SEMI-REGULARITY OF PAIRS OF BOOLEAN POLYNOMIALS

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ABSTRACT. Semi-regular sequences over $\mathbb{F}_2$ are sequences of homogeneous elements of the algebra $B^{(n)} = \mathbb{F}_2[X_1, ..., X_n]/(X_1^2, ..., X_n^2)$, which have a given Hilbert series and can be thought of as having as few relations between them as possible. It is believed that most such systems are semi-regular and this property has important consequences for understanding the complexity of Gröbner basis algorithms such as F4 and F5 for solving such systems. We investigate the case where the sequence has length two and give an almost complete description of the number of semi-regular sequences for each n.

1. INTRODUCTION

The concept of $\mathbb{F}_2$-*semi-regularity* (which we will here shorten to *semi-regularity*) was introduced in [1, 2] in order to assess the complexity of certain Gröbner basis algorithms, such as the XL algorithm [16] or Faugère's F4 or F5 algorithms [7, 8], applied to solving systems of non-linear equations over the Galois field $\mathbb{F}_2$. Heuristically, semi-regular systems of equations are systems for which there are no non-trivial relations between the equations. Bardet, Faugère, Salvy and Yang were able to compute the asymptotic complexity of these algorithms in the case of semi-regular systems, proving that the complexity was exponential [2]. The motivation behind this work was to understand the security of certain multi-variate cryptosystems such as Patarin's Hidden Field Equation system, since the decryption of such systems could be performed by solving such systems of equations. Unfortunately, it soon became clear that the systems of equations that arose in multivariate cryptography were not semi-regular. Moreover, despite a belief that "generic" systems of quadratic equations are semi-regular, little progress has been made on proving even the existence of semi-regular systems of equations. Since systems of polynomial equations over $\mathbb{F}_2$ arise

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naturally in many diverse settings (such as the solution of the discrete logarithm problem [13]), it remains an important goal to understand whether indeed most such systems are semi-regular.

Set $B = \mathbb{F}_2[X_1, \ldots, X_n]/(X_1^2, \ldots, X_n^2)$. Let V be an m-dimensional subspace of the space B_2 of homogeneous quadratic elements of B. The space V is semi-regular if the Hilbert series of the graded quotient ring B/BV is given by the polynomial

$$T_{n,m}(z) = \left[\frac{(1+z)^n}{(1+z^2)^m}\right]$$

where $[\sum_{i=0}^{\infty} a_i z^i]$ denotes the series $\sum_{i=0}^{\infty} a_i z^i$ truncated at the first i for which $a_i \leq 0$.

The question we would like to answer in general is: *What proportion of such spaces are semi-regular*? The total number of subspaces of dimension m is well-known - it is the cardinality of the Grassmanian $\mathrm{Gr}(m, B_2)$. Let

$$sr(n,m) = |\{V \in \mathrm{Gr}(m, B_2) \mid V \text{ is semi-regular}\}|$$

and let

$$p_{n,m} = \frac{sr(n,m)}{|\mathrm{Gr}(m, B_2)|}$$

That is, $p_{n,m}$ is the proportion of m dimensional spaces that are semi-regular. It is conjectured that for $m = m(n)$ sufficiently large compared to n (say $m(n) > n/4$), this proportion tends to 1 as n tends to infinity. Very little is known about this conjecture. In particular, it is not even known whether there are infinitely many n for which $p_{n,n} \neq 0$. It was shown in [11] that for any fixed m, we must have that $p_{n,m} = 0$ for sufficiently large n. The case when $m = 1$ was described in Kruglov's PhD thesis [12, Lemma 3.12]. We give a brief review of this case in Section 5.

The purpose of this paper is to describe in detail the case when $m = 2$ and to give a fairly exact description of which 2-dimensional subspaces are semi-regular for all possible values of n. In Section 4 we show that no semi-regular two dimensional subspaces exist for $n \geq 9$ (and in more generality that no semi-regular two dimensional subspaces exist for $n \geq 4(m+1)$). In Section 6 we deal with the easy cases when $n = 3, 4, 5$ and 7. In the last two sections we consider the more complicated situations when $n = 6$ and 8. In all cases except $n = 8$ we are able to give the exact value of $p_{n,2}$; for $p_{8,2}$ we give a fairly tight bound for the value. Our hope was that analysis of the special case $m = 2$ would give insight into the more important general case. Our results were mixed. It is clear that the rank type (as defined in Section 3) is an important invariant which may help provide more general families of semi-regular spaces. On the other hand, as the degree of $T_{n,2}(z)$ increased, the problem of determining each of the coefficients of the Hilbert series became increasingly intricate and frequently required more ad-hoc methods. This suggests that substantial results on the existence and/or ubiquity of semi-regular sequences may still be elusive.

Related work: The case when $m = 1$ was described in Kruglov's PhD thesis [12, Lemma 3.12]. We give a brief review of this case in Section 5. In this case it is relatively easy to give an exact value for $p_{n,1}$ for all n. At the opposite extreme, recent work by Semaev and Tenti [15] describes the behavior in the overdetermined case when m is sufficiently large compared to n. When $m > \lceil (n-1)(n-2)/6 \rceil$, Theorem 1.1 of [15] provides a lower bound on the proportion of m-dimensional spaces which are semi-regular and this bound tends to 1 as n tends to infinity. Some general results on the existence of semi-regular subspaces of B_k for $k \geq 2$ are given in [11]. In particular it is shown that for all m there exists an N_m such that $p_{n,m} = 0$ for all $n \geq N_m$. A homological characterization of semi-regularity was given by Hodges and Molina in [10].

2. Background and Basics

Let $\mathbb{F} = \mathbb{F}_2$ be the field with two elements. Set

$$B = B^n = \mathbb{F}[X_1, \ldots, X_n]/(X_1^2, \ldots, X_n^2)$$

(we shall drop the superscript when there is no need to emphasize the number n); and let x_i denote the image of X_i in B. This ring inherits the structure of a strongly graded ring from the polynomial ring $\mathbb{F}[X_1, \ldots, X_n]$. That is, if we denote by B_k^n the span of the monomials $x_{i_1} \ldots x_{i_k}$ of degree k, then $B^n = \bigoplus_{k=0}^n B_k^n$ and $B_k^n B_m^n = B_{k+m}^n$. It is easy to see that $\dim B_k^n = \binom{n}{k}$ and that $\dim B^n = 2^n$. The monomials $x_{\mathbf{i}} = x_{i_1} \ldots x_{i_k}$ form a basis for B^n so an arbitrary element of B can be written as $b = \sum_{\mathbf{i}} a_{\mathbf{i}} x_{\mathbf{i}}$ for some $a_{\mathbf{i}} \in \mathbb{F}$. We define the support of b to be

$$\mathrm{Supp}(b) = \{x_{\mathbf{i}} \mid a_{\mathbf{i}} \neq 0\}$$

For $m \leq n$ we will identify the graded subalgebra of B^n generated by $x_1, \ldots, x_m$ with B^m.

In [2], the concept of a semi-regular sequence of elements of B was defined in the following iterative fashion.

Definition 2.1. Let $f_1, \ldots, f_m \in B$ be a sequence of homogeneous polynomials with $\deg f_i = d_i$. Let

$$D_{f_1,\ldots,f_m} = \min \left\{ k \mid \sum_{i=1}^m B_{k-d_i} f_i = B_k \right\}$$

The sequence $f_1, \ldots, f_m \in B$ is *semi-regular* if for all $i = 1, 2, \ldots, m$ and homogeneous $g \in B$

$$g f_i \in (f_1, \ldots, f_{i-1}) \quad \text{and} \quad \deg(g) + \deg(f_i) < D_{f_1,\ldots,f_m}$$

implies $g \in (f_1, \ldots, f_i)$.

For any series $\sum_i a_i z^i \in \mathbb{F}[[z]]$, we denote by $\left[\sum_i a_i z^i\right]$ the truncated series $\sum_i b_i z^i$ where $b_i = a_i$ if $a_j > 0$ for $j = 0, \ldots, i$ and $b_i = 0$ otherwise.

Proposition 2.2. [2] *Let $f_1, \ldots, f_m \in B$ be a sequence of homogeneous polynomials with $\deg f_i = d_i$. The sequence $f_1, \ldots, f_m$ is semi-regular if and only if the Hilbert series of the graded ring $B/(f_1, \ldots, f_m)$ is given by*

$$HS_{B/(f_1,\ldots,f_m)}(z) = \left[\frac{(1+z)^n}{\prod_{i=1}^m (1+z^{d_i})}\right]$$

This shows that the number $D_{f_1,\ldots,f_m}$ is the same for any semi-regular sequence of given multi-degree $\mathbf{d} = (d_1, \ldots, d_m)$. We call this number the *degree of regularity* of a semi-regular sequence of degree $\mathbf{d}$.

We are interested here in the case where all the f_i are quadratic (that is $d_i = 2$ for all i). In this case, Proposition 2.2 implies that if we restrict our attention to linearly independent sequences, then the semi-regularity of the sequence depends only on the subspace $V \subset B_2$ that they generate and not on the choice of f_i (note that if the sequence is linearly dependent, then it is never semi-regular so we may disregard this situation). For this reason, we find it more natural to discuss the semi-regularity of subspaces, rather than of sequences. Thus a quadratic subspace V of dimension m is semi-regular if

$$HS_{B/BV}(z) = \left[\frac{(1+z)^n}{(1+z^2)^m}\right]$$

Set

$$T_{m,n}(z) = \left[\frac{(1+z)^n}{(1+z^2)^m}\right], \text{ and } D_{n,m} = 1 + \deg T_{m,n}(z)$$

So $D_{n,m}$ is the degree of regularity of an m-dimensional semi-regular space of homogeneous quadratic elements.

Another way of characterizing semi-regularity is that the only relation between the f_i's are the trivial ones in degrees less than $D_{n,m}$. Consider the linear maps $\phi_j \colon B_{j-2} \otimes V \to B_j$ given by $\phi_j(\sum_i b_i \otimes v_i) = \sum_i b_i v_i$. Let $R_j(V) = \ker \phi_j$. Inside $R_j(V)$ there is a subspace of "trivial relations" $T_j(V)$ spanned by the elements

(1) $b(v \otimes w - w \otimes v)$ where $v, w \in V$ and $b \in B_{j-4}$;
(2) $b(v \otimes v)$ where $v \in V$ and $b \in B_{j-4}$.

Theorem 2.3. [10, Theorem 3.8] *Let V be an m-dimensional subspace of B_2 and let $D = D_{n,m}$. Then V is semi-regular if and only if*

(1) $R_j(V) = T_j(V)$ for all $3 \leq j < D$, and
(2) $B_{D-2}V = B_D$

If $\{v_1, \ldots, v_m\}$ is a basis for V, then it can be easily shown that

$$T_k(V) = \sum_{i \neq j} B_{k-4}(v_i \otimes v_j - v_j \otimes v_i) + \sum_i B_{k-4}(v_i \otimes v_i)$$

We are interested in understanding the proportion of such spaces which are semi-regular. Note that the set of all m-dimensional subspaces is the

Grassmannian $\mathrm{Gr}(m, B_2)$ and that the size of the Grassmanian of m-dimensional subspaces of the t-dimensional space W for $t \geq m$ is given by the formula

$$|\mathrm{Gr}(m, W)| = \frac{(2^t - 1)(2^t - 2)\dots(2^t - 2^{m-1})}{(2^m - 1)(2^m - 2)\dots(2^m - 2^{m-1})}$$

Let

$$sr(n, m) = |\{V \in \mathrm{Gr}(m, B_2^n) \mid V \text{ is semi-regular}\}|$$

and let

$$p_{n,m} = \frac{sr(n, m)}{|\mathrm{Gr}(m, B_2^n)|}$$

be the proportion of m-dimensional subspaces which are semi-regular. It is generally believed that if $m(n)$ is a function of n that is sufficiently large relative to n, then

$$\lim_{n\to\infty} p_{n,m(n)} = 1.$$

For instance, one can conjecture for $c > 1/4$, that $\lim_{n\to\infty} p_{n,\lfloor cn \rfloor} = 1$. By contrast, we show here that for $c < 1/4$

$$\lim_{n\to\infty} p_{n,\lfloor cn \rfloor} = 0$$

At the other extreme, we can consider spaces such that $m = \dim V$ is large enough large that $\deg T_{n,m} = 2$. This is the case if

$$\frac{(n-1)(n-2)}{6} \leq m < \binom{n}{2}$$

since

$$\frac{(1+z)^n}{(1+z^2)^m} = 1 + nx + \left(\binom{n}{2} - m\right)x^2 + \left(\binom{n}{3} - mn\right)x^3 + \cdots$$

In this case Theorem 2.3 implies that V is semi-regular if and only if the map $B_1 \otimes V \to B_3$ is surjective. In this case, Semaev and Tenti give a lower bound [15, Theorem 1.1] for $p_{n,m}$ from which one deduces easily that

$$\lim_{n\to\infty} p_{n,m(n)} = 1$$

if $m(n) > \lceil (n-1)(n-2)/6 \rceil$. However, little is known about the behavior of $p_{n,m}$ when m is between these bounds; in particular, it is not even known if $p_{n,n}$ is non-zero for infinitely many n.

The general linear group $\mathrm{GL}(B_1)$ acts naturally as graded automorphisms of the algebra B. It therefore acts as permutations of $\mathrm{Gr}(m, B_2^n)$. Thus we can decompose the Grassmannian as a union of $\mathrm{GL}(B_1)$-orbits and semi-regularity is an invariant of these orbits. Under the action of $\mathrm{GL}(B_1)$ every element $\mu \in B_2$ is equivalent to an element of the form $x_1x_2 + \cdots + x_{2i-1}x_{2i}$, as shown in Corollary 3.2 below. We call the number $2i$ the rank of μ and we often denote it as $\mathrm{rk}(\mu)$ or $\mathrm{rk}\,\mu$. There is an important connection between the rank and failure of semi-regularity due to the following result.

Theorem 2.4. [5, Corollary 2.2] *If $\mu \in B_2$ has rank k, then*

$$\dim \frac{\mathrm{Ann}(\mu) \cap B_d}{B_{d-2}\mu} = \binom{n-k}{d-k/2} 2^{k/2}$$

In particular, $\mathrm{Ann}(\mu) \cap B_d \supsetneq B_{d-2}\mu$ *when* $k/2 \leq d \leq n - k/2$.

This immediately yields the following condition on the ranks of elements of a semi-regular space.

Corollary 2.5. *If V is a semi-regular subspace of B_2^n, then V contains no elements of rank k if $k/2 + 2 < D_{n,m}$. In particular, in order for there to exist semi-regular subspaces of dimension m, we must have $D_{n,m} \leq n/2 + 2$.*

Proof. Let $\mu \in V$ be an element of rank k and let $V' = \langle \mu \rangle$. Suppose $d < D_{m,n}$; so $R_d(V)/T_d(V) = 0$. Then by [9, Theorem 2.7]

$$R_d(V') = R_d(V') \cap R_d(V) = R_d(V') \cap T_d(V) = T_d(V')$$

so $R_d(V')/T_d(V') = 0$. But $R_d(V')/T_d(V') = (\mathrm{Ann}(\mu) \cap B_{d-2})/B_{d-4}\mu$. So by Theorem 2.4 $R_{k/2+2}(V')/T_{k/2+2}(V') \neq 0$. Hence, $k/2 + 2 \geq D_{n,m}$. □

3. Alternating Matrices and Rank Type

An important invariant of a non-zero homogeneous quadratic element $\mu \in B_2$ is its rank: the number m such that we can write

$$\mu = y_1 y_2 + \cdots + y_{m-1} y_m.$$

for some $y_1, \ldots y_m \in B_1$ (see Corollary 3.2). From this we can define the *rank type* of a two-dimensional subspace to be the set of ranks of its three non-zero elements. In order to calculate the proportion of subspaces that are semi-regular we need to count the number of subspaces of each rank type. Fortunately these numbers can be deduced from some work of Pott, Schmidt and Zhou [14, Theorem 5] on triples of alternating matrices $(A, B, A + B)$. In this section we establish the connection between elements of B_2 and alternating matrices which enables us to count the subspaces by rank type.

Recall that a matrix $A = (a_{ij}) \in M_n(\mathbb{F})$ is *alternating* if

(1) $a_{ii} = 0$ for all $i = 1, \ldots, n$;
(2) $a_{ij} = a_{ji}$ for all $1 \leq i, j \leq n$

Denote by $\mathrm{Alt}_n(\mathbb{F})$ the space of all alternating matrices. Define $\Gamma : \mathrm{Alt}_n(\mathbb{F}) \to B_2$ by: for $A = (a_{ij})$,

$$\Gamma(A) = \sum_{i<j} a_{ij} x_i x_j$$

Define $\Delta : B_2 \to \mathrm{Alt}_n(\mathbb{F})$ by: for $\mu = \sum_{i<j} b_{ij} x_i x_j$,

$$\Delta(\mu) = (a_{ij}) \text{ where } a_{ij} = \begin{cases} b_{ij} & \text{if } i < j \\ b_{ji} & \text{if } i > j. \end{cases}$$

For any $P \in GL_n(\mathbb{F})$, denote by σ_P the automorphism of B obtained by extending the linear isomorphism of B_1 defined by P with respect to the standard basis $x_1, \ldots, x_n$.

Theorem 3.1. *The maps Γ and Δ are mutually inverse linear isomorphisms. Moreover, if $P \in GL_n(\mathbb{F})$, $B \in M_n(\mathbb{F})$ and $A = P^T BP$, then*

$$\Gamma(A) = \Gamma(P^T BP) = \sigma_P(\Gamma(B))$$

Proof. Routine. □

Corollary 3.2. *For any $\mu \in B_2$ there exist linearly independent elements $y_1, \ldots y_{2i} \in B_1$ such that*

$$\mu = y_1 y_2 + \cdots + y_{2i-1} y_{2i}.$$

Moreover the number $2i$ is equal to the matrix rank of $\Delta(\mu)$, and is independent of the choice of $y_1, \ldots y_{2i}$.

Proof. For any alternating matrix A there exists a matrix B which consists of diagonal 2×2 blocks of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and invertible matrix P such that $A = P^T BP$ [3, Lemma 10]. □

Definition 3.3. We call the number $2i$ the *rank* of the element μ, and we often denote it as $\operatorname{rk} \mu$.

Let $\nu(m)$ be the number of elements $\mu \in B_2$ of rank m. Then $\nu(m)$ is the number of alternating matrices in $M_n(\mathbb{F})$ of rank m and this number is given by the formula [14, Equation 4]

$$\nu(2i) = \begin{bmatrix} t \\ i \end{bmatrix}_4 \prod_{k=0}^{i-1} \left(2^{\frac{n(n-1)}{2t}} - 2^{2k} \right)$$

where

$$t = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } \begin{bmatrix} t \\ i \end{bmatrix}_q = \prod_{j=1}^{i} \frac{q^{t-j+1} - 1}{q^j - 1}$$

Table 1 gives the number of elements of each possible rank r for $n = 3, 4, 5$ and 6

$r \backslash n$	3	4	5	6
2	7	35	155	651
4	0	28	868	18228
6	0	0	0	13888

TABLE 1. The number of elements of rank r in B_2^n for $n = 3, 4, 5$ and 6

Let $V \subset B_2$ be a subspace of dimension 2. In this case $V = \{0, \mu, \mu', \mu+\mu'\}$ for some $\mu, \mu' \in B_2^n$. An important invariant of this space is the triple

$$\mathrm{Rk}(V) = [\mathrm{rk}\,\mu, \mathrm{rk}\,\mu', \mathrm{rk}\,\mu + \mu'] \in \mathbb{N}^3/\Sigma_3$$

(that is, the equivalence class of the triple under the action of the symmetric group S_3). We call $\mathrm{Rk}(V)$ the *rank type* of V.

Since $\Delta(V)$ is a two-dimensional subspace of the space of alternating matrices, we may use the formula of Pott, Schmidt and Zhou [14, Theorem 5] which counts the number of triples of alternating matrices $(A, B, A+B)$ of given matrix rank. Using the $\Delta - \Gamma$ correspondence this also counts the number of two-dimensional spaces with given rank type. For instance the numbers of subspaces of the different rank types when $n = 6$ are given in Table 2.

Type	Number
$[2,2,2]$	9,765
$[2,2,4]$	182,280
$[2,4,4]$	3,417,750
$[2,4,6]$	4,666,368
$[2,6,6]$	2,187,360
$[4,4,4]$	30,902,536
$[4,4,6]$	69,995,520
$[4,6,6]$	54,246,528
$[6,6,6]$	13,332,480
Total	178,940,587

TABLE 2. The number of subspaces of B_2^6 of each rank type

4. An Upper Bound on n

We begin by giving an explicit bound on n above which there are no m-dimensional semi-regular subspaces of B_2^n. This improves upon the result in [11, Theorem 5.1] which established that such a bound always existed. A version of this result which fully extends [11, Theorem 5.1] is given in the Appendix.

Lemma 4.1. *Given any $0 \neq a \in B$, there exists $b \in B$ such that $ab = x_1 \dots x_n$. In particular, if a is homogeneous of degree k, we may pick b to be homogeneous of degree $n-k$.*

Proof. Take a monomial m of smallest length in $\mathrm{Supp}\, a$. Say after renumbering, that $m = x_1 \dots x_k$. Then $m' = x_{k+1} \dots x_n$ must annihilate all the other elements of $\mathrm{Supp}\, a$. So $am' = mm' = x_1 \dots x_n$. □

Lemma 4.2. *If $n \geq t + j$, then $B_j \cap \operatorname{Ann} B_t = 0$. Equivalently,* $\operatorname{Ann} B_t \cap \sum_{i=0}^{n-t} B_i = 0$.

Proof. Let $0 \neq a \in B_j \cap \operatorname{Ann} B_t$ where $j \leq n - t$, and suppose that $a \neq 0$. Then by Lemma 4.1 there exists an element $b \in B_{n-j}$ such that $ab = x_1 \dots x_n$. But $b \in B_{n-j} = B_t B_{n-j-t}$, so

$$ab \in aB_{n-j} = aB_tB_{n-j-t} = 0$$

contradicting $ab \neq 0$. □

Theorem 4.3. *Let V be a subspace of B_2 of dimension m and let $D = D_{n,m}$. If $n \geq D + 2m$, then $B_{D-2}V \neq B_D$; in particular V is not semi-regular.*

Proof. Let $B = \{\mu_1, \dots, \mu_m\}$ be a basis for V. Choose a subset $\{\mu_{i_1}, \dots, \mu_{i_s}\}$ which is maximal with respect to

$$\mu_{i_1} \dots \mu_{i_s} \neq 0.$$

Then for any $i = 1, \dots, m$, $\mu_{i_1} \dots \mu_{i_s}\mu_i = 0$, so $\mu_{i_1} \dots \mu_{i_s}V = 0$. Suppose that $B_{D-2}V = B_D$. Then

$$\mu_{i_1}.\dots.\mu_{i_s}B_D = \mu_{i_1} \dots \mu_{i_s}B_{D-2}V = B_{D-2}\mu_{i_1} \dots \mu_{i_s}V = 0$$

This implies that $\mu_{i_1} \dots \mu_{i_s} \in B_{2s} \cap \operatorname{Ann} B_D$. So Lemma 4.2 implies that $n < D + 2s \leq D + 2m$. Thus if $n \geq D + 2m$, then $B_{D-2}V \neq B_D$ and V is not semi-regular. □

Unfortunately the behavior of $D_{n,m}$ is too erratic for this result to give us an upper bound (for instance, even though $D_{n,m}$ grows slower than n for any fixed m, the difference $n - D_{n,m}$ is not an increasing function). This can be rectified somewhat using the following result.

Theorem 4.4. *There are no semi-regular m-dimensional subspaces of B_2^n when $n \geq 4(m + 1)$.*

Proof. Suppose that $n \geq 4(m+1)$; this implies that $n/2+2 \leq n-2m$. Suppose that there exist semi-regular subspaces of dimension m. By Corollary 2.5, we must have that $D_{n,m} \leq n/2 + 2$. So $D_{n,m} \leq n - 2m$ contradicting Theorem 4.3. □

For small n one can always backfill the difference to get more exact answers.

Corollary 4.5. *There are no semi-regular subspaces of B_2^n*

- *of dimension one for $n \geq 7$;*
- *of dimension two for $n \geq 9$;*
- *of dimension three for $n \geq 12$;*
- *of dimension four for $n \geq 14$.*

Proof. For instance when $m = 2$, Theorem 4.4 tells us that there are no semi-regular 2-dimensional subspaces for $n \geq 12$. For the cases $n = 9, 10, 11$, one can directly check that $D_{n,2} = 5, 6, 6$ respectively. Thus $D_{n,2} \leq n - 4$

in these cases and so by Theorem 4.3 there are no 2-dimensional semi-regular subspaces. The cases $m = 3$ and 4 follow similarly by observing that $D_{n,3} = 6, 7, 7, 8$ for $n = 12, 13, 14, 15$; and that $D_{n,4} = 6, 7, 7, 8, 8, 9$ for $n = 14, 15, 16, 17, 18, 19$ □

This leads to the following interesting conjecture:

Conjecture 4.6. For $m \neq 2$, there exist m-dimensional semi-regular subspaces of B_2^n if and only if $n \leq D_{n,m} + 2m$.

As we shall see, this conjecture is not true for $m = 2$. However, this would seem to be an exceptional case. The validity of this conjecture when $m = 3, 4$ can be seen in [11, Table 1].

Theorem 4.4 also confirms the need for some condition on c in the Conjecture that $\lim_{n\to\infty} p_{n,\lfloor cn \rfloor} = 1$.

Corollary 4.7. *If $c < 1/4$, then $\lim_{n\to\infty} p_{n,\lfloor cn \rfloor} = 0$.*

Proof. If $c < 1/4$ then there exists an N such that for $n > N$, $cn \leq n/4 - 1$. So Theorem 4.4 implies that $p_{n,\lfloor cn \rfloor} = 0$ for $n > N$. □

5. The Case $m = 1$

Let us start by briefly reviewing the case when $m = 1$. If one-dimensional semi-regular subspaces exist for small n, their Hilbert series and degree of regularity would be as in Table 1.

n	$T_{n,1}(z)$	$D_{n,1}$
3	$1 + 3z + 2z^2$	3
4	$1 + 4z + 5z^2$	3
5	$1 + 5z + 9z^2 + 5z^3$	4
6	$1 + 6z + 14z^2 + 14z^3 + z^4$	5
7	$1 + 7z + 20z^2 + 28z^3 + 15z^4$	5

Table 3. The polynomials $T_{n,1}(z)$ and $D_{n,1} = \deg T_{n,1}(z)+1$ for $n = 2, \ldots, 7$

Lemma 5.1. *Suppose $n \geq 2$ and let $\mu \in B_2$. Then*

$$\dim B_1\mu = \begin{cases} n-2 & \text{if } \operatorname{rk}\mu = 2 \\ n & \text{if } \operatorname{rk}\mu \geq 4 \end{cases}$$

and

$$\dim B_2\mu = \begin{cases} \binom{n-2}{2} & \text{if } \operatorname{rk}\mu = 2 \\ \binom{n}{2} - 5 & \text{if } \operatorname{rk}\mu = 4 \\ \binom{n}{2} - 1 & \text{if } \operatorname{rk}\mu \geq 6 \end{cases}$$

Proof. Note that $\dim B_d\mu = \dim B_d - \dim \mathrm{Ann}(\mu) \cap B_d$. Hence by Theorem 2.4,

$$\dim B_d\mu = \binom{n}{d} - \binom{n-k}{d-k/2}2^{k/2} - \dim B_{d-2}\mu$$

The result then follows by direct calculation. □

Whether or not $V = \{0, \mu\}$ is semi-regular depends purely on the rank of μ.

Theorem 5.2. *Let $V = \{0, \mu\}$ be a one dimensional subspace of B_2.*

1. *When $n = 3$, all one dimensional spaces are semi-regular. So $p_{3,1} = 1$.*
2. *When $n = 4$, V is semi-regular if and only if $\mathrm{rk}\,\mu = 4$. So $p_{4,1} = 28/63 \approx 0.44$.*
3. *When $n = 5$, V is semi-regular if and only if $\mathrm{rk}\,\mu = 4$. So $p_{5,1} = 868/1023 \approx 0.85$*
4. *When $n = 6$, V is semi-regular if and only if $\mathrm{rk}\,\mu = 6$. So $p_{6,1} = 13888/32767 \approx 0.42$*
5. *When $n \geq 7$, V cannot be semi-regular. Thus $p_{n,1} = 0$ for $n \geq 7$.*

Proof. In the cases $n = 3, 4$, we have $D_{n,1} = 3$, so it suffices to verify the equality $B_1V = B_3$. Since $\dim B_3^3 = 1$ and $\dim B_3^4 = 4$, the result follows immediately from Lemma 5.1. In the case $n = 5$, we have $D_{5,1} = 4$, so we need to verify that the map $\phi_5 : B_1^5 \otimes V \to B_3^5$ is injective and the map $\phi_4 : B_2^5 \otimes V \to B_4^5$ is surjective. Lemma 5.1 implies that these conditions hold precisely when $\mathrm{rk}\,\mu = 4$. Finally, for $n = 6$, we need that $\dim B_1^6V = 6$, $\dim B_2^6V = 14$ and $B_3^6V = B_5^3$. Lemma 5.1 implies that first two conditions hold only when $\mathrm{rk}\,\mu = 6$. The last condition is easily verified directly when $\mathrm{rk}\,\mu = 6$.

The figures for the proportions follow from the numbers of elements of each rank given in Table 1. □

6. The Case $m = 2$ - Preliminaries

We now consider the situation where $m = \dim V = 2$. Then, $V = \{0, \mu, \mu', \mu + \mu'\}$ for some $\mu, \mu' \in B_2^n$, and recall the definition of the *rank type* $\mathrm{Rk}(V) = [\mathrm{rk}\,\mu, \mathrm{rk}\,\mu', \mathrm{rk}\,\mu + \mu']$. Unfortunately the rank type of a space V does not determine its equivalence class under the action of $\mathrm{GL}(B_1)$. For instance, Examples 7.5 and 7.6 show two subspaces of B_2^6 of equal rank type $[4, 4, 4]$, one of which is not semi-regular while the other is semi-regular; since automorphisms preserve the linear relations between elements in a subspace, $GL(B_1)$ maps semi-regular spaces to semi-regular spaces, so the latter examples are in different orbits under the action of $GL(B_1)$. However the rank type does provide an important and useful decomposition of the Grassmanian $\mathrm{Gr}(2, B_2)$.

6.1. **The cases $n = 3, 4, 5$ and 7.** From the table below we see that for $n = 3, 4$, and 5 the degree of regularity of a semi-regular 2-dimensional subspace of B_2 would be 3. By Theorem 2.3, when $D_{n,2} = 3$, semi-regularity

n	$T_{n,2}(z)$	$D_{n,2}$
3	$1 + 3z + z^2$	3
4	$1 + 4z + 4z^2$	3
5	$1 + 5z + 8z^2$	3
6	$1 + 6z + 13z^2 + 8z^3$	4
7	$1 + 7z + 19z^2 + 21z^3$	4
8	$1 + 8z + 26z^2 + 40z^3 + 17z^4$	5
9	$1 + 9z + 34z^2 + 66z^3 + 57z^4$	5

TABLE 4. The polynomials $T_{n,2}(z)$ and $D_{n,2} = \deg T_{n,2}(z)+1$ for $n = 3, \ldots, 9$

is equivalent to condition (2), $B_1V = B_3$, since condition (1) is null.

Theorem 6.1. *If $n = 3$, then all two dimensional subspaces are semi-regular.*

Proof. In this case $\dim B_3 = 1$ and $B_1V \neq 0$ by Lemma 4.1, so we must always have $B_3 = B_1V$. □

Theorem 6.2. *Let $n = 4$ and let $V \subset B_2$ be a two dimensional subspace. Then V is semi-regular if and only if V contains an element of rank 4.*

Proof. In this case $\dim B_3 = 4$. If $\operatorname{rk} \mu = 4$, then by Lemma 5.1, $\dim B_1\mu = 4$. So if V contains an element of rank 4, we have $B_1V = B_3$. On the other hand, suppose that V does not contain an element of rank 4. Then V has rank type $[2, 2, 2]$. Let μ, μ' be a basis for V. Then $\mu = \lambda_1\lambda_2$ and $\mu' = \lambda_1'\lambda_2'$ for some $\lambda_1, \lambda_2, \lambda_1', \lambda_2' \in B_1$. Let $\Lambda = \langle \lambda_1, \lambda_2, \lambda_1', \lambda_2' \rangle$. If $\dim \Lambda = 4$, then the λ's are linearly independent and $\mu + \mu'$ would have rank 4; on the other hand, if $\dim \Lambda = 2$, then $\dim \Lambda^2 = 1$ and $V \subset \Lambda^2$, a contradiction. Therefore we must have $\dim \Lambda = 3$. Hence we can find a one dimensional subspace $V_0 \subset B_1$ such that $B_1 = \Lambda \oplus V_0$. But then

$$B_1V = \Lambda V + V_0V \subset \Lambda^3 + V_0V$$

Hence $\dim B_1V \leq \dim \Lambda^3 + \dim V_0V \leq 1 + 2 = 3$ and therefore $B_1V \neq B_3$. So if V is semi-regular, it must contain an element of rank 4. □

Corollary 6.3. *In the case $n = 4$, the proportion of subspaces of B_2^4 that are semi-regular is $p_{4,2} = 546/651 \approx 0.84$.*

Proof. The total number of two-dimensional subspaces is $|\operatorname{Gr}(m, B_2^4)| = 651$. From [14, Theorem 5], we have the number of subspaces of type $[2, 2, 2]$ is 105. So $p_{4,2} = (651 - 105)/651$. □

Now consider the case when $n = 5$. Note that $\dim B_3^5 = 10$ so $B_1V = B_3$ if and only if the map $\phi_3 : B_1^5 \otimes V \to B_3^5$ is an isomorphism.

Theorem 6.4. *The map $\phi_3 : B_1^5 \otimes V \to B_3^5$ is not surjective for any two dimensional subspace $V \subset B_2^5$. Hence there are no semi-regular two dimensional subspaces of B_2^5.*

Proof. We may assume, after appropriate change of variables, that $V = \{0, \mu, \mu', \mu + \mu'\}$ where $\mu \in B_2^4$ and $\mu' = \mu_0 + \lambda x_5$ for $\mu_0 \in B_2^4$ and $\lambda \in B_1^4$. Then

$$\begin{aligned} B_1V &= B_1\mu + B_1\mu' = (B_1^4 + \mathbb{F}x_5)\mu + (B_1^4 + \mathbb{F}x_5)(\mu_0 + \lambda x_5) \\ &\subset (B_1^4\mu + B_1^4\mu_0) + (\mathbb{F}\mu + \mathbb{F}\mu_0 + B_1^4\lambda)x_5 \end{aligned}$$

Now $B_3^5 = B_3^4 + B_2^4x_5$, so for this map to be surjective we must have $B_1^4\mu + B_1^4\mu_0 = B_3^4$ and $\mathbb{F}\mu + \mathbb{F}\mu_0 + B_1^4\lambda = B_2^4$. However $\dim B_1^4\lambda \leq 3$, so

$$\dim(\mathbb{F}\mu + \mathbb{F}\mu' + B_1^4\lambda) \leq 5 < 6 = \dim B_2^4$$

Thus $B_1V \neq B_3^5$ and V is not semi-regular. □

Next we jump ahead to consider the case when $n = 7$. Here the degree of regularity is four. So in order for the space V to be semi-regular we need the map $\phi_4 : B_2^7 \otimes V \to B_4^7$ to be surjective.

Theorem 6.5. *The map $\phi_4 : B_2^7 \otimes V \to B_4^7$ is not surjective for any 2-dimensional subspace $V \subset B_2^7$. Hence there are no semi-regular two dimensional subspaces of B_2^7.*

Proof. Pick a basis for V, say $\{\mu, \mu'\}$. After a suitable choice of generators we can assume that

$$\mu \in B_2^6, \quad \mu' = \mu_0 + \lambda x_7, \text{ where } \mu_0 \in B_2^6, \lambda \in B_1^6$$

Then

$$\begin{aligned} B_2^7V &= B_2^7\mu + B_2^7\mu' \\ &= (B_2^6 + B_1^6x_7)\mu + (B_2^6 + B_1^6x_7)(\mu_0 + \lambda x_7) \\ &\subset (B_2^6\mu + B_2^6\mu_0) + (B_1^6\mu + B_1^6\mu_0 + B_2^6\lambda)x_7 \end{aligned}$$

Suppose that ϕ_4 is surjective. Then we must have

$$B_1^6\mu + B_1^6\mu_0 + B_2^6\lambda = B_3^6$$

If $\lambda = 0$, then we would have $B_1^6\mu + B_1^6\mu_0 = B_3^6$ which is impossible because the left hand side has dimension at most 12 and $\dim B_3^6 = 20$. So $\lambda \neq 0$. Consider the map $B^6 \to \tilde{B} = B^6/(\lambda) \cong B^5$. Denote the images of μ and μ_0 by $\tilde{\mu}$ and $\tilde{\mu}_0$. Then we would have

$$\tilde{B}_1\tilde{\mu} + \tilde{B}_1\tilde{\mu}_0 = \tilde{B}_3$$

But this contradicts Theorem 6.4. □

This yields an exact value for $p_{n,2}$ in all cases except $n = 6$ or 8. In the next two sections we consider these two remaining cases which are considerably more complicated.

7. The Case $m = 2$, $n = 6$

Since $D_{6,2} = 4$, a two-dimensional space $V \subset B_2^6$ is semi-regular if

(1) the map $\phi_3 : B_1^6 \otimes V \to B_3^6$ is injective; and

(2) the map $\phi_4 : B_2^6 \otimes V \to B_4^6$ is surjective

Note that $\dim B_1^6 = 6$, $\dim B_2^6 = 15$, $\dim B_3^6 = 20$, and $\dim B_4^6 = 15$.

Proposition 7.1. *If V contains an element of rank 2, then V is not semi-regular. In particular if V has rank type $[2,2,2], [2,2,4], [2,4,4]$ or $[2,4,6]$, then V is not semi-regular.*

Proof. By Corollary 2.5 a semi-regular two-dimensional subspace of B_2^6 can contain no elements of rank less than $2(D_{6,2} - 2) = 4$. □

This leaves the cases where V has rank type $[4,4,4], [4,4,6], [4,6,6]$ and $[6,6,6]$. In the case where V contains an element of rank 6 the surjectivity condition is easily established.

Lemma 7.2. *If V contains an element of rank 6, then the map $\phi_4 : B_2^6 \otimes V \to B_4^6$ is surjective.*

Proof. We may assume that the element of rank 6 is $\mu = x_1x_2 + x_3x_4 + x_5x_6$. Then $B_2^6\mu$ contains all the monomials of B_4^6 except

$$x_1x_2x_3x_4, x_1x_2x_5x_6, x_3x_4x_5x_6$$

In addition it contains

$$(x_1x_2 + x_3x_4)x_5x_6, (x_1x_2 + x_5x_6)x_3x_4, (x_3x_4 + x_5x_6)x_1x_2$$

Let μ' be another non-zero element of V. Suppose that we have a monomial $x_ix_j \in \mathrm{Supp}(\mu')$ which is not one of x_1x_2, x_3x_4, x_5x_6. Without loss of generality suppose it is x_1x_3. Then $x_1x_2x_3x_4 \in \mathrm{Supp}(x_2x_4\mu')$. Since $B_2^6\mu$ contains all the other monomials involving x_2x_4, B_2^6V must contain $x_1x_2x_3x_4$ and so $B_2^6V = B_4^6$. Now suppose that $\mathrm{Supp}(\mu') \subset \{x_1x_2, x_3x_4, x_5x_6\}$ and $\mu' \neq \mu$ so μ' is the sum of one or two of these terms. It is easily verified that in this case again $B_2^6V = B_4^6$. □

Lemma 7.3. *Suppose that $n \geq 6$ and let V be a 2-dimensional subspace of B_2^n. If V contains an element of rank at least 6, then $\mathrm{Ann}\, V \cap B_2^n = 0$. If, in addition, V has no elements of rank 2, then the map $\phi_3 : B_1^n \otimes V \to B_3^n$ is injective.*

Proof. Suppose that $V = \langle \mu, \mu' \rangle$ where $\mathrm{rk}\, \mu \geq 6$ and $\mu' \neq \mu$. Since $\mathrm{rk}\, \mu \geq 6$, we know from Lemma 5.1 that $\mathrm{Ann}\, \mu \cap B_2 = \{0, \mu\}$. Therefore $\mu'\mu \neq 0$ and

$\mu \notin \operatorname{Ann}\mu' \cap B_2$. Hence

$$\begin{aligned}\operatorname{Ann} V \cap B_2 &= (\operatorname{Ann}\mu \cap B_2) \cap (\operatorname{Ann}\mu' \cap B_2)\\ &= \{0,\mu\} \cap (\operatorname{Ann}\mu' \cap B_2) = \{0\}\end{aligned}$$

Now assume that $\operatorname{rk}\mu'$ and $\operatorname{rk}(\mu+\mu')$ are both at least 4. An element of $\operatorname{Ker}\phi_3$ is of the form $a\otimes\mu + b\otimes\mu'$ where $a,b \in B_1$ and

$$a\mu + b\mu' = 0$$

In this case $ab\mu = 0$ and $ab\mu' = 0$ so $ab \in \operatorname{Ann} V \cap B_2 = \{0\}$. Hence $a \in \operatorname{Ann} b \cap B_1 = \{0, b\}$. If $a = 0$, then $b\mu' = 0$, so $b = 0$ since $\operatorname{rk}\mu' \geq 4$. If $a = b$ then $a(\mu+\mu') = 0$, so $a = b = 0$ since $\operatorname{rk}(\mu+\mu') \geq 4$. Thus $\operatorname{Ker}\phi_3 = 0$. □

Theorem 7.4. *If V is a a 2-dimensional subspace of B_2^6 of rank type $[4,4,6],[4,6,6]$ or $[6,6,6]$ then V is semi-regular.*

Proof. The injectivity condition follows from Lemma 7.3. The surjectivity condition follows from Lemma 7.2. □

7.1. **Spaces of rank type** $[4,4,4]$**.** If V contains a rank four element we can assume this element is of the form $\mu = x_1x_2 + x_3x_4$. Thus we may assume that $V = \langle\mu,\mu'\rangle = \{0,\mu,\mu',\mu+\mu'\}$ where

$$\begin{aligned}\mu &= x_1x_2 + x_3x_4\\ \mu' &= \mu_0 + \lambda_1x_5 + \lambda_2x_6 + \epsilon x_5x_6\end{aligned}$$

and $\mu_0 \in B_2^4$, $\lambda_1,\lambda_2 \in B_1^4$ and $\epsilon \in \{0,1\}$.

Example 7.5. If $\mu_0 = 0, \lambda_1 = x_1, \lambda_2 = x_3$ and $\epsilon = 0$, we get

$$\begin{aligned}\mu &= x_1x_2 + x_3x_4\\ \mu' &= x_1x_5 + x_3x_6\\ \mu+\mu' &= x_1(x_2+x_5) + x_3(x_4+x_6)\end{aligned}$$

Note that in this example $V \subset B_1\langle x_1,x_3\rangle$. Thus $B_2V \subset B_3\langle x_1,x_3\rangle$ and so B_2V does not contain $x_2x_4x_5x_6$. Thus V is not semi-regular.

Example 7.6. If $\mu_0 = x_1x_2, \lambda_1 = \lambda_2 = 0$ and $\epsilon = 1$, we get

$$\begin{aligned}\mu &= x_1x_2 + x_3x_4\\ \mu' &= x_1x_2 + x_5x_6\\ \mu+\mu' &= x_3x_4 + x_5x_6\end{aligned}$$

One can verify directly that $B_1V = B_1x_1x_2 \oplus B_1x_3x_4 \oplus B_1x_5x_6$, which has dimension 12, so the map ϕ_3 is injectve. Thus $B_2V = B_2x_1x_2 + B_2x_3x_4 + B_2x_5x_6 = B_4$ since every monomial of length 4 contains one of the subwords x_1x_2, x_3x_4 or x_5x_6. Hence ϕ_4 is surjective and V is semi-regular.

Lemma 7.7. *Let V be a two-dimensional subspace of rank type $[4,4,4]$. If either*

(1) V is induced (there is a proper subspace $W \subset B_1$ such that $V \subset W^2$); or
(2) there is a two-dimensional subspace $\Lambda \subset B_1$ such that $V \subset B_1\Lambda$,

then V is not semi-regular.

Proof. (1) Without loss of generality, we can assume that $V \subset B_2^5$. In this case,

$$B_2^6 V = (B_2^5 + B_1^5 x_6)V \subset B_2^5 V + B_1^5 V x_6 \subset B_4^5 + B_1^5 V x_6$$

Since $B_4^6 = B_4^5 \oplus B_3^5 x_6$ and $B_1^5 V \subsetneq B_3^5$ by Theorem 6.4, we cannot have $B_2^6 V = B_4^6$.

(2) In this case, as in Example 7.5, $B_2 V \subset B_3\Lambda \subsetneq B_4$ so V is not semi-regular. □

Theorem 7.8. *Let V be a two-dimensional subspace of rank type $[4,4,4]$. Then V is semi-regular if and only if it is equivalent to a space of the form given in Example 7.6*

Proof. Suppose that V is semi-regular. We may assume that V is generated by μ and μ' of the form

$$\mu = x_1x_2 + x_3x_4$$
$$\mu' = \mu_0 + \lambda_1 x_5 + \lambda_2 x_6 + \epsilon x_5 x_6$$

where $\mu_0 \in B_2^4$, $\lambda_1, \lambda_2 \in B_1^4$ and $\epsilon \in \{0,1\}$. Let $\Lambda = \langle \lambda_1, \lambda_2 \rangle$.

Suppose that $\epsilon = 0$. If $\dim \Lambda = 1$, then $\mu' = \mu_0 + \lambda x$ for some $\lambda \in \Lambda$ and $x \in \langle x_5, x_6 \rangle$. So we are in case (1) of Lemma 7.7 with $W = B_1^4 + \langle x \rangle$, contradicting the semi-regularity of V. Hence we must have $\dim \Lambda = 2$. Extend $\{\lambda_1, \lambda_2\}$ to a basis $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ for B_1^4. Note that $B_2^4 = \lambda_1 B_1^4 + \lambda_2 B_1^4 + \mathbb{F}\lambda_3\lambda_4$. Therefore, since $V \not\subset \Lambda B_1$, we must have that either μ_0 or $\mu_0 + \mu$ is of the form $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3\lambda_4$ for some $a_1, a_2 \in B_1^4$. Assuming without loss of generality that it is μ_0, we have that

$$\begin{aligned}\mu' &= \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3\lambda_4 + \lambda_1 x_5 + \lambda_2 x_6 \\ &= \lambda_1(x_5 + a_1) + \lambda_2(x_6 + a_2) + \lambda_3\lambda_4\end{aligned}$$

which is of rank 6 because $\lambda_1, \lambda_2, \lambda_3, \lambda_4, (x_5 + a_1), (x_6 + a_2)$ form a basis for B_1^6. This contradicts the assumption that $\operatorname{rk} \mu' = 4$.

Thus we must have $\epsilon = 1$. In this case after an appropriate change of basis, we may assume that $\lambda_1 = \lambda_2 = 0$ and $\mu' = \mu_0 + x_5x_6$. In this case $\operatorname{rk} \mu' = \operatorname{rk} \mu_0 + 2$, so $\operatorname{rk} \mu_0 = 2$; similarly $\operatorname{rk}(\mu + \mu_0) = 2$. Thus $\mu = \mu_0 + (\mu + \mu_0)$ and up to a linear change of variables we are in the case of Example 7.6. □

Theorem 7.9. *There are* $153,129,088$ *semi-regular 2-dimensional subspaces of B_2^6. Thus the proportion of such subspaces that are semi-regular is*

$$p_{6,2} = \frac{153,129,088}{178,940,587} \approx 0.86$$

Proof. From Proposition 7.1 and Theorem 7.4 it suffices to calculate the number of spaces of rank type $[4,4,4]$ that are semi-regular. By Theorem 7.8, such spaces are precisely the orbit of the space given in Example 7.6. The stabilizer of this space in $GL_6(\mathbb{F})$ is isomorphic to $(GL_2(\mathbb{F}) \times GL_2(\mathbb{F}) \times GL_2(\mathbb{F})) \rtimes \Sigma_3$ which has order 6^4. Hence the size of the orbit is

$$\frac{20,158,709,760}{1,296} = 15,554,560$$

Adding this number to the total number of subspaces of type $[4,4,6]$, $[4,6,6]$ or $[6,6,6]$ given in the table, yields the claimed conclusion. □

8. The Case $n = 8$

In this case $D_{8,2} = 5$, so semi-regularity of a two-dimensional quadratic subspace V is equivalent to the following properties

- The map $\phi_3 : B_1 \otimes V \to B_3$ is injective
- The kernel of $\phi_4 : B_2 \otimes V \to B_4$ is the trivial kernel $T_4(V)$.
- The map $\phi_5 : B_3 \otimes V \to B_5$ is surjective.

Note that $\dim B_2 = 28$, $\dim B_3 = 56$, $\dim B_4 = 70$, and $\dim B_5 = 56$.

Throughout this section, unless stated otherwise, V will denote a two-dimensional subspace of B_2^8.

Lemma 8.1. *Let V be a semi-regular two-dimensional subspace of B_2^8. Then V contains no non-zero elements of rank less than or equal to 4.*

Proof. By Corollary 2.5 a semi-regular two-dimensional subspace of B_2^8 can contain no elements of rank less than $2(D_{8,2} - 2) = 6$. □

Thus it remains to investigate semi-regularity when the rank of V is $[6,6,6]$, $[6,6,8]$, $[6,8,8]$ or $[8,8,8]$. Table 5 below gives the numbers of subspaces of the different rank types [14, Theorem 5].

Type	Number
$[6,6,6]$	2,093,462,703,144,960
$[6,6,8]$	4,719,790,074,101,760
$[6,8,8]$	3,567,475,986,923,520
$[8,8,8]$	888,431,072,772,096

Table 5. The number of two dimensional subspaces of B_2^8 of Rank Type $[6,6,6]$, $[6,6,8]$, $[6,8,8]$ or $[8,8,8]$.

Note that the injectivity of the map $\phi_3 : B_1 \otimes V \to B_3$ holds in all such cases by Lemma 7.3. We can easily eliminate the following special case.

Theorem 8.2. *Suppose that there exists a proper subspace $W \subset B_1$ such that $V \subset W^2$. Then the map $\phi_5 : B_3 \otimes V \to B_5$ is not surjective. Hence V is not semi-regular.*

Proof. Without loss of generality, we may assume $W = B_1^7$ and $V \subset W^2$. Now $B_3 = B_3^7 \oplus B_2^7 x_8$, so

$$B_3 V = B_3^7 V + B_2^7 V x_8$$

By Theorem 6.5, $B_2^7 V \subsetneq B_4^7$. Since $B_5 = B_5^7 \oplus B_4^7 x_8$ we must have $B_3 V \subsetneq B_5$ and the map is not surjective. □

In this situation (there exists a proper subspace $W \subset B_1$ such that $V \subset W^2$), we shall say that the space V is *induced from* W (or just *induced* if W is not specified).

Lemma 8.3. *Suppose that $V = \langle \mu, \mu' \rangle$. The map $\phi_4 : B_2 \otimes V \to B_4$ has trivial kernel if and only if* $\operatorname{rk} \mu$ *and* $\operatorname{rk} \mu'$ *are at least* 6 *and*

$$B_2\mu \cap B_2\mu' = \{0, \mu\mu'\}$$

Proof. The trivial kernel of the map $\phi_4 : B_2 \otimes V \to B_4$ is three dimensional with basis $\{\mu \otimes \mu, \mu' \otimes \mu - \mu \otimes \mu', \mu' \otimes \mu'\}$. Thus the kernel is trivial if and only if $\dim B_2 V = \dim(B_2 \otimes V) - 3 = 53$.

If $\operatorname{rk} \mu \leq 4$, then by Lemma 5.1 the kernel of the map $B_2 \otimes \mathbb{F}\mu \to B_2\mu$ has dimension at least 5 and so $\ker \phi_4$ cannot be trivial. So suppose that μ and μ' both have rank at least 6. Then $\dim B_2\mu = \dim B_2\mu' = 27$ by Lemma 5.1. On the other hand $B_2 V = B_2\mu + B_2\mu'$ so the kernel is trivial if and only if $\dim B_2 V = 54 - \dim(B_2\mu \cap B_2\mu') = 53$; that is, $\dim B_2\mu \cap B_2\mu' = 1$. Since $\mu\mu' \neq 0$ (by Lemma 5.1 again), this is equivalent to $B_2\mu \cap B_2\mu' = \{0, \mu\mu'\}$. □

We now look in detail at the situation where V contains an element of rank 6.

Lemma 8.4. *Let $\mu = y_1 y_2 + \cdots + y_{m-1} y_m$ be an element of rank m in B_2^n. Then the space $U(\mu) = \langle y_1, \ldots, y_m \rangle$ is independent of the choice of $y_1, \ldots y_m$.*

Proof. Suppose that

$$\mu = y_1 y_2 + \cdots + y_{m-1} y_m = y_1' y_2' + \cdots + y_{m-1}' y_m'$$

for some $y_1, \ldots, y_m$ and $y_1', \ldots, y_m'$ in B_1. Since $\operatorname{rk} \mu = m$, the $y_1, \ldots, y_m$ and $y_1', \ldots, y_m'$ must be linearly independent; hence it suffices to show that $y_1, \ldots, y_m \in \langle y_1', \ldots, y_m' \rangle$.

Extend $y_1', \ldots, y_m'$ to a basis $y_1', \ldots, y_m', y_{m+1}', \ldots y_n'$ for B_1^n. Write

$$y_i = \sum_{j=1}^{n} a_{ij} y_j'$$

for some $a_{ij} \in \mathbb{F}$. Suppose that $y_k \notin \langle y_1', \ldots, y_m' \rangle$; that is, $a_{kl} \neq 0$ for some $m + 1 \leq l \leq n$. After renumbering of the $y_1, \ldots, y_m$ and $y_{m+1}', \ldots, y_n'$ if necessary, we may assume that $a_{1n} = 1$. The coefficient of the monomial $y_j' y_n'$ in $y_1 y_2 + \cdots + y_{m-1} y_m$ is

$$0 = a_{1j} a_{2n} + a_{1n} a_{2j} + \cdots + a_{m-1,j} a_{mn} + a_{m-1,n} a_{mj}$$

Hence

$$a_{2j} = a_{1j}a_{2n} + \sum_{k=2}^{m/2} (a_{2k-1,j}a_{2k,n} + a_{2k-1,n}a_{2k,j})$$

Therefore

$$\begin{aligned} y_2 &= \sum_{j=1}^{n} a_{2j}y'_j \\ &= \sum_{j=1}^{n} \left(a_{1j}a_{2n} + \sum_{k=2}^{m/2} (a_{2k-1,j}a_{2k,n} + a_{2k-1,n}a_{2k,j}) \right) y'_j \\ &= \sum_{j=1}^{n} a_{1j}a_{2n}y'_j + \sum_{k=2}^{m/2} \left(\sum_{j=1}^{n} a_{2k-1,j}a_{2k,n}y'_j + \sum_{j=1}^{n} a_{2k-1,n}a_{2k,j}y'_j \right) \\ &= a_{2n}y_1 + \sum_{k=2}^{m/2} a_{2k,n}y_{2k-1} + \sum_{k=2}^{m/2} a_{2k-1,n}y_{2k} \end{aligned}$$

contradicting the linear independence of the y_i. Hence we must have all $y_1, \ldots, y_m \in \langle y'_1, \ldots, y'_m \rangle$, as required. □

Definition 8.5. Let V be a non-induced 2-dimensional subspace of B_2^8 containing an element μ of rank 6. We say that V is of

(A) Type A with respect to μ if $V \not\subset U(\mu)B_1$.
(B) Type B with respect to μ if $V \subset U(\mu)B_1$

Note that the definition of type is dependent on the choice of μ, as the following example illustrates.

Example 8.6. Let

$$\mu = x_1x_2 + x_3x_4 + x_5x_6$$
$$\mu' = x_1x_2 + x_3x_7 + x_4x_8$$

and let $V = \langle \mu, \mu' \rangle$. Then $\mu' \in U(\mu)B_1$, so V is of Type B with respect to μ but $\mu \notin U(\mu')B_1$ so V is of Type A with respect to μ'.

The following proposition gives a more explicit description of spaces of these different types.

Proposition 8.7. *Let V be a non-induced two dimensional subspace of B_2 containing an element μ of rank six.*

(1) If V is of Type A with respect to μ then there exists a basis $\{y_1, y_2, \ldots, y_8\}$ of B_1 such that $V = \langle \mu, \mu' \rangle$ where $\mu = y_1y_2 + y_3y_4 + y_5y_6$ and $\mu' = \mu_0 + y_7y_8$ for some $\mu_0 \in B_2^6$.
(2) If V is of Type B with respect to μ then there exists a basis $\{y_1, y_2, \ldots, y_8\}$ of B_1 such that $V = \langle \mu, \mu' \rangle$ where $\mu = y_1y_2 + y_3y_4 + y_5y_6$ and $\mu' = \mu_0 + \lambda y_7 + \lambda' y_8$ for some $\mu_0 \in B_2^6$ and some linearly independent $\lambda, \lambda' \in B_1^6$.

Proof. Since μ has rank six we may choose $y_1, \ldots, y_6$ such that $\mu = y_1y_2 + y_3y_4 + y_5y_6$ and the y_i are linearly independent. Extend $\{y_1, \ldots, y_6\}$ to a basis $\{y_1, \ldots, y_8\}$ for B_1. Pick $\mu' \in V \backslash \{0, \mu\}$. Then $\mu' = \mu_0 + \lambda y_7 + \lambda' y_8 + \eta y_7 y_8$ where $\mu_0 \in B_2^6$, $\lambda, \lambda' \in B_1^6$ and $\eta \in \mathbb{F}$. Clearly V is of Type A with respect to μ if $\eta = 1$ and of type B if $\eta = 0$. If $\eta = 1$, then

$$\mu' = (\mu_0 + \lambda\lambda') + (\lambda' + y_7)(\lambda + y_8)$$

So replacing y_7 with $\lambda' + y_7$ and y_8 with $\lambda' + y_8$ yields the required form. If $\eta = 0$ and $\dim\langle \lambda, \lambda' \rangle \leq 1$, then V is induced. So if V is non-induced and of Type B we must have $\dim\langle \lambda, \lambda' \rangle = 2$. □

Note that if $V \ni \mu$ is a Type A space of the form given in (1) above, then V has rank type $[6, \mathrm{rk}(\mu_0) + 2, \mathrm{rk}(\mu + \mu_0) + 2]$.

We now proceed to count the semi-regular subspaces of each possible rank type containing a fixed element μ of rank 6. We break this into analysis of the two different possible types, beginning with Type A.

Theorem 8.8. *Let V be a subspace of rank type $[6, 6, 6]$, $[6, 6, 8]$ or $[6, 8, 8]$. If V is of Type A with respect to a rank 6 element $\mu \in V$, then V is semi-regular.*

Proof. By Proposition 8.7 we can assume that $V = \langle \mu, \mu' \rangle$ where

$$\mu = x_1x_2 + x_3x_4 + x_5x_6 \text{ and } \mu' = \mu_0 + x_7x_8$$

for some $\mu_0 \in B_2^6$; the assumption on the rank type of V implies that the rank of μ_0 and $\mu + \mu_0$ are both at least 4. We need to prove (i) $B_2\mu \cap B_2\mu' = \{0, \mu\mu'\}$ and (ii) $B_3V = B_5$.

(i) $B_2\mu \cap B_2\mu' = \{0, \mu\mu'\}$. Suppose that $a\mu = b\mu' \in B_2\mu \cap B_2\mu'$, for some $a, b \in B_2$. Let

$$b = \mu_1 + \lambda_1 x_7 + \lambda_2 x_8 + \epsilon x_7 x_8, \quad a = \mu_2 + \lambda_3 x_7 + \lambda_4 x_8 + \epsilon' x_7 x_8$$

where $\mu_1, \mu_2 \in B_2^6$, $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in B_1^6$ and $\epsilon, \epsilon' \in \mathbb{F}$. Then

$$\begin{aligned} 0 &= a\mu + b\mu' \\ &= (\mu_0\mu_1 + \mu_2\mu) + x_7(\mu_0\lambda_1 + \lambda_3\mu) \\ &\quad + x_8(\mu_0\lambda_2 + \lambda_4\mu) + x_7x_8(\mu_0\epsilon + \mu_1 + \epsilon'\mu) \end{aligned}$$

So

$$\epsilon\mu_0 + \mu_1 = \epsilon'\mu, \quad \mu_0\mu_1 = \mu_2\mu, \quad \lambda_3\mu = \lambda_1\mu_0, \quad \lambda_4\mu = \lambda_2\mu_0$$

Then $\lambda_1\lambda_3\mu = \lambda_1^2\mu_0 = 0$. Therefore $\lambda_1\lambda_3 \in \mathrm{Ann}(\mu) \cap B_2 = \{0, \mu\}$. But $\mu \neq \lambda_1\lambda_3$ since $\mathrm{rk}\,\mu = 6$, so $\lambda_1\lambda_3 = 0$. Suppose $\lambda_1 = \lambda_3 \neq 0$. Then $\lambda_1(\mu + \mu_0) = 0$; but this is impossible since $\mathrm{rk}(\mu + \mu_0) \geq 4$. If $\lambda_1 \neq 0$ and $\lambda_3 = 0$ then we would have $\lambda_1\mu_0 = 0$ which is again impossible because $\mathrm{rk}(\mu_0) \geq 4$. A similar argument works for the case $\lambda_1 = 0$ and $\lambda_3 \neq 0$. Thus we must have $\lambda_1 = \lambda_3 = 0$. An analogous argument shows that $\lambda_2 = \lambda_4 = 0$ also. Therefore, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$.

Now consider the first two constraints: $\epsilon\mu_0 + \mu_1 = \epsilon'\mu, \mu_0\mu_1 = \mu_2\mu$. Consider the two cases:

$\epsilon' = 1$: Then $\epsilon\mu_0 + \mu_1 = \mu$. So $\mu_2\mu = \mu_0\mu_1 = \mu_0\mu$. Hence $\mu(\mu_0 + \mu_2) = 0$ and so $\mu_0 + \mu_2 \in \mathrm{Ann}(\mu) \cap B_2 = \{0, \mu\}$; that is, $\mu_2 \in \{\mu_0, \mu_0 + \mu\}$. So $a \in \{\mu', \mu' + \mu\}$ and $a\mu = \mu'\mu$ as required.

$\epsilon' = 0$: Then $\mu_1 = \epsilon\mu_0$, so $\mu_2\mu = \mu_0\mu_1 = 0$. Hence $\mu_2 \in \{0, \mu\}$ and $a\mu = 0$.

This proves that $B_2\mu \cap B_2\mu' = \{0, \mu\mu'\}$.

(ii) $B_3V = B_5$. Recall that $B_3 = B_3^6 \oplus B_2^6x_7 \oplus B_2^6x_8 \oplus B_1^6x_7x_8$ so

$$B_3\mu = B_3^6\mu \oplus B_2^6\mu x_7 \oplus B_2^6\mu x_8 \oplus B_1^6\mu x_7x_8$$

Also

$$B_5 = B_5^6 \oplus B_4^6x_7 \oplus B_4^6x_8 \oplus B_3^6x_7x_8$$

It is easily seen that any degree 5 monomial is a multiple of μ, so $B_3^6\mu = B_5^6$. Since $x_7\mu' = x_7\mu_0$,

$$B_3V \supset x_7B_2^6\mu + x_7B_2^6\mu' = (B_2^6\mu + B_2^6\mu_0)x_7 = B_4^6x_7$$

by Lemma 7.2. Similarly $B_3V \supset B_4^6x_8$.

Finally, if $a \in B_3^6$ then $a\mu' = a\mu_0 + ax_7x_8 \in B_3V$. But $a\mu_0 \in B_5^6 \subset B_3V$, so $ax_7x_8 \in B_3V$ also. Hence $B_3^6x_7x_8 \subset B_3V$. Putting all this together yields $B_5 = B_5^6 \oplus B_4^6x_7 \oplus B_4^6x_8 \oplus B_3^6x_7x_8 \subset B_3V$, so $B_3V = B_5$ as claimed. Hence, all such Type A spaces are semi-regular. □

This enables us to count exactly the number of Type A spaces with respect to a fixed μ, by rank type.

Theorem 8.9. *Let $\mu = x_1x_2 + x_3x_4 + x_5x_6$. Then*

1. *There are 11,796,480 Type A semi-regular subspaces of B_2^8 containing μ which are of type $[6, 8, 8]$.*
2. *There are 31,997,952 Type A semi-regular subspaces of B_2^8 containing μ which are of type $[6, 8, 6]$.*
3. *There are 20,643,840 Type A semi-regular subspaces of B_2^8 containing μ which are of type $[6, 6, 6]$.*

Proof. (1) If V is of Type A with respect to μ, then there exist $\lambda, \lambda' \in \langle x_1, \ldots, x_6\rangle$ such that $V = \langle \mu, \mu'\rangle$ and

$$\mu' = \mu_0 + (\lambda' + x_7)(\lambda + x_8)$$

for some $\mu_0 \in \langle x_1, \ldots, x_6\rangle$. If V is of rank type $[6, 8, 8]$, then $\langle \mu, \mu_0\rangle$ must be of rank type $[6, 6, 6]$. From Tables 1 and 2 the number of $[6, 6, 6]$ subspaces of B_2^6 is 13,332,480 and the number of elements of B_2^6 of rank 6 is 13,880. So the number of $[6, 6, 6]$ subspaces of B_2^6 containing μ is

$$3 * 13,332,480/13,888 = 2,880$$

For each such subspace there are 2^{12} choices for λ and λ', yielding a total of

$$2,880 * 2^{12} = 11,796,480$$

$[6, 6, 6]$ subspaces of Type A containing μ. The numbers in (2) and (3) are found by a similar calculation using the number of [6,4,6] and [6,4,4] subspaces (54,246,528 and 69,995,520 respectively). □

This completes our analysis of the Type A case. We now move to the Type B case, which requires a little more work.

Lemma 8.10. *Suppose $\lambda, \lambda', \kappa, \kappa' \in B_1$ and λ and λ' are linearly independent. If $\lambda\kappa + \lambda'\kappa' = 0$, then $\kappa, \kappa' \in \langle \lambda, \lambda' \rangle$.*

Proof. We may assume that $\lambda = x_1$ and $\lambda' = x_2$. Let $\kappa = \sum_i \epsilon_i x_i$ and $\kappa' = \sum_i \epsilon'_i x_i$ where $\epsilon_i = 0, \epsilon'_i \in \mathbb{F}$. If $x_1\kappa + x_2\kappa' = 0$, then we must have $\epsilon_i = 0$ and $\epsilon'_i = 0$ for $i \neq 1, 2$. □

Lemma 8.11. *Consider elements of the form $\mu' = \mu_0 + \lambda x_7 + \lambda' x_8$ where $\mu_0 \in B_2^6$ and $\lambda, \lambda' \in B_1^6$ are linearly independent. Then*

(1) $\operatorname{rk} \mu' \geq 6$ if and only if $\lambda\lambda'\mu_0 \neq 0$;
(2) there are $63 * 62 * 2^9 * 28$ such elements μ' of rank 8.

Proof. Choose a complementary subspace $W \subset B_1^6$ such that $B_1^6 = W \oplus \langle \lambda, \lambda' \rangle$. In this case we can write $\mu_0 = \nu_0 + \kappa\lambda + \kappa'\lambda' + \epsilon\lambda\lambda'$ where $\nu_0 \in W^2$, $\kappa, \kappa' \in W$ and $\epsilon \in \mathbb{F}$. Then

$$\mu' = \nu_0 + \lambda(x_7 + \kappa + \epsilon\lambda') + \lambda'(x_8 + \kappa')$$

and so $\operatorname{rk} \mu' = \operatorname{rk} \nu + 4$. So $\operatorname{rk} \mu' = 8$ if and only if $\operatorname{rk} \nu_0 = 4$. In each case there are $63 * 62$ choices for λ, λ', 2^8 choices for κ and κ' and 28 choices for ν_0 yielding a total of $63 * 62 * 2^9 * 28$ choices for μ'.

For part (1) observe that $\operatorname{rk} \mu' \geq 6$ if and only if $\nu_0 \neq 0$; and this is equivalent to $\lambda\lambda'\mu_0 \neq 0$. □

We can now give a useful characterization of when Type B spaces are semi-regular.

Theorem 8.12. *Suppose that $V = \langle \mu, \mu' \rangle$ where*

$$\mu = x_1x_2 + x_3x_4 + x_5x_6 \text{ and } \mu' = \mu_0 + \lambda x_7 + \lambda' x_8$$

for some $0 \neq \mu_0 \in B_2^6$ and some linearly independent $\lambda, \lambda' \in B_1^6$. Then V is semi-regular if and only if $\lambda\lambda'\mu_0 \notin B_2^6\mu$.

Proof. Suppose that $\lambda\lambda'\mu_0 \in B_2\mu$. We want to show that V is not semi-regular. We may assume that μ and μ' have rank at least 6 because otherwise V is not semi-regular by Theorem 8.1. Clearly $\lambda\lambda'\mu' \in B_2\mu \cap B_2\mu'$. We want to show that $\lambda\lambda'\mu' \notin \{0, \mu\mu'\}$. Suppose that $\lambda\lambda'\mu' = \mu\mu'$. Then $\lambda\lambda' + \mu \in \operatorname{Ann}(\mu') \cap B_2^6 = \{0, \mu'\}$ by Lemma 5.1. Hence $\lambda\lambda' \in \{\mu, \mu + \mu'\}$, contradicting the fact that both μ and $\mu + \mu'$ have rank at least 6. If $\lambda\lambda'\mu' = 0$, then $\lambda\lambda' \in \operatorname{Ann}(\mu') \cap B_2^6 = \{0, \mu'\}$, again yielding a contradiction because the linear independence property of λ and λ' implies that $\lambda\lambda' \neq 0$. So $B_2\mu \cap B_2\mu' \supsetneq \{0, \mu\mu'\}$ and V is not semi-regular by Lemma 8.3.

Now assume that $\lambda\lambda'\mu_0 \notin B_2\mu$. Lemma 8.11 implies that the ranks of μ' and $\mu' + \mu$ are at least 6. As before, we need to prove (i) $B_2\mu \cap B_2\mu' = \{0, \mu\mu'\}$ and (ii) $B_3V = B_5$. Set $\Lambda = \langle \lambda, \lambda' \rangle$.

(i) $B_2\mu \cap B_2\mu' = \{0, \mu\mu'\}$: Suppose $a\mu = b\mu' \neq 0$ where

$$a = \mu_2 + \lambda_1 x_7 + \lambda_2 x_8 + \epsilon' x_7 x_8, \quad b = \mu_1 + \lambda_7 x_7 + \lambda_8 x_8 + \epsilon x_7 x_8,$$

and $\mu_2, \mu_1 \in B_2^6, \lambda_1, \lambda_2, \lambda_7, \lambda_8 \in B_1^6, \epsilon, \epsilon' \in \mathbb{F}$. Equating the coefficients of x_7x_8 on both sides of $a\mu = b\mu'$, yields

$$\epsilon'\mu = \epsilon\mu_0 + \lambda\lambda_8 + \lambda'\lambda_7.$$

So

$$\epsilon'\mu + \epsilon\mu' = \lambda(\epsilon x_7 + \lambda_8) + \lambda'(\epsilon x_8 + \lambda_7).$$

Since the right hand side has rank at most 4 and the rank of μ, μ' and $\mu + \mu'$ are all at least 6, this implies that $\epsilon = \epsilon' = 0$. Hence $\lambda\lambda_8 + \lambda'\lambda_7 = 0$, so by Lemma 8.10, $\lambda_7, \lambda_8 \in \Lambda$.

Thus

$$a = \mu_2 + \lambda_1 x_7 + \lambda_2 x_8, \quad b = \mu_1 + \lambda_7 x_7 + \lambda_8 x_8,$$

Comparing the coefficients of x_7, x_8 and the term that is purely contained in B_4^6 yields

$$\mu\lambda_1 = \mu_0\lambda_7 + \mu_1\lambda$$
$$\mu\lambda_2 = \mu_0\lambda_8 + \mu_1\lambda'$$
$$\mu_0\mu_1 = \mu\mu_2$$

Since $\lambda_7 \in \Lambda$, $\lambda_7\lambda \in \Lambda^2 = \{0, \lambda\lambda'\}$. If $\lambda_7\lambda = \lambda'\lambda$, then

$$\mu\lambda_1\lambda = \mu_0\lambda_7\lambda = \mu_0\lambda'\lambda$$

contradicting our assumption that $\mu_0\lambda\lambda' \notin B_2^6\mu$. Therefore $\lambda_7\lambda = 0$ and so $\lambda_7 \in \{0, \lambda\}$. Similarly we obtain $\lambda_8 \in \{0, \lambda'\}$ and $\lambda_7 + \lambda_8 \in \{0, \lambda + \lambda'\}$. Therefore

$$(\lambda_7, \lambda_8) = (0, 0) \text{ or } (\lambda, \lambda')$$

Since $\lambda_7\lambda = 0$, we also have $\mu\lambda_1\lambda = 0$. Since $\operatorname{rk}\mu = 6$, this implies $\lambda_1\lambda = 0$, and so $\lambda_1 \in \{0, \lambda\}$. Similarly we obtain $\lambda_2 \in \{0, \lambda'\}$ and $\lambda_1 + \lambda_2 \in \{0, \lambda + \lambda'\}$. Thus

$$(\lambda_1, \lambda_2) = (0, 0) \text{ or } (\lambda, \lambda')$$

Suppose $\lambda_1 = \lambda_2 = 0$. If $\lambda_7 = \lambda_8 = 0$, then $\lambda, \lambda' \in \operatorname{Ann}(\mu_1)$, so $\mu_1 \in \{0, \lambda\lambda'\}$. Since $b \neq 0$, we must have $\mu_1 \neq 0$, so $\lambda\lambda'\mu_0 = \mu_1\mu_0 = \mu\mu_2 \in B_2^6\mu$, a contradiction.

Now suppose that $(\lambda_7, \lambda_8) = (\lambda, \lambda')$. Then

$$(\mu_0 + \mu_1)\lambda = \mu_0\lambda_7 + \mu_1\lambda = 0 \text{ and } (\mu_0 + \mu_1)\lambda' = \mu_0\lambda_8 + \mu_1\lambda' = 0$$

so $\lambda, \lambda' \in \operatorname{Ann}(\mu_0 + \mu_1)$ and $\mu_0 + \mu_1 \in \{0, \lambda\lambda'\}$. If $\mu_0 + \mu_1 = 0$, then $b = \mu'$ and $b\mu' = 0$, contradicting our assumption. Thus $\mu_0 + \mu_1 = \lambda\lambda'$ so $b = \mu' + \lambda\lambda'$ and $\lambda\lambda'\mu_0 = (b + \mu')\mu' = a\mu \in B_2^6\mu$, again a contradiction.

Hence we must have $(\lambda_1, \lambda_2) = (\lambda, \lambda')$. In this case

$$\mu\lambda = \mu_0\lambda_7 + \mu_1\lambda$$
$$\mu\lambda' = \mu_0\lambda_8 + \mu_1\lambda'$$

If $(\lambda_7, \lambda_8) = (0, 0)$, then $\operatorname{Ann}(\mu + \mu_1)$ contains Λ and therefore $\mu + \mu_1 \in \{0, \lambda\lambda'\}$. If $\mu + \mu_1 = \lambda\lambda'$, then $\mu_1 = \mu + \lambda\lambda'$ and so $\mu\mu_2 = \mu_0(\mu + \lambda\lambda')$ which

would imply $\lambda\lambda'\mu_o \in B_2^6\mu$, a contradiction. So $\mu + \mu_1 = 0$, in which case $b = \mu$ and $b\mu' = \mu\mu'$ as required.

If $\lambda_7 = \lambda$ and $\lambda_8 = \lambda'$, then $\mathrm{Ann}(\mu + \mu_1 + \mu_0)$ contains Λ and therefore $\mu + \mu_1 + \mu_0 \in \{0, \lambda\lambda'\}$. If $\mu + \mu_1 + \mu_0 = \lambda\lambda'$, then $\mu_1 = \mu + \mu_0 + \lambda\lambda'$ and so $\mu\mu_2 = \mu_0(\mu + \mu_0 + \lambda\lambda')$ which again implies $\lambda\lambda'\mu_o \in B_2^6\mu$, a contradiction. So $\mu + \mu_1 + \mu_0 = 0$, or $\mu = \mu_0 + \mu_1$. Since

$$b = \mu_1 + \lambda x_7 + \lambda' x_8 = \mu + \mu_0 + \lambda x_7 + \lambda' x_8 = \mu + \mu'$$

we have that $b\mu' = \mu\mu'$. Thus we have proved that $B_2\mu \cap B_2\mu' = \{0, \mu\mu'\}$

In this case $\{\lambda, \lambda'\}$ is linearly independent so we may extend $\{\lambda, \lambda'\}$ to a basis $\{\lambda, \lambda', y_1, y_2, y_3, y_4\}$ for B_2^6. Let $Y = \langle y_1, y_2, y_3, y_4\rangle$. Then we have that after a possible change of the x_i basis, $\mu' = \mu_0 + \lambda x_7 + \lambda' x_8$ where $\mu_0 \in Y$.

(ii) $B_3V = B_5$: Recall, as in the previous proof, that $B_3 = B_3^6 \oplus x_7B_2^6 \oplus x_8B_2^6 \oplus x_7x_8B_1^6$; that

$$B_5 = B_5^6 \oplus B_4^6x_7 \oplus B_4^6x_8 \oplus B_3^6x_7x_8$$

and that $B_5^6 = B_3^6\mu \subset B_3V$. The assumption that $\lambda\lambda'\mu' = \lambda\lambda'\mu_0 \notin B_2^6\mu$ implies that $B_2^6V \cap B_4^6 \supsetneq B_2^6\mu$. Since $\dim B_2^6\mu = 14 = \dim B_4^6 - 1$, this implies that $B_2^6V \supset B_4^6$. So

$$B_3V \supset (B_2^6x_7 + B_2^6x_8)V = B_2^6Vx_7 + B_2^6Vx_8 \supset B_4^6x_7 + B_4^6x_8.$$

Thus it remains to show that $B_3V \supset B_3^6x_7x_8$. For $b \in B_2^6$ we have that

$$bx_7\mu' = b\mu_0x_7 + b\lambda'x_7x_8$$

Since $b\mu_0x_7 \in B_4x_7 \subset B_3V$, this implies that $b\lambda'x_7x_8 \in B_3V$. A similar argument for λ yields that $B_3V \supset (B_2^6\lambda + B_2^6\lambda')x_7x_8$. Also $B_3V \supset B_1^6x_7x_8\mu' = B_1^6\mu_0x_7x_8$, and $B_3V \supset B_1^6x_7x_8\mu$. Hence

$$B_3V \supset (B_1^6\mu_0 + B_2^6\lambda + B_2^6\lambda' + B_1^6\mu)x_7x_8$$

Thus it suffices to show that $B_1^6\mu_0 + B_2^6\lambda + B_2^6\lambda' + B_1^6\mu = B_3^6$. Then we may write $\mu = \nu + \lambda a + \lambda a'$ where $\nu \in Y$ and $a, a' \in B_1^6$. Then $B_1^6\mu_0 + B_2^6\lambda + B_2^6\lambda' + B_1^6\mu = B_3^6$ is equivalent to $Y\mu_0 + Y\nu = Y^3$. Suppose that $Y\mu_0 + Y\nu \neq Y^3$. Then μ_0, ν and $\mu_0 + \nu$ all have rank 2, so we may assume that, after an appropriate change of basis for Y, that $\mu_0 = y_1y_2$ and $\nu = y_1y_3$. In this case

$$\mu = y_1y_3 + \lambda a + \lambda' a'$$

and since μ has rank 6, $y_1, y_3, \lambda, a, \lambda', a'$ must form a basis for B_1^6. Let $A = \langle a, a'\rangle$. Then

$$\Lambda y_1\mu = \Lambda y_1(\lambda a + \lambda' a') = y_1\lambda\lambda' A$$

Since $\lambda\lambda'\mu = y_1\lambda\lambda' y_3$, this yields that

$$B_2^6\mu \supset \Lambda y_1\mu + \mathbb{F}\lambda\lambda'\mu = y_1\lambda\lambda'(A + \mathbb{F}y_3) = y_1\lambda\lambda' B_1^6 = (y_1B_1^6)\lambda\lambda'$$

Hence $\mu_0\lambda\lambda' = y_1y_2\lambda\lambda' \in B_2^6\mu$, contrary to assumption. □

Given this condition for Type B sequences to be semi-regular, we now enumerate such sequences. The following are auxiliary results to help do this.

Lemma 8.13. *Let $\mu = x_1x_2 + x_3x_4 + x_5x_6$ and let λ, λ' be linearly independent elements of B_1^6. Then there exists a $\mu_1 \in B_2^6$ such that $\lambda\lambda'\mu_1 \notin B_2\mu$.*

Proof. Recall that $\dim B_2^6\mu = 15-1 = 14$ by Lemma 5.1 and $\dim B_4^6 = 15$, so $B_2^6\mu \subsetneq B_4^6$. On the other hand if $W = \langle \mu, \lambda\lambda' \rangle$, then by Lemma 7.2 we have that $B_2^6 W = B_4^6$. Hence there must exist a $\mu_1 \in B_2^6$ with $\mu_1\lambda\lambda' \notin B_2^6\mu$. □

Theorem 8.14. *Let $\mu = x_1x_2 + x_3x_4 + x_5x_6$. There are $63 * 62 * 2^{13} = 31,997,952$ semi-regular subspaces containing μ which are of Type B with respect to μ.*

Proof. Recall from Theorem 8.2 that if a space V is induced then it is not semi-regular. Let $\mathcal{T}_B$ be the set of all non-induced two dimensional subspaces of B_2^8 that are Type B with respect to μ. These are spaces of the form $V = \langle \mu, \mu' \rangle$ where $\mu' = \mu_0 + \lambda x_7 + \lambda' x_8$ and $\lambda, \lambda' \in B_1^6$ are linearly independent. For each pair of linearly independent elements λ, λ' choose a $\mu_1 \in B_2^6$ such that $\lambda\lambda'\mu_1 \notin B_2\mu$. Since $\dim B_4^6/B_2^6\mu = 1$, we have that $\lambda\lambda'\mu_0 \in B_2^6\mu$ if and only if $\lambda\lambda'(\mu_0 + \mu_1) \notin B_2^6\mu$. Define $\Phi : \mathcal{T}_B \to \mathcal{T}_B$ by

$$\Phi(\langle \mu, \mu_0 + \lambda x_7 + \lambda' x_8 \rangle) = \langle \mu, \mu_0 + \mu_1 + \lambda x_7 + \lambda' x_8 \rangle$$

Then $\Phi^2 = I$ and $\Phi(V)$ is semi-regular if and only if V is not semi-regular. Hence exactly half of the spaces in $\mathcal{T}_B$ are semi-regular.

The number of choices for λ and λ' is $63*62$; the number of choices for μ_0 is 2^{15} and these come in pairs, $\{\mu_0, \mu_0 + \mu\}$ which generate the same subspace. So the total number of non-induced Type B subspaces is $63 * 62 * 2^{14}$, and half of these are semi-regular. □

At this point we know how many semi-regular spaces there are containing a given μ of rank 6. In order to pass from this local result to a global result about the number of semi-regular subspaces we need to have a precise breakdown of these spaces by rank type (because each space can contain 1,2 or 3 elements of rank 6). Finding this breakdown for Type B spaces required a little more work.

Theorem 8.15. *Let $\lambda, \lambda', \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, x_7, x_8$ be a basis for B_1^8 and let $W = \langle \lambda, \lambda', \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6 \rangle$. Let $\mu' = \epsilon_3\epsilon_4 + \epsilon_5\epsilon_6 + \lambda x_7 + \lambda' x_8$ and let $\mu = \nu + a\lambda + b\lambda' + \eta\lambda\lambda' \in W^2$ be an element of rank 6 for some $\eta \in \mathbb{F}$, $a, b \in \langle \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6 \rangle$ and $\nu \in \langle \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6 \rangle^2$. Then the two dimensional vector space $V = \langle \mu, \mu' \rangle$ is semi-regular if and only if $\nu(\epsilon_3\epsilon_4 + \epsilon_5\epsilon_6) \neq 0$.*

Proof. Let $\mu_0 = \epsilon_3\epsilon_4 + \epsilon_5\epsilon_6$. Suppose that V is not semi-regular. then by an earlier result, we know that $\lambda\lambda'\mu_0 = \gamma\mu$ for some $\gamma = e + c\lambda + d\lambda' + \eta'\lambda\lambda' \in W$. Now

$$\begin{aligned}\gamma\mu &= (\nu + a\lambda + b\lambda' + \eta\lambda\lambda')(e + c\lambda + d\lambda' + \eta'\lambda\lambda') \\ &= \nu e + (ae + \nu c)\lambda + (d\nu + eb)\lambda' + (\eta'\nu + \eta e + cb + ad)\lambda\lambda'\end{aligned}$$

Comparing coefficients yields

$$\begin{aligned}\mu_0 &= \eta'\nu + \eta e + cb + ad\\ 0 &= \nu e\\ 0 &= ae + c\nu\\ 0 &= be + d\nu\end{aligned}$$

So

$$\begin{aligned}\nu\mu_0 &= (\eta'\nu + \eta e + cb + ad)\nu\\ &= bc\nu + ad\nu = b(ae) + a(be) = 0\end{aligned}$$

Conversely assume that $\nu\mu_0 = 0$. Suppose first that $\eta = 1$. Then $\lambda\lambda' = \mu + \nu + a\lambda + b\lambda'$ and so

$$\lambda\lambda'\mu_0 = (\mu + \nu + a\lambda + b\lambda')\mu_0 = \mu_0\mu + (a\lambda + b\lambda')\mu_0$$

Now $\mu = (\nu + ab) + (\lambda + b)(\lambda' + a)$ and so $\operatorname{rk}(\nu + ab) = 4$, since $\operatorname{rk}\mu = 6$. Let $W = \langle\epsilon_3, \ldots, \epsilon_6\rangle$. Since $\operatorname{rk}\mu_0 = 4$ also we have $W(\nu + ab) = W\mu_0$. So there exist $c, d \in U$ such that $a\mu_0 = c(\nu + ab)$ and $b\mu_0 = d(\nu + ab)$. But then

$$\begin{aligned}[(\lambda + b)c + (\lambda' + a)d]\mu &= (\lambda + b)c(\nu + ab) + (\lambda' + a)d(\nu + ab)\\ &= (\lambda + b)a\mu_0 + (\lambda' + a)b\mu_0\\ &= (a\lambda + b\lambda')\mu_0\end{aligned}$$

So $\lambda\lambda'\mu_0 \in B_2\mu$ and V is not semi-regular.

Now suppose $\eta = 0$. Then, $\mu = \nu + a\lambda + b\lambda'$. If $a = b$, $\mu = \nu + a(\lambda + \lambda')$ is expressible in five variables, but $\operatorname{rk}(\mu) = 6$, so $a \neq b$. Then we can extend a, b to a basis a, b, c, d for $\langle\epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6\rangle$. Since μ is rank 6, it must have a term not divisible by a or b, so $cd \in \operatorname{Supp}(\nu) \subset \operatorname{Supp}(\mu)$ in the a, b, c, d basis. Depending on if ac, ad, bc, bd, ab are in $\operatorname{Supp}(\nu)$, we have $\mu = \nu + a\lambda + b\lambda' = (c + \epsilon_1 a + \epsilon_1' b)(d + \epsilon_2 a + \epsilon_2' b) + \epsilon ab + a\lambda + b\lambda'$, for some $\epsilon_1, \epsilon_1', \epsilon_2, \epsilon_2', \epsilon \in \mathbb{F}$. Making a coordinate transformation $c \to c + \epsilon_1 a + \epsilon_1' b, d \to d + \epsilon_2 a + \epsilon_2' b$, we get $\mu = cd + \epsilon ab + a\lambda + b\lambda'$, where $\langle a, b, c, d\rangle = \langle\epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6\rangle$ and $\epsilon \in \{0, 1\}$ with $\nu = cd + \epsilon ab$. Note that

$$(\lambda' + \epsilon a)c\mu = \lambda\lambda' ac \in B_2\mu$$

Likewise,

$$\lambda\lambda'\langle ac, ad, bc, bd\rangle \in B_2\mu$$

Also,

$$\lambda\lambda'\mu = \lambda\lambda'(\epsilon ab + cd) \in B_2\mu$$

In both cases, whether $\epsilon = 0$ or 1, we see therefore that

$$B_2\mu \supseteq \lambda\lambda'\langle ac, ad, bc, bd, \epsilon ab + cd\rangle = \lambda\lambda' \operatorname{Ann}(\epsilon ab + cd) = \lambda\lambda' \operatorname{Ann}(\nu) \ni \lambda\lambda'\mu_0$$

The last inclusion is because $\nu\mu_0 = 0$ implies $\mu_0 \in \operatorname{Ann}(\nu)$. Hence, $\lambda\lambda'\mu_0 \in B_2\mu$, and V is not semi-regular. □

Lemma 8.16. *Let $a, b \in B_1^4$. Then the following are equivalent*

(1) $\operatorname{rk}(x_1x_2 + ax_5 + bx_6) = 6$

(2) x_1, x_2, a, b, x_5, x_6 *are a basis for* B_2^6
(3) x_1, x_2, a, b *are a basis for* B_2^4
(4) $\mathrm{rk}(x_1x_2 + ab) = 4$
(5) $\mathrm{rk}(x_1x_2 + ax_5 + bx_6 + x_5x_6) = 6$

Moreover there are 96 possible such choices for the pair a, b.

Proof. The equivalence of the first four conditions is straightforward. For the last equivalence we note that

$$x_1x_2 + ax_5 + bx_6 + x_5x_6 = x_1x_2 + ab + (a + x_6)(b + x_5)$$

Clearly the number of choices of a and b satisfying (3) is $(2^4 - 4)(2^4 - 8) = 96$. □

Lemma 8.17. *Let* $\mu \in B_2^4$ *be an element of rank 4 and let* $N = \{\nu \in B_2^4 \mid \nu\mu \neq 0\}$. *Then* $|N| = 32$ *and* N *contains 12 elements of rank 4 and 20 elements of rank 2. Moreover, if* $\nu \in N$, *then* $\mathrm{rk}\,\nu = \mathrm{rk}(\nu + \mu)$.

Proof. Let $V = \langle \mu, \nu \rangle$. Then $\nu \in N$ if and only if $V^2 \neq 0$. The two dimensional subspaces of B_2^4 of types $[4, 4, 4]$, $[4, 4, 2]$ and $[4, 2, 2]$ are equivalent up to change of basis to the spaces

$$\begin{aligned}
[4,4,4]: &\quad \{0, x_1x_2 + x_3x_4, x_1x_2 + x_1x_3 + x_2x_4, x_3x_4 + x_1x_3 + x_2x_4\} \\
[4,4,2]: &\quad \{0, x_1x_2 + x_3x_4, x_1x_3, x_1x_3 + x_1x_2 + x_3x_4\} \\
[4,2,2]: &\quad \{0, x_1x_2 + x_3x_4, x_1x_2, x_3x_4\}.
\end{aligned}$$

To see this, note first that the latter case [4,2,2] is clear by the rank decomposition of an element. For [4,4,2], $\mathrm{Supp}(\mu')$ and $\mathrm{Supp}(\mu + \mu')$ cannot contain only elements of $\{x_1x_2, x_3x_4\}$, so without loss of generality suppose μ' is the rank 2 element and that $x_1x_3 \in \mathrm{Supp}(\mu')$. Then, since μ' is rank 2, we can write it as $\mu' = (x_1 + a)(x_3 + b)$ for $a, b \in \langle x_2, x_4 \rangle$. In fact, if necessary we can make a change of basis $x_1 \to x_1 + x_2$ or likewise $x_3 \to x_3 + x_4$ without changing the form of μ, so also assume $a \in \mathbb{F}x_4$ and $b \in \mathbb{F}x_2$. If $a = b = 0$, we're done. If $a = x_4$ and $b = x_2$, changing $x_1 \to x_1 + x_4$ and $x_3 \to x_3 + x_2$ again preserves the form of $\mu = x_1x_2 + x_3x_4$ while yielding $\mu' = x_1x_3$, so we're done again. On the other hand, if $a = x_4$ and $b = 0$, $\mu + \mu' = x_1x_2 + x_3x_4 + (x_1 + x_4)x_3 = x_1(x_2 + x_3)$ which is rank 2, contradicting the rank type $\mathrm{Rk}(V) = [4, 4, 2]$, and likewise for $a = 0$ and $b = x_3$.
The last case is [4,4,4]. In this case, again we can write $\mu' = (x_1 + a)(x_3 + b) + x_2x_4$ for $a \in \mathbb{F}x_4, b \in \mathbb{F}x_2$ and each of the 4 choices for (a, b) gives either the equivalent form of V above or a contradiction to the rank type.
Thus $V^2 \neq 0$ if and only if V is of type $[4, 4, 4]$ or $[4, 2, 2]$. It follows immediately that $\mathrm{rk}\,\nu = \mathrm{rk}(\nu + \mu)$. There are 6 subspaces of type $[4, 4, 4]$ containing a given μ and 10 of type $[4, 2, 2]$. Thus N contains 12 elements of rank 4 and 20 elements of rank 2. □

Lemma 8.18. *Let* ν *be a rank 4 element of* B_2^4 *such that* $\nu \notin B_1^4x_1 + B_1^4x_2$. *Then there exists a basis* x_1, x_2, y_3, y_4 *of* B_1^4 *such that* $\nu = x_1x_2 + y_3y_4$.

Proof. Extend x_1, x_2 to an arbitrary basis x_1, x_2, z_3, z_4 of B_2^4. Write $\nu = \epsilon x_1x_2 + z_3a + z_4b + \epsilon' z_3z_4$ for $a, b \in \langle x_1, x_2\rangle$. Since $\nu \notin B_1^4x_1 + B_1^4x_2$, $\epsilon' = 1$. So, $\nu = (\epsilon x_1x_2 + ab) + (z_3 + b)(z_4 + a)$ with $\epsilon x_1x_2 + ab \in \{0, x_1x_2\}$. Then, since ν is rank 4, $\epsilon x_1x_2 + ab = x_1x_2$ and $\nu = x_1x_2 + (z_3+b)(z_4+a)$. Setting $y_3 = z_3+b$ and $y_4 = z_4+a$, x_1, x_2, y_3, y_4 is a basis of B_1^4 (since $a, b \in \langle x_1, x_2\rangle$) such that $\nu = x_1x_2 + y_3y_4$. □

Theorem 8.19. *Let $\mathcal{V}$ be the set of semi-regular two dimensional subspaces of $B_1^6B_1^8$ which contain an element of B_1^6 of rank 6. Then there are*

(1) $63 * 62 * 2^9 * 28 * 12 * 128$ *spaces in $\mathcal{V}$ of rank type* $[6, 8, 8]$.
(2) $63 * 62 * 2^9 * 28 * 20 * 192$ *spaces in $\mathcal{V}$ of rank type* $[6, 6, 8]$*;*

Proof. Let us fix an element $\mu' \in B_1^6B_1^8$ of rank 8 and count the number of $V \in \mathcal{V}$ containing μ'. As in the proof of Lemma 8.11, we may assume that μ' has the form

$$\mu' = \mu_0 + \lambda x_7 + \lambda' x_8$$

where $B_1^6 = W \oplus \langle \lambda, \lambda'\rangle$, $\dim W = 4$ and $\mu_0 \in W^2$ has rank 4. Suppose that $V = \langle \mu, \mu'\rangle$ where $\mu \in B_2^6$ has rank 6. In this case $\mu = \nu + a\lambda + b\lambda' + \epsilon\lambda\lambda'$ for some $\nu \in W^2$, $a, b \in W$ and $\epsilon \in \mathbb{F}$. By Theorem 8.15, V is semi-regular if and only if $\nu\mu_0 \neq 0$. By Lemma 8.17, there are 12 choices for ν of rank 4 and 20 choices of rank 2. Again by Lemma 8.17, we see that if $\operatorname{rk}\nu = 4$, then $\operatorname{rk}(\nu + \mu_0) = 4$ also, so

$$\mu' + \mu = (\mu_0 + \nu) + \lambda(x_7 + a) + \lambda'(x_8 + b) + \epsilon\lambda\lambda'$$

has rank 8. Thus V is of type $[6, 8, 8]$. Similarly, if $\operatorname{rk}\nu = 2$, then V is of type $[6, 6, 8]$. Now we use Lemma 8.16 to count the number of possible μ for which V has each of the two rank types. First consider the case $\operatorname{rk}\nu = 2$.

(i) If $\epsilon = 0$, then $\mu = \nu + a\lambda + b\lambda'$ and by Lemma 8.16 there are 96 choices for a and b which yield $\operatorname{rk}\mu = 6$.

(ii) If $\epsilon = 1$, then $\mu = \nu + a\lambda + b\lambda' + \lambda\lambda'$. Again by Lemma 8.16 there are 96 choices for a and b which yield $\operatorname{rk}\mu = 6$..

Thus in the $[6, 6, 8]$ case, for any given μ' there are 20 choices for ν and 192 choice for a, b and ϵ. The number of choices for μ' is given by Lemma 8.11 and combining these two results yields (2).

Now consider the case $\operatorname{rk}\nu = 4$.

(i) If $\epsilon = 0$, then $\mu = \nu + a\lambda + b\lambda'$. Note that $\operatorname{rk}\mu = 6$ implies that a and b are linearly independent and $\nu \notin B_1^4a + B_1^4b$. So by Lemma 8.18 $\nu = ab + y_3y_4$ where a, b, y_3, y_4 is a basis for W. Thus $\langle \nu, ab\rangle$ is a $[4, 2, 2]$ space containing ν. There are ten such subspaces for each ν; each space yields two choices for ab and each such choice of ab yields 6 choices for a and b. This yields a total of 120 choices for μ.

(ii) If $\epsilon = 1$, then $\mu = \nu + a\lambda + b\lambda' + \lambda\lambda' = \nu + ab + (\lambda + b)(\lambda' + a)$.. In this case $\operatorname{rk}\mu = 6$ if and only if $\operatorname{rk}(\nu + ab) = 4$. If $ab = 0$, this is always true and there are 46 ways to choose a and b such that $ab = 0$. The latter is because B_1^4 has $2^4 = 16$ elements and $ab = 0$ implies either $a = b$ (16 choices) or if $a \neq b$, either $a = 0$ and $b \neq 0$ (16-1 = 15 choices) or $b = 0$ and $a \neq 0$ (again

15 choices), yielding $15 + 15 + 16 = 46$ choices. If $ab \neq 0$, this holds if and only if $\langle \nu, ab \rangle$ is of type $[4, 4, 2]$. There are 15 such spaces containing a given element of rank four, so 15 choices for ab; and for each of these, there are 6 different ways of choosing a and b. This yields 136 possibilities for μ in this case.

Thus there is a total of 256 choices for μ. Finally there is a 2-1 correspondence between choices for μ and spaces of rank type $[6, 8, 8]$ □

Corollary 8.20. *Let $\mu \in B_2^8$ have rank 6. Then*

1. *There are $6,193,152$ two-dimensional semi-regular subspaces of B_2^8 of type $[6, 8, 8]$ which are Type B with respect to μ.*
2. *There are $15,482,880$ two-dimensional semi-regular subspaces of B_2^8 of type $[6, 8, 6]$ which are Type B with respect to μ.*
3. *There are $10,321,920$ two-dimensional semi-regular subspaces of B_2^8 of type $[6, 6, 6]$ which are Type B with respect to μ.*

Proof. Without loss of generality we can assume that $\mu \in B_2^6$. In this case the semi-regular subspaces of Type B with respect to μ are exactly the spaces $V \in \mathcal{V}$ containing μ.

(1) Let

$$\mathcal{V}' = \{V \in \mathcal{V} \mid V \text{ has rank type } [6, 8, 8]\}$$

and let $\mathcal{V}'_\mu = \{V \in \mathcal{V}' \mid \mu \in V\}$ be the subset of $\mathcal{V}'$ consisting of such spaces containing our fixed element μ. Note that if $\sigma \in GL(B_1^6)$, then $\sigma(\mathcal{V}'_\mu) = \mathcal{V}'_{\sigma(\mu)}$. Since $GL(B_1^6)$ acts transitively on the set of all rank 6 elements of B_2^6, we have that $\mathcal{V} = \bigsqcup \mathcal{V}'_{\sigma(\mu)}$. Thus since B_2^6 contains 13888 elements of rank 6, $|\mathcal{V}'| = 13888 * |\mathcal{V}'_\mu|$. So by Theorem 8.19

$$|\mathcal{V}'_\mu| = |\mathcal{V}'|/13888 = 63 * 62 * 2^9 * 28 * 12 * 128/13888 = 6,193,152$$

A similar argument proves (2). For part (3), notice that the number of semi-regular subspaces containing μ and of Type B with respect to μ is $31,997,952$ by Theorem 8.14. Since these have either rank type $[6, 8, 8], [6, 6, 8]$ or $[6, 6, 6]$, the number of the latter type is

$$31,997,952 - 6,193,152 - 15,482,880 = 10,321,920$$

□

Corollary 8.21. *Let $\mu = x_1x_2 + x_3x_4 + x_5x_6$. Then*

1. *There are $17,989,632$ two-dimensional semi-regular subspaces of B_2^8 of type $[6, 8, 8]$ containing μ.*
2. *There are $47,480,832$ two-dimensional semi-regular subspaces of B_2^8 of type $[6, 8, 6]$ containing μ.*
3. *There are $30,965,760$ two-dimensional semi-regular subspaces of B_2^8 of type $[6, 6, 6]$ containing μ.*

Proof. The number of such spaces is just the sum of the number of spaces which are Type A and Type B with respect to μ. Thus we just add the numbers in Theorem 8.9, and Corollary 8.20. □

Corollary 8.22. *There are*

(1) $2,697,022,899,486,720$ *two-dimensional semi-regular subspaces of* B_2^8 *of type* $[6,8,8]$
(2) $3,559,185,957,519,360$ *two-dimensional semi-regular subspaces of* B_2^8 *of type* $[6,8,6]$
(3) $1,547,472,155,442,200$ *two-dimensional semi-regular subspaces of* B_2^8 *of type* $[6,6,6]$

Proof. For any element of B_2^8 of rank 6, there is an automorphism $\sigma \in \mathrm{GL}(B_1^8)$ such that $\sigma(\tilde{\mu}) = \mu$. This automorphism then induces a bijection between the set of semi-regular subspaces of B_2^8 of type $[6,8,6]$ containing $\tilde{\mu}$ and the set of semi-regular subspaces of B_2^8 of type $[6,8,6]$ containing μ. Since there are $149,920,960$ elements of B_2^8 of rank 6, the total number of semi-regular subspaces of B_2^8 of type $[6,8,6]$ is

$$\frac{47,480,832 * 149,920,960}{2} = 3,559,185,957,519,360$$

The other cases are handled similarly. □

8.1. **Approximation of $p_{8,2}$.** The case when $\mathrm{Rk}\, V = [8,8,8]$ seems to be even more complex than the Type B case above. Thus we content ourselves with an approximation of $p_{8,2}$ in this case.

Theorem 8.23. *Let* $p_{8,2}$ *be the proportion of two dimensional subspaces of* B_2^8 *which are semi-regular. Then*

$$0.65 \leq p_{8,2} \leq 0.72$$

Proof. We are able to determine the semi-regularity of all but the $888,431,072,772,096$ spaces of rank type $[8,8,8]$. Using Corollary 8.22 we obtain that the number $sr(8,2)$ of semi-regular 2 dimensional subspaces satisfies

$$7,803,681,012,449,280 \leq sr(8,2) \leq 8,692,112,085,221,376$$

Dividing by the total number of 2 dimensional subspaces, $12,009,598,872,103,595$ yields the claimed bounds. □

9. Hilbert Polynomials

An even more fine-grained understanding can be obtained by looking at the possible Hilbert polynomials that can arise for B/BV. We list here (without proof) a complete description of the Hilbert polynomials that can arise in the cases $n = 4, 5$ and 6. The main determining factor is the rank-type and whether or not the space is induced.

When $n = 4$ the situation is simple. When $n = 5$ we begin to see the distinction between the induced and non-induced cases. When $n = 6$, more subtle distinctions begin to appear. In the types column we have

- i4: V is induced from a 4 dimensional subspace
- i5: V is induced from a 5 dimensional subspace

Type	Number	$H_V(z)$
$[2,2,2]$	105	$1+4z+4z^2+z^3$
$[2,2,4]$	280	$1+4z+4z^2$
$[2,4,4]$	210	$1+4z+4z^2$
$[4,4,4]$	56	$1+4z+4z^2$
Total	651	

TABLE 6. Hilbert Polynomials of B/BV by Rank when $n=4$

Rank	Type	Number	$H_V(z)$
$[2,2,2]$		1,085	$1+5z+8z^2+5z^3+z^4$
$[2,2,4]$		8,680	$1+5z+8z^2+4z^3$
$[2,4,4]$	i	6,510	$1+5z+8z^2+4z^3$
$[2,4,4]$	ni	52,080	$1+5z+8z^2+2z^3$
$[4,4,4]$	i	1,736	$1+5z+8z^2+4z^3$
$[4,4,4]$	ni	104,160	$1+5z+8z^2+z^3$
Total		174,251	

TABLE 7. Hilbert Polynomials of B/BV by Rank and Type when $n=5$

- nin: V is not induced but not semi-regular
- nis: V is not induced and is semi-regular

10. Conclusion

We conducted a detailed study of the semi-regularity of two dimensional quadratic spaces. We found the following values for $p_{n,2}$, the proportion of quadratic subspaces that were semi-regular.

Our hope was that this study would shed some light which would enable progress towards two of the most glaring open questions concerning semi-regularity: a) do there exist semi-regular sequences of homogeneous quadratic elements for all n? and b) is $\lim_{n\to\infty} p_{n,n} = 1$; i.e., are most sequences of n homogeneous quadratic elements in n variables semi-regular? On the positive side, the rank type is an invariant which can be used to establish certain results easily. It seems possible that the answer to a) can be found by considering specific spaces of high rank type. On the other hand the table of Hilbert series in the case $n=6$ suggest that getting the Hilbert series exactly right is a hard thing to control. While most spaces seem to be close to being semi-regular (in the sense that their Hilbert series are close to $T_{n,m}(z)$), it appears that it will be a highly non-trivial problem to prove the exact match of dimensions in each degree.

Rank	Type	Number	$H_V(z)$
$[2,2,2]$		9,765	$1+6z+13z^2+13z^3+6z^4+z^5$
$[2,2,4]$		182,280	$1+6z+13z^2+13z^3+4z^4$
$[2,4,4]$	i4	136,710	$1+6z+13z^2+13z^3+4z^4$
$[2,4,4]$	i5	3,281,040	$1+6z+13z^2+10z^3+2z^4$
$[2,4,6]$		4,666,368	$1+6z+13z^2+10z^3$
$[2,6,6]$		2,187,360	$1+6z+13z^2+10z^3$
$[4,4,4]$	i4	36,456	$1+6z+13z^2+13z^3+4z^4$
$[4,4,4]$	i5	6,562,080	$1+6z+13z^2+9z^3+z^4$
$[4,4,4]$	nin	8,749,440	$1+6z+13z^2+8z^3+z^4$
$[4,4,4]$	nis	15,554,560	$1+6z+13z^2+8z^3$
$[4,4,6]$		69,995,520	$1+6z+13z^2+8z^3$
$[4,6,6]$		54,246,528	$1+6z+13z^2+8z^3$
$[6,6,6]$		13,332,480	$1+6z+13z^2+8z^3$
Total		178,940,587	

TABLE 8. Hilbert Polynomials of B/BV by Rank and Type when $n = 6$

n	3	4	5	6	7	8	≥ 9
$p_{n,2}$	1.00	0.84	0.00	0.86	0.00	$[0.65, 0.72]$	0.00

TABLE 9. The proportion $p_{n,2}$ of 2-dimensional subspaces of B_2 that are semi-regular

For most applications, it is sufficient to show that the degree of the Hilbert polynomial is the same as that of a semi-regular system. Proving this should be significantly easier and would give a more useful result from the point of view of applications. Thus a weaker but more accessible conjecture would be that for "most" m-dimensional subspaces $B_{D-2}V = B_D$ for $D = D_{n,m}$. For instance we are able to prove this result in the one case that we were not able to establish semi-regularity - spaces of rank type $[8,8,8]$ when $n = 8$.

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Appendix A. The General Upper Bound

Let V be an m-dimensional graded subspace of B. Let $\{\mu_1, \ldots, \mu_m\}$ be a homogeneous basis for V and set $d_i = \mu_i$. If we assume that $d_1 \leq \cdots \leq d_m$ then the vector $\underline{d} = (d_1, \ldots, d_m)$ is independent of the choice of homogeneous basis. For such a vector $\underline{d} = (d_1, \ldots, d_m)$ we define

$$T_{n,\underline{d}}(z) = \left[\frac{(1+z)^n}{\prod_i (1+z^{d_i})}\right]$$

and

$$D_{n,\underline{d}} := \deg T_{n,\underline{d}}(z)$$

Denote the Hilbert series of the quotient ring B/BV by $HS_V(z)$. We say the space V is semi-regular if $HS_V(z) = T_{n,\underline{d}}(z)$.

Theorem A.1. *Let V be a graded subspace of B^n with degree vector $\underline{d}$ and let $d = \sum_i d_i$. If $n \geq D_{n,\underline{d}} + d$, then V is not semi-regular.*

Proof. Let $B = \{\mu_1, \ldots, \mu_m\}$ be a basis for V. Choose an element ξ of BV of maximal degree. Clearly $\deg \xi \leq d$ and $\xi\mu_i = 0$ for all i. Let $D = D_{n,\underline{d}}$. If V is semi-regular, then

$$B_D = \sum_i B_{D-d_i}\mu_i$$

But then

$$\xi B_d = \xi \sum_i B_{D-d_i}\mu_i = \sum_i B_{D-d_i}\xi\mu_i = 0$$

This implies that $\xi \in B_{\deg \xi} \cap \operatorname{Ann} B_D = 0$. So Lemma 4.2 implies that $n < D + \deg \xi \leq D + d$. Thus if $n \geq D + d$, V can not be semi-regular. □

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