pointers without reading or handling any secret VAS bits, each of P_0's configurations can be computed by F from 1^k (which is given to F as an input).

After so simulating Stage 1, F starts observing the computation of $\mathcal{P}(x)$. At each step, F is allowed a measurement M, and the measurement it chooses coincides with the one F' wants, thus F can easily forward to F' the obtained result. At the end of Stage 2 F receives $y_1 = f_{\mathcal{P}}(x)$ (which it stores but does not forward to F').

Stage 3 is as easily simulated as Stage 1.

During Stage 4, F "virtually runs" physical computer $\mathcal{P}(y_1)$, that is, it runs the corresponding abstract computer $A(y_1)$. At each step, if A_i is the active machine in configuration C, and F' specifies a measurement M, then F returns the leakage $L_i(C, M, R)$ for a random R.

Upon simulating stage 5 (as easily as Stage 1), F computes $y_2 = f_{\mathcal{P}}(y_1)$, and gives it to F' to receive x.

□

The Axioms in Action

Let us show that all our axioms for physically observable computation are already reflected in the very simple proof of Claim 1.

- The simulation of Stages 1, 3, and 5 relies on Axiom 1. In fact, F can simulate P_0 only because P_0 does not access the VAS, and unaccessed VAS leaks no information.
- The simulation of Stage 2 relies on Axiom 4. Specifically, we relied on the fact that $\mathcal{P}(x)$ run in "isolation" has the same leakage distribution as $\mathcal{P}(x)$ "introduced" by P_0, and more generally in the "context" of $\mathcal{P}'$.

  Similarly, also the simulation of Stage 4 relies on Axiom 4: the leakage of running $\mathcal{P}$ from scratch on a string y_1 is guaranteed to be the same as the leakage of running $\mathcal{P}$ after y_1 is computed as $\mathcal{P}(x)$.
- The simulation of Stage 4 relies on Axiom 5. In fact F was not observing the real $\mathcal{P}$, but rather was running $\mathcal{P}$ on its own and simulating $\mathcal{P}$'s leakage which therefore had to be polynomial-time computable.
- Axiom 2 is implicitly relied upon. In a sense, Axiom 2 says that the same algorithm can have different leakage distributions, depending on the different physical machines which run it. In particular, therefore, it makes the very existence of a physically observable one-way permutation *plausible*. Trivial machines that leak everything certainly exist, and using them to compute $f(x)$ from x would make it easy to find an inverse of $f(x)$. Thus, if the f's

leakage were the same for every machine, PO one-way permutations would not exist, making the entire theory moot.

- Axiom 3 has been incorporated into the model, by giving adversary F' the power of choosing its own measurements at every step of the computation.

E Proof Sketch of Theorem 1

Proof sketch of Theorem 1. Let $\mathcal{P}$ be a durable function. To construct out of $\mathcal{P}$ a PO unpredictable generator $\mathcal{G}$ with expansion p, we will mimic the iterative construction of Blum and Micali [7], combining it with the Goldreich-Levin [13] hardcore bit. For this construction, it is crucial that the bits are output in reverse order, as in [7]: namely, that all computations of $\mathcal{P}$ take place before any Goldreich-Levin bits are computed (because we are not assuming a secure machine for computing Goldreich-Levin bits, and hence the hardcore bit computation will leak everything about its inputs).

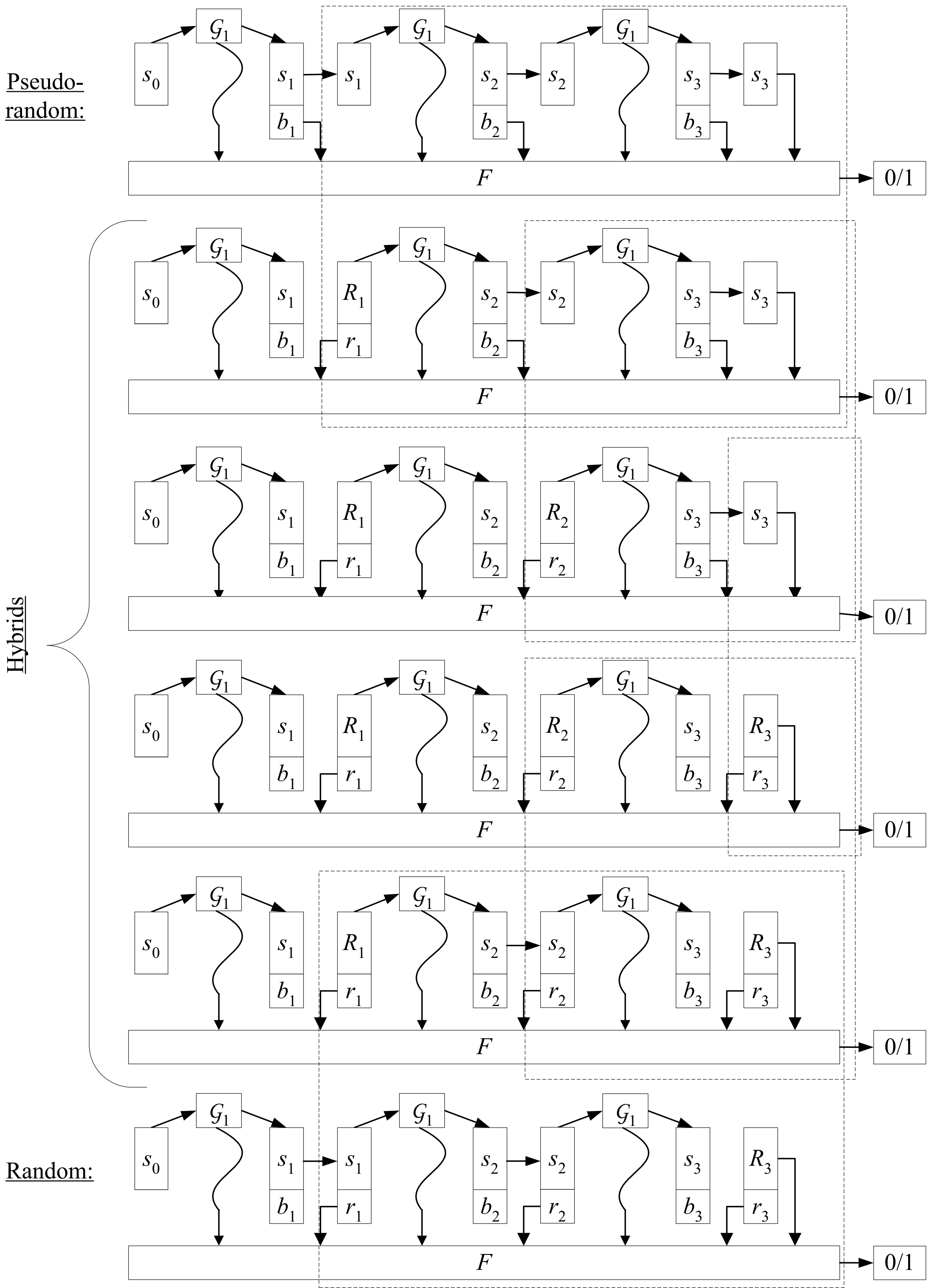
Specifically, given a random seed (x_0, r), to output $\ell = p(|x|+|r|)$ bits, $\mathcal{G}$ computes $x_1 = \mathcal{P}(x_0)$, $x_2 = \mathcal{P}(x_1)$, …, $x_\ell = \mathcal{P}(x_{\ell-1})$, and outputs $b_1 = x_{\ell-1} \cdot r, b_2 = x_{\ell-2} \cdot r, \ldots, b_\ell = x_0 \cdot r$, where "$\cdot$" denotes the dot product modulo 2 (i.e., the Goldreich-Levin bit). Formally, this is done by constructing a trivial physical VTM to "drive" this process and compute the hardcore bits; we omit the details here, as they are straightforward and similar to the proof of Claim 1.

To prove that this is indeed unpredictable, consider first a simpler situation. Starting from a random x, compute $\mathcal{P}(x)$, letting the adversary observe the computation. Now provide the adversary with $\mathcal{P}(x)$ and a random r, and have it predict $r \cdot x$. If the adversary is successful with probability significantly better than 1/2, then it is successful for significantly more than 50% of all possible values for r. Thus, we can run it for multiple different values of r, and reconstruct x using the same techniques as in [13], which would contradict the minimal one-wayness of $\mathcal{P}$. Note that even though we use the adversary to predict $x \cdot r$ for multiple values r, the adversary needs to observe $\mathcal{P}(x)$ only *once*. This is because the observation takes place before r is provided to the adversary, and therefore the choices made by the adversary during the observation are independent of r.

The actual generator, of course, is more complex than the above scenario. To prove that bit b_i is unpredictable, first note that $x_{\ell-i}$ is not computable by the adversary even if the adversary observes the computation until b_{i-1} is output (this can be shown by a properly constructed hybrid argument based on the definition of durable, similarly to the hybrid argument in the proof of Theorem 2). Also observe that the previous bits, $b_1, \ldots, b_{i-1}$ are all efficiently computable from $x_{\ell-i+1} = \mathcal{P}(x_{\ell-i})$, which the adversary receives anyway when it observes the computation of b_{i-1}. Thus, proving that b_i is unpredictable reduces to the simpler case already proven above. □

F Proof Sketch of Theorem 2

Proof sketch of Theorem 2. Let $\mathcal{G}_1$ be a PO indistinguishable generator with one-bit expansion that, on input s_0 of length k, outputs s_1 of length k followed by a single bit b. To construct out of $\mathcal{G}_1$ a PO indistinguishable generator $\mathcal{G}$ with expansion p, we will simply mimic the iterative construction of [7]: to generate $\ell = p(k)$ pseudorandom bits on a k-bit input seed s_0, compute $(s_1, b_1) = \mathcal{G}_1(s_0)$ and output b_1; then compute $(s_2, b_2) = \mathcal{G}_1(s_1)$ and output b_3, and so on for ℓ times (note that there is no need here to output bits in reverse order). Formally, this is done by


Pseudo-
random:
Hybrids
Random:
$\mathcal{G}_1$
s_0
s_1
b_1
s_2
b_2
s_3
b_3
R_1
r_1
R_2
r_2
R_3
r_3
F
0/1

constructing a trivial physical VTM to "drive" this process; we omit the details here, as they are straightforward and similar to the proof of Claim 1.

The proof that the resulting $\mathcal{G}$ is PO indistinguishable is by a hybrid argument, somewhat similar to (but more complex than) the hybrid argument that shows that unpredictability implies indistinguishability for traditional pseudorandom generators ([30]; see [14] for an excellent exposition). We recall the essence of that hybrid argument here to prepare for the more complex hybrid argument in our case. Suppose that the pseudorandom string $b_1 b_2 \dots b_\ell$ is unpredictable (i.e., b_i cannot be predicted given $b_1 \dots b_{i-1}$), but can be distinguished from a truly random string $r_1 r_2 \dots r_\ell$. Then consider the $\ell - 1$ "hybrid" strings, the i-th string h_i being $b_1 b_2 \dots b_{\ell-i} r_{\ell-i+1} r_{i+2} \dots r_\ell$ (then $h_0 = b_1 b_2 \dots b_\ell$ and $h_\ell = r_1 r_2 \dots r_\ell$). If the i-th string can be distinguished from the $(i+1)$-th, then the bit $b_{\ell-i+1}$ can be distinguished from $r_{\ell-i+1}$ in the presence of $b_1 b_2 \dots b_{\ell-i}$, i.e., the bit $b_{\ell-i+1}$ can be predicted (which is a contradiction).

In our proof, our hybrids are not just strings. Rather, because we have to also deal with the leakage, our hybrids are *processes*. The pseudorandom process consists of running $\mathcal{G}_1(s_0)$ and then giving the adversary $b_1 \dots b_\ell$ (actually, we can give s_ℓ as well, it will not change the proof, just like in the hybrid argument above). The random process consists of running $\mathcal{G}_1(s_0)$ and then giving the adversary random bits $r_1 \dots r_\ell$ (and a random R_ℓ in place of s_ℓ). There are $2(n-1)$ hybrid processes, depicted in the figure on page 22 and defined as follows.

The i-th hybrid process H_i for $i \leq n$ is the process that runs $\mathcal{G}_1(s_0)$ to obtain (s_1, b_1); then replaces (s_1, b_1) with new random (R_1, r_1), outputs r_1 and runs $\mathcal{G}_1(R_1)$ to obtain (s_2, b_2); then replaces (s_2, b_2) with new random (R_2, r_2), outputs r_2 and runs $\mathcal{G}_1(R_2)$ to obtain (s_3, b_3); it continues in this manner until it replaces (s_i, b_i) with (R_i, r_i), at which point it proceeds properly as $\mathcal{G}$ would. Thus, H_0 is the pseudorandom process. The i-th hybrid process H_i for $i > n$ is the same as the $(2n-1-i)$-th hybrid process, except that it always outputs truly random bits and a random R_l (even where $(2n-1-i)$-th hybrid process would have output a pseudorandom bit b_j and the actual s_l) (see figure on page 22). Thus, H_{2n-1} is the random process from the definition of PO indistinguishability.

If any two consecutive hybrids were distinguishable, then the output of $\mathcal{G}_1$ on a random input would be distinguishable from random, by a simple reduction, which we omit here (but depict via large rectangles in the figure on page 22). This is a contradiction, however, because $\mathcal{G}_1$ is a PO indistinguishable generator. □