Cougar: Cubic Root Verifier Inner Product Argument under Discrete Logarithm Assumption

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Abstract. An inner product argument (IPA) is a cryptographic primitive used to construct a zero-knowledge proof (ZKP) system, which is a notable privacy-enhancing technology. We propose a novel efficient IPA called Cougar. Cougar features cubic root verifier and logarithmic communication under the discrete logarithm (DL) assumption. At Asiacrypt2022, Kim et al. proposed two square root verifier IPAs under the DL assumption. Our main objective is to overcome the limitation of square root complexity in the DL setting. To achieve this, we combine two distinct square root IPAs from Kim et al.: one with pairing (Protocol3) and one without pairing (Protocol4). To construct Cougar, we first revisit Protocol4 and reconstruct it to make it compatible with the proof system for the homomorphic commitment scheme. Next, we utilize Protocol3 as the proof system for the reconstructed Protocol4. Furthermore, we provide a soundness proof for Cougar in the DL assumption.

1 Introduction

Zero-Knowledge Proof (ZKP) is one of the privacy enhancement technologies that international organizations and others are focusing on [34]. ZKP is a protocol that allows a prover to convince a verifier that a statement is true without leaking any additional information [24]. ZKP schemes are employed as foundational components in various cryptographic applications, including identification [19, 16], verifiable computation [6, 7, 39, 9], range proofs [12, 17], confidential transactions [37, 12, 26, 17, 23], and incrementally verifiable computation [11, 13].

Our main goal is to construct an efficient inner product argument (IPA), which is an argument of knowledge for the inner product relation between two vectors. For constructing ZKP convincing the satisfiability of the arithmetic circuit (AC), one notable approach is to utilize IPA as a building block of the ZKP scheme [8, 12, 14, 31, 17]. Bootle et al. [8] first proposed an IPA with logarithmic proof size under the discrete logarithm (DL) assumption, and later, Bünz et al. [12] improved the IPA, which is called Bulletproofs. In [14], Bünz et al. proposed a paradigm for constructing ZKPs by applying a polynomial commitment scheme (PCS), which can be seen as a specialized form of IPA, to a polynomial

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interactive oracle proof (PIOP) system. Following this paradigm, the complexity of ZKPs heavily relies on that of the IPA. Hence, the efficient construction of an IPA is crucial for designing efficient ZKPs.

Bulletproofs is a widely known IPA because of its efficient proof size and lack of reliance on trusted parties. However, one of the main drawbacks of Bulletproofs is its linear verification cost, which makes it challenging to apply in certain applications, such as verifiable computation and incrementally verifiable computation. To avoid linear verification, Daza et al. [18] proposed a sublinear verifier IPA using bilinear pairing. However, the sublinear IPA [18] requires a trusted setup, which means that a trusted third party is necessary to generate a common reference string (CRS), whereas Bulletproofs does not. After, Lee [31] proposed a sublinear pairing-based IPA, called Dory, without a trusted setup. However, Dory depends on stronger cryptographic assumptions, such as the symmetric external Diffie-Hellman (SXDH) assumption.

Without relying on more cryptographic assumptions than Bulletproofs, Kim et al. [29] proposed two square root verifier IPAs, pairing-based IPA (Protocol3) and pairing-free IPA (Protocol4). Both IPAs provide linear prover and logarithm communication, equivalent to Bulletproofs. In [28], Kim et al. presented optimizations and a concrete implementation of Protocol3, which is called Leopard.

In this paper, we introduce the first cubic root verifier IPA, called Cougar, under the DL assumption. Our IPA maintains equivalent assumptions and setup to previous works such as [8, 12, 29], which rely on the DL assumption without requiring a trusted setup. In Table 1, we provide a comparison between previous IPA proposals and ours.

Schemes	Comm.	Prover	Verifier	Assumption	Setup	Pairing
Updatable IPA[18]	$O(\log_2 N)$	$O(N)$	$O(\log_2 N)$	DL, DPair	Trusted	Yes
Dory[31]	$O(\log_2 N)$	$O(N)$	$O(\log_2 N)$	SXDH	Trustless	Yes
Bulletproofs[8, 12]	$O(\log_2 N)$	$O(N)$	$O(N)$	DL	Trustless	No
Leopard[29, 28]	$O(\log_2 N)$	$O(N)$	$O(\sqrt{N})$	DL	Trustless	Yes
Protocol4[29]	$O(\log_2 N)$	$O(N)$	$O(\sqrt{N}\log_2 N)$	DL	Trustless	No
TENET[30]	$O(\sqrt{\log_2 N})$	$O(N \cdot 2^{\sqrt{\log_2 N}})$	$O(N/2^{\sqrt{\log_2 N}})$	DL, DPair	Trustless	Yes
This Work	$O(\log_2 N)$	$O(N)$	$O(\sqrt[3]{N}\sqrt{\log_2 N})$	DL	Trustless	Yes

Comm., Prover, and Verifier mean cost of communication, prover computation, and verifier computation, respectively. Pairing means requirement of pairing-friendly groups.

Table 1. Comparison Table of IPAs for length-N vectors

1.1 Technical Overview

Two-tier Commitment with Proof. We first revisit Protocol4, pairing-free square root verifier IPA [29]. The main idea of Protocol4 is a two-tier commitment scheme with a proof for the second layer. The two-tier commitment scheme comprises two layers. In the first layer, mn-length vectors are compressed into n elliptic curve points using a parallel Pedersen commitment scheme with a m-dimensional commitment key. Subsequently, these n elliptic curve points are interpreted as $3n$-length vectors in the embedding field. In the second layer, these vectors are further compressed into a single elliptic curve point through a

Pedersen commitment scheme with a $3n$-dimensional commitment key.
The proof of the second layer is intricately connected to the commitment scheme used in the second layer. Concretely, the proof should ensure knowledge of the first layer results and the elliptic curve relation between them. To address this issue, homomorphic commitment and a related proof system are required. From this viewpoint, we generalize the second layer commitment from the Pedersen commitment to any homomorphic commitments.
Proof for Elliptic Curve Relation. The second layer proof is about the elliptic curve relation. The proof is constructed by decomposing the elliptic curve relation to the arithmetic relation over the embedding field and then adapting the proof system for the arithmetic relation. In [29], they adopted a projective representation of an elliptic curve for the arithmetization of the elliptic curve operation because of the simple expression of complete addition. In [40], the authors proposed an efficient proof for the complete addition of the affine formula using the Plonk proof system. Because the affine representation has an efficiency advantage over the projective representation, we adopt the idea of [40] to construct a proof for the elliptic curve relation.
Plonk-Friendly Extended Polynomial Commitment Scheme. In the proof of the second layer, the prover claims knowledge of vectors and the corresponding elliptic curve relation. Constructing a proof system that satisfies both conditions is intricate because the elliptic curve relation is associated with all committed vectors. To address this, we propose a Plonk-friendly extended PCS constructed from a homomorphic PCS compatible with Plonkish elliptic curve proof system [40]. Concretely, the new PCS helps to show the consistency of committed vectors and wire polynomials from Plonkish arithmetization.
Cubic Root Verifier Inner Product Argument. From the above results, we conclude that the protocol features $O(\log_2 mn)$ communication and $O(m + \|\mathcal{V}_{\mathsf{Eval}}(n \log_2 m)\|)$, where $\|\mathcal{V}_{\mathsf{Eval}}(n \log_2 m)\|$ is the verifier complexity of Eval, the evaluation protocol of the PCS for degree $O(n \log_2 m)$ polynomials. Then, we apply Leopard evaluation protocol, which features square root verifier complexity. Finally, we set the parameters $m = \sqrt[3]{N}$ and $n = \sqrt[3]{N^2}$, where N is the length of the witness vectors. Then, the total verifier complexity is $O(m + \|\mathcal{V}_{\mathsf{Eval}}(n \log_2 m)\|) = O(\sqrt[3]{N} + \sqrt[3]{N}\sqrt{\log_2 N})$, which is the cubic root of N.

1.2 Related Works

ZK Argument based on Discrete Logarithm Setting. Groth [25] first proposed a sublinear ZK argument for AC under the DL assumption, and Seo [38] improved it. These works feature constant round complexity as well. Later works [8, 12, 14, 31, 17, 29] focus on reducing communication complexity (to logarithmic scale) rather than round complexity (allowing logarithmic complexity). Starting from Bulletproofs [8, 12], various works have been proposed to improve the verifier complexity of Bulletproofs [31, 17, 29]. In a different view point, Kim et al. [29] proposed sublogarithmic communication ZK argument for the first time, and then Lee et al. [30] enhanced it from linear verifier cost to sublinear one with sublogarithm communication.

ZK Argument based Other Settings. There are other approaches for constructing ZKP. Unknown order group [14, 2] based schemes features logarithmic verifier complexity but the prover complexity is quasi-linear.

To overcome vulnerability against quantum computer-aided attacks, cryptographic hash-based ZK scheme [5, 41] and lattice-based ZK scheme [3, 33, 10] are proposed. However, both schemes feature large communication complexity, at least $O(\log_2^2 N)$, compared with the DL setting, $O(\log_2 N)$.

2 Preliminary

2.1 Definitions and Notations

We first define the notations used in the paper. $[\ell]$ denotes a set of integers from 1 to ℓ. We denote a negligible function as $\mathsf{negl} : \mathbb{N} \to \mathbb{R}$, which satisfies that: for any $c \in \mathbb{N}$, there exists N_c such that $\mathsf{negl}(\lambda) < 1/\lambda^c$ for all $\lambda > N_c$. For a prime p, we denote asymmetric bilinear groups of order p, $\mathbb{G}_1, \mathbb{G}_2$, and $\mathbb{G}_t$ with a non-degenerated bilinear map $e : \mathbb{G}_1 \times \mathbb{G}_2 \to \mathbb{G}_t$. We use additive notation to describe group operations on $\mathbb{G}_1$, $\mathbb{G}_2$, and $\mathbb{G}_t$. To denote a scalar multiplication, we denote $[k]G$ for a scalar $k \in \mathbb{Z}_p$ and $G \in \mathbb{G}$. We prefer to use upper and lowercase letters to denote group elements and field elements, respectively. We use bold font to represent vectors in $\mathbb{Z}_p^m$ or $\mathbb{G}^m$. For a vector $\boldsymbol{a} \in \mathbb{Z}_p^m$ and $i \in [m]$, we use a_i (non-bold style letter with a subscript i) to denote the i-th element of $\boldsymbol{a}$. We use $\|$ notation to represent concatenation of two vectors, *i.e.*, for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}_p^m$, $\boldsymbol{a} \parallel \boldsymbol{b} = (a_1, \ldots, a_m, b_1, \ldots, b_m)$.

For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}_p^m$ and $\boldsymbol{G}, \boldsymbol{H} \in \mathbb{G}^m$, we use the following vector notations:

- **Component-wise addition** : $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, \ldots, a_m + b_m) \in \mathbb{Z}_p^m$ and $\boldsymbol{G} + \boldsymbol{H} = (G_1 + H_1, \ldots, G_m + H_m) \in \mathbb{G}^m$.
- **Component-wise product** : $\boldsymbol{a} \circ \boldsymbol{b} = (a_1 b_1, \ldots, a_m b_m) \in \mathbb{Z}_p^m$.
- **Multi-Scalar Multiplication** : $[\boldsymbol{x}]\boldsymbol{G} = \sum_{i \in [m]} [x_i] G_i \in \mathbb{G}$.
- **Inner Pairing Product** : $\boldsymbol{E}(\boldsymbol{G}, \boldsymbol{H}) = \sum_{i \in [m]} e(G_i, H_i) \in \mathbb{G}_t$, where $\boldsymbol{G} \in \mathbb{G}_1^m$ and $\boldsymbol{H} \in \mathbb{G}_2^m$.

Parallel Multi-Scalar Multiplication. Let $\boldsymbol{a} \in \mathbb{Z}_p^{m \times n}$ be a matrix and $\boldsymbol{G} \in \mathbb{G}^m$ be group elements. We denote $[\boldsymbol{a}]\boldsymbol{G} := ([\boldsymbol{a}_1]\boldsymbol{G}, \ldots, [\boldsymbol{a}_n]\boldsymbol{G})$, where $\boldsymbol{a}_i \in \mathbb{Z}_p^m$ is the i-th column vector of matrix $\boldsymbol{a}$.

Argument System for Relation $\mathcal{R}$. Let $\mathcal{R}$ be a polynomial-time verifiable relation consisting of common reference string (CRS), statement, and witness, denoted by σ, x, and w respectively. An interactive argument system for relation $\mathcal{R}$ consists of three probabilistic polynomial-time algorithms (PPTs), a key generation algorithm $\mathcal{K}$, a prover algorithm $\mathcal{P}$, and a verifier algorithm $\mathcal{V}$. The $\mathcal{K}$ algorithm takes the security parameter λ and outputs CRS, which is the input of $\mathcal{P}$ and $\mathcal{V}$. $\mathcal{P}$ and $\mathcal{V}$ generate a transcript interactively, denoted by $tr \leftarrow \langle \mathcal{P}(\sigma, x, w), \mathcal{V}(\sigma, x) \rangle$. At the end of the transcript, $\mathcal{V}$ outputs a bit, 0 or 1, which means reject or accept, respectively.

Argument of Knowledge. An argument of knowledge (AoK) is a special case of an argument system that satisfies the properties of completeness and witness extractability. As previous works did [12, 29], we consider witness-extended emulation [32] for the latter, which is equivalent to knowledge soundness.

Definition 1 (Perfect Completeness). *Let $(\mathcal{K}, \mathcal{P}, \mathcal{V})$ be an argument system and $\mathcal{R}$ be a polynomial-time verifiable relation. We say that the argument system $(\mathcal{K}, \mathcal{P}, \mathcal{V})$ for the relation $\mathcal{R}$ has perfect completeness if, for every PPT adversary $\mathcal{A}$, the following probability equation holds:*

$$\Pr\left[\begin{array}{l|l} tr \leftarrow \langle \mathcal{P}(\sigma, x, w), \mathcal{V}(\sigma, x)\rangle & \sigma \leftarrow \mathcal{K}(1^\lambda); \\ tr \textsf{ is accepting} & (x, w) \leftarrow \mathcal{A}(\sigma) \\ & \wedge(\sigma, x; w) \in \mathcal{R} \end{array}\right] = 1$$

Definition 2 (Computational Witness Extended Emulation). *Let $(\mathcal{K}, \mathcal{P}, \mathcal{V})$ be an argument system and $\mathcal{R}$ be a polynomial-time verifiable relation. We say that the argument $(\mathcal{P}, \mathcal{V})$ has computational witness-extended emulation if, for every deterministic polynomial prover $\mathcal{P}^*$, which may not follow $\mathcal{P}$, and all pairs of interactive polynomial-time adversaries $(\mathcal{A}_1, \mathcal{A}_2)$, there exists a polynomial time emulator $\mathcal{E}$, the following probability equation holds:*

$$\left|\begin{array}{l} \Pr\left[\mathcal{A}_1(tr) = 1 \left| \begin{array}{l} \sigma \leftarrow \mathcal{K}(1^\lambda);\ (x, s) \leftarrow \mathcal{A}_2(\sigma); \\ tr \leftarrow \langle \mathcal{P}^*(\sigma, x, s), \mathcal{V}(\sigma, x)\rangle \end{array}\right.\right] - \\ \Pr\left[\begin{array}{l} \mathcal{A}_1(tr) = 1 \wedge \\ (\sigma, w, x) \in \mathcal{R} \end{array} \left| \begin{array}{l} \sigma \leftarrow \mathcal{K}(1^\lambda);\ (x, s) \leftarrow \mathcal{A}_2(\sigma); \\ (tr, w) \leftarrow \mathcal{E}^{\mathcal{O}}(\sigma, x), tr \textsf{ is accepting} \end{array}\right.\right] \end{array}\right| < \mathsf{negl}(\lambda)$$

The emulator $\mathcal{E}$ can access the oracle $\mathcal{O} = \langle \mathcal{P}^*(\sigma, x, s), \mathcal{V}(\sigma, x)\rangle$, which outputs the transcript between $\mathcal{P}^*$ and $\mathcal{V}$. $\mathcal{E}$ permits to rewind $\mathcal{P}^*$ at a specific round and rerun $\mathcal{V}$ using fresh randomness. s can be considered as the state of $\mathcal{P}^*$, which includes randomness.

Definition 3. *We say that the argument system $(\mathcal{K}, \mathcal{P}, \mathcal{V})$ is an argument of knowledge for relation $\mathcal{R}$ if the argument has (perfect) completeness and (computational) witness-extended emulation.*

Trusted Setup. In some arguments, the key generation algorithm takes a trapdoor that should not be revealed to anyone, including the prover and verifier. In this case, CRS generation should be run by a trusted third party. A setting requiring a trusted party is called the trusted setup.

Non-interactive Argument in the Random Oracle Model. We call an interactive argument a public coin if $\mathcal{V}$ outputs without decision bits constituting a uniformly random message without dependency of $\mathcal{P}$'s messages. Fiat and Shamir [20] proposed a method to transform any public coin interactive argument into a non-interactive one using the random oracle model. The approach involves replacing $\mathcal{V}$'s random messages with random oracle outputs, where the inputs are derived from previous messages at that point.

Assumptions. Let $\mathcal{G}$ be a group generator that takes security parameters λ and then outputs $\mathbb{G}$, describing a group of order p.

Definition 4 (Discrete Logarithm Relation Assumption). *We say that* $\mathbb{G}$ *satisfies the discrete logarithm relation (DLR) assumption if, for all non-uniform polynomial-time adversaries* $\mathcal{A}$*, the following inequality holds:*

$$\Pr\left[\boldsymbol{a} \neq \boldsymbol{0} \wedge \boldsymbol{g}^{\boldsymbol{a}} = 1_{\mathbb{G}} \;\middle|\; \begin{array}{l}(p, g, \mathbb{G}) \leftarrow \mathcal{G}(1^{\lambda}), \boldsymbol{g} \overset{\$}{\leftarrow} \mathbb{G}^{n};\\ \boldsymbol{a} \leftarrow \mathcal{A}(\boldsymbol{g}, p, g, \mathbb{G})\end{array}\right] \leq \mathsf{negl}(\lambda)$$

It is well-known that the discrete logarithm relation (DLR) assumption is equivalent to the discrete logarithm (DL) assumption [12, 29].

Definition 5 (Commitment Scheme). *A commitment scheme* $\mathcal{C}$ *consists of three PPT algorithms: a key generation* Gen*, a commitment* Com*, and an open* Open*. A commitment scheme* $\mathcal{C} = (\mathsf{Gen}, \mathsf{Com}, \mathsf{Open})$ *over a message space* M*, a random space* R*, and a commitment space* C *is defined by:*

– $\underline{\mathsf{Gen}(1^{\lambda}, \ell) \to \mathsf{ck}}$ *: On input security parameter* λ *and dimension of message space* ℓ*, sample commitment key* ck
– $\underline{\mathsf{Com}(\mathsf{ck}, m; r) \to C}$ *: Take commitment key* ck*, message* $m \in \mathsf{M}$*, and randomness* $r \in \mathsf{R}$*, output commitment* $C \in \mathsf{C}$
– $\underline{\mathsf{Open}(\mathsf{ck}, m, r, C) \to 0/1}$ *: Take commitment key* ck*, message* $m \in \mathsf{M}$*, randomness* $r \in \mathsf{R}$*, and commitment* $C \in \mathsf{C}$ *output* 1 *if* $\mathsf{Com}(\mathsf{ck}, m; r) = C$*,* 0 *otherwise.*

Since the Open *algorithm can be described by using* Com *algorithm, we omit the* Open *algorithm from the commitment scheme* $\mathcal{C}$*. Now, we call* $\mathcal{C} = (\mathsf{Gen}, \mathsf{Com})$ *a commitment scheme if the following properties hold:*
Binding: *For any expected PPT adversary* $\mathcal{A}$*,*

$$\Pr\left[m_0 \neq m_1 \;\middle|\; \begin{array}{l}\mathsf{ck} \leftarrow \mathsf{Gen}(1^{\lambda}, \ell); (m_0, r_0, m_1, r_1) \leftarrow \mathcal{A}(\mathsf{ck})\\ \wedge C_0 = C_1 \, where \; C_i = \mathsf{Com}(\mathsf{ck}, m_i; r_i)\end{array}\right] \leq \mathsf{negl}(\lambda)$$

Hiding: *For any expected PPT adversary* $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$

$$\left|\Pr\left[\mathsf{b} = \mathsf{b}' \;\middle|\; \begin{array}{c}\mathsf{ck} \leftarrow \mathsf{Gen}(1^{\lambda}, \ell); (m_0, m_1, \mathsf{state}) \leftarrow \mathcal{A}_1(\mathsf{ck});\\ \mathsf{b} \overset{\$}{\leftarrow} \{0, 1\}, r \overset{\$}{\leftarrow} \mathcal{R},\\ C \leftarrow \mathsf{Com}(\mathsf{ck}, m_{\mathsf{b}}; r); \mathsf{b}' \leftarrow \mathcal{A}_2(\mathsf{ck}, C, \mathsf{state}),\end{array}\right] - \frac{1}{2}\right| \leq \mathsf{negl}(\lambda)$$

Additionally, we call a commitment scheme $\mathcal{C}$ *is (additively) homomorphic if the following property holds:*
(Additive) Homomorphic: *For any commitment key* $\mathsf{ck} \leftarrow \mathsf{Gen}(1^{\lambda}, \ell)$ *and pairs of message-randomness* $(m_0, r_0), (m_1, r_1) \in \mathsf{M} \times \mathsf{R}$*, the following equality holds:* $\mathsf{Com}(\mathsf{ck}, m_0; r_0) + \mathsf{Com}(\mathsf{ck}, m_1; r_1) = \mathsf{Com}(\mathsf{ck}, m_0 + m_1; r_0 + r_1)$

Homomorphic Vector Commitment Schemes. A homomorphic vector commitment scheme is a homomorphic commitment for N-dimensional message, etc. $\mathbb{Z}_p^N$ or $\mathbb{G}^N$. We introduce two widely used homomorphic vector commitment schemes: *Pedersen vector commitment* [35] and *AFGHO group commitment* [1].

– $\underline{\mathsf{Gen}_{\mathsf{Ped}}(1^\lambda, N) \to (\boldsymbol{G}, H)}$:
  1. Sample $\boldsymbol{G} \xleftarrow{\$} \mathbb{G}^N$ and $H \xleftarrow{\$} \mathbb{G}$
  2. Output $\mathsf{ck} = (\boldsymbol{G}, H) \in \mathbb{G}^N \times \mathbb{G}$

– $\underline{\mathsf{Com}_{\mathsf{Ped}}((\boldsymbol{G}, H), \boldsymbol{a}; r) \to C}$:
  1. Compute $C = [\boldsymbol{a}]\boldsymbol{G} + [r]H$
  2. Output $C \in \mathbb{G}$

– $\underline{\mathsf{Gen}_{\mathsf{GC}}(1^\lambda, N) \to (\boldsymbol{F}, K)}$:
  1. Sample $\boldsymbol{F} \xleftarrow{\$} \mathbb{G}_2^N$ and $K \xleftarrow{\$} \mathbb{G}_t$
  2. Output $\mathsf{ck} = (\boldsymbol{F}, K) \in \mathbb{G}_2^N \times \mathbb{G}_t$

– $\underline{\mathsf{Com}_{\mathsf{GC}}((\boldsymbol{F}, K), \boldsymbol{G}; r) \to C}$:
  1. Compute $C = \boldsymbol{E}(\boldsymbol{G}, \boldsymbol{F}) + [r]K$
  2. Output $C \in \mathbb{G}_t$

Fig. 1. Homomorphic Vector Commitment Schemes

Pedersen vector commitment. Pedersen vector commitment $\mathcal{C}_{\mathsf{Ped}} = (\mathsf{Gen}_{\mathsf{Ped}}, \mathsf{Com}_{\mathsf{Ped}})$ is a commitment scheme over message space $\mathbb{Z}_p^N$. Pedersen vector commitment provides perfect hiding and computational binding under the DL assumption. Specially, we sometimes use subscript Ped, p for Pedersen commitment over group $\mathbb{G}_p$ of order p to distinguish base group.

AFGHO group commitment. AFGHO group commitment $\mathcal{C}_{\mathsf{GC}} = (\mathsf{Gen}_{\mathsf{GC}}, \mathsf{Com}_{\mathsf{GC}})$ is a commitment scheme over message space $\mathbb{G}_1^N$. $\mathcal{C}_{\mathsf{GC}}$ uses a bilinear pairing for the commitment algorithm.

Two-tier Commitment Scheme. A two-tier commitment is a commitment scheme for a two-dimensional array, *e.g.* $\mathbb{Z}_p^{m \times n}$. The use of the two-tier commitment scheme has some merits. To construct an IPA with a two-tier commitment, the size of the common reference string (CRS) can be reduced sublinear of $N = mn$, concretely, $O(n + m)$. This reduced CRS leads to a reduction in the verification cost of IPA [15, 31, 29, 28]. A two-tier commitment scheme is constructed by combining two distinct commitment schemes $\mathcal{C}_1 = (\mathsf{Gen}_1, \mathsf{Com}_1)$ and $\mathcal{C}_2 = (\mathsf{Gen}_2, \mathsf{Com}_2)$. For a matrix in $\mathbb{Z}_p^{m \times n}$, commit m row vectors using the first commitment algorithm Com_1 in parallel. After then, with regard m commitments from Com_1 as a message of Com_2, use the second commitment algorithm Com_2, and output it.

Definition 6 (Two-tier Commitment Scheme). *Let* $\mathcal{C}_1 = (\mathsf{Gen}_1, \mathsf{Com}_1)$ *and* $\mathcal{C}_2 = (\mathsf{Gen}_2, \mathsf{Com}_2)$ *be commitment schemes over (message,commitment,randomness) space* $(\mathbb{Z}_p^n, \mathsf{C}_1, \mathsf{R}_1)$ *and* $(\mathsf{C}_1^m, \mathsf{C}_2, \mathsf{R}_2)$ *respectively. Then, the commitment scheme* $\mathcal{C} = (\mathsf{Gen}, \mathsf{Com})$ *over space* $((\mathbb{Z}_p^{m \times n}, \mathsf{C}_2, \mathsf{R}_1 \times \mathsf{R}_2)$ *is called as a two-tier commitment scheme based on* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *defined by:*

– $\underline{\mathsf{Gen}(1^\lambda, mn) \to \mathsf{ck} = (\mathsf{ck}_1, \mathsf{ck}_2)}$:
  1. Run $\mathsf{Gen}_1(1^\lambda, n) \to \mathsf{ck}_1$
  2. Run $\mathsf{Gen}_2(1^\lambda, m) \to \mathsf{ck}_2$
  3. Return $\mathsf{ck} = (\mathsf{ck}_1, \mathsf{ck}_2)$

– $\underline{\mathsf{Com}(\mathsf{ck}, M; (\boldsymbol{r}, r_f)) \to \mathrm{C}}$:
  1. Compute $\mathsf{Com}_1(\mathsf{ck}_1, M_i; r_i) \to C_i, \forall i$
  2. Compute $\mathsf{Com}_2(\mathsf{ck}_2, \boldsymbol{C}; r_f) \to \mathrm{C}$
  3. Return C

Specially, we use roman-style to denote commitment from two-tier commitment schemes. In terms of IPA, the binding property of the commitment is sufficient for ensuring soundness. Since our main focus is the construction of IPA, we omit the randomness r in the commitment algorithm, which does not affect the binding property. Hereafter, we simply write $\mathsf{Com}(\mathsf{ck}, M)$ to describe the commitment algorithm for a message M.

Pairing-based Two-tier Commitment Scheme. From two commitment schemes $\mathcal{C}_1 = (\mathsf{Gen}_{\mathsf{Ped}}, \mathsf{Com}_{\mathsf{Ped}})$ and $\mathcal{C}_2 = (\mathsf{Gen}_{\mathsf{GC}}, \mathsf{Com}_{\mathsf{GC}})$ over spaces $(\mathbb{Z}_p^{mn}, \mathbb{G}_1^n, \mathbb{G}_1)$ and $(\mathbb{G}_1^m, \mathbb{G}_2^m, \mathbb{G}_t)$ respectively, one can construct a homomorphic two-tier commitment scheme. The homomorphic two-tier commitment is widely used for constructing sublinear verifier IPA schemes [15, 31, 29]. The homomorphic property helps to apply the folding technique in Bulletproofs; however, this construction is restricted to a choice of a base group: pairing-friendly elliptic curves.
Polynomial Commitment Scheme. A polynomial commitment scheme (PCS) [27, 14] is a special case of the commitment scheme that commits the given polynomial within the specific degree bound d. PCS allows convincing polynomial evaluation without opening the polynomial itself. Concretely, PCS contains an argument system $\mathsf{Eval} = (\mathcal{K}, \mathcal{P}, \mathcal{V})$ for the following relation:

$$\mathcal{R}_{\mathsf{Eval}} = \left\{ \begin{array}{c} (\mathsf{ck}_{\mathsf{PC}}, C \in \mathsf{C}, z, y \in \mathbb{Z}_p, d \in \mathbb{N}; f \in \mathbb{Z}_p^{\leq d}[X]) : \\ C = \mathsf{Com}(\mathsf{ck}_{\mathsf{PC}}, f(X)) \wedge y = f(z) \end{array} \right\} \tag{1}$$

The formal definition of PCS is given as below:

Definition 7 (Polynomial Commitment Scheme). *A polynomial commitment scheme* $\mathsf{PCS} = (\mathsf{Gen}, \mathsf{Com}, \mathsf{Eval})$ *consists of key generation algorithms* Gen*, commitment algorithm* Com*, and argument system* Eval *for the relation* $\mathcal{R}_{\mathsf{Eval}}$*. We call* $\mathsf{PCS} = (\mathsf{Gen}, \mathsf{Com}, \mathsf{Eval})$ *is a polynomial commitment scheme if the following properties hold:*

- *The commitment scheme* $(\mathsf{Gen}, \mathsf{Com})$ *satisfies the binding property.*
- *The argument system* Eval *is an AoK for the relation* $\mathcal{R}_{\mathsf{Eval}}$ *in Eq.* (1)

2.2 Plonkish: Proof for Elliptic Curve Relation

In Protocol4, one of the main bottlenecks was checking the relation between elliptic curve points. More precisely, for elliptic curve points $L_i, R_i, P_{i+1}, P_i \in E(\mathbb{Z}_p)$ which are in fact commitments of corresponding messages, and a scalar $x_i \in \mathbb{Z}_p$, the relation of the form $P_{i+1} \overset{?}{=} [x_i^{-1}]L_i + P_i + [x_i]R_i$ should be ensured during the protocol. However, due to its construction, the commitment scheme to produce each curve point is no longer *homomorphic*, so [29] took a strategy to convert the relation into the AC. To this end, rather than using the affine coordinate representation, they attempted to represent each elliptic curve point as the projective coordinate representation, where the complete point addition formula is known [36] for prime order short Weierstrass curves. But, this increases the number of input gates by a factor of 3 on the number of elliptic curve points.

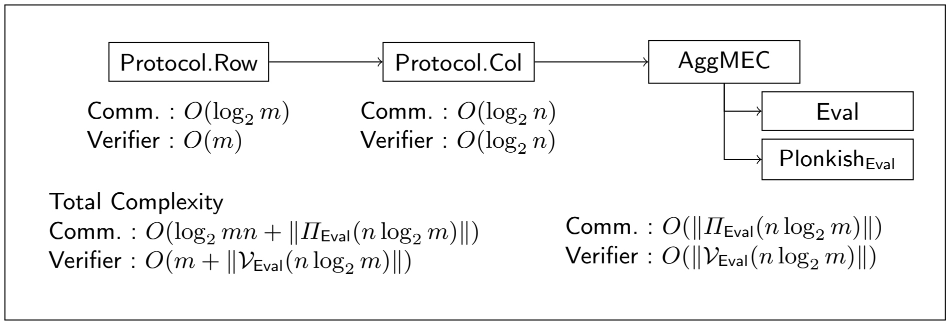
Plonk [22] is one of the well-known methods to represent the circuit satisfiability of the given AC as the constraints system. By Lagrange interpolation, the latter can be converted to showing the equality of two polynomials, which can be proved efficiently by PIOP instantiated by PCS [14]. As shown in [21, 40], Plonk-style arithmetization can cope with *custom gates*, which are arithmetic gates other than addition or multiplication, efficiently. Hence, by utilizing an appropriate custom gate for elliptic curve addition in affine coordinates, we can reduce the blow-up factor from representing the elliptic curve point to 2, alleviating the above problem on the size of the CRS and the circuit. For this reason, we use $\mathsf{Plonkish}$ [40], which is an extension of Plonk by constructing a constraint system about the execution trace acquired from running the given AC. $\mathsf{Plonkish}$ supports custom gates and look-up operations. For constructing the custom gate of the elliptic curve operation, we follow the method from [40].

Throughout this paper, we will denote $\mathsf{Plonkish}_{\mathsf{Eval}}$ as the proof system for $\mathsf{Plonkish}$ supporting the custom gate for elliptic curve addition. $\mathsf{Plonkish}_{\mathsf{Eval}}$ takes a commitment key $\mathsf{ck}_{\mathsf{PC}}$ for the underlying PCS as a public input. For witnesses, $\mathsf{Plonkish}_{\mathsf{Eval}}$ takes 6 wire polynomials $w_L^{(1)}$, $w_L^{(2)}$, $w_R^{(1)}$, $w_R^{(2)}$, $w_O^{(1)}$, $w_O^{(2)}$ corresponding to the 1st, 2nd coordinates of curve points in each wire and 5 auxiliary polynomials α, β, γ, δ, λ required for the elliptic curve point addition. The detailed explanation about $\mathsf{Plonkish}_{\mathsf{Eval}}$ and the construction of the custom gate are given in Appendix D.

3 Main Results

3.1 Reconstruction of Protocol4

In this section, we generalize the IPA $\mathsf{Protocol4}$ [29]. Before describing the protocol, we focus on the structure of the commitment scheme used in $\mathsf{Protocol4}$.
Doubly-Pedersen Two-tier Commitment Scheme. To remove reliance on the pairing operation, Kim et al. proposed Pedersen commitment for the elliptic curve points, which are already committed by the Pedersen commitment scheme. This approach can be viewed as a two-tier commitment scheme using the Pedersen commitment on both the first and second layers. For convenience, we call this commitment scheme as the *Doubly-Pedersen two-tier commitment scheme*. The doubly-Pedersen two-tier commitment process for $\boldsymbol{a} \in \mathbb{Z}_p^{m\times n}$ is as follows: First, commit each row vector of $\boldsymbol{a}$ using Pedersen vector commitment on the group of elliptic curve points $\mathbb{G} = E(\mathbb{Z}_q)$ over the field $\mathbb{Z}_q$. After the first layer commitment, one gets n distinct elliptic curve points. For the second layer commitment, one considers n elliptic curve points in $E(\mathbb{Z}_q)$ as coordinates of the field elements in $\mathbb{Z}_q$ and then recommits them using the Pedersen vector commitment on the elliptic curve $\mathbb{G}_q$ of order q.
Homomorphic Vector Commitment in Second Layer. Contrary to the homomorphic commitment schemes, such as Pedersen commitment and AFGHO group commitment, the doubly-Pedersen two-tier commitment scheme does not have a homomorphic property [29]. For this reason, to apply the folding technique [8, 12] on the doubly-Pedersen commitment-based IPA, the prover should send additional proofs to ensure the validity of group operations, which are brought by Pedersen commitment in the first layer. Because the homomorphic property of second commitments helps to construct additional proofs efficiently, we prefer to use homomorphic commitment at the second layer. Additionally, the role of the second commitment is compressing a large message to single commitment, e.g. from $\mathbb{Z}_q^N$ to C, so that the second commitment satisfies the *compression* property; converts N-dimensional message into a single element.



Protocol.Row
Protocol.Col
AggMEC
Eval
Plonkish$_{\mathsf{Eval}}$
Comm. : $O(\log_2 m)$
Verifier : $O(m)$
Comm. : $O(\log_2 n)$
Verifier : $O(\log_2 n)$
Total Complexity
Comm. : $O(\log_2 mn + \|\Pi_{\mathsf{Eval}}(n \log_2 m)\|)$
Verifier : $O(m + \|\mathcal{V}_{\mathsf{Eval}}(n \log_2 m)\|)$
Comm. : $O(\|\Pi_{\mathsf{Eval}}(n \log_2 m)\|)$
Verifier : $O(\|\mathcal{V}_{\mathsf{Eval}}(n \log_2 m)\|)$

Fig. 2. Process of Protocol

For a precise description, let us consider the Pedersen commitment scheme $\mathcal{C}_1 = (\mathsf{Gen}_{\mathsf{Ped}}, \mathsf{Com}_{\mathsf{Ped}})$ over $(\mathbb{Z}_p^m, \mathbb{G}_p = E(\mathbb{Z}_q))$ at the first layer and a homomorphic commitment scheme $\mathcal{C}_2 = (\mathsf{Gen}_2, \mathsf{Com}_2)$ over $(\mathbb{Z}_q^{2n}, \mathsf{C})$ at the second layer. At the second commitment, we consider group elements (elliptic curve points) as pair of $\mathbb{Z}_q$ elements following affine coordinates. Then, we can construct two-tier commitment scheme $\mathcal{C}_{\mathsf{TC}} = (\mathsf{Gen}_{\mathsf{TC}}, \mathsf{Com}_{\mathsf{TC}})$ as follows:

– $\underline{\mathsf{Gen}_{\mathsf{TC}}(1^\lambda, mn) \to \mathsf{ck} = (\boldsymbol{G}, \mathsf{ck}_2)}$:
  1. Run $\mathsf{Gen}_{\mathsf{Ped},p}(1^\lambda, n) \to \boldsymbol{G} \in \mathbb{G}_p^n$
  2. Run $\mathsf{Gen}_2(1^\lambda, 2m) \to \mathsf{ck}_2 \in \mathbb{G}_q^{2m}$
  3. Return $\mathsf{ck} = (\boldsymbol{G}, \mathsf{ck}_2)$

– $\underline{\mathsf{Com}_{\mathsf{TC}}(\mathsf{ck} = (\boldsymbol{G}, \mathsf{ck}_2), \boldsymbol{a} \in \mathbb{Z}_p^{m \times n}) \to \mathrm{C} \in \mathbb{G}_q}$:
  1. Compute $\mathsf{Com}_{\mathsf{Ped},p}(\boldsymbol{G}, \boldsymbol{a}_i) \to C_i \in \mathbb{G}_p, \forall i \in [m]$
  2. Compute $\mathsf{Com}_2(\mathsf{ck}_2, \boldsymbol{C}) \to \mathrm{C} \in \mathsf{C}$
  3. Return C

Using the commitment $\mathcal{C}_{\mathsf{TC}}$, we consider IPA for the following relation:

$$\mathcal{R}^{m,n}_{\mathsf{GenPT4}} = \left\{ \begin{matrix} (\boldsymbol{G}, \boldsymbol{H} \in \mathbb{G}_p^m, \mathsf{ck}_2, \mathrm{P} \in \mathbb{G}_q, c \in \mathbb{Z}_p; \boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}_p^{m \times n}) : \\ \mathrm{P} = \mathsf{Com}_{\mathsf{TC}}((\boldsymbol{G} \parallel \boldsymbol{H}, \mathsf{ck}_2), \boldsymbol{a} \parallel \boldsymbol{b}) \wedge c = \langle \boldsymbol{a}, \boldsymbol{b} \rangle \end{matrix} \right\} \quad (2)$$

We intend to construct an IPA in two parts: the reduction part and the proof of the multi-elliptic curve (MEC) operation part. The reduction part reduces the argument from the relation $\mathcal{R}^{m,n}_{\mathsf{GenPT4}}$ to $\mathcal{R}^{m/2,n}_{\mathsf{GenPT4}}$(Row-reduction) or $\mathcal{R}^{1,n}_{\mathsf{GenPT4}}$ to $\mathcal{R}^{1,n/2}_{\mathsf{GenPT4}}$(Column-reduction). The overall process of the proposed IPA is as follows: first, the prover and verifier run row-wise reduction Protocol.Row recursively until the row of the witness reaches $m = 1$. Then, they run column-wise reduction Protocol.Col recursively until the column of the witness reaches $n = 1$. Next is proof for the MEC operation part. In this part, the prover and verifier run AggMEC for ensuring elliptic curve relation between witness vectors. In this phase, Eval and Plonkish$_{\mathsf{Eval}}$ are used as subroutines. Notice that both have verifier complexity $\|\mathcal{V}_{\mathsf{Eval}}(n \log_2 m)\|$. We illustrate the overall process in Fig. 2.

Reduction and Store the States. In the reduction protocol, the prover and verifier recursively run the reduction process: reduction from an argument for vectors to those for half-sized vectors. Contrary to Bulletproofs [8, 12] or Leopard [28], the prover and verifier store the history of reduction processes because the verifier has not been convinced of the group operation relation between received commitments yet. The states st_V and st_P role recording the history of the verifier and prover, respectively. st_V and st_P are used to run the aggregated multi-elliptic curve operation, AggMEC, which guarantees the validity of the inner value of commitment for every round. We illustrate states st_V and st_P in Fig. 3. Hereafter, we denote $\mu = \log_2 m$ and $\nu = \log_2 n$.

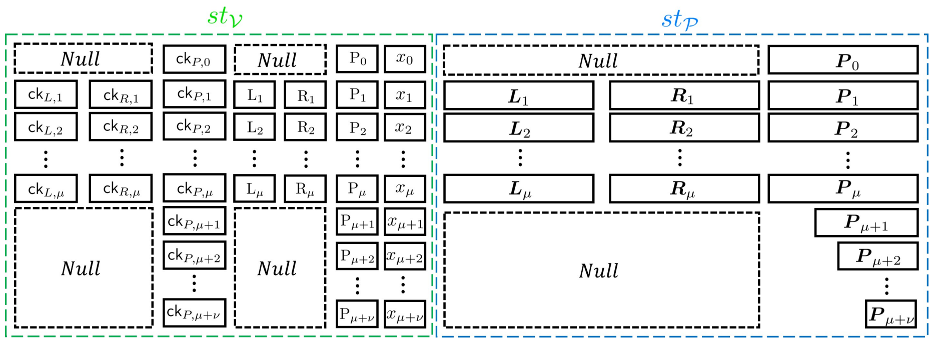


Fig. 3. Format of st_V and st_P

Row-wise Reduction: Algorithm 1. In row-wise reduction, the $\mathcal{P}$ sends crossed inner product values c_L, c_R and commitments L, R, whose messages are pairs of half-sized witness vectors, to $\mathcal{V}$. Then, $\mathcal{V}$ sends challenge x to $\mathcal{P}$. Contrary to other IPAs based on homomorphic commitments, $\mathcal{V}$ cannot update the instance $\widehat{\mathrm{P}}$ for the next round. To resolve this issue, $\mathcal{P}$ sends an updated instance $\widehat{\mathrm{P}}$ to $\mathcal{V}$. In this phase, $\mathcal{V}$ should verify the well-construction of $\widehat{\mathrm{P}}$, but we postpone the verification of it and run the row-wise reduction recursively until $m = 1$. At $m = 1$, $\mathcal{P}$ and $\mathcal{V}$ run Protocol.Col. The description of Protocol.Row is given in Algorithm 1.

Column-wise Reduction: Algorithm 2. In column-wise reduction, the $\mathcal{P}$ sends crossed inner product values c_L and c_R. Then, $\mathcal{V}$ sends a challenge x to $\mathcal{P}$. The update process is different from that of row-wise reduction because the first commitment key is already compressed to a single element, G and H. In order to update the instance, $\mathcal{P}$ parses the vector $\boldsymbol{P}$ to 4 parts and then constructs the half-length updated vector $\widehat{\mathrm{P}}$. At the end of Protocol.Col, $\mathcal{P}$ and $\mathcal{V}$ additionally run AggMEC for knowledge of tuples of $(\boldsymbol{L}, \boldsymbol{R}, \boldsymbol{P})$, which guarantees well-construction of $\widehat{\mathrm{P}}$ for all rounds in both row-wise and column-wise reduction. The description of Protocol.Col is given in Algorithm 2.

Theorem 1. *Assume that both* Protocol.Col *and* AggMEC *provide perfect completeness and computational witness-extended emulation. Then,* Protocol.Row *in Algorithm 1 has perfect completeness and computational witness-extended emulation under the DL assumption.*

Theorem 2. *Assume that* AggMEC *provides perfect completeness and computational witness-extended emulation. Then,* Protocol.Col *in Algorithm 2 has perfect completeness and computational witness-extended emulation under the DL assumption.*

Algorithm 1 Protocol.Row

```
    $\mathsf{Protocol.Row}(\boldsymbol{G}, \boldsymbol{H}, (\mathsf{ck}_k)_{k=s}^{\mu}, \mathsf{ck}_{\mathsf{Col}}, \mathrm{P}, c, st_V; \boldsymbol{a}, \boldsymbol{b}, st_p)$
    where $\mathsf{ck}_k = (\mathsf{ck}_{L,k}, \mathsf{ck}_{R,k}, \mathsf{ck}_{P,k})$, $\mathsf{ck}_{\mathsf{Col}} = (\mathsf{ck}_{P,k})_{k=\mu+1}^{\mu+\nu+1}$
 1: if $m = 1$, base case $s = \mu$ then:
 2:     $\mathcal{P}$ and $\mathcal{V}$ run $\mathsf{Protocol.Col}(G, H, \mathsf{ck}_{\mathsf{Col}}, \mathrm{P}, c, st_V; \boldsymbol{a}, \boldsymbol{b}, st_P)$
 3: else
 4:     if $st_P = \perp$ and $st_V = \perp$ then
 5:         $\mathcal{P}$ sets $\boldsymbol{P} = [\boldsymbol{a}]\boldsymbol{G} \parallel [\boldsymbol{b}]\boldsymbol{H}$ and adds $(\cdot, \cdot, \boldsymbol{P})$ into the bottom row of $st_P$.
 6:         $\mathcal{V}$ adds $(\mathsf{ck}_{P,0}, \cdot, \cdot, \mathrm{P}, \cdot)$ into the bottom row of $st_V$.
 7:     else
 8:         $\mathcal{P}$ refers $\boldsymbol{P}$ in the bottom row of $st_P$
 9:     end if
    Set $\widehat{m} = \frac{m}{2}$ and $\boldsymbol{a} = [\boldsymbol{a}_L \| \boldsymbol{a}_R]$, $\boldsymbol{b} = [\boldsymbol{b}_L \| \boldsymbol{b}_R]$, $\boldsymbol{G} = \boldsymbol{G}_L \| \boldsymbol{G}_R$, $\boldsymbol{H} = \boldsymbol{H}_L \| \boldsymbol{H}_R$
10:     $\mathcal{P}$ computes $c_L$, $c_R$ and L, R and sends them to $\mathcal{V}$:
        $\boldsymbol{L} = [\boldsymbol{a}_L]\boldsymbol{G}_R \parallel [\boldsymbol{b}_R]\boldsymbol{H}_L$, $\boldsymbol{R} = [\boldsymbol{a}_R]\boldsymbol{G}_L \parallel [\boldsymbol{b}_L]\boldsymbol{H}_R \in \mathbb{G}_p^{2n}$,
        $c_L = \langle \boldsymbol{a}_L, \boldsymbol{b}_R \rangle$, $c_R = \langle \boldsymbol{a}_R, \boldsymbol{b}_L \rangle \in \mathbb{Z}_p$,
        $\mathrm{L} = \mathsf{Com}_2(\mathsf{ck}_{L,s}, \boldsymbol{L})$, $\mathrm{R} = \mathsf{Com}_2(\mathsf{ck}_{R,s}, \boldsymbol{R}) \in \mathbb{G}_q$
11:     $\mathcal{V}$ chooses $x \overset{\$}{\leftarrow} \mathbb{Z}_p^*$ and returns it to $\mathcal{P}$
12:     $\mathcal{P}$ computes $\widehat{\mathrm{P}}$ and sends it to $\mathcal{V}$:
        $\widehat{\boldsymbol{P}} = [x^{-1}]\boldsymbol{L} + \boldsymbol{P} + [x]\boldsymbol{R} \in \mathbb{G}_p^{2n}$,    $\widehat{\mathrm{P}} = \mathsf{Com}_2(\mathsf{ck}_{P,s}, \widehat{\boldsymbol{P}}) \in \mathbb{G}_q$
13:     Both $\mathcal{P}$ and $\mathcal{V}$ update:
        $\widehat{\boldsymbol{G}} = \boldsymbol{G}_L + [x^{-1}]\boldsymbol{G}_R$,    $\widehat{\boldsymbol{H}} = \boldsymbol{H}_L + [x]\boldsymbol{H}_R \in \mathbb{G}_p^{\widehat{m}}$,    $\widehat{c} = x^{-1}c_L + c + xc_R \in \mathbb{Z}_p$
14:     $\mathcal{P}$ updates $\widehat{\boldsymbol{a}} = \boldsymbol{a}_L + x\boldsymbol{a}_R$, $\widehat{\boldsymbol{b}} = \boldsymbol{b}_L + x^{-1}\boldsymbol{b}_R \in \mathbb{Z}_p^{\widehat{m} \times n}$.
15:     $\mathcal{V}$ adds $(\mathsf{ck}_s, \mathrm{L}, \mathrm{R}, \widehat{\mathrm{P}}, x)$ into the bottom row of $st_V$.
16:     $\mathcal{P}$ adds $(\boldsymbol{L}, \boldsymbol{R}, \widehat{\boldsymbol{P}})$ into the bottom row of $st_P$.
17:     Both $\mathcal{P}$ and $\mathcal{V}$ run $\mathsf{Protocol.Row}(\widehat{\boldsymbol{G}}, \widehat{\boldsymbol{H}}, (\mathsf{ck}_k)_{k=s+1}^{\mu}, \mathsf{ck}_{\mathsf{Col}}, \widehat{\mathrm{P}}, \widehat{c}, st_V; \widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}, st_P)$
18: end if
```

The perfect completeness and computational witness-extended emulation for the overall reduction process, *i.e.*, the sequential combination of Protocol.Row and Protocol.Col, relies on those of AggMEC, along with the DL assumption. We formally state these in Theorem 1 for Protocol.Row and Theorem 2 for Protocol.Col, whose proofs are presented in Appendix A and B, respectively.

Proof of Multi-Elliptic Curve Operation: Algorithm 3. In this section, we explain how to construct multi-elliptic curve operation arguments AggMEC. Contrary to [29], we unify and aggregate row-wise and column-wise multi-elliptic curve operation proofs into a single protocol. That is, AggMEC guarantees the well-constructed updated instances $\widehat{\mathrm{P}}$ from every round of both row-wise and column-wise reduction. Concretely, AggMEC checks that the k-th row of state tuples $(st_V; st_P)_k = (\mathsf{ck}_k, (\mathrm{L}_k, \mathrm{R}_k, \mathrm{P}_k, x_k); (\boldsymbol{L}_k, \boldsymbol{R}_k, \boldsymbol{P}_k))$ satisfy the following:

1. Commitment

$$\begin{aligned} &\mathrm{L}_k = \mathsf{Com}_2(\mathsf{ck}_{L,k}, \boldsymbol{L}_k), \mathrm{R}_k = \mathsf{Com}_2(\mathsf{ck}_{R,k}, \boldsymbol{R}_k) \text{ for } k = 1, \ldots, \mu \\ &\mathrm{P}_k = \mathsf{Com}_2(\mathsf{ck}_{P,k}, \boldsymbol{P}_k) \text{ for } k = 0, \ldots, \mu + \nu - 1 \\ &\mathrm{P}_{\mu+\nu} = \mathsf{Com}_2(\mathsf{ck}, [a]G \parallel [b]H) \end{aligned} \tag{3}$$

Algorithm 2 Protocol.Col

```
    $\mathsf{Protocol.Col}(G, H, (\mathsf{ck}_{P,k+\mu})_{k=s}^{\nu}, \mathrm{P}, c, st_V; \boldsymbol{a}, \boldsymbol{b}, st_P)$
 1: if $n = 1$, base case $s = \nu$ then:
 2:     $\mathcal{P}$ sends $a$ and $b$ to $\mathcal{V}$
 3:     $\mathcal{V}$ checks $c \overset{?}{=} a \cdot b$ and set $\boldsymbol{P}_{\mathsf{Pub}} = [a]G \parallel [b]H \in \mathbb{Z}_q^4$
 4:     $\mathcal{P}$ and $\mathcal{V}$ run $\mathsf{AggMEC}(\boldsymbol{P}_{\mathsf{Pub}}, st_V; st_P)$
 5: else
 6:     if $st_P = \perp$ and $st_V = \perp$ then
 7:         $\mathcal{P}$ sets $\boldsymbol{P} = [\boldsymbol{a}]G \parallel [\boldsymbol{b}]H$ and adds $(\boldsymbol{P})$ into the bottom row of $st_P$
            $\mathcal{V}$ adds $(\mathsf{ck}_{P,\mu}, \mathrm{P}, \cdot)$ into the bottom row of $st_V$.
 8:     else
 9:         $\mathcal{P}$ refers $\boldsymbol{P}$ in the bottom row of $st_P$
10:     end if
    Set $\widehat{n} = \frac{n}{2}$ and $\boldsymbol{a} = \boldsymbol{a}_L \| \boldsymbol{a}_R$, $\boldsymbol{b} = \boldsymbol{b}_L \| \boldsymbol{b}_R$, $\boldsymbol{P} = \boldsymbol{P}^{(q_1)} \parallel \boldsymbol{P}^{(q_2)} \parallel \boldsymbol{P}^{(q_3)} \parallel \boldsymbol{P}^{(q_4)}$
11:     $\mathcal{P}$ computes $c_L$ and $c_R$ and sends them to $\mathcal{V}$:
        $c_L = \langle \boldsymbol{a}_L, \boldsymbol{b}_R \rangle \in \mathbb{Z}_p, \quad c_R = \langle \boldsymbol{a}_R, \boldsymbol{b}_L \rangle \in \mathbb{Z}_p$.
12:     $\mathcal{V}$ chooses $x \overset{\$}{\leftarrow} \mathbb{Z}_p^*$ and returns it to $\mathcal{P}$
13:     $\mathcal{P}$ computes $\widehat{\mathrm{P}}$ and sends it to $\mathcal{V}$:
        $\widehat{\boldsymbol{P}} = (\boldsymbol{P}^{(q_1)} + [x]\boldsymbol{P}^{(q_2)} \parallel \boldsymbol{P}^{(q_3)} + [x^{-1}]\boldsymbol{P}^{(q_4)}) \in \mathbb{G}_p^{2\widehat{n}}$, $\widehat{\mathrm{P}} = \mathsf{Com}_2(\mathsf{ck}_{P,\mu+s}, \widehat{\boldsymbol{P}}) \in \mathbb{G}_q$
14:     Both $\mathcal{P}$ and $\mathcal{V}$ compute $\widehat{c} = x^{-1} c_L + c + x c_R \in \mathbb{Z}_p$
15:     Additionally, $\mathcal{P}$ computes $\widehat{\boldsymbol{a}} = \boldsymbol{a}_L + x\boldsymbol{a}_R$, $\widehat{\boldsymbol{b}} = \boldsymbol{b}_L + x^{-1}\boldsymbol{b}_R \in \mathbb{Z}_p^{\widehat{n}}$.
16:     $\mathcal{V}$ adds $(\mathsf{ck}_{P,\mu+s}, \widehat{\mathrm{P}}, x)$ into the bottom row of $st_V$.
17:     $\mathcal{P}$ adds $(\widehat{\boldsymbol{P}})$ into the bottom row of $st_P$.
18:     Both $\mathcal{P}$ and $\mathcal{V}$ run $\mathsf{Protocol.Col}(G, H, (\mathsf{ck}_{P,k+\mu})_{k=s+1}^{\nu}, \widehat{\mathrm{P}}, \widehat{c}, st_V; \widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}, st_P)$
19: end if
```

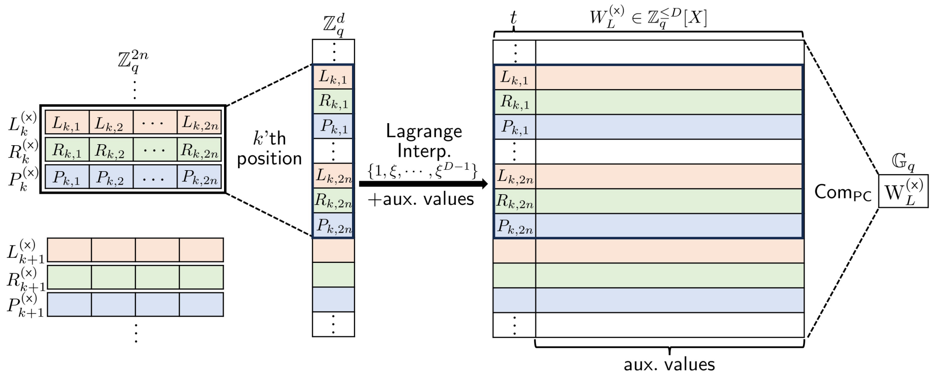
2. Elliptic Curve Operation on $\mathbb{G}_p = E(\mathbb{Z}_q)$

$$\bigwedge_{k=0}^{\mu-1} \left(\boldsymbol{P}_{k+1} = [x_k^{-1}]\boldsymbol{L}_{k+1} + \boldsymbol{P}_k + [x_k]\boldsymbol{R}_{k+1} \in \mathbb{G}_p^{2n} \right) \tag{4}$$

$$\bigwedge_{k=\mu}^{\mu+\nu} \left(\boldsymbol{P}_{k+1} = (\boldsymbol{P}_k^{(q_1)} + [x_k]\boldsymbol{P}_k^{(q_2)} \parallel \boldsymbol{P}_k^{(q_3)} + [x_k^{-1}]\boldsymbol{P}_k^{(q_4)}) \in \mathbb{G}_p^{n/2^{k-\mu}}) \right) \tag{5}$$

Two Roots of Unity. To construct the protocol, we consider two roots of unity: one for the commitment part and the other for the execution trace of the elliptic operation. Using the two roots of unity, we encode vectors to interpolated polynomial on power of unities. First, we consider total number d of elements consisting message vectors $\boldsymbol{L}_k$, $\boldsymbol{R}_k$, and $\boldsymbol{P}_k$ of L_k, R_k, and P_k. Since each $\boldsymbol{L}_k$, $\boldsymbol{R}_k$ consist of $2n$ elements for all $k \in [\mu]$, and $\boldsymbol{P}_k$ consists of $2n$ elements for $k = 0, \dots \mu$ and $n/2^{k-\mu-1}$ for all $k = \mu + 1 \dots \mu + \nu$, the total number d should be $6n\mu + 4n - 2$. We denote d-th root of unity ζ.

Next, we consider the root of unity for the execution trace. In Eq. (4) and (5), the elliptic curve operation consists of $4n\mu + n - 1$ complete additions and $4n\mu + 4n - 2$ multi-scalar multiplications. Each multi-scalar multiplication can be represented as $2 \log_2 q$ complete additions. Then, the total number of complete additions for Eq. (4) and (5) is at most $8n(\mu+1)\log_2 q$. We choose a sufficiently large integer D that satisfies $D \geq 8n(\mu+1)\log_2 q$ and $d|D$ (d is a divisor of D). Next, we define the D-root of unity ξ, which will be used for interpolating the wire polynomial in Plonkish. Note that $\zeta = \xi^t$ for some t and each ζ^i and ξ^i is the root of the polynomial $X^d - 1$ and $X^D - 1$ respectively.



$\mathbb{Z}_q^{2n}$
$L_k^{(\times)}$
$R_k^{(\times)}$
$P_k^{(\times)}$
$L_{k,1}$ $L_{k,2}$ $\cdots$ $L_{k,2n}$
$R_{k,1}$ $R_{k,2}$ $\cdots$ $R_{k,2n}$
$P_{k,1}$ $P_{k,2}$ $\cdots$ $P_{k,2n}$
$L_{k+1}^{(\times)}$
$R_{k+1}^{(\times)}$
$P_{k+1}^{(\times)}$
k'th position
$\mathbb{Z}_q^d$
Lagrange Interp.
$\{1, \xi, \cdots, \xi^{D-1}\}$
+aux. values
t
$W_L^{(\times)} \in \mathbb{Z}_q^{\leq D}[X]$
$\mathsf{Com}_{\mathsf{PC}}$
$\mathbb{G}_q$
$\mathrm{W}_L^{(\times)}$
aux. values

Fig. 4. Structure of Wire Polynomial. Best viewed in color.

Plonk-Friendly Extended Polynomial Commitment Scheme. To prove the consistency of the wire polynomial and commitments $\mathrm{L}_k, \mathrm{R}_k, \mathrm{P}_k$, we construct a commitment scheme for the message vectors $\boldsymbol{L}_k, \boldsymbol{R}_k$ and $\boldsymbol{P}_k$ considering compatibility with the polynomial commitment scheme. To this end, we first encode vectors $\boldsymbol{L}_k, \boldsymbol{R}_k$ and $\boldsymbol{P}_k$ into polynomials $F_{L,k}, F_{R,k}, F_{P,k}$ and then commit them. The encoding function $\mathsf{Enc}_{\mathsf{type}}$ takes ξ, index k and a vector $\boldsymbol{a}$, returning a polynomial $F_{\mathsf{type},k}$ in $\mathbb{Z}_q[X]$, where $\mathsf{type} \in \{L, R, P\}$. The encoding process extends $2n$ vectors to D-degree polynomials. We intend that each encoded function is *activated* at different positions. That is, for two encoded functions F_{type_1,k_1} and F_{type_2,k_2} with $(\mathsf{type}_1, k_1) \neq (\mathsf{type}_2, k_2)$, $F_{\mathsf{type}_1,k_1}(\xi^i)F_{\mathsf{type}_2,k_2}(\xi^i) = 0$ holds for all $i \in [D]$. In our setting, decoding of a polynomial $F_{\mathsf{type},k}$ can be performed uniquely when the type type and position k are determined. Furthermore, the sum of two encoded functions preserves their original non-zero evaluations at ξ^i.

- $\mathsf{Enc}_L(\xi, k \in [\mu], \boldsymbol{a} \in \mathbb{Z}_q^{\leq 2n}) \rightarrow F_{L,k} \in \mathbb{Z}_q[X]$
  Construct degree D polynomial $F_{L,k}(X)$ such that:

$$F_{L,k}(\xi^i) = \begin{cases} \boldsymbol{a}[j - 2n(k-1)], & \text{if } i = (3j-2)t \text{ for } 2n(k-1) < j \leq 2nk \\ 0, & \text{otherwise} \end{cases}$$

- $\mathsf{Enc}_R(\xi, k \in [\mu], \boldsymbol{a} \in \mathbb{Z}_q^{\leq 2n}) \rightarrow F_{R,k} \in \mathbb{Z}_q[X]$
  Construct degree D polynomial $F_{R,k}(X)$ such that:

$$F_{R,k}(\xi^i) = \begin{cases} \boldsymbol{a}[j - 2n(k-1)], & \text{if } i = (3j-1)t \text{ for } 2n(k-1) < j \leq 2nk \\ 0, & \text{otherwise} \end{cases}$$

- $\mathsf{Enc}_P(\xi, k \in \{0, \dots, \mu+\nu+1\}, \boldsymbol{a} \in \mathbb{Z}_q^{\leq 2n}) \rightarrow F_{P,k} \in \mathbb{Z}_q[X]$
  Construct degree D polynomial $F_{P,k}(X)$ such that:

$$F_{P,k}(\xi^i) = \begin{cases} \boldsymbol{a}[j-2nk], & \text{if } i = 3jt \text{ for } 2nk < j \leq 2n(k+1) \\ 0, & \text{otherwise} \end{cases}$$

Using the encoding function, we define the commitment Com_2 based on the homomorphic polynomial commitment $\mathsf{Com}_{\mathsf{PC}}$. The $\mathsf{ck}_{\mathsf{type},k}$ consists of four tuples: $(\mathsf{ck}_{\mathsf{PC}}, \xi, \mathsf{type}, k)$. We describe the commitment Com_2 for message $\boldsymbol{a}$ as follows:

- $\mathsf{Com}_2(\mathsf{ck}_{\mathsf{type},k}, \boldsymbol{a} = (\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}) \in \mathbb{Z}_q^{4n}) \rightarrow \mathrm{A}$
  1. $\mathsf{Enc}_{\mathsf{type}}(\xi, k, \boldsymbol{a}^{(i)}) \rightarrow F^{(i)}_{\mathsf{type},k}$ for $i \in \{1,2\}$
  2. $\mathsf{Com}_{\mathsf{PC}}(\mathsf{ck}_{\mathsf{PC}}, F^{(i)}_{\mathsf{type},k}) \rightarrow \mathrm{A}^{(i)}$ for $i \in \{1,2\}$
  3. Output $\mathrm{A} = (\mathrm{A}^{(1)}, \mathrm{A}^{(2)})$

Designated Execution Table. The execution table contains all values $\{\boldsymbol{L}_k, \boldsymbol{R}_k, \boldsymbol{P}_k\}$ some position. To construct AggMEC, we allocate each value $\{\boldsymbol{L}_k, \boldsymbol{R}_k, \boldsymbol{P}_k\}$ at a specific position in left position w_L following polynomial encoding Enc. That is, the non-zero evaluation of the encoding polynomial $F_{\mathsf{type},k}$ at ξ^i is equal to the evaluation of the wire polynomial $w_L(\xi^i)$ for all k and type.
Consistency Proof. Now we explain how to construct proof for the relations Eq. (3) and Eq. (4), (5). Each relation can be ensured by Eval and $\mathsf{Plonkish}_{\mathsf{Eval}}$. To ensure consistency of $\boldsymbol{L}_k, \boldsymbol{R}_k, \boldsymbol{P}_k$, we first merge every commitment $\mathrm{L}_k, \mathrm{R}_k, \mathrm{P}_k$ to one commitment A, whose message polynomial is the sum of encoding polynomials, $a(X) = \sum F_{\mathsf{type},k}(X)$. Then the difference polynomial $w_L(X) - a(X)$ is divided by $X^d - 1$ due to $w_L(\xi^i) - a(\xi^i) = 0$ for all i. The verifier can check it by using Eval after receiving a commitment of the wire polynomial $w_L(X)$.

Theorem 3. *Assume that the polynomial commitment scheme* $\mathsf{PCS} = (\mathsf{Gen}, \mathsf{Com}_{\mathsf{PC}}, \mathsf{Eval})$ *satisfies property of Definition 7 and homomorphic property. Then,* AggMEC *in Algorithm 3 has perfect completeness and computational witness-extended-emulation.*

The proof of Theorem 3 is presented in Appendix C.

3.2 Cougar: Cubic Root Verifier IPA

From the above construction, we adopt the homomorphic polynomial commitment scheme $\mathsf{Leopard}_{\mathsf{PC}}$ in place of $\mathsf{Com}_{\mathsf{PC}}$. We call this IPA Cougar. The full description of $\mathsf{Leopard}_{\mathsf{PC}}$ is given in Appendix E.
Complexity Analysis. We provide a complexity analysis of Cougar.
1. Row-reduction, Algorithm 1
`[Prover Cost]`: For commitments L, R and $\widehat{\mathrm{P}}$ at i-th round, $\mathcal{P}$ computes $O(\frac{N}{2^i})$ $\mathbb{G}_p$ operations and $O(n \log_2 m)$ $\mathbb{G}_q$ operations. For updating $\widehat{\boldsymbol{G}}, \widehat{\boldsymbol{H}}$ and $\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}, \widehat{c}$ at i-th round, $\mathcal{P}$ computes $O(\frac{m}{2^i})$ $\mathbb{G}_p$ operation and $O(n \cdot \frac{m}{2^i})$ $\mathbb{Z}_p$ respectively. Then, the total prover cost is $O(N)$ $\mathbb{Z}_p$ and $O(N)$ $\mathbb{G}_p$ operations.

Algorithm 3 AggMEC

```
    $\mathsf{AggMEC}(\boldsymbol{P}_{\mathsf{Pub}}, \mathsf{ck}_k, (\mathrm{L}_k, \mathrm{R}_k, \mathrm{P}_k, x_k); (\boldsymbol{L}_k, \boldsymbol{R}_k, \boldsymbol{P}_k))$
    $\mathsf{ck}_k = (\mathsf{ck}_{L,k}, \mathsf{ck}_{R,k}, \mathsf{ck}_{P,k})$, each $\mathsf{ck}_k$ contains $\mathsf{ck}_{\mathsf{PC}}$
 1: $\mathcal{P}$ and $\mathcal{V}$ set $\mathrm{A}^{(i)} = \sum_{k=1}^{\mu}(\mathrm{L}_k^{(i)} + \mathrm{R}_k^{(i)}) + \sum_{k=0}^{\mu+\nu} \mathrm{P}_k^{(i)}$ for $i \in \{1, 2\}$
 2: $\mathcal{P}$ sets $a^{(i)} = \sum_{k=1}^{\mu}(F_{L,k}^{(i)} + F_{R,k}^{(i)}) + \sum_{k=0}^{\mu+\nu} F_{P,k}^{(i)}$ for $i \in \{1, 2\}$:
        $F_{L,k}^{(i)} = \mathsf{Enc}_L(\xi, k, \boldsymbol{L}_k^{(i)})$, $F_{R,k}^{(i)} = \mathsf{Enc}_R(\xi, k, \boldsymbol{R}_k^{(i)})$, $F_{P,k}^{(i)} = \mathsf{Enc}_P(\xi, k, \boldsymbol{P}_k^{(i)})$
 3: $\mathcal{P}$ construct left wire polynomials $\{w_L^{(i)}(X)\}$ from execution table with public
    in/out $\boldsymbol{P}_{\mathsf{Pub}}$ and then computes $\mathrm{W}_L^{(1)}, \mathrm{W}_L^{(2)}, \mathrm{Q}^{(1)}, \mathrm{Q}^{(2)}$ and sends them to $\mathcal{V}$:
    $q^{(i)}(X) = \frac{w_L^{(i)}(X) - a^{(i)}(X)}{X^d - 1}$, $\mathrm{W}_L^{(i)} = \mathsf{Com}_{\mathsf{PC}}(\mathsf{ck}_{\mathsf{PC}}, w_L^{(i)})$, $\mathrm{Q}^{(i)} = \mathsf{Com}_{\mathsf{PC}}(\mathsf{ck}_{\mathsf{PC}}, q^{(i)})$
 4: $\mathcal{V}$ chooses $z, \rho \xleftarrow{\$} \mathbb{Z}_q$ and sends them to $\mathcal{P}$.
 5: $\mathcal{P}$ and $\mathcal{V}$ compute:
    $\mathrm{V} = \sum_{\mathrm{i}=1}^{2}(\sum_{k=1}^{\mu}([\rho^{4k-2-i}]\mathrm{L}_k^{(i)} + [\rho^{4k-i}]\mathrm{R}_k^{(i)}) + \rho^{4\mu}(\sum_{k=0}^{\mu+\nu}[\rho^{2k-1+i}]\mathrm{P}_k^{(i)}))$
 6: $\mathcal{P}$ computes $F_V(X)$:
    $F_V = \sum_{\mathrm{i}=1}^{2}(\sum_{k=1}^{\mu}(\rho^{4k-2-i}F_{L,k}^{(i)} + \rho^{4k-i}F_{R,k}^{(i)}) + \rho^{4\mu}(\sum_{k=0}^{\mu+\nu}\rho^{2k-1+i}F_{P,k}^{(i)}))$
 7: $\mathcal{P}$ sends $s, t^{(1)}, t^{(2)}, r^{(1)}, r^{(2)}$ to $\mathcal{V}$: $s = F_V(z)$, $t^{(i)} = q^{(i)}(z)$, $r^{(i)} = w_L^{(i)}(z)$
 8: $\mathcal{V}$ chooses $\tau \xleftarrow{\$} \mathbb{Z}_q$ and sends them to $\mathcal{P}$.
 9: $\mathcal{P}$ and $\mathcal{V}$ set $\mathrm{P} = \mathrm{V} + \sum_{i=1}^{2}([\tau^i]\mathrm{A}^{(i)} + [\tau^{2+i}]\mathrm{Q}^{(i)})$ and
    $y = s + \sum_{i=1}^{2}(\tau^i(s^{(i)} - t^{(i)}(z^d - 1)) + \tau^{2+i}t^{(i)})$
10: $\mathcal{P}$ set $F_P = F_V + \sum_{i=1}^{2}(\tau^i a^{(i)} + \tau^{2+i}q^{(i)})$
11: $\mathcal{P}$ sets wire/aux polynomials $w_L^{(1)}, w_L^{(2)}, w_R^{(1)}, w_R^{(2)}, w_O^{(1)}, w_O^{(2)}, \alpha, \beta, \gamma, \delta, \lambda \in \mathbb{Z}_q[X]$
12: $\mathcal{P}$ and $\mathcal{V}$ set run $\mathsf{Eval}(\mathsf{ck}_{\mathsf{PC}}, \mathrm{P}, z, y; F_P)$ and $\mathsf{Eval}(\mathsf{ck}_{\mathsf{PC}}, \mathrm{W}_L^{(i)}, z, r^{(i)}; w_L^{(i)})$
13: $\mathcal{P}$ and $\mathcal{V}$ run $\mathsf{Plonkish}_{\mathsf{Eval}}(\mathsf{ck}_{\mathsf{PC}}; w_L^{(1)}, w_L^{(2)}, w_R^{(1)}, w_R^{(2)}, w_O^{(1)}, w_O^{(2)}, \alpha, \beta, \gamma, \delta, \lambda)$
```

`[Verifier Cost]`: For updating $\widehat{\boldsymbol{G}}, \widehat{\boldsymbol{H}}$ and $\widehat{c}$ at i-th round, $\mathcal{V}$ computes $O(\frac{m}{2^i})$ $\mathbb{G}_p$ operation and 2 multiplication in $\mathbb{Z}_p$. Then, the total verifier cost is $O(m)$ $\mathbb{G}_p$ and $O(\log_2 m)$ $\mathbb{Z}_p$ operations.

`[Communication Cost]`: For each round, $\mathcal{P}$ sends L, R, $\widehat{\mathrm{P}}$, c_L, and c_R. Then, the total communication cost is $3\log_2 m|\mathbb{G}_q| + 2\log_2 m|\mathbb{Z}_p|$.

2. Column-reduction, Algorithm 2

`[Prover Cost]`: For a inner product c_L and c_R at i-th round, the prover computes $O(\frac{n}{2^i})$ $\mathbb{Z}_p$ operations. For updating $\widehat{\mathrm{P}}$, $\widehat{\boldsymbol{a}}$, $\widehat{\boldsymbol{b}}$, and $\widehat{c}$ at i-th round, $\mathcal{P}$ computes $O(\frac{n}{2^i})$ $\mathbb{G}_p$ and $\mathbb{Z}_p$ operations, $O(\frac{n}{2^i}\log_2 m)$ $\mathbb{G}_q$ operations. Then, the total prover cost is $O(n\log_2 m)$ $\mathbb{G}_q$ operations.

`[Verifier Cost]`: For updating $\widehat{c}$ at each round except the final round, $\mathcal{V}$ computes 2 multiplication in $\mathbb{Z}_p$. In the final round, $\mathcal{V}$ computes one $\mathbb{Z}_p$ operation for verification. Then, the total verifier cost is $O(\log_2 n)$ $\mathbb{Z}_p$ operations.

`[Communication Cost]`: For each round, the prover sends $\widehat{\mathrm{P}}$, c_L, and c_R. The total communication cost is $\log_2 n|\mathbb{G}_q| + 2\log_2 n|\mathbb{Z}_p|$.

3. Aggregated MEC, Algorithm 3

`[Prover Cost]`: From line 1 to 11, $\mathcal{P}$ treats at most $\log_2 N$ polynomials of degree D. Then, $\mathcal{P}$ computes $O(D\log_2 N) = O(n\log_2 N)$ operations, including $\mathbb{Z}_p$, $\mathbb{G}_p$, and $\mathbb{G}_q$. And the cost of Eval and $\mathsf{Plonkish}_{\mathsf{Eval}}$ is $O(\|\mathcal{P}_{\mathsf{Eval}}(D)\|)$. Then, total prover cost is $O(n\log_2 N + \|\mathcal{P}_{\mathsf{Eval}}(D)\|)$.

`[Verifier Cost]`: From line 1 to 11, $\mathcal{V}$ computes $O(\log_2 N)$ $\mathbb{G}_q$ operations. And the cost of Eval and $\mathsf{Plonkish}_{\mathsf{Eval}}$ is $O(\|\mathcal{V}_{\mathsf{Eval}}(D)\|)$. Then the total verifier cost is $O(\log_2 N + \|\mathcal{V}_{\mathsf{Eval}}(D)\|)$.
`[Communication Cost]`: From line 1 to 11, $\mathcal{P}$ sends $\mathcal{V}$ 4 $\mathbb{G}_q$ elements and 5 field elements. Additionally, for Eval and $\mathsf{Plonkish}$ the prover sends $O(\|\mathit{\Pi}_{\mathsf{Eval}}(D)\|)$. Then total communication cost is $O(\|\mathit{\Pi}_{\mathsf{Eval}}(D)\|)$
Cubic Root Verifier IPA from Parameter Setting. Let consider $N = mn$ be the length of the witness vectors with $n = \sqrt[3]{N^2}$ and $m = \sqrt[3]{N}$. Since Leopard features $(\|\mathcal{P}_{\mathsf{Eval}}(D)\|, \|\mathcal{V}_{\mathsf{Eval}}(D)\|, \|\mathit{\Pi}_{\mathsf{Eval}}(D)\|) = (O(D), O(\sqrt{D}), O(\log_2 D))$, we counclude that the Cougar features $O(N)$ prover cost, $O(\log_2 N)$ communication cost and $O(\sqrt[3]{N}\sqrt{\log_2 N})$ verifier cost, which is the cubic root of N.

Theorem 4. Cougar *is an IPA, which features* $O(\log_2 N)$ *communication cost,* $O(N)$ *prover cost and* $O(\sqrt[3]{N}\sqrt{\log_2 N})$ *verifier cost where* N *is length of witness.* Cougar *provides perfect completeness and computational witness extended emulation under the DL assumption.*

Proof. The prover, verifier and communication costs can be checked in the above analysis. By Theorem 1, Theorem 2, and Theorem 3 and the soundness of Leopard under the DL assumption [28], Cougar satisfies perfect completeness and computational witness-extended-emulation under the DL assumption. □

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A Proof of Theorem 1

Proof. (Completeness) For a base case $m = 1$, the completeness is held by the completeness of $\mathsf{Protocol.Col}$ and AggMEC. Let us consider the case $m > 1$. In this case, we show that if the input $(\boldsymbol{G}, \boldsymbol{H}, \mathsf{ck}_{P,s}, \mathrm{P}, c; \boldsymbol{a}, \boldsymbol{b})$ belongs to $\mathcal{R}^{m,n}_{\mathsf{GenPT4}}$, then the updated input $(\widehat{\boldsymbol{G}}, \widehat{\boldsymbol{H}}, \mathsf{ck}_{P,s+1}, \widehat{\mathrm{P}}, \widehat{c}; \widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}})$ belongs to $\mathcal{R}^{m/2,n}_{\mathsf{GenPT4}}$. Following the $\mathcal{P}$ algorithm, we get the following equations:

$$\begin{aligned}
\widehat{c} &= x^{-1}c_L + c + xc_R = \langle \boldsymbol{a}_L, x^{-1}\boldsymbol{b}_R\rangle + \langle \boldsymbol{a}, \boldsymbol{b}\rangle + \langle x\boldsymbol{a}_R, \boldsymbol{b}_L\rangle = \langle \widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}\rangle \\
\widehat{\boldsymbol{P}} &= [x^{-1}]\boldsymbol{L} + \boldsymbol{P} + [x]\boldsymbol{R} \\
&= x^{-1}[\boldsymbol{a}_L]\boldsymbol{G}_R \parallel [x^{-1}\boldsymbol{b}_R]\boldsymbol{H}_L + [\boldsymbol{a}]\boldsymbol{G} \parallel [\boldsymbol{b}]\boldsymbol{H} + [x\boldsymbol{a}_R]\boldsymbol{G}_L \parallel x[\boldsymbol{b}_L]\boldsymbol{H}_R \\
&= \widehat{\boldsymbol{a}}\widehat{\boldsymbol{G}} \parallel \widehat{\boldsymbol{b}}\widehat{\boldsymbol{H}} \\
\widehat{\mathrm{P}} &= \mathsf{Com}_2(\mathsf{ck}_{P,s+1}, \widehat{\boldsymbol{P}}) = \mathsf{Com}_{\mathsf{TC}}((\widehat{\boldsymbol{G}} \parallel \widehat{\boldsymbol{H}}, \mathsf{ck}_{P,s+1}), \widehat{\boldsymbol{a}} \parallel \widehat{\boldsymbol{b}})
\end{aligned}$$

Therefore, we can conclude that updated input $(\widehat{\boldsymbol{G}}, \widehat{\boldsymbol{H}}, \mathsf{ck}_{P,s+1}, \widehat{\mathrm{P}}, \widehat{c}; \widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}})$ belongs to $\mathcal{R}^{m/2,n}_{\mathsf{GenPT4}}$.
(Witness-extended-emulation) For the computational witness-extended emulation, we construct an expected polynomial time extractor $\mathcal{E}_{Row}$ whose goal is to extract a witness by using a polynomially bounded tree of accepting transcripts. To this end, we utilize the general forking lemma [8], which is stated as follows:

Theorem 5 (General Forking Lemma). *Let $(\mathcal{K}, \mathcal{P}, \mathcal{V})$ be a $(2\mu + 1)$-move, public coin interactive protocol with μ challenges $x_1, \ldots, x_\mu$ in sequence. Let $n_i \geq 1$ for $i \in [\mu]$. Consider an $(n_1, \ldots, n_\mu)$-tree of accepting transcripts with challenges in the following format. The tree has depth μ and $N = \prod_{i=1}^{\mu} n_i$ leaves. The root of the tree is labeled with the statement. Each node of depth i has exactly n_i children, each labeled with a distinct value of the i-th challenge x_i.*

Let $\mathcal{E}$ be a witness extraction algorithm that succeeds with probability $1 - \mathsf{negl}(\lambda)$ for some negligible function $\mathsf{negl}(\lambda)$ in extracting a witness from an $(n_1, \ldots, n_\mu)$-tree of accepting transcripts in probabilistic polynomial time. Assume that $\prod_{i=1}^{\mu} n_i$ is bounded above by a polynomial in the security parameter λ. Then, $(\mathcal{K}, \mathcal{P}, \mathcal{V})$ has witness-extended emulation.

$\mathcal{E}_{Row}$ takes public inputs $(\boldsymbol{G}, \boldsymbol{H}, (\mathsf{ck}_k)_{k=1}^{\mu}, \mathsf{ck}_{\mathsf{Col}}, \mathrm{P}, c, st_V; \boldsymbol{a}, \boldsymbol{b}, st_P)$. By premise, $\mathcal{E}_{Row}$ exploits two PPT extractors $\mathcal{E}_{Col}$ and $\mathcal{E}_{MEC}$, that extract witness $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}_p^{1\times n}$ and st_P respectively. Note that st_P consists of tuples of commitments $(\boldsymbol{L}_k, \boldsymbol{R}_k, \boldsymbol{P}_k)$, which satisfies the Eq. (3) and (4).

We show how to extract witness $\boldsymbol{a}, \boldsymbol{b}$ from accepting transcripts. By the general forking lemma, it is sufficient to construct an extractor $\mathcal{E}_{Row}$ that extracts a witness from a suitable tree of accepting transcripts in probabilistic polynomial time. We begin with $(\underbrace{4, \ldots, 4}_{\log_2 m})$-tree of accepting transcripts. Since the number of leaves of the tree is polynomially bound, $4^{\log_2 m}$, we can apply the general forking lemma.

First, for the base case $m = 1$, the extracted witness $\boldsymbol{a}, \boldsymbol{b}$ from $\mathcal{E}_{Col}$ satisfies the desire condition so that $\mathcal{E}_{Row}$ outputs the $\boldsymbol{a}, \boldsymbol{b}$ in polynomial time.

In the case $m > 1$, we construct extractor $\mathcal{E}_{Row}$ by inductively extraction. That is, retrieves s-round witness $\boldsymbol{a}^{(s)}, \boldsymbol{b}^{(s)} \in \mathbb{Z}_p^{m/2^s \times n}$ from next steps $\boldsymbol{a}^{(s+1)}, \boldsymbol{b}^{(s+1)} \in \mathbb{Z}_p^{m/2^{s+1} \times n}$ recursively.

First, $\mathcal{E}_{Row}$ run $\mathcal{E}_{Col}$ and get extracted witnesses $\boldsymbol{a}^{(\mu)}, \boldsymbol{b}^{(\mu)} \in \mathbb{Z}_p^{1\times n}$, which is valid witness for the relation $\mathcal{R}^{1,n}_{\mathsf{GenPT4}}$. Now, we assume that $\widehat{\boldsymbol{a}}_i, \widehat{\boldsymbol{b}}_i \in \mathbb{Z}_p^{m/2^{s+1} \times n}$ is valid witness of instance $(\widehat{\boldsymbol{G}}_i, \widehat{\boldsymbol{H}}_i, \widehat{\mathrm{P}}_i, \widehat{c}_i)$, that are updated instance using challenge x_i for the relation $\mathcal{R}^{m/2^{s+1},n}_{\mathsf{GenPT4}}$. From the tree of accepting transcript, we can get 4 instance-witness pairs:$(\widehat{\boldsymbol{G}}_i, \widehat{\boldsymbol{H}}_i, \widehat{\mathrm{P}}_i, \widehat{c}_i; \widehat{\boldsymbol{a}}_i, \widehat{\boldsymbol{b}}_i)$. Furthermore, the $\mathcal{E}_{Row}$ can get s-round prover's commitments $\mathrm{L}, \mathrm{R}, \mathrm{P}, \widehat{\mathrm{P}}$ and their messages $\boldsymbol{L}, \boldsymbol{R}, \boldsymbol{P}, \widehat{\boldsymbol{P}}$ from s-round transcripts and $\mathcal{E}_{MEC}$ respectively. From 3 distinct tuples, $\mathcal{E}_{Row}$ can construct the following linear system:

$$\begin{bmatrix} x_1^{-1} & 1 & x_1 \\ x_2^{-1} & 1 & x_2 \\ x_3^{-1} & 1 & x_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{L} \\ \boldsymbol{P} \\ \boldsymbol{R} \end{bmatrix} = \begin{bmatrix} \widehat{\boldsymbol{P}}_1 \\ \widehat{\boldsymbol{P}}_2 \\ \widehat{\boldsymbol{P}}_3 \end{bmatrix} = \begin{bmatrix} [\widehat{\boldsymbol{a}}_1]\widehat{\boldsymbol{G}}_1 \parallel [\widehat{\boldsymbol{b}}_1]\widehat{\boldsymbol{H}}_1 \\ [\widehat{\boldsymbol{a}}_2]\widehat{\boldsymbol{G}}_2 \parallel [\widehat{\boldsymbol{b}}_2]\widehat{\boldsymbol{H}}_2 \\ [\widehat{\boldsymbol{a}}_3]\widehat{\boldsymbol{G}}_3 \parallel [\widehat{\boldsymbol{b}}_3]\widehat{\boldsymbol{H}}_3 \end{bmatrix} \tag{6}$$

Since the right-hand side of Eq. (6) is decomposed by $\widehat{\boldsymbol{G}} = \boldsymbol{G}_L + [x^{-1}]\boldsymbol{G}_R$ and $\widehat{\boldsymbol{H}} = \boldsymbol{H}_L + [x]\boldsymbol{H}_R$ and each $\boldsymbol{G}$ and $\boldsymbol{H}$ are not effected by challenge x, $\mathcal{E}_{Row}$ can get represented vectors $\boldsymbol{l}, \boldsymbol{r}, \boldsymbol{p} \in \mathbb{Z}_p^{m/2^s \times 2n}$ of $\boldsymbol{L}, \boldsymbol{R}, \boldsymbol{P} \in \mathbb{G}^{2n}$ under base $\boldsymbol{G} \parallel \boldsymbol{H}$. Let $\mathcal{E}_{Row}$ parse $\boldsymbol{l}, \boldsymbol{r}, \boldsymbol{p}$ to 4 segments $\boldsymbol{l}^{(t)}, \boldsymbol{p}^{(t)}, \boldsymbol{r}^{(t)} \in \mathbb{Z}_p^{m/2^s \times n/2}$ where $t \in [4]$. Let the representation vectors put on Eq. (6). Then

$$[x_i^{-1}\boldsymbol{l}^{(1)} + \boldsymbol{p}^{(1)} + x_i\boldsymbol{r}^{(1)}]\boldsymbol{G}_L = [\widehat{\boldsymbol{a}}]\boldsymbol{G}_L \tag{7}$$

$$[x_i^{-1}\boldsymbol{l}^{(2)} + \boldsymbol{p}^{(2)} + x_i\boldsymbol{r}^{(2)}]\boldsymbol{G}_R = [x_i^{-1}\widehat{\boldsymbol{a}}]\boldsymbol{G}_R \tag{8}$$

$$[x_i^{-1}\boldsymbol{l}^{(3)} + \boldsymbol{p}^{(3)} + x_i\boldsymbol{r}^{(3)}]\boldsymbol{H}_L = [\widehat{\boldsymbol{b}}]\boldsymbol{H}_L \tag{9}$$

$$[x_i^{-1}\boldsymbol{l}^{(4)} + \boldsymbol{p}^{(4)} + x_i\boldsymbol{r}^{(4)}]\boldsymbol{H}_R = [x_i\widehat{\boldsymbol{b}}]\boldsymbol{H}_R \tag{10}$$

By DL assumption on $\mathbb{G}$, the representation vectors of both side should be equivalent. From Eq.(7), Eq.(8) and Eq.(9), Eq.(10), we get

$$-\boldsymbol{l}^{(1)} + x_i(\boldsymbol{l}^{(2)} - \boldsymbol{p}^{(1)}) + x_i^2(\boldsymbol{p}^{(2)} - \boldsymbol{r}^{(1)}) + x_i^3\boldsymbol{r}^{(2)} = 0$$
$$-\boldsymbol{l}^{(4)} + x_i(\boldsymbol{l}^{(3)} - \boldsymbol{p}^{(4)}) + x_i^2(\boldsymbol{p}^{(3)} - \boldsymbol{r}^{(4)}) + x_i^3\boldsymbol{r}^{(3)} = 0$$

for $x_1, \ldots, x_4$. Then each terms of x_i^k should be zero. Let $\boldsymbol{p}^{(i)}$ denote $\tilde{\boldsymbol{a}}_L = \boldsymbol{p}^{(1)}$, $\tilde{\boldsymbol{a}}_R = \boldsymbol{p}^{(2)}$, $\tilde{\boldsymbol{b}}_L = \boldsymbol{p}^{(3)}$, $\tilde{\boldsymbol{b}}_R = \boldsymbol{p}^{(4)}$. Then, we can obtain the following equation:

$$\begin{aligned} x_i^{-1}c_L + c + x_ic_R = \widehat{c} &= \langle \widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}} \rangle \\ &= x_i^{-1}\langle \tilde{\boldsymbol{a}}_L, \tilde{\boldsymbol{b}}_R \rangle + \langle \tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}} \rangle + x_i\langle \tilde{\boldsymbol{a}}_R, \tilde{\boldsymbol{b}}_L \rangle \end{aligned} \tag{11}$$

Similarly, $x_1, \ldots, x_4$ guarantees $c = \langle \tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}} \rangle$. Therefore, the extracted witnesses $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{b}}$ is valid witness for the relation $\mathcal{R}^{m/2^s,n}_{\mathsf{GenPT4}}$. By inductively retrieving process and general forking lemma, $\mathcal{E}_{Row}$ can extract witness vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. $\square$

B Proof of Theorem 2

Proof. (Completeness) For a base case $m = 1$, the completeness can get straightforward by our premise: completeness of AggMEC and $(G, H, \mathsf{ck}_1, \mathrm{P}, c; \boldsymbol{a}, \boldsymbol{b}) \in \mathcal{R}^{1,1}_{\mathsf{GenPT4}}$. Let consider the case $m > 1$. In this case, we show that if the input $(G, H, \mathsf{ck}_{P,\mu+s}, \mathrm{P}, c; \boldsymbol{a}, \boldsymbol{b})$ belongs to $\mathcal{R}^{1,n}_{\mathsf{GenPT4}}$, then the updated input $(G, H, \mathsf{ck}_{P,\mu+s+1}, \widehat{\mathrm{P}}, \widehat{c}; \widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}})$ belongs to $\mathcal{R}^{1,n/2}_{\mathsf{GenPT4}}$. Following the $\mathcal{P}$ algorithm, we get the following equations:

$$\begin{aligned}
\widehat{c} &= x^{-1}c_L + c + xc_R = \langle \boldsymbol{a}_L, x^{-1}\boldsymbol{b}_R\rangle + \langle \boldsymbol{a}, \boldsymbol{b}\rangle + \langle x\boldsymbol{a}_R, \boldsymbol{b}_L\rangle = \langle \widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}}\rangle \\
\widehat{\boldsymbol{P}} &= (\boldsymbol{P}^{(q_1)} \parallel [x]\boldsymbol{P}^{(q_4)}) + (\boldsymbol{P}^{(q_2)} \parallel [x^{-1}]\boldsymbol{P}^{(q_3)}) \\
&= [\boldsymbol{a}_L]G \parallel [x^{-1}\boldsymbol{b}_R]H + [x\boldsymbol{a}_R]G \parallel [\boldsymbol{b}_L]H = [\widehat{\boldsymbol{a}}]G \parallel [\widehat{\boldsymbol{b}}]H \\
\widehat{\mathrm{P}} &= \mathsf{Com}_2(\mathsf{ck}_\nu, \widehat{\boldsymbol{P}}) = \mathsf{Com}_{\mathsf{TC}}((G \parallel H, \mathsf{ck}_\nu), \boldsymbol{a} \parallel \boldsymbol{b})
\end{aligned}$$

Therefore, we can conclude that updated input $(G, H, \mathsf{ck}_{P,\mu+s+1}, \widehat{\mathrm{P}}, \widehat{c}; \widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}})$ belongs to $\mathcal{R}^{1,n/2}_{\mathsf{GenPT4}}$.

(Witness-extended-emulation) For the computational witness-extended emulation, we construct an expected polynomial time extractor $\mathcal{E}_{Col}$ whose goal is to extract a witness by using a polynomially bounded tree of accepting transcripts. $\mathcal{E}_{Col}$ takes public inputs $(\mathsf{pp}_\nu, G, H, \mathrm{P}, c, st_V; \boldsymbol{a}, \boldsymbol{b}, st_P)$. By premise, $\mathcal{E}_{Col}$ exploits a PPT extractor $\mathcal{E}_{MEC}$, that extract st_P which consists of commitments $(\boldsymbol{P}_k)^{\mu+\nu+1}_{k=\mu+1}$, which satisfies Eq. (5) and $\mathrm{P}_k = \mathsf{Com}_2(\mathsf{ck}_\nu, \boldsymbol{P}_k)$ In the similar way in proof of Theorem 1, we show that how to extract witness $\boldsymbol{a}, \boldsymbol{b}$ from accepting transcripts. By the general forking lemma, it is sufficient to construct an extractor $\mathcal{E}_{Row}$ that extracts a witness from a suitable tree of accepting transcripts in probabilistic polynomial time. We begin with $(\underbrace{3, \ldots, 3}_{\log_2 n})$-tree of accepting transcripts. Since the number of leaves of the tree is polynomially bound, $3^{\log_2 n}$, we can apply the general forking lemma.

First, in the base case $n = 1$, the $\mathcal{P}$ sends witnesses a, b to $\mathcal{V}$ and $\mathcal{V}$ check the relation directly. That means the witness a and b belongs to transcripts and $\mathcal{E}_{Col}$ can extract them.

Now we consider the case $n > 1$. We construct extractor $\mathcal{E}_{Col}$ by inductively extraction. That is, retrieves s-round witness $\boldsymbol{a}^{(s)}, \boldsymbol{b}^{(s)} \in \mathbb{Z}_p^{1\times n/2^s}$ from next round witnesses $\boldsymbol{a}^{(s+1)}, \boldsymbol{b}^{(s+1)} \in \mathbb{Z}_p^{1\times n/2^{s+1}}$ recursively.

First, $\mathcal{E}_{Col}$ can extract final round witnesses $a^{(\nu+1)}$ and $b^{(\nu+1)}$. We assume that $\widehat{\boldsymbol{a}}, \widehat{\boldsymbol{b}} \in \mathbb{Z}_p^{1\times n/2^{s+1}}$ is valid witness of instance $(G, H, \widehat{\mathrm{P}}_i, \widehat{c}_i)$, that are affected by challenge x_i for the relation $\mathcal{R}^{1,n/2^{s+1}}_{\mathsf{GenPT4}}$. From the tree of accepting transcript, we can get 3 instance-witness pairs:$(G, H, \widehat{\mathrm{P}}_i, \widehat{c}_i; \widehat{\boldsymbol{a}}_i, \widehat{\boldsymbol{b}}_i)$. Furthermore, $\mathcal{E}_{Col}$ can get k-round prover's commitments $(\mathrm{P}, \widehat{\mathrm{P}})$ and their message $(\boldsymbol{P}, \widehat{\boldsymbol{P}})$ from transcript and $\mathcal{E}_{MEC}$ respectively. From 2 distinct tuples, $\mathcal{E}_{Col}$ can construct following linear system:

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^{(q_1)} \\ \boldsymbol{P}^{(q_2)} \end{bmatrix} = \begin{bmatrix} [\widehat{\boldsymbol{a}}_1]G \\ [\widehat{\boldsymbol{a}}_2]G \end{bmatrix}, \begin{bmatrix} 1 & x_1^{-1} \\ 1 & x_2^{-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{P}^{(q_3)} \\ \boldsymbol{P}^{(q_4)} \end{bmatrix} = \begin{bmatrix} [\widehat{\boldsymbol{b}}_1]H \\ [\widehat{\boldsymbol{b}}_2]H \end{bmatrix} \qquad (12)$$

By DL assumption, $\mathcal{E}_{Col}$ solves the linear equation and then get the representation $\boldsymbol{p}^{(q_1)}, \boldsymbol{p}^{(q_2)} \in \mathbb{Z}_p^{n/2^{s+1}}$ of $\boldsymbol{P}^{(q_1)}, \boldsymbol{P}^{(q_2)} \in \mathbb{G}^{n/2^{s+1}}$ under base G and $\boldsymbol{p}^{(q_3)}, \boldsymbol{p}^{(q_4)} \in \mathbb{Z}_p^{n/2^{s+1}}$ of $\boldsymbol{P}^{(q_3)}, \boldsymbol{P}^{(q_4)} \in \mathbb{G}^{n/2^{s+1}}$ under base H respectively. Then $\boldsymbol{p} = \boldsymbol{p}^{(q_1)} \parallel \boldsymbol{p}^{(q_2)} \parallel \boldsymbol{p}^{(q_3)} \parallel \boldsymbol{p}^{(q_4)}$ is naturally representation of $\boldsymbol{P}$. Let $\boldsymbol{p}^{(q_i)}$ denote $\tilde{\boldsymbol{a}}_L = \boldsymbol{p}^{(q_1)}$, $\tilde{\boldsymbol{a}}_R = \boldsymbol{p}^{(q_2)}$, $\tilde{\boldsymbol{b}}_L = \boldsymbol{p}^{(q_3)}$, $\tilde{\boldsymbol{b}}_R = \boldsymbol{p}^{(q_4)}$. In the similar way in Eq. (11) of Theorem 1, 3 distinct challenges guarantee extracted vectors $\langle \tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}} \rangle$ is equal to the value c. Therefore, the extracted witnesses $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{b}}$ is valid witness for the relation $\mathcal{R}^{1,n/2^s}_{\mathsf{GenPT4}}$. By inductively retrieving process and general forking lemma, $\mathcal{E}_{Col}$ can extract witness vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. □

C Proof of Theorem 3

Proof. (Completeness) Assume that the input $\mathsf{ck}_k, (\mathrm{L}_k, \mathrm{R}_k, \mathrm{P}_k, x_k); (\boldsymbol{L}_k, \boldsymbol{R}_k, \boldsymbol{P}_k)$ satisfies Eq. (3), (4), and (5). By the homomorphic property of polynomial commitment scheme and perfect completeness of Eval and $\mathsf{Plonkish}_{\mathsf{Eval}}$, $\mathcal{V}$ accepts both Eval and $\mathsf{Plonkish}_{\mathsf{Eval}}$. Therefore, we are shown the completeness of AggMEC.
(Witness-Exetended Emulation) For the computational witness-extended emulation, we construct an expected polynomial-time extractor $\mathcal{E}_{MEC}$ whose goal is to extract a witness by using a polynomially bounded tree of accepting transcripts. $\mathcal{E}_{MEC}$ takes public inputs $\mathsf{ck}_k, (\mathrm{L}_k, \mathrm{R}_k, \mathrm{P}_k, x_k)$ and returns witness vectors $(\boldsymbol{L}_k, \boldsymbol{R}_k, \boldsymbol{P}_k)$ satisfying Eq. (3), Eq. (4), and Eq. (5).

By the general forking lemma, it is sufficient to construct an extractor $\mathcal{E}_{MEC}$ that extracts a witness from a suitable tree of accepting transcripts in probabilistic polynomial-time. We begin with a $(6\log m + 2\log n + 2, 5)$-tree of accepting transcripts. Since the number of leaves in the tree is polynomially bounded, we can apply the general forking lemma [12].

By our premise, one can construct a PPT extractor $\mathcal{E}_{\mathsf{Eval}}$ for $\mathsf{PCS.Eval}$. In addition, since the above premise implies that the $\mathsf{Plonkish}_{\mathsf{Eval}}$ is an AoK, one can construct a PPT extractor $\mathcal{E}_{\mathsf{Plonkish}}$ for $\mathsf{Plonkish}_{\mathsf{Eval}}$ that extracts wire polynomials $w_L^{(i)}$, $w_R^{(i)}$, $w_O^{(i)}$ and auxiliary polynomials $\alpha, \beta\, \gamma, \delta, \lambda$. The $\mathcal{E}_{MEC}$ uses them as sub-routines.

First, the $\mathcal{E}_{MEC}$ gets $F_P(X)$ and $w_L^{(i)}(X)$ by feeding $\mathcal{E}_{\mathsf{Eval}}$ with $(\mathsf{ck}_{\mathsf{PC}}, \mathrm{P}, z, y)$ and $(\mathsf{ck}_{\mathsf{PC}}, \mathrm{W}_L^{(i)}, z, r^{(i)})$, respectively. From the 5 transcripts from distinct challenge τ, the $\mathcal{E}_{MEC}$ extracts F_V, a polynomial $a^{(i)}(X)$, and quotient polynomials $q^{(i)}(X)$ by regarding the following relation as a polynomial with respect to τ of degree 4: $F_P = F_V + \sum_{i=1}^{2}(\tau^i a^{(i)} + \tau^{2+i} q^{(i)})$. Note that $w_L^{(i)}(X)$, $a^{(i)}(X)$, and $q^{(i)}(X)$ satisfy $a^{(i)}(z) = w_L^{(i)}(z) - q^{(i)}(z)(z^d - 1)$.

From the $6\log m + 2\log n + 2$ transcripts from distinct challenge ρ, the $\mathcal{E}_{MEC}$ extracts polynomials $F_{L,k}^{(i)}, F_{R,k}^{(i)}, F_{P,k}^{(i)}$ by regarding the following relation as a polynomial with respect to ρ of degree $6\log m + 2\log n + 1$:

$$F_V = \sum_{\mathrm{i}=1}^{2} \left(\sum_{k=1}^{\mu} \left(\rho^{4k-2-i} F_{L,k}^{(i)} + \rho^{4k-i} F_{R,k}^{(i)}\right) + \rho^{4\mu} \Big(\sum_{k=0}^{\mu+\nu} \rho^{2k-1+i} F_{P,k}^{(i)}\Big) \right).$$

The extracted polynomials satisfy the following relation:

$$\mathrm{L}_k^{(i)} = \mathsf{Com}_{\mathsf{PC}}(\mathsf{ck}_{\mathsf{PC}}, F_{L,k}),\ \mathrm{R}_k^{(i)} = \mathsf{Com}_{\mathsf{PC}}(\mathsf{ck}_{\mathsf{PC}}, F_{R,k}),\ \mathrm{P}_k^{(i)} = \mathsf{Com}_{\mathsf{PC}}(\mathsf{ck}_{\mathsf{PC}}, F_{P,k}) \quad (13)$$

Finally, $\mathcal{E}_{MEC}$ outputs $\boldsymbol{L}_k^{(i)}, \boldsymbol{R}_k^{(i)}, \boldsymbol{P}_k^{(i)}$ by decoding $F_{L,k}^{(i)}, F_{R,k}^{(i)}, F_{P,k}^{(i)}$ respectively.

It remains to check that these extracted vectors are valid witnesses satisfying all relations from Eq. (3) to (5). First, by the extraction process, the extracted vectors $\boldsymbol{L}_k^{(i)}, \boldsymbol{R}_k^{(i)}, \boldsymbol{P}_k^{(i)}$ satisfy Eq. (13), so does Eq. (3). In addition, by the construction of $\mathrm{A}^{(i)}$, the polynomial $a^{(i)}$ is equal to the sum of $F_{L,k}^{(i)}, F_{R,k}^{(i)}, F_{P,k}^{(i)}$, *i.e.* $a^{(i)} = \sum_{k=1}^{\mu}(F_{L,k}^{(i)} + F_{R,k}^{(i)}) + \sum_{k=0}^{\mu+\nu} F_{P,k}^{(i)}$. This implies that the evaluations of wire polynomial $w_L^{(i)}$ at appropriate ξ^i contain those of $F_{L,k}^{(i)}, F_{R,k}^{(i)}, F_{P,k}^{(i)}$, where each value matches with the values of $\boldsymbol{L}_k^{(i)}, \boldsymbol{R}_k^{(i)}, \boldsymbol{P}_k^{(i)}$ at the corresponding positions. Furthermore, the wire polynomials $w_L^{(i)}$, $w_R^{(i)}$, $w_O^{(i)}$ and auxiliary polynomials $\alpha, \beta, \gamma, \delta, \lambda$ extracted from $\mathcal{E}_{\mathsf{Plonkish}}$ ensure that $\boldsymbol{L}_k^{(i)}, \boldsymbol{R}_k^{(i)}, \boldsymbol{P}_k^{(i)}$ satisfy the relation Eq. (4) and Eq. (5).

To sum up, the extracted vectors $\boldsymbol{L}_k^{(i)}, \boldsymbol{R}_k^{(i)}, \boldsymbol{P}_k^{(i)}$ are actually the valid witnesses, concluding that AggMEC satisfies computational witness extended emulation. □

D Plonkish Arithmetization with Custom Gates

In this section, we provide supplementary description of Plonkish for elliptic curve operations.

D.1 Plonkish arithmetization

We first provide a basic idea of Plonk arithmetization. For each gate in the circuit, Plonk constructs a constraint equation according to the type, *e.g.*, addition or multiplication, of the gate. To represent this, Plonk adopts an auxiliary variable called the *selector* that indicates which types of gates are enabled or not in the current gate. To ensure that the given two gates are connected, Plonk exploits the constraints using a permutation, which ensures that the values in the wires that connecting gates do not change after permutation on them. With Lagrange Interpolation for left inputs, right inputs, outputs, and selectors separately, using a cyclic group generated by the N-th root of unity ζ of $\mathbb{Z}_q$, all constraint equations except the permutation constraints can be expressed as a single polynomial equation.

Plonkish generalizes Plonk by handling all the values occurred in the execution of the circuit as the *execution trace* $\mathbb{Z}_q^{N \times M}$. Each row represents the inputs, outputs, or auxiliary values occurred in the corresponding execution step. This execution trace can be represented as a sequence of polynomials by applying Lagrange interpolation with respect to each column. Each gate can be written as polynomial comprised of column polynomials that are engaged to the current gate. After then, arguments for gate identity and permutation can be constructed using these polynomials. Formally, let $\{v_i(X)\}_{i=1}^{M}$ be the polynomials that represents the execution trace of the given circuit, which correspond to the column polynomials mentioned. For the number N_g of the types of gates in the circuit, we denote $\{c_i(X)\}_{i=1}^{N_g}$ as the gate polynomials for the circuit. Each gate polynomial can be represented as $c_i(X) = g_i(v_1(X), v_2(X), \dots, v_M(X))$ for some M-variate polynomial g_i. Let us define $\{s_i(X)\}_{i=1}^{N_g}$ as the selector polynomials.

In addition, for the permutation argument, we denote a permutation $\sigma : [N] \times [M] \to [N] \times [M]$. $\sigma(i,j) = (\sigma(i,j)_1, \sigma(i,j)_2)$ is equivalent to $v_i(\zeta^j) = v_{\sigma(i,j)_1}(\zeta^{\sigma(i,j)_2})$. Suppose $N = 2^k$ and δ is a T-th root of unity, where $T \cdot 2^S + 1 = q$ with odd T and $k \le S$. We use $\delta^i \cdot \zeta^j$ as the label for a value corresponding to $(\sigma(i,j)_1, \sigma(i,j)_2)$, as mentioned in [40]. Define $\mathsf{ID}_i(\zeta^j) = \delta^i \cdot \zeta^j$ that is an identity polynomial of $v_i(\zeta^j)$ and $r_i(\zeta^j) = \delta^{\sigma(i,j)_1} \cdot \zeta^{\sigma(i,j)_2}$. The idea behind the permutation argument technique is the fact that $\prod_{h=1}^{N} \prod_{i=1}^{M} \frac{v_i(\zeta^h) + u_1 \mathsf{ID}_i(\zeta^h) + u_2}{v_i(\zeta^h) + u_1 r_i(\zeta^h) + u_2}$ is equal to 1 when $v_i(\zeta^j) = v_{\sigma(i,j)_1}(\zeta^{\sigma(i,j)_2})$ for random values u_1, u_2. We can check the details for the technique in [4].

Plonkish is a protocol for arithmetic circuit satisfiability, and the circuit satisfiability is ensured when (1) $v_i(\zeta^j) = v_{\sigma(i,j)_1}(\zeta^{\sigma(i,j)_2})$ for $i \in [N], j \in [M]$ and (2) $\sum_{i=1}^{N_g} s_i(X) c_i(X) = 0 \mod X^N - 1$. As shown in several studies [21, 22, 40], the relations can be efficiently proved by the Polynomial IOP instantiated by PCS [14]. In short, to check polynomial relations, the prover commits polynomial and then the verifier sends random point as challenge. After then, the prover responds evaluations. To verify the responds, the prover and verifier run Eval interactive proof. Thanks to the Fiat-Shamir transform, interactive proof Eval can be converted to non-interactive proof system. We describe Plonkish protocol in Algorithm 4.

We provide a brief idea of [22] to construct the polynomial relation covering both (1) and (2) as follows: First, in line 3, the prover computes $z(X)$, which is the interpolation of the values obtained by multiplying $\prod_{i=1}^{M} \frac{v_i(\zeta^h) + u_1 \mathsf{ID}_i(\zeta^h) + u_2}{v_i(\zeta^h) + u_1 r_i(\zeta^h) + u_2}$ one by one for $h \in [N]$. Then, as shown by [4], $z(X)$ satisfies $z(\zeta X)/z(X) = \sum_{i=1}^{M}(v_i(X) + u_1 \mathsf{ID}_i(X) + u_2) / \sum_{i=1}^{M}(v_i(X) + u_1 r_i(X) + u_2) \mod X^N - 1$ and $z(\zeta) = 1$ for random challenges u_1, u_2. Hence, by combining these constraints and the gate constraints by another random challenge u_3, the prover computes $t(x)$, as described in line 5. Now, checking that $t(\zeta^i) = 0$ for $i \in [N]$ is sufficient to convince the relations (1) and (2), which can be done by several runs of Eval.

D.2 Custom Gate for Elliptic Curve Addition

We provide the detailed construction of custom gate for elliptic curve addition in affine coordinates introduced by [40].

Let $E(\mathbb{Z}_q)$ be a prime elliptic curve group with $q \ge 5$ given by the short Weierstrass equation and $|E(\mathbb{Z}_q)| = p$. $E(\mathbb{Z}_q)$ is given by the form $\{(X, Y) \in \mathbb{Z}_q^2 | Y^2 = X^3 + aX + b\} \cup \{\mathcal{O}\}$ for a, b in $\mathbb{Z}_q$. We require an additional requirement $X^3 + aX + b = 0$ and $X^2 - b = 0$ have no solutions in $\mathbb{Z}_q$, *i.e.*, each coordinate of non-identity points is all nonzero, to represent the point at infinity $\mathcal{O}$ as $(0, 0)$. For curve points $W_L = (w_L^{(1)}, w_L^{(2)})$, $W_R = (w_R^{(1)}, w_R^{(2)})$ and $W_O = (w_O^{(1)}, w_O^{(2)})$, the formula for point addition is categorized into six cases: (1) $W_L +_e W_R$ for

Algorithm 4 $\mathsf{Plonkish}_{\mathsf{Eval}}$

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   $\mathsf{Plonkish}_{\mathsf{Eval}}(\mathsf{ck}_{\mathsf{PC}}, \{s_i(X), g_i(X_1,\ldots,X_M)\}_{i=1}^{N_g}, \{r_i(X)\}_{i=1}^{M}; \{v_i(X)\}_{i=1}^{M})$
   **Precompute:** $C_{\mathsf{ID}_i} = \mathsf{Com}_{\mathsf{PC}}(\mathsf{ck}_{\mathsf{PC}}, \mathsf{ID}_i(X)),\ C_{r_i} = \mathsf{Com}_{\mathsf{PC}}(\mathsf{ck}_{\mathsf{PC}}, r_i(X)),\ i \in [M]$
1: $\mathcal{P}$ sends $V_i = \mathsf{Com}_{\mathsf{PC}}(\mathsf{ck}_{\mathsf{PC}}, v_i(X))$ to $\mathcal{V}$
2: $\mathcal{V}$ chooses $u_1, u_2 \xleftarrow{\$} \mathbb{Z}_q$ and sends it to $\mathcal{P}$
3: $\mathcal{P}$ sends $Z = \mathsf{Com}_{\mathsf{PC}}(\mathsf{ck}_{\mathsf{PC}}, z(X))$ to $\mathcal{V}$ where
   $$z(X) = H_1(X) + \sum_{j=1}^{N-1}\left(H_{j+1}(X)\prod_{h=1}^{j}\prod_{i=1}^{M}\frac{v_i(\zeta^h) + u_1\mathsf{ID}_i(\zeta^h) + u_2}{v_i(\zeta^h) + u_1 r_i(\zeta^h) + u_2}\right).$$
   $$H_j(X) = \prod_{i \neq j, i \in [N]} (X - \zeta^i)/(\zeta^j - \zeta^i) \text{ for all } j \in [N].$$
4: $\mathcal{V}$ chooses $u_3 \xleftarrow{\$} \mathbb{Z}_q$ and sends it to $\mathcal{P}$.
5: $\mathcal{P}$ sends $T = \mathsf{Com}_{\mathsf{PC}}(\mathsf{ck}_{\mathsf{PC}}, t(X)), Q = \mathsf{Com}_{\mathsf{PC}}(\mathsf{ck}_{\mathsf{PC}}, q(X))$ to $\mathcal{V}$ where
   $$t(X) = \sum_{i=1}^{N_g} s_i(X) g_i(v_1(X), \ldots v_M(X)) + u_3 \cdot z(X) \sum_{i=1}^{M}(v_i(X) + u_1\mathsf{ID}_i(X) + u_2)$$
   $$- u_3 \cdot z(\zeta X)\sum_{i=1}^{M}(v_i(X) + u_1 r_i(X) + u_2) + u_3^2 \cdot (z(X) - 1)H_1(X)$$
   $$q(X) = t(X)/z_H(X), \text{ where } z_H(X) = \prod_{i=1}^{N}(X - \zeta^i).$$
6: $\mathcal{V}$ chooses $u_4 \xleftarrow{\$} \mathbb{Z}_q$ and sends it to $\mathcal{P}$.
7: $\mathcal{P}$ sends $\{\alpha_i = v_i(u_4)\}_{i=1}^{M}$, $\beta = z(u_4)$, $\gamma = z(\zeta u_4)$,
       $\{\phi_i = \mathsf{ID}_i(u_4)\}_{i=1}^{M}$, and $\{\psi_i = r_i(u_4)\}_{i=1}^{M}$ to $\mathcal{V}$.
8: $\mathcal{V}$ evaluates $\rho_1$ and $\rho_2 = \rho_1 / z_H(u_4)$ using the values received from $\mathcal{P}$
   $$\rho_1 = \sum_{i=1}^{N_g} s_i(u_4) \cdot g_i(\alpha_i, \ldots, \alpha_M) + u_3 \cdot \beta \sum_{i=1}^{M}(\alpha_i + u_1 \cdot \phi_i + u_2)$$
   $$- u_3 \cdot \gamma \sum_{i=1}^{M}(\alpha_i + u_1 \cdot \psi_i + u_2) + u_3^2 \cdot (\beta - 1)H_1(u_4)$$
9: $\mathcal{P}$ and $\mathcal{V}$ set run $\mathsf{Eval}(\mathsf{ck}_{\mathsf{PC}}, T, u_4, \rho_1; t(X))$, $\mathsf{Eval}(\mathsf{ck}_{\mathsf{PC}}, V_i, u_4, \alpha_i; v_i(X))_{i \in [M]}$,
   $\mathsf{Eval}(\mathsf{ck}_{\mathsf{PC}}, Q, u_4, \rho_2; q(X))$, $\mathsf{Eval}(\mathsf{ck}_{\mathsf{PC}}, Z, u_4, \beta; z(X))$, $\mathsf{Eval}(\mathsf{ck}_{\mathsf{PC}}, Z, \zeta u_4, \gamma; z(X))$,
   $\mathsf{Eval}(\mathsf{ck}_{\mathsf{PC}}, C_{\mathsf{ID}_i}, u_4, \phi_i; \mathsf{ID}_i(X))_{i \in [M]}$, and $\mathsf{Eval}(\mathsf{ck}_{\mathsf{PC}}, C_{r_i}, u_4, \psi_i; r_i(X))_{i \in [M]}$.
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$w_L^{(1)} \neq w_R^{(1)}$ and $w_L^{(2)} \neq \pm w_R^{(2)}$, (2) $W_L +_e W_L$, (3) $W_L +_e (-W_L)$, (4) $W_L +_e \mathcal{O}$, (5) $\mathcal{O} +_e W_R$ and (6) $\mathcal{O} +_e \mathcal{O}$. To represent each case on arithmetic circuit, we first define a function $\mathsf{Inv}(x)$ that returns the inverse of x if $x \neq 0$, otherwise returns 0. Using this, if we define witnesses$\alpha, \beta, \gamma, \delta$ and λ such that

$$\alpha = \mathsf{Inv}(w_R^{(1)} - w_L^{(1)}), \quad \beta = \mathsf{Inv}(w_L^{(1)}), \quad \gamma = \mathsf{Inv}(w_R^{(1)}),$$
$$\delta = \begin{cases} \mathsf{Inv}(w_R^{(2)} + w_L^{(2)}), & w_L^{(1)} = w_R^{(1)} \\ 0 & \text{otherwise} \end{cases}$$
$$\lambda = \begin{cases} (w_R^{(2)} - w_L^{(2)})/(w_R^{(1)} - w_L^{(1)}), & w_L^{(1)} \neq w_R^{(1)} \\ 3{w_L^{(1)}}^2/2w_L^{(2)}, & w_L^{(1)} = w_R^{(1)} \wedge w_L^{(2)} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

We can figure out that (1) occurs iff $\alpha \neq 0$, (2) occurs $\alpha = 0$ and $\delta \neq 0$, (3) occurs iff $\alpha = 0$ and $\delta = 0$, (4) occurs iff $\beta \neq 0$ and $\gamma = 0$, (5) occurs iff $\beta = 0$ and $\gamma \neq 0$, and (6) occurs iff $\beta = 0$ and $\gamma = 0$. In addition, λ indicates the slope that occurs for case (1) and (2).

From these auxiliary variables, we can construct a bunch of constaint equations that covers all cases (1)-(6) as follows. For simplicity, we will denote the straight line passing through W_L and W_R as $\mathcal{L}$. First of all, for (1), if we set

(a) $((w_R^{(1)}(X) - w_L^{(1)}(X))((w_R^{(1)}(X) - w_L^{(1)}(X))\lambda - (w_R^{(2)}(X) - w_L^{(2)}(X))) = 0$
(b) $w_L^{(1)}(X)w_R^{(1)}(X)(w_R^{(1)}(X) - w_L^{(1)}(X))(\lambda^2 - w_L^{(1)}(X) - w_R^{(1)}(X) - w_O^{(1)}(X)) = 0$

(c) $w_L^{(1)}(X)w_R^{(1)}(X)(w_R^{(1)}(X) - w_L^{(1)}(X))(\lambda(w_L^{(1)}(X) - w_O^{(1)}(X)) - w_L^{(2)}(X) - w_O^{(2)}(X)) = 0$,

each (a), (b) and (c) ensures that the slope of $\mathcal{L}$, $w_O^{(1)}$ and $w_O^{(2)}$ are computed correctly, otherwise $w_L^{(1)} = w_R^{(1)}$ or at least one of them is $\mathcal{O}$.

For (2), if we set

(d) $(1 - (w_R^{(1)}(X) - w_L^{(1)}(X))\alpha)(2w_L^{(2)}(X)\lambda - 3{w_L^{(1)}(X)}^2) = 0$
(e) $w_L^{(1)}(X)w_R^{(1)}(X)(w_R^{(2)}(X) + w_L^{(2)}(X))(\lambda^2 - w_L^{(1)}(X) - w_R^{(1)}(X) - w_O^{(1)}(X)) = 0$

(f) $w_L^{(1)}(X)w_R^{(1)}(X)(w_R^{(2)}(X) + w_L^{(2)}(X))(\lambda(w_L^{(1)}(X) - w_O^{(1)}(X)) - w_L^{(2)}(X) - w_O^{(2)}(X)) = 0$,

each (d),(e) and (f) ensures the slope of $\mathcal{L}$, $w_O^{(1)}$ and $w_O^{(2)}$ are computed correctly, otherwise $w_L^{(1)} \neq w_R^{(1)}$, $w_L^{(2)} \neq w_R^{(2)}$, or at least one of them is $\mathcal{O}$.

For considering the case (3), it suffices to ensure if $\alpha = \delta = 0$, then $W_O = \mathcal{O}$. To this end, we set

(g) $(1 - (w_R^{(1)} - w_L^{(1)})\alpha - (w_R^{(2)} + w_L^{(2)})\delta)w_O^{(1)} = 0$
(h) $(1 - (w_R^{(1)} - w_L^{(1)})\alpha - (w_R^{(2)} + w_L^{(2)})\delta)w_O^{(2)} = 0.$

Finally, for cases (4)-(6), we need to ensure that $\beta = 0$ implies $W_O = W_R$, and $\gamma = 0$ implies $W_O = W_L$. If we set,

(i) $(1 - w_L^{(1)}\beta)(w_O^{(1)} - w_R^{(1)}) + (1 - w_L^{(1)}\beta)(w_O^{(2)} - w_R^{(2)}) = 0$
(j) $(1 - w_R^{(1)}\gamma)(w_O^{(1)} - w_L^{(1)}) + (1 - w_R^{(1)}\gamma)(w_O^{(2)} - w_L^{(2)}) = 0$,

each (i), (j) guarantees the above requirement, otherwise $\beta \neq 0$ and $\gamma \neq 0$.

$\mathsf{Plonkish}_{\mathsf{Eval}}$. By merging all constraints from (a) to (j), we can obtain the polynomial g_{CA} for elliptic curve point addition in affine coordinates, which takes polynomials corresponding to $w_L^{(1)}$, $w_L^{(2)}$, $w_R^{(1)}$, $w_R^{(2)}$, $w_O^{(1)}$, $w_O^{(2)}$ and α, β, γ, δ, λ as inputs. We will call $\mathsf{Plonkish}_{\mathsf{Eval}}$ as an instantiation of $\mathsf{Plonkish}$ in Section D.1, with the custom gate polynomial g_{CA}.

E Polynomial Commitment Scheme from Leopard

In this section, we provide details about the $\mathsf{Leopard}_{\mathsf{PC}}$, which is a key ingredient to instantiate Cougar. Remark that [29, Section E.1.] provided a basic idea for constructing this; we present the full description for the sake of completeness.

$\mathsf{Leopard}_{\mathsf{PC}}$ is a natural tweak of $\mathsf{Protocol3}$ [29] as a PCS. The construction idea is basically the same as the PCS introduced by [11], which was built upon BulletProofs. More precisely, we can construct PCS from IPA by regarding the point evaluation of the polynomial as an inner product between the coefficient vector and the vector comprised by the powers of the evaluation point. The asymptotic communication and computation complexities of Eval from this approach are the same as those of the underlying IPA. Note that $\mathsf{Protocol3}$ features square root verifier cost and logarithmic communication cost; hence so does $\mathsf{Leopard}_{\mathsf{PC}}.\mathsf{Eval}$.

Following the above approach, we provide the full description of $\mathsf{Leopard}_{\mathsf{PC}}$ as follows: Let $(\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_t)$ be a bilinear group, where $\mathbb{G}_1 = E(\mathbb{Z}_p)$. For a polynomial $a(X) \in \mathbb{Z}_p^{<mn}[X]$ and positive integers $m, n \in \mathbb{N}$, we will denote its coefficient vector as $\boldsymbol{a} \in \mathbb{Z}_p^{mn}$, namely, $\boldsymbol{a} = (a_0, \ldots, a_{mn-1})$ such that $a(X) = \sum_{i=0}^{mn-1} a_i X^i$. $\mathsf{Leopard}_{\mathsf{PC}} = (\mathsf{Gen}, \mathsf{Com}, \mathsf{Eval})$ over a message space $\mathbb{Z}_p^{<mn}[X]$ and a commitment space $\mathbb{G}_t$ is defined as follows[1]:

- $\mathsf{Gen}(1^\lambda) \to \mathsf{ck}_{\mathsf{PC}} \in \mathbb{G}_1^m \times \mathbb{G}_2^n$.
- $\mathsf{Com}(\mathsf{ck}_{\mathsf{PC}} = (\boldsymbol{G}, \boldsymbol{H}), a(X)) \to P := (\boldsymbol{G} \otimes \boldsymbol{H})^{\boldsymbol{a}} \in \mathbb{G}_t$.

In addition, $\mathsf{Eval} = (\mathcal{K}, \mathcal{P}, \mathcal{V})$ is an interactive argument system for the following relation:

$$\mathcal{R}_{\mathsf{Leopard}_{\mathsf{PC}}.\mathsf{Eval}} = \left\{ \begin{pmatrix} \mathsf{ck}_{\mathsf{PC}} = (\boldsymbol{G}, \boldsymbol{H}) \in \mathbb{G}_1^m \times \mathbb{G}_2^n, \\ C \in \mathbb{G}_t, z, y \in \mathbb{Z}_p, d \in [mn]; \\ a(X) \in \mathbb{Z}_p^{<mn}[X] \end{pmatrix} : \begin{array}{c} C = (\boldsymbol{G} \otimes \boldsymbol{H})^{\boldsymbol{a}} \\ \wedge \\ y = a(z) \end{array} \right\}$$

[1] We will not consider hiding property because zero-knowledge property is unnecessary in our context.

A typical strategy to cope with the above relation is to modify the above relation into that for IPA: For $\boldsymbol{z} = (1, z, \ldots, z^{mn-1})$, we can rewrite $a(z) = \langle \boldsymbol{a}, \boldsymbol{z} \rangle$. For this reason, the construction of Eval is almost identical to $\mathsf{Protocol3}$ except for some modifications regarding the fact that $\boldsymbol{z}$ is also known to the verifier. The precise description of $\mathsf{Leopard_{PC}.Eval}$ is given in Algorithm 5. Here, $\mathsf{bit}(k)$ refers to the bit decomposition of a number k.

We now show that $\mathsf{Leopard_{PC}}$ is indeed the PCS, *i.e.*, satisfying the conditions in Definition 7, under the DL assumption. In fact, $\boldsymbol{G} \otimes \boldsymbol{H}$ in the above relation can be seen as the commitment key of the Pedersen vector commitment defined over the group $\mathbb{G}_t$, along with a certain structure. Since the binding property of the Pedersen vector commitment depends on the DLR assumption, one can expect that the same holds for $\mathsf{Leopard_{PC}}$ under a structure-aware version of the DLR assumption.

For this reason, we first provide a definition of generalized discrete logarithm relation (GDLR) assumption, which was previously defined in [29, Definition 8]. For simplicity, we denote $\mathcal{G}_b$ as a bilinear group generator that takes the security parameter λ and outputs a bilinear group $(\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_t)$ of order p, generators g, h for $\mathbb{G}_1$ and $\mathbb{G}_2$, respectively, and a pairing operator e.

Definition 8. *For $m, n \in \mathbb{N}$ and the security parameter $\lambda \in \mathbb{N}$, let* GDLRsp *be a sampler defined by*

$$\begin{aligned} \mathsf{GDLRsp}(1^\lambda) : {} & (p, g, h, \mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_t, e) \leftarrow \mathcal{G}_b(1^\lambda); \boldsymbol{G} \xleftarrow{\$} \mathbb{G}_1^m; \boldsymbol{H} \xleftarrow{\$} \mathbb{G}_2^n; \\ & \textit{Output } (p, \boldsymbol{G} \otimes \boldsymbol{H}, \mathbb{G}_t), \end{aligned}$$

Then, we say that GDLRsp *satisfies the general discrete logarithm relation (GDLR) assumption if all non-uniform polynomial-time adversaries $\mathcal{A}$, the following inequality holds:*

$$\Pr\left[\boldsymbol{a} \neq \boldsymbol{0} \ \wedge \ \boldsymbol{g}^{\boldsymbol{a}} = 1_{\mathbb{G}_t} \middle| \begin{array}{c} (p, \boldsymbol{g} \in \mathbb{G}_t^{m \times n}, \mathbb{G}_t) \leftarrow \mathsf{GDLRsp}(1^\lambda) \\ \boldsymbol{a} \leftarrow \mathcal{A}(p, \boldsymbol{g}, \mathbb{G}_t) \end{array}\right]$$

where $1_{\mathbb{G}_t}$ is the identity of $\mathbb{G}_t$ and $negl(\lambda)$ is a negligible function in λ.

As shown by [29, Theorem 5], if the DL assumption on both $\mathbb{G}_1$ and $\mathbb{G}_2$ hold, then the GDLR assumption also holds. In addition, by assuming the GDLR assumption, the binding property of $\mathsf{Leopard_{PC}}$ holds immediately.

Now it remains to check that $\mathsf{Leopard_{PC}.Eval}$ is an AoK for the relation $\mathcal{R}_{\mathsf{Leopard_{PC}.Eval}}$. As we mentioned, this relation can be understood as a special case of that for $\mathsf{Protocol3}$, and Algorithm 5 is in fact almost identical to $\mathsf{Protocol3}$. We note that $\mathsf{Protocol3}$ satisfies perfect completeness and computational witness-extended emulation under the GDLR assumption [28]. In fact, we made the same modifications as [11] for constructing $\mathsf{Leopard_{PC}.Eval}$, without considering zero-knowledge. That is, the proof strategies for computational witness-extended emulation of ours and theirs are identical, except for replacing the DLR assumption with the GDLR assumption. We refer to [28] and [11] for more detailed information.

Algorithm 5 $\mathsf{Leopard}_{\mathsf{PC}}.\mathsf{Eval}$

```
    $\mathsf{Leopard}_{\mathsf{PC}}.\mathsf{Eval}(\mathsf{ck}_{\mathsf{PC}} = (\boldsymbol{G}, \boldsymbol{H}) \in \mathbb{G}_1^m \times \mathbb{G}_2^n, P \in \mathbb{G}_t, z, y \in \mathbb{Z}_p; \boldsymbol{a} \in \mathbb{Z}_p^{mn})$
    where $m = 2^\mu$ and $n = 2^\nu$
 1: $\mathcal{V}$ picks $U \xleftarrow{\$} \mathbb{G}_t$ and sends it to $\mathcal{P}$
 2: $\mathcal{P}$ and $\mathcal{V}$ set $P_0 = P + [y]U$, $\boldsymbol{G}_0 = \boldsymbol{G}, \boldsymbol{H}_0 = \boldsymbol{H}$.
    Additionally, $\mathcal{P}$ set $\boldsymbol{a}_0 = \boldsymbol{a}$ and $\boldsymbol{z}_0 = [z^{m(i-1)+(j-1)}] \in \mathbb{Z}_p^{m \times n}$
 3: for $i = 0, \ldots, \mu - 1$ do
 4:     $\mathcal{P}$ parses $\boldsymbol{a}_i$, $\boldsymbol{z}_i$, and $\boldsymbol{G}_i$ to
        $\boldsymbol{a}_i = [\boldsymbol{a}_{i,L} \parallel \boldsymbol{a}_{i,R}]$,   $\boldsymbol{z}_i = [\boldsymbol{z}_{i,L} \parallel \boldsymbol{z}_{i,R}]$,   $\boldsymbol{G}_i = \boldsymbol{G}_{i,L} \| \boldsymbol{G}_{i,R}$
 5:     $\mathcal{P}$ computes:
        $L_i = [\boldsymbol{a}_{i,L}](\boldsymbol{G}_{i,R} \otimes \boldsymbol{H}) + [\langle \boldsymbol{a}_{i,L}, \boldsymbol{z}_{i,R} \rangle]U \in \mathbb{G}_t$
        $R_i = [\boldsymbol{a}_{i,R}](\boldsymbol{G}_{i,L} \otimes \boldsymbol{H}) + [\langle \boldsymbol{a}_{i,R}, \boldsymbol{z}_{i,L} \rangle]U \in \mathbb{G}_t$
 6:     $\mathcal{P}$ sends $L_i, R_i$ to $\mathcal{V}$
 7:     $\mathcal{V}$ chooses $r_i \xleftarrow{\$} \mathbb{Z}_p^*$ and sends it to $\mathcal{P}$
 8:     $\mathcal{P}$ computes:
        $\boldsymbol{a}_{i+1} = \boldsymbol{a}_{i,L} + r_i^{-1} \boldsymbol{a}_{i,R}$,   $\boldsymbol{z}_{i+1} = \boldsymbol{z}_{i,L} + r_i \boldsymbol{z}_{i,R} \in \mathbb{Z}_p^{m/2^{i+1} \times n}$
        $\boldsymbol{G}_{i+1} = \boldsymbol{G}_{i,L} + [r_i] \boldsymbol{G}_{i,R} \in \mathbb{G}_1^{m/2^{i+1}}$
        $P_{i+1} = [r_i]L_i + P_i + [r_i^{-1}]R_i \in \mathbb{G}_t$
 9: end for
10: for $j = 0, \ldots, \nu - 1$ do
11:     $\mathcal{P}$ sets $i = j + \mu$ and then parses $\boldsymbol{a}_i$, $\boldsymbol{z}_i$, and $\boldsymbol{H}_j$ to
        $\boldsymbol{a}_i = \boldsymbol{a}_{i,L} \parallel \boldsymbol{a}_{i,R}$,   $\boldsymbol{z}_i = \boldsymbol{z}_{i,L} \parallel \boldsymbol{z}_{i,R}$,   $\boldsymbol{H}_j = \boldsymbol{H}_{j,L} \| \boldsymbol{H}_{j,R}$
12:     $\mathcal{P}$ computes:
        $L_i = [\boldsymbol{a}_{i,L}](G_\mu \otimes \boldsymbol{H}_{j,R}) + [\langle \boldsymbol{a}_{i,L}, \boldsymbol{z}_{i,R} \rangle]U \in \mathbb{G}_t$
        $R_i = [\boldsymbol{a}_{i,R}](G_\mu \otimes \boldsymbol{H}_{j,L}) + [\langle \boldsymbol{a}_{i,R}, \boldsymbol{z}_{i,L} \rangle]U \in \mathbb{G}_t$
13:     $\mathcal{V}$ chooses $r_i \xleftarrow{\$} \mathbb{Z}_p^*$ and sends it to $\mathcal{P}$
14:     $\mathcal{P}$ computes:
        $\boldsymbol{a}_{i+1} = \boldsymbol{a}_{i,L} + s_j^{-1} \boldsymbol{a}_{i,R}$,   $\boldsymbol{z}_{i+1} = \boldsymbol{z}_{i,L} + s_j \boldsymbol{z}_{i,R} \in \mathbb{Z}_p^{n/2^{j-1}}$
        $\boldsymbol{H}_{j+1} = \boldsymbol{H}_{j,L} + [s_j] \boldsymbol{H}_{j,R} \in \mathbb{G}_2^{n/2^{j+1}}$
        $P_{i+1} = [r_i]L_i + \cdot P_i + [r_i^{-1}]R_i \in \mathbb{G}_t$
15: end for
16: $\mathcal{P}$ sends $a = a_{\mu+\nu} \in \mathbb{Z}_p$ to $\mathcal{V}$
17: $\mathcal{V}$ computes:
    $\boldsymbol{r}[k+1] = \langle \mathsf{bit}(k), (r_0, \ldots, r_{\mu+\nu-1}) \rangle$ for $k = 0, \ldots m + n - 1$
    Parse $\boldsymbol{r}$ to $\boldsymbol{r}_{row} \parallel \boldsymbol{r}_{col}$ where $\boldsymbol{r}_{row} \in \mathbb{Z}_p^m$ and $\boldsymbol{r}_{col} \in \mathbb{Z}_p^n$
    $G = \langle \boldsymbol{r}_{row}, \boldsymbol{G}_0 \rangle, H = \langle \boldsymbol{r}_{col}, \boldsymbol{H}_0 \rangle, z = \boldsymbol{r}_{row} \boldsymbol{z}_0 \boldsymbol{r}_{col}$
18: $\mathcal{V}$ checks:
    $P_0 + \sum_{i \in [\mu+\nu]} ([r_i]L_i + [r_i^{-1}]R_i) = e([az]G, H)$
```

To sum up, $\mathsf{Leopard}_{\mathsf{PC}}$ satisfies all conditions in Definition 7 under the DL assumption on $\mathbb{G}_1$ and $\mathbb{G}_2$. In addition, it does not require the trusted setup and features squared root verification cost and logarithmic communication cost with respect to the length of the witness. Therefore, it is a desirable PCS for instantiating Cougar.