

# On asymptotically optimal tests for random number generators

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**Abstract**—The problem of constructing effective statistical tests for random number generators (RNG) is considered. Currently, statistical tests for RNGs are a mandatory part of cryptographic information protection systems, but their effectiveness is mainly estimated based on experiments with various RNGs. We find an asymptotic estimate for the p-value of an optimal test in the case where the alternative hypothesis is a known stationary ergodic source, and then describe a family of tests each of which has the same asymptotic estimate of the p-value for any (unknown) stationary ergodic source.

**keywords:** randomness testing, statistical test for cryptographic applications, random number generators, Shannon entropy.

## I. INTRODUCTION

Random number generators (RNG) and pseudo-random number generators (PRNG) are widely used in many information technologies, including cryptographic information protection systems. The goal of RNG and PRNG is to generate sequences of binary digits, which are distributed as a result of throwing an "honest" coin or, more precisely, obey the Bernoulli distribution with parameters  $(1/2, 1/2)$ . As a rule, for practically used RNG and PRNG this property is verified experimentally with the help of statistical tests intended for this purpose, see for a review [1], [2]. Nowadays, there are more than a hundred applicable statistical tests and some of them is a mandatory part of cryptographic information protection systems [3]. Besides, there are dozens RNGs based on different physical processes [4], and an even greater number of PRNGs based on different mathematical algorithms; see for review [1]. In such a situation, the natural question is how to compare different tests? Currently, the main method of such a comparison is numerical experiments in which different tests are practically applied to different RNGs, see for review [1]–[3].

Here we consider the problem of finding optimal tests in the case when the RNG is modeled by stationary ergodic sources. We found the following asymptotic solution to this problem: we first described the asymptotic behaviour of the p-value of the optimal test for the case where the probability distribution of the RNG is a priori known, and then described a family of statistical tests that have the same asymptotic estimates of the p-value for any distribution (which is not known in advance). More precisely, we showed that in both cases, with

probability 1,  $\lim_{n \rightarrow \infty} -\frac{1}{n} \log \pi_\tau(x_1 x_2 \dots x_n) = 1 - h(\nu)$ , where  $x_1 x_2 \dots x_n$  is the sample,  $\tau$  is the test,  $\pi_\tau(x_1 x_2 \dots x_n)$  is the p-value, and  $h(\nu)$  is the Shannon entropy of the (unknown) RNG distribution  $\nu$ . It turns out that asymptotically optimal tests with the required properties are known [5], [6], and are deeply connected with so-called universal codes. Note that nowadays there are many universal codes which are based on different ideas and approaches, among which we note the PPM universal code [7], the arithmetic code [8]), the Lempel-Ziv (LZ) codes [9], the Burrows-Wheeler transform [10] which is used along with the book-stack (or MTF) code [11]–[13], the class of grammar-based codes [14], [15] and some others [16]–[18]. All these codes are universal. This means that, asymptotically, the length of the compressed file goes to the smallest possible value, i.e. the Shannon entropy ( $h(\nu)$ ) per letter.

The main idea of randomness tests based on universal codes is rather natural: try to "compress" a test sequence by a universal code: if the sequence is significantly compressed, then it is not random, see [5], [6] and a short description below.

The rest of the paper is organised as follows. The next section contains the necessary definitions and some basic facts used in the following. Sections III, IV are devoted to investigation of the Neyman-Pearson test and tests based on universal codes, correspondingly. The proofs are given in the appendix.

## II. DEFINITIONS

1) *The main notations:* We consider RNG which generates a sequence of letters  $x = x_1 x_2 \dots x_n$ ,  $n \geq 1$ , from the alphabet  $\{0, 1\}^n$ . There are two following statistical hypothesis  $H_0 = \{x \text{ obeys uniform distribution } (\mu_U) \text{ on } \{0, 1\}^n\}$  and the alternative hypothesis  $H_1 = \bar{H}_0$ , that is,  $H_1$  is negation of  $H_0$ . Let  $T$  be a test. Then, by definition, a significance level  $\alpha$  equals probability of the Type I error. (Recall, that Type I error occurs if  $H_0$  is true and  $H_0$  is rejected. Type II error occurs if  $H_1$  is true, but  $H_0$  is accepted.) Denote a critical region of the test  $T$  for the significance level  $\alpha$  by  $C_T(\alpha)$  and let  $\bar{C}_T(\alpha) = \{0, 1\}^n \setminus C_T(\alpha)$ . (Recall, that for a certain  $x = x_1 x_2 \dots x_n$  the hypothesis  $H_0$  is rejected if and only if  $x \in C_T(\alpha)$ .)

Let us assume that  $H_1$  is true and the investigated sequence  $x = x_1x_2\dots x_n$  is generated by (unknown) source  $\nu$ . By definition, the test  $T$  is consistent (for  $\nu$ ), if for any significance level  $\alpha \in (0, 1)$  the probability of Type II error goes to 0, that is

$$\lim_{n \rightarrow \infty} \nu(\bar{C}_T(\alpha)) = 0. \quad (1)$$

Let us give a definition of a so-called p-value, which plays an important role in the randomness testing. Let there be a statistic  $\tau$  (that is, a function on  $\{0, 1\}^n$ ) and  $x$  be a word from  $\{0, 1\}^n$ . A p-value ( $\pi_\tau(x)$ ) of  $\tau$  and  $x$  is defined by the equation

$$\pi_\tau(x) = \mu_U\{y : \tau(y) \geq \tau(x)\} = |\{y : \tau(y) \geq \tau(x)\}|/2^n. \quad (2)$$

(Here and below  $|X|$  is a number of elements  $X$ , if  $X$  is a set, and the length of  $X$ , if  $X$  is a word.)

Informally,  $\pi_\tau(x)$  is the probability to meet a random point  $y$  which is worse than the observed when considering the null hypothesis.

2) *The consistent tests for stationary ergodic sources and universal codes:* First let us give a short informal description of the universal codes. For any integer  $m$  a lossless code  $\phi$  is defined as such a map from the set of  $m$ -letter words to the set of all binary words that for any sequence of encoded  $m$ -letter words  $\phi(v_1)\phi(v_2)\dots$  the initial sequence  $v_1v_2\dots$  can be found without mistakes; the formal definition can be found, for example, in [19].

We will consider so-called universal codes which have the following property: for any stationary ergodic  $\nu$  defined on the set of all infinite binary words  $x = x_1x_2\dots$ , with probability one

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\phi(x_1x_2\dots x_n)|/n = h(\nu), \quad (3)$$

where  $h(\nu)$  is the Shannon entropy of  $\nu$  (see for definition [19]). Such codes exist, see, for example, [5], [6]. Note, that a goal of codes is to "compress" sequences, i.e. make a length of the codeword  $\phi(x_1x_2\dots x_n)$  as small as possible. The property (3) shows that the universal codes are asymptotically optimal, because the Shannon entropy is a low bound of the length of the compressed sequence (per letter), see [19].

Let us back to considered problem of hypothesis testing. Suppose, it is known that a sample sequence  $x = x_1x_2\dots$  was generated by stationary ergodic source and, as before, we consider the same  $H_0$  against the same  $H_1$ . Let  $\phi$  be a universal code. The following test is a particular case of a goodness-of-fit test suggested in [5], [6]:

*If  $n - |\phi(x_1\dots x_n)| \geq -\log_2 \alpha$  then  $H_0$  is rejected, otherwise accepted. Here, as before,  $\alpha$  is the significance level,  $|\phi(x_1\dots x_n)|$  is the length of encoded ("compressed") sequence.*

We denote this test by  $T_\phi$  and its statistic by  $\tau_\phi$ , i.e.

$$\tau_\phi(x_1\dots x_n) = n - |\phi(x_1\dots x_n)|. \quad (4)$$

It turns out that this test is consistent for any stationary ergodic source. More precisely, the following theorem is proven in [5], [6]:

*For each stationary ergodic  $\nu$ ,  $\alpha \in (0, 1)$  and a universal code  $\phi$ , the Type I error of the described test is not larger than  $\alpha$  and the Type II error goes to 0, when  $n \rightarrow \infty$ .*

### III. ASYMPTOTIC BEHAVIOUR OF A P-VALUE OF THE NEYMAN-PEARSON TEST.

Suppose, that  $H_1$  is true and sequences  $x \in \{0, 1\}^n$  are obey a certain distribution  $\nu$ . It is well-known in mathematical statistics that the optimal test ( $NP$ -test or likelihood-ratio test) is described by Neyman-Pearson lemma and the critical region of this test is defined as follows:

$$C_{NP}(\alpha) = \{x : \mu_U(x)/\nu(x) \leq \lambda_\alpha\},$$

where  $\alpha \in (0, 1)$  is the significance level and the constant  $\lambda_\alpha$  is chosen in such a way that  $\mu_U(C_{NP}(\alpha)) = \alpha$ , see [20]. (We did not take into account that the set  $\{0, 1\}^n$  is finite. Strictly speaking, in such a case a randomized test should be used, but in what follows we will consider asymptotic behaviour of the tests for large  $n$  and this effect will be negligible). The p-value for the  $NP$ -test can be derived from the definition (2), if we put  $\tau(x) = \nu(x)$  and take into account that by definition,  $\mu_U(x) = 2^{-n}$  for any  $x \in \{0, 1\}^n$ . So,

$$\pi_{NP}(x) = \mu_U\{y : \nu(y) \geq \nu(x)\} = |\{y : \nu(y) \geq \nu(x)\}|/2^n. \quad (5)$$

The following theorem describes an asymptotic behaviour of p-values for stationary ergodic sources for  $NP$  test.

*Theorem 1:* If  $\nu$  is a stationary ergodic measure, then, with probability 1,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \pi_{NP}(x) = 1 - h(\nu), \quad (6)$$

where  $h(\nu)$  is the Shannon entropy of  $\nu$ , see for definition [19].

The  $NP$ -test is optimal in the sense that its probability of a Type II error is minimal, but when testing RNG the alternative distribution is unknown, and, hence, some other tests should be used. It turns out that the above described test  $T_\phi$  has the same asymptotic behaviour as  $NP$ -test.

### IV. ASYMPTOTICALLY OPTIMAL TESTS FOR RANDOMNESS.

The following theorem describes an asymptotic behaviour of p-values for stationary ergodic sources for tests which are based on universal codes.

*Theorem 2:* Let  $\phi$  be a universal code and the test  $T_\phi$  with statistic  $\tau_\phi$  (4) is applied. Then for any stationary ergodic measure  $\nu$ , with probability 1,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \pi_{\tau_\phi}(x) = 1 - h(\nu), \quad (7)$$

where  $\pi_{\tau_\phi}$  is the p-value.

Note that this theorem gives some idea of the relation between the Shannon entropy of the (unknown) process  $\nu$  and the required sample size. Indeed, suppose that a  $NP$  test is used and the desired significance level is  $\alpha$ . Then, we can see that (asymptotically)  $\alpha$  should be less than  $\pi_{NP}(x)$  and from (6)

we obtain  $n > -\log \alpha / (1 - h(\nu))$  (for the most powerful test). It is known that the Shannon entropy is 1 if and only if  $\nu$  is the uniform measure  $\mu_u$ . Therefore, in a certain sense, the difference  $1 - h(\nu)$  estimates the distance between the distributions, and the last inequality shows that the required sample size goes to infinity if  $\nu$  approaches the uniform distribution.

The next simple example illustrates the theorems. Let there be a test  $\kappa$  and a generator (a measure  $\nu$ ) that generates sequences of independent binary digits with, say,  $\nu(0) = 0.501, \nu(1) = 0.499$ . Suppose that  $\lim_{n \rightarrow \infty} -\frac{1}{n} \log \pi_\kappa(x) = c$ , where  $c$  is a positive constant. Let us consider the following ‘‘decimation test’’  $\kappa^{1/2}$ : an input sequence  $x_1 x_2 \dots x_n$  is transformed into  $x_1 x_3 x_5 \dots x_{2 \lfloor n/2 \rfloor - 1}$  and then  $\kappa$  is applied to this transformed sequence. Obviously, for this test  $\lim_{n \rightarrow \infty} -\frac{1}{n/2} \log \pi_{\kappa^{1/2}}(x) = c$ , and, hence,  $\lim_{n \rightarrow \infty} -\frac{1}{n} \log \pi_{\kappa^{1/2}}(x) = c/2$ . Thus, the value  $-\frac{1}{n} \log \pi_\kappa(x_1 \dots x_n)$  seems to be a reasonable estimate of the power of the test for large  $n$ .

## V. APPENDIX

*Proof of Theorem 1.* The well-known Shannon-McMillan-Breiman (SMB) theorem states that for the stationary ergodic source  $\nu$  and any  $\epsilon > 0, \delta > 0$  there exists such  $n'(\epsilon, \delta)$  that

$$\nu\{x \in \{0, 1\}^n : h(\nu) - \epsilon < -\frac{1}{n} \log \nu(x) < h(\nu) + \epsilon\} > 1 - \delta \quad \text{for } n > n'(\epsilon, \delta), \quad (8)$$

see [19]. From this we obtain

$$\nu\{x \in \{0, 1\}^n : 2^{-n(h(\nu) - \epsilon)} > \nu(x) > 2^{-n(h(\nu) + \epsilon)}\} > 1 - \delta \quad (9)$$

for  $n > n'(\epsilon, \delta)$ . It will be convenient to define

$$\Phi_{\epsilon, n} = \{x \in \{0, 1\}^n : h(\nu) - \epsilon < -\frac{1}{n} \log \nu(x) < h(\nu) + \epsilon\} \quad (10)$$

From this definition and (9) we obtain

$$(1 - \delta) 2^{n(h(\nu) - \epsilon)} \leq |\Phi_{\epsilon, n}| \leq 2^{n(h(\nu) + \epsilon)}. \quad (11)$$

For any  $x \in \Phi_{\epsilon, n}$  define

$$\Lambda_x = \{y : \nu(y) \geq \nu(x)\} \cap \Phi_{\epsilon, n}. \quad (12)$$

Note that, by definition,  $|\Lambda_x| \leq |\Phi_{\epsilon, n}|$  and from (11) we obtain

$$|\Lambda_x| \leq 2^{n(h(\nu) + \epsilon)}. \quad (13)$$

For any  $\rho \in (0, 1)$  we define  $\Psi_\rho \subset \Phi_{\epsilon, n}$  such that

$$\nu(\Psi_\rho) = \rho \ \& \ \forall u \in \Psi_\rho, \forall v \in (\Phi_{\epsilon, n} \setminus \Psi_\rho) : \nu(u) \geq \nu(v). \quad (14)$$

(That is,  $\Psi_\rho$  contains the most probable words whose total probability equals  $\rho$ . If there are several such sets we can take any of them.) Let us consider any  $x \in (\Phi_{\epsilon, n} \setminus \Psi_\rho)$ . Taking

into account the definition (14) and (11) we can see that for this  $x$

$$|\Lambda_x| \geq \rho |\Phi_{\epsilon, n}| \geq \rho(1 - \delta) 2^{n(h(\nu) - \epsilon)}. \quad (15)$$

So, from this inequality and (13) we obtain

$$\rho(1 - \delta) 2^{n(h(\nu) - \epsilon)} \leq |\Lambda_x| \leq 2^{n(h(\nu) + \epsilon)}. \quad (16)$$

From equation (9), (10) and (14) we can see that  $\nu(\Phi_{\epsilon, n} \setminus \Psi_\rho) \geq (1 - \delta)(1 - \rho)$ . Taking into account (16) and this inequality, we can see that

$$\nu\{x : h(\nu) - \epsilon + \log(\rho(1 - \delta))/n \leq \log |\Lambda_x|/n \leq h(\nu) + \epsilon\} \geq (1 - \delta)(1 - \rho). \quad (17)$$

From the definition (5) of  $\pi_{NP}(x)$  and the definition (12) of  $\Lambda_x$ , we can see that  $\pi_{NP}(x) = |\Lambda_x|/2^n$ . Taking into account this equation and (17) we obtain the following:

$$\nu\{x : 1 - (h(\nu) - \epsilon + \log(\rho(1 - \delta))/n) \geq -\log \pi_{NP}(x)/n \geq 1 - (h(\nu) + \epsilon)\} \geq (1 - \delta)(1 - \rho). \quad (18)$$

Clearly, there exists such  $n^*(\rho)$  that for  $n > n^*(\rho) - \log(\rho(1 - \delta))/n < \epsilon$ . Taking into account (8) we can see that

$$\nu\{x : 1 - (h(\nu) - 2\epsilon) \geq -\log \pi_{NP}(x)/n \geq 1 - (h(\nu) + \epsilon)\} \geq (1 - \delta)(1 - \rho) \quad (19)$$

for  $n > \max(n'(\epsilon, \delta), n^*(\rho))$ . This inequality is valid for any  $\rho \in (0, 1)$  and, in particular, for  $\rho = \delta$ . So, from (19) we obtain

$$\nu\{x : 1 - (h(\nu) - 2\epsilon) \geq -\log \pi_{NP}(x)/n \geq 1 - (h(\nu) + \epsilon)\} \geq (1 - 2\delta).$$

for  $n > \max(n'(\epsilon, \delta), n^*(\delta))$ .

Having taken into account that this inequality is valid for all positive  $\epsilon$  and  $\delta$ , we obtain the statement of the theorem.

*Proof of Theorem 2* is similar to the previous one. First, for any  $\epsilon > 0, \delta > 0$  we define

$$\hat{\Phi}_{\epsilon, n} = \{x : h(\nu) - \epsilon < |\phi(x_1 \dots x_n)|/n < h(\nu) + \epsilon\}. \quad (20)$$

Note that from (3) we can see that there exists such  $n''(\epsilon, \delta)$  that, for  $n > n''(\epsilon, \delta)$ ,

$$\nu(\hat{\Phi}_{\epsilon, n}) > 1 - \delta. \quad (21)$$

We will use the set  $\hat{\Phi}_{\epsilon, n}$  (see (20)). Having taken into account the SMB theorem (8) and (21), we can see that

$$\nu(\hat{\Phi}_{\epsilon, n} \cap \Phi_{\epsilon, n}) > 1 - 2\delta, \quad (22)$$

if  $n > \max(n'(\epsilon, \delta), n''(\epsilon, \delta))$ .

From this moment, the proof begins to repeat the proof of the first theorem, if we use the set  $(\hat{\Phi}_{\epsilon, n} \cap \Phi_{\epsilon, n})$  instead of  $\Phi_{\epsilon, n}$ . Namely, define

$$\hat{\Lambda}_x = \{y : |\phi(y)| \leq |\phi(x)|\} \cap (\hat{\Phi}_{\epsilon, n} \cap \Phi_{\epsilon, n}) \quad (23)$$

and  $\hat{\Psi}_\rho$  is such a subset of  $(\hat{\Phi}_{\epsilon, n} \cap \Phi_{\epsilon, n})$  that

$$\nu(\hat{\Psi}_\rho) = \rho \ \& \ \forall u \in \hat{\Psi}_\rho, \forall v \in ((\hat{\Phi}_{\epsilon, n} \cap \Phi_{\epsilon, n}) \setminus \hat{\Psi}_\rho) :$$

$$|\phi(u)| \leq |\phi(v)|. \quad (24)$$

Let us consider any  $x \in ((\hat{\Phi}_{\epsilon,n} \cap \Phi_{\epsilon,n}) \setminus \hat{\Psi}_\rho)$ . Taking into account the definition (23) and (22), we obtain

$$\rho(1-2\delta)2^{n(h(\nu)-\epsilon)} \leq |\hat{\Lambda}_x| \leq 2^{n(h(\nu)+\epsilon)}. \quad (25)$$

From equations (22) and (24) we can see that  $\nu((\hat{\Phi}_{\epsilon,n} \cap \Phi_{\epsilon,n}) \setminus \hat{\Psi}_\rho) \geq (1-2\delta)(1-\rho)$ . Taking into account (25) and this inequality, we can see that

$$\begin{aligned} & \nu\{x : h(\nu) - \epsilon + \log(\rho(1-2\delta))/n \\ & \leq \log|\hat{\Lambda}_x|/n \leq h(\nu) + \epsilon\} \geq (1-2\delta)(1-\rho). \end{aligned} \quad (26)$$

From the definition of p-value (2) and the definition (23), we can see that  $\pi_{\tau_\phi}(x) = |\hat{\Lambda}_x|/2^n$ . Taking into account this equation and (26) we obtain the following:

$$\begin{aligned} & \nu\{x : 1 - (h(\nu) - \epsilon + \log(\rho(1-\delta)))/n \geq \\ & -\log \pi_{\tau_\phi}(x)/n \geq 1 - (h(\nu) + \epsilon)\} \geq (1-2\delta)(1-\rho). \end{aligned} \quad (27)$$

Clearly, there exists such  $n^{**}(\rho)$  that for  $n > n^{**}(\rho) - \log(\rho(1-2\delta))/n < \epsilon$ . Taking it account we can see from (27) that

$$\begin{aligned} & \nu\{x : 1 - (h(\nu) - 2\epsilon) \geq \\ & -\log \pi_{\tau_\phi}(x)/n \geq 1 - (h(\nu) + \epsilon)\} \geq (1-2\delta)(1-\delta) \end{aligned} \quad (28)$$

for  $n > \max(n'(\epsilon, \delta), n''(\epsilon, \delta), n^{**}(\rho))$ . So, from (28) we obtain

$$\begin{aligned} & \nu\{x : 1 - (h(\nu) - 2\epsilon) \geq \\ & -\log \pi_{\tau_\phi}(x)/n \geq 1 - (h(\nu) + \epsilon)\} \geq (1-3\delta). \end{aligned}$$

for  $n > \max(n'(\epsilon, \delta), n''(\epsilon, \delta), n^{**}(\delta))$ .

Having taken into account that this inequality is valid for all positive  $\epsilon$  and  $\delta$ , we obtain the statement of the theorem.

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