Dual Support Decomposition in the Head: Shorter Signatures from Rank SD and MinRank

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Abstract

The MPC-in-the-Head (MPCitH) paradigm is widely used for building post-quantum signature schemes, as it provides a versatile way to design proofs of knowledge based on hard problems. Over the years, the MPCitH landscape has changed significantly, with the most recent improvement coming from *VOLE-in-the-Head* (VOLEitH) and *Threshold-Computation-in-the-Head* (TCitH).

While a straightforward application of these frameworks already improve the existing MPCitH-based signatures, we show in this work that we can adapt the arithmetic constraints representing the underlying security assumptions (here called the *modeling*) to achieve smaller sizes using these new techniques. More precisely, we explore existing modelings for the rank syndrome decoding (RSD) and MinRank problems and we introduce a new modeling, named *dual support decomposition*, which achieves better sizes with the VOLEitH and TCitH frameworks by minimizing the size of the witnesses. While this modeling is naturally more efficient than the other ones for a large set of parameters, we show that it is possible to go even further and explore new areas of parameters. With these new modeling and parameters, we obtain low-size witnesses which drastically reduces the size of the "arithmetic part" of the signature.

We apply our new modeling to both TCitH and VOLEitH frameworks and compare our results to RYDE, MiRitH, and MIRA signature schemes. We obtain signature sizes below 4 kB for 128 bits of security with $N = 256$ parties (a.k.a. leaves in the GGM trees) and going as low as ≈ 3.5 kB with $N = 2048$, for both RSD and MinRank. This represents an improvement of more than 1.5 kB compared to the original submissions to the 2023 NIST call for additional signatures. We also note that recent techniques optimizing the sizes of GGM trees are applicable to our schemes and further reduce the signature sizes by a few hundred bytes, bringing them arround 3 kB (for 128 bits of security with $N = 2048$).

1 Introduction

The MPC-in-the-Head (MPCitH) paradigm is a popular framework to build post-quantum signatures. After sharing the secret key, the signer emulates "in his head" an MPC protocol and commits each party's view independently. He then reveals the views of a pseudo-random subset of parties, where this subset is given by the hash digest of the commitments (in the setting of the Fiat-Shamir heuristic). By the privacy of the MPC protocol, nothing is revealed about the secret key, which implies the zero-knowledge property. On the other hand, a malicious signer needs to cheat for at least one party, which shall be discovered by the verifier with high probability, hence ensuring the unforgeability property.

In the new NIST call for additional post-quantum signatures [33], many submissions rely on the MPCitH paradigm applied on a large range of security assumptions. Three MPCitH candidates fall in the rank-based cryptography category:

- RYDE [4], for which the security relies on the hardness of solving the rank syndrome decoding problem;
- MIRA [5] and MiRitH [1], for which the security relies on the hardness of solving the MinRank problem (MIRA and MiRitH rely on the same security assumption, but use different modelings and MPC protocols).

Recently, new techniques of MPC-in-the-Head have been proposed:

- the VOLE-in-the-Head (VOLEitH) framework [12] released in Summer 2023;[1]
- the TC-in-the-Head (TCitH) framework [21] released in Autumn 2023.[2]

As shown in [21] a simple application of these frameworks leads to shorter and faster signature schemes compared to those submitted to the NIST call (for similar underlying security assumption).

For MPCitH-based schemes (including those based on VOLEitH and TCitH), the signatures are composed of two parts, a "symmetric part" made of seeds and hash digests and an "arithmetic part" composed of the open party views and broadcast shares of the MPC protocol. While for a given security level the symmetric part is of rather fixed size (for the considered MPCitH framework), the arithmetic part depends on the modeling of the used security assumption and the associated MPC protocol. In the traditional broadcast-based MPCitH framework (*i.e.* the MPCitH framework widely used before VOLEitH and TCitH), to minimize the signature size, the designers had minimize the *sum of the sizes of the MPC input and of the broadcasted values* while considering only *linear* multiparty computation. With the VOLEitH and TCitH frameworks, the game rules

[1] While VOLEitH has not been introduced as an MPCitH technique, [21] showed that it can be considered as such.

[2] The original version of the TCitH framework was released in Autumn 2022 [22] (and published at Asiacrypt 2023), we refer here to the improved version of the TCitH framework [21].

have changed. These frameworks enable quadratic (or higher degree) multiparty computation, which implies that minimizing the signature size is achieved by minimizing the MPC protocol input (i.e., the witness of the modeling).

In rank-based cryptography, several modelings for the rank syndrome decoding problem and the MinRank problem have been proposed. The first one is derived from [37] and consists in working with a permuted version and an additively-masked version of the secret. The best scheme relying on it is proposed in [15]. The second modeling is based on q-polynomials and is first used in such a context in [19]. The last modeling consists in writing the low-rank object as the product of two small matrices and is first used in such a context in [2] and [19]. We sum up the different techniques to handle the rank metric in Table 1.

Problem	**Permuted Secret**	**q-Polynomial Evaluation (q-pol)**	**Matrix Rank Decomposition (MRD)**	**Kipnis Shamir (KS)**	**Dual Support Decomposition (DSD)**
RSD	BG23 [15]	RYDE [4, 19]	Fen24 [19]	-	This work
MinRank	-	MIRA [5, 19]	Fen24 [19]	MiRitH [1]	This work

Table 1: Techniques used in MPCitH-based signatures for RSD and MinRank.

In this work, we explore modelings for the rank syndrome decoding problem and the MinRank problems to identify the best option with the new VOLEitH and TCitH techniques. We show that the shortest signatures with RSD and MinRank are obtained thanks to the *dual support decomposition* modeling, which consists in finding a basis $(e_1, \dots, e_r)$ and coefficients $c_{1,1}, \dots, c_{n,r}$ such that

$$y = Hx \quad \text{and} \quad \forall i,\ x_i = \sum_{j=1}^{r} c_{i,r} \cdot e_j.$$

While this modeling is quite natural for the rank syndrome decoding problem, it requires to work in a dual version of the MinRank problem: we need to consider the syndrome decoding problem for matrix codes, while the MinRank problem is the message decoding problem for such codes. Working in the dual has the advantage to remove the encoded message from the witness of the code-based problem, leading to a shorter witness. With the dual support decomposition modeling, the witness size (and thus the signature size) is independent of the code dimension. This enables us to optimize the parameters by taking codes of larger dimensions.

We then apply the TCitH and VOLEitH frameworks on the optimal modeling, yielding new signature schemes with smaller sizes as summarized in Table 2. We also put the signature sizes of the NIST candidates based on the same security assumptions (namely RYDE, MIRA and MiRitH) in the column

"MPCitH" and their signature sizes when performing a straightforward application of VOLEitH and TCitH. We observe that the difference in signature sizes between VOLEitH and TCitH tends to disappear while increasing the parameter N, i.e., the number of leaves in GGM seed trees used for the commitment (a.k.a. the number of parties in standard MPCitH schemes). Since these two frameworks are faster than previous MPCitH schemes, it becomes natural to consider larger values of N. We obtain signature sizes down to 3.8 kB for TCitH with $N = 512$ leaves, and down to respectively 3.5 and 3.6 kB for VOLEitH and TCitH with $N = 2048$ leaves (more details are given in tables 13 and 15). The ranges of sizes reported in Table 2 correspond to a parameter N ranging between 32 and 2048. Very recently, new generic optimizations for MPCitH-based signatures have been proposed in [11]. We can apply these optimisations to our new signature schemes, enabling us to save an additional few hundred bytes. The obtained sizes are reported in Table 2 with the label "optimized".

Security Assumption	Scheme	MPCitH	VOLEitH	TCitH
Rank SD	RYDE (q-pol)	5 956 B	4 133–4 720 B	4 274–5 281 B
	Our scheme (DSD)	-	3 502–3 880 B	3 636–4 357 B
	Our scheme (DSD), optimized	-	2 912–3 592 B	3 035–3 619 B
MinRank	MIRA (q-pol)	5 640 B	4 170–4 770 B	4 314–5 340 B
	MiRitH-Ia (KS)	5 665 B	3 762–4 226 B	3 873–4 694 B
	MiRitH-Ib (KS)	6 298 B	4 110–4 690 B	4 250–5 245 B
	Our scheme (DSD)	-	3 462–3 826 B	3 548–4 219 B
	Our scheme (DSD), optimized	-	2 875–3 538 B	2 994–3 561 B

Table 2: Comparison of our schemes based on dual support decomposition (DSD) with the NIST candidates based on the same security assumptions. The sizes in the column "MPCitH" are given when using seed trees with 256 leaves, while the size range in columns "VOLEitH" and "TCitH" are given when using seed trees with between 256 and 2048 leaves.

Paper organization. The paper is organized as follows: In Section 2, we introduce the necessary background on the rank metric and sharing schemes. We present the existing attacks against RSD and MinRank in Section 3. We explore the possible modelings for rank-based cryptography in Section 4. We recall the TCitH and VOLEitH frameworks in Section 5 and we apply these frameworks to the dual support decomposition modeling to obtain new signature schemes in Section 6.

2 Preliminaries

2.1 Notations

We denote by $\mathbb{F}_q$ the finite field of size q. The set of vectors with n coordinates in $\mathbb{F}_q$ is referred as $\mathbb{F}_q^n$, the set of matrices with m rows and n columns in $\mathbb{F}_q$ is referred as $\mathbb{F}_q^{m\times n}$. We use lowercase bold letters to represent vectors and uppercase bold letters for matrices ($\boldsymbol{E} \in \mathbb{F}_q^{m\times n}$, $\boldsymbol{x} \in \mathbb{F}_q^k$, $x \in \mathbb{F}_q$). The subset of integers from 1 to n is represented with $[1, n]$. If S is a set, we write $x \xleftarrow{\$} S$ the uniform sampling of a random element x in S. We note the $\mathbb{F}_q$-linear subspace of $\mathbb{F}_{q^m}$ generated by $(x_1, \ldots, x_n) \in \mathbb{F}_{q^m}^n$ as $\langle x_1, \ldots, x_n\rangle$. Let us define the gaussian coefficient $\begin{bmatrix} m \\ r \end{bmatrix}_q = \prod_{i=0}^{r-1} \frac{q^m - q^i}{q^r - q^i} \approx q^{r(m-r)}$, it corresponds to the number of different dimension-r $\mathbb{F}_q$-linear subspaces of $\mathbb{F}_{q^m}$.

2.2 Secret Sharing

A threshold secret sharing scheme is a method to share a value v into a sharing $[\![v]\!] := ([\![v]\!]_1, \ldots, [\![v]\!]_N)$ such that v can be reconstructed from any ℓ+1 shares while no information is revealed on the secret from the knowledge of ℓ shares. We note by $[\![x]\!]_i$ the i^{th} share of $[\![x]\!]$ (*i.e.* the share of the i^{th} party). We can also note $[\![x]\!]_I$ where I is a set of indices, to denote all the shares of the parties in the set I.

Let us define Shamir's secret sharing scheme [36], since the frameworks we will consider rely on it. Let ℓ and N two integers such that $1 \leq \ell \leq N$. Let $e, \omega_1, \ldots, \omega_N$ be $N + 1$ distinct elements of $\mathbb{F} \cup \{\infty\}$. To share a value $v \in \mathbb{F}$ using Shamir's secret sharing scheme, one should

1. sample ℓ randoms values $r_1, \ldots, r_\ell$ of $\mathbb{F}$;
2. compute the polynomial P by interpolation such that
$$P(e) = v \quad \text{and} \quad \forall i \in [1, \ell], P(\omega_i) = r_i;$$
3. build the N shares $[\![v]\!]_1, \ldots, [\![v]\!]_N$ as
$$\forall i \in [1, N],\ [\![v]\!]_i := P(\omega_i).$$

To recover the secret value from $\ell + 1$ shares, we re-compute the polynomial P by interpolation and we just deduce $P(e)$. Let us stress that $P(\infty)$ refers to the leading coefficient of the polynomial P. The most classical choice is to set e to zero but we may consider alternative choices depending on the context (and in particular $e = \infty$).

We define the *degree* of a Shamir's secret sharing as the degree of the underlying polynomial. A sharing generated using the above process is of degree ℓ. The sum of a d_1-degree sharing and a d_2-degree sharing is of degree $\max(d_1, d_2)$, while the multiplication is of degree $d_1 + d_2$.

2.3 Rank Metric and Hard Problems for Cryptography

We will first recall some background on the Rank Metric, and we will then define hard problems we will use (RSD and MinRank).

Definition 1 (Rank Metric over $\mathbb{F}_{q^m}^n$) *Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{F}_{q^m}^n$, and $\mathcal{B} = (b_1, \ldots, b_m) \in \mathbb{F}_{q^m}^m$ an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^m}$. Each coordinate x_j can be associated with a vector $(x_{j,1}, \ldots, x_{j,m}) \in \mathbb{F}_q^m$ such that $x_j = \sum_{i=1}^m x_{j,i} b_i$. Let us define the following notations:*

- $\boldsymbol{M_x} = (x_{i,j})_{(i,j)\in[1,m]\times[1,n]}$ *is the* matrix associated to the vector $\boldsymbol{x}$*;*
- *the* rank weight *is defined as:* $w_R(\boldsymbol{x}) = \mathsf{rank}(\boldsymbol{M_x})$*;*
- *the* distance *between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{F}_{q^m}^n$ is:* $d(x, y) = w_R(\boldsymbol{x} - \boldsymbol{y})$*;*
- *the* support *of a vector* $\mathrm{Supp}(\boldsymbol{x})$ *is the $\mathbb{F}_q$-linear subspace of $\mathbb{F}_{q^m}$ generated by its coordinates:* $\mathrm{Supp}(\boldsymbol{x}) = \langle x_1, \ldots, x_n \rangle$.

Definition 2 *A linear code $\mathcal{C}$ over $\mathbb{F}_{q^m}$ of dimension k and length n is a linear subspace of $\mathbb{F}_{q^m}^n$ of dimension k. The elements of $\mathcal{C}$ are called codewords. The code $\mathcal{C}$ can be represented in two ways:*

- *by a generator matrix $\boldsymbol{G}$, where $\mathcal{C} = \{\boldsymbol{mG}, \boldsymbol{m} \in \mathbb{F}_{q^m}^k\}$, or*
- *by a parity-check matrix $\boldsymbol{H} \in \mathbb{F}_{q^m}^{(n-k)\times n}$ where $\mathcal{C} = \{\boldsymbol{x} \in \mathbb{F}_{q^m}^n : \boldsymbol{H}\boldsymbol{x}^\top = \boldsymbol{0}^\top\}$*

We now continue by formally recalling the definition of the rank syndrome decoding (RSD) problem.

Definition 3 (RSD problem) *Let q, m, n, k and r be positive integers. Let $\boldsymbol{H} \xleftarrow{\$} \mathbb{F}_{q^m}^{(n-k)\times n}$ and $\boldsymbol{x} \xleftarrow{\$} \mathbb{F}_{q^m}^n$ such that $w_R(\boldsymbol{x}) = r$. Let $\boldsymbol{y}^\top = \boldsymbol{H}\boldsymbol{x}^\top$. Given $(\boldsymbol{H}, \boldsymbol{y})$, the computational $\mathsf{RSD}(q, m, n, k, r)$ problem asks to find a vector $\tilde{\boldsymbol{x}} \in \mathbb{F}_{q^m}^n$ such that $\boldsymbol{H}\tilde{\boldsymbol{x}}^\top = \boldsymbol{y}^\top$ and $w_R(\tilde{\boldsymbol{x}}) = r$.*

We now introduce a variant of the above problem, the $\mathsf{RSD_s}$ problem and later argue that it is as hard as the standard RSD problem.

Definition 4 ($\mathsf{RSD_s}$ problem) *Let q, m, n, k and r be positive integers. Let $\boldsymbol{H} \xleftarrow{\$} \mathbb{F}_{q^m}^{(n-k)\times n}$ and $\boldsymbol{x} = (x_i) \xleftarrow{\$} \mathbb{F}_{q^m}^n$ such that $w_R(\boldsymbol{x}) = r$, $x_1 = 1 \in \mathbb{F}_{q^m}$ and $\langle x_1, \ldots, x_r \rangle_{\mathbb{F}_q} = \mathrm{Supp}(\boldsymbol{x})$. Let $\boldsymbol{y}^\top = \boldsymbol{H}\boldsymbol{x}^\top$. Given $(\boldsymbol{H}, \boldsymbol{y})$, the computational $\mathsf{RSD}_s(q, m, n, k, r)$ problem asks to find a vector $\tilde{\boldsymbol{x}} \in \mathbb{F}_{q^m}^n$ such that $\boldsymbol{H}\tilde{\boldsymbol{x}}^\top = \boldsymbol{y}^\top$ and $w_R(\tilde{\boldsymbol{x}}) = r$.*

The last problem we will rely on is the well-known MinRank problem:

Definition 5 (MinRank problem) *Let q, m, n, k and r be positive integers. Let $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_k, \boldsymbol{E} \in \mathbb{F}_q^{m \times n}$ and $\boldsymbol{x} := (x_1, \ldots, x_k) \in \mathbb{F}_q^k$ be uniformly sampled such that*

$$\mathsf{rank}(\boldsymbol{E}) \leq r \quad \textit{with} \quad \boldsymbol{M} := \boldsymbol{E} - \sum_{i=1}^{k} x_i \boldsymbol{M}_i.$$

Given $\boldsymbol{M}, \boldsymbol{M}_1 \ldots, \boldsymbol{M}_k$, the computational $\mathsf{MinRank}(q, m, n, k, r)$ problem asks to retrieve the vector $\boldsymbol{x}$.

The last notion to recall is the Gilbert-Varshamov bound for the rank metric and for MinRank. This bound in rank metric has been introduced in [30]. It can be seen as the probable minimum weight of a random code.

Definition 6 (Rank Gilbert-Varshamov Bound) *Let S_r be the number of elements of the sphere in $\mathbb{F}_{q^m}^n$ of radius r centered in 0, i.e, the number of elements in $\mathbb{F}_{q^m}^n$ of weight exactly r. We have $S_0 = 1$, and for $r \geq 1$,*

$$S_r = \prod_{j=0}^{r-1} \frac{(q^n - q^j)(q^m - q^j)}{q^r - q^j}.$$

Let $B_r := \sum_{i=0}^{r} S_r$ be the number of elements of the ball in $\mathbb{F}_{q^m}^n$ of radius r centered in 0. The Rank Gilbert-Varshamov (RGV) bound for an $[n, k]$ linear code over $\mathbb{F}_{q^m}$ is the smallest integer r such that

$$q^{m(n-k)} \leq B_r$$

Using the approximation $B_r \approx q^{(m+n-r)r}$, one can say the RGV bound is the smallest r such that $m(n-k) \leq (m-r)r + nr$. We call this value d_{RGV}. The same bound exists for matrix codes (i.e, for MinRank) as they are simply $\mathbb{F}_q$-linear codes. Courtois described this bound in [16, Section 24.2], and it can also be derived from the one above easily (consider a $[m \times n, k]$ linear code over $\mathbb{F}_q$ instead of $[n, k]$ linear over $\mathbb{F}_{q^m}$). This bound is also mentioned in attacks on MinRank ([9], [8] for instance). Concretely, this states that, for an instance of MinRank with parameters (q, m, n, k, r), we do not expect to obtain more than one solution if r is chosen such that $k + 1 \leq (m-r)(n-r)$.

Complexity of attacks for parameters on the GV bound. For RSD, the parameter r is taken as $d_{\mathsf{RGV}} - 1$, i.e, the highest r such that $(m-r)r + nr < m(n-k)$. With this parameter, if $\boldsymbol{H}$ and $\boldsymbol{y}$ were to be randomly sampled, one would expect to have a solution with probability $q^{(m+n-r)r - m(n-k)}$. Since $\boldsymbol{y}$ is set so there is a solution and since we are below RGV, it is not expected to have an other solution. For MinRank, we take parameters on the RGV bound, with $k+1 = (m-r)(n-r)$. For $k+1$ matrices randomly sampled $(\boldsymbol{M}, \boldsymbol{M}_1, \ldots, \boldsymbol{M}_k)$, the probability to have a solution to the MinRank instance is $q^{(m+n-r)r - (mn-k)}$. Since $\boldsymbol{M}$ is set so that there is a solution and since we are on GV, it is not expected to have an other solution for the instance. Let us now explain why in

addition to having only one solution, it is important to take parameters according to these bounds. Since the combinatorial attacks from [34] for RSD and [26] for MinRank, very few improvements have been made in the complexity. For MinRank, the kernel attack is still the best combinatorial attack, and for RSD, the exponential part of the complexities is still quadratic and has known almost no improvement over 20 years (with the exception of [6], which slightly improved the complexity). Regarding the algebraic attacks, introduced in [7] and improved in [10] and [8], they managed to greatly reduce the complexity for the RQC and LRPC schemes. However, this came from the fact that these parameters were not on RGV. The attacked parameters were in $\mathcal{O}\left(\sqrt{n-k}\right)$, which made them easier to attack, whereas we will consider parameters around the RGV bound, in $\mathcal{O}(n)$. In practice, for parameters taken at the RGV bound, or just below, the algebraic attacks have roughly the same complexity as the combinatorial ones ([8]). Overall, this means that, for parameters taken on the Rank Gilbert-Varshamov bound, the attacks have known no significant amelioration since over 20 years.

3 Security and Parameters for RSD and MinRank

We give here the well known reduction from RSD to $\mathsf{RSD}_{\mathrm{s}}$, and then the attacks considered against RSD and MinRank, which we will use in order to establish parameters for the signature schemes. We will also use these attacks in order to establish parameters to compare the different modelings in Section 4.

3.1 Security of the Rank Syndrome Decoding Problem

We deal here with the RSD problem, first by explaining the relation between RSD and $\mathsf{RSD}_{\mathrm{s}}$, and then the attacks on RSD.

3.1.1 Security Reduction

The $\mathsf{RSD}_{\mathrm{s}}$ problem was most notably used in the RQC scheme in order to optimize it [31]. In the following, we show that the $\mathsf{RSD}_{\mathrm{s}}$ problem is as hard as the standard RSD problem. More precisely, we show that any RSD instance can be solved by an $\mathsf{RSD}_{\mathrm{s}}$ solver. This is the same reduction as in [34], [6], [7], and others, used to specialize some variables. We exhibit below the reduction which has not formally been described in previous works (as part of the folklore of rank-based cryptography).

Proposition 1 *Let q, m, n, k, r be positive integers such that $n > k$. Let $\mathcal{A}_s$ be an algorithm which solves a $(q, m, n, k+1, r)$-instance of the RSD_s problem in time t with success probability ε_s. Then there exists an algorithm $\mathcal{A}$ which solves a (q, m, n, k, r)-instance of the RSD problem in time t with probability ε, where*

$$\varepsilon \geq \left(\prod_{i=0}^{r-1} \frac{q^n - q^{n-r+i}}{q^n - q^i} \right) \cdot \varepsilon_s$$

under the assumption that the code $\mathcal{C}$ associated to the parity-check matrix $\boldsymbol{H}$ of the RSD *instance contains no words of weight r.*

Proof. To prove the theorem, we build below an algorithm $\mathcal{A}$ to solve the RSD problem of parameters (q, m, n, k, r) using an algorithm $\mathcal{A}_{\text{s}}$ which solves the RSD_{s} problem with parameters $(q, m, n, k+1, r)$, assuming that the code that corresponds to the input instance does not contain words of weight r.

Algorithm $\mathcal{A}$ (on input an RSD instance $(\boldsymbol{H}, \boldsymbol{y})$):

1. Sample an invertible matrix $\boldsymbol{U} \in \mathbb{F}_q^{n \times n}$.
2. Compute $\hat{\boldsymbol{H}}^\top \in \mathbb{F}_{q^m}^{(n-k)\times n}$ as $\boldsymbol{U}\boldsymbol{H}^\top$.
3. Find $\boldsymbol{z}$ such that $\boldsymbol{y} = \boldsymbol{z}\hat{\boldsymbol{H}}^\top$.
4. Build $\hat{\boldsymbol{H}}' \in \mathbb{F}_{q^m}^{(n-k-1)\times n}$ as the parity check matrix of $\mathcal{C} + \langle \boldsymbol{z} \rangle$, where $\mathcal{C}$ is the linear code which has $\hat{\boldsymbol{H}}$ as parity-check matrix.
5. Run $\mathcal{A}_{\text{s}}$ on input $(\hat{\boldsymbol{H}}', \boldsymbol{0})$ to get $\hat{\boldsymbol{x}}$.
6. If $\hat{\boldsymbol{x}} = \perp$, return $\perp$.
7. Compute $\alpha \in \mathbb{F}_{q^m}$ such that $\hat{\boldsymbol{x}}\hat{\boldsymbol{H}}^\top = \alpha \cdot \boldsymbol{y}$.
8. Compute $\hat{\hat{\boldsymbol{x}}}$ as $\alpha^{-1} \cdot \hat{\boldsymbol{x}} \cdot \boldsymbol{U}$.
9. Return $\hat{\hat{\boldsymbol{x}}}$.

By definition, we know that the RSD instance $(\boldsymbol{H}, \boldsymbol{y})$ has a solution, meaning that there exists a vector $\boldsymbol{x}$ such that $\boldsymbol{y} = \boldsymbol{x}\boldsymbol{H}^\top$ and $w_R\big(\boldsymbol{x}\big) = r$. First, we define $\boldsymbol{x}'$ as $\boldsymbol{x}\boldsymbol{U}^{-1}$. The probability that $\boldsymbol{x}'$ has its r first coordinates which are full rank (under the randomness of $\boldsymbol{U}$) is

$$\varepsilon_1 := \frac{\prod_{i=0}^{r-1}\left(q^n - q^{n-r+i}\right)\prod_{j=r}^{n}(q^n - q^j)}{\#\{\text{invertible matrices of } \mathbb{F}_q^{n\times n}\}} = \prod_{i=0}^{r-1}\frac{q^n - q^{n-r+i}}{q^n - q^i}.$$

We now detail how we obtain this probability. Let $\mathsf{Ker}(\boldsymbol{x})$ be the right kernel of x, *i.e.* $\mathsf{Ker}(\boldsymbol{x}) := \{\boldsymbol{v} \in \mathbb{F}_q^n : \boldsymbol{x}\boldsymbol{v}^\top = 0\}$. It is a $\mathbb{F}_q$-linear subspace of dimension $n-r$ of $\mathbb{F}_q^n$. To obtain $\boldsymbol{x}'$ where the first r coordinates are of rank r, $\boldsymbol{U}^{-1}$ must be as follows (we write the i-th column of $\boldsymbol{U}^{-1}$ as $\boldsymbol{u}_i$):

- $\boldsymbol{u}_1 \notin \mathsf{Ker}(\boldsymbol{x})$;
- $\boldsymbol{u}_2 \notin (\mathsf{Ker}(\boldsymbol{x}) + \langle \boldsymbol{u}_1 \rangle)$;
- More generally, $\boldsymbol{u}_i \notin (\mathsf{Ker}(\boldsymbol{x}) + \langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{i-1} \rangle)$.

Let us count the number of successful $\boldsymbol{U}^{-1}$: there are $q^n - q^{n-r}$ choices for $\boldsymbol{u}_1$, $q^n - q^{n-r+1}$ choices for $\boldsymbol{u}_2$, and more generally $q^n - q^{n-r+i}$ choices for $\boldsymbol{u}_i$. In total, there are $\prod_{i=0}^{r-1}\left(q^n - q^{n-r+i}\right)$ choices for the first r columns of $\boldsymbol{U}^{-1}$. The $n-r$ last ones need to be such that $\boldsymbol{U}^{-1}$ is of full rank. Because each additional column should not be included in the subspace spanned by the previous ones, there are $\prod_{j=r}^{n}(q^n - q^j)$ choices for them. By combining the two products, we obtain the probability ε_1.

We assume that the event in which the r first coordinates of $\boldsymbol{x'}$ are full rank occurs. Let us define $\boldsymbol{c} := \boldsymbol{x'} - \boldsymbol{z}$. We have that

$$\boldsymbol{c}\hat{\boldsymbol{H}}^\top = (\boldsymbol{x'} - \boldsymbol{z})\hat{\boldsymbol{H}}^\top = \boldsymbol{x}\boldsymbol{U}^{-1}\boldsymbol{U}\boldsymbol{H}^\top - \boldsymbol{z}\hat{\boldsymbol{H}}^\top = \boldsymbol{x}\boldsymbol{H}^\top - \boldsymbol{y} = \boldsymbol{0},$$

so $\boldsymbol{c}$ is a codeword of $\mathcal{C}$. By defining $\boldsymbol{x''} := (x'_1)^{-1} \cdot \boldsymbol{x'}$ (x'_1 is not zero because the r first coordinates of $\boldsymbol{x'}$ are full rank by assumption), we have that $\boldsymbol{x''} = (x'_1)^{-1} \cdot \boldsymbol{c} + (x'_1)^{-1} \cdot \boldsymbol{z}$ is a codeword of $\mathcal{C} + \langle \boldsymbol{z} \rangle$. Therefore $\boldsymbol{x''}\hat{\boldsymbol{H}}'$ is equal to $\boldsymbol{0}$. Moreover, the first coordinate of $\boldsymbol{x''}$ is equal to $(x'_1)^{-1} \cdot x'_1 = 1$ and the r first coordinates of $\boldsymbol{x''}$ are full rank (because those of $\boldsymbol{x'}$ are full rank). We thus have that $(\hat{\boldsymbol{H}}', \boldsymbol{0})$ is a RSD_s instance with probability ε_1.

Let us consider that $\mathcal{A}_\mathrm{s}$ outputs $\hat{\boldsymbol{x}}$ such that $\hat{\boldsymbol{x}} \neq \perp$. We have $\hat{\boldsymbol{x}}\hat{\boldsymbol{H}}'^T = \boldsymbol{0}$ and $w_R(\hat{\boldsymbol{x}}) = r$. Since $\hat{\boldsymbol{x}}$ belongs to $\mathcal{C} + \langle \boldsymbol{z} \rangle$ (because $\hat{\boldsymbol{x}}\hat{\boldsymbol{H}}'^T = \boldsymbol{0}$), $\hat{\boldsymbol{x}}$ can be written as

$$\hat{\boldsymbol{x}} := \gamma_1 \cdot \boldsymbol{c_1} + \ldots + \gamma_k \cdot \boldsymbol{c_k} + \alpha \cdot \boldsymbol{z}$$

for some $\gamma_1, \ldots, \gamma_k, \alpha \in \mathbb{F}_{q^m}$, where $(\boldsymbol{c_1}, \ldots, \boldsymbol{c_k})$ is a basis of $\mathcal{C}$. In that case, we have that

$$\begin{aligned}\hat{\boldsymbol{x}}\hat{\boldsymbol{H}}^T &= \gamma_1 \cdot \boldsymbol{c_1}\hat{\boldsymbol{H}}^T + \ldots + \gamma_k \cdot \boldsymbol{c_k}\hat{\boldsymbol{H}}^T + \alpha \cdot \boldsymbol{z}\hat{\boldsymbol{H}}^T \\ &= \boldsymbol{0} + \ldots + \boldsymbol{0} + \alpha \cdot \boldsymbol{y}\end{aligned}$$

If $\alpha = 0$, then there would be a codeword of weight r in the code $\mathcal{C}$. Since we assume this is not the case, we get that $\alpha \neq 0$ and so $\hat{\hat{\boldsymbol{x}}}$ is well-defined in Step 8. We thus obtain that

$$\hat{\hat{\boldsymbol{x}}}\boldsymbol{H}^\top = \alpha^{-1} \cdot \hat{\boldsymbol{x}}\boldsymbol{U}\boldsymbol{H}^\top = \alpha^{-1} \cdot \hat{\boldsymbol{x}}\hat{\boldsymbol{H}}^T = \boldsymbol{y}.$$

Moreover, since multiplying by an invertible matrix over $\mathbb{F}_q$ does not change the support, we have $\mathrm{Supp}\left(\hat{\hat{\boldsymbol{x}}}\right) = \alpha^{-1} \cdot \mathrm{Supp}(\hat{\boldsymbol{x}})$, implying that $w_R(\hat{\hat{\boldsymbol{x}}}) = w_R(\hat{\boldsymbol{x}}) = r$. The algorithm $\mathcal{A}$ outputs a valid RSD solution or $\perp$, and the probability that $\mathcal{A}$ does not output $\perp$ is lower bounded by

$$\begin{aligned}\varepsilon &:= \Pr[\mathcal{A}(\boldsymbol{H}, \boldsymbol{y}) \neq \perp] = \Pr[\mathcal{A}_\mathrm{s}(\hat{\boldsymbol{H}}', \boldsymbol{0}) \neq \perp] \\ &\geq \Pr[(\hat{\boldsymbol{H}}', \boldsymbol{0}) \text{ is a } \mathsf{RSD}_\mathrm{s} \text{ instance} \cap \mathcal{A}_\mathrm{s}(\hat{\boldsymbol{H}}', \boldsymbol{0}) \neq \perp] \\ &= \varepsilon_1 \cdot \Pr[\mathcal{A}_\mathrm{s}(\hat{\boldsymbol{H}}', \boldsymbol{0}) \neq \perp \mid (\hat{\boldsymbol{H}}', \boldsymbol{0}) \text{ is a } \mathsf{RSD}_\mathrm{s} \text{ instance}] \\ &= \varepsilon_1 \cdot \varepsilon_s \ .\end{aligned}$$

□

Remark 1. In practice, the value of ε_1 tends to 1 when q grows. For our considered parameters, with $q = 2$, its value is around 0.3. Moreover, one can get the average number of codewords of $\mathcal{C}$ of weight r to justify our assumption. Let $\mathcal{S}_r = \prod_{i=0}^{r-1} \frac{(q^n - q^i)(q^m - q^i)}{q^r - q^i}$ be the number of words in $\mathbb{F}_{q^m}^n$ of weight exactly r. Then, on average, there are $\frac{S_r}{q^{m(n-k)}}$ words of rank r in the code. When below RGV, this makes the probability that a random code $\mathcal{C}$ contains no codeword

of weight r close to 1.

Remark 2. The best known attacks on RSD use the reduction to $\mathsf{RSD}_{\mathrm{s}}$ in order to solve the instance ([34], [6], [10], [8]), meaning that in practice we consider the best attacks on RSD to evaluate the security of $\mathsf{RSD}_{\mathrm{s}}$.

3.1.2 Complexities of the Best Known Attacks

Ourivski-Johansson. The attack first uses the above reduction, and exhibits a system of quadratic equations. The aim of this attack is to linearize the equations, which is done after fixing a number of values (see [34]). This algorithm solves the problem in

$$\mathcal{O}\left((rm)^{\omega} q^{(r-1)(k+1)}\right).$$

AGHT: improved GRS. The idea of the GRS attack, from [24], is to sample a subspace E' of dimension $r' \geq r$, and hope that it includes $E = \mathrm{Supp}(\boldsymbol{x})$. Then, one solves a linear system, when $r' \leq \lfloor \frac{(n-k)m}{n} \rfloor$.
The improvement, from [6], uses the reduction above, where the success condition is if E' contains αE for any $\alpha \in \mathbb{F}_{q^m}^*$. The resulting complexity is

$$\mathcal{O}\left((n-k)^{\omega} m^{\omega} q^{r\lfloor \frac{(k+1)m}{n} \rfloor - m}\right).$$

Algebraic attacks. There are two main algebraic attacks for RSD. The first one is the *MaxMinors* modeling [7]. It consists in solving the minors of size r of the matrix $\boldsymbol{C}\boldsymbol{H}^{\top}$, where $\boldsymbol{x} = \boldsymbol{s}\boldsymbol{C}$ for $\boldsymbol{s} \in \mathbb{F}_{q^m}^r$ and $\boldsymbol{C} \in \mathbb{F}_q^{r \times n}$. The system is then solved, and yields a complexity of

$$\mathcal{O}\left(q^{ar} \binom{n-a-p}{r}^{\omega}\right)$$

where a is the parameter of the *Hybrid method* (see [8]), and p is the number of positions punctured.

The second algebraic attack [9] [8] is the *Support Minors*. In this modeling, one constructs a vector $\boldsymbol{v} = -\boldsymbol{m}\boldsymbol{G} + \boldsymbol{x}$ where $-\boldsymbol{m}\boldsymbol{G} \in \mathcal{C}$, and write it as a product $\boldsymbol{s}\boldsymbol{C}$ where $\boldsymbol{s} \in \mathbb{F}_{q^m}^r, \boldsymbol{C} \in \mathbb{F}_q^{r \times n}$. The equations come from $\begin{pmatrix} \boldsymbol{r}_i \\ \boldsymbol{C} \end{pmatrix}$ where $\boldsymbol{r}_i$ is the i-th row of $\boldsymbol{x} - \boldsymbol{m}\boldsymbol{G}$. When applying this modeling, by computing

$$N = \sum_{i=1}^{k} \binom{n-a-i}{r}\binom{k-a+b-1-i}{b-1} - \binom{n-k-1}{r}\binom{k-a+b-1}{b}$$
$$- (m-1)\sum_{i=1}^{b}(-1)^{i+1}\binom{k-a+b-i-1}{b-i}\binom{n-k-1}{r+i}$$

and

$$M = \binom{k-a+b-1}{b}\left(\binom{n-a}{r} - m\binom{n-k-1}{r}\right),$$

as soon as $N \geq M-1$, we obtain the complexity of

$$\mathcal{O}\left(q^{ar} m^2 N M^{\omega-1}\right)$$

where, as before, a is the parameter of the hybrid attack, and the parameter b minimizes the above quantities.

According to these attacks, we give in Table 3 the parameters which we will use for our RSD_s instances.

NIST Security level	q	m	n	k	r
I	2	53	53	45	4
III	2	79	75	67	4
V	2	97	95	87	4

Table 3: Choice of parameters for RSD_s

3.2 Security of the MinRank Problem

We now recall the attacks on MinRank. In this case, the *Hybrid method* works well for both combinatorial and algebraic attacks. In particular, for a cost of q^{ar} repetitions, it is possible to reduce a (q, m, n, k, r) MinRank instance into a $(q, m, n-a, k-am, r)$ one.

Kernel attack. The attack, introduced by Goubin and Courtois [26], consists in sampling randomly a matrix vectors of $\mathbb{F}_q^n$, and hoping they are in the right kernel of the matrix $\boldsymbol{E} = \boldsymbol{M} + \sum_{i=1}^{k} x_i \boldsymbol{M}_i$. Since the kernel is of dimension $n-r$, the probability to sample a vector in the kernel is $\frac{1}{q^r}$. When sampling l vectors and multiplying $\boldsymbol{E}$ by these vectors on the right, we obtain k unknowns and $m \cdot l$ equations. We are able to solve it when $l = \lceil \frac{k}{m} \rceil$. The overall complexity is thus

$$\mathcal{O}\left(k^\omega q^{r\lceil \frac{k}{m} \rceil}\right).$$

Algebraic attacks. As for RSD, the first algebraic attack is *MaxMinors* [9]. The modeling is simply to write $\boldsymbol{E} = \boldsymbol{M} + \sum_{i=1}^{k} x_i \boldsymbol{M}_i$, and to compute its minors of rank $r+1$. The complexity of the attack depends on the Hilbert series

$$HS(t)\left[(1-t)^{(m-r)(n-r)-(k+1)} \frac{\det(A(t))}{t^{\binom{r}{2}}}\right],$$
$$\text{with } A(t) = \left(\sum_{\ell=0}^{\max(m-i,n-j)} \binom{m-i}{\ell}\binom{n-j}{\ell} t^\ell\right)_{1\leq i\leq r, 1\leq j\leq r}$$

The total complexity is

$$\mathcal{O}\left(\binom{k+D}{D}^{\omega}\right)$$

where D is the degree of regularity of the system.

The second modeling, the *Support Minors* modeling [9] [8], allows to obtain equations by setting $\boldsymbol{E} = \boldsymbol{SC}$ where $\boldsymbol{S} \in \mathbb{F}_q^{m \times r}, \boldsymbol{C} \in \mathbb{F}_q^{r \times n}$, setting $\boldsymbol{r}_i$ the i-th row of $\boldsymbol{M} + \sum_{i=1}^{k}$, and computing the maximal minors of $\begin{pmatrix} \boldsymbol{r}_i \\ \boldsymbol{C} \end{pmatrix}$ The final complexity is

$$\mathcal{O}\left(NM^{\omega-1}\right)$$

where

$$N = \sum_{i=1}^{b} (-1)^{i+1} \binom{n}{r+i} \binom{k+b-1-i}{b-i} \binom{m+i-1}{i}$$

and

$$M = \binom{k+b-1}{b} \binom{n}{r},$$

with $N \geq M - 1$ and $b \leq \min(q-1, r+1)$.
When $q = 2$, the complexity is slightly different, with

$$N = \sum_{j=1}^{b} \sum_{i=1}^{j} (-1)^{i+1} \binom{n}{r+i} \binom{k}{j-i} \binom{m+i-1}{i}$$

and

$$M = \sum_{j=1}^{b} \binom{k}{j} \binom{n}{r},$$

with $b < r + 2$.
According to these attacks, we give in Table 4 the parameters which we will use for our MinRank instances.

NIST Security level	q	m	n	k	r
I	2	43	43	1520	4
III	2	60	60	3135	4
V	2	75	75	5040	4

Table 4: Choice of parameters for MinRank

4 MPCitH Modeling for $\mathsf{RSD_s}$ and MinRank

A zero-knowledge proof constructed using the MPCitH paradigm is composed of two parts, a "symmetric part" made of GGM trees (or Merkle trees) and an "arithmetic part" composed of the open party views and broadcast shares of the MPC protocol. While for a given security level the symmetric part is of rather fixed size (e.g., around 2kB for GGM trees and 4kB for Merkle trees at a 128-bit security level), the arithmetic part depends on the modeling (i.e., the way the problem instance is verified) and the associated MPC protocol. For the recent TCitH and VOLEitH techniques, the arithmetic part is actually mainly impacted by the size of the witness, which favors modelings with low-size witnesses.

In this section, we study different modelings for RSD and MinRank with respect to the witness size criterion. For the RSD problem, we recall the permuted secret, q-polynomial and Kipnis-Shamir modelings. We propose an other modeling, named *dual support decomposition*, which can be seen as an improvement of the rank decomposition from [19]. We also slightly improve all the modelings by relying on the $\mathsf{RSD_s}$ variant. For the MinRank problem, we recall the q-polynomial and Kipnis-Shamir modelings and propose an adaptation of the dual support decomposition modeling for MinRank.

4.1 Modelings for the $\mathsf{RSD_s}$ Problem

Permuted Secret. We start by recalling the permuted secret technique, which was used for RSD in [15]. The idea of this technique consists in revealing a "permuted" and a "masked" versions of the secret: let us denote σ an isometry in the rank metric (such a isometry consists of multiplying the secret matrix by a invertible matrices on both sides) and $\boldsymbol{u}$ a vector of the left kernel of $\boldsymbol{H}$, one reveals $\boldsymbol{v} := \sigma(\boldsymbol{x})$ and $\tilde{\boldsymbol{x}} := \boldsymbol{x} + \boldsymbol{u}$ and the goal is to find such values σ and $\boldsymbol{u}$. More precisely, the rank syndrome decoding problem consists, from two vectors $\boldsymbol{v}, \tilde{\boldsymbol{x}} \in \mathbb{F}_{q^m}^n$ satisfying $w_R(\boldsymbol{v}) = r$ and $\boldsymbol{H}\tilde{\boldsymbol{x}}^\top = \boldsymbol{y}^\top$, in finding an isometry σ and a vector $\boldsymbol{u} \in \mathbb{F}_{q^m}^n$ such that

$$\begin{cases} \boldsymbol{H}\boldsymbol{u}^\top = \boldsymbol{0}^\top, \\ \sigma(\tilde{\boldsymbol{x}}) = \boldsymbol{v} + \sigma(\boldsymbol{u}). \end{cases}$$

Indeed, if we get both σ and $\boldsymbol{u}$, we can easily restore the initial secret as $\boldsymbol{x} := \tilde{\boldsymbol{x}} - \boldsymbol{u}$: we have $\boldsymbol{H}\boldsymbol{x}^\top = \boldsymbol{y}^\top - \boldsymbol{0}^\top$ and $w_R(\boldsymbol{x}) = w_R(\sigma(\boldsymbol{x})) = w_R(\sigma(\tilde{\boldsymbol{x}}) - \sigma(\boldsymbol{u})) = w_R(\boldsymbol{v}) = r$.

Unfortunately, this modeling is not compatible with the recent MPCitH techniques as TCitH or VOLEitH. Such techniques requires at least additive sharings over a commutative group (or for the more recent techniques, Shamir's secret sharing over a ring). However, the isometry σ lives in a non-commutative group, so it requires to rely on a special form of MPCitH named the shared-permutation framework [15, 20].

q-Polynomial. The q-polynomial technique proposed in [19] to check the rank metric constitutes an improvement compared to a number of previous methods. Let us first recall the definition of a q-polynomial.

Definition 7 (q-polynomial) *A q-polynomial of q-degree r is a polynomial in $\mathbb{F}_{q^m}[X]$ of the form:*

$$P(X) = X^{q^r} + \sum_{i=0}^{r-1} p_i \cdot X^{q^i} \qquad \textit{with } p_i \in \mathbb{F}_{q^m}.$$

The roots of a q-polynomial of q-degree r form a linear subspace of $\mathbb{F}_{q^m}$ of dimension at most r. Moreover, for each linear subspace of $\mathbb{F}_{q^m}$ of dimension at most r, there exists a unique monic q-polynomial of q-degree r annihilating all the elements of the subspace. Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{F}_{q^m}^n$ of rank $w_R(\boldsymbol{x}) = r$ and let $P_{\boldsymbol{x}}(X)$ the monic q-polynomial annihilating $\mathrm{Supp}(\boldsymbol{x})$. In this modeling, the rank syndrome decoding problem consists in finding a vector $\boldsymbol{x} \in \mathbb{F}_{q^m}^n$ and a q-polynomial $P_{\boldsymbol{x}} \in \mathbb{F}_{q^m}[X]$ of q-degree r such that

$$\boldsymbol{H}\boldsymbol{x}^\top = \boldsymbol{y}^\top \quad \text{and} \quad \forall i \ , P_{\boldsymbol{x}}(x_i) = 0.$$

Concretely, the MPC protocol based on the q-polynomial technique takes as input some shares of $\boldsymbol{x}$ and some shares of $P_{\boldsymbol{x}}(X)$. The protocol then checks that $P_{\boldsymbol{x}}(x_i) = 0$ for all $i \in [1, n]$. Using the standard representation $\boldsymbol{H} = \left(\boldsymbol{I}_{n-k} || \boldsymbol{H}'\right)$, one can send only the right part of $\boldsymbol{x}$ of size k, denoted as $\boldsymbol{x}_B$. Furthermore, it is possible to send one less coefficient of the polynomial $P_{\boldsymbol{x}}$, since $1 \in \mathrm{Supp}(\boldsymbol{x})$ (see [4] for the optimization) and as a result the size of witness is (in bits):

$$(\underbrace{k \cdot m}_{\boldsymbol{x}_B} + \underbrace{(r-1) \cdot m}_{P_{\boldsymbol{x}}}) \cdot \log_2(q)$$

We give in Table 5 the $\mathsf{RSD_s}$ parameters that minimize the witness size of this modeling.

q	m	n	k	r	$(km + (r-1)m) \cdot \log_2(q)$
2	31	33	15	10	96 B

Table 5: Optimized parameters for $\mathsf{RSD_s}$ q-polynomials modeling.

This modeling based on q-polynomials currently leads to the shortest communications for RSD when considering *linear* multiparty computation, but it is not the best one when considering non-linear multiparty computation as in the new MPCitH frameworks.

Kipnis-Shamir. Historically, the Kipnis-Shamir modeling was introduced in the cryptanalysis of the MinRank problem [29]. We can use the same idea to

have a modeling of RSD. It consists in giving the right-kernel of the matrix of $\boldsymbol{x}$. We denote this matrix in $\mathbb{F}_q^{m\times n}$ by $\boldsymbol{M}_{\boldsymbol{x}}$. If $w_R(\boldsymbol{x}) = r$, then the right-kernel of $\boldsymbol{M}_{\boldsymbol{x}}$ is of dimension $n-r$ and can be represented by an $r \times (n-r)$ matrix.

In the $\mathsf{RSD}_{\mathrm{s}}$ case, the witness is composed of $\boldsymbol{x}$ and of the matrix $\boldsymbol{A} \in \mathbb{F}_q^{r\times(n-r)}$. The MPC protocol takes in input $\boldsymbol{K} = \begin{pmatrix} \boldsymbol{I}_{n-r} \\ \boldsymbol{A} \end{pmatrix}$, and then checks that $\boldsymbol{M}_{\boldsymbol{x}}\boldsymbol{K} = \boldsymbol{0}$. It is possible to send only $\boldsymbol{x}_B$, as previously with q-polynomials, and since 1 is in the support, the size of the witness is:

$$(\underbrace{k\cdot m}_{\boldsymbol{x}_B} + \underbrace{(r-1)\cdot(n-r)}_{\boldsymbol{A}})\cdot\log_2(q)\ .$$

Note that transmitting $\boldsymbol{A}$ costs $(r-1)\cdot(n-r)$ only since we know that 1 is in $\boldsymbol{x}$. This approach is slightly better than the q-polynomial technique in terms of witness size. We give in Table 6 the $\mathsf{RSD}_{\mathrm{s}}$ parameters that minimize the witness size of this modeling.

q	m	n	k	r	$((r-1)(n-r)+km)\cdot\log_2(q)$
2	31	33	15	10	86 B

Table 6: Optimized parameters for $\mathsf{RSD}_{\mathrm{s}}$ Matrix Rank Decomposition modeling

Dual Support Decomposition. Finally, we introduce an other modeling for $\mathsf{RSD}_{\mathrm{s}}$, using only the support and the coordinates. This can be seen as an improvement of the rank decomposition from [19]. To that end, one has as inputs:

- The support of $\boldsymbol{x}$, $\mathrm{Supp}(\boldsymbol{x}) = \langle 1, x_2, \ldots, x_r\rangle$;
- The coordinates of $\boldsymbol{x}$ in this basis, i.e, $\boldsymbol{C} \in \mathbb{F}_q^{r\times(n-r)}$ such that

$$(1, x_2, \ldots, x_r)\cdot\begin{pmatrix}\boldsymbol{I}_r & \boldsymbol{C}\end{pmatrix} = (1, x_2, \ldots, x_n) = \boldsymbol{x}$$

More precisely, in this modeling, the $\mathsf{RSD}_{\mathrm{s}}$ problem consists in finding $x_2, \ldots, x_r \in \mathbb{F}_{q^m}$ and $\boldsymbol{C} \in \mathbb{F}_q^{r\times(n-r)}$ such that

$$\boldsymbol{H}\boldsymbol{x}^\top = \boldsymbol{y}^\top \quad \text{where} \quad \boldsymbol{x} := (1, x_2, \ldots, x_r)\cdot\begin{pmatrix}\boldsymbol{I}_r & \boldsymbol{C}\end{pmatrix}.$$

Concretely, after computing $\boldsymbol{x} = (x_1, \ldots, x_r)\cdot\begin{pmatrix}\boldsymbol{I}_r & \boldsymbol{C}\end{pmatrix}$, one verifies that $\boldsymbol{H}\boldsymbol{x}^T$ is indeed equal to $\boldsymbol{y}^T$. Since 1 is in the support of $\boldsymbol{x}$, it is possible to transmit only $r-1$ elements for $\mathrm{Supp}(\boldsymbol{x})$, and we can have a gain on the matrix $\boldsymbol{C}$ as well since the r first coordinates are linearly independent. This results in an efficient protocol, where the inputs are of size

$$(\underbrace{(r-1)\cdot m}_{\mathrm{Supp}(\boldsymbol{x})} + \underbrace{r\cdot(n-r)}_{\boldsymbol{C}})\cdot\log_2(q)$$

We see here that the input size does not depend on k anymore, allowing us to take more efficient parameters. We give in Table 7 the $\mathsf{RSD}_{\mathrm{s}}$ parameters that minimize the witness size of this modeling.

q	m	n	k	r	$((r-1)m + r(n-r)) \cdot \log_2(q)$
2	53	53	45	4	45 B

Table 7: Optimized parameters for $\mathsf{RSD}_{\mathrm{s}}$ Support Decomposition modeling.

Global Comparison. Table 8 provides a global comparison of the different modelings in terms of witness size for the RSD problem. For each of the described modelings, we provide the size formula as well as the obtained concrete size for optimized parameters reaching a 128-bit security according to the attacks in Section 3.1.2.

Modeling	**Witness size**	**Size for $\lambda = 128$**
q-polynomial	$[\,km + (r-1)m\,] \cdot \log_2(q)$	93 B
Kipnis-Shamir	$[\,km + (r-1)(n-r)\,] \cdot \log_2(q)$	86 B
Dual Support decomposition	$[\,(r-1)m + r(n-r)\,] \cdot \log_2(q)$	45 B

Table 8: Witness size for different MPCitH modelings for the $\mathsf{RSD}_{\mathrm{s}}$ problem.

4.2 Modelings for the MinRank Problem

The MinRank problem is closely related to the RSD problem. The two problems indeed share a number of similarities as evidence of the algebraic attacks applying to both problems (see, e.g., [8, 10]). Quite naturally, most of the above modelings for RSD can be adapted for MinRank.

q-Polynomial. The q-polynomial technique of [19] can be also applied to MinRank: the witness is composed of the shares of $\boldsymbol{x} \in \mathbb{F}_q^k$ and the coefficients $\boldsymbol{\beta} \in \mathbb{F}_{q^m}^r$ of the q-polynomial associated to $\boldsymbol{E}$. The MPC protocol computes $\boldsymbol{E} = \boldsymbol{M} + \sum_{i=1}^{k} x_i \boldsymbol{M}_i$ and verifies that $P_{\boldsymbol{E}}(X) := \sum_{i=0}^{r-1} \beta_i X^{q^i} + X^{q^r}$ is the annihilator polynomial of $\boldsymbol{E}$. This verification relies on the isomorphism between $\mathbb{F}_{q^m}$ and $\mathbb{F}_q^m$, and associates each column of $\boldsymbol{E}$, denoted as $\boldsymbol{e}_i$, to an element of $\mathbb{F}_{q^m}$, e_i. The protocol hence simply checks that $P_{\boldsymbol{E}}(e_i) = 0$ for $i \in [1, n]$.

With this modeling, the size of the witness size is (in bits):

$$(\underbrace{k}_{\boldsymbol{x}} + \underbrace{r \cdot m}_{P_{\boldsymbol{E}}}) \cdot \log_2(q) \ .$$

q	m	n	k	r	$(rm+k)\cdot\log_2(q)$
16	15	15	78	6	76 B

Table 9: Optimized parameters for MinRank q-polynomial modeling.

Kipnis-Shamir. This is the modeling used in MiRitH [1], which is an improvement of MinRank-in-the-Head [2]. The goal of this modeling is to use the right kernel of $\boldsymbol{E}$ in order to prove its rank. Let $\boldsymbol{K} = \begin{bmatrix} \boldsymbol{I}_{(n-r)} \\ \boldsymbol{A} \end{bmatrix}$ a matrix of rank $n-r$ representing the right kernel of $\boldsymbol{E}$. The witness is composed of $\boldsymbol{x}$ and $\boldsymbol{A} \in \mathbb{F}_q^{r\times(n-r)}$. The protocol recomputes $\boldsymbol{E} = \boldsymbol{M} + \sum_{i=1}^{k} x_i \boldsymbol{M}_i$ and verifies that $\boldsymbol{E}\cdot\boldsymbol{K} = \boldsymbol{0}$. If the verification succeeds, one deduces that $\boldsymbol{E}$ is indeed of rank r since it has a kernel of rank $n-r$.

With this modeling, the witness is of size:

$$(\underbrace{k}_{\boldsymbol{x}} + \underbrace{r\cdot(n-r)}_{\boldsymbol{A}})\cdot\log_2(q)\ .$$

As for $\mathsf{RSD_s}$, the witness is smaller with this modeling than with the q-polynomials technique.

q	m	n	k	r	$(r(n-r)+k)\cdot\log_2(q)$
16	15	15	78	6	66 B

Table 10: Optimized parameters for MinRank Kipnis-Shamir modeling.

New Modeling for the MinRank Problem: Dual Support Decomposition. We introduce hereafter a new MPCitH modeling for the MinRank problem which achieves smaller witness sizes than the previous modelings.

By definition of the problem, we know that solving $\boldsymbol{E} = \boldsymbol{M} + \sum_{i=1}^{k} x_i \boldsymbol{M}_i$ with unknowns x_i is the same as solving the instance $\boldsymbol{M} = \boldsymbol{E} + \sum_{i=1}^{k} x'_i \boldsymbol{M}_i$ where each $x'_i = -x_i$. The goal is to try to get the notion of dual, in order to apply the same idea of modeling as for $\mathsf{RSD_s}$. First, one can define the map

$$\begin{array}{rccc} \rho: & \mathbb{F}_q^{m\times n} & \to & \mathbb{F}_q^{mn} \\ & \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix} & \mapsto & \big(a_{1,1},\dots,a_{1,n},\dots,a_{m,1},\dots,a_{m,n}\big) \end{array}$$

Let $\mathcal{C} = \langle \boldsymbol{M}_1,\dots,\boldsymbol{M}_k\rangle$. Then, one can consider $\boldsymbol{G} \in \mathbb{F}_q^{k\times mn}$ where the i-th line of $\boldsymbol{G}$, $\boldsymbol{G}_i = \rho\big(\boldsymbol{M}_i\big)$ for i from 1 to k. With such a construction, we see that $\boldsymbol{G}$

is the generator matrix of $\mathcal{C}$ since

$$\boldsymbol{G} = \begin{pmatrix} \rho(\boldsymbol{M}_1) \\ \vdots \\ \rho(\boldsymbol{M}_k) \end{pmatrix} .$$

This matrix $\boldsymbol{G}$ is a $k \times mn$ matrix, generating an $[mn, k]$ code. It follows that we can easily build $\mathcal{C}^{\perp}$, a $[mn, mn-k]$ code, using the usual inner product on vectors, with a generator matrix $\boldsymbol{H} \in \mathbb{F}_q^{(mn-k)\times mn}$ such that $\boldsymbol{G}\boldsymbol{H}^T = \boldsymbol{0}$. Then, it is easy to see that

$$\rho(\boldsymbol{E})\boldsymbol{H}^T = \rho(\boldsymbol{M})\boldsymbol{H}^T \tag{1}$$

as

$$\left(\sum_{i=1}^{k} x_i \rho\left(\boldsymbol{M}_i\right) \right) \cdot \boldsymbol{H}^T = \boldsymbol{0}$$

We thus obtain

$$\big(\rho(\boldsymbol{E}) - \rho(\boldsymbol{M})\big)\boldsymbol{H}^T = \boldsymbol{0} \tag{2}$$

Since we can compute $\rho(\boldsymbol{M})\boldsymbol{H}^T$ easily, all that is left to do is to prove that we know $\boldsymbol{E}$ of rank r verifying Equation (2).

As in the rank decomposition method from [19], one can view $\boldsymbol{E}$ as a product of two matrices, $\boldsymbol{E} = \boldsymbol{S}\boldsymbol{C}$, with $\boldsymbol{S} \in \mathbb{F}_q^{m\times r}$ and $\boldsymbol{C} \in \mathbb{F}_q^{r\times n}$. Furthermore, one can write without loss of generality $\boldsymbol{S}$ as $\begin{bmatrix} \boldsymbol{I}_r \\ \boldsymbol{S'} \end{bmatrix}$ for some matrix $\boldsymbol{S'} \in \mathbb{F}_q^{(m-r)\times r}$ (this is always possible up to a permutation of the lines). Then, one can simply set $\boldsymbol{E} = \begin{bmatrix} \boldsymbol{I}_r \\ \boldsymbol{S'} \end{bmatrix} \cdot \boldsymbol{C}$. Taking in inputs $\boldsymbol{C}$ and $\boldsymbol{S}$, one must simply verify that $\boldsymbol{E}$ verifies the equation above.

Overall, the inputs are of size

$$(\underbrace{r \cdot (m-r)}_{\boldsymbol{S}} + \underbrace{r \cdot n}_{\boldsymbol{C}}) \cdot \log_2(q)$$

considering we use the identity matrix in the support.
Exactly as for RSD, the size does not depend on k anymore, which allows a better selection of parameters.

q	m	n	k	r	$(r(m-r)+rn)\cdot\log_2(q)$
2	43	43	1520	4	41 B

Table 11: Optimized parameters for MinRank Dual Support Decomposition modeling for $\lambda = 128$.

Global comparison. Table 12 provides a global comparison of the different modelings in terms of witness size for the MinRank problem. For each of the described modelings, we provide the size formula as well as the obtained concrete size for optimized parameters reaching a 128-bit security for the attacks described in section 3.2.

Modeling	**Witness size**	**Size for $\lambda = 128$**
q-polynomial	$[\,k + rm\,] \cdot \log_2(q)$	76 B
Kipnis-Shamir	$[\,k + r(n-r)\,] \cdot \log_2(q)$	66 B
Dual support decomposition	$[\,r(m-r) + rn\,] \cdot \log_2(q)$	41 B

Table 12: Modeling for MinRank and resulting witness size in MPC protocols.

5 The TCitH and VOLEitH Frameworks

The MPCitH paradigm [27] is a versatile method introduced in 2007 to build zero-knowledge proof systems using techniques from secure multi-party computation (MPC). This paradigm has been drastically practically improved in recent years (see, e.g., [3, 18, 22, 28]) and is particularly efficient to build zero-knowledge proofs for small circuits such as those involved in (post-quantum) signature schemes. The MPCitH paradigm can be summarized as follows. The prover emulates "in his head" an ℓ-private MPC protocol with N parties and commits each party's view independently. The verifier then challenges the prover to reveal the views of a random subset of ℓ parties. By the privacy of the MPC protocol, nothing is revealed about the plain input, which implies the zero-knowledge property. On the other hand, a malicious prover needs to cheat for at least one party, which shall be discovered by the verifier with high probability, hence ensuring the soundness property.

In what follows, we describe two recently introduced MPCitH-based frameworks, namely the *VOLE-in-the-Head* (VOLEitH) framework from [12] and the *Threshold-Computation-in-the-Head* (TCitH) framework from [21, 22].

5.1 Threshold-Computation-in-the-Head Framework

The TCitH framework has been recently introduced in [21] as an extension of a previous work [22] published at Asiacrypt 2023. While almost all the former MPCitH-based proof system relied on additive sharings, the TCitH framework shows how using Shamir's secret sharings (instead of additives sharings) lead to faster schemes with shorter communication.

We refer the reader to [21, 22] for a detailed exposition of the TCitH framework which is only briefly abstracted here. In a nutshell, the TCitH framework relies on MPC protocols with broadcasting, randomness oracle and hint oracle

(as previous MPCitH schemes) but using Shamir's secret sharing unlock the use of non-linear multiparty computation (whereas previous MPCitH schemes are based on linear multiparty computation). More precisely, in the considered MPC protocols, one can compute a sharing $[\![a \cdot b]\!]$ of a product $a \cdot b$ from the sharings $[\![a]\!]$ and $[\![b]\!]$ of the operands by share-wise multiplication (for all i, $[\![a \cdot b]\!]_i \leftarrow [\![a]\!]_i \cdot [\![b]\!]_i$).

The TCitH framework comes with two variants depending on how one commits the input shares: either relying on GGM trees [25] or on Merkle trees [32]. In the present work, we focus on the GGM-tree variant which leads to shorter signature sizes for the considered statements. Moreover, we only consider 1-private Shamir's secret sharings, *i.e.* $\ell = 1$, which gives the best results in our context.

Given some degree-d polynomials $f_1, \ldots, f_m$ from $\mathbb{F}[X_1, \ldots, X_{|w|}]$, we want a zero-knowledge proof of knowledge of a witness w satisfying

$$\forall j \in [1, m],\ f_j(w) = 0.$$

We shall use the proof system TCitH-Π_{PC} described in [21, Section 5.2]. We recall the underlying MPC protocol Π_{PC} in Protocol 1. The sharing $[\![0]\!]$ used in Step 4 of the MPC protocol is a publicly-known degree-1 sharing of zero (for example, $[\![0]\!]_i = \omega_i$ when $e = 0$). This MPC protocol is ℓ-private and sound with false positive probability $\frac{1}{|\mathbb{F}|}$ (see [21, Lemma 2]). In practice, the MPC protocol is repeated ρ times in parallel to achieve a false positive probability of $\frac{1}{|\mathbb{F}|^\rho}$. The soundness error of TCitH-Π_{PC} (when $\ell = 1$) is

$$\epsilon = \frac{1}{|\mathbb{F}|^\rho} + \left(1 - \frac{1}{|\mathbb{F}|^\rho}\right) \cdot \frac{d}{N}\ .$$

To obtain a signature scheme, we first transform the above MPC protocol into a proof of knowledge (PoK) of soundness error ϵ by applying the TCitH transform. We then perform τ parallel repetitions of this PoK and apply the Fiat-Shamir transform [23]. To achieve a λ-bit security, we take the number ρ of MPC repetitions such that $\frac{1}{|\mathbb{F}|^\rho} \leq 2^{-\lambda}$ and the number τ of PoK repetitions such that $\left(\frac{d}{N}\right)^\tau \leq 2^{-\lambda}$.

The proof transcript (*i.e.* the signature) includes:

- The opened shares $[\![w]\!]_I$ of the witness $w \in \mathbb{F}^{|w|}$, for each of the τ PoK repetitions. In practice, the sent values are the auxiliary values Δw.
- The opened shares of $[\![v]\!]_I$: because v is uniformly-sampled, these shares are communication-free since we rely on the TCitH-GGM variant.
- The degree-d sharing $[\![\alpha]\!]$, for each of the ρ MPC repetitions of the τ PoK repetitions. Since $[\![\alpha]\!]_I$ can be recomputed by the verifier and since the α should be zero, the prover just needs to send $(d+1) - 1 - 1 = (d-1)$ shares.

1. The parties receive a sharing $[\![w]\!]$, with $\deg[\![w]\!] = 1$.
2. The parties get a uniformly-random degree-$(d-1)$ sharing $[\![v]\!]$ of a random value $v \in \mathbb{F}$ from O_H.
3. The parties receive random values $\gamma_1, \ldots, \gamma_m \in \mathbb{F}$ from O_R.
4. The parties locally compute
$$[\![\alpha]\!] = [\![v]\!] \cdot [\![0]\!] + \sum_{j=1}^{m} \gamma_j \cdot f_j([\![w]\!]) \ .$$
5. The parties open $[\![\alpha]\!]$ to publicly recompute α.
6. The parties output Accept if and only if $\alpha = 0$.

Protocol 1: Π_{PC} – Verification of polynomial constraints. O_R is an oracle which provides public trusted randomness to the parties: in a MPCitH setting, this randomness is provided by the verifier. O_H is an oracle which provides sharings of untrusted values named hints: in a MPCitH setting, these sharings are provided by the prover.

- The sibling paths in the GGM trees, together with the unopened seed commitments.

Moreover, the signature includes a 2λ-bit salt and a 2λ-bit commitment digest that correspond to the last verifier challenge (in the Fiat-Shamir heuristic). Therefore, the signature size when using the TCitH framework in the above setting is (in bits):

$$\textsc{Size}_{\textsc{TCitH}} = 4\lambda + \tau \cdot \left(\underbrace{|w| \cdot \log_2 |\mathbb{F}|}_{[\![w]\!]_I} + \underbrace{(d-1) \cdot \rho \cdot \log_2 |\mathbb{F}|}_{[\![\alpha]\!]} + \underbrace{\lambda \cdot \log_2 N}_{\text{GGM tree}} + 2\lambda \right) .$$

5.2 VOLE-in-the-Head Framework

The VOLEitH framework has been introduced at Crypto 2023 [12]. This work provides a way to compile any zero-knowledge protocol in the VOLE-hybrid model into a publicly verifiable protocol. While it has not been introduced as a MPCitH construction, it can yet be interpreted as such. Specifically, [21] shows that the VOLEitH framework can be described in the TCitH syntax. Indeed, this framework is similar to the TCitH framework with $\ell = 1$ and GGM trees, up to several details:

- The secret is stored at $P(\infty)$ when sharing, meaning that $e = \infty$. As a result, to share a value v, one samples a random value r and builds the Shamir's polynomial P as $P(X) := vX + r$. While multiplying two

Shamir's sharings when $e = \infty$ is similar than when $e \neq \infty$, the addition operation is slightly different: to add two Shamir's sharings $[\![a]\!]$ and $[\![b]\!]$ of degrees respectively d_1 and d_2 (such that $d_1 \leq d_2$) when $e = \infty$, the parties can compute the following d_2-degree sharing

$$\forall i,\ [\![a+b]\!]_i \leftarrow [\![a]\!]_i \cdot \omega_i^{d_2-d_1} + [\![b]\!]_i,$$

where ω_i is the evaluation point of the i^{th} party.

- The VOLEitH framework relies on a *large field embedding*: in the commitment phase, the prover commits τ N-sharings $[\![w]\!]^{(1)}, \ldots, [\![w]\!]^{(\tau)}$ of the witness w. In the basic TCitH framework, the prover runs τ MPC protocols in parallel, each of them on a different sharing $[\![w]\!]^{(j)}$. In the VOLEitH framework, these N sharings are merged to obtain a N^τ-sharing $[\![w]\!]^{(\phi)}$ living in a large field extension $\mathbb{K}$ such that the extension degree $[\mathbb{K} : \mathbb{F}]$ is ρ, then the prover runs a unique MPC protocol which takes as input this N^τ-sharing. More precisely, the i^{th} share of $[\![w]\!]^{(\phi)}$ is computed as

$$[\![w]\!]_i^{(\phi)} \leftarrow \phi\left([\![w]\!]_{i_1}^{(1)}, \ldots, [\![w]\!]_{i_\tau}^{(\tau)}\right)$$

  where $i_1, \ldots, i_\tau \in [1, N]$ satisfy $(i-1) = (i_1-1) + (i_2-1)\cdot N + \ldots + (i_\tau - 1)\cdot N^{\tau-1}$ and ϕ is an one-to-one ring homomorphism between $\mathbb{F}^\tau$ and $\mathbb{K}$ ($\rho \geq \tau$). If the sharings $[\![w]\!]^{(1)}, \ldots, [\![w]\!]^{(\tau)}$ encode the *same* witness w, then we get that $[\![w]\!]^{(\phi)}$ is a valid Shamir's secret sharing of w for which the evaluation point of the i^{th} party is $\phi(\omega_{i_1}, \ldots, \omega_{i_\tau})$ (with ω_i the i^{th} party evaluation point in the standard TCitH setting). The main advantage of this large field embedding is that the resulting soundness error of the proof system is $\frac{d}{N^\tau}$ instead of being $\left(\frac{d}{N}\right)^\tau$ (up to the false positive probability).

- The above optimisation requires that the prover ensures that the τ sharings encode the same value (without revealing this value). To ensure this property, the VOLEitH framework introduces an additional prover-verifier pair of rounds. After committing the input shares (including the hint sharings),

  - the prover commits τ additional uniformly-random sharings $[\![u]\!]^{(1)}, \ldots, [\![u]\!]^{(\tau)}$ of the *same* random value $u \in \mathbb{F}^{\rho+B}$, for $B \geq 0$ an additional parameter,
  - the verifier sends a challenge $(H_1|H_2) \in \mathbb{F}^{(\rho+B)\times(n+\rho)}$,
  - for all $j \in [1, \tau]$, the prover reveals the *digest sharing* $[\![\alpha']\!]^{(j)} := H_1[\![w]\!]^{(j)} + H_2[\![v]\!]^{(j)} + [\![u]\!]^{(j)}$, where $\alpha' \in \mathbb{F}^{\rho+B}$.

  The idea behind this process is that the prover computes the digests of all the plain values encoded in $[\![w]\!]^{(1)}, \ldots, [\![w]\!]^{(\tau)}$ (and in $[\![v]\!]^{(1)}, \ldots, [\![v]\!]^{(\tau)}$) and compares them. If $([\![w]\!]^{(i)}, [\![v]\!]^{(i)})$ and $([\![w]\!]^{(j)}, [\![v]\!]^{(j)})$ encode different values, then their digests $[\![\alpha']\!]^{(i)}$ and $[\![\alpha']\!]^{(j)}$ will differ with high probability. In practice, the parameters ρ and B are chosen such that the probability

that two different plain values lead to the same digest is negligible. We further note that taking $(H_1|H_2)$ uniformly at random gives the smallest probability but requires to perform matrix-vector multiplications. Other strategies are possible for $(H_1|H_2)$ such as relying on a polynomial-based hash: this increases a bit the collision probability (so one needs to increase B to compensate) but lightens the computation. This strategy is used in the FAEST signature scheme [13].

We use the VOLEitH framework with the same MPC protocol than with the TCitH framework, namely the MPC protocol Π_{PC} described in Protocol 1, which is equivalent to the QuickSilver VOLE-based protocol [39] in the VOLE setting. The publicly-known degree-1 sharing $[\![0]\!]$ in Protocol 1 when $e = \infty$ can be built as $[\![0]\!]_i = 1$ for all i.

To achieve a PoK with λ-bit security (i.e. $2^{-\lambda}$ soundness error), we take the field extension $\mathbb{K}$ of degree ρ such that $\frac{1}{|\mathbb{F}|^\rho} \leq 2^{-\lambda}$, the number τ of sharings $[\![w]\!]^{(j)}$ such that $\frac{d}{N^\tau} \leq 2^{-\lambda}$ and the additional parameter B such[3] that $B \cdot \log_2 |\mathbb{F}| \geq 16$ (the latter choice corresponds to the choice in the specification of FAEST [13]). Then we obtain a signature scheme by applying the Fiat-Shamir transform [23] as previously.

The proof transcript (*i.e.* the signature) includes:

- The opened shares $[\![w]\!]_I$ of the witness $w \in \mathbb{F}^{|w|}$. In practice, one sends the auxiliary values of the sub-sharings $[\![w]\!]^{(1)}, \ldots, [\![w]\!]^{(\tau)}$.
- The opened shares of $[\![v]\!]_I$. When v is uniformly-sampled, the shares are usually communication-free. However, we need τ sub-sharings of the same (uniformly-random) value v. While the first sharing is communication-free, the $\tau - 1$ others require an auxiliary value to ensure that all the sub-sharings encode the same value.
- The degree-d sharing $[\![\alpha]\!]$, for the single MPC execution. Since $[\![\alpha]\!]_I$ can be recomputed by the verifier and since the α should be zero, the prover just needs to send $(d+1) - 1 - 1 = d - 1$ shares.
- The sibling paths in the GGM trees, together with the unopened seed commitments.
- The opened shares $[\![u]\!]_I$. As for $[\![v]\!]_I$, since all the τ sub-sharings must encode the same random value u, only the first sharing is communication-free and the $\tau - 1$ others require an auxiliary value.
- The degree-1 sharings $[\![\alpha']\!]^{(1)}, \ldots, [\![\alpha']\!]^{(\tau)}$. Since the plaintext value α' is the same for all these sharings and since $[\![\alpha]\!]_I^{(j)}$ can be recomputed by the verifier for all j, sending all these sharings costs only $(\rho + B)$ field elements.

[3] As explained previously, the parameter B aims to compensate the security loss due to the use of a polynomial-based hash. Such a hash consists in evaluating in a large domain the polynomial which has the hashed values as coefficients. Thanks to the Schwartz-Zippel lemma, we get that the security loss is of a factor $n + \rho$ (which is the length of the hashed vector). By taking $B \cdot \log_2 |\mathbb{F}| \geq 16$ as in the specification of FAEST, we can securely hash vectors of length at most 2^{16}.

Moreover, the signature includes a 2λ-bit salt and a 2λ-bit commitment digest that correspond to the last verifier challenge (in the Fiat-Shamir heuristic). Therefore, the signature size when using the VOLEitH framework is (in bits):

$$\begin{aligned}\textsc{Size}_{\text{VOLEitH}} = 4\lambda \\ + \tau \cdot \left(\underbrace{|w| \cdot \log_2 |\mathbb{F}|}_{[\![w]\!]_I} + \underbrace{\lambda \cdot \log_2 N}_{\text{GGM tree}} + 2\lambda \right) + \underbrace{(d-1)\rho \cdot \log_2 |\mathbb{F}|}_{[\![\alpha]\!]} \\ + (\tau - 1) \cdot \left(\underbrace{\rho \cdot \log_2 |\mathbb{F}|}_{[\![v]\!]_I} + \underbrace{(\rho + B) \log_2 |\mathbb{F}|}_{[\![u]\!]_I} \right) + \underbrace{(\rho + B) \cdot \log_2 |\mathbb{F}|}_{[\![\alpha']\!]}.\end{aligned}$$

6 New Signatures Based on $\mathsf{RSD_s}$ and MinRank

In this section, we propose new signature schemes based on the rank syndrome decoding problem and on the MinRank problem. To proceed, we rely on the TCitH and VOLEitH frameworks to obtain non-interactive zero-knowledge proofs of knowledge for these two problems using the new Dual Support Decomposition model described in Section 4.

6.1 New Signatures Based on $\mathsf{RSD_s}$

The TCitH and VOLEitH frameworks enable us to prove the knowledge of a witness that satisfies some polynomial constraints. In order to get a signature scheme based on the rank syndrome decoding problem, one just needs to exhibit the polynomial constraints which is satisfied by a rank syndrome decoding solution. As shown in Section 4.1, solving an $\mathsf{RSD_s}$ instance for $\boldsymbol{y}$ and $\boldsymbol{H}$ is equivalent to finding $\boldsymbol{s} = (1, x_2, \ldots, x_r)$ where $x_i \in \mathbb{F}_{q^m}$ for $i \in \{2, \ldots, r\}$ and $\boldsymbol{C} \in \mathbb{F}_q^{r \times (n-r)}$ such that

$$\boldsymbol{x}\boldsymbol{H}^T - \boldsymbol{y} = \boldsymbol{0} \quad \text{with} \quad \boldsymbol{x} := \boldsymbol{s} \cdot \begin{pmatrix} \boldsymbol{I_r} & \boldsymbol{C} \end{pmatrix} \tag{3}$$

Equation 3 directly gives degree-2 polynomial constraints into the coefficients of $\boldsymbol{s}$ and $\boldsymbol{C}$.

The signature size using the TCitH framework is (in bits):

$$\begin{aligned}\textsc{Size}_{\text{TCitH}} = 4\lambda + \tau \cdot \left(\underbrace{[(r-1)m + r(n-r)] \cdot \log_2 q}_{[\![\boldsymbol{s}]\!]_I, [\![\boldsymbol{C}]\!]_I} \right. \\ \left. + \underbrace{(d-1) \cdot \rho \cdot \log_2 q}_{[\![\alpha]\!]} + \underbrace{\lambda \cdot \log_2 N}_{\text{GGM tree}} + 2\lambda \right),\end{aligned}$$

while the signature size using the VOLEitH framework is (in bits):

$$\begin{aligned}\textsc{Size}_{\text{VOLEitH}} &= 4\lambda \\ &+ \tau \cdot \left(\underbrace{[(r-1)m + r(n-r)] \cdot \log_2 q}_{[\![\boldsymbol{s}]\!]_I, [\![\boldsymbol{C}]\!]_I} + \underbrace{\lambda \cdot \log_2 N}_{\text{GGM tree}} + 2\lambda \right) + \underbrace{(d-1)\rho \cdot \log_2 q}_{[\![\alpha]\!]} \\ &+ (\tau - 1) \cdot \left(\underbrace{\rho \cdot \log_2 q}_{[\![v]\!]_I} + \underbrace{(\rho + B) \log_2 q}_{[\![u]\!]_I} \right) + \underbrace{(\rho + B) \cdot \log_2 q}_{[\![\alpha']\!]} .\end{aligned}$$

We present in Table 13 the sizes obtained for the signature scheme.

Comparison. We provide in Table 14 a comparison of our scheme with previous works. We include in the comparison only short parameters, i.e, with $N = 256$ for MPCitH based signatures, and $N = 32$ for [15]. We include [37] and [38] applied to the rank metric. These two schemes obtain around 30 kB. This size is halved by [20] and [15]. Finally, [19] managed to reduce it below 6 kB, and this work manages to get it below 4 kB.

Resilience Property. One should note that the scheme is highly resilient to attacks on $\mathsf{RSD_s}$: if we were to take the set of parameters for $\mathsf{RSD_s}$ corresponding to NIST III, applied to the proof of knowledge for NIST I, i.e, a security of $\lambda = 192$ for $\mathsf{RSD_s}$ and $\lambda = 128$ for the protocol, we have an increase of only 0.4 kB for $N = 512$ and 0.3 kB for $N = 2048$. This could allow to take a large margin of security for the parameters, while still being competitive.

6.2 New Signatures Based on MinRank

The TCitH and VOLEitH frameworks enable us to prove the knowledge of a witness that satisfies some polynomial constraints. In order to get a signature scheme based on the rank syndrome decoding problem, one just needs to exhibit the polynomial constraints which is satisfied by a MinRank solution. As shown in Section 4.2, solving a MinRank problem for matrices $\boldsymbol{M}, \boldsymbol{M}_1, \ldots, \boldsymbol{M}_k$ is equivalent in finding $\boldsymbol{S'} \in \mathbb{F}_q^{(m-r)\times r}$ and $\boldsymbol{C} \in \mathbb{F}_q^{r\times n}$ such that

$$[\rho(\boldsymbol{E}) - \rho(\boldsymbol{M})] \cdot \boldsymbol{H}^T = \boldsymbol{0} \quad \text{with} \quad \boldsymbol{E} := \begin{pmatrix} \boldsymbol{I_r} \\ \boldsymbol{S'} \end{pmatrix} \cdot \boldsymbol{C}, \tag{4}$$

where $\boldsymbol{H}$ is the parity-check matrix of the linear code defined by the generator matrix

$$\begin{pmatrix} \rho(\boldsymbol{M}_1) \\ \vdots \\ \rho(\boldsymbol{M}_k) \end{pmatrix}.$$

Security	RSDs Parameters	Trade-off	N	Framework	τ	ρ	Size
NIST I	$q = 2$ $m = 53$ $n = 53$ $k = 45$ $r = 4$	Fast	32	TCitH	32	3	5 736 B
				VOLEitH	26	128	5 016 B
		Short	256	TCitH	19	3	4 357 B
				VOLEitH	16	128	3 880 B
		Shorter	512	TCitH	16	3	3 940 B
				VOLEitH	15	128	3 882 B
		Shortest	2048	TCitH	13	3	3 636 B
				VOLEitH	12	128	3 503 B
NIST III	$q = 2$ $m = 79$ $n = 75$ $k = 67$ $r = 4$	Fast	32	TCitH	48	3	12 756 B
				VOLEitH	39	192	11 140 B
		Short	256	TCitH	28	3	9 517 B
				VOLEitH	24	192	8 621 B
		Shorter	512	TCitH	24	3	8 754 B
				VOLEitH	22	192	8 439 B
		Shortest	2048	TCitH	20	3	8 279 B
				VOLEitH	18	192	7 787 B
NIST V	$q = 2$ $m = 97$ $n = 95$ $k = 87$ $r = 4$	Fast	32	TCitH	64	3	22 096 B
				VOLEitH	52	256	19 468 B
		Short	256	TCitH	37	3	16 408 B
				VOLEitH	32	256	15 102 B
		Shorter	512	TCitH	32	3	15 240 B
				VOLEitH	29	256	14 627 B
		Shortest	2048	TCitH	26	3	14 083 B
				VOLEitH	24	256	13 663 B

Table 13: Parameters and resulting sizes for the new signature scheme based on $\mathsf{RSD}_{\mathrm{s}}$.

Equation (4) directly gives degree-2 polynomial constraints into the coefficients of $\boldsymbol{S'}$ and $\boldsymbol{C}$.

RSD Parameters	Scheme	N	M	τ	η	ρ	Signature Size
$q = 2$ $m = 31$ $n = 33$ $k = 15$ $r = 10$	[37]	-	-	219	-	-	33 886 B
	[38]	-	-	219	-	-	28 794 B
	[20]	32	389	28	-	-	14 792 B
	[15]	32	389	28	-	-	12 816 B
	[19] RD	256	-	21	24	-	8 990 B
	[19] LP and [4] ($\mathsf{RSD_s}$)	256	-	20	1	-	5 956 B
$q = 2, m = 53, n = 53$	Our scheme (TCitH)	256	-	19	-	3	4 357 B
$k = 45, r = 4$	Our scheme (VOLEitH)	256	-	16	-	128	3 880 B

Table 14: Comparison of the signatures relying on RSD, restricting to the schemes using the Fiat-Shamir transform.

The signature size using the TCitH framework is (in bits):

$$\textsc{Size}_{\textsc{TCitH}} = 4\lambda + \tau \cdot \left(\underbrace{[r(m-r)+rn] \cdot \log_2 q}_{[\![\boldsymbol{S'}]\!]_I, [\![\boldsymbol{C}]\!]_I} + \underbrace{(d-1) \cdot \rho \cdot \log_2 q}_{[\![\alpha]\!]} + \underbrace{\lambda \cdot \log_2 N}_{\text{GGM tree}} + 2\lambda \right),$$

while the signature size using the VOLEitH framework is (in bits):

$$\begin{aligned}\textsc{Size}_{\textsc{VOLEitH}} = 4\lambda &+ \tau \cdot \left(\underbrace{[r(m-r)+rn] \cdot \log_2 q}_{[\![\boldsymbol{S'}]\!]_I, [\![\boldsymbol{C}]\!]_I} + \underbrace{\lambda \cdot \log_2 N}_{\text{GGM tree}} + 2\lambda \right) + \underbrace{(d-1)\rho \cdot \log_2 q}_{[\![\alpha]\!]} \\ &+ (\tau - 1) \cdot \left(\underbrace{\rho \cdot \log_2 q}_{[\![v]\!]_I} + \underbrace{(\rho + B) \log_2 q}_{[\![u]\!]_I} \right) + \underbrace{(\rho + B) \cdot \log_2 q}_{[\![\alpha']\!]}.\end{aligned}$$

We present in Table 15 the sizes obtained for the signature scheme.

Comparison. We provide in Table 16 a comparison of our scheme with previous works. We include in the comparison only short parameters, i.e, with $N = 256$ for MPCitH based signatures, and $N = 32$ for [15]. For the MinRank parameters, we will compare to the parameters $q = 16, m = 16, n = 16, k = 142, r = 4$. Historically, the first schemes from [17], [35], and [14] obtained no less than 26 kB. Then, the technique from [15] applied to MinRank managed to

Security	MinRank Parameters	Trade-off	N	Framework	τ	ρ	Size
NIST I	$q = 2$ $m = 43$ $n = 43$ $k = 1520$ $r = 4$	Fast	32	TCitH	32	130	5 512 B
				VOLEitH	26	128	4 928 B
		Short	256	TCitH	19	128	4 219 B
				VOLEitH	16	128	3 826 B
		Shorter	512	TCitH	16	135	3 838 B
				VOLEitH	15	128	3 831 B
		Shortest	2048	TCitH	13	132	3 548 B
				VOLEitH	12	128	3 462 B
NIST III	$q = 2$ $m = 60$ $n = 60$ $k = 3135$ $r = 4$	Fast	32	TCitH	48	195	12 162 B
				VOLEitH	39	192	10 862 B
		Short	256	TCitH	28	192	9 160 B
				VOLEitH	24	192	8 450 B
		Shorter	512	TCitH	24	198	8 466 B
				VOLEitH	22	192	8 282 B
		Shortest	2048	TCitH	20	198	8 039 B
				VOLEitH	18	192	7 658 B
NIST V	$q = 2$ $m = 75$ $n = 75$ $k = 5040$ $r = 4$	Fast	32	TCitH	64	260	21 280 B
				VOLEitH	52	256	19 006 B
		Short	256	TCitH	37	256	15 917 B
				VOLEitH	32	256	14 818 B
		Shorter	512	TCitH	32	261	14 836 B
				VOLEitH	29	256	14 369 B
		Shortest	2048	TCitH	26	264	13 764 B
				VOLEitH	24	256	13 450 B

Table 15: Parameters and resulting sizes for the new signature scheme based on MinRank.

obtain a size with less than half this size, and [2] reduced it even lower to 7 kB. The recent work from [19] reduces it below 6 kB, and the submissions to the NIST additional signature project MIRA and MiRitH obtains sizes below 6 kB as well. Finally, this work manages to reduces it even further below 4 kB.

Resilience property. One should note that the scheme is highly resilient to attacks on MinRank, as for $\mathsf{RSD}_{\mathrm{s}}$: if we were to take the set of parameters for MinRank corresponding to NIST III, applied to the proof of knowledge for NIST I, i.e, a security of $\lambda = 192$ for MinRank and $\lambda = 128$ for the protocol, we have an increase of only 0.4 kB for $N = 512$ and 0.3 kB for $N = 2048$. This could allow to take a large margin of security for the parameters, while still being

MinRank Parameters	Scheme	N	M	τ	η	ρ	Signature Size
$q = 16$ $m = 16$ $n = 16$ $k = 142$ $r = 4$	[17]	-	-	219	-	-	28 575 B
	[35]	-	-	128	-	-	28 128 B
	[14]	-	256	128	-	-	26 405 B
	[15]	32	389	28	-	-	10 937 B
	[2]	256	-	18	-	-	7 422 B
	[19] RD	256	-	19	9	-	7 122 B
$q = 16, m = 16, n = 16$ $k = 120, r = 5$	[19] LP and MIRA [5]	256	-	18	1	-	5 640 B
$q = 16, m = 15, n = 15$ $k = 78, r = 6$	MiRitH [1]	256	-	19	9	-	5 673 B
$q = 2, m = 43, n = 43$ $k = 1520, r = 4$	Our scheme (TCitH) Our scheme (VOLEitH)	256 256	- -	19 16	- -	130 128	4 219 B 3 826 B

Table 16: Comparison of the signatures relying on MinRank, restricting to the schemes using the Fiat-Shamir transform.

competitive.

6.3 Additional MPCitH Optimisations

New generic optimizations for MPCitH-based schemes relying on GGM trees have been proposed in a recent work [11]. The improvements are threefold:

1. Instead of considering τ independent GGM trees of N leaves in parallel, the authors propose to rely on a unique large GGM tree of $\tau \cdot N$ leaves where the i^{th} share of the e^{th} PoK repetition is associated to the $(e \cdot N + i)^{\text{th}}$ leaf of the large GGM tree. As explained in [11], "opening all but τ leaves of the big tree is more efficient than opening all but one leaf in each of the τ smaller trees, because with high probability some of the active paths in the tree will merge relatively close to the leaves, which reduces the number of internal nodes that need to be revealed."

2. The authors further propose to improve the previous approach using the principle of *grinding*. When the last Fiat-Shamir challenge is such that the number of revealed nodes in the revealed sibling paths exceed a threshold T_{open}, the signer rejects the challenge and recompute the hash with an incremented counter. This process is done until the number of revealed nodes is $\leq T_{\text{open}}$. For example, if we consider $N = 256$ and $\tau = 16$, the number of revealed nodes is smaller than (or equal to) $T_{\text{open}} := 110$ with probability ≈ 0.2. The selected value of T_{open} induces a rejection probability $p_{\text{rej}} = 1 - 1/\theta$, for some $\theta \in (0, \infty)$, and the signer hence needs to perform an average of θ hash computations for the challenge (instead of 1). While this strategy decreases the challenge space by a factor θ, it

does not change the average number of hashes that must be computed to succeed an attack (since the latter is multiplied by θ). As noticed by the authors of [11], this strategy can be thought of as loosing $\log_2 \theta$ bit of security (because of a smaller challenge space) which are regained thanks to a proof-of-work (performing an average of θ hash computations before getting a valid challenge).

3. Finally, [11] proposes to add another explicit proof-of-work to the Fiat-Shamir hash computation of the last challenge. The signer must get a hash digest for which the w last bits are zero, for w a parameter of the scheme. The same counter as for the previous improvement is used as a nonce in this hash and increased until the w-zeros property is satisfied. This strategy increases the cost of hashing the last challenge by a factor 2^w and hence increases the security of w bits. This thus allows to take smaller parameters (N, τ) for the large tree, namely parameters achieving $\lambda - w$ bits of security instead of λ.

While the authors of [11] focus on VOLEitH, the same optimisations also apply to TCitH. In summary, for a given w, one picks parameters (N, τ) ensuring $\lambda - w$ bits of security. Then fixing T_{open} for these (N, τ) yields a rejection probability $p_{\text{rej}} = 1 - 1/\theta$. The gain in size comes from the smaller parameters (N, τ) on the one hand, and the smaller sibling paths (of size $\leq T_{\text{open}}$ instead of $\approx \tau \log_2 N$) on the other hand. This gain in size is traded for an increased number of Fiat-Shamir hash attempts ($\theta \cdot 2^w$ on average instead of 1).

We can apply these optimisations to our new signature schemes, which enables us to save a few hundred bytes in the signature sizes. As an illustration, we apply the optimisations on our instances with $N \in \{256, 2048\}$ for the first security level. The results are given in Table 17.

	N	Framework	T_{open}	w	τ	Signature
Rank SD	256	TCitH	122	9	17	3 619 B
		VOLEitH	110	8	16	3 592 B
	2048	TCitH	114	8	12	3 035 B
		VOLEitH	102	7	11	2 912 B
MinRank	256	TCitH	122	9	17	3 561 B
		VOLEitH	110	8	16	3 538 B
	2048	TCitH	114	8	12	2 994 B
		VOLEitH	102	7	11	2 875 B

Table 17: Signature sizes of our schemes when using recent MPCitH optimisations of [11], for 128-bit security level.

We rely on the same range of parameters as for the VOLEitH-based signatures proposed in [11]. For the TCitH instances we need to take $\left(\frac{2}{N}\right)^\tau \leq 2^{-(\lambda-w)}$ so we select w such that $\tau := (\lambda - w)/(\log_2(N) - 1)$ is an integer. This result

in increasing w of 1 compared to the VOLEitH instances which we compensate with larger values of T_{open} to get similar proof-of-work overheads for both types of instances. Let us also note that for the VOLEitH instances, some repetitions are actually with $N/2$ leaves instead of N. Namely, one takes $\tau = \tau_1 + \tau_2$ so that $N^{\tau_1} \cdot \left(\frac{N}{2}\right)^{\tau_2} = 2^{\lambda - w}$ and has τ_1 repetitions with N leaves and τ_2 with $N/2$.

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