How to Improve Rebound Attacks*

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Abstract. Rebound attacks are a state-of-the-art analysis method for hash functions. These cryptanalysis methods are based on a well chosen differential path and have been applied to several hash functions from the SHA-3 competition, providing the best known analysis in these cases. In this paper we study rebound attacks in detail and find for a large number of cases that the complexities of existing attacks can be improved.
This is done by identifying problems that optimally adapt to the cryptanalytic situation, and by using better algorithms to find solutions for the differential path. Our improvements affect one particular operation that appears in most rebound attacks and which is often the bottleneck of the attacks. This operation, which varies depending on the attack, can be roughly described as *merging* large lists. As a result, we introduce new general purpose algorithms for enabling further rebound analysis to be as performant as possible. We illustrate our new algorithms on real hash functions. More precisely, we demonstrate how to reduce the complexities of the best known analysis on four SHA-3 candidates: JH, Grøstl, ECHO and LANE and on the best known rebound analysis on the SHA-3 candidate Luffa.

Keywords: hash functions, SHA-3 competition, rebound attacks, algorithms

1 Introduction

The rebound attack is a recent technique introduced in [13] by Mendel *et al.* It was conceived to analyze AES-like hash functions (like Grøstl [7] in [14, 8, 15], Echo [2] in [14, 8, 17], Whirlpool [1] in [11]). A rebound attack is composed of two parts: the inbound phase and the outbound phase. The aim of the inbound phase is to find, at a low cost, a large number of pairs of values that satisfy a part of a differential path that would be very expensive to satisfy in a probabilistic way. The outbound phase then uses these values to perform an attack.

This technique has been applied to other algorithms with inner permutations which are not AES-like; for instance it has been applied to JH [20] (reduced to 22 rounds) in [16] and Luffa [4] (reduced to 7 rounds) in [10]; both of those hash functions use Sboxes of size 4×4 and have a linear part in which the mixing is done in a very different way than in the AES. The hash function LANE [9], which includes several AES states, each treated by the AES round transformation, and a different transformation for mixing these states has also been analysed in [12, 21] using rebound attacks.

*This is the extended version of the article published at CRYPTO 2011.

†Supported by the National Competence Center in Research on Mobile Information and Communication Systems (NCCR-MICS), a center of the Swiss National Science Foundation under grant number 5005-67322.

In these cryptanalysis results, the rebound attack technique needs to be refined and adapted to each case, but all of them follow the same scheme: first find a differential path, then find solutions verifying this differential path. This paper focuses on optimizing the latter part. In all the previously mentioned cryptanalysis, that part involves enumerating, from a very large set of possible candidates represented as a cross product of lists, all those that verify a given relation. We call this operation *"merging" the lists.* The merging problem can be described more formally as follows.

Merging problem with respect to t: Let t be a Boolean function taking N k-bit words as input, *i.e.* $t : (\{0,1\}^k)^N \rightarrow \{0,1\}$. Let $L_1, \ldots, L_N$ be N given lists of k-bit words drawn uniformly and independently at random from $\{0,1\}^k$. We assume that the probability over all N-tuples X in $L_1 \times \ldots \times L_N$ that $t(X) = 1$ is P_t. For any given function t and any given N-tuple of lists $(L_1, \ldots, L_N)$ the *merging problem* consists in finding the list $\mathcal{L}_{sol}$ of all $X \in L_1 \times \ldots \times L_N$ satisfying $t(X) = 1$. We call this operation *merging* the lists $L_1, \ldots, L_N$ to obtain $\mathcal{L}_{sol}$.

It is assumed that the image of a given input under t can be easily computed. In the following, the size of a list L is denoted by $|L|$. A brute force method for solving this problem therefore consists in enumerating all the $|L_1| \times \ldots \times |L_N|$ inputs, in computing t on all of them and in keeping the ones verifying $t = 1$. Note that, in the lack of any additional information on t, it is theoretically impossible to do better. However, in practice, the function t often has a set of properties which can be exploited to optimize this approach. We aim at reducing the number of candidates which have to be examined, in some cases by a preliminary sieving similar to the one used in [5]. This paper presents such optimization techniques, that, when applied to most of the rebound attacks published on the SHA-3 candidates, yield significant improvements in the overall time and/or memory complexities of the attack, as shown on Table 1. In this table we can see that we have considered the best existing attacks against four hash functions and the best rebound attack on a fifth (two of them are finalists and two are second-round candidates of the SHA-3 competition), where by best attack we denote the one on the highest number of rounds. We have been able to improve their complexities by scrutinizing the original attack and finding a more efficient algorithm for obtaining the solutions for the differential path. Most of the time the improvement relies on a better *merging* of the lists, and sometimes it is due to the use of more adequate conditions in the general algorithm. Let us recall here that the aim is to find *all* the N-tuples that verify $t = 1$ for a complex function t, which is significantly different from finding just *one* (or few) of them for a linear t such as in [19, 18, 6, 3]. As in the previous rebound analysis, we will work throughout the paper with average values in the probabilistic cases.

In Section 2, we define *Problem 1* that corresponds to functions t with a particular form, and we propose three generic algorithms to solve it. These 3 algorithms have different optimal scenarios. Some examples of applications are given. In Section 3 we define *Problem 2* and propose the *stop-in-the-middle algorithms* for solving it. We also present two concrete algorithms in this family applied to the scenarios of ECHO and LANE. In Section 4 we show

how applying these algorithms combined with an appropriate definition and decomposition of the problem in each case, allows us to improve the complexities of the best known rebound attacks on 5 SHA-3 candidates.

Table 1. Improvements on best known attacks. The highlighted values are the improved complexities. For Luffa we consider the best known rebound attack where the complexities presented in the second row have already been obtained in [10] by a dedicated algorithm similar to our general approach.

Hash function	SHA3 Round	Best Known Analysis	Rounds / total	Previous Time	Previous Memory	Ref.	This paper Time	This paper Memory
JH	Final	semi-free-start coll.	16 / 42	2^{190}	2^{104}	[16]	$\mathbf{2^{97}}$	$\mathbf{2^{97}}$
JH		semi-free-start near coll.	22 / 42	2^{168}	$2^{143.70}$	[16]	$\mathbf{2^{96}}$	$\mathbf{2^{96}}$
Grøstl-256	Final*	(compr. function property)	10 / 10	2^{192}	2^{64}	[15]	$\mathbf{2^{182}}$	2^{64}
Grøstl-256		(internal permutation dist.)	10 / 10	2^{192}	2^{64}	[15]	$\mathbf{2^{175}}$	2^{64}
Grøstl-512		(compr. function property)	11 / 14	2^{640}	2^{64}	[15]	$\mathbf{2^{630}}$	2^{64}
ECHO-256	2nd	internal permutation dist.	8 / 8	2^{182}	2^{37}	[17]	$\mathbf{2^{151}}$	2^{67}
Luffa	2nd	semi-free-start coll.	7 / 8	2^{132}	$2^{68.8}$	[10]	$\mathbf{2^{112.9}}$	$2^{68.8}$
							$\mathbf{(2^{104})}$	(2^{102})
LANE-256	1st	semi-free-start coll.	6+3 / 6+3	2^{96}	2^{88}	[12]	$\mathbf{2^{80}}$	$\mathbf{2^{66}}$
LANE-512		semi-free-start coll.	8+4 / 8+4	2^{224}	2^{128}	[12]	2^{224}	$\mathbf{2^{66}}$

* The Grøstl analysis does not apply after the final round tweak.

Besides the results in Table 1, the main interest of this paper is to present a general framework for improving rebound attacks. We introduce several new algorithms that considerably improve the overall effectiveness when the attack needs to *merge* large lists. We provide a formal definition of the field of application of those algorithms, and describe them as a set of constraints on t, in hope that designers of rebound attacks will be able to easily *identify* scenarios where one of these algorithms, or variants, may be applied. This was motivated by our own research path, when we realized that a generalization of the techniques leveraged in specific cases allowed us to find similar improvements in almost all of the rebound attacks that we have studied so far.

2 When t is Group-Wise

In some cases we can considerably reduce the complexity of the *merging* problem by redefining it into a more concrete one. We consider here a very common case that will appear in many rebound scenarios, as we will later show with the examples. This case corresponds to a function t that can be decomposed in smaller functions. After introducing the general problem, we will illustrate it with an example. Though we preferred to state the problem in full generality for any possible N, in the concrete rebound examples that we studied, the number of lists N was either 2, 4 or 6. Also, the elements of each list can be decomposed in sets of small size s, where s is typically the size of the involved Sbox; and z is the number of such sets involved [1] in the function t.

Problem 1: Let $L_1, \dots, L_N$ be N lists of size $2^{l_1}, \dots, 2^{l_N}$ respectively, where the elements are drawn uniformly and independently at random from $\{0,1\}^k$.
Let t be a Boolean function, $t : \left(\{0,1\}^k\right)^N \rightarrow \{0,1\}$ for which there exists $N' < N$, an integer z and some triples of functions $t_j : \{0,1\}^{2s} \rightarrow \{0,1\}$, $f_j : (\{0,1\}^k)^{N'} \rightarrow \{0,1\}^s$ and $f'_j : (\{0,1\}^k)^{(N-N')} \rightarrow \{0,1\}^s$ for $j = 1, \dots, z$ such that, $\forall\, (\boldsymbol{x_1}, \dots, \boldsymbol{x_N}) \in L_1 \times \dots \times L_N$:

$$t(\boldsymbol{x_1}, \dots, \boldsymbol{x_N}) = 1 \quad \Leftrightarrow \quad \begin{array}{l} \forall j = 1, \dots, z, \\ \begin{cases} t_j(v_j, v'_j) = 1 \\ \text{with } v_j = f_j(\boldsymbol{x_1}, \dots, \boldsymbol{x_{N'}}) \\ \text{and } \; v'_j = f'_j(\boldsymbol{x_{N'+1}}, \dots, \boldsymbol{x_N}) \end{cases} \end{array}$$

Let P_t be the probability that $t = 1$ for a random input.
Problem 1 consists in *merging* these N lists to obtain the set $\mathcal{L}_{sol}$, of size $P_t 2^{\sum_{i=1}^{N} l_i}$, of all N-tuples of $(L_1 \times \dots \times L_N)$ verifying $t = 1$.

Reduction from N to 2: For any $N \geq 2$ *Problem 1* can be reduced to an equivalent and simplified problem with $N = 2$, *i.e.* merging two lists L_A and L_B, which consist of elements in $(\{0,1\}^s)^z$ corresponding to $\boldsymbol{x_A} = \boldsymbol{v} = (v_1, \dots, v_z)$ and $\boldsymbol{x_B} = \boldsymbol{v'} = (v'_1, \dots, v'_z)$, with respect to the function $\boldsymbol{x_A}, \boldsymbol{x_B} \mapsto \Pi_{j=1}^{z} t_j(v_j, v'_j)$. The reduction is performed as follows:

1. Build a table T_A^* of size $2^{\sum_{i=1}^{N'} l_i}$ storing each element $\boldsymbol{e_A} = (\boldsymbol{x_1}, \dots, \boldsymbol{x_{N'}})$ of $L_1 \times \dots \times L_{N'}$, indexed[2] by the value of $(f_1(\boldsymbol{e_A}), \dots, f_z(\boldsymbol{e_A}))$, *i.e.* $(v_1, \dots, v_z)$. Store the corresponding $(v_1, \dots, v_z)$ in a list L_A. Note that several $\boldsymbol{e_A}$ may lead to the same value of $(v_1, \dots, v_z)$.
2. Build a similar table T_B^* of size $2^{\sum_{i=N'+1}^{N} l_i}$ storing each element $\boldsymbol{e_B} = (\boldsymbol{x_{N'+1}}, \dots, \boldsymbol{x_N})$ of $L_{N'+1} \times \dots \times L_N$, indexed by $(f_1(\boldsymbol{e_B}), \dots, f_z(\boldsymbol{e_B}))$, *i.e.* $(v'_1, \dots, v'_z)$. Store $(v'_1, \dots, v'_z)$ in a list L_B.
3. *Merge* L_A and L_B with respect to $\Pi_{j=1}^{z} t_j$ and obtain $\mathcal{L}_{sol}$.
4. Build $\mathcal{L}_{sol}^*$ by iterating over each pair $((v_1, \dots, v_z), (v'_1, \dots, v'_z))$ of $\mathcal{L}_{sol}$, and adding the set of all $(\boldsymbol{x_1}, \dots, \boldsymbol{x_{N'}}, \boldsymbol{x_{N+1}}, \dots, \boldsymbol{x_N}) \in T_A^*\left[(v_1, \dots, v_z)\right] \times T_B^*\left[(v'_1, \dots, v'_z)\right]$. $\mathcal{L}_{sol}^*$ is the solution to the original problem.

[1] Sometimes, elements are only partially involved in t.

[2] Here and in the following sections we can use standard hash tables for storage and lookup in constant time, since the keys are integers.

Let $2^{T_{\text{merge}}}, 2^{M_{\text{merge}}}$ be the time and memory complexities of step 3. The total time complexity of solving *Problem 1* is $\mathcal{O}(sz2^{\sum_{i=1}^{N'} l_i} + sz2^{\sum_{i=N'+1}^{N} l_i} + 2^{T_{\text{merge}}} + P_t 2^{\sum_{i=1}^{N} l_i})$ where the last term comes from the fact that only the N-tuples satisfying $t = 1$ are examined at step 4 because of the sieve applied at step 3. The proportion of such tuples is then P_t. The memory complexity [3] is $\mathcal{O}((zs + N'k)2^{\sum_{i=1}^{N'} l_i} + (zs + (N - N')k)2^{\sum_{i=N'+1}^{N} l_i} + 2^{M_{\text{merge}}} + P_t 2^{\sum_{i=1}^{N} l_i})$, where the last term appears only when the solutions must be stored.

Using the brute force approach, $2^{T_{\text{merge}}}$ would be $2^{l_A + l_B}$ where 2^{l_A} (respectively 2^{l_B}) denotes the size of L_A (L_B), and $2^{M_{\text{merge}}}$ would be negligible. We present in the following sections some algorithms for solving *Problem 1* considering $N = 2$ with L_A and L_B, that provide better complexities than the brute force approach. Note that the roles of L_A and L_B are assigned by choice to obtain the best overall complexity. Those algorithms can be applied for obtaining a smaller $2^{T_{\text{merge}}}$ when $N > 2$.

2.1 Basic Algorithm for Solving *Problem 1*: Instant Matching

As s is typically very small we can enumerate the solutions (v_j, v'_j) of $t_j(v_j, v'_j) = 1$ and store them in tables T_j of size $\leq 2^{2s}$, indexed by v'_j. This costs $\mathcal{O}(z \cdot 2^{2s})$ in time and memory. We propose in Fig. 1 a first algorithm for solving *Problem 1*, which has lower complexity than the brute-force approach. Although being the simplest algorithm presented in this paper, it has not been applied in critical steps of some of the previously mentioned attacks, though it could yield significant improvements.

Fig. 1 Instant matching algorithm.

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Require: Two lists $L_A, L_B$ and a Boolean function $t$ as described in Problem 1.
Ensure: The returned list $\mathcal{L}_{sol}$ will contain all elements of $L_A \times L_B$ verifying $t$.
 1: for $j$ from 1 to $z$ do
 2:     for all $(v_j, v'_j)$ in $\{0,1\}^s \times \{0,1\}^s$ do
 3:         if $t_j(v_j, v'_j) = 1$, then add $v_j$ to $T_j[v'_j]$.
 4: for each $(v'_1, \ldots, v'_z) \in L_B$ do
 5:     Empty $L_{aux}$.
 6:     for $j$ from 1 to $z$ do
 7:         if $T_j[v'_j]$ is empty, then go to 4.
 8:     Add all tuples $(v_1, \ldots, v_z)$ verifying $\forall j \; v_j \in T_j[v'_j]$ to $L_{aux}$.
 9:     for each $(v_1, \ldots, v_z)$ in $L_{aux}$ do
10:         if $(v_1, \ldots, v_z) \in L_A$ then
11:             Add $(v_1, \ldots, v_z, v'_1, \ldots, v'_z)$ to $\mathcal{L}_{sol}$.
12: Return $\mathcal{L}_{sol}$.
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Let 2^{-p_j} be the probability over all pairs (v_j, v'_j) that $t_j(v_j, v'_j) = 1$. The relationship between t and the $(t_j)_{1 \leq j \leq z}$ implies that $\sum_{j=1}^{z} p_j = -\log_2(P_t)$ where P_t is the probability

[3] The first two terms, corresponding to the storage of T_A^* and T_B^* could be avoided if they were the bottleneck by slightly increasing the time complexity by a factor of 2.

that $t = 1$.
Let us determine the average size of L_{aux}. The average size of $T_j[v'_j]$ over all v'_j is 2^{s-p_j}. Then the average size of L_{aux} is $2^{zs-\sum_{j=1}^{z} p_j} = P_t 2^{zs}$. It follows that the time complexity of the algorithm is $\mathcal{O}(z2^s + zP_t 2^{l_B+zs})$ and is proportional to the product of the size of L_B by the average size of [4] L_{aux}. The memory complexity is $\mathcal{O}(z2^s + 2^{l_A} + 2^{l_B} + P_t 2^{l_A+l_B})$. In some cases, the last term can disappear, namely if we do not need to store the list $\mathcal{L}_{sol}$, but just use each solution as soon as it is obtained. The same way, the list L_B does not need to be stored, if it can be given on the fly.

We now describe a concrete example of application of the *instant-matching algorithm* in a case included in a particular rebound attack, improving its complexity. In Appendix A we provide two more examples where it clearly appears that identifying and isolating the most appropriate problem (or problems) to solve is of major importance. These two last examples might help also to understand the role of f_j and f'_j.

Example 1: Application of the Instant Matching Algorithm We use here a case presented in the analysis of JH [16] which is the attack on 8 rounds using one inbound when the dimension of a block of bits denoted by d is 4. Here we improve step 3 of the attack, which is also the bottleneck in time complexity. Two lists are given, L_A and L_B of size $2^{24.18}$ elements each. The aim of step 3 is to *merge* those lists, *i.e.* find all pairs $(\boldsymbol{v}, \boldsymbol{v'}) \in L_A \times \mathrm{L}_B$ verifying 10 conditions on groups of $s = 4$ bits of $(\boldsymbol{v}, \boldsymbol{v'})$.
In [16] this is solved by exhaustive search, *i.e.* all possible pairs are examined and only the ones that verify the 10 conditions are kept, which has cost $2^{48.36}$. We can improve this complexity by applying the instant-matching algorithm: first, we notice that 6 out of these 10 conditions can be written as

$$t_j(v_j, v'_j) = 1, \forall j \in \{1, \ldots, 6\},$$

where variables v_j and v'_j represent groups of differences of 4 bits. The functions t_j return 1 when the linear function of JH, L, applied to v_j and v'_j produces 4 bits out of 8 without difference in the wanted positions. Those functions t_j can be computed directly by using a precomputed table of size 2^8.
This is an instance of *Problem 1* with the parameters: $z = 6$ (corresponding to the number of relations $t_1, \ldots, t_6$), and $p_j = 3.91\ \forall j$. Hence $P_t 2^{zs} = 2^{0.09 \cdot 6} = 2^{0.54} \simeq 1.45$. The instant-matching algorithm allows us to find all pairs satisfying these 6 conditions with a complexity of $2^{27.8}$ in time and no additional memory. We then obtain $2^{24.9}$ pairs of elements that pass the first 6 conditions. To complete step 3 of the attack, we evaluate the 4 remaining conditions for each pair, for a global complexity of $2^{24.9}$.
To summarize, we were able to resolve step 3 of the attack with a time complexity of about $2^{27.8}$, improving significantly the complexity of $2^{48.36}$ given in [16].

[4]The cost of building and storing the lists $T_j[v'_j]$ is negligible.

2.2 Solving Problem 1 when $P_t 2^{zs} > 2^{l_A}$: Gradual Matching

In Fig. 2 we present an algorithm for solving *Problem 1* that is useful in cases where the average size of L_{aux} exceeds the size of L_A, *i.e.*[5] $P_t 2^{zs} > 2^{l_A}$. In this case the instant-matching algorithm has a higher complexity than the exhaustive search. This is why here, instead of directly matching the z groups that appear in relation t, we will first match the $z' < z$ ones, and next, the $z - z'$ remaining ones. We present here how to use one step of the *gradual-matching algorithm* for solving Problem 1. This algorithm reminds the method used in Example 1 where the problem is first solved with only 6 relations. But the difference is that the remaining $z - z'$ relations can also be written in the form needed for *Problem 1* and $P_t 2^{zs} > 2^{l_A}$. Let us suppose that we choose z' so that $z's < l_A$ (the best value for z' depends on the situation).

Fig. 2 Gradual matching algorithm.

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Require: Two lists $L_A$ and $L_B$ and a function $t$ as described in Problem 1.
Ensure: List $\mathcal{L}_{sol} \subset L_A \times L_B$ of all elements verifying $t$.
 1: for $j$ from 1 to $z$ do
 2:   for all $(v_j, v'_j)$ in $\{0,1\}^s \times \{0,1\}^s$ do
 3:     if $t_j(v_j, v'_j) = 1$, then add $v_j$ to $T_j[v'_j]$.
 4: for each $\alpha = (\alpha_1, \dots, \alpha_{z'})$ in $(\{0,1\}^s)^{z'}$ do
 5:   Empty $L_{aux}$.
 6:   Consider the sublist $L_B(\alpha)$ of all elements in $L_B$ with $(v'_1, \dots, v'_{z'}) = \alpha$.
 7:   for each $(v_1, \dots, v_{z'})$ in $T_1[\alpha_1] \times \dots \times T_{z'}[\alpha_{z'}]$ do
 8:     add $(v_1, \dots, v_{z'})$ to $L_{aux}$.
 9:   for each $\gamma = (\gamma_1, \dots, \gamma_{z'})$ in $L_{aux}$ do
10:     Consider the sublist $L_A(\gamma)$ of all elements of $L_A$ with $(v_1, \dots, v_{z'}) = \gamma$.
11:     Merge $L_A(\gamma)$ with $L_B(\alpha)$ with respect to $t' = \Pi_{j=z'+1}^{z} t_j$.
12:     Add the solutions to $\mathcal{L}_{sol}$.
13: Return $\mathcal{L}_{sol}$, containing about $P_t 2^{l_A + l_B}$ elements.
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Let 2^{merge} be the time complexity of *merging* once lists $L_B(\alpha)$ and $L_A(\gamma)$ as defined in Fig. 2. Since their respective average sizes are $2^{l_A - z's}$ and $2^{l_B - z's}$ the complexity of the brute force is $2^{l_A + l_B - 2z's}$. It can be improved by using one of the proposed algorithms from this section but it cannot be smaller than the size of the resulting merged list, *i.e.* $2^{l_A + l_B - 2z's - \sum_{j=z'+1}^{z} p_j}$. Now the average size of L_{aux} [6] is $\mathcal{S} = 2^{z's - \sum_{j=1}^{z'} p_j}$. Then, the time complexity of this algorithm is $\mathcal{O}(z2^s + 2^{z's}(z' + \mathcal{S}2^{\text{merge}}))$. It is worth noticing that this complexity corresponds to $z'2^{z's} + 2^{l_A + l_B - \sum_{j=1}^{z'} p_j}$ when the intermediate lists are merged by the brute force algorithm and to $z'2^{z's} + P_t 2^{l_A + l_B}$ if they are merged by an optimal algorithm. The memory complexity is $\mathcal{O}(z2^s + 2^{l_A} + 2^{l_B} + \mathcal{S} + P_t 2^{l_A + l_B})$. Again, in some

[5]When $P_t 2^{zs}$ is close to 2^{l_A} this algorithm might also outperform the instant-matching technique.

[6]Here and in the previous section, there is no need for storing L_{aux}, as each element can be treated as soon as it is obtained, but these auxiliary lists are very useful for describing the complexities.

cases, the last term can disappear, if we do not need to store the list $\mathcal{L}_{sol}$, but just use the solutions on the fly.

2.3 Time-Memory Trade-Offs when $P_t 2^{zs} > 2^{l_A}$: Parallel Matching

The *parallel-matching algorithm* improves the time complexity of the gradual-matching by a time-memory trade-off and can be applied in the same situations. It is a generalization of an algorithm proposed in [10]. As the gradual-matching algorithm this algorithm first finds elements that verify $t_j = 1$ for $j \in \{1, \ldots, z'\}$ and then, for each of them, it checks if the remaining $(z - z')$ relations are also verified. However, in this algorithm, the matching of the z' relations is done in parallel for n and m relations, so that $z' = m+n$. The motivation of choosing different variables for n and m is showing that there is no need for them to be the same when applying the algorithm.

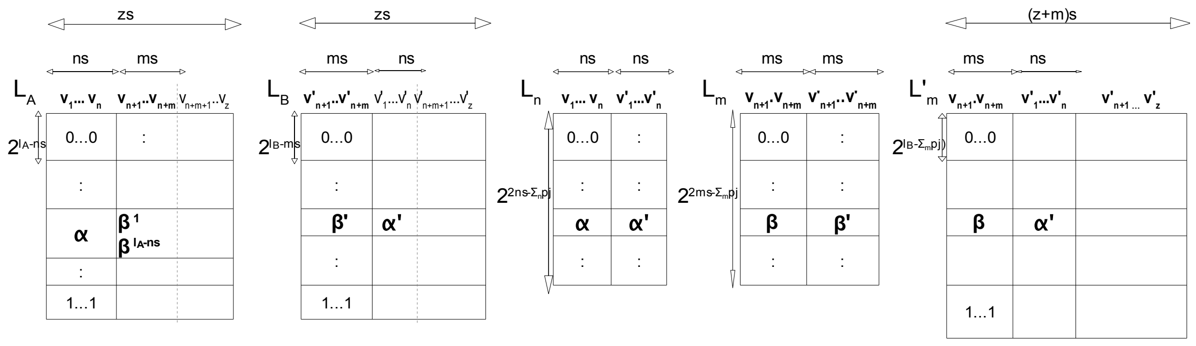

Fig. 3: Representation of the parallel-matching algorithm.

We choose n so that $n < z$, $ns < l_A$ and $ns < l_B$, and in the same way, we choose m $(n + m = z' \leq z)$. This algorithm will be explained with ordered lists, as it is more graphical and helps the understanding. However, since we can perform it with hash tables indexed by the values we want to have ordered, we do not need to take into account the logarithmic terms for ordering and searching in the final complexity. First we build the lists that we will use and that are represented in Fig. 3:

- We order the list L_A by the first n groups $(v_1, \ldots, v_n)$. L_A has $2^{l_A - sn}$ elements in average corresponding to a given value of these n groups.
- We order the list L_B by the next m groups $(v'_{n+1}, \ldots, v'_{n+m})$. L_B has $2^{l_B - sm}$ elements in average corresponding to a given value of these m groups.

- We build the list L_n of size $2^{2ns-\sum_{j=1}^{n} p_j}$ formed by all $(v_1, \ldots, v_n, v'_1, \ldots, v'_n)$ with $v_j \in T_j[v'_j]$ for all $1 \leq j \leq n$. All the elements from this list satisfy $t_j(v_j, v'_j) = 1$ for $j \in [1, \ldots, n]$.
- We build the list L_m of size $2^{2ms-\sum_{j=n+1}^{n+m} p_j}$ formed by all $(v_{n+1}, \ldots, v_{n+m}, v'_{n+1}, \ldots, v'_{n+m})$ with $v_j \in T_j[v'_j]$ for all $(n+1) \leq j \leq (n+m)$. All the elements from this list satisfy $t_j(v_j, v'_j) = 1$ for $j \in [n+1, \ldots, n+m]$.
- From L_m and L_B we build L'_m as follows: for each (β, β') in L_m, we add to L'_m all elements $(\beta, v'_1, \ldots, v'_z)$ of L_B such that $(v'_{n+1}, \ldots, v'_{n+m}) = \beta'$ and we store them ordered by the values of $(\beta, v'_1, \ldots, v'_n)$. The average size of L'_m is $2^{l_B+sm-\sum_{j=n+1}^{n+m} p_j}$. Then we perform the algorithm given in Fig. 4.

Fig. 4 Parallel matching algorithm.

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1: for each $(\alpha, \alpha')$ in $L_n$ do
2:   for each $(v_1, \ldots, v_z)$ in $L_A$ with $(v_1, \ldots, v_n) = \alpha$ do
3:     if $L'_m$ contains any element $(v_{n+1}, \ldots, v_{n+m}, v'_1, \ldots, v'_z)$ starting by $(v_{n+1}, \ldots, v_{n+m}, \alpha')$ then
4:       if $(v_1, \ldots, v_z, v'_1, \ldots, v'_z)$ satisfies the remaning $(z-n-m)$ conditions then
5:         Add $(v_1, \ldots, v_z, v'_1, \ldots, v'_z)$ to $\mathcal{L}_{sol}$.
6: Return $\mathcal{L}_{sol}$ containing about $P_t 2^{l_A+l_B}$ elements.
```

As already mentioned, the respective average sizes of L_n and L_m are $2^{l_n} = 2^{2ns-\sum_{j=1}^{n} p_j}$ and $2^{l_m} = 2^{2ms-\sum_{j=n+1}^{n+m} p_j}$, and the average size of L'_m is $2^{l_B+ms-\sum_{j=n+1}^{n+m} p_j}$. In total we will find the $2^{l_A+l_B-\sum_{j=1}^{z} p_j}$ matches that exist, with a complexity in time $\mathcal{O}(2^{l_n} + 2^{l_m} + 2^{l_A+l_B-\sum_{j=1}^{n+m} p_j} + 2^{l_A+ns-\sum_{j=1}^{n} p_j} + 2^{l_B+ms-\sum_{j=n+1}^{m} p_j})$ and $\mathcal{O}(2^{l_n} + 2^{l_m} + 2^{l_B} + 2^{l_B+ms-\sum_{j=n+1}^{m} p_j} + 2^{l_A+l_B-\sum_{j=1}^{z} p_j})$ in memory, where the last term corresponds to the storage of all solutions, not always needed. In this case, the storage of L_A is not necessary.

2.4 Example 2: Gradual Matching vs Parallel Matching

We are going to apply both previous algorithms to the analysis of Luffa presented in [10]. We are given two lists L_A and L_B of size 2^{67} and $2^{65.6}$. These lists contain elements formed by $z = 52$ groups of differences of $s = 4$ bits. List L_A contains the possible differences for the input of 52 Sboxes. List L_B contains the possible differences for the output of the same 52 Sboxes. For the j-th Sbox, the probability that one input difference can be associated to one output difference is $2^{-p_j} = 2^{-1.23}$. The average size of L_{aux} if we apply the instant-matching algorithm is then $P_t 2^{zs} = 2^{144.04}$. In this case t can be decomposed in 52 t_j, one per Sbox. So $t_j(v_j, v'_j) = 1$ if there exists $x \in \{0,1\}^s$ such that

$$\text{Sbox}(x) \oplus \text{Sbox}(x \oplus v_j) = v'_j.$$

The brute force algorithm for solving this problem has a time complexity $2^{65.6+67} = 2^{132.6}$ and a memory complexity of $2^{68.8}$. If we apply the gradual-matching algorithm with $z' = 16$

we have $\mathcal{S} = 2^{44.32}$, and we obtain the $2^{68.8}$ solutions with a time complexity of $2^{112.9}$ and the same memory as before as no additional memory is needed. If instead we apply the parallel-matching algorithm with $m = n = 13$, we can obtain the solutions with a time complexity of 2^{104} and a memory complexity of 2^{102}. Different choices of parameters allow many other time-memory trade-offs, but we just show here the one that provides the lowest time complexity, and so the highest memory needs, for contrast with the gradual matching algorithm.

3 Stop-in-the-Middle Algorithms

In this section we present another case that allows to reduce the complexity of solving the basic problem. It is described in *Problem 2*. Then, we describe the main lines of the *stop-in-the-middle algorithms*, that we use for solving *Problem 2*. Next, we present such an algorithm that solves *Problem 2* in the scenario of LANE-256. Then a more complex variant of this algorithm is applied to a ECHO-256 scenario. But we believe that, in particular, this kind of algorithms can be adapted and applied to functions that use several AES (like) states in parallel which are then merged at the end of each round. In the following, we consider a permutation F from $\{0,1\}^{sk}$ to $\{0,1\}^{sk}$ and we assume that there exist a decomposition function ϕ (respectively ψ) of the input of F (respectively the output) in k elements of $\{0,1\}^s$. These two decompositions may be different. Then, instead of the original function F we will now focus on the function $f = \psi \circ\ F \circ \phi^{-1}$ which is a function over $(\{0,1\}^s)^k$ (see Fig. 5). In the following (u, w) denotes the word corresponding to the concatenation of the vectors u and w.

Problem 2: Let z_A and z_B be two integers less than or equal to k . Let L_A be a list of elements in $(\{0,1\}^s)^{z_A}$ and L_B be a list of elements on $(\{0,1\}^s)^{z_B}$. The *Problem 2* consists on finding all triples (a, b, c) with $a \in L_A$, $b \in L_B$ and $c \in L_C = (\{0,1\}^s)^k$ such that

$$f(c) \oplus f(c \oplus (a, 0^{s(k-z_A)})) = (b, 0^{s(k-z_B)}),$$

where there exists the function $F_1 : (\{0,1\}^s)^k \to (\{0,1\}^s)^k$ and some permutations of $\{0,1\}^s$, $g_1, \ldots, g_k$ and $h_1, \ldots, h_k$ over $\{0,1\}^s$ such that

$$f = H \circ F_1 \circ G$$

where

$$\begin{aligned} G : \qquad & (\{0,1\}^s)^k \to (\{0,1\}^s)^k \\ & (x_1, \ldots, x_k) \to (g_1(x_1), \ldots, g_k(x_k)) \end{aligned}$$

and

$$\begin{aligned} H : \qquad & (\{0,1\}^s)^k \to (\{0,1\}^s)^k \\ & (x_1, \ldots, x_k) \to (h_1(x_1), \ldots, h_k(x_k)) \end{aligned}$$

It is worth noting that we assume that both decompossitions ϕ and ψ have been chosen in an appropriate way such that the z_A words of a (respectively the z_B words of b) correspond to the first words of the input state (respectively of the output state). We call

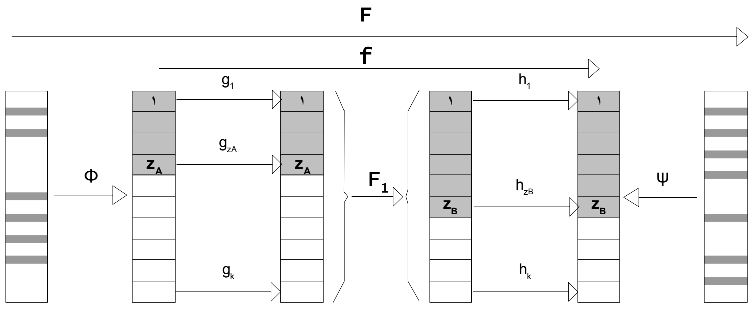


Fig. 5: Representation of F from *Problem 2*.

stop-in-the-middle algorithms those that solve *Problem 2* following the main general scheme described in Fig. 6. The complexities associated depend on the particular form of F_1, as we show in the next sections.

Fig. 6 General scheme of stop-in-the-middle algorithms.

```
1: for each $b$ in $L_B$ do
2:     for each $j \in [1, \ldots, z_B]$ do
3:         for each $y_j \in \{0,1\}^s$ do
4:             add $(h_j^{-1}(y_j), h_j^{-1}(y_j) \oplus h_j^{-1}(y_j \oplus b_j))$ to $L_{j,b}$.
5: for each $a$ in $L_A$ do
6:     for each $i \in [1, \ldots, z_A]$ do
7:         for each $x_i$ in $\{0,1\}^s$ do
8:             add $(g_i(x_i), g_i(x_i) \oplus g_i(x_i \oplus a_i))$ to $L_i$.
9:     Using the previous lists $L_i$ and $L_{j,b}$, match in the middle using $F_1$, i.e. construct the list
       $L_{aux} = \{(x, b_1, \ldots, b_{z_B}), x \in (\{0,1\}^s)^k\}$ such that
       $((F_1[g_1(x_1), \ldots, g_{z_A}(x_{z_A}), x^*)], F_1[g_1(x_1 \oplus a_1), \ldots, g_{z_A}(x_{z_A} \oplus a_{z_A}), x^*)]) =$
       $\big((h_1^{-1}(y_1), \ldots, h_{z_B}^{-1}(y_{z_B}), y^*), (h_1^{-1}(y_1 \oplus b_1), \ldots, h_{z_B}^{-1}(y_{z_B} \oplus b_{z_B}), y^*)\big)$
       for some $x^* \in (\{0,1\}^s)^{k-z_A}$ and $y^* \in (\{0,1\}^s)^{k-z_B}$.
10:    for all $(x, b_1, \ldots, b_{z_B})$ in $L_{aux}$ do
11:        if $b = (b_1, \ldots, b_{z_B}) \in L_B$ then
12:            add $(a, b, x)$ to $\mathcal{L}_{sol}$.
13: Return $\mathcal{L}_{sol}$.
```

In the cases we have studied and that we detail below, the function f is formed by several AES transformations in parallel. We then expect $2^{l_A+l_B}$ solutions, as for each $a \in L_A$ and each $b \in L_B$ there exists one $c \in L_C$ so that the condition of *Problem 2* holds. The match-in-the-middle step is assumed to be simple due to the simple form of F_1 (typical functions F_1 are linear diffusion layers). For the same reason, L_{aux} can typically be written in a compact way, for example, in several independent lists.

3.1 Algorithm for Lane-256

Each lane of the internal state of Lane-256 is composed of two AES states. An AES state is a state of size 128 bits that can be seen as a 4×4 matrix of bytes. The AES transformations are noted: SB for SubBytes, SR for ShiftRows and MC for MixColumn. The transformation SC mixes the two AES states at the end of each round by interchanging their columns. We consider Fig. 7 that represents a part of the differential path used in [12]. In that attack it was treated as the merging of two inbounds and 2^{64} solutions were found with a complexity of 2^{96} in time and 2^{88} in memory. We consider the scheme represented in Fig. 7 where we have swapped lines and columns for a more easy intuitive understanding (so SR is applied to the columns and MC is applied to the lines).

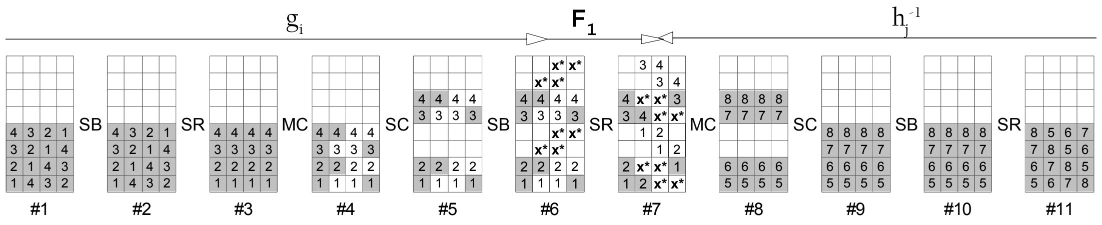


Fig. 7: Differential path associated to the first improvement on the Lane analysis.

Using the example from [12], $l_A = 32$ and $l_B = 32$ and L_C is the list of all possible input values and needs to be neither stored nor computed. We consider that the input state (respectively the output state) of the function f presented in Fig. 7 is decomposed into eight 32-bit words (*i.e.* $s = 32$ and $k = 8$). The input differences and output differences that we consider in L_A and L_B correspond to the first $z_A = z_B = 4$ 32-bit words of the state. In Fig. 7 each one of the $4 + 4 = 8$ 32-bits active word corresponds to the four active bytes with the same number written on them (1 to 4 for the four active input words and 5 to 8 for the 4 active output words).

With the algorithm described in Fig. 8 we find the 2^{64} solutions with a complexity of 2^{66} in time and 2^{65} in memory. The time complexity associated to the studied path is $z_B 2^{l_B+32} + 2^{l_A+32}$. This comes from the fact that each L_i has average size 2^{16}. Then, $L_{5,6}$ and $L_{7,8}$ have size 2^{l_B+32}. Then the size of both L^0_{aux} and L^1_{aux} is 2^{l_B} since in each we keep

Fig. 8 Algorithm for solving two inbounds of LANE-256.

```
Require: Function $f$ and lists $L_A$ and $L_B$ of differences in #1 and #11 respectively.
Ensure: List $\mathcal{L}_{sol} = \{(a, b, c)$ such that $f(c \oplus (a, 0^{s(k-z_A)})) \oplus f(c) = (b, 0^{s(k-z_B)})\}$.
 1: for each $b$ in $L_B$ do
 2:   for $i$ from 5 to 8 do
 3:     for each $y \in \{0,1\}^{32}$ do
 4:       if $h_i^{-1}(y) \oplus h_i^{-1}(y \oplus b_i)$ has only the two wanted bytes active (see #7 of Fig. 7) then
 5:         Store $(y, b_i, h_i^{-1}(y), h_i^{-1}(y \oplus b_i))$ in $L_i$, where the last two terms are truncated to the 2 active
            bytes.
 6:   for each $(y_5, b_5, u_5, w_5)$ from $L_5$ and $(y_6, b_6, u_6, w_6)$ from $L_6$ do
 7:     Add $(u_5, w_5, u_6, w_6, y_5, y_6, b_5, b_6)$ in $L_{5,6}$ indexed by the values of the $u_5, w_5, u_6, w_6$ operations.
 8:   for each $(y_7, b_7, u_7, w_7)$ from $L_7$ and $(y_8, b_8, u_8, w_8)$ from $L_8$ do
 9:     Add $(u_7, w_7, u_8, w_8, y_7, y_8, b_7, b_8)$ in $L_{7,8}$ indexed by the values of the $u_7, w_7, u_8, w_8$ operations.
10:   Empty $L_5$, $L_6$, $L_7$ and $L_8$.
11: for each $a$ in $L_A$ do
12:   for $i$ from 1 to 4 do
13:     for each $x_i \in \{0,1\}^{32}$ do
14:       if $g_i(x_i) \oplus g_i(x_i \oplus a_i)$ has only the two wanted bytes active (see #4 of Fig. 7) then
15:         Store $(x_i, g_i(x_i), g_i(x_i \oplus a_i))$ in $L_i$, where the two last terms are truncated to the 2 active
            bytes.
16:   for $i$ from 0 to 1 do
17:     for each $(x_{2i+1}, u_{2i+1}, w_{2i+1})$ in $L_{2i+1}$ and $(x_{2i+2}, u_{2i+2}, w_{2i+2})$ in $L_{2i+2}$ do
18:       if there exists an element in $L_{5+2i,6+2i}$ indexed by $(u_{2i+1}, w_{2i+1}, u_{2i+2}, w_{2i+2})$ then
19:         Add $(x_{2i+1}, x_{2i+2}, b_{5+2i}, b_{6+2i})$ to $L^i_{aux}$ indexed by $(b_{5+2i}, b_{6+2i})$.
20:   for each $(x_1, x_2, b_5, b_6)$ in $L^0_{aux}$ do
21:     for each $(b_7, b_8)$ such that $(b_5, b_6, b_7, b_8) \in L_B$ do
22:       if there exists an element in $L^1_{aux}$ indexed by $(b_7, b_8)$ then
23:         add $(a, (b_5, b_6, b_7, b_8), (x_1, x_2, x_3, x_4))$ to $\mathcal{L}_{sol}$.
24: Return $\mathcal{L}_{sol}$.
```

the pairs of elements that match on 4 active bytes, and this happens with a probability of 2^{-64} (32 values and 32 differences); and the number of possible pairs is $2^{16+16+l_B+32}$. The memory complexity is $2^{l_B+32+1} + 2^{32+1} + 2^{l_A+l_B}$ for obtaining $2^{l_A+l_B}$ solutions. We explain in Section 4.5 how this algorithm allows to considerably reduce the complexity of the LANE-256 semi-free-start collision presented in [12] when applied jointly with other improvements concerning other steps of the attack.

3.2 Algorithm for ECHO-256

An ECHO-256 state is a state of size 2048 bits that can be seen as a 4×4 matrix of AES states. The ECHO operations BigSR, BigMC and BigSB are similar to the AES ones, but they operate on AES states instead of bytes. A SuperSbox is an Sbox defined by SR ∘ SB ∘ MC ∘ SR ∘ SB. Applied on an AES state, it can be seen as a 32×32 Sbox. We define a *SuperSbox set* as each one of the 4 (in the AES state) sets of bits that act as input and output of the SuperSbox. We define a *BigSuperSbox* as an Sbox defined by

BigSR ∘ BigSB ∘ BigMC ∘ BigSR ∘ BigSB. Applied to ECHO it defines 4 sets of size 4 AES-states.

We consider Fig. 9, where each column represents the four AES states that form a BigSuperSbox at a certain state $\#i$, for i from 1 to 13. Each possible differences in #1 in L_A consist of $z_A = 12$ 32-bit words and the possible differences in #13 consist of $z_B = 8$ 32-bit words, where L_B can be written as $L_B = L_{B_1} \times L_{B_2}$ with both L_{B_1} (associated to AES state B_1 in Fig. 9) and L_{B_2} (associated to AES state B_4) are subsets of $(\{0,1\}^{32})^4$ each of size 2^{32} (this is a particular case which has to be adapted in other cases). Finding solutions for this differential path with the previously mentioned conditions is a problem proposed in [17] and was solved in such a way that 2^{32} solutions could be found with a complexity of 2^{128} in time and 2^{37} in memory. We propose here a new algorithm that can solve it for obtaining 2^{64} solutions with the same time complexity and a memory of 2^{67}. Variants of our algorithm can be applied in several cases, like when the transition in #7 to #8 is from 2 active states to 3, or from 1 to 4 or from 4 to 1. Additionally we believe that it can improve the complexity of other future attacks on ECHO-256.

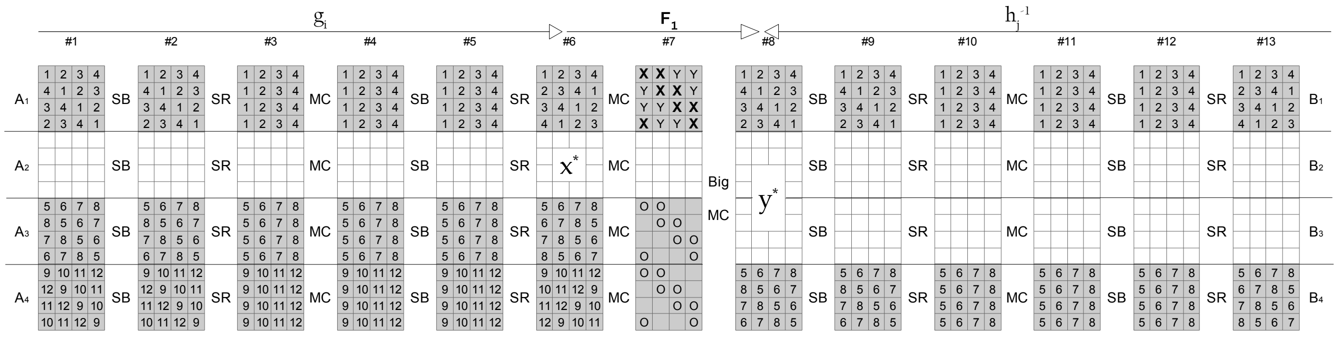


Fig. 9: Differential path on a BigSuperSbox of ECHO-256.

The list L_C contains all the possible values for the input state. This list needs to be neither computed nor stored. Here the aim is to find for each possible (a, b^1, b^2) in $L_A \times L_{B_1} \times L_{B_2}$ the associated c so that $f(c) \oplus f(c \oplus (a, 0^{s(k-z_A)}) = (b^1, b^2, 0^{s(k-z_B)})$. In Fig. 9 we can see how the function f can be written in the way requested by *Problem 2*. We omit the operation BigSR as it does not affect the states, as well as the round keys that are taken into account in the different g_i and h_j. For the sake of simplicity we consider in Fig. 9 the list L_B of possible differences before the last MC of the BigSuperSbox. This can be done by a simple transformation MC^{-1} of the differences in #B' (see Fig. 12). The grey bytes represent the bytes with differences. We can observe that, from #1 to #6 there are $z_A = 12$ independent active SuperSbox sets ($s = 32$), denoted in Fig. 9 by a number from 1 to 12. To each of these groups we can associate a difference from L_A and a value from L_C at state #1 and we can apply independently g_i, $i \in [1, \ldots, 12]$ to obtain the value and the difference of the group in #6. The same way, from #8 to #13 there are $z_B = 8$ independent active SuperSbox sets and the corresponding functions $h_i^{-1}, i \in [1, \ldots 8]$ that link state #13 with state #8. The function F_1 =MC∘BigMC takes a complete internal state in #6 and computes the corresponding state in #8. Let $f(x) = y$, and let $d_i^{\#7}$ the ith active diagonal in state #7. Without knowing the values of x^* nor of y^* represented in Fig. 9 we can still write the following equations that have to be verified, that are obtained from BigMC, and that are used in the algorithm:

$$2 \times d_i^{\#7} \oplus d_{i+4}^{\#7} \oplus d_{i+8}^{\#7} \oplus 9 \times d_i^{\#7} \oplus 3 \times d_{i+4}^{\#7} \oplus 6 \times d_{i+8}^{\#7} = h_i^{-1}(y_i) \oplus 3 \times h_{i+4}^{-1}(y_{i+4}) \qquad (1)$$

for $i \in 1, \ldots, 4$ where the multiplication corresponds to the one in the definition of MC applied independently to each byte of the diagonal.

We consider that the input state (respectively the output state) of the function presented in Fig. 9 is decomposed into sixteen 32-bit words (*i.e.* $s = 32$ and $k = 16$). The input differences (respectively output differences) that we consider in L_A (L_B) correspond to the first $z_A = 12$ ($z_B = 8$) 32-bit words of the state. In Fig. 9 each one of the 12 (respectively 8) 32-bits active word from the input (respectively the output) corresponds to the four active bytes with the same number written on them (1 to 12 for the twelve active input words and 1 to 8 for the eight active output words).

Let V_X (V_Y, V_O respectively) be the values at the positions in #7 marked with an X (Y, O) and Δ_X (Δ_Y, Δ_O) their differences. Let $\Delta_{j'}^{\#r}$ be an auxiliary variable denoting the difference for the SuperSbox set j' in state $\#r$. The algorithm is described in Fig. 10.

So the time complexity is $\mathcal{O}(z_B 2^{l_{B_1}+s} + z_B 2^{l_{B_2}+s} + z_A 2^s + 2^{l_A+64}(2^{l_{B_1}} + 2^{l_{B_2}} + 2^{l_{B_1}} 2^{l_{B_2}} + z_A 2^{64}))$. The memory complexity is $\mathcal{O}(z_B 2^{l_{B_1}+s} + z_B 2^{l_{B_2}+s} + 2^{l_{B_1}+l_{B_2}} + |\mathcal{L}_{sol}|)$. In the case of $l_A = 0$, we will obtain a complexity of 2^{129} in time and 2^{66} in memory for obtaining 2^{64} solutions. This algorithm proposes several trade-offs when changing the values of $|\Delta_X|$, and can be adapted for other forms of L_B.

4 How to improve the best known attacks on five SHA-3 candidates

In this section we first enumerate briefly the main algorithms or ideas that we use to improve the best known attacks on each of the hash functions JH, Grøstl, ECHO, Luffa and Lane as shown on Table 1. Then, we provide more detailed descriptions.

- **JH:** To improve the complexities over the ones in [16] we use the instant-matching (as in Section 2.1) and gradual-matching algorithms as well as the fact that we do not merge the lists until we really have to (to keep lists of smaller sizes, with a smaller complexity).
- **Grøstl:** Instead of the initial lists used in [15], we can define them so that we erase the elements that for sure won't verify the outbound part. Having lists of smaller size translates to a smaller complexity.

Fig. 10 Algorithm for finding solutions for one ECHO BigSuperSbox.

```
Require: Function $f$, list $L_A$ of differences in #1 and lists $L_{B_1}$ and $L_{B_2}$ of differences in #13.
Ensure: List $\mathcal{L}_{sol} = \{(a, b^1, b^2, c),$ such that $f(c) \oplus f(c \oplus (a, 0^{s(k-z_A)})) = (b^1, b^2, 0^{s(k-z_B)})\}$ .
 1: for $j$ from 1 to 4 do
 2:   for each $y_j \in \{0,1\}^{32}$ and for each $b^1$ from $L_{B_1}$ do
 3:     Store $(h_j^{-1}(y_j), h_j^{-1}(y_j) \oplus h_j^{-1}(y_j \oplus b_j^1))$ in $L^j_{\#8,b^1}$ (one of $4 \times 2^{32}$ lists of size $2^{32}$).
 4: for $j$ from 5 to 8 do
 5:   for each $y_j \in \{0,1\}^{32}$ and for each $b^2$ from $L_{B_2}$ do
 6:     Store $(h_j^{-1}(y_j), h_j^{-1}(y_j) \oplus h_j^{-1}(y_j \oplus b_j^2))$ in $L^j_{\#8,b^2}$ (one of $4 \times 2^{32}$ lists of size $2^{32}$).
 7: for each $a$ in $L_A$ do
 8:   for $i$ from 1 to 12 do
 9:     for each $x_i \in \{0,1\}^{32}$ do
10:       Store $(g_i(x_i), g_i(x_i) \oplus g_i(x_i, a_i))$ in $L^i_{\#6}$.
11:   for $\Delta_X$ from 0 to $2^{64} - 1$ (and not the 128 bits as done in [17]) do
12:     Compute $\Delta_O$ (with linear conditions of inactive states in #8) and $\Delta^{\#8}_{j'}$ for $j' \in \{1, 2, 5, 6\}$ (with
        BigMC).
13:     for each $b^1$ in $L_{B_1}$ and for $j = [1, 2]$ do
14:       Find an element in $L^j_{\#8,b^1}$ such that $h_j^{-1}(y_j) \oplus h_j^{-1}(y_j \oplus b_j^1) = \Delta^{\#8}_j$ and store
          $(h_1^{-1}(y_1), \Delta^{\#8}_1, h_2^{-1}(y_2), \Delta^{\#8}_2, b^1)$ in $L_{aux_1}$.
15:     for each $b^2$ in $L_{B_2}$ and for $j = [5, 6]$ do
16:       Find an element in $L^j_{\#8,b^2}$ such that $h_j^{-1}(y_j) \oplus h_j^{-1}(y_j \oplus b_j^2) = \Delta^{\#8}_j$ and store
          $(h_5^{-1}(y_5), \Delta^{\#8}_5, h_6^{-1}(y_6), \Delta^{\#8}_6, b^2)$ in $L_{aux_2}$.
17:     for each $(h_1^{-1}(y_1), \Delta^{\#8}_1, h_2^{-1}(y_2), \Delta^{\#8}_2, b^1)$ in $L_{aux_1}$ and for each $(h_5^{-1}(y_5), \Delta^{\#8}_5, h_6^{-1}(y_6), \Delta^{\#8}_6, b^2)$
        in $L_{aux_2}$ do
18:       Compute $V_1' = h_1^{-1}(y_1) \oplus 3 \times h_5^{-1}(y_5)$ and $V_2' = h_2^{-1}(y_2) \oplus 3 \times h_6^{-1}(y_6)$, and store
          $((h_1^{-1}(y_1), \Delta^{\#8}_1, h_2^{-1}(y_2), \Delta^{\#8}_2, b^1), (h_5^{-1}(y_5), \Delta^{\#8}_5, h_6^{-1}(y_6), \Delta^{\#8}_6, b^2))$ in a hash table $T$ indexed
          by these $(V_1', V_2')$.
19:     for $\Delta_Y$ from 0 to $2^{64} - 1$ do
20:       Determine by BigMC $\Delta^{\#8}_{j'}$ for $j' = 3, 4, 7, 8$; and $\Delta^{\#6}_j$ for $j \in [1, \ldots, 12]$.
21:       for $i$ from 1 to 12 do
22:         Find the element from $L^i_{\#6}$ such that $g_i(x_i) \oplus g_i(x_i, a_i) = \Delta^{\#6}_i$.
23:       Compute with them by MC the values $d_i^{\#7}$ of the active diagonals in #7 and then
          $V_j = 2 \times d_j^{\#7} \oplus d_{j+4}^{\#7} \oplus d_{j+8}^{\#7} \oplus 9 \times d_j^{\#7} \oplus 3 \times d_{j+4}^{\#7} \oplus 6 \times d_{j+8}^{\#7}$ for $j = 1, 2$.
24:       if there is an element such that $V_1' = V_1$ and $V_2' = V_2$ in $T$ (one on average, determines $b^1$ and
          $b^2$) then
25:         Find from $L^{j'}_{\#8,b^1}$ the element $(h_{j'}^{-1}(y_{j'}), \Delta^{\#8}_{j'})$ for $j' = 3, 4$. This determines $y_3$ and $y_4$.
26:         Find from $L^{j'}_{\#8,b^2}$ the element $(h_{j'}^{-1}(y_{j'}), \Delta^{\#8}_{j'})$ for $j' = 7, 8$. This determines $y_7$ and $y_8$.
27:         if with these values of $(h_{j'}^{-1}(y_{j'}),\ j' = 3, 4, 7, 8$ and the ones obtained in step 22 of $g_i(x_i)$ for
            $i = 3, 4, 7, 8, 11, 12$ that we have not used yet, the equation (1) for $i = 3, 4$ derived from $F_1$ can
            be verified (happens with a probability of $2^{-64}$) then
28:           The value $x_*$ is determined. Add the element $(x_1, \ldots, x_{z_A}, x^*, a, b^1, b^2)$ to $\mathcal{L}_{sol}$
29: Return $\mathcal{L}_{sol}$, containing about $2^{64+l_A}$ elements.
```

– **ECHO:** Using conviniently the algorithm from Section 3.2 we provide better trade-offs improving the time complexity from [17].
– **Luffa:** The parallel-matching algorithm is applied in [10], improving the time complexity over the brute force merging method by increasing the memory requirements. If we apply instead the gradual-matching algorithm (with three layers), the time complexity can still be better than the brute force one while the memory needs are not increased.
– LANE**:** In the cases of LANE-256 and LANE-512 several improvements are applied at different steps of the attacks from [12]. They use the instant-matching algorithm, as well as some more appropriate ways to formulate the problem, as shown in Appendix A, and the algorithms from Section 3.1 and from Appendix B.

For a detailed description of the hash functions, we refer to their SHA-3 submission documents. As those attacks are quite complex, we do not explain here all the details, but we give the information needed for identifying the problem, referring in each case to the corresponding attack. These improvements are based on the algorithms that we have described in this paper as well as on recognizing the situations where they can be applied. This way we are able to reduce the overall complexity of the attacks.

4.1 JH

For simplicity, we consider here the attack on JH with $d = 4$ for 8 rounds when using the three-inbound attack given in [16] with a complexity of $2^{32.09}$ in time and $2^{24.18}$ in memory. We shall see here how, when we apply one of the previously introduced algorithms, this complexity can be significantly improved. For $d = 8$ the improvement is performed the same way for the three-inbound attack on 19 and 22 rounds, and it is simpler in the case of one-inbound for 16 rounds. The three-inbound attack for $d = 4$ uses the differential path represented in Fig. 11, where #0 represents the initial internal state and #8 the final one. The colored parts are the parts with a difference. Each small square represents the 4×4 Sbox for rows from 0 to 15, and each rectangle represents the linear permutation on 8 bits. Each wire conrresponds to 4 bits.

To improve this attack, we use the algorithms from Sections 2.1 and 2.2. Besides, we consider the same three inbounds as in [16] but we sometimes keep two non-overlaping groups of solutions per inbound (instead of merging them into one). Without this, we could still improve the time complexity, but this would be limited by the size of the intermediate lists stored, $2^{24.18}$. Keeping two lists instead of one means that their size will be smaller. We start the attack as in [16] by finding the possible solutions for the first inbound (from round #0 to the beginning of round #2), storing a list L_A of $2^{11.36}$ solutions with a cost of 2^{16}.

We consider the third inbound, from round #5 to the beginning of round #7. In this part, we obtain two sets of 2^{16}, each one associated to a list: $L^5_{0,1,8,9}$ and $L^5_{2,3,10,11}$. This is done by first building the lists, $L_{0,1}$, $L_{8,9}$, $L_{2,3}$ and $L_{10,11}$ of size $2^{11.91}$ each verifying the

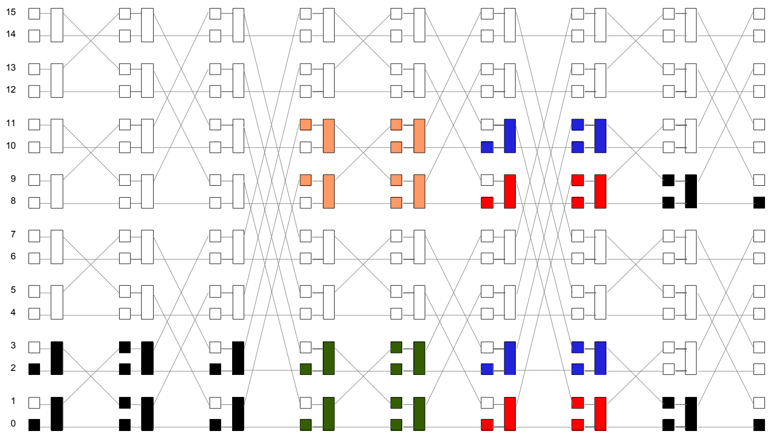


Fig. 11: Differential path for $d = 4$ of JH of the three-inbound attack.

conditions from #6. Next the two first ones are merged in the same way as the Example 1 of Section 2.1 using the instant-matching algorithm. The list $L^5_{0,1,8,9}$ is obtained. We do the same with lists $L_{2,3}$ and $L_{10,11}$ to obtain $L^5_{2,3,10,11}$, both of size 2^{16}. The cost of this phase is 2^{17}.

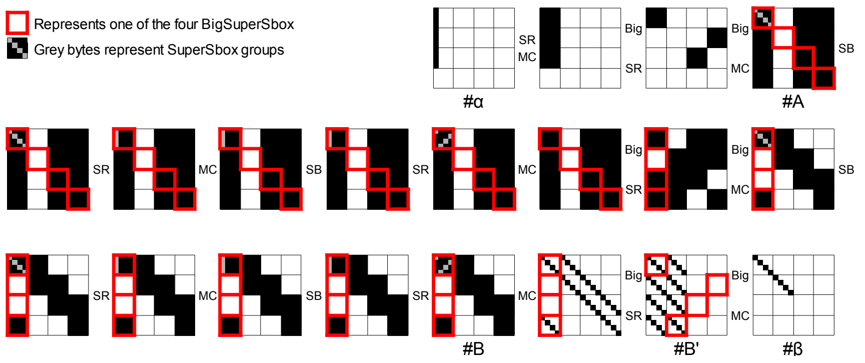
Next, for rounds #3 and #4 we will repeat the same procedure at the same cost for obtaining two sets of solutions for these two rounds: $L^3_{0,1,2,3}$ and $L^3_{8,9,10,11}$ of size also 2^{16}. We will now merge these two lists and the list L_A. Merging $L^3_{0,1,2,3}$ and $L^3_{8,9,10,11}$ determines 3.91×2 bit conditions and, for merging both of them with L_A, 16 bit conditions need to be verified (from the two active 4-bit words where they collide). Merging these three lists can be done by first applying a gradual-matching to $L^3_{0,1,2,3}$ and L_A using the groups of differences. Next, we can apply an instant-matching with the partial solutions and list $L^3_{8,9,10,11}$. As a result, a new list L_B of size $2^{19.54}$ is obtained with a cost of $2^{19.54}$. The elements of this list are solutions for the rounds #0 to #5. Next, in a similar way we will merge L_B with $L^5_{0,1,8,9}$ and $L^5_{2,3,10,11}$ (here there are 32 bit conditions to verify), and we will obtain $2^{19.54}$ solutions that verify the merge (and so rounds from #0 to #7) with a cost of $2^{19.54}$. For each solution, we check if it also verifies round #8 (3.91×2 bit conditions), obtaining $2^{11.72}$ solutions (as in [16], before taking the symmetries into account). The complexity of the attack using our algorithm is then $2^{19.54}$ in time and $2^{19.54}$ in memory, improving the previous complexity of $2^{32.09}$ in time and $2^{24.18}$ in memory. Similarly as we have shown for $d = 4$ and 8 rounds, we can identify the same problem and apply the algorithms of Sections 2.1 and 2.2 to the attack on 19 and 22 rounds of [16] that uses three-inbound attacks and has a complexity of $2^{168.02}$ in time and $2^{143.70}$ in memory, so that it can also be improved using the same algorithm, and having a final complexity of $2^{95.63}$ in time and memory. The 16-round attack with one-inbound attack of [16], can also be improved to $2^{96.12}$ in time and memory, while its complexity was 2^{190} in time and 2^{104} in memory.

4.2 *Grøstl*

In this case we do not apply one of the algorithms but we state again the importance of identifying the best problem to solve. Here, we consider the results on Grøstl-256 presented in [15], where, in particular, distinguishers are given for the full compression function as well as for the internal permutation. We can improve by a factor of 2^{10} or 2^{17} (depending on the differential path considered) their time complexities. In this case, instead of finding a new algorithm (the corresponding part of the path can be directly solved with a SuperSbox precomputation) we have identified a better problem to solve: the lists L_A and L_B, representing differences in the input and in the output of the SuperSbox phase respectively, can have a smaller size than considered in [15]. They were built with all the possible differences, but we noticed that they can be smaller by just storing the differences that we know for sure might also satisfy the outbound phase. The factor that we are going to gain will depend on the number of active columns in the input (N_i) and the number of active columns in the output (N_o). So instead of merging two lists of size $2^{l_A} = 2^{64N_i}$ and $2^{l_B} = 2^{64N_o}$, we have to merge one list of size $2^{l_A} = 2^{63N_i}$ and one list of size $2^{l_B} = 2^{56N_o}$. The algorithm applied to merge these lists is the same one as in [15], obtaining a complexity in time of $2^{63N_i+56N_o}$ instead of $2^{64(N_i+N_o)}$. This is possible because in this attack the one byte differences introduced by the constants additions have a fixed value, implying that the number of possible differences at the input and output of the SuperSbox will be smaller. In the 10-round compression function analysis this improves from 2^{192} to 2^{182} and in the permutation distinguisher from 2^{192} to 2^{175}. In the case of Grøstl-512 we can improve time complexity of the analysis on 11 rounds of the compression function from 2^{640} to 2^{630}.

4.3 ECHO-256

In [17] an analysis of the whole ECHO-256 permutation is provided which has complexity 2^{182} in time and 2^{37} in memory. By studying in detail this analysis we have been able to provide some trade-offs that were previously unknown and that allow to improve the time complexity. For example, we can perform the same attack with a complexity 2^{151} in time and 2^{67} in memory. We consider the differential path given in [17]. In Fig. 12, the inbound part is represented. We need to find 2^{86} solutions of this part in order to satisfy also the outbound part. In Fig. 12, the BigSB are decomposed into the AES operations (2 rounds, where we omit operations that do not influence the differential path) and we can see how two BigSB can be seen as a BigSuperSbox (from #A to #B'), where the sets formed by it have the form of the highlighted sets of four AES states. For finding solutions for each one of this 4 BigSuperSbox we can apply the algorithm from Section 3.2. As with one element from L_A we obtain 2^{64} solutions, we will have to iterate the algorithm over 2^{22} different a. For reducing the memory needs, we will find solutions for the whole inbound considering $l_A = 0$ and next we will repeat the process for 2^{22} different a. The elements in L_{B_1} (L_{B_2}) are generated by the 2^{32} possible differences in the AES state $(0, 0)$ $(1, 1$ respectively) in $\#\beta$. We then apply the algorithm from Section 3.2 to each BigSuperSbox



Represents one of the four BigSuperSbox
Grey bytes represent SuperSbox groups
SR
MC
Big
SR
Big
MC
SB
#α
#A
SR
MC
SB
SR
MC
Big
SR
Big
MC
SB
SR
MC
SB
SR
MC
Big
SR
Big
MC
#B
#B'
#β

Fig. 12: Inbound part of the differential path on ECHO. A number of 2^{86} solutions needs to be found for satisfying the outbound part.

set, obtaining 4 associated sets of solutions of size $2^{64+l_A} = 2^{64}$. Each element from one set will be associated to an unique 3-tuple of elements from the other groups: the ones that were generated by the same difference in $\#\beta$ (that define the b_1, b_2 differences). This gives in total 2^{64} solutions for the whole inbound phase. As said previously, if we repeat this procedure for 2^{22} distinct differences in $\#\alpha$ (that define the a differences) we will obtain the 2^{86} needed solutions with a time complexity of 2^{151} and memory of 2^{67}.

4.4 *Luffa*

In [10], a way of finding a semi-free-start collision is provided for 7 rounds out of 8. This is done by using the differential path represented in Fig. 13, where each small square represents one bit, and the colored ones are the ones with differences. This path is solved by first, finding solutions for the possible differences of the path from #1 to #7 (in L_A). In parallel the possible differences are found for the part of the path from #8 to #14 (in L_B). State #7 is separated from #8 by 64 4×4 Sboxes. Among them, 52 are active. We want to keep the possible differences for the whole path from #1 to #14. In this case, the problem is very easy to identify: we have two lists of differences, one of differences of the inputs of 52 active Sboxes, the other one of the outputs of these 52 active Sboxes. We can apply the gradual-matching or the parallel-matching algorithms, as we did in Example 2. In [10] the parallel-matching was applied, reducing the time complexity from $2^{132.6}$ to 2^{104}, while the memory complexity increases to 2^{102}. We can also apply the gradual-matching algorithm with $z' = 16$ and obtain an improved time complexity of $2^{112.9}$ while the memory complexity stays the same ($2^{68.8}$).

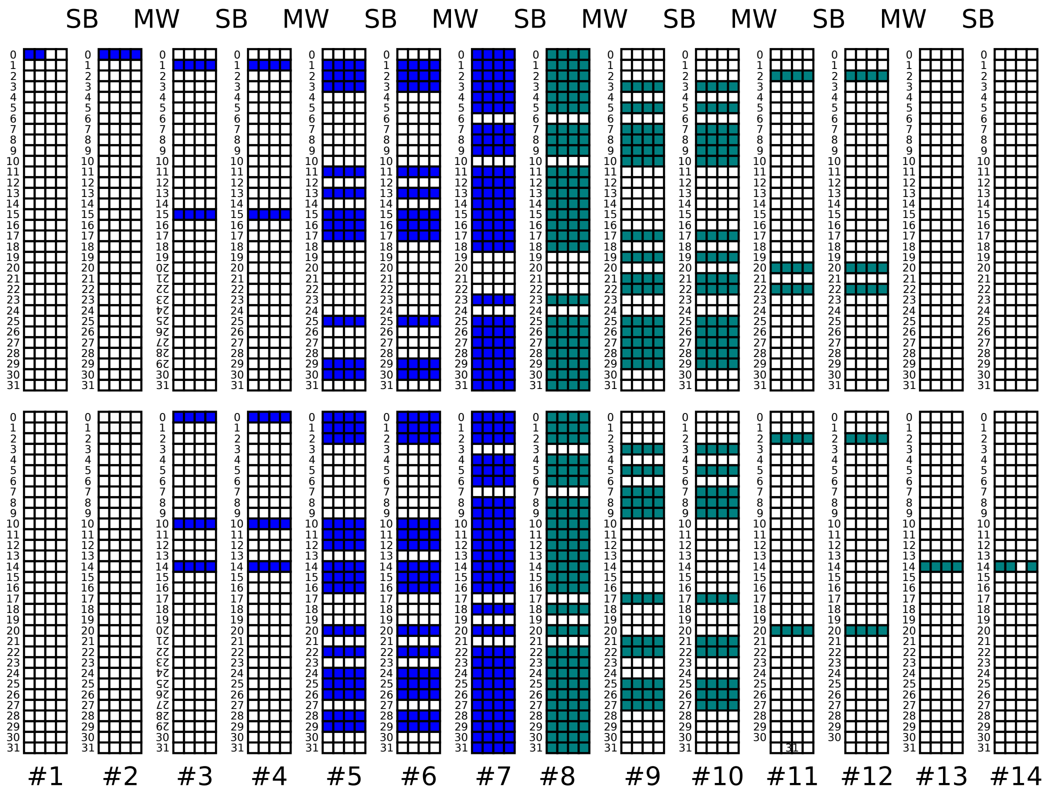


Fig. 13: Differential path used in the rebound attack from [10] on Luffa.

4.5 Lane-256

The analysis in [12] provides a way of finding a semi-free-start collision for the complete compression function of Lane-256 with a complexity of 2^{96} in time and 2^{88} in memory. In this section we are going to identify 2 concrete problems extracted from this attack, and by applying two of the previously described algorithms, we are able to reduce the total complexity of the attack to 2^{80} in time and 2^{66} in memory, or more precisely, to 2^{80} in time and 2^{58} in memory + 2^{64} in time and 2^{66} in memory. We are not going to describe in detail here the analysis from [12], but we give the information needed for identifying and defining the problem to be treated by the corresponding algorithm.

First problem: In this attack, the first three steps aim at finding 2^{56} solutions for two inbounds in 4 independent lanes. Each one of the four lanes represents an independent and similar problem. Instead of looking at it as three steps, we are going to unify it in just one, and we will use the differential path from Fig. 7. We can now build the list of possible differences in the input: from five active bytes, we can obtain 2^{40} possible differences in the input, before the first SB considered. We store 2^{32} out of these 2^{40} and this forms the list L_A. We can do the same with the possible differences in the output: out of a totally full active AES state, we want to reach a position with only 4 active bytes. The list L_B will be formed by all the 2^{32} possible differences in the output after the last Sbox considered (in the two inbounds). We want to merge these two lists keeping the differences that can verify the whole path defined by the two inbounds and to recover the associated values. So we want to obtain a total of 2^{64} values and differences as solutions. There is an extra condition of one byte before the differential path from Fig. 7, so we finally obtain 2^{56} solutions. We will directly apply the algorithm from Section 3.1. The cost of this step was 2^{96} in time and 2^{88} in memory (it was the bottleneck for time and memory). Now, we can perform these two inbound phases with a complexity of 2^{66} in time complexity and 2^{65} in memory. As this step is not bottleneck anymore for the attack, we can now try to reduce the rest of the complexities.

Second problem: Once the previous step is finished we have obtained 2^{56} solutions for the first two inbounds, for each lane (four lists of values and differences). They need to be merged so that they verify the message expansion. In [12] this is done in steps 4 and 5 with a complexity of 2^{80} in time and memory. This memory complexity can be reduced to 2^{48} by directly applying the example from Section A.1, obtaining 2^{64} solutions for this step and giving the new bottleneck of the time complexity of the attack: 2^{80}. The last part of the attack is the same as in [12], and corresponds to the bottleneck in memory: $2^{64} \times 4$.

4.6 Lane-512

A semi-free-start collision attack is given in [12] for the whole compression function of Lane-512 with a complexity of 2^{224} in time and 2^{128} in memory. Applying three of the previously described algorithms we can reduce this memory complexity from 2^{128} to 2^{66}. At this aim, we have to identify 3 problems.

First problem: The original first step in the attack on Lane-512 leads to 4 lists of 2^{68} solutions for a first inbound. We realized that, as it is possible to change the number of active bytes at the beginning of each lane from 6 to 4, obtaining 2^{56} solutions is enough. Steps 2 and 3 merge these 4 lists for finding one solution that verifies also the message expansion. We can apply, as we did before, the example from Section A.1. We obtain one solution with complexity 2^{56} in memory and 2^{80} in time, instead of the previous 2^{88} in time and memory.

Second problem: In the attack on LANE-512, in the Starting Points phase, four lists of values are built, of size 2^{64}. The Merge Lanes and Message Expansion phases need a complexity of 2^{128} in time and 2^{128} in memory. We can instead apply the example from Section A.2. With a complexity of 2^{192} in time and $2^{64} \times 4$ in memory we can obtain the 2^{128} starting points needed for repeating the rest of the attack enough times until we find one solution for the whole path (and so a semi-free-start collision). We do not need to store these 2^{128} starting points, because we can perform the rest of the attack as soon as we find one. This way, the memory complexity does not go beyond 2^{64}, and the time complexity, though higher, is not the bottleneck.

Third problem: In [12], the second merge of inbound phases (that finds a collision between two lanes) needs a memory of $2^{96} \times 4$. With the previous improvements, the memory needed is 2^{64}, so we want to reduce the memory needs of this last phase to 2^{64}. We have three lists of 2^{32} elements for each of the two lanes of the same branch (6 in total). Instead of merging the three lists into a new one of size 2^{96}, as done in [12], we can apply the algorithm from Appendix B. This way we will only store a list of 2^{64} elements.

5 Conclusion

The main contributions of this paper can be classified in three groups. First, we propose several algorithms for solving the problem which constitutes the bottleneck of most rebound attacks, leading to improvements of the previously known complexities.

Secondly, we highlight with some examples the importance of identifying the situations that could help improving the complexity of this type of attacks and we show how to find the problems in each particular case that will provide the best overall complexity. This is often a difficult task due to the high technicality of the attacks and algorithms.

Finally, the previous two contributions lead to improvements of most of the best known rebound attacks applied to the SHA-3 candidates JH, Grøstl, Luffa, ECHO-256 and LANE. It is important to point out that we just tried to improve the complexities of existing attacks. However, the work presented in this paper can be very useful for future rebound attacks, in particular we believe that the attacks on JH and on the compression function of ECHO can be improved (extending the number of rounds attacked) by exploiting the algorithms and ideas presented here. Finally, we believe that some of these algorithms, specially those of Section 2, will be applicable in other contexts besides rebound attacks.

Acknowledgements

The author would like to thank Willi Meier, Simon Knellwolf, Marine Minier, Thomas Peyrin, Martin Schläffer, Joana Treger and Fabien Viger for many helpful comments and discussions. A special mention is needed for Anne Canteaut and Andrea Röck for all the help and suggestions to improve this paper.

References

1. Barreto, P.S.L.M., Rijmen, V.: The Whirlpool Hashing Function, revised in 2003.
2. Benadjila, R., Billet, O., Gilbert, H., Macario-Rat, G., Peyrin, T., Robshaw, M., Seurin, Y.: Sha-3 proposal: ECHO. Submission to NIST (updated) (2009)
3. Camion, P., Patarin, J.: The knapsack hash function proposed at Crypto'89 can be broken. In: EUROCRYPT. Lecture Notes in Computer Science, vol. 547, pp. 39–53. Springer (1991)
4. Canniere, C.D., Sato, H., Watanabe, D.: Hash Function Luffa: Specification. Submission to NIST (Round 2) (2009)
5. Canteaut, A., Naya-Plasencia, M.: Structural weaknesses of permutations with low differential uniformity and generalized crooked functions. In: Finite Fields: Theory and Applications - Selected papers from the 9th International Conference Finite Fields ans applications. Contemporary Mathematics, vol. 518, pp. 55–71. AMS (2010), http://www-rocq.inria.fr/secret/Maria.Naya_Plasencia/papers/canteaut-nayaplasencia.pdf
6. Chose, P., Joux, A., Mitton, M.: Fast correlation attacks: An algorithmic point of view. In: EUROCRYPT. Lecture Notes in Computer Science, vol. 2332, pp. 209–221. Springer (2002)
7. Gauravaram, P., Knudsen, L.R., Matusiewicz, K., Mendel, F., Rechberger, C., Schläffer, M., Thomsen, S.S.: Grøstl – a SHA-3 candidate. Submitted to the SHA-3 competition, NIST (2008), http://www.groestl.info
8. Gilbert, H., Peyrin, T.: Super-Sbox Cryptanalysis: Improved Attacks for AES-like permutations. In: FSE. Lecture Notes in Computer Science (2010), to appear
9. Indesteege, S.: The Lane hash function. Submitted to the SHA-3 competition, NIST (2008), http://www.cosic.esat.kuleuven.be/publications/article-1181.pdf
10. Khovratovich, D., Naya-Plasencia, M., Røck, A., Schläffer, M.: Cryptanalysis of Luffa v2 components. In: SAC. Lecture Notes in Computer Science, vol. 6544, pp. 388–409 (2010)
11. Lamberger, M., Mendel, F., Rechberger, C., Rijmen, V., Schläffer, M.: Rebound Distinguishers: Results on the Full Whirlpool Compression Function. In: ASIACRYPT. Lecture Notes in Computer Science, vol. 5912, pp. 126–143. Springer (2009)
12. Matusiewicz, K., Naya-Plasencia, M., Nikolic, I., Sasaki, Y., Schläffer, M.: Rebound Attack on the Full Lane Compression Function. In: ASIACRYPT. Lecture Notes in Computer Science, vol. 5912, pp. 106–125. Springer (2009)
13. Mendel, F., Rechberger, C., Schläffer, M., Thomsen, S.S.: The Rebound Attack: Cryptanalysis of Reduced Whirlpool and Grøstl. In: Fast Software Encryption - FSE 2009. Lecture Notes in Computer Science, vol. 1008. Springer (5665)
14. Mendel, F., Peyrin, T., Rechberger, C., Schläffer, M.: Improved cryptanalysis of the reduced Grøstl compression function, ECHO permutation and AES block cipher. In: Jacobson, Jr., M.J., Rijmen, V., Safavi-Naini, R. (eds.) Selected Areas in Cryptography. Lecture Notes in Computer Science, vol. 5867, pp. 16–35. Springer (2009)
15. Peyrin, T.: Improved Differential Attacks for ECHO and Grøstl. In: Advances in Cryptology - CRYPTO 2010, 30th Annual Cryptology Conference, Santa Barbara, CA, USA, August 15-19, 2010. Proceedings. Lecture Notes in Computer Science, vol. 6223, pp. 370–392. Springer (2010)

16. Rijmen, V., Toz, D., Varici, K.: Rebound Attack on Reduced-Round Versions of JH. In: FSE. Lecture Notes in Computer Science, vol. 6147, pp. 286–303 (2010)
17. Sasaki, Y., Li, Y., Wang, L., Sakiyama, K., Ohta, K.: Non-Full-Active Super-Sbox Analysis Applications to ECHO and Grøstl. In: ASIACRYPT. Lecture Notes in Computer Science, vol. 6477, pp. 38–55 (2010), to appear
18. Schroeppel, R., Shamir, A.: A T=O($2^{n/2}$), S=O($2^{n/4}$) algorithm for certain NP-complete problems. SIAM J. Comput. 10(3), 456–464 (1981)
19. Wagner, D.: A generalized birthday problem. In: CRYPTO. Lecture Notes in Computer Science, vol. 2442, pp. 288–303. Springer (2002)
20. Wu, H.: The hash function JH. Submission to NIST (updated) (2009), http://icsd.i2r.a-star.edu.sg/staff/hongjun/jh/jh_round2.pdf
21. Wu, S., Feng, D., Wu, W.: Cryptanalysis of the Lane hash fonction. In: SAC 2009 - Selected Areas in Cryptography. Lecture Notes in Computer Science, Springer (2009)

A Importance of Identifying the Appropriate Problems

Here we present some improvements of two different steps of the attacks against Lane presented in [12], which result from a formulation of the underlying problem which is more appropriate for applying the instant-matching algorithm.

A.1 Example 3: Using Equalities for Dividing the Problem

We improve the memory complexity needed for steps 4 and 5 of the attack in [12], which was the bottleneck of the attack. At the beginning of step 4, four lists have been obtained (L_1, L_2, L_3 and L_4), each one with 2^{56} elements. These elements can be represented in 20 groups of size $s = 8$ bits. Among these 20 bytes, 4 correspond to differences and 16 to values. Let $(v^i_j)_{1\leq j\leq 20}$ be the 20 bytes of an element v in list L_i. We denote by ℓ_1 and ℓ_2 two linear permutations. We want to find all 4-tuples (v^1, v^2, v^3, v^4) from the four lists that satisfy the following relations:

$$v^1_i = v^2_i = v^3_i = v^4_i \text{ for } 1 \leq i \leq 4$$

$$\ell_1(v^1_i, v^2_i) = \ell_2(v^3_i, v^4_i) \text{ for } 5 \leq i \leq 20.$$

In [12], this problem was solved with a complexity of 2^{80} in time and 2^{80} in memory. Their approach was to first *merge* lists L_1 and L_2, as well as L_3 and L_4 by using the equations involving the differences. This lead to two new lists $L_{1,2}$ and $L_{3,4}$ of size $2^{56+56-8\cdot 4} = 2^{80}$ each. Next, these two lists were *merged* using the remaining equations, obtaining one solution on average since $2^{l_{1,2}+l_{3,4}-p_t} = 2^{80+80-20\cdot 8} = 1$. However this memory complexity can be improved. First we can separate this problem in smaller ones: by considering the first equation, we know that we won't find a 4-tuple that will be a solution unless all the elements have the same first four bytes. We can then separate each one of the four lists in $2^{4s} = 2^{32}$ sublists, L^γ_i for γ from 0 to $2^{32} - 1$, so that each sublist L^γ_i contains the elements from L_i that had a difference γ in the 4 bytes. Now the initial problem can be seen as

2^{32} independent problems, where the *merge* is determined by the conditions on the last 16 bytes. Each problem can be solved with the instant-matching algorithm with parameters: $N = 4$, $N' = 2$, $z = 16$, $s = 8$, $p_t = 128$, $l_i = 56 - 32 = 24$ and t_j is the $\oplus$ operation. The memory complexity now is $4 \cdot 2^{56} + 2^{24+24} \simeq 2^{58}$ instead of 2^{80} while the time complexity stays the same.

A.2 Example 4: Increasing Time to Reduce Memory

We present here another application example. In this case, we suppose that the memory complexity is the bottleneck of the attack instead of the time complexity (so we are allowed to increase it). We study a case we find when improving the overall complexity of the attack on LANE-512 presented in [12]. We consider steps 7 and 8 of the attack. In this particular case, the time complexity, of 2^{224}, was imposed by another step of the attack. The concrete problem is the following: we have four lists, $L_1, \ldots, L_4$ of size 2^{64} elements. These elements are defined by $s = 8$, $z = 8$. Let ℓ_1 and ℓ_2 be linear permutations. Let $v_1^i, \ldots, v_8^i$ denote an element of L_i. Then we want to find all the 4-tuples of values (v^1, v^2, v^3, v^4) in $L_1 \times L_2 \times L_3 \times L_4$ that verify the following relation over $(\{0,1\}^8)^2$:

$$\ell_1(v_j^1, v_j^2) = \ell_2(v_j^3, v_j^4),\ j \in [1, z].$$

Here we also have $P_t 2^{zs} = 1$. In the attack presented in [12], this part was solved with a complexity of 2^{128} in time and memory. With the improvements that we present in this paper of other steps of the attack in [12], this step would be the bottleneck in memory, but we show here how to reduce this memory complexity to 2^{66} with a correct use of the instant-matching algorithm. We notice that this relation can also be written the following way ($N' = 1$):

$$v_j^1 = \ell'(v_j^2, v_j^3, v_j^4).$$

We can then directly apply the instant-matching algorithm obtaining $2^{192+64-8s} = 2^{128}$ solutions. In this case, each time we obtain a match we can use it instead of storing it. Hence the memory complexity is now 2^{66} and the time complexity $2^{64 \cdot 3} = 2^{192}$, which is still below the bottleneck in time complexity. This way, memory needs are reduced from 2^{128} to 2^{66} while the overall time complexity stays 2^{224}.

B Algorithm for Improving the Complexity of the Third Problem in LANE-512

We consider the path given by Fig. 14. In this case, we are given 6 lists, L_A, L_B, L_C, $L_{A'}$, $L_{B'}$ and $L_{C'}$ of size 2^{32}, where each list L_i contains possible values for the AES state marked in Fig. 14 with an i on state #1. The black bytes represent bytes with differences and have been completely determined in previous inbound phases, *i.e.* the value and the difference in each black byte is fixed. We want to find all the elements from $L_A \times L_B \times L_C \times L_{A'} \times L_{B'} \times L_{C'}$

such that, when we consider the values $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ in the three corresponding states of #1, and we compute the state #15, the difference Δ obtained is the same as the Δ' obtained when we consider the values $\boldsymbol{a'}, \boldsymbol{b'}, \boldsymbol{c'}$ in the parallel lane. The final operations have been omitted for the sake of simplicity, as they are linear and colliding in #15 is equivalent to colliding at the end. We expect to find $2^{32\times 6-128}$ solutions. In [12] these 6-tuples are found by computing a list L_{ABC} with all values in $L_A \times L_B \times L_C$ and the corresponding Δ, and then by checking for all triples $L_{A'} \times L_{B'} \times L_{C'}$ the resulting Δ' belongs to L_{ABC}. The complexity for finding the 2^{64} solutions is 2^{97} in time and 2^{96} in memory. With the other improvements presented in Section 4.6, this memory requirement would be the bottleneck of the attack. We show here how to apply an algorithm for solving this problem that would need the same time complexity but with a memory complexity of only 2^{64}.

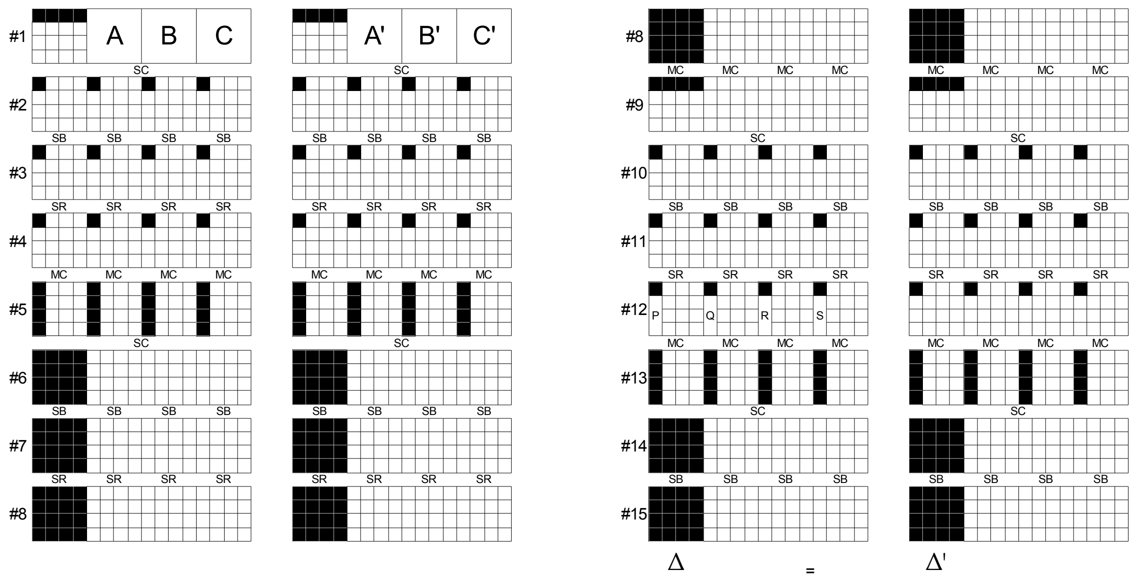


Fig. 14: Part of the differential path on Lane-512 representing the third improved part from Section 4.6

First we remark that if we go through all the 2^{24} possible values for the three bytes in #12 marked with a P, then we will generate all the possible values for the differences in the first column of Δ. That means that this column can only take 2^{24} possible differences among the 2^{32}. The same happens with the groups Q, R and S for the second, third and fourth columns of Δ, respectively. We proceed as follows:

1. We store four tables of size 2^{24}, L_P, L_Q, L_R and L_S, of possible differences in each of the columns of $\varDelta$.
2. For each one of the 2^{96} elements in $L_{A'} \times L_{B'} \times L_{C'}$ we compute the associated $\varDelta'$ and we check if each of its four columns is included in the corresponding list L_P, L_Q, L_R or L_S. For each column, this will be the case with probability $2^{24-32} = 2^{-8}$. Then the probability that $\varDelta'$ is valid is 2^{-32}.
3. If $\varDelta'$ is valid we add an element $(\varDelta', \boldsymbol{a'}, \boldsymbol{b'}, \boldsymbol{c'})$ to the list $L_{A'B'C'}$. At the end, the size of this list is $2^{96-32} = 2^{64}$.
4. Once $L_{A'B'C'}$ is computed, we can try for all the 3-tuples from $L_A \times L_B \times L_C$ if the $\varDelta$ they generate belongs to $L_{A'B'C'}$. This will happen with a probability of $2^{-24\times 4} = 2^{-96}$.

The number of solutions is 2^{64}. With our algorithm we can find them with the same time complexity as before and with a reduced memory complexity of 2^{64} instead 2^{96} as was the case in [12].