On a new invariant of Boolean functions

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Abstract

A new invariant of the set of n-variable Boolean functions with respect to the action of $AGL(n,2)$ is studied. Application of this invariant to prove affine nonequivalence of two Boolean functions is outlined. The value of this invariant is computed for PS_{ap} type bent functions.

1 Introduction

A function from $\mathbb{F}_{2^n}$ into $\mathbb{F}_2$ is called a Boolean function on n variables. The set of all such functions is denoted by $\mathcal{B}_n$. Two Boolean function f and g are said to be affine equivalent if there exists an $A \in GL(n,2)$ - the group of all invertible $\mathbb{F}_2$-linear transformations over $\mathbb{F}_{2^n}$ - and $b \in \mathbb{F}_{2^n}$ such that $g(x) = f(Ax+b)$ for all $x \in \mathbb{F}_{2^n}$. The group of transformations containing all the invertible affine transformations of the form $x \mapsto Ax+b$ where $A \in GL(n,2)$ and $b \in \mathbb{F}_{2^n}$ is denoted by $AGL(n,2)$ and a general element in the group is written as (A,b). The $\{Tr_1^n(\lambda x)|\lambda \in \mathbb{F}_{2^n}\}$ is the set of linear functions from $\mathbb{F}_{2^n}$ into $\mathbb{F}_2$, where $Tr_1^n(x) = x + x^2 + x^{2^2} + \ldots + x^{2^{n-1}}$ is called the trace function. Walsh-Hadamard transformation of $f \in \mathcal{B}_n$ at $\lambda \in \mathbb{F}_{2^n}$ is $\hat{f}(\lambda) = \sum_{x\in\mathbb{F}_{2^n}}(-1)^{f(x)+Tr_1^n(\lambda x)}$. The multiset $[\hat{f}(\lambda)|\lambda \in \mathbb{F}_{2^n}]$ is called the Walsh-Hadamard spectrum of f. A function is called bent if and only if its Walsh-Hadamard spectrum takes the values $\pm 2^{\frac{n}{2}}$. The autocorrelation of a Boolean function is defined by $C_f(\lambda) = \sum_{x\in\mathbb{F}_{2^n}}(-1)^{f(x)+f(x+\lambda)}$. The autocorrelation spectrum of a Boolean function the multiset $[C_f(\lambda)|\lambda \in \mathbb{F}_{2^n}]$. For any bent function all the entries in the autocorrelation spectrum is zero - in fact is a characterization of a bent function. Probably the most well known invariants of a Boolean function are its Walsh-Hadamard spectrum and autocorrelation spectrum along with algebraic degree. These may be used for partial solution to the problem of deciding whether any two given Boolean functions are not affine equivalent. However in case of bent functions the first two invariants are identical over the complete class and therefore if two bent fucntions have the same algebraic degree we cannot distinguish them by using these invariants. The action of $AGL(m,2)$ on bent functions has been studied by Hou [5, 6]. An algorithm to decide whether two Boolean functions are affine equivalent or not and in case so to compute the matrix A and the element b is proposed by Meng, Yang and Zhang [7]. It is to be noted

that introduction of new invariants of Boolean functions has the potential of improving any algorithm in this direction. Moreover given any bent function to decide that it is not affine equivalent to a partial spreads bent is a particularly difficult problem [1, 3, 4]. In this paper we introduce a new invariant and provide a partial solution to this problem.

2 Description of the invariant

Suppose $f \in \mathcal{B}_n$ and $supp(f) = \{x \in \mathbb{F}_{2^n} | f(x) = 1\}$. Consider the set

$$S_n(f(x)) = \{\{a, b\} | f(a) = f(b) = f(a + b) = 1\}.$$

Define $M_n : \mathcal{B}_n \longrightarrow \mathbb{Z}$ by $M_n(f(x)) = |S_n(f(x))|$.

Lemma 1 *Suppose $g(x) = f(Ax)$ for some $A \in GL(n, 2)$. Then $M_n(f) = M_n(g)$.*

Proof : Consider $\phi : S_n(f(x)) \longrightarrow S_n(g(x))$, defined by $\phi : (a, b) \mapsto (A^{-1}a, A^{-1}b)$. Clearly this map is bijective hence $M_n(f(x)) = M_n(g(x))$. ■

Theorem 1 *If $g(x) = f(Ax + b)$ for some $A \in GL(n, 2)$ and $b \in \mathbb{F}_{2^n}$ then $M_n(g(x + A^{-1}b)) = M_n(f(x))$.*

Proof : Putting $x = y + A^{-1}b$ we obtain $g(y + A^{-1}b) = f(A(y + A^{-1}b) + b) = f(Ay)$. Since $M_n(f(Ay)) = M_n(f(y))$ by lemma 1, we obtain, $M_n(g(y + A^{-1}b)) = M_n(f(y))$. ■

By using the above theorem we outline a method of deciding whether two given Boolean functions $f, g \in \mathcal{B}_n$ are not affine equivalent. Suppose ζ is a primitive element of $\mathbb{F}_{2^n}$.

- **Step 1:** Construct the following multiset:
  $[M_n(g(x)), M_n(g(x + 1)), M_n(g(x + \zeta)), M_n(g(x + \zeta^2)), \ldots, M_n(g(x + \zeta^{2^n-2}))]$
  Let us call this M-spectrum of $g(x)$.
- **Step 2:** Compute $M_n(f(x))$.
- **Step 3:** If $M_n(f(x)) \neq M_n(g(x+\zeta^i))$ for all $i = 0, 1, \ldots, 2^n-2$ along with $M_n(f(x)) \neq M_n(g(x))$ we conclude that $f(x)$ is not affine equivalent to $g(x)$.

Clearly above is a necessary but not sufficient condition for affine equivalence.

Remark 1 *It is to be noted that $M_n(f(x))$ is the number of triangles formed at each vertex of the Cayley graph corresponding to the function f.*

3 The invariant M_n on PS_{ap}

In this section the value of $M_n(f(x))$ for $f \in PS_{ap}$ is computed. It is to be noted that this is the only subclass of PS type bent functions which can be effectively constructed and till date no PS type bent function is known which is not affine equivalent to a PS_{ap} type function. The class PS_{ap} is partitioned into two subclasses PS_{ap}^- and PS_{ap}^+ such that $PS_{ap} = PS_{ap}^- \cup PS_{ap}^+$. Their definitions are given below: For $n = 2p$ and ζ a primitive element of $\mathbb{F}_{2^n}$ define $V_i = \zeta^i \mathbb{F}_{2^p}$, $V_i^* = \zeta^i \mathbb{F}_{2^p} \backslash \{0\}$. $\mathbb{F}_{2^n} = \cup_{i \in C} V_i$ where $C = \{0, 1, 2, \ldots, 2^p\}$.

Definition 1 *A function $f \in \mathcal{B}_n$ is a PS_{ap}^- (PS_{ap}^+) type bent if and only if $supp(f) = \cup_{i \in I} V_i^*$ ($supp(f) = \cup_{i \in I} V_i$) where $I \subseteq C$ and $|I| = 2^{p-1}$ ($|I| = 2^{p-1} + 1$)*

We prove the following theorem.

Theorem 2 *Suppose $n = 2p$ and f is a PS_{ap} type bent function on n variables. Then*

1. *If $f \in PS_{ap}^-$ then*

$$M_n(f(x)) = \binom{2^p - 1}{2} 2^{p-1} + \binom{2^{p-1}}{2} (2^{p-1} - 2)(2^p - 1).$$

2. *If $f \in PS_{ap}^+$ then*

$$M_n(f(x)) = \binom{2^p - 1}{2} (2^{p-1} + 1) + \binom{2^{p-1} + 1}{2} (2^{p-1} - 1)(2^p - 1) + (2^p - 1)(2^{p-1} + 1).$$

Proof :
(a) We have to count the number of sets $\{a, b\}$ $(a \neq b)$ such that $f(a) = f(b) = f(a+b) = 1$. Suppose $f \in PS_{ap}^-$, $V_i^* \subseteq supp(f)$. If $a, b \in V_i^*$ then $a + b \in V_i^*$. For each $i \in I$ there are $\binom{2^p - 1}{2}$ such $\{a, b\}$. Then the total number of such pairs is $\binom{2^p - 1}{2} 2^{p-1}$.

Suppose $i, j \in I$ and $i \neq j$. For any $a \in V_i^*$ we have $|\{a + x | x \in V_j^*\} \cap V_k^*| = 1$ if $i \neq k$ and $j \neq k$, otherwise $|\{a + x | x \in V_j^*\} \cap V_k^*| = 0$. $|I \setminus \{i, j\}| = 2^{p-1} - 2$, Thus if a is fixed in V_i^* and x varies over V_j^* then $a + x$ is 1 exactly $2^{p-1} - 2$ times. Number of ways i, j can be chosen is $\binom{2^{p-1}}{2}$ and the number of ways a can be chosen is $2^p - 1$. Thus value of $|S_n(f(x))| = M_n(f(x))$ is

$$M_n(f(x)) = \binom{2^p - 1}{2} 2^{p-1} + \binom{2^{p-1}}{2} (2^{p-1} - 2)(2^p - 1).$$

(b) Similar argument proves this part. It is to be remembered that in this case $f(0) = 1$. This explains the third term $(2^p - 1)(2^{p-1} + 1)$ in the expression for $M_n(f(x))$. ∎

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