Efficient Permutation Correlations and Batched Random Access for Two-Party Computation

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Abstract. In this work we define the notion of a permutation correlation (π, A, B, C) s.t. $\pi(A) = B + C$ for a random permutation π of n elements and vectors $A, B, C \in \mathbb{F}^n$. We demonstrate the utility of this correlation for a wide range of applications. The correlation can be derandomized to obliviously shuffle a secret-shared list, permute a secret-shared list by a secret-shared permutation, and more. Similar techniques have emerged as a popular building block for the honest majority protocols when efficient batched random access is required, e.g. collaborative filtering, sorting, database joins, graph algorithms, and many more. We present the highly flexible notion of permutation correlation and argue that it should be viewed as a first class primitive in the MPC practitioner's toolbox.

We give two novel protocols for efficiently generating a random permutation correlation. The first makes use of recent advances in MPC-friendly PRFs to obtain a protocol requiring $O(n\ell)$ OTs/time and constant rounds to permute n ℓ-bit strings. Unlike the modern OT extension techniques we rely on, this was previously only achievable from relatively more expensive public-key cryptography, e.g. Paillier or LWE. We implement this protocol and demonstrate that it can generate a correlation for $n = 2^{20}, \ell = 128$ in 19 seconds and $\sim 2\ell n$ communication, a 15 & $1.1\times$ improvement over the LWE solution of Juvekar at al. (CCS 2018). The second protocol is based on pseudo-random correlation generators and achieves an overhead that is *sublinear* in the string length ℓ, i.e. the communication and number of OTs is $O(n \log \ell)$. The latter protocol is ideal for the setting when you need to repeatedly permute secret-shared data by the same permutation, e.g. in graph algorithms.

Finally, we present a suite of highly efficient protocols for performing various batched random access operations. These include a class of protocols we refer to as *extraction*, which allow a user to *mark* a subset of X and have this subset obliviously extracted into an output list. Additionally, the parties can specify an *arbitrary* selection function $\sigma : [n] \to [n]$ and obtain shares of $\sigma(X) = (X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ from X. We implement these protocols and report on their performance. [4]

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1 Introduction

Secure multi-party computation (MPC) is increasingly used to perform complex data intensive tasks while maintaining strong privacy guarantees. Examples include machine learning & data analytics, database joins, sorting and many more. A common thread to most of these complex tasks is the need to perform some type of random access into the processed data. Unfortunately, this is at odds with how MPC and similar technologies work. Typically, MPC protocols require the target function to be expressed as a circuit where all memory accesses are fixed and independent of the input. One line of work that overcomes this is oblivious RAM for MPC [LO13,WCS15,HKO23,PLS23]. However, while very flexible, this approach introduces a poly log overhead for each access.

Many of the most important applications can be made efficient in a circuit model with access to *permutation gates*. In this model, the main part of the circuit is expressed with addition and multiplication gates. The circuit also has access to a permutation gate that takes as input a permutation, possibly expressed as a vector, and a list of values. The output of the permutation gate is the list permuted according to the input permutation. The most ubiquitous example of such a gate would be to permute a list of values by the sorting permutation [AHI+22]. We show that it is possible for these permutation gates to be implemented in concretely efficient linear time and constant depth/round, sidestepping the poly log overhead of oblivious RAM.

1.1 Applications of Secret-shared Permutation

Secret-shared permutation is an essential building block of many MPC protocols. Thus, reducing the cost of secret-shared permutation will lead to significant cost reductions in these protocols. We now present a non-exhaustive list of applications.

Shuffle. Two parties jointly shuffle an array and returns additive secret sharing of the result. The security guarantee is that no party learns the random permutation corresponding to the shuffle. This technique has primarily been used in the three-party setting [AKK+23,BDG+22,AHI+22] but has also been generalized to n-party shuffle. [FO20] explicates the use of shuffle in MPC.

Merging and Sorting. The most efficient secure sorting algorithms rely on a shuffle-then-sort paradigm [HKI+13] or Radix Sort[CHI+19]. For the former, the idea is that after the shuffle, the (comparison base) sorting algorithms can *reveal* the result of each comparison and move data based on the result without compromising security. This is because the comparison bit is independent of the underlying value after shuffling. Radix sort uses different techniques and permutes the data several times, once per bit of the sorting key. Due to the lack of efficient two-party permutation, these protocols have remained costly.

Similarly, permutations are a useful component for secure merging protocols, where two sorted lists are merged so that the final secret-shared list is ordered. E.g., [FO20]'s merge relies on secure shuffle in many subprotocols.

Graph Algorithms. Our improvement is particularly significant in frameworks that rely heavily on oblivious permutations. E.g., GraphSC [NWI+15] is a framework that enables efficient secure implementation of graph-based algorithms. It is inspired by parallelization techniques from Pregel [MAB+10], a programming model for developing parallel algorithms on large-scale graphs. More specifically, it securely implements Pregel's scatter, gather, and apply operations, which can be used to solve several important parallel data mining and machine learning algorithms. Without going into the details of the gather and scatter operations, invoking them requires oblivious permutations on the entire graph. As the graph is large and the operations are called repeatedly, we can get considerable savings by using our permutation.

Extraction and Filtering. An immediate application of a permutation protocol is to extract elements from an array that satisfy some condition. Consider a secret shared vector where each entry indicates if it has been flagged, e.g. satisfies a condition or possible in a set intersection. To extract the flagged elements (in secret-shared form without compromising security), we can obliviously permute the secret-shared arrays together with their secret-shared flag array. Then parties can reveal the flag array and discard elements not in the intersection. We note that many other applications also make use of this functionality, e.g. [FO20].

Database Joins & Private Set Intersection. Secure database joins have now become an active research area [MRR20,BDG+22]. This line of research can be viewed as a generalization of Private Set Intersection, where the input sets are secret-shared, have associated values, and can contain duplicates. They are essentially SQL tables. One can then consider various joins between these tables. [MRR20,BDG+22] demonstrate that one can compute such joins with $O(n \log n)$ overhead given access to efficient permutation protocol, or in $O(n)$ time if the join keys are unique. As there are were no efficient two-party protocols, [MRR20,BDG+22] focus on the honest majority setting.

Random Access Memory in Secure Multi-party Computation. Commonly, MPC frameworks work in the circuit model where the function to be computed is expressed as a circuit. This is in contrast with the RAM model where memory accesses can be performed in an input-dependent way. To access an (encrypted) value at position i in memory, the parties executing the protocol must know i. If i is computed as a function of the input, then i can leak information about the input.

To mitigate this, one can equip a circuit base MPC protocol with a special "random-access" gate, e.g. via garbled RAM [LO13]. This gate takes as input the index i and returns the memory value m_i, without leaking any information about i. A trivial solution for such a gate is linear scan, which reads all $O(n)$ memory locations and saves m_i when it is read. The gate then returns m_i obliviously. However, this solution adds an $O(n)$ factor to the running time. Specialized garbled RAM protocols are able to obliviously access m_i with poly log

overhead [LO13,HKO23,PLS23]. Despite significant improvements, they remain expensive and have not been implemented until recently [YPHK23]. For example, the state-of-the-art garbled RAM of [HKO23] requires $O(T \log^3 n \cdot \log \log n)$ time to perform T RAM accesses into a memory of size n.

An alternative, proposed by [DS17], leverages a private information retrieval (PIR) scheme based on point functions [GI14] to construct a random-access gate. Their construction requires an amortized $O(\sqrt{n})$ overhead within the MPC protocol and additionally $O(n)$ work per access that is "outside" the MPC with small constants. Despite having linear overhead this scheme can perform very well for some n. Another downside of this approach is that it requires interaction between the parties, and therefore is not compatible with constant-round garbled circuit protocols.

Most schemes that support a random-access gate require the memory accesses to be sequential. Therefore, if the random-access gate is implemented using secret sharing, this introduces a poly log multiplicative overhead to the round complexity of the protocol. This is always true for [DS17] due to its inherent interactiveness. Garbled RAM schemes, such as [HKO23], are non-interactive due to their use of garbled circuits as opposed to secret sharing. However, this adds an additional security parameter overhead to the communication, compared to some secret-sharing based schemes. One could consider combining parallel garbled RAM [BCP16,LO17] techniques with secret sharing. These techniques allow for the evaluation of $t = O(n)$ RAM gates in parallel. However, these schemes thus far have only been proposed for garbled circuit and are not as optimized as, for example, [HKO23]. Moreover, they still impose a poly log overhead in communication.

We observe that many of the most important applications for MPC that require random access can be efficiently implemented in a batched manner. In particular, instead of reading just one memory location at a time, the circuit generates a set of indices $\sigma \subseteq [n]$ with $|\sigma| = O(n)$ and the batched RAM gate accesses and returns all memory locations $\{m_{\sigma(i)}\}$. In this amortized setting one can hope to avoid the poly log overhead lower bounds inherent to the former schemes. Indeed, when σ is a permutation of size n, our permutation protocols can implement the batched RAM gate with an amortized $O(1)$ overhead[5].

1.2 Our Contributions

In our work, we design and present a comprehensive suite of highly efficient protocols that make it feasible to use permutations in real-world two-party computation applications. We begin with our permutation protocols:

- We present two new protocols for the $\mathcal{F}_{\mathsf{Gen\text{-}Perm}}$ functionality, which generates a random permutation correlation. In particular, one party learns a permutation $\pi : [n] \to [n]$ while the other party learns a random list $A \in \mathbb{F}_{2^\ell}^n$. The parties obtain a secret sharing of $\pi(A) = (A_{\pi_1}, \ldots, A_{\pi_n})$, .i.e., $B + C = \pi(A)$.

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[5] We assume the read memory is of size at least κ-bits.

- $\Pi_{\mathsf{Prf\text{-}Perm}}$ makes use of recent advances in MPC-friendly PRFs to construct a protocol that requires $O(n\ell)$ OTs (binary OLEs). We implement this protocol and observe that it can permute a list of size $n = 2^{20}, \ell = 128$ in less than 20 seconds on a single thread.
- $\Pi_{\mathsf{Pcg\text{-}Perm}}$ is the first protocol to achieve "sub-linear" communication with the help of pseudo-random correlation generators. In particular, for a chosen π this protocol can generate a *random permutation correlation* with $O(n \log \ell)$ communication/OTs. We believe this protocol to be ideal when a large amount of data needs to be permuted.

We then present various permutation protocols that make use of random permutation correlations. In some cases similar protocols have been previously been proposed, with our contribution being a systematization of the knowledge.

- Building on the work of [CGP20], we formalize the notation of random permutation correlation and its related protocols. In particular, we consider the functionality of $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ that on input permutation π and list X, outputs shares of $\pi(X)$. The protocol $\Pi_{\mathsf{Derand\text{-}Msg}}$ allows one to first generate a random permutation correlation via $\mathcal{F}_{\mathsf{Gen\text{-}Perm}}$ and then derandomize the correlation to output a secret sharing of $\pi(X)$. $\Pi_{\mathsf{Derand\text{-}Msg}}$ is information-theoretic and is extremely efficient; it requires one round, $n\ell$ bits of communication and $2n$ $\mathbb{F}_{2^\ell}$ additions to derandomize X.
- Building on the sublinear nature of $\Pi_{\mathsf{Pcg\text{-}Perm}}$, we observe that it is possible to use a random permutation correlation for ℓ-bit strings to derandomize many lists $X, Y, Z, \ldots$ in an "on-demand" manner given that the string length of $X, Y, Z, \ldots$ adds to at most ℓ bits.
- A natural extension of these protocols also allows the input list X to be secret-shared between the parties. The $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ protocol achieves this with virtually no overhead as the party with π can locally permute their shares of X.
- We then generalize $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ to the setting where π is secret-shared as the composition of two random permutations π_1, π_2 such that $\pi = \pi_2 \circ \pi_1$. The efficiency of this protocol is essentially two invocations of $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$.
- Given a random permutation correlation for π (possibly secret-shared), we show that it is possible to compute shares of $\pi^{-1}(X)$ with the same efficiency as $\pi(X)$. Moreover, one can generate shares for $\pi^{-1}(X)$ using the correlated randomness for π, and vice versa.
- To facilitate the ability to *programmatically* generate a permutation within an MPC computation, we give a new protocol Π_{A2C} that allows converting between an additive secret sharing of π, e.g. XOR shares $\pi = \pi_1 \oplus \pi_2$, to a permutation composition sharing π_1', π_2' such that $\pi = \pi_1' \circ \pi_2'$. We also give the protocol Π_{C2A} for converting composition shares of π into an additive share. The efficiency of both protocols is essentially one invocation of $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ of n strings of length $\ell = \log n$ or the derandomization of a similar-sized correlated randomness.

Finally, we build more expressive functionalities such as sorting and general batched RAM gate using our permutation correlations. In some cases, e.g. sorting, we build on prior art, with our contribution being a systematization of the techniques.

- We present a class of protocols $\Pi_{\mathsf{Ext\text{-}Ord\text{-}Pad}}, \Pi_{\mathsf{Ext\text{-}Unord}}, \ldots$ referred to as extraction. Instead of specifying an input permutation, these protocols allow the users to input a secret-shared bitvector $f \in \{0,1\}^n$ and a secret-shared list $X \in \mathbb{F}^n$ and output a secret-shared list $\{X_i \mid f_i = 1\}$. The output can be padded to a fixed length c and can have the same order as the input. We also give methods for inverting these transformations to map the output back to the input. These protocols are linear time and mostly constant round with one exception requiring $O(\log n)$ rounds.
- We then show that our protocols can be used to implement a class of protocols for sorting, $\Pi_{\mathsf{Partition}}, \Pi_{\mathsf{Radix\text{-}Sort}}, \Pi_{\mathsf{Quick\text{-}Sort}}$. In particular, we show that one can recast most existing honest majority sorting protocols into our two-party setting with the same asymptotic complexity, i.e. sorting in $O(n \log n)$ time and $O(\log n)$ rounds.
- Lastly, our protocols $\Pi_{\mathsf{Batched\text{-}RAM\text{-}Read}}, \Pi_{\mathsf{Batched\text{-}RAM\text{-}Write}}$ generalize our permutation protocols to allow the parties to input a sharing of an arbitrary function $\sigma : [n] \rightarrow [n]$. In particular, when performing a read operation, multiple output positions may read the same input location. This protocol runs in time $O(n \log n)$ and $O(\log n)$ rounds. Similarly, we also present $\Pi_{\mathsf{Batched\text{-}RAM\text{-}Write}}$ that allows the parties to write to memory in a generalized manner.

2 Related Work

Few solutions for permuting secret-shared values in the two-party setting exist. The oldest is the folklore solution using additive homomorphic encryption (AHE). The core idea is relatively simple, the sender party holds a permutation π while the other party, the receiver, holds an input list $X = (X_1, \ldots, X_n)$. The receiver with X first encrypts each entry X_i with their own AHE key k and sends the ciphertexts to the sender. The sender then permutes the ciphertexts by π, randomizes them, and adds a random mask C_i to each ciphertext. The resulting ciphertexts are then sent back to the receiver who decrypts them to obtain $B_1, \ldots, B_n$. Observe that $C + B = \pi(X)$, which can be viewed as a secret sharing of $\pi(X)$. This basic solution can easily be extended to the setting where X is secret-shared. The Paillier scheme is an example of such an AHE scheme. However, it has some disadvantages. The first being that many MPC protocols work with binary secret sharing or with some other small modulus p. The Paillier scheme has a large modulus, which necessitates converting between these representations and incurs an additional overhead. Moreover, Paillier encryption is relatively slow resulting is poor practical performance [CGP20].

[JVC18] proposed the Gazelle protocol that makes use of lattice-based (LWE) AHE to implement permutations. It follows a similar outline as above along

with some additional complexities. In particular, LWE AHE schemes support SIMD/batching operations where many, e.g. $n = 2048$, plaintext values, modulo some p, can be packed into a single ciphertext. This results in improved efficiency, i.e. an $n\times$ reduction is communication/computes. However, to support permutations, one must then be able to permute values within and between ciphertexts. [JVC18] proposes such techniques. We compare our efficiency to both Paillier and Gazelle in Section 12.

An alternative is to make use of a Benes permutation network [Ben64] and any generic MPC protocol, for example GMW or a garbled circuit. The input list X consists of n strings, each of length ℓ bits. A Benes permutation network is a circuit of depth $\log n$ and consists of $\ell n \log n$ swap gates. Each gate takes two ℓ bit string as inputs and output them in order or in swapped order. The party with π can program each of the swap gates. Each way to program the $\ell n \log n$ switch gates corresponds to exactly one permutation. This approach benefits from not making extensive use of public key cryptography but still requires significant communication.

Recently, Chase et al. proposed a different scheme for permutations [CGP20]. Their scheme makes clever use of a function secret sharing and punctured PRF to generate small permutations of size $T \leq 256$. They show that these can then be combined using a Benes permutation network where the swap gates are replaced with small permutation gates with T inputs. Their overall running time is $O(\kappa n \log n + \ell n \log n / \log T)$. In practice this gives a sizable improvement over the classic Benes network constructions.

Finally, in the honest majority setting there is a very simple and efficient protocol for implementing permutations. For example, for three parties, each pair of two parties can jointly hold a random permutation (π_1, π_2, π_3) such that the overall permutation is $\pi = \pi_3 \circ \pi_2 \circ \pi_1$. When using replicated secret sharing [AFL+16] each pair of two parties also holds a secret sharing of the input X. The pair of parties holding π_1 can locally permute their shares of X by π_1 and then secret share the result. This can be repeated for π_2 and π_3. The result is then a secret sharing of $\pi(X)$. [CHI+19] make use of this well-known construction to implement sorting. Recently, [AHI+22, AKK+23] extended it to the malicious honest majority setting.

3 Preliminaries

3.1 Notation

Let $\{a, b, c\}$ denote the set containing elements a, b, c. Let (a, b, c) denote the vector with elements a, b, c. The ith element of a set or a vector S is denoted as S_i. For integers a, b, let the notation $[a, b]$ denote the ordered set $(a, a+1, \ldots, b)$. Let $[n]$ be shorthand for $[1, n]$. We denote κ and λ as the computational and statistical security parameters, respectively.

3.2 Permutations & Injective Functions

We define a permutation $\pi : [n] \to [n]$ of size n as a bijection between the set $[n]$ and itself, i.e. $\pi(i)$ returns a distinct value for each $i \in [n]$. There are several ways to represent such a function. Typically, we consider π to be a vector from the space $[n]^n$ such that $\pi(i) = \pi_i$. One can also view π as a matrix Π in the space $\{0,1\}^{n \times n}$ where $\Pi_{i,\pi(i)} = 1$ and otherwise is zero. This is referred to as a permutation matrix and is convenient in the context of linear algebra. In particular, for a vector $X \in \mathbb{F}^n$, let $\pi(X) := \Pi \cdot X$ denote the vector X permuted by π. Equivalently, $\pi(X) = (X_{\pi(1)}, \dots, X_{\pi(n)})$.

The set of permutations forms a group under function composition. That is, for permutations $\pi, \rho : [n] \to [n]$, the composition $\pi \circ \rho$ is a permutation $\phi : [n] \to [n]$ such that $\phi(i) = \pi(\rho(i))$. Equivalently, let $\Pi, P, \Phi \in \{0,1\}^{n \times n}$ be the matrix representations of π, ρ, ϕ respectively; then $\Phi = \Pi \cdot P$. A permutation π also has an inverse π^{-1} such that $\pi \circ \pi^{-1} = (1, 2, \dots, n)$ or equivalently $\Pi \cdot \Pi^{-1} = I$ where I is the identity matrix.

Most of our protocols also work for the case of an injective function $\nu : [n] \to [m]$ for some $m > n$. For ν to be an injection, we require that $\nu(i)$ outputs a distinct value for each i. The vector, matrix representation, and function composition are defined in the same way. However, as ν is injective, there is no inverse function/matrix.

3.3 Secret Sharing & Functionalities

We denote a secret sharing of x as $[\![x]\!]$. For concreteness, we assume binary secret sharing where the two parties respectively hold $[\![x]\!]_1, [\![x]\!]_2 \in \mathbb{F}_{2^\ell}$ such that $x = [\![x]\!]_1 \oplus [\![x]\!]_2$. In some places, our protocols will call for the *integer* addition of several shares. We assume this is achieved by converting to secret sharing over an integer modulus, performing the summation, and switching back to binary sharing, see [MR18] for the relevant techniques.

We define ideal functionalities with the $\mathcal{F}$ notation. One can think of calling $\mathcal{F}$ as sending the inputs to a trusted third party that computes the function and returns the result to the parties. These functionalities will be securely realized by cryptographic protocols, which we denote with the Π notation. A protocol is considered secure if there exists a simulator for it that can only interact with the functionality, see [Lin16] for details.

3.4 Oblivious PRF with Shared Output

We make use of an oblivious weak PRF functionality with secret-shared output, $\mathcal{F}_{\mathsf{Sowprf}}$. The definition of a weak PRF $F : \mathcal{K} \times \mathcal{X} \to \mathcal{Y}$ states that for a set of random inputs $x_1, \dots, x_n \in \mathcal{X}$, and for a random key $k \in \mathcal{K}$, the distribution of $F_k(x_1), \dots, F_k(x_n)$ should be pseudo-random. We formalize in Definition 1.

Definition 1 (Weak Pseudo-random Function). *A function $f : \mathcal{K} \times \mathcal{X} \to \mathcal{Y}$ is a weak pseudo-random function if*

$$\{(x_1, \ldots, x_n, F_k(x_1), \ldots, F_k(x_n)) : x_1, \ldots, x_n \leftarrow \mathcal{X}, k \leftarrow \mathcal{K}\}$$
$$\approx \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \qquad : x_1, \ldots, x_n \leftarrow \mathcal{X}, y_1, \ldots, y_n \leftarrow \mathcal{Y}\}.$$

The $\mathcal{F}_{\mathsf{Sowprf}}$ functionality evaluates a weak PRF F where the sender inputs a key k while the receiver inputs one or more x. For each x, the parties output a secret sharing $[\![F_k(x)]\!]$. This functionality is meant to model the parties evaluating the PRF within an MPC protocol.

This functionality differs from a traditional OPRF protocol in that the output $F_k(x)$ is secret-shared between the parties instead of revealed in the clear to the receiver. To instantiate this building block we make use of recent advances in so-called MPC-Friendly symmetric key primitives. One of the first popular examples is the LowMC pseudo-random permutation [ARS+15] by Albrecht et al. This block cipher is specifically designed to minimize the number of AND gates in its binary circuit representation.

Another prominent class of MPC-Friendly primitives is referred to as alternating moduli first proposed by Boneh et al. [BIP+18] and later optimized in [DGH+21,APRR24]. This weak PRF takes a radically different design. The function can be divided into two phases, one that performs a linear transformation of the state mod 3. The state is then reinterpreted mod 2 and then another linear transformation is applied. The security of this construction relies on the assumption that linear options over two different moduli result in a highly non-linear and unpredictable function. To efficiently implement this in MPC, [APRR24] designs a custom protocol that performs secret sharing over the two different moduli and then performs share conversion between the different moduli. This effectively reduces the number of AND gates per output bit to 2.

In particular, the protocol provided by [APRR24], which implements $\mathcal{F}_{\mathsf{Sowprf}}$, must first perform a key-specific setup protocol. By allowing the key to be reused, we can save on the cost of performing the setup. In this case, the round complexity of the protocol is 2 plus the cost to preprocess $4n\kappa$ OTs where κ is the security parameter. These OTs can be preprocessed and performed in 3 additional rounds. The total communication complexity is $\approx 16n\ell$ bits.

Functionality $\mathcal{F}_{\mathsf{Sowprf}}$:

PARAMETERS: Let $F : \mathcal{K} \times \mathcal{X} \to \mathcal{Y}$ be a weak pseudo-random function.

$\mathcal{F}_{\mathsf{Sowprf}}(\text{SENDER} : k, \text{RECEIVER} : x) :$
Upon input k from the sender and input x from the receiver, return a secret sharing $[\![F_k(x)]\!]$ to the parties.

Fig. 1. The $\mathcal{F}_{\mathsf{Sowprf}}$ functionality that returns secret shares of a weak PRF with a key from the sender and input from the receiver.

Function Secret Sharing. We will make use of *function secret sharing* (FSS) [BGI15] that allows two parties to generate secret shares k_0, k_1 of a function f in some class $\mathcal{F}$. Given that the two parties respectively hold $(k_0, k_1) \leftarrow \mathsf{Gen}(1^\kappa, f)$, it is possible for them to efficiently and non-interactively generate a secret share $[\![f(x)]\!] := (\mathsf{Eval}(k_0, x), \mathsf{Eval}(k_1, x))$ for any public x. The security guarantee is that given either k_0 or k_1, nothing about the function f is revealed, apart from it being a member of $\mathcal{F}$. Formally,

Definition 2 (Function Secret Sharing). *Let $\mathcal{F}$ be a class of functions. For all $f \in \mathcal{F}$ and $p \in \{0,1\}$, a FSS scheme $(\mathsf{Gen}, \mathsf{Eval})$ is private if there exists a PPT simulator* sim *s.t.:*

$$\{(k_p, \mathcal{F}) : (k_0, k_1) \leftarrow \mathsf{Gen}(1^\kappa, f, \mathcal{F})\}$$
$$\approx \{(k_p, \mathcal{F}) : k_p \quad \leftarrow \mathsf{Sim}(p, 1^\kappa, \mathcal{F})\}.$$

and for all x and all $(k_0, k_1) \in \mathsf{Gen}(1^\kappa, \mathsf{f}, \mathcal{F}) : f(x) = \mathsf{Eval}(k_0, x) + \mathsf{Eval}(k_1, x)$

A trivial solution would be to simply define k_0, k_1 to be the secret sharing of the truth table of f. However, we desire k_0, k_1 be small, i.e. sublinear in the truth table size. The most prominent example of this technique has been FSS for point functions, a function $f : [n] \to \{0, 1\}^\ell$ that is zero for all input but a for which $f(a) = b$. [GI14,BGI15] describe an efficient scheme based on a length-doubling PRG. Their construction has each party expand a binary tree where the children of a node are defined as the outputs of the length-doubling PRG. The leaves of the tree are indexed by the inputs to f. I.e., at leaf a, the random leaf values differ by b, while all other leaves hold the same random values. Each key k_0, k_1 consists of a single κ-bit element per level of the tree that assists in its generation. That is, the size of k_0, k_1 is $O(\kappa \log_2(n) + \ell)$. In the event that $\ell = 1$, it is possible to reformulate the problem as $n' = n/\kappa$ and $\ell' = \kappa$ by defining b as a unit vector of length κ. Overall, the same function is computed but at a cost $O(\kappa \log_2(n/\kappa) + \kappa)$. Doerner and Shelat [DS17] later gave an efficient two-party key generation protocol that allows the parties to input secret shares of a, b and securely generate k_0, k_1. Their protocol requires $O(\log n)$ rounds and makes only black box calls to the PRG, i.e. does not evaluate it in an MPC circuit.

Functionality $\mathcal{F}_{\mathsf{FSS\text{-}Gen}}$:

PARAMETERS: Let $(\mathsf{Gen}, \mathsf{Eval})$ be a function secret sharing scheme for point functions $f_{\alpha,\beta}$ s.t. $f(\alpha) = \beta$ and otherwise $f(*) = 0$.

$\mathcal{F}_{\mathsf{FSS\text{-}Gen}}([\![\alpha]\!], [\![\beta]\!])$:
Upon input $[\![\alpha]\!], [\![\beta]\!]$, compute $(k_1, k_2) \leftarrow \mathsf{Gen}(f_{\alpha,\beta})$. Return k_1 to the first party and k_2 to the second party.

Fig. 2. Function Secret Sharing key generation protocol $\mathcal{F}_{\mathsf{FSS\text{-}Gen}}$ for point functions.

Syndrome Decoding. The syndrome decoding assumption with regular noise states that for a class of matrices $\mathcal{C}$, one can sample a weight $t = O(\kappa)$ vector $e \in \mathbb{G}^n$ that when compressed by a matrix $A \in \mathcal{C}$, results in a shorter pseudo-random vector $r := Ae$. Syndrome decoding can be shown to be equivalent to the Learning Parity with Noise assumption, where e is the error vector. For reasons of efficiency, we restrict the noise distribution of e to be the concatenation of t random unit vectors. More formally, the Regular Syndrome Decoding assumption is stated in Definition 3

Definition 3 (Regular Syndrome Decoding [BCG⁺19a]). *For some ring $\mathbb{G}$, let $\mathcal{R}$ denote the uniform distribution over $\mathbb{G}^n$ s.t. the all samples are the concatenation of t regular-sized unit vectors. Let C be a probabilistic code generation algorithm such that $\mathsf{C} \to \mathbb{G}^{k \times n}$. For weight $t = t(\kappa)$, dimension $k = k(\kappa)$, number of samples (or block length) $n = n(\kappa)$, and ring $\mathbb{G} = \mathbb{G}(\kappa)$, the regular syndrome decoding assumption $(\mathcal{R}, \mathsf{C}, \mathbb{G})$-RSD states that*

$$\{(A, \vec{b}) : A \leftarrow \mathsf{C}, \vec{e} \leftarrow \mathcal{R}, \vec{b} := A \cdot \vec{e}\}$$
$$\approx \{(A, \vec{b}) : A \leftarrow \mathsf{C}, \qquad \vec{b} \leftarrow \mathbb{G}^n\}.$$

We refer to [BCG⁺19a,BCG⁺22,RRT23] for parameter selection.

Aggregation Trees. [BDG⁺22] presents a useful functionality called an aggregation tree $\mathcal{F}_{\mathsf{Agg}}$. This functionality takes as input a shared list X and a shared bit vector B. The list is logically divided up into blocks with the start of a block being denoted by $B_i = 0$. For each block, the functionality will independently apply a prefix sum to the block for some associative sum operator $\star$. For example, if a block begins at i and is of size 3, then the output X' will contain $X'_i := X_i, X'_{i+1} := X_i \star X_{i+1}, X'_{i+2} := X_i \star X_{i+1} \star X_{i+2}$. The operator $\star$ can be any associative operator. [BDG⁺22] give a protocol for implementing this that takes $O(n \cdot \star_{\mathsf{time}})$ time and $O(\log n \cdot \star_{\mathsf{rounds}})$ rounds, where $\star_{\mathsf{time}}, \star_{\mathsf{rounds}}$ are the time and round complexity of computing the $\star$ circuit. We will make use of a duplication tree where $\star$ is defined as $\star(x_1, x_2) := x_1$. That is, it simply returns the first argument.

Functionality $\mathcal{F}_{\mathsf{Agg}}$:

$\mathcal{F}_{\mathsf{Agg}}(\llbracket X \rrbracket, \llbracket B \rrbracket, \star)$:
Upon input vector $\llbracket X \rrbracket \in \mathbb{D}^n$ and control bits $\llbracket B \rrbracket \in \{0, 1\}^n$ and associative operator $\star$ from the parties, define $\mathsf{pre\text{-}ind}(i) \in [i]$ to be the maximum value such that $B_{\mathsf{pre\text{-}ind}(i)} = 0$.
Output $\llbracket X' \rrbracket \leftarrow \mathsf{share}(X')$ where $X'_i := \bigstar_{j=\mathsf{pre\text{-}ind}(i)}^i X_j$

Fig. 3. Functionality $\mathcal{F}_{\mathsf{Agg}}$ for secret-shared aggregation.

4 Technical Overview

We are now ready to give a high-level overview of our techniques. We begin with $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}, \mathcal{F}_{\mathsf{Gen\text{-}Perm}}$, our most foundational functionalities for performing permutations on secret-shared data. The former, $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$, provides the most natural interface. It takes as input a permutation $\pi : [n] \to [n]$ from a *sender* party and vector $X \in \mathbb{F}^n$ from a *receiver* party. The parties output a secret sharing $[\![\pi(X)]\!] = ([\![X_{\pi(1)}]\!], \ldots, [\![X_{\pi(n)}]\!])$. However, our most primitive protocols do not implement this functionality. Instead, they are designed to output random permutation correlations as specified by $\mathcal{F}_{\mathsf{Gen\text{-}Perm}}$. These can then be derandomized to realize the $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ functionality.

For most of our protocols we give two running times which typically differ by a multiplicative security parameter κ. The first we refer to as *in practice*. This one has the additional κ overhead but achieves better concrete performance (at the time of writing this document). The second setting we refer to as *in theory*. The primary cause of the difference is the overhead of generating a bit OT / binary OLE [BCG+23]. We defer a more detailed discussion to Section 4.7.

4.1 Permutation Correlation Generators

The functionality $\mathcal{F}_{\mathsf{Gen\text{-}Perm}}$ generates the random correlation

$$(A, B), (C, \pi) \qquad \text{s.t.} \qquad \pi(A) = B + C.$$

where the receiver holds A, B, the sender holds C, π, and $A, B, C \in \mathbb{F}^n$ are uniform vectors subject to the constraint above. We note that this correlation is the permutation equivalent to correlated randomness used by multiplication gates in traditional MPC, i.e. a random OLE correlation $(a, b), (c, d)$ s.t. $d \cdot a = b + c$. Looking forward, we will then use this correlation to realize the $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ functionality.

MPC-Friendly PRF Permutation. $\Pi_{\mathsf{Prf\text{-}Perm}}$, our first protocol that implements the $\mathcal{F}_{\mathsf{Gen\text{-}Perm}}$ functionality, achieves the best performance for typical use cases where the strings to be permuted are of small to moderate sizes. The core building block is MPC-Friendly PRF. In particular, we make use of a weak PRF F that allows for efficient evaluation in MPC when one party knows the key k and the other party knows the input x. The output is a secret sharing of $F_k(x)$, where the secret sharing is over the same group as the permutation correlation.

The protocol can be described as follows. The receiver first samples a key k for the weak PRF F and defines their A vector as the output of the PRF on the identity permutation. I.e., $A_i = F_k(i)$. The parties then engage in n PRF evaluations, where the $\mathcal{F}_{\mathsf{Gen\text{-}Perm}}$ receiver acts as the $\mathcal{F}_{\mathsf{Sowprf}}$ sender with key k. The $\mathcal{F}_{\mathsf{Gen\text{-}Perm}}$ sender, which acts as the $\mathcal{F}_{\mathsf{Sowprf}}$ receiver, provides $\pi(i)$ as his input to the ith evaluation. $\mathcal{F}_{\mathsf{Sowprf}}$ outputs a secret sharing of $T_i := F_k(\pi(i))$.

The first observation is that $T_i = A_{\pi(i)}$, and therefore $T = \pi(A)$. In other words, T is A permuted by π. Therefore, what remains is to take the secret share

of T and generate the B, C vectors. However, we make the observation that this step comes for free in the case that the secret sharing output by $\mathcal{F}_{\mathsf{Sowprf}}$ uses the same group as the permutation correlation. In this case, the receiver defines B_i as their share of T_i, while the sender defines C_i as theirs. Observe that this is precisely what we want,

$$A_{\pi(i)} = T_i = B_i + C_i$$
$$\pi(A) = B + C$$

One deficit of this description is that the input to the weak PRF are the integers between 1 and n. However, weak PRFs require the input values to be sampled uniformly at random. We resolve this issue by first running the input $\pi(i)$ through a random oracle H. In particular, the receiver will define $A_i := F_k(H(i))$ while the sender will input $H(\pi(i))$ to $\mathcal{F}_{\mathsf{Sowprf}}$. In this way, the simulator will be able to program H to output the input values associated with the weak PRF challenge.

As an additional optimization, instead of resampling the k each time $\Pi_{\mathsf{Prf\text{-}Perm}}$ is invoked, it is possible to reuse it and instead sample a new random oracle H. This will ensure that the inputs to the PRF will remain unique and random while amortizing any key-specific cost of $\mathcal{F}_{\mathsf{Sowprf}}$. Overall, the running time is $O(n\ell'\kappa)$ in practice or $O(n\ell')$ in theory, where $\ell' := \lceil \ell/\kappa \rceil \kappa$ is the bit length of the element ℓ rounded up to κ. The communication complexity is $\sim 5n\ell'$ bits [APRR24].

PCG Permutation. As an alternative to our PRF-based construction, we show that one can build a permutation correlation generator $\Pi_{\mathsf{Pcg\text{-}Perm}}$ with communication that is *sublinear* in the length of the strings ℓ. Specifically, for a permutation π of size n, with correlations $A, B, C \in \mathbb{F}_2^{n \times \ell}$, we give a protocol with $O(n\kappa^2 \log \ell)$ bits of communication. At the heart of this construction is the ability to use LPN/syndrome decoding to get a succinct PRG seed [BCG+19b,BCG+19a]. This seed can then be non-interactively expanded in the MPC context. At a high level, the construction works by first permuting n seeds, $s_1, ..., s_n$ by the permutation π, with the result $\pi(s)$ secret-shared. Once permuted, the parties can non-interactively expand these secret-shared seeds to get secret shares of $\mathsf{PRG}(s_{\pi(i)})$. As before, these shares form the C, B components of the correlation.

In more detail, let P_1 hold permutation π. Let P_2 sample a key k for the wPRF F and define A such that $(p_{i,1}, ..., p_{i,t}) := F_k(H(i))$ where $p_{i,j} \in \log_2(\ell) + 1$. Let $\vec{e}_i \in \{0,1\}^{2\ell}$ be the syndrome decoding weight-t vector such that $e_{i,p_{i,j}} = 1$. Let $G \in \{0,1\}^{\ell \times 2\ell}$ be a matrix such that syndrome decoding is hard. Then define $A_i := G\vec{e}_i$.

The parties compute $[\![p'_{i,1}, ..., p'_{i,t}]\!] := F_{[\![k]\!]}([\![H(\pi(i))]\!])$ for each $i \in [n]$, where $[\![H(\pi(i))]\!]$ is input by P_1 and $[\![k]\!]$ is input by P_2. For $j \in [t]$, use a distributed point function key generation protocol to generate keys $K_{i,j,1}, K_{i,j,2}$ for point $p'_{i,j}$, where P_1 learns the former and P_2 learns the latter. The parties each expand

their key to get shares $[\![\vec{e}'_{i,j}]\!]$. The parties compute $[\![A']\!] := G[\![\vec{e}'_{i,j}]\!]$. Define B, C as the shares of $[\![A']\!]$.

The running time of this protocol consists of generating nt distributed point functions with bit length σ, where $\sigma = \log(\ell/t\kappa), t = O(\kappa)$. The inputs to the point functions can be generated with $nt\sigma/\kappa = O(n\sigma)$ invocations of $\mathcal{F}_{\mathsf{Sowprf}}$, where each cost $O(\kappa)$ in theory or $O(\kappa^2)$ in practice. The key generation for each distributed point function requires $O(\kappa\sigma + \ell/t)$ work. Finally, we can instantiate G as a linear time code, or more commonly, an $O(\ell \log \ell)$ time code. Overall, the running time is $O(n\kappa^2 \log(\ell/\kappa^2) + n\ell)$ and $O(n\kappa^2 \log(\ell/\kappa^2))$ bits of communication.

Generalized Mapping Functions. We note that these protocols do not require π to be a permutation. The $\Pi_{\mathsf{Prf-Perm}}, \Pi_{\mathsf{Pcg-Perm}}$ naturally support an arbitrary function $\pi : [m] \to [n]$. However, when π is not a permutation, the function over $([n] \to [m])$ no longer form a group. This can lead to certain limitations for secret sharing the permutation itself as we will see later.

4.2 Derandomization

In Section 6, we define the $\mathcal{F}_{\mathsf{Basic-Perm}}$ functionality, which allows the user to specify the permutation along with the input vector X to be permuted. We then present the $\Pi_{\mathsf{Basic-Perm}}$ protocol in the $\mathcal{F}_{\mathsf{Gen-Perm}}$-hybrid model, which, given a random permutation correlation, can derandomize it to give the secret shares of $\pi(X)$ as output. More generally, we present several derandomization techniques that allow the parties to derandomize π, and then generate shares of $\pi(x)$ and/or $\pi^{-1}(X)$ from a single random permutation correlation in an on-demand manner.

$\Pi_{\mathsf{Derand-Perm}}$**: Choosing π.** We begin with our protocol for derandomizing the permutation for a random permutation correlation (ρ, A, B, C). In particular, the sender holds a random $\rho : [n] \to [n]$ and C, while the receiver holds A, B such that $B + C = \rho(A)$. The sender party with π computes $\delta := \pi^{-1} \circ \rho$ and sends it to the other party who computes $A' := \delta(A)$. Observe that $A = \rho^{-1}(B + C)$, and therefore:

$$A' = \delta(A)$$
$$= (\pi^{-1} \circ \rho)(\rho^{-1}(B + C))$$
$$\pi(A') = B + C$$

As can be seen by the $\mathcal{F}_{\mathsf{Gen-Perm}}$ functionality, it is also possible to directly generate a permutation correlation for the desired permutation π. However, in some cases it is desirable to be able to generate the correlation before π is known, such as during a prepossessing phase or in a "just in time" manner a few rounds before π is determined. This allows the parties to not have to pay for the round complexity associated with the protocol implementing $\mathcal{F}_{\mathsf{Gen-Perm}}$, since it can be performed concurrently with other useful work. Moreover, this derandomization

is extremely efficient, requiring $O(n)$ time and sending $n \log n$ bits in a single message from the sender to the receiver.

$\Pi_{\text{Derand-Msg}}$**: Sharing $\pi(X)$.** Now, assume we already hold a permutation correlation (π, A, B, C) for the desired permutation π such that $B + C = \pi(A)$. The receiver party with (X, A, B) sends $\Delta := A + X$ and outputs B. The other party outputs $C' := C + \pi(\Delta)$. Observe that

$$
\begin{aligned}
B + C' &= C + \pi(\Delta) + B \\
&= C + \pi(A + X) + B \\
&= C + \pi(\pi^{-1}(B + C) + X) + B \\
&= \pi(X)
\end{aligned}
$$

This protocol is extremely efficient and requires applying the permutation π to the input Δ, performing $2n$ additions, and sending a single message of the same size as X. This derandomization protocol was first observed by [CGP20]. We propose a natural extension to this protocol where the $A, B, C \in \mathbb{F}^{n \times \ell}$ correlations are larger than $X \in \mathbb{F}^{n \times 1}$. One can simply use the first column of the correlation to mask X, saving the rest of the correlation for later use on some other input X'. This observation can be very useful in applications, where we wish to permute by π several times, such as in the GraphSC framework [NWI+15].

$\Pi_{\text{Derand-Inv-Msg}}$**: Sharing $\pi^{-1}(X)$.** Our last derandomization technique allows us to also permute by the inverse permutation. Let (π, A, B, C) be the correlated randomness such that $B + C = \pi(A)$. The party with (X, A, B) sends $\Delta := B + X$ and outputs A. The other party computes outputs $C' := \pi^{-1}(C + \Delta)$. Observe that

$$
\begin{aligned}
B + C' &= \pi^{-1}(C + \Delta) + A \\
&= \pi^{-1}(\pi(A) + X) + A \\
&= \pi^{-1}(X)
\end{aligned}
$$

To the best of our knowledge, we are the first to observe that one can use the same correlation for π to permute by both π and π^{-1}. As in $\Pi_{\text{Derand-Msg}}$, we can use a *single* correlation to permute and unpermute data in an on-demand manner, saving part of the correlation for later. When combined with our sublinear correlation generator $\Pi_{\text{Pcg-Perm}}$, this results in high performance if in need to permute by π, π^{-1} multiple times. A prime example is the aforementioned GraphSC framework [NWI+15], where for many iterations, the algorithm permutes by π and then π^{-1}.

Secret-Shared Input X. It is trivial to extend $\Pi_{\text{Derand-Msg}}$ and $\Pi_{\text{Derand-Inv-Msg}}$ to the setting where the X to be permuted is secret-shared over the same group as the random permutation correlation. In particular, the party with π can simply permute their own share and add the result to the output share that was obtained previously in $\Pi_{\text{Derand-Msg}}, \Pi_{\text{Derand-Inv-Msg}}$.

4.3 Secret-shared Permutations

Composed Permutations. We now present the relatively standard technique for allowing π to be secret-shared. We denote a composed sharing of π as $\langle\!\langle\pi\rangle\!\rangle$ and it consists of two permutations $\langle\!\langle\pi\rangle\!\rangle_1, \langle\!\langle\pi\rangle\!\rangle_2$ such that $\pi = \langle\!\langle\pi\rangle\!\rangle_2 \circ \langle\!\langle\pi\rangle\!\rangle_1$. Each party holds exactly one of these permutations. The parties can then generate a secret sharing of $Z = \pi(X)$ by first invoking $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ on $[\![X]\!]$ to obtain $[\![Y]\!] = \langle\!\langle\pi\rangle\!\rangle_1([\![X]\!])$ and then invoking $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ again to compute $[\![Z]\!] = \langle\!\langle\pi\rangle\!\rangle_2([\![Y]\!])$. The cost of this protocol is two invocations of $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ and local additions.

This protocol inherits all of the extensions we present to the core protocols for $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$. In particular, one can first generate the correlations for $\langle\!\langle\pi\rangle\!\rangle_1, \langle\!\langle\pi\rangle\!\rangle_2$ and then derandomize these on demand. Similarly, one can apply the invert π^{-1} to a secret-shared input by first applying $\langle\!\langle\pi\rangle\!\rangle_2^{-1}$ and then $\langle\!\langle\pi\rangle\!\rangle_1^{-1}$.

Additive Permutations. Finally, we present a new protocol for efficiently converting the representation of π between a composition format $\langle\!\langle\pi\rangle\!\rangle$ and an additive secret sharing format $[\![\pi]\!]$ where each party holds a share $[\![\pi]\!]_1, [\![\pi]\!]_2 \in \mathbb{Z}_n^n$ such that $\pi = [\![\pi]\!]_1 + [\![\pi]\!]_2$. If π is additively shared, we can get composed sharing by having one party sample $\langle\!\langle\pi\rangle\!\rangle_1$ at random. We can then permute the shares of $[\![\pi]\!]$ by $\langle\!\langle\pi\rangle\!\rangle_1$ to get shares of $\langle\!\langle\pi\rangle\!\rangle_2$ which is then revealed to the other party. When compared to the natural extension of [CHI+19] to the two-party setting, our construction requires permuting π only once as opposed to twice. Therefore, our protocol is twice as efficient. Additionally, instead of directly converting $[\![\pi]\!]$ into $\langle\!\langle\pi\rangle\!\rangle$, [CHI+19] converts $[\![\pi]\!]$ into $\rho, \langle\!\langle\pi'\rangle\!\rangle$ such that ρ is public and $\pi = \rho \circ \pi'$. As such, [CHI+19] requires the parties to perform an additional plaintext permutation by ρ.

Generalized Mapping Functions. The protocols in this section can be made to support any injective function $\pi : [m] \to [n]$ (i.e. $\pi(x)$ does not duplicate any of its inputs). The main alteration that is required is to define $\langle\!\langle\pi\rangle\!\rangle_2 : [m] \to [n]$ as injective, while $\langle\!\langle\pi\rangle\!\rangle_1$ remains a bijective permutation. The conversion protocols $\Pi_{\mathsf{A2C}}, \Pi_{\mathsf{C2A}}$ can then be defined for injective functions in the natural way.

Composition. We can also achieve composition of permutations. If one permutation is public, then the parties can simply locally permute their shares. Given two secret-shared permutations $\langle\!\langle\pi\rangle\!\rangle, [\![\rho]\!]$, one can compute the composition $\theta = \pi \circ \rho$ by invoking $\mathcal{F}_{\mathsf{Comp\text{-}Perm}}$ on input permutation $\langle\!\langle\pi\rangle\!\rangle$ and list $[\![\rho]\!]$.

4.4 Extraction

Now that we introduced our permutation protocols, we present a class of protocols called extractions. In these protocols, the parties input a secret-shared bitvector $f \in \{0,1\}^n$ and return a permutation that if applied to X, returns the subset $\{X_i \mid f_i = 1\}$.

We further refine this basic idea in two ways:

- **Ordering.** The first concerns the *order* of elements in the output. $\Pi_{\text{Ext-Unord}}$ outputs X_i (s.t. $f_i = 1$) in random order; $\Pi_{\text{Ext-Ord}}$ outputs X_i in the original order within X.
- **Padding.** The second concerns the *size* of the output. The output can either contain only the X_i s.t. $f_i = 1$ ($\Pi_{\text{Ext-Unord}}$, $\Pi_{\text{Ext-Ord}}$) or it can be padded to a fixed length ($\Pi_{\text{Ext-Unord-Pad}}$, $\Pi_{\text{Ext-Ord-Pad}}$). In this case, the protocols receive one additional input c, which specifies the output size. This is especially useful when the parties do not know the size $|\{X_i \mid f_i = 1\}|$. They can determine the upper bound on the output size to ensure obliviousness in secure computation.

All of these protocols are based on the same fundamental idea that if we permute f according to a random permutation π (unknown to parties), then it is secure to reveal $f' = \pi(f)$, or a padded version of $\pi(f)$, in the clear. This is because now we cannot correlate any f_i' with f_j. Once f' is revealed we simply compose the (injective) permutation that returns all $[\![X_i]\!]$ such that $f_i' = 1$ with π. This is exactly how $\Pi_{\text{Ext-Unord}}$ is implemented. The remaining extraction protocols use $\Pi_{\text{Ext-Unord}}$ as a subprotocol, and hence are derived from the same idea.

To implement $\Pi_{\text{Ext-Unord-Pad}}$ (unordered extraction with padding), we pad f with c elements. We mark a subset of these elements so that the total number of marked elements (in X and the c appended elements) is c. Once padded, we use $\Pi_{\text{Ext-Unord}}$ to extract them.

To implement $\Pi_{\text{Ext-Ord}}$ (i.e. output the marked elements in their original order), we additionally mark each X_i with the number of elements with $f_{j<i} = 1$. This mark stores the original order of X_i with $f_i = 1$ and can be computed locally via simple additions. Then we invoke $\Pi_{\text{Ext-Unord}}$ to extract all the X_i with $f_i = 1$. Recall the output is in random order as $\Pi_{\text{Ext-Unord}}$ uses a random permutation. We place the extracted elements in their original order by opening the additional mark and ordering the elements based on this mark.

$\Pi_{\text{Ext-Ord-Pad}}$ combines the ideas from $\Pi_{\text{Ext-Ord}}$ and $\Pi_{\text{Ext-Unord-Pad}}$. The main difference is that we need to do some extra accounting to ensure the padded elements are placed at the end of the output list after calling $\Pi_{\text{Ext-Unord}}$.

4.5 Sorting

Additionally, we show that many of the existing ideas for implementing sorting in the honest-majority three-party setting [AHI+22,CHI+19,HKI+13] directly translate to our two-party framework. As with our previous protocols, the output of these functionalities is a secret-shared permutation $\langle\!\langle \pi \rangle\!\rangle$ that when applied to the input would sort it. $\mathcal{F}_{\text{Stable-Sort}}$ defines the ideal functionality.

We begin with our $\Pi_{\text{Partition}}$ protocol that can be viewed as a sorting protocol for single-bit elements. This protocol takes inspiration from [CHI+19] and makes the following observation. Let $[\![X]\!]$ be the input vector. One can generate the inverse sorting permutation by constructing a $2 \times n$ matrix f, where the first

row is $\neg X$ and the second is X. For example, if $X = (0, 1, 1, 0, 1)$ then f will be:

$$1, 0, 0, 1, 0$$
$$0, 1, 1, 0, 1$$

Now, apply a prefix sum over each element of the first row followed by the second. We obtain a new $2 \times n$ matrix c:

$$1, 1, 1, 2, 2$$
$$2, 3, 4, 4, 5$$

One can then take the component-wise mutiplication between f and c to obtains:

$$1, 0, 0, 2, 0$$
$$0, 3, 4, 0, 5$$

Observe that if one sums the two row vectors, we obtain a permutation:

$$\pi = (1, 3, 4, 2, 5)$$

This is precisely the inverse permutation that sorts X. We make the observation that many advanced protocols make use of similar techniques. I.e., they have a specialized protocol that allows each share to compute the index of the position to which it should be mapped, i.e. the inverse permutation. The $\Pi_{\mathsf{Partition}}$ protocol concludes by converting $[\![\pi^{-1}]\!]$ into $\langle\!\langle\pi^{-1}\rangle\!\rangle$ using $\mathcal{F}_{\mathsf{A2C}}$. A second benefit of this approach is that the sorting permutation is stable, meaning that items with equal values maintain the same order after the sort.

This protocol can be extended to sort inputs with b bits, where $b = O(1)$, by defining f as having 2^b rows, where the ith column has a 1 in the position X_i. Overall, the time of this protocol is dominated by the cost of generating the permutation correlation, i.e. $O(n\kappa)$ time and bits of communication, assuming $\mathcal{F}_{\mathsf{Sowprf}}$ is computed in $O(\kappa)$ time [APRR24], otherwise $O(n\kappa^2)$ time.

Additionally, one can extend $\Pi_{\mathsf{Partition}}$ to implement our radix sort protocol Π_{Sort} for arbitrary bit length items. The idea is to call $\Pi_{\mathsf{Partition}}$ for each bit position and compose the resulting permutations. I.e., first sort the most significant bit, then the next most significant bit, and on down to the least significant bit. After each $\Pi_{\mathsf{Partition}}$, one can compose the permutations, apply the result to the next bit, and recurse. As with $\Pi_{\mathsf{Partition}}$, this radix sort returns a stable sorting permutation. Overall, the protocol requires $O(n\ell\kappa)$ time and communication, assuming $\mathcal{F}_{\mathsf{Sowprf}}$ is computed in $O(\kappa)$ time [APRR24], otherwise $O(n\ell\kappa^2)$ time.

We also present the (non-stable) sorting protocol Π_{Sort} using the so-called shuffle-and-reveal model [HKI+13].

4.6 Batched Random Access

Finally, we present the $\Pi_{\mathsf{Batched-RAM-Read}}$, which can be viewed as a generalization of our permutation protocols. In particular, the parties input a list X of length n

and an additive secret sharing of an arbitrary function $\sigma : [m] \to [n]$. The output is a secret sharing of Y such that $Y_i := X_{\sigma(i)}$. This differs from our permutation functionality in that X_j could be mapped to *multiple* positions in Y. To support this functionality the protocol makes use of more sophisticated techniques such as sorting and aggregation trees. We will model these using the $\mathcal{F}_{\mathsf{Stable\text{-}Sort}}$ and $\mathcal{F}_{\mathsf{Agg}}$ functionalities that we presented in Section 3. The protocol works by first calling stable sort $\mathcal{F}_{\mathsf{Stable\text{-}Sort}}$ on $\sigma = (\sigma(1), ..., \sigma(n))$. If σ contains duplicates, they now each form a group. We mark the first element of each group and map the corresponding element of X to the marked position. All other positions are given a dummy value. The parties invoke $\mathcal{F}_{\mathsf{Agg}}$ on this vector with the markings as the aggregation tree control bits, resulting is the mapped elements are duplicates across its group. The final result is obtained by permuting vector by the inverse of the sorting permutation for σ. The running time of this protocol is dominated by the cost of sorting. If implemented with the radix sort protocol $\varPi_{\mathsf{Radix\text{-}Sort}}$, the overhead is $O(n \log n\kappa)$ time and bits of communication.

4.7   Computational Overheads

Figure 11 in Section 7 summarizes the overheads of our protocols. We present two sets of asymptotic running times. The first is referred to as *Time (theory)* and corresponds to the running time when one can compute $\mathcal{F}_{\mathsf{Sowprf}}$ in $O(\kappa)$ time and communication. While not practical as of the time of writing this work, we note that [APRR24] combined with [BCG+23] gives a theoretical construction that can achieve the desired overhead. However, it is more practical to assume $\mathcal{F}_{\mathsf{Sowprf}}$ requires $O(\kappa^2)$ time and $O(\kappa)$ communication. We denote this setting as *Time (practice)*. More generally, the overhead of most of our constructions are proportional to the overhead of invoking $\mathcal{F}_{\mathsf{Sowprf}}$ on an input size that is proportional to the input being permuted or otherwise manipulated. For radix sorting, the primary overhead is a permutation for each bit of the key while quick sort requires one permutation and $n \log n$ comparisons. Finally, the overhead of our batched RAM protocols is essentially proportional to sorting the selection vector and two permutations of the data.

5   Permutation Correlation Generators

We formally define the ideal functionality $\mathcal{F}_{\mathsf{Gen\text{-}Perm}}$ in Figure 4. It is parameterized by public inputs $n, \ell, \mathbb{F}$, which are the permutation size, the correlation string length, and the group that the correlation is over, respectively. The functionality takes as input a permutation and outputs uniform $A, C, D \in \mathbb{F}^{n \times \ell}$ such that $\pi(A) = C + D$. We give two implementations of this functionality, $\varPi_{\mathsf{Prf\text{-}Perm}}$ and $\varPi_{\mathsf{Pcg\text{-}Perm}}$. The former is based on any weak PRF such as [APRR24]. The latter additionally makes use of syndrome decoding.

MPC-Friendly PRF-based Permutation Correlation Generator. We begin with the $\varPi_{\mathsf{Prf\text{-}Perm}}$ protocol and prove that is is secure in the semi-honest setting. We refer to Section 4.1 for the intuition of the protocol.

Functionality $\mathcal{F}_{\mathsf{Gen\text{-}Perm}}(\textsc{Sender} : \pi, \textsc{Receiver})$:

PUBLIC PARAMETERS: Permutation size n, group $\mathbb{F}$ and string length ℓ.
INPUT: The sender party inputs a permutation $\pi : [n] \to [n]$.
OUTPUT: The functionality samples uniformly random $A, B, C \in \mathbb{F}^{n \times \ell}$ s.t. $\pi(A) = B + C$. (A, B) are output to the receiver, C is output to the sender.

Fig. 4. The ideal functionality $\mathcal{F}_{\mathsf{Gen\text{-}Perm}}$ for generating random permutation correlations.

Protocol $\Pi_{\mathsf{Prf\text{-}Perm}}(\textsc{Sender} : \pi, \textsc{Receiver})$:

1. Let $F : \mathcal{K} \times \mathcal{X} \to \mathbb{F}^w$ be a weak PRF and $H : \{0,1\}^* \to \mathcal{X}$ be a random oracle.
2. The receiver samples $k \leftarrow \mathcal{K}$ and the sender samples $t \in \{0,1\}^\kappa$. The sender sends t to the receiver.
3. Let $x_{i,j} := H(t, i, j)$ for $i \in [n], j \in [m]$ where $m := \lceil \ell/w \rceil$.
4. The receiver computes $A_{i,j} := F_k(x_{i,j})$ for $i \in [n], j \in [m]$.
5. The parties invoke $\mathcal{F}_{\mathsf{Sowprf}}$ nm times with the receiver inputting k and sender inputting $\pi(x)$. The parties receive shares $[\![y_{i,j}]\!]$ where $y_{i,j} = F_k(x_{\pi(i),j})$ for $i \in [n], j \in [m]$.
6. The sender sets $C_i := [\![y_i]\!]_{\mathsf{s}}$ and the receiver sets $B_i := [\![y_i]\!]_{\mathsf{r}}$.
7. The receiver returns (A, B) and the sender returns C.

Fig. 5. The $\Pi_{\mathsf{Prf\text{-}Perm}}$ protocol that implements the $\mathcal{F}_{\mathsf{Gen\text{-}Perm}}$ functionality.

Theorem 1. *The* $\Pi_{\mathsf{Prf\text{-}Perm}}$ *protocol of Figure 5 realizes* $\mathcal{F}_{\mathsf{Gen\text{-}Perm}}$ *functionality with semi-honest security in the* $\mathcal{F}_{\mathsf{Sowprf}}$*-hybrid model, assuming F is weak PRF.*

Proof. **Corrupt Sender:** The view of the sender consists of their shares of $[\![y]\!]$, which are uniformly distributed due to the output distribution of $\mathcal{F}_{\mathsf{Sowprf}}$, which evaluates the weak PRF F. Now consider the output distribution of the honest receiver. B is uniform subject to $\pi(A) = B + C$, as required.

To show that A is uniform, consider the weak PRF game of Definition 1. For sake of a contradiction, let us assume a distinguisher that can distinguish A from uniform. The simulator first queries the weak PRF challenger to obtain the instance $(x_1, ..., x_{nm}, y_1, ..., y_{nm})$ and programs the random oracle H to output x_{in+j} on input (t, i, j). Since t is picked at random, the probability that such a query has previously been made is negligible. Note that the output distribution of H remains uniformly random. Therefore, A is precisely the elements $y_1, ..., y_{nm}$, and therefore our distinguisher can also distinguish the weak PRF game.

Corrupt Receiver: The view of the receiver consists of the random nonce t and the output of $\mathcal{F}_{\mathsf{Sowprf}}$ which is uniformly random. Similarly, the output distribution of the honest sender is uniform subject to the desired correlation. $\square$

PCG-based Permutation Correlation Generator. We now turn our attention to our permutation correlation generator based on the techniques underlying the recent developments on pseudo-random correlation generators [BCG+19b].

Protocol $\Pi_{\mathsf{Pcg-Perm}}(\text{SENDER} : \pi, \text{RECEIVER})$:

1. Let F be a weak PRF and H be a random oracle.
2. The receiver samples $k \leftarrow \{0,1\}^\kappa$ and the sender samples $t \in \{0,1\}^\kappa$. The sender sends t to the receiver.
3. Let $x_i := H(t, i)$ for $i \in [n]$.
4. The receiver defines $(p_{i,1}, ..., p_{i,t}) := F_k(x_i)$ where $p_{i,j} \in [\log_2(\ell/t) + 1)$.
5. Let $\vec{e}_{i,j} \in \{0,1\}^{2\ell/t}$ be the unit vector such that $e_{i,j,p_{i,j}} = 1$. Let $\vec{e}_i$ denote the concatenation of $\vec{e}_{i,1}, ..., \vec{e}_{i,t}$.
6. Let $G \in \{0,1\}^{\ell \times 2\ell}$ be a matrix such that syndrome decoding is hard with regular weight t noise.
7. The receiver defines $A_i := G\vec{e}_i$.
8. For each $i \in [n]$, the parties compute $[\![p'_{i,1}, ..., p'_{i,t}]\!] := F_{[\![k]\!]}([\![H(\pi(i))]\!])$ by invoking $\mathcal{F}_{\mathsf{Sowprf}}$, where $[\![H(\pi(i))]\!]$ is input by the sender and $[\![k]\!]$ is input by the receiver.
9. For $i \in [n], j \in [t]$, the parties invoke the distributed point function key generation functionality $\mathcal{F}_{\mathsf{FSS-Gen}}$ on input $([\![p'_{i,j}]\!], [\![1]\!])$ to generate keys $K_{i,j,1}, K_{i,j,2}$, where the sender learns the former and the receiver learns the latter.
10. For $i \in [n], j \in [t]$, the parties each expand their key to get shares $[\![\vec{e}'_{i,j}]\!]$.
11. For $i \in [n]$, the parties compute $[\![\vec{e}'_i]\!] := ([\![\vec{e}'_{i,1}]\!] || ... || [\![\vec{e}'_{i,t}]\!])$.
12. For $i \in [n]$, the parties compute $[\![A']\!] := G[\![\vec{e}'_i]\!]$. Define B, C as the shares of $[\![A']\!]$.
13. The receiver returns (A, B) and the sender returns C.

Fig. 6. The $\Pi_{\mathsf{Pcg-Perm}}$ protocol that implements the $\mathcal{F}_{\mathsf{Gen-Perm}}$ functionality.

Theorem 2. *The $\Pi_{\mathsf{Pcg-Perm}}$ protocol of Figure 6 realizes the $\mathcal{F}_{\mathsf{Gen-Perm}}$ functionality with semi-honest security in the $\mathcal{F}_{\mathsf{Sowprf}}$ and $\mathcal{F}_{\mathsf{FSS-Gen}}$-hybrid model, assuming F is a weak PRF and RSD is hard.*

Proof. **Corrupt Sender:** The view of the corrupt sender is the output of $\mathcal{F}_{\mathsf{Sowprf}}$, which is uniformly random. This is fed as input to $\mathcal{F}_{\mathsf{FSS-Gen}}$, which returns $K_{i,j,1}$. By Definition 2, this key can be simulated.

Consider the hybrid model where the weak PRF is replaced by a random function. Following the same argument as in Theorem 1, the ability of the distinguisher to distinguish implies a distinguisher for the weak PRF game of Definition 1.

The output distribution of the honest receiver consists of A, B. Fixing A, then B is fully determined by π, C. For the sake of a contradiction, let us assume the distinguisher can distinguish A (and therefore B) from uniformly random, i.e. the simulator replaces A, B such that A is uniform. Given that each e_i was sampled as specified by Definition 3, this implies that the distinguisher can distinguish $G\vec{e}_i$ from uniformly random. However, this contradicts that RSD is hard.

Corrupt Receiver: The view of the corrupt receiver includes the random nonce t, the output of $\mathcal{F}_{\mathsf{Sowprf}}$ and $\mathcal{F}_{\mathsf{FSS-Gen}}$. These can all be simulated in a straightforward way. Finally, the output distribution of the honest sender is correct, i.e. C such that $C = B - \pi(A)$. $\qquad\square$

6 Derandomization

We are now ready to present our derandomization protocols. The final goal of these protocols is to efficiently implement the $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ functionality in various settings. In particular, our goal is to enable the parties to call $\mathcal{F}_{\mathsf{Gen\text{-}Perm}}$ either during a preprocessing phase or in the online phase, and then derandomize the correlation to generate a secret sharing of $\pi(X)$ where both $[\![X]\!]$ and π are chosen by the parties. $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ functionality is relatively straightforward. It takes as input a permutation of size n from the sender and a shared input list $X \in \mathbb{F}^{n \times \ell}$; the output is a secret sharing $[\![Y]\!]$ of the rows of X permuted by the permutation π, i.e. $Y = \pi(X)$.

Functionality $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}([\![X]\!], \mathrm{SENDER} : \pi) :$

PUBLIC PARAMETERS: Permutation of size n, group $\mathbb{F}$, and string length ℓ.
INPUT: The parties input a sharing of $X \in \mathbb{F}^{n \times \ell}$ and the sender party inputs a permutation $\pi : [n] \to [n]$.
OUTPUT: The functionality samples uniformly random sharing $[\![Y]\!]$ s.t. $Y = \pi(X)$.

Fig. 7. The $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ functionality.

We begin with our protocol $\mathit{\Pi}_{\mathsf{Derand\text{-}Perm}}$ in Figure 8 for transforming a random permutation correlation (A, B, C, ρ) into a correlation (A', B, C, π), where π is chosen by the sender. Given that ρ is uniformly distributed in the view of the receiver, the resulting correlation (A', B, C, π) is indistinguishable from the one returned by $\mathcal{F}_{\mathsf{Gen\text{-}Perm}}$ on input π.

Protocol $\mathit{\Pi}_{\mathsf{Derand\text{-}Perm}}(\mathrm{SENDER} : \pi, (\rho, C), \mathrm{RECEIVER}(A, B)) :$

1. The sender sends $\delta := \pi^{-1} \circ \rho$ to the receiver.
2. The receiver computes $A' := \delta(A)$.
3. The sender outputs (π, C) and the receiver outputs (A', B).

Fig. 8. The $\mathit{\Pi}_{\mathsf{Derand\text{-}Perm}}$ protocol that derandomizes a random permutation correlation to a chosen permutation correlation.

Theorem 3. *The composition of $\mathcal{F}_{\mathsf{Gen\text{-}Perm}}$ for random ρ and $\mathit{\Pi}_{\mathsf{Derand\text{-}Perm}}$ for arbitrary π securely realizes $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ for input π in the semi-honest setting.*

Proof. Observe that the protocol is correct. Moreover, the only message sent is $\delta = \pi^{-1} \circ \rho$. Given that permutations under composition form a group and that ρ is uniformly distributed, it is straightforward to see the distribution of δ is uniform. $\qquad\square$

Protocol $\Pi_{\text{Derand-Msg}}$ in Figure 9 takes as input a list $X \in \mathbb{F}^{n \times \ell}$ and a permutation correlation (A, B, C, π). The protocol derandomizes the correlation to return secret shares of X permuted by π. Given that (A, B, C, π) is a uniformly distributed permutation correlation, the result is indistinguishable from the parties calling $\mathcal{F}_{\text{Basic-Perm}}$ on input π and X.

Protocol $\Pi_{\text{Derand-Msg}}$($\text{SENDER} : (\pi, C), \text{RECEIVER} : X, (A, B))$:

1. The receiver sends $\Delta := A + X$ to the sender.
2. The sender computes $C' := \pi(\Delta) + C$.
3. The sender outputs C' and the receiver outputs B.

Fig. 9. The $\Pi_{\text{Derand-Msg}}$ protocol that derandomizes a permutation correlation to generate shares of X permuted by π.

Theorem 4. *The composition of $\mathcal{F}_{\text{Gen-Perm}}$ for arbitrary π and $\Pi_{\text{Derand-Msg}}$ for input $[\![X]\!]$ securely realizes $\mathcal{F}_{\text{Basic-Perm}}$ on input $[\![X]\!], \pi$ in the semi-honest setting.*

Proof. The only message sent is $\Delta = A + X$. Given that A is uniformly distributed, so is Δ. $\qquad\square$

Protocol $\Pi_{\text{Derand-Inv-Msg}}$ in Figure 10 takes as input a list $X \in \mathbb{F}^{n \times \ell}$ and a permutation correlation (A, B, C, π). The protocol derandomizes the correlation to return secret shares of X permuted by π^{-1}. Given that (A, B, C, π) is a uniformly distributed permutation correlation, the result is indistinguishable from the parties calling $\mathcal{F}_{\text{Basic-Perm}}$ on input π^{-1} and X.

Protocol $\Pi_{\text{Derand-Inv-Msg}}$($\text{SENDER} : (\pi, C), \text{RECEIVER} : X, (A, B))$:

1. The receiver sends $\Delta := B + X$ to the sender.
2. The sender computes $C' := \pi^{-1}(\Delta + C)$.
3. The sender outputs C' and the receiver outputs A.

Fig. 10. The $\Pi_{\text{Derand-Inv-Msg}}$ protocol that derandomizes a permutation correlation to generate shares of X permuted by π^{-1}.

Theorem 5. *The composition of $\mathcal{F}_{\text{Gen-Perm}}$ for arbitrary π and $\Pi_{\text{Derand-Inv-Msg}}$ for input $[\![X]\!]$ securely realizes $\mathcal{F}_{\text{Basic-Perm}}$ on input $[\![X]\!]$ and π^{-1} in the semi-honest setting.*

Proof. Similar to Theorem 4. $\qquad\square$

Definition 4. *Let $\Pi_{\text{Basic-Perm}}(\pi, [\![X]\!])$ be defined as the composition of $\mathcal{F}_{\text{Gen-Perm}}(\pi)$ and $\Pi_{\text{Derand-Msg}}$.*

7 Protocol Asymptotics

Figure 11 summarizes the overheads of our protocols. *Time (theory)* refers to the running time when one can compute $\mathcal{F}_{\mathsf{Sowprf}}$ in $O(\kappa)$ time and communication. While not practical as of the time of writing this work, we note that [APRR24] combined with [BCG⁺23] gives a theoretical construction that can achieve the desired overhead. However, it is more practical to assume $\mathcal{F}_{\mathsf{Sowprf}}$ requires $O(\kappa^2)$ time and $O(\kappa)$ communication. We denote this setting as *Time (practice)*.

Protocol	Time (theory)	Time (practice)	Comm. (bits)	Rounds
$\Pi_{\mathsf{Prf\text{-}Perm}}$	$n\ell'$	$n\ell'\kappa$	$n\ell'$	2
$\Pi_{\mathsf{Pcg\text{-}Perm}}$	$n\kappa^2\log(\ell/\kappa^2)+n\ell$	$n\kappa^2\log(\ell/\kappa^2)+n\ell\log\ell$	$n\kappa^2\log(\ell/\kappa^2)+n\ell$	$\log(\ell/\kappa^2)$
$\Pi_{\mathsf{Derand\text{-}Perm}}$	n	n	$n\log n$	1
$\Pi_{\mathsf{Derand\text{-}Msg}}$	$n\ell$	$n\ell$	$n\ell$	1
$\Pi_{\mathsf{Derand\text{-}Inv\text{-}Msg}}$	$n\ell$	$n\ell$	$n\ell$	1
$\Pi_{\mathsf{Basic\text{-}Perm}}$	$n\ell'$	$n\ell'\kappa$	$n\ell'$	1
$\Pi_{\mathsf{Comp\text{-}Perm}}$	$n\ell'$	$n\ell'\kappa$	$n\ell'$	2
Π_{A2C}	$n\kappa$	$n\kappa^2$	$n\kappa$	1
Π_{C2A}	$n\kappa$	$n\kappa^2$	$n\kappa$	1
$\Pi_{\mathsf{Ext\text{-}Unord}}$	$n\kappa$	$n\kappa^2$	$n\kappa$	3
$\Pi_{\mathsf{Ext\text{-}Ord}}$	$n\kappa$	$n\kappa^2$	$n\kappa$	3
$\Pi_{\mathsf{Ext\text{-}Unord\text{-}Pad}}$	$n\kappa$	$n\kappa^2$	$n\kappa$	$3+\log n$
$\Pi_{\mathsf{Ext\text{-}Ord\text{-}Pad}}$	$n\kappa$	$n\kappa^2$	$n\kappa$	$3+\log n$
$\Pi_{\mathsf{Partition}}$	$n\kappa$	$n\kappa^2$	$n\kappa$	3
$\Pi_{\mathsf{Radix\text{-}Sort}}$	$n\ell\kappa$	$n\ell\kappa^2$	$n\kappa$	3ℓ
$\Pi_{\mathsf{Quick\text{-}Sort}}$	$n\ell^*\log n\cdot\log\ell^*+n\kappa$	$n\ell^*\kappa\log n\cdot\log\ell^*+n\kappa^2$	$n\ell^*\log n\cdot\log\ell^*+n\kappa$	$3+\log n\log\ell^*$
$\Pi_{\mathsf{Batched\text{-}RAM\text{-}Read}}$	$n\log n\log\log n+n\kappa$	$n\kappa\log n\log\log n+n\kappa^2$	$n\log n\log\log n+n\kappa$	$3+\log n$

Fig. 11. Performance metrics of our various protocols. n is the input length, ℓ is the element bit length, κ is the security parameter, $\ell':=\lceil\ell/\kappa\rceil\kappa$ is the element bit length rounded up to κ, and $\ell^*:=\ell+\log n$. For round complexity, we do not count any rounds that can be preprocessed. (theory) refers to the time required if [APRR24] is implemented in $O(\kappa)$ time and we make use of $O(1)$ time amortized bit OTs[BCG⁺23]. For the extraction protocols, we omit the cost of permuting the data, which can be done seperately via $\Pi_{\mathsf{Basic\text{-}Perm}}$.

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8 Secret-Shared Permutations

In functionalities $\mathcal{F}_{\mathsf{Comp\text{-}Perm}}$ and $\mathcal{F}_{\mathsf{Add\text{-}Perm}}$, we extend the core $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ permutations functionality to allow the permutation to be secret-shared. The former $\mathcal{F}_{\mathsf{Comp\text{-}Perm}}$ allows the parties to input a composed permutation $\langle\!\langle \pi \rangle\!\rangle$ and a secret-shared list $[\![X]\!]$, and then receive secret shares of $\pi(X)$. This is achieved by each party holding a permutation, which when composed together, equals π. In particular, we denote the two permutations $\langle\!\langle \pi \rangle\!\rangle_1, \langle\!\langle \pi \rangle\!\rangle_2$ (the first party holds $\langle\!\langle \pi \rangle\!\rangle_1$, the second $\langle\!\langle \pi \rangle\!\rangle_2$). The latter functionality $\mathcal{F}_{\mathsf{Add\text{-}Perm}}$ achieves the same result, but allows the permutation π to be additively secret-shared, i.e. $\pi = [\![\pi]\!]_1 + [\![\pi]\!]_2$.

Functionality $\mathcal{F}_{\mathsf{Comp\text{-}Perm}}$ $(\langle\!\langle \pi \rangle\!\rangle, [\![X]\!])$:

INPUT: Composed permutation $\langle\!\langle \pi \rangle\!\rangle$ and secret-shared list $[\![X]\!]$.
OUTPUT: Secret-shared list $[\![Y]\!]$ such that $Y := \pi(X)$.

Protocol $\Pi_{\mathsf{Comp\text{-}Perm}}(\langle\!\langle \pi \rangle\!\rangle, [\![X]\!])$:

1. The parties invoke $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ on $\langle\!\langle \pi \rangle\!\rangle_1$ and $[\![X]\!]$, and receive $[\![Y]\!]$ as the result.
2. The parties invoke $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ on $\langle\!\langle \pi \rangle\!\rangle_2$ and $[\![Y]\!]$, and receive $[\![Z]\!]$ as the result.
3. The parties output $[\![Z]\!]$.

Fig. 12. The $\mathcal{F}_{\mathsf{Comp\text{-}Perm}}$ functionality and $\Pi_{\mathsf{Comp\text{-}Perm}}$ protocol that permute a secret shared list $[\![X]\!]$ by a composed permutation $\langle\!\langle \pi \rangle\!\rangle$.

We implement $\mathcal{F}_{\mathsf{Comp\text{-}Perm}}$ with the $\Pi_{\mathsf{Comp\text{-}Perm}}$ protocol in the $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$-hybrid model. We note that one can permute by π^{-1} simply by running the protocol in reverse.

Theorem 6. *Protocol* $\Pi_{\mathsf{Comp\text{-}Perm}}$ *securely realizes the* $\mathcal{F}_{\mathsf{Comp\text{-}Perm}}$ *functionality in the semi-honest setting.*

Proof. The protocol is trivial to simulate given that only $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ is invoked and the parties do not directly send messages to each other. $\square$

Functionality $\mathcal{F}_{\mathsf{Add\text{-}Perm}}$ $([\![\pi]\!], [\![X]\!])$:

INPUT: Additively shared permutation $[\![\pi]\!]$ and secret-shared list $[\![X]\!]$.
OUTPUT: Secret-shared list $[\![Y]\!]$ such that $Y := \pi(X)$.

Fig. 13. The $\mathcal{F}_{\mathsf{Add\text{-}Perm}}$ functionality that permutes a secret-shared list $[\![X]\!]$ by an additively shared permutation $[\![\pi]\!]$.

To implement $\mathcal{F}_{\mathsf{Add\text{-}Perm}}$, we must first convert $[\![\pi]\!]$ into $\langle\!\langle \pi \rangle\!\rangle$. We achieve this with the $\mathcal{F}_{\mathsf{A2C}}$ functionality, which is realized by the Π_{A2C} protocol. Once the

parties hold $\langle\!\langle\pi\rangle\!\rangle$, they can simply invoke $\mathcal{F}_{\mathsf{Comp\text{-}Perm}}$ to realize the $\mathcal{F}_{\mathsf{Add\text{-}Perm}}$ functionality. We also present the $\mathcal{F}_{\mathsf{C2A}}$ functionality for converting $\langle\!\langle\pi\rangle\!\rangle$ into an additive sharing $[\![\pi]\!]$. $\mathcal{F}_{\mathsf{C2A}}$ is realized by the $\mathit{\Pi}_{\mathsf{C2A}}$ protocol.

Functionality $\mathcal{F}_{\mathsf{A2C}}$ $([\![\pi]\!])$:

INPUT: Additively shared permutation $[\![\pi]\!]$.
OUTPUT: Composed permutation $\langle\!\langle\pi\rangle\!\rangle$.

Protocol $\mathit{\Pi}_{\mathsf{A2C}}([\![\pi]\!])$:

1. The second party samples a random permutation $\langle\!\langle\pi\rangle\!\rangle_2$.
2. The parties invoke $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$ on permutation $\langle\!\langle\pi\rangle\!\rangle_2^{-1}$ and input $[\![\pi]\!]$. They receive $[\![\rho]\!]$ as the result.
3. The parties reveal $[\![\rho]\!]$ to the first party who computes $\langle\!\langle\pi\rangle\!\rangle_1 := \rho$.
4. The parties output $\langle\!\langle\pi\rangle\!\rangle$.

Fig. 14. The $\mathcal{F}_{\mathsf{A2C}}$ functionality and $\mathit{\Pi}_{\mathsf{A2C}}$ protocol that converts an additively secret-shared permutation $[\![\pi]\!]$ into a composed permutation $\langle\!\langle\pi\rangle\!\rangle$.

Theorem 7. *Protocol* $\mathit{\Pi}_{A2C}$ *securely realizes the* $\mathcal{F}_{A2C}$ *functionality in the semi-honest setting.*

Proof. Observe that permutation composition $\gamma := \theta \circ \omega$ can be computed by considering the vector representation and computing $\gamma := \theta(\omega)$, where γ, ω are vectors. Therefore $\mathit{\Pi}_{A2C}$ computes shares of $\rho = \langle\!\langle\pi\rangle\!\rangle_2^{-1} \circ \pi$. If we multiply from the left by $\langle\!\langle\pi\rangle\!\rangle_2$ we obtain

$$\langle\!\langle\pi\rangle\!\rangle_2 \circ \langle\!\langle\pi\rangle\!\rangle_1 = \pi$$

as desired. Therefore, the protocol is correct. Privacy follows from a straightforward simulation of $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$. $\qquad\square$

Theorem 8. *Protocol* $\mathit{\Pi}_{C2A}$ *securely realizes the* $\mathcal{F}_{C2A}$ *functionality in the semi-honest setting.*

Proof. As detailed in the proof of Theorem 7, $[\![\pi]\!]$ is correctly computed due to $\pi = \langle\!\langle\pi\rangle\!\rangle_2 \circ \langle\!\langle\pi\rangle\!\rangle_1 = \langle\!\langle\pi\rangle\!\rangle_2(\langle\!\langle\pi\rangle\!\rangle_1)$. Therefore, the protocol is correct. Privacy follows from a straightforward simulation of $\mathcal{F}_{\mathsf{Basic\text{-}Perm}}$. $\qquad\square$

Corollary 1. *As implied in the proof of Theorem 7, the parties can compute composition of two permutations when the first is shared in the permutation group and the second is additively shared.*

Theorem 9. *The composition of* $\mathcal{F}_{A2C}$ *and* $\mathcal{F}_{Comp\text{-}Perm}$ *realizes the* $\mathcal{F}_{Add\text{-}Perm}$ *functionality in the semi-honest setting.*

Proof. Correctness follows by inspection. Similarly, privacy can be demonstrated via straightforward simulation. $\qquad\square$

Functionality $\mathcal{F}_{\text{C2A}}$ ($\langle\!\langle \pi \rangle\!\rangle$) :

INPUT: Composed permutation $\langle\!\langle \pi \rangle\!\rangle$.
OUTPUT: Additively shared permutation $[\![pi]\!]$.

Protocol $\Pi_{\text{C2A}}(\langle\!\langle \pi \rangle\!\rangle)$:

1. The parties invoke $\mathcal{F}_{\text{Basic-Perm}}$ on permutation $\langle\!\langle \pi \rangle\!\rangle_2$ and for input a sharing of $\langle\!\langle \pi \rangle\!\rangle_1$ [a].
   They receive $[\![\pi]\!]$ as the result.
2. The parties output $[\![\pi]\!]$.

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[a] $\langle\!\langle \pi \rangle\!\rangle_1$ can be considered an additive sharing by defining the first share as $\langle\!\langle \pi \rangle\!\rangle_1$ and the second share as 0. We also note that $\mathcal{F}_{\text{Basic-Perm}}$ can trivially be extended to accept a plaintext input list.

Fig. 15. The $\mathcal{F}_{\text{C2A}}$ functionality and Π_{C2A} protocol that converts composed permutation $\langle\!\langle \pi \rangle\!\rangle$ into an additively secret-shared permutation $[\![\pi]\!]$.

9 Extraction Protocols

We present our extraction protocols at a high level in Section 4.4. We now present them in formal detail:

9.1 Extract Unordered

$\mathcal{F}_{\text{Ext-Unord}}$ (Figure 16) receives as input $[\![X]\!]$ and a bitvector $[\![f]\!]$ such that $|X| = |f| = n$. f_i indicates if X_i is marked. Then it extracts and outputs all marked elements $[\![Y]\!] = \{[\![X_i]\!] \mid f_i = 1\}$ in a uniform order. Additionally, it outputs the permutation π and size c such that $Y = \pi(X)_{[c]}$. Therefore, one can unextract by simply computing $\pi^{-1}(Y\|0^{n-c})$.

The protocol $\Pi_{\text{Ext-Unord}}$ first associates each X_i with its corresponding f_i and permutes them by a random π. (step 2). Given the output size $c = \sum_i f_i$, observe that the $f' = \pi(f)$ is a uniformly random weight c vector. We can thus safely reveal the permuted f'. Then we select all elements of the permuted $[\![X]\!]' = \pi([\![X]\!])$ where $f'_i = 1$ (step 4,5) and output them (step 6). The resulting permutation is also returned by computing the "selection" permutation ρ with π.

Theorem 10. *Protocol $\Pi_{\text{Ext-Unord}}$ realizes the $\mathcal{F}_{\text{Ext-Unord}}$ functionality in the semi-honest setting.*

Proof. Correctness can be verified by inspection. The simulation of $\mathcal{F}_{\text{Comp-Perm}}$ is straightforward. To simulate f', observe that the ideal output includes the weight c of f. Therefore, the simulator can simulate f' by sampling a uniformly random f' with weight c and then sampling a consistent π. This distribution is identical, and therefore f' reveals no information beyond the desired output. □

Functionality $\mathcal{F}_{\text{Ext-Unord}}(\llbracket X \rrbracket, \llbracket f \rrbracket)$:

INPUT: Secret-shared list $\llbracket X \rrbracket \in \mathbb{F}^n$, flags $\llbracket f \rrbracket \in \{0,1\}^n$.
OUTPUT: $(\llbracket Y \rrbracket, \langle\!\langle \pi \rangle\!\rangle, c)$. Count $c := \sum_i f_i$, permutation $\langle\!\langle \pi \rangle\!\rangle$ where $\pi : [n] \to [n]$ is random subject to $\pi(f) = 1^* || 0^*$, and $Y = \pi(X)$, i.e. the first c items of Y are $\{X_i \mid f_i = 1\}$ and are in random order.

Protocol $\Pi_{\text{Ext-Unord}}(\llbracket X \rrbracket, \llbracket f \rrbracket)$:

1. Locally sample random $\langle\!\langle \pi \rangle\!\rangle$.
2. $\llbracket X' \rrbracket \bowtie \llbracket f' \rrbracket := \mathcal{F}_{\text{Comp-Perm}}(\langle\!\langle \pi \rangle\!\rangle, \llbracket X \rrbracket \bowtie \llbracket f \rrbracket)$
3. $f' := \mathcal{F}_{\text{Open}}(\llbracket f' \rrbracket), c := \sum_i f'_i$
4. Sample an arbitrary permutation ρ s.t. $\rho(f') = 1^c 0^{n-c}$
5. $\llbracket Y \rrbracket := \rho(\llbracket X'_i \rrbracket)$
6. return $(\llbracket Y \rrbracket, \rho \circ \langle\!\langle \pi \rangle\!\rangle, c)$

Fig. 16. $\Pi_{\text{Ext-Unord}}$ implements $\mathcal{F}_{\text{Ext-Unord}}$. It outputs the extracted elements in random order. $\Pi_{\text{Ext-Unord}}$ runs in $O(n)$ time & communication and $O(1)$ rounds.

9.2 Extract Ordered

$\mathcal{F}_{\textbf{Ext-Ord}}$ (Figure 17) is also similar to $\mathcal{F}_{\text{Ext-Unord}}$, but outputs the extracted X_i in the order they appear in X.

The protocol $\Pi_{\text{Ext-Ord}}$ uses $\Pi_{\text{Ext-Unord}}$ as a subprotocol. As the output of $\Pi_{\text{Ext-Unord}}$ is in random order, we mark X with additional information that will enable to place the extracted X_i in their original order in X. I.e., we compute $\llbracket \rho \rrbracket$, the number of marked elements before each X_i, via simple local additions (step 1), mark X with ρ and invoke $\Pi_{\text{Ext-Unord}}$ (step 2), open the extracted ρ' to both parties (step 3), and permute the extracted X' according to ρ' (step 4). We then output the resulting list (step 5) along with the shared permutation and output size c.

Functionality $\mathcal{F}_{\text{Ext-Ord}}$ $(\llbracket X \rrbracket, \llbracket f \rrbracket)$:

INPUT: Secret-shared list $\llbracket X \rrbracket \in \mathbb{F}^n$, $\llbracket f \rrbracket \in \{0,1\}^n$ of size n.
OUTPUT: $(\llbracket Y \rrbracket, \langle\!\langle \pi \rangle\!\rangle, c)$. Count $c := \sum_i f_i$, permutation $\langle\!\langle \pi \rangle\!\rangle$ where $\pi : [n] \to [n]$ is subject to $\pi(f) = 1^c || 0^*$ and $\pi(i) < \pi(i')$ for $0 \le i < i' < c$, and $Y = \pi(X)$, i.e. the first c items of Y are $\{X_i \mid f_i = 1\}$ and are in order.

Protocol $\Pi_{\text{Ext-Ord}}(\llbracket X \rrbracket, \llbracket f \rrbracket)$:

1. $\llbracket \rho_i \rrbracket := \Sigma_{j \le i} \llbracket f_j \rrbracket$
2. $(\llbracket X' \rrbracket \bowtie \llbracket \rho' \rrbracket, \langle\!\langle \pi' \rangle\!\rangle) := \mathcal{F}_{\text{Ext-Unord}}(\llbracket X \rrbracket \bowtie \llbracket \rho \rrbracket, \llbracket f \rrbracket, c)$
3. $\rho' := \mathcal{F}_{\text{Open}}(\llbracket \rho' \rrbracket_{[c]}) || [c, n]$
4. $\llbracket Y \rrbracket := \rho'^{-1}(\llbracket X' \rrbracket)$
5. return $(\llbracket Y \rrbracket, \rho'^{-1} \circ \langle\!\langle \pi' \rangle\!\rangle, c)$

Fig. 17. $\Pi_{\text{Ext-Ord}}$ implements $\mathcal{F}_{\text{Ext-Ord}}$. It outputs the extracted elements in the original order they appear in the input list X. $\Pi_{\text{Ext-Unord}}$ runs in $O(n)$ time & communication and $O(1)$ rounds.

Theorem 11. *Protocol $\Pi_{\text{Ext-Ord}}$ realizes the $\mathcal{F}_{\text{Ext-Ord}}$ functionality in the semi-honest setting.*

Proof. Correctness can be verified by inspection. Simulating $\mathcal{F}_{\text{Ext-Unord}}$ is straightforward. For simulating ρ', observe that ρ contains $[c]$ in monotonically increasing order. Moreover, for i with $f_i = 1$, ρ_i is one larger than its predecessor. Therefore, $\mathcal{F}_{\text{Ext-Unord}}$ will return the ρ_i for $f_i = 1$ in a random order and these ρ_i will all be unique. This can be simulated simply by returning a random permutation of $[c]$. $\qquad\square$

9.3 Extract Unordered Padded

$\mathcal{F}_{\text{Ext-Unord-Pad}}$ (Figure 18) is similar to $\mathcal{F}_{\text{Ext-Unord}}$ but pads the output to fixed length t. If fewer than t flags are one, then the output will include padding elements from a second input $[\![P]\!]$. Let $c := \Sigma_{i<n} f_i$ denote the number of flagged inputs. The protocol $\Pi_{\text{Ext-Unord-Pad}}$ first invokes $\Pi_{\text{IndexToOneHot}}$ (step 1), which returns a one-hot vector $[\![h]\!]$ of size t. If $c \leq t$, then $h_c = 1$ and otherwise is zero. In step 2, we extend f with the prefix sum of h. Note that after this step $|\{i \mid f_i = 1\}| = n + t$. In step 3, we extend $[\![X]\!]$ with the padding elements and then invoke $\mathcal{F}_{\text{Ext-Unord}}$ to extract $X_i | f_i = 1$ (step 4) along with any padding elements.

Now that we explained $\Pi_{\text{Ext-Unord-Pad}}$, we look more closely at $\Pi_{\text{IndexToOneHot}}$ invoked in step 1. $\Pi_{\text{IndexToOneHot}}$ constructs a complete binary tree of depth $w := \lceil \log_2(c) \rceil$ (step 1). Our invariant is that the node values at each level represent a one-hot vector. At the root we have a size 1 one-hot vector. At the next levels i, we have size 2^i one-hot vectors. This results in the last level having c leaves, all zeros but at one point. Our technique carefully arranges that the nonzero value is at point I. The values at the leaves represent the output of $\Pi_{\text{IndexToOneHot}}$. We denote each node as $t_{i,j}$, where i is the current level of the tree and j is the node index at that level (from the left). We bit decompose I into $I_{w-1}, \ldots, I_0$ (I_{w-1} is the MSB), which represents the path to $t_{w-1,I} = 1$ (step 2).

To maintain our invariant, we set the root of the tree $t_{0,0} := 1$ (step 3) and next proceed with setting the remaining $t_{i,j}$. We iterate over each level i and over each node j in that level (step 4). In step a we compute the position (i,j) of the current parent p and the position of its children c_0, c_1.

In steps b-c, we set the node values of the children t_{c_0} and t_{c_1}. We set them such that the one-hot position at the children's level is in exactly one of the children of the one-hot position in the parent's level. We use I_{w-1-i} to determine if the left child is 1 ($I_{w-1-i} = 0$) or the right child ($I_{w-1-i} = 1$). This ensures that we hold 1 on the path to position I in the leaf level. More specifically, we look at all the *right* children of a given level. It will be 1 if and only if the parent $t_p = 1$ AND $I_{w-1-i} = 1$ (step b). Then we look at all the *left* children of a given level. It will be 1 if and only if the parent $t_p = 1$ AND $I_{w-1-i} = 0$. We can optimize this logic. We know that $t_{c_0} \oplus t_{c_1} \oplus t_p = 0$. I.e., if the parent $t_p = 0$, then both children $t_{c_0} = t_{c_1} = 0$. If the parent $t_p = 1$, then at most one child t_{c_0} or t_{c_1} is 1. Hence, we can simply define the right child as $t_{c_0} \oplus t_p$ (step c). Now that we computed all $t_{i,j}$, we output the leaves t_{w-1} (step 5).

Functionality $\mathcal{F}_{\mathsf{Ext\text{-}Unord\text{-}Pad}}(\llbracket f \rrbracket, \llbracket X \rrbracket, \llbracket P \rrbracket)$:

INPUT: Secret-shared flags $\llbracket f \rrbracket \in \{0,1\}^n$. Optionally, shared list $\llbracket X \rrbracket \in \mathbb{F}^n$, padding list $\llbracket P \rrbracket \in \mathbb{F}^t$.

OUTPUT: $(\llbracket Y \rrbracket, \langle\!\langle \pi \rangle\!\rangle, c)$. Count $c := \mathsf{max}(f^*, t)$ where $f^* = \sum_i f_i$. Shared permutation $\langle\!\langle \pi \rangle\!\rangle$ where $\pi : [n+t] \to [n+t]$ is random subject to $\pi(f||1^r||0^{t-r}) = 1^*||0^*$ where $r := c - f^*$. Additionally, if the optional $\llbracket X \rrbracket, \llbracket P \rrbracket$ are provided, output a sharing of $Y = \pi(X||P)$; the first c positions of Y contain $\{X_i \mid f_i = 1\} \cup \{P_i \mid i \in [r]\}$ in a random order.

Protocol $\varPi_{\mathsf{Ext\text{-}Unord\text{-}Pad}}(\llbracket f \rrbracket, \llbracket X \rrbracket, \llbracket P \rrbracket)$:

1. $\llbracket h \rrbracket := \varPi_{\mathsf{IndexToOneHot}}(\Sigma_{i<n}\llbracket f_i \rrbracket, t)$
2. $\llbracket f_{n+i} \rrbracket := \bigoplus_{j \leq i} \llbracket h_j \rrbracket$ for $i \in [t]$.
3. $\llbracket X \rrbracket := \llbracket X \rrbracket || \llbracket P \rrbracket$
4. return $\mathcal{F}_{\mathsf{Ext\text{-}Unord}}(\llbracket X \rrbracket, \llbracket f \rrbracket)$

Protocol $\varPi_{\mathsf{IndexToOneHot}}(\llbracket I \rrbracket, t)$:

1. $w := \lceil \log_2(t) \rceil$
2. $\llbracket I_{w-1} \rrbracket, \ldots, \llbracket I_0 \rrbracket := \mathcal{F}_{\mathsf{Bit\text{-}Decomp}}(\llbracket I \rrbracket, w)$.
3. $\llbracket t_{0,0} \rrbracket := 1$
4. for $i = [w-1]$ and parallel for $j \in [2^i]$:
    a. $p := (i,j), c_0 := (i+1, 2j), c_1 := (i+1, 2j+1)$
    b. $\llbracket t_{c_0} \rrbracket := \llbracket t_p \rrbracket \wedge \llbracket I_{w-1-i} \rrbracket$
    c. $\llbracket t_{c_1} \rrbracket := \llbracket t_{c_0} \rrbracket \oplus \llbracket I_{w-1-i} \rrbracket$
5. return $\llbracket t_{w-1} \rrbracket$

Fig. 18. $\varPi_{\mathsf{Ext\text{-}Unord\text{-}Pad}}$ implements $\mathcal{F}_{\mathsf{Ext\text{-}Unord\text{-}Pad}}$. Like $\varPi_{\mathsf{Ext\text{-}Unord}}$, it extracts elements in random order. Additionally, the output list is padded to size c so that the number of non-dummies $c' \leq c$ stays private. The dummies are interspersed with the output. It runs in $O(n)$ time & communication and $O(1)$ rounds.

Theorem 12. *Protocol $\varPi_{\mathsf{Ext\text{-}Unord\text{-}Pad}}$ realizes the $\mathcal{F}_{\mathsf{Ext\text{-}Unord\text{-}Pad}}$ functionality in the semi-honest setting.*

Proof. Correctness can be verified by inspection. Privacy can be demonstrated by invoking the simulator for ideal functionalities and that of a generic MPC protocol. $\qquad\square$

9.4 Extract Ordered Padded

$\mathcal{F}_{\mathsf{Ext\text{-}Ord\text{-}Pad}}$ (Figure 19) is the equivalent of $\mathcal{F}_{\mathsf{Ext\text{-}Unord\text{-}Pad}}$, but outputs extracted elements $\{X_i \mid f_i = 1\}$ in their original order (followed by dummies to pad to size t). The protocol $\varPi_{\mathsf{Ext\text{-}Ord\text{-}Pad}}$ combines the ideas from $\varPi_{\mathsf{Ext\text{-}Ord}}$ and $\varPi_{\mathsf{Ext\text{-}Unord\text{-}Pad}}$. Like $\varPi_{\mathsf{Ext\text{-}Ord}}$, $\varPi_{\mathsf{Ext\text{-}Ord\text{-}Pad}}$ computes $\llbracket t \rrbracket$ in step 1 to remember the original order in X. But since our protocol also pads (similarly to $\varPi_{\mathsf{Ext\text{-}Unord\text{-}Pad}}$), it additionally computes $\llbracket t \rrbracket$ for the c appended elements so that after extraction they are placed

at the end of the list (step 4). Steps 2-3 are equivalent to those in $\Pi_{\mathsf{Ext\text{-}Unord\text{-}Pad}}$, and output a bitvector of size c that is non-zero only for the number of elements that should be padded. This ensures that the extracted output is of size c. This bitvector is then used in step 4 to extend t for the padded elements. Step 5 sets all values of X in the c appended elements to zeros (to mark them as dummies). Now X contains c elements with $f_i = 1$ and their associated order t. Hence, we are ready to extract $[\![X']\!] \bowtie [\![t']\!]$ (step 6). The remaining steps 7-9 are now identical to $\Pi_{\mathsf{Ext\text{-}Unord\text{-}Pad}}$. We open t, permute $[\![X']\!]$ based on t, and output.

Functionality $\mathcal{F}_{\mathsf{Ext\text{-}Ord\text{-}Pad}}([\![f]\!], [\![X]\!], [\![P]\!])$:

INPUT: Secret-shared flags $[\![f]\!] \in \{0,1\}^n$. Optionally, shared list $[\![X]\!] \in \mathbb{F}^n$, padding list $[\![P]\!] \in \mathbb{F}^t$.

OUTPUT: $([\![Y]\!], \langle\!\langle \pi \rangle\!\rangle, c)$. Count $c := \mathsf{max}(f^*, t)$ where $f^* = \sum_i f_i$. Shared permutation $\langle\!\langle \pi \rangle\!\rangle$ where $\pi : [n+t] \to [n+t]$ is random subject to $\pi(f||1^r||0^{t-r}) = 1^*||0^*$ where $r := c - f^*$. Additionally, if the optional $[\![X]\!], [\![P]\!]$ are provided, output a sharing of $Y = \pi(X||P)$; the first c positions of Y contain $\{X_i \mid f_i = 1\} \cup \{P_i \mid i \in [r]\}$ in their original order.

Protocol $\Pi_{\mathsf{Ext\text{-}Ord\text{-}Pad}}([\![f]\!], [\![X]\!], [\![P]\!])$:

1. $[\![d_i]\!] := \Sigma_{j<i}[\![f_j]\!]$ for $i \in [n]$
2. $[\![h]\!] := \Pi_{\mathsf{IndexToOneHot}}(\Sigma_{i<n}[\![f_i]\!], t)$
3. $[\![f_{n+i}]\!] := \bigoplus_{j \leq i}[\![h_j]\!]$ for $i \in [c]$
4. $[\![d_{n+i}]\!] := [\![f_{n+i}]\!] \cdot i$ for $i \in [t]$
5. $[\![X]\!] := [\![X]\!]||[\![P]\!]$
6. $([\![X']\!] \bowtie [\![d']\!], [\![\pi]\!], c) := \mathcal{F}_{\mathsf{Ext\text{-}Unord}}([\![X]\!] \bowtie [\![d]\!], [\![f]\!])$
7. $\rho := \mathcal{F}_{\mathsf{Open}}([\![d']\!]_{[c]})||[c, n+t]$
8. $[\![Y]\!] := \rho([\![X'_i]\!])$
9. return $([\![Y]\!], \rho \circ \langle\!\langle \pi \rangle\!\rangle, c)$.

Fig. 19. $\Pi_{\mathsf{Ext\text{-}Ord\text{-}Pad}}$ implements $\mathcal{F}_{\mathsf{Ext\text{-}Ord\text{-}Pad}}$. Like $\Pi_{\mathsf{Ext\text{-}Ord}}$, it extracts elements in the order they appear in X. Additionally, the output list is padded to size c so that the number of non-dummies $c' \leq c$ stays private. The dummies are (obliviously) placed at the end of the output list. It runs in $O(n)$ time & communication and $O(1)$ rounds.

Theorem 13. *Protocol $\Pi_{\mathsf{Ext\text{-}Ord\text{-}Pad}}$ realizes the $\mathcal{F}_{\mathsf{Ext\text{-}Ord\text{-}Pad}}$ functionality in the semi-honest setting.*

Proof. Correctness can be verified by inspection. The simulation of $\Pi_{\mathsf{IndexToOneHot}}$ and $\mathcal{F}_{\mathsf{Ext\text{-}Unord}}$ directly follows from their simulators. For simulating ρ, observe that ρ contains $[c]$ in monotonically increasing order. Moreover, for i with $f_i = 1$, ρ_i is one larger than its predecessor. Any missing ρ_i values up to t are then manually included in step 4. Therefore, $\mathcal{F}_{\mathsf{Ext\text{-}Unord}}$ will return the ρ_i for $f_i = 1$ in a random order and these ρ_i will all be unique. This can be simulated simply by returning a random permutation of $[c]$. $\square$

10 Sorting

We demonstrate that several prior works [CHI$^+$19,AHI$^+$22,HKI$^+$13] can be efficiently implemented in our framework. The ideal functionality $\mathcal{F}_{\mathsf{Stable\text{-}Sort}}$ is presented in Figure 20. $\mathcal{F}_{\mathsf{Stable\text{-}Sort}}$ is stable, and hence ensures that equal values maintain their order. It outputs a permutation that sorts the inputs.

Functionality $\mathcal{F}_{\mathsf{Stable\text{-}Sort}}$:

$\mathcal{F}_{\mathsf{Stable\text{-}Sort}}(\llbracket X \rrbracket)$:
 Upon input $\llbracket X \rrbracket$, compute stable sorting permutation π of X. Return $\langle\!\langle \pi \rangle\!\rangle$.

Fig. 20. The random and stable sorting functionality $\mathcal{F}_{\mathsf{Stable\text{-}Sort}}$. We note that one could consider outputting π in additive format $\llbracket \pi \rrbracket$.

10.1 Partition

Most prior works were presented in the three-party honest-majority setting due to the existence of efficient permutations. [CHI$^+$19,AHI$^+$22] implement radix sort and generate a sorting permutation of a single bit using a circuit. In particular, they consider a subprotocol, which we denote as $\Pi_{\mathsf{Partition}}$. It takes as input a bit vector X and returns the stable sorting permutation. We refer to 4.5 for the intuition of this protocol. We note that compared to [CHI$^+$19], we optimize this protocol with the use of $\Pi_{\mathsf{IndexToOneHot}}$ which requires significantly less communication.

Protocol $\Pi_{\mathsf{Partition}}$:

$\Pi_{\mathsf{Partition\text{-}Add}}(\llbracket X \rrbracket)$:

1. $i \in [n] : (\llbracket f_{1,i} \rrbracket, ..., \llbracket f_{2^\ell,i} \rrbracket) := \Pi_{\mathsf{IndexToOneHot}}(\llbracket X_i \rrbracket, 2^\ell)$
2. $j \in [2^\ell], i \in [n] : \llbracket s_{j,i} \rrbracket = \sum_{i',j' \text{ s.t. } j' < j \vee (j'=j \wedge i' \leq i)} \llbracket f_{i',j'} \rrbracket$
3. $i \in [n] : \llbracket \pi_i \rrbracket = \sum_{j \in [2^\ell]} \llbracket s_{i,j} \rrbracket$
4. return $\llbracket \pi \rrbracket$

$\Pi_{\mathsf{Partition}}(\llbracket X \rrbracket)$:

1. return $\mathcal{F}_{\mathsf{A2C}}(\Pi_{\mathsf{Partition\text{-}Add}}(\llbracket X \rrbracket))$

Fig. 21. Protocol $\Pi_{\mathsf{Partition}}$ that implements the $\mathcal{F}_{\mathsf{Stable\text{-}Sort}}$ functionality. Let n be the length of X and ℓ be the bit-length of the elements of X. It makes use of the subprotocol $\Pi_{\mathsf{Partition\text{-}Add}}$, which implements $\mathcal{F}_{\mathsf{Stable\text{-}Sort}}$ with additive secret-shared output.

Theorem 14. *Protocol $\Pi_{\mathsf{Partition}}$ realizes the $\mathcal{F}_{\mathsf{Stable\text{-}Sort}}$ functionality in the semi-honest setting.*

Proof. Correctness can be verified by inspection. Simulation follows from only making use of generic computation for circuits and the simulator for $\mathcal{F}_{\mathsf{A2C}}$. □

10.2 Radix Sort

To sort multiple bits, one can invoke $\Pi_{\mathsf{Partition}}$ multiple times, starting with the most significant bit and then compose the resulting permutations to get the overall sorting permutation. Given access to $O(n)$ time permutation gates, the radix sort protocol $\Pi_{\mathsf{Radix\text{-}Sort}}$ of Figure 22 requires $O(n\ell \log \ell)$ communication and $O(\ell)$ rounds to stable sort n items of length ℓ.

Protocol $\Pi_{\mathsf{Radix\text{-}Sort}}$:

PARAMETER: Let $t \in \mathbb{N}$ be a hyperparameter with default value 2.

$\Pi_{\mathsf{Radix\text{-}Sort}}(\llbracket X \rrbracket)$:

1. Let $\llbracket X_i' \rrbracket := \llbracket X_{i,[\ell-t,\ell]} \rrbracket$ be the t most significant bits of $\llbracket X_i \rrbracket$ for $i \in [n]$
2. $\langle\!\langle \pi \rangle\!\rangle := \Pi_{\mathsf{Partition}}(\llbracket X' \rrbracket)$
3. For $j \in [1, \lceil \ell/t \rceil]$:
   - (a) $\ell' := \ell - jt$
   - (b) $\llbracket X_i' \rrbracket := \llbracket X_{i,[\ell'-t,\ell']} \rrbracket$ for $i \in [n]$
   - (c) $\llbracket \rho \rrbracket := \Pi_{\mathsf{Partition\text{-}Add}}(\llbracket X' \rrbracket)$
   - (d) $\llbracket \pi \rrbracket := \mathcal{F}_{\mathsf{Comp\text{-}Perm}}(\langle\!\langle \pi \rangle\!\rangle, \llbracket \rho \rrbracket)$
   - (e) $\langle\!\langle \pi \rangle\!\rangle := \mathcal{F}_{\mathsf{A2C}}(\llbracket \pi \rrbracket)$
4. return $\langle\!\langle \pi \rangle\!\rangle$
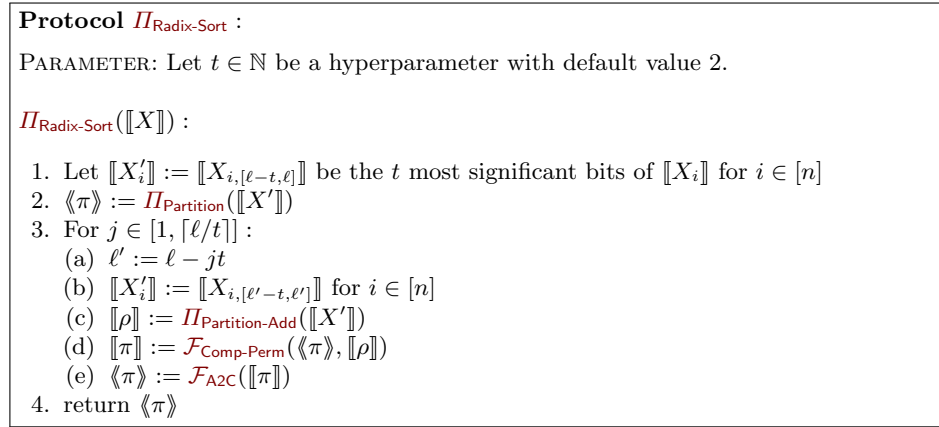
Fig. 22. Protocol $\Pi_{\mathsf{Radix\text{-}Sort}}$ implements the $\mathcal{F}_{\mathsf{Stable\text{-}Sort}}$ functionality.

Theorem 15. *Protocol $\Pi_{\mathsf{Radix\text{-}Sort}}$ realizes the $\mathcal{F}_{\mathsf{Stable\text{-}Sort}}$ functionality in the semi-honest setting.*

Proof. Correctness can be verified by inspection. Simulation follows from the simulators for $\Pi_{\mathsf{Partition}}, \mathcal{F}_{\mathsf{Comp\text{-}Perm}}, \mathcal{F}_{\mathsf{A2C}}$. □

10.3 Quick Sort

The latter, [HKI+13], take a different approach in the so-called shuffle-reveal model. The idea is that if the input lists are first shuffled, one can run most insecure comparison-based sorting algorithms, e.g. quick sort, where each comparison is replaced by a protocol that only reveals the result of the comparison. This protocol makes use of $O(n \log n)$ comparisons over $O(\log n)$ steps. The running time of this protocol is $O(n\ell \log n \log \ell)$ and $O(\log n\ell \log \ell)$ rounds assuming a comparison requires $O(\ell \log \ell)$ time and $O(\ell)$ rounds.

However, one shortcoming of a direct implementation of this paradigm is that the inputs must be totally ordered, i.e. with *no duplicate values*. Let us consider quick sort and consider the view of the parties if all elements are the same or all are different. If they are the same, the result of the comparisons will all be the same, while different values implies that the comparisons will be uniformly random. [HKI+13] proposed a simple solution to make any input list totally ordered by appending the index of the item as its least significant bit, e.g. $X_i' := X_i \cdot n + i$. Then it is possible to invoke the shuffle-reveal compiler and get secure sort. One benefit of this approach is that the resulting sort is stable while quick sort is usually unstable. We present this protocol as $\Pi_{\mathsf{Quick\text{-}Sort}}$ in Figure 23. The overall running time is $O(n\ell' \log n \log \ell')$ and $O(\log n + \log \ell')$ rounds where $\ell' := \ell + \log n$.

Protocol $\Pi_{\mathsf{Quick\text{-}Sort}}$:

$\Pi_{\mathsf{Quick\text{-}Sort\text{-}Impl}}(\llbracket X \rrbracket)$:

1. The parties jointly sample $i \leftarrow [n]$
2. $\llbracket p \rrbracket := \llbracket X_i \rrbracket$ and $c_i := 1$
3. for $j \in [n] \setminus \{i\} : c_j := \mathcal{F}_{\mathsf{Open}}((\llbracket X_j \rrbracket \cdot 2 + \llbracket t \rrbracket) < (\llbracket p \rrbracket \cdot 2 + \llbracket s \rrbracket)) \cdot 2$ where $\llbracket t \rrbracket$ is sampled as a uniform bit and $\llbracket s \rrbracket := 1 - \llbracket t \rrbracket = 1 \oplus \llbracket t \rrbracket$.
4. Let ρ be the sorting permutation for c
5. $\llbracket X' \rrbracket := \rho(\llbracket X \rrbracket)$
6. $i' := \rho^{-1}(i)$
7. $\llbracket Z \rrbracket := \llbracket X'_{[i'-1]} \rrbracket$
8. $\llbracket Y \rrbracket := \llbracket X'_{[i'+1,n]} \rrbracket$
9. $(\theta, \gamma) := (\Pi_{\mathsf{Quick\text{-}Sort\text{-}Impl}}(\llbracket Z \rrbracket), \Pi_{\mathsf{Quick\text{-}Sort\text{-}Impl}}(\llbracket Y \rrbracket))$
10. return $(\theta || i' || \gamma + i') \circ \rho$

$\Pi_{\mathsf{Quick\text{-}Sort}}(\llbracket X \rrbracket)$:

1. The parties sample $\langle\!\langle \pi \rangle\!\rangle$ uniformly at random
2. $\llbracket X' \rrbracket := \mathcal{F}_{\mathsf{Comp\text{-}Perm}}(\langle\!\langle \pi \rangle\!\rangle, \llbracket X \rrbracket)$
3. return $\Pi_{\mathsf{Quick\text{-}Sort\text{-}Impl}}(\llbracket X' \rrbracket, [n]) \circ \langle\!\langle \pi \rangle\!\rangle$
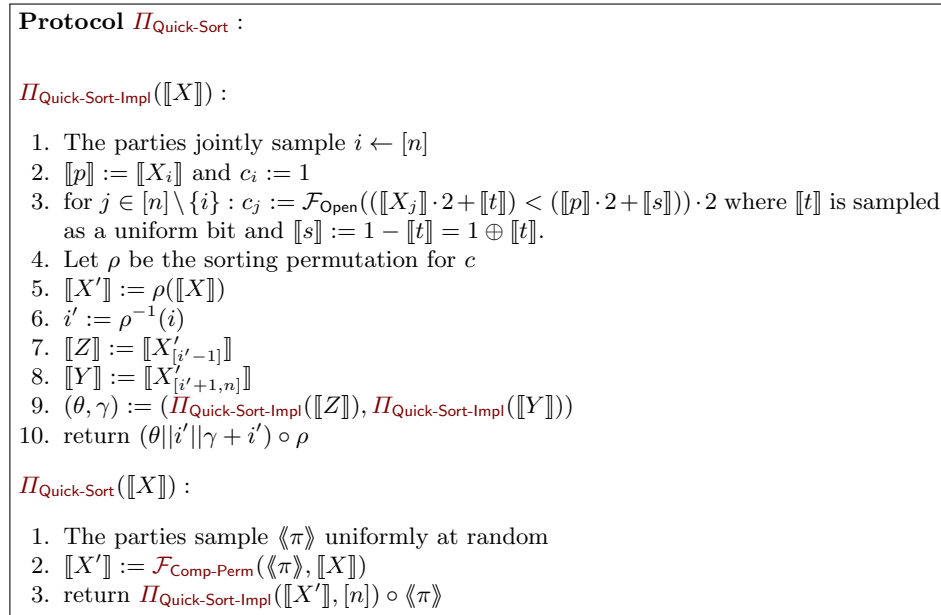
Fig. 23. Protocol $\Pi_{\mathsf{Quick\text{-}Sort}}$ that implements the $\mathcal{F}_{\mathsf{Stable\text{-}Sort}}$ functionality.

Theorem 16. *Protocol* $\Pi_{\mathsf{Quick\text{-}Sort}}$ *realizes the* $\mathcal{F}_{\mathsf{Stable\text{-}Sort}}$ *functionality in the semi-honest setting.*

Proof. Correctness can be verified by inspection. The simulator works by sampling a random permutation $\theta : [n] \to [n]$. For the first invocation of $\Pi_{\mathsf{Quick\text{-}Sort\text{-}Impl}}$, the c_j values are computed as

$$c_j := \theta^{-1}(i) < \theta^{-1}(j)$$

Subsequent recursions of $\Pi_{\mathsf{Quick\text{-}Sort\text{-}Impl}}$ are computed the same way with the indices appropriately updated based on the subrange in question.

Observe that this perfectly simulates the protocol. The c_j collectively specify θ exactly. In the real protocol, the overall sorting permutation is $\rho = \theta \circ \pi$ where π is uniformly distributed. Therefore, $\theta := \rho \circ \pi^{-1}$ is also uniformly distributed. □

11 Batched Random Access Memory

We now present a generalization of our permutation functionalities to allow the parties to input an arbitrary selection vector σ and list X. The protocol results in the parties performing batched random access into X based on σ. I.e. it returns the sharing $\llbracket Y \rrbracket := (\llbracket X_{\sigma(1)} \rrbracket, ..., \llbracket X_{\sigma(m)} \rrbracket)$. This protocol makes use of the aggregation tree technique first presented by [BDG+22] in the three-party honest majority setting and summarized in Section 3.4.

The protocol begins by applying a stable sort to the set of indices $[n]$ extended with the selection vector σ to obtain the sorting permutation π. Intuitively, we will use π to place each X_i before the "output" positions that want to access X_i. More specifically, π is used to permute both the values X concatenated with m dummy elements to obtain X'. Additionally, the parties compute flags $b = \pi(0^n || 1^m)$ to denote which items are dummies, i.e. X_i' is a dummy if $b_i = 1$. The critical property of X' is that each non-dummy X_i' is followed by t dummies where t is the number of times it is being selected.

The parties then invoke the aggregation tree functionality $\mathcal{F}_{\mathsf{Agg}}$ with the flag bits b and the permuted list X'. $\mathcal{F}_{\mathsf{Agg}}$ will duplicate each non-dummy X_i' into the next set of contiguous dummy positions [BDG+22]. We refer to Section 3.4 or [BDG+22] for more details on how this is achieved but overall it requires $O(n\ell)$ time and $O(\log n)$ rounds. Let Z denote the result of $\mathcal{F}_{\mathsf{Agg}}$. The parties can then unpermute the aggregated vector Z to Y. The first n positions of Z will be X while the last m positions will be the desired output Y.

Functionality $\mathcal{F}_{\mathsf{Batched\text{-}RAM\text{-}Read}}(\llbracket \sigma \rrbracket, \llbracket X \rrbracket)$:

INPUT: A shared selection $\llbracket \sigma \rrbracket$ where $\sigma \in [n]^m$ and shared list $\llbracket X \rrbracket$ where $X \in \mathbb{F}^{n \times \ell}$
OUTPUT: A shared vector $\llbracket Y \rrbracket$ such that $Y_i = X_{\sigma(i)}$

Protocol $\Pi_{\mathsf{Batched\text{-}RAM\text{-}Read}}(\llbracket \sigma \rrbracket, \llbracket X \rrbracket)$:

1. $\langle\!\langle \pi \rangle\!\rangle := \mathcal{F}_{\mathsf{Stable\text{-}Sort}}([n] || \llbracket \sigma \rrbracket)$
2. $\llbracket b \rrbracket := \mathcal{F}_{\mathsf{Comp\text{-}Perm}}(\langle\!\langle \pi \rangle\!\rangle, 0^n || 1^m)$
3. $\llbracket X' \rrbracket := \mathcal{F}_{\mathsf{Comp\text{-}Perm}}(\langle\!\langle \pi \rangle\!\rangle, \llbracket X \rrbracket || 0^m)$
4. $\llbracket Z \rrbracket := \mathcal{F}_{\mathsf{Agg}}(\llbracket b \rrbracket, \llbracket X' \rrbracket, \mathsf{dup})$ where $\mathsf{dup}(x, y) := x$.
5. $\llbracket Y \rrbracket := \mathcal{F}_{\mathsf{Comp\text{-}Perm}}(\langle\!\langle \pi \rangle\!\rangle^{-1}, \llbracket Y \rrbracket)$
6. return $\llbracket Y \rrbracket_{[n, n+m]}$

Fig. 24. Protocol $\Pi_{\mathsf{Batched\text{-}RAM\text{-}Read}}$ that implements the $\mathcal{F}_{\mathsf{Batched\text{-}RAM\text{-}Read}}$ functionality.

Theorem 17. *The protocol $\Pi_{\mathsf{Batched\text{-}RAM\text{-}Read}}$ realizes the functionality $\mathcal{F}_{\mathsf{Batched\text{-}RAM\text{-}Read}}$ in the semi-honest setting.*

Proof. Correctness can be verified by inspection. Simulation follows from a simple composition argument. □

We note that the inverse write operation of $\Pi_{\mathsf{Batched\text{-}RAM\text{-}Read}}$ has the following challenges. First, we need to define what happens when multiple inputs are written to the same output position. We consider several options, the output could take the first or the last value to be written. More generally, the caller can define their own associative operator $\star(x, y)$ and the values mapped to an output position j are computed as $\bigstar_{i \in \{i | \sigma_j = i\}} X_i$.

Lastly, we need to define the value of output position j for which σ has no mapping, i.e. $|\{i \mid \sigma_j = i\}| = 0$. The most natural option is to provide a default value D_j. Overall, the jth output position will be the summation of the jth default value and the input values written to it, i.e. $Y_j := D_j \star \bigstar_{i \in \{i | \sigma_j = i\}} X_i$. We note that simple modifications of the protocol can select the default only if no X_i are mapped to it. The full protocol $\Pi_{\mathsf{Batched\text{-}RAM\text{-}Write}}$ is presented in Figure 25. It follows a similar logic as $\Pi_{\mathsf{Batched\text{-}RAM\text{-}Read}}$.

Functionality $\mathcal{F}_{\mathsf{Batched\text{-}RAM\text{-}Write}}(\llbracket \sigma \rrbracket, \llbracket X \rrbracket, \llbracket D \rrbracket, \star)$:

INPUT: A shared selection $\llbracket \sigma \rrbracket$ where $\sigma \in [n]^m$ and shared list $\llbracket X \rrbracket, \llbracket D \rrbracket$ where $X \in \mathbb{F}^{n \times \ell}, D \in \mathbb{F}^{m \times \ell}$ and associative operator $\star : \mathbb{F}^\ell \times \mathbb{F}^\ell \to \mathbb{F}^\ell$.
OUTPUT: A shared vector $\llbracket Y \rrbracket$ such that $Y_j = D_j \star \bigstar_{i \in \{i | \sigma_j = i\}} X_i$ for $j \in [m]$.

Protocol $\Pi_{\mathsf{Batched\text{-}RAM\text{-}Write}}(\llbracket \sigma \rrbracket, \llbracket X \rrbracket, \llbracket D \rrbracket, \star)$:

1. $\langle\!\langle \pi \rangle\!\rangle := \mathcal{F}_{\mathsf{Stable\text{-}Sort}}([n] || \llbracket \sigma \rrbracket)$
2. $\llbracket b \rrbracket := \mathcal{F}_{\mathsf{Comp\text{-}Perm}}(\langle\!\langle \pi \rangle\!\rangle, 1^n || 0^m)$
3. $\llbracket X' \rrbracket := \mathcal{F}_{\mathsf{Comp\text{-}Perm}}(\langle\!\langle \pi \rangle\!\rangle, \llbracket D \rrbracket || \llbracket X \rrbracket)$
4. $\llbracket Z \rrbracket := \mathcal{F}_{\mathsf{Agg}}(\llbracket b \rrbracket, \llbracket X' \rrbracket, \star)$.
5. $\llbracket Y \rrbracket := \mathcal{F}_{\mathsf{Comp\text{-}Perm}}(\langle\!\langle \pi \rangle\!\rangle^{-1}, \llbracket Y \rrbracket)$
6. return $\llbracket Y \rrbracket_{[n]}$
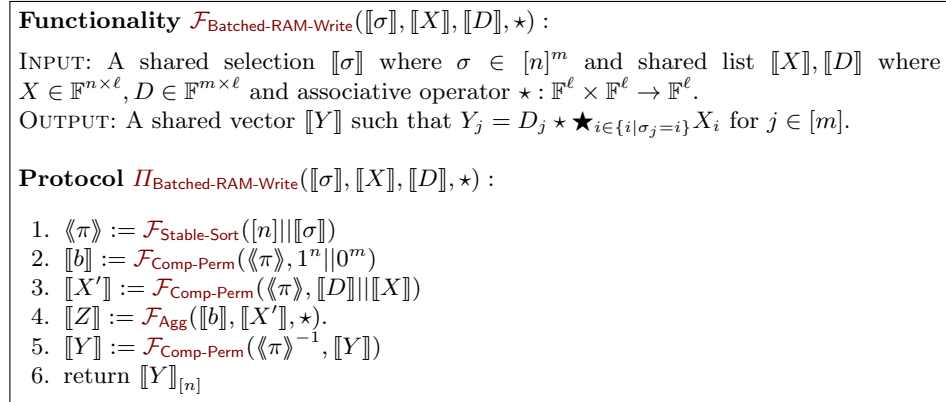
Fig. 25. Protocol $\Pi_{\mathsf{Batched\text{-}RAM\text{-}Write}}$ that implements the $\mathcal{F}_{\mathsf{Batched\text{-}RAM\text{-}Write}}$ functionality.

Theorem 18. *The $\Pi_{\mathsf{Batched\text{-}RAM\text{-}Write}}$ protocol realizes the functionality $\mathcal{F}_{\mathsf{Batched\text{-}RAM\text{-}Write}}$ in the semi-honest setting.*

Proof. Correctness can be verified by inspection. Simulation follows from a simple composition argument. □

12   Evaluation

We implement several of our protocols and report on their performance. The implementation is written in C++ and primarily focuses on binary secret shar-

	Time (ms)			Comm. (MB)		
Protocol \ n	2^{12}	2^{16}	2^{20}	2^{12}	2^{16}	2^{20}
⋆ Gazelle [JVC18] ($\Pi_{\text{Basic-Perm}}$)	2,135	19,846	303,219	0.8	13.4	215
Paillier Permutations ($\Pi_{\text{Basic-Perm}}$)	143,030	572,120	2,288,480	4.2	67	1,075
$\Pi_{\text{Basic-Perm}} + \Pi_{\text{Prf-Perm}}$ w/ [ARS$^+$15]	970	13,498	250,689	3.8	61	975
$\Pi_{\text{Basic-Perm}} + \Pi_{\text{Prf-Perm}}$ w/ [APRR24]	**77**	**1,117**	**19,622**	**0.7**	**11.4**	**182**
$\Pi_{\text{Partition}}$	86	1,396	24,242	8.0	20	324
$\Pi_{\text{Radix-Sort}}$ $\ell = 32$	1,208	17,890	299,702	16.9	270	4,333
$\Pi_{\text{Batched-RAM-Read}}$	592	10,846	221,214	7.9	163	3,183

Fig. 26. Performance metrics for running our protocols on lists of size n. The string length is $\ell = 128$ bits (except for sort). Time is measured in milliseconds and communication in MB. ⋆ Gazelle numbers are taken from [JVC18] and do not account for ciphertext privacy.

ing. We intend to open source the implementation. Where more efficient, e.g. $\Pi_{\text{Partition}}$, the protocols switch to arithmetic secret sharing and then back to binary. We employ the binary GMW [GMW87] protocol to evaluate circuits with correlated randomness generated using Silent OT [BCG$^+$19b] with optimization provided by [RRT23]. Each AND gate requires 4 bits of communication. We consider two implementations of a weak PRF. The first is LowMC PRP [ARS$^+$15] and the second is alternating moduli-based PRF [APRR24]. For the latter, we make use of their implementation.

All performance numbers for our protocols and Paillier were obtained by running the protocol on a laptop computer with 8GB of RAM. Both parties run on a *single shared* thread. As such, real deployments with each party running on a separate machine could see up to a 2x improvement in running time, plus network latency. More generally, most of the protocols can be parallelized and we would expect a new linear speedup (at least for a moderate number of threads). Communication is performed by directly passing in memory buffers between the parties. Our core permutation protocol $\Pi_{\text{Basic-Perm}} + \Pi_{\text{Prf-Perm}}$ is constant-round, and therefore one can estimate the additional cost of sending the data by dividing the total communication by the bandwidth. For the non-constant round protocols $\Pi_{\text{Radix-Sort}}, \Pi_{\text{Batched-RAM-Read}}$, we note that they require only a relatively small number of rounds, and therefore are well-approximated by the bandwidth.

We compare our protocols implementing $\mathcal{F}_{\text{Gen-Perm}}$ to the lattice AHE based Gazelle protocol [JVC18] and the folklore Paillier AHE protocol. Both of these protocols follow the same outline. The receiver holds an input vector A and the sender holds a permutation π. The receiver encrypts the components of A using AHE, i.e. $[\![A_i]\!] = \text{AHE.enc}_k(A_i)$, sends $[\![A]\!]$ to the sender who sends $[\![C]\!] = \pi([\![A]\!]) - B$ back where B is a random vector. The receiver can decrypt the result as C such that $B + C = \pi(A)$. Unlike our binary correlation, Gazelle uses a 20-bit integer modulus and Paillier requires a larger modulus, e.g. a 2000-bit integer. Although we do not report this overhead, one can convert such

correlations to binary with additional overhead, e.g. $b \log b$ OTs and $\log b$ rounds for a b-bit modulus.

The Gazelle protocol presents various optimizations such as lattice-based AHE packing/SIMD where the multiplied plaintexts are packed into a single ciphertext. This results in better communication and encryption times, but adds additional complexity due to the need to permute values within a single ciphertext. We refer to [JVC18] for details but note that ciphertexts must be decomposed, permuted, masked and then recombined. We *do not* include the time to mask or recombine. Moreover, to prevent leaking information about π via the distribution of LWE ciphertext noise, one must use an LWE/AHE scheme with circuit privacy [Klu22]. However, Gazelle does not consider this, and therefore the overhead of a fully secure scheme will be noticeably higher, e.g. due to the need to perform noise flooding. In particular, in Figure 26 we only report the times provided by [JVC18] to encrypt, decrypt, decompose and permute values within single ciphertext without circuit privacy.

Our protocol $\Pi_{\mathsf{Basic\text{-}Perm}} + \Pi_{\mathsf{Prf\text{-}Perm}}$ with the alternating moduli PRF [APRR24] outperforms the alternative of using Paillier additive homomorphic encryption by $100\times$ in terms of running time and $5\times$ for communication. We also report the performance metrics of our permutation protocol with the LowMC PRF [ARS+15] and observe that is is about an order of magnitude slower than [APRR24] and requires $7\times$ more rounds. The lattice based AHE scheme of Gazelle [JVC18] requires only slightly more communication[6] but noticable more computation time, e.g. $15\times$ more for $n = 2^{20}, \ell = 128$. We do not compare to Chase et al. [CGP20] due to no implementation and that they require $O(n \log n)$ overhead.

We additionally implement the $\Pi_{\mathsf{Partition}}$ protocol and observe that the main overhead is generating the permutation correlation. We report the performance of our radix sorting protocol $\Pi_{\mathsf{Radix\text{-}Sort}}$ for 32-bit strings. We observe that the majority of the overhead comes from the generation of the 16 permutation correlations that the protocol requires. This suggests that quick sort $\Pi_{\mathsf{Quick\text{-}Sort}}$ would likely result in better performance (more comparisons but only 1 permutation).

Lastly, we implement our batched RAM protocol and report the performance of reading n memory locations out of a list of size n. The elements being read are of size $\ell = 128$. The vast majority of the overhead associated with this protocol is the invocation of $\mathcal{F}_{\mathsf{Stable\text{-}Sort}}$ on n elements of size $\log(n)$ bits. We implement $\mathcal{F}_{\mathsf{Stable\text{-}Sort}}$ using $\Pi_{\mathsf{Radix\text{-}Sort}}$. As discussed before, we suspect that we could get improved performance by replacing $\Pi_{\mathsf{Radix\text{-}Sort}}$ with $\Pi_{\mathsf{Quick\text{-}Sort}}$. We leave this to future work.

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